SPECTRAL THEORY OF CLOSED DISTRIBUTIVE OPERATORS.

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1. **Introduction.** Let X be a complex Banach space, and T a closed distributive operator with domain and range both in X. Let $[X]$ denote the set of all continuous distributive (bounded linear) operators which map X into itself. This set $[X]$ is a ring, and in fact an algebra. Next suppose that we have an algebra, related in some way to T, whose elements are complexvalued functions of the complex variable λ, and that we are able to define a mapping of the algebra of functions into the algebra $[X]$ in such a way that we have a homomorphism. If $f(\lambda)$ is the complex function, we shall denote the corresponding member of $[X]$ by $f(T)$. The fact that we have a homomorphism is then expressed by the equations

$$(af+bg)(T) = af(T)+bg(T)\ , \quad (fg)(T) = f(T)g(T)\ . \tag{1.1}$$

When such a homomorphism has been established we shall speak of the application of formulas (1.1) and other related results flowing out of the homomorphism as an operational calculus for T.

Some years ago (Dunford, [1 and 2]; Taylor [2])[1] an operational calculus was developed for bounded operators T by choosing as the algebra of functions the set of functions $f(\lambda)$, each singlevalued and analytic in some open set containing the spectrum $\sigma(T)$ of T. The homomorphism was established by defining

$$f(T) = \frac{1}{2\pi i}\int f(\lambda)(\lambda I-T)^{-1}d\lambda\ , \tag{1.2}$$

the integral being extended over the boundary of a suitable bounded domain containing $\sigma(T)$. Dunford [1] used the resulting operational calculus to develop syste-

[1] All references are to the bibliography at the end of the paper.

matically the spectral theory of a bounded linear operator; he obtained generalizations of the principal features of the corresponding theory for finite matrices.

In the present paper we show how to develop an operational calculus which is effective for any closed distributive operator whose spectrum does not cover the entire plane. Furthermore, this development is made in such a way that the operational calculus based on (1.2) for a bounded operator is obtained as part of the general theory for a closed operator. Likewise we develop the spectral theory of the closed operator. We have thus a uniform theory which includes that developed by Dunford, and one in which the role of boundedness at particular places is made clear.

The difficulty with (1.2) in the case when T is not bounded lies principally in the fact that $\sigma(T)$ need not be a bounded point set in the complex plane. We get around this difficulty by considering functions $f(\lambda)$ which are regular at $\lambda = \infty$ as well as being analytic in some open set containing $\sigma(T)$. For such functions we have

$$f(\xi) = f(\infty) + \frac{1}{2\pi i} \int \frac{f(\lambda)}{\lambda - \xi} d\lambda ,$$

the integration being over the boundary of a suitable domain which contains $\lambda = \xi$, $\sigma(T)$, and a neighborhood of $\lambda = \infty$. The homomorphism is then established by the formula

$$f(T) = f(\infty)I + \frac{1}{2\pi i} \int f(\lambda)(\lambda I - T)^{-1} d\lambda . \tag{1.3}$$

Historical references to the use of formula (1.2) in the study of finite matrices may be found in the addresses delivered by Dunford [2] and the author (Taylor [2]). During the preparation of the present paper, while the author was in England, his attention was called by H. Hamburger to what are perhaps the earliest anticipations of the formula. Frobenius [1], considering a matrix A, refers to $f(A)$ (though not explicitly in the form (1.2)) as the residue of $(\lambda I - A)^{-1} f(\lambda)$ with respect to the roots of the characteristic equation. Frobenius ascribes an important share of credit for this idea to Stickelberger, whose dissertation, "Zur Theorie der linearen Differentialgleichungen", appeared in Leipzig in 1881. The same idea for $f(A)$ occurs in Bromwich [1]. Frobenius and Bromwich both refer to related work by Sylvester and Bucheim.

We shall mention briefly some features of the paper.

In § 3 we introduce the idea of a Cauchy domain, and prove a theorem (Theorem 3.3) about the existence of Cauchy domains meeting certain specifications. This theorem enables us to avoid an awkward construction based on the Heine-Borel theorem. Its use in § 4 makes our development of the operational calculus much simpler than Dunford's original exposition of the corresponding material in the case of a bounded operator.

In § 5 we consider the resolvent of the resolvent of T, for a fixed value of the parameter in the latter resolvent. This device permits us to transform the study of a closed distributive operator into the study of a bounded linear operator, in somewhat the same way that certain types of differential equations are transformed into integral equations. By means of this device we could obtain the operational calculus and spectral theory for closed operators from the already known theory for bounded operators. We prefer not to do this, however, since it is better, methodologically, to develop the theory as generally and comprehensively as possible from the beginning. The device of studying the resolvent of the resolvent is accompanied by inversion in the complex plane. On this account the device is useful in studying the resolvent of T in the neighborhood of the point $\lambda = \infty$.

In § 6 we study polynomials in T, and fit the results into the operational calculus. The theorems in this section make it possible to carry through many of the arguments used by Dunford for bounded operators, although with different justification. Inverses of polynomial operators are also considered, in § 7. All of this is entirely new, since the problems of § 6, 7 do not arise as distinct problems when $T \in [X]$.

In § 8 we introduce the notion of the extended spectrum of T; it coincides with $\sigma(T)$ if T is bounded over all of X, and otherwise consists of $\sigma(T)$ together with the point $\lambda = \infty$. With the aid of this notion we extend the concept of spectral set and discuss the corresponding projections. The recently published Colloquium Lectures of E. Hille appeared during the final stages of preparation of the present paper. Hille's book contains theorems about bounded spectral sets and their associated projections for closed operators; these theorems (Hille [1], pp. 110–112.) overlap to some extent with our Theorems 8.2 and 10.1.

The appropriate formulations of Theorems 9.1, 11.1, 12.1 (the spectral mapping theorem, the 'Sylvester' theorem, and the minimal equation theorem) all require proper recognition of the spectral character of the point $\lambda = \infty$.

Some of our theorems furnish new criteria for T to belong to $[X]$: Theorems 8.5, 9.2, and 11.4. Finally we mention that if T is not in $[X]$ there can be no minimal polynomial: we can never have $P(T) = 0$ (Theorem 12.2).

2. **Definitions and notation.** Let X be a complex Banach space having at least two elements. We deal with operators which map a specified subset of X into all or part of X. The set $\mathfrak{D}(T)$ on which an operator T is defined is called the domain

of T; the set $\mathfrak{R}(T)$ onto which $\mathfrak{D}(T)$ is mapped by T is called the range of T. The operator is called distributive when $\mathfrak{D}(T)$ is a linear subspace of X and $T(ax+by) = aTx+bTy$ for every linear combination $ax+by$ in $\mathfrak{D}(T)$. The operator T is called closed if, whenever $\{x_n\}$ is a sequence such that $x_n \in \mathfrak{D}(T)$, $x_n \to x, Tx_n \to y$, then it is true that $x \in \mathfrak{D}(T)$ and $Tx = y$. Throughout the paper the symbol T will denote a closed and distributive operator; this will be understood without explicit mention of these properties at each occurrence of T.

When $\mathfrak{D}(T) = X$ the operator T is continuous, or bounded (Banach [1], p. 41). The class of such operators becomes a Banach space if we define the norm of T as

$$||T|| = \sup_{||x||=1} ||Tx|| .$$

We shall denote this Banach space of operators by $[X]$.

For a general T we classify all complex numbers λ into two sets: (1) the resolvent set $\varrho(T)$, consisting of all λ such that $\lambda I-T$ sets up a $1-1$ correspondence between $\mathfrak{D}(T)$ and X (in which case the inverse $(\lambda I-T)^{-1}$ belongs to $[X]$), and (2) the spectrum $\sigma(T)$, consisting of all λ not in $\varrho(T)$. If $\lambda \in \varrho(T)$ we denote $(\lambda I-T)^{-1}$ by $R_\lambda(T)$ and call it the resolvent of T. The spectrum is divided into three parts: (1) the point spectrum $p(T)$, consisting of those λ for which $(\lambda I-T)^{-1}$ does not exist, (2) the continuous spectrum $c(T)$, consisting of those λ not in $\varrho(T)$ or $p(T)$ for which $\mathfrak{R}(\lambda I-T)$ is dense in X, and (3) the residual spectrum $r(T)$, consisting of those λ not in $\varrho(T)$, $p(T)$ or $c(T)$.

It is known that $\varrho(T)$ is open and hence that $\sigma(T)$ is closed. When $T \in [X]$ it is known that $\sigma(T)$ is bounded and nonempty (Taylor [1], p. 74). In the general case it is possible for $\sigma(T)$ to be empty, or unbounded, or even for it to cover the entire plane.

When $\varrho(T)$ is not empty it is known that $R_\lambda(T)$ is analytic in $\varrho(T)$ as a function with values in $[X]$. It satisfies the functional equation

$$R_\lambda - R_\mu = (\mu-\lambda) R_\lambda R_\mu . \tag{2.1}$$

When $T \in [X]$ and $|\lambda| > ||T||$ we have $\lambda \in \varrho(T)$ and

$$R_\lambda(T) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} . \tag{2.2}$$

3. **Topological considerations in the plane.** We shall have to deal with Cauchy's integral theorem for analytic functions of a complex variable, the function values lying in a complex Banach space. The functions with which we shall be

concerned will be singlevalued, but the domains on which they are defined may (and often will) consist of more than one component (a component of an open set is a maximal connected subset).

Definition. *A set D in the complex plane is called a Cauchy domain if the following conditions are satisfied: (a) D is open; (b) D has a finite number of components, the closures of any two of which are disjoint; (c) the boundary of D is composed of a finite positive number of closed rectifiable Jordan curves, no two of which intersect.*

A component of a Cauchy domain is a Cauchy domain. If the Cauchy domain D is unbounded, it has just one unbounded component; this component contains a neighborhood of the point at infinity (i. e. all points outside a sufficiently large circle), and has as its boundary a finite number of closed rectifiable Jordan curves, nonintersecting and no one inside any other. The nature of a bounded Cauchy domain with a single component will be clear to the reader after a moment's thought.

If D is a Cauchy domain and C is one of the curves composing its boundary, we follow the usual practice in defining the positive orientation of C as part of the boundary $B(D)$ of D. We denote the positively oriented boundary of D by $+B(D)$; when it is given the reverse orientation we denote it by $-B(D)$.

We denote the closure of D by $\bar{D}$ and the complement of D by $C(D)$. The two following theorems will be used later.

Theorem 3.1. *Let D_1 and D_2 be Cauchy domains, and suppose that $\bar{D}_1 \subset D_2$. Let $D = D_2 - \bar{D}_1$. Then D is a Cauchy domain, and $+B(D)$ consists of $+B(D_2)$ and $-B(D_1)$. The domain D is bounded if both D_1 and D_2 are unbounded, or if D_2 is bounded.*

Theorem 3.2. *Let D_1 be a Cauchy domain, and let $D_2 = C(\bar{D}_1)$. Then D_2 is a Cauchy domain, and $+B(D_2) = -B(D_1)$.*

We omit the proofs.

We often need to know of the existence of Cauchy domains lying in prescribed sets and containing other prescribed sets. The following theorem is designed to meet such needs.

Theorem 3.3. *Let F and Δ be point sets in the plane. Let F be closed, Δ open, and $F \subset \Delta$. Suppose that $B(\Delta)$ is nonempty and bounded. Then there exists a Cauchy domain D such that:* (1) $F \subset D$, (2) $\bar{D} \subset \Delta$, (3) *the curves forming $B(D)$ are polygons,* (4) *D is unbounded if Δ is unbounded.*

Proof: Since F is closed and $B(\Delta)$ is compact, there exists a positive δ such that the distance from a point of F to a point of $B(\Delta)$ is never less than δ. Let us cover the plane by a honeycomb network of nonoverlapping congruent hexagons of side $\delta/4$. The boundary plus the interior of one of the hexagons will be called a cell. A cell cannot contain a point of F and also a point of $C(\Delta)$. For such a cell would contain a point of $B(\Delta)$, and there would then be a point of F and a point of $B(\Delta)$ at a distance at most $\delta/2$ apart.

Let S be the sum of all the cells which meet $C(\Delta)$. Then $F \subset C(S)$, for $FS = 0$ (the empty set). Also, $C(\Delta) \subset S$, and therefore $C(S) \subset \Delta$. The set S is closed and nonempty. Since $B(\Delta)$ is bounded, S is bounded if Δ is unbounded; also S is unbounded if Δ is bounded. In either case $B(S)$ is bounded. Finally, $B(S) \subset \Delta$. For, if a point is in $B(S)$, it is on the boundary of at least one cell which does not meet $C(\Delta)$, so that the point in question must be in Δ.

Let us now define $D = C(S)$. Then D is open, and $\overline{D} = D + B(D) = D + B(S) \subset \Delta$. That D has properties (1), (2) and (4) is now clear. It remains to prove that D is a Cauchy domain with property (3).

Let V be a corner point of a cell. Suppose that $V \in B(D)$. Since $B(D) = B(S)$, V is a corner point of a cell belonging to S. Now just three cells meet at V, and at least one of these cells is not in S. Hence either one or two of the three cells meeting at V belong to S. In either case exactly two of the three cell edges meeting at V belong to $B(S)$. Since $B(S)$ is bounded and composed of cell edges, it follows that $B(S)$ consists of a finite number of disjoint simple closed polygons.

The set D can have only a finite number of components. For, it can have at most one unbounded component; the sum of all the bounded components is bounded; and each component must contain the interior of at least one cell. If a point P is on the boundary of a component of D, it is a boundary point of a cell belonging to S. It is also a boundary point of either one or two cells whose interiors do not belong to S. But if two such cells are adjacent all the non-end points of their common edge are in D. Hence the point P lies on the boundary of just one component of D.

The proof of Theorem 3.3 is now complete. For the idea of using a network of hexagons, which makes the proof much simpler than by using a network of squares, I am indebted to Mr. William Gustin.

4. **The Operational Calculus.** If $f(\lambda)$ is a complex function of the complex variable λ, we denote by $\Delta(f)$ the set on which f is defined. We shall always assume that $\Delta(f)$ is a nonempty open set, not necessarily connected, and that f is singlevalued and analytic on $\Delta(f)$.

Definition. *By the class $\mathcal{G}(T)$ we mean the family of all analytic functions f which* *are such that*: (1) $\sigma(T) \subset \Delta(f)$, (2) $\Delta(f)$ *contains a neighborhood of* $\lambda = \infty$ *and* f *is regular at* $\lambda = \infty$.

When $f \in \mathcal{G}(T)$ we denote the limiting value of $f(\lambda)$ as $\lambda \to \infty$ by $f(\infty)$.

When T is such that $\sigma(T)$ covers the entire plane we are unable to carry out the developments of the paper. *Hence we assume once and for all that* $\varrho(T)$ *is not empty in all subsequent parts of the paper.*

Theorem 4.1. *Suppose that* $f \in \mathcal{G}(T)$. *There exists an unbounded Cauchy domain* D *such that* $\sigma(T) \subset D$ *and* $\overline{D} \subset \Delta(f)$. *The integral*

$$\frac{1}{2\pi i}\int_{+B(D)} f(\lambda)R_\lambda(T)d\lambda \tag{4.1}$$

defines an element of $[X]$ *which is the same for any choice of* D *satisfying the above conditions.*

Proof: The existence of D follows from Theorem 3.3 by taking $F = \sigma(T)$ and choosing Δ to be the intersection of $\Delta(f)$ and the exterior of a sufficiently small circle with center in $\varrho(T)$ (this last precaution being necessary only in case $\Delta(f)$ is the entire plane).

To prove that the integral (4.1) is independent of D, suppose that D_1 and D_2 are two Cauchy domains of the type specified. Then their intersection D_1D_2 is open and unbounded, with a bounded, nonempty boundary; also, $\sigma(T) \subset D_1D_2$. Hence, by Theorem 3.3, there exists an unbounded Cauchy domain D_3 such that $\sigma(T) \subset D_3$ and $\overline{D}_3 \subset D_1D_2 \subset \Delta(f)$. Let $D = D_1 - \overline{D}_3$. The domain D is bounded, and

$$\frac{1}{2\pi i}\int_{+B(D)} f(\lambda)R_\lambda(T)d\lambda = \frac{1}{2\pi i}\int_{+B(D_1)} f(\lambda)R_\lambda(T)d\lambda - \frac{1}{2\pi i}\int_{+B(D_3)} f(\lambda)R_\lambda(T)d\lambda\,,$$

by Theorem 3.1. But $\overline{D} \subset \varrho(T)$; hence $f(\lambda)R_\lambda(T)$ is regular in $\overline{D}$, and the integral on the left of the equality is zero, by Cauchy's theorem. It follows that the integral over $+B(D_1)$ is equal to the integral over $+B(D_3)$. The same argument applies with D_2 in place of D_1, so that the proof is complete.

Definition. *When* $f \in \mathcal{G}(T)$ *we define*

$$f(T) = f(\infty)I + \frac{1}{2\pi i}\int_{+B(D)} f(\lambda)R_\lambda(T)d\lambda\,, \tag{4.2}$$

the integral here being the same as in (4.1) *and* I *being the identity operator.*

Definition. *The class* $\mathcal{G}_0(T)$ *is defined as the subset of all* $f \in \mathcal{G}(T)$ *such that* $f(\infty) = 0$.

Theorem 4.2. *If $f \in \mathcal{G}(T)$ and $f(\lambda) \equiv c$ (a constant) on $\Delta(f)$, we have $f(T) = cI$.*

Proof: The integral in (4.2) vanishes, for it is equal to

$$\frac{c}{2\pi i}\int_{+B(D)} R_\lambda(T)\,d\lambda = -\frac{c}{2\pi i}\int_{+B(D_1)} R_\lambda(T)\,d\lambda\,,$$

where $D_1 = C(\bar{D})$ is a bounded Cauchy domain (by Theorem 3.2). Since $\bar{D}_1 \subset \varrho(T)$ the integral vanishes.

When f and g belong to $\mathcal{G}(T)$ we define the functions $f+g$ and fg in the obvious way, taking $\Delta(f)\Delta(g)$ as their domain of definition. It is clear that $f+g$ and fg belong to $\mathcal{G}(T)$. The basic rules of the operational calculus are obtained in the next theorem.

Theorem 4.3. *Suppose that $f, g \in \mathcal{G}(T)$. Then*

(a) $$(f+g)(T) = f(T)+g(T)\,,$$

(b) $$(fg)(T) = f(T)g(T)\,.$$

Proof: The assertion (a) is obvious. To prove (b) we observe by Theorem 3.3 that there exist unbounded Cauchy domains D_1, D_2 such that $\sigma(T) \subset D_1$, $\bar{D}_1 \subset D_2$, $\bar{D}_2 \subset \Delta(f)\Delta(g)$. Then

$$f(T)-f(\infty)I = \frac{1}{2\pi i}\int_{+B(D_1)} f(\lambda)R_\lambda(T)\,d\lambda\,,$$

$$g(T)-g(\infty)I = \frac{1}{2\pi i}\int_{+B(D_2)} g(\mu)R_\mu(T)\,d\mu\,.$$

By using the functional equation (2.1) we readily find that the product $\big(f(T)-f(\infty)I\big)\big(g(T)-g(\infty)I\big)$ is given by the expression

$$\frac{1}{2\pi i}\int_{+B(D_1)} f(\lambda)R_\lambda(T)\,d\lambda\,\frac{1}{2\pi i}\int_{+B(D_2)}\frac{g(\mu)}{\mu-\lambda}\,d\mu$$

$$+\frac{1}{2\pi i}\int_{+B(D_2)} g(\mu)R_\mu(T)\,d\mu\,\frac{1}{2\pi i}\int_{+B(D_1)}\frac{f(\lambda)}{\lambda-\mu}\,d\lambda\,.$$

Now let C be a circle of large radius so chosen that $B(D_1)$ and $B(D_2)$ lie inside C. Let D be the part of D_2 that lies inside C. Then D is a bounded Cauchy domain, and $+B(D)$ consists of $+B(D_2)$ and the circle C oriented counterclockwise. Since $B(D_1) \subset D$ we have, if λ is on $B(D_1)$,

$$g(\lambda) = \frac{1}{2\pi i}\int_{+B(D)}\frac{g(\mu)}{\mu-\lambda}\,d\mu = \frac{1}{2\pi i}\int_{+B(D_2)}\frac{g(\mu)}{\mu-\lambda}\,d\mu + \frac{1}{2\pi i}\int_{C}\frac{g(\mu)}{\mu-\lambda}\,d\mu\,.$$

Upon letting the radius of C become infinite, the last integral is seen to have the value $g(\infty)$. A similar analysis yields the result

$$\frac{1}{2\pi i}\int_{+B(D_1)} \frac{f(\lambda)}{\lambda-\mu}\, d\lambda = -f(\infty)$$

when μ is on $B(D_2)$. Thus we have

$$\big(f(T)-f(\infty)I\big)\big(g(T)-g(\infty)I\big) =$$

$$\frac{1}{2\pi i}\int_{+B(D_1)} f(\lambda)\big(g(\lambda)-g(\infty)\big)R_\lambda(T)\,d\lambda - \frac{1}{2\pi i}\int_{+B(D_2)} g(\mu)f(\infty)R_\mu(T)\,d\mu =$$

$$(fg)(T)-f(\infty)g(\infty)I-g(\infty)\big(f(T)-f(\infty)I\big)-f(\infty)\big(g(T)-g(\infty)I\big)\,.$$

Formula (b) of the theorem now follows at once.

We shall now discuss the connection between the operational calculus based on the foregoing definitions and theorems and the operational calculus for bounded operators which was introduced by Dunford and the author several years ago.

Definition. *Suppose that $T \in [X]$, so that $\sigma(T)$ is bounded and nonempty. By $\mathcal{F}(T)$ we mean the class of analytic functions f such that $\sigma(T) \subset \Delta(f)$.*

We observe that if $f, g \in \mathcal{F}(T)$, $\Delta(f)\Delta(g)$ is nonempty, so that $f+g$, $fg \in \mathcal{F}(T)$. The class $\mathcal{G}(T)$ is a proper subset of $\mathcal{F}(T)$, for any integral function f, with $\Delta(f)$ the whole plane, belongs to $\mathcal{F}(T)$.

Definition. *When $T \in [X]$ and $f \in \mathcal{F}(T)$, we write*

$$f^*(T) = \frac{1}{2\pi i}\int_{+B(D)} f(\lambda)R_\lambda(T)\,d\lambda\,, \tag{4.3}$$

where D is a bounded Cauchy domain such that $\sigma(T) \subset D$ and $\bar{D} \subset \Delta(f)$.

Theorem 4.4. *Suppose that $T \in [X]$ and $f \in \mathcal{F}(T)$. Then $f^*(T)$, as defined in* (4.3), *is an element of $[X]$ which is independent of D. Suppose that $g \in \mathcal{G}(T)$ and that $f(\lambda) \equiv g(\lambda)$ on some open set containing $\sigma(T)$. Then $f^*(T) = g(T)$.*

Proof: We prove that $f^*(T)$ is independent of D by an argument like that used in proving Theorem 4.1. Let G be a bounded open set such that $\sigma(T) \subset G \subset \Delta(f)\Delta(g)$ and $f(\lambda) = g(\lambda)$ when $\lambda \in G$. By Theorem 3.3 we may assume that $\bar{D} \subset G$, so that the integral (4.2) is unchanged in value when we replace $f(\lambda)$ by $g(\lambda)$. Let C be a circle of radius r, so chosen that G lies inside C and that C and the region exterior to it lie in $\Delta(g)$. Let D_1 be the domain consisting of D and the exterior of C. Then D_1 is an unbounded Cauchy domain with $\sigma(T) \subset D_1$, $\bar{D}_1 \subset \Delta(g)$; further, $+B(D_1)$ consists of $+B(D)$ and C taken clockwise.

Hence

$$g(T)-g(\infty)I = f^*(T)+\frac{1}{2\pi i}\int_C g(\lambda)R_\lambda(T)\,d\lambda .$$

The proof will be complete when we have shown that the integral on the right is equal to $-g(\infty)I$. It is equal to

$$\frac{1}{2\pi i}\int_C \big(g(\lambda)-g(\infty)\big)R_\lambda(T)\,d\lambda+\frac{1}{2\pi i}\int_C g(\infty)R_\lambda(T)\,d\lambda . \tag{4.4}$$

Now it follows from (2.2) that

$$||R_\lambda(T)|| \leqq \frac{1}{|\lambda|-||T||}$$

when $|\lambda| > ||T||$. It is therefore easily seen that the first integral in (4.4) tends to zero as $r\to\infty$, and must therefore vanish, since it is independent of r. The second integral is found to have the value $-g(\infty)I$, by integrating the series (2.2) termwise (recall that C is oriented clockwise). Thus the proof is complete.

A function $g(\lambda)$ of the type described in Theorem 4.4 will be called a $\mathcal{G}(T)$ extension of $f(\lambda)$. If $g(\infty)=0$ we call the function a $\mathcal{G}_0(T)$ extension of $f(\lambda)$. It is evident upon a moment's consideration that any f in $\mathcal{F}(T)$ has an infinite number of $\mathcal{G}(T)$ extensions, among them an infinite number of $\mathcal{G}_0(T)$ extensions.

Theorem 4.5. *Suppose that $T\in[X]$ and that $f, g\in\mathcal{F}(T)$. Then*

(a) $(f+g)^*(T) = f^*(T)+g^*(T)$;

(b) $(fg)^*(T) = f^*(T)g^*(T)$;

(c) *if $f(\lambda) = \Sigma_0^\infty a_n(\lambda-\lambda_0)^n$ is a complex power series whose circle of convergence contains $\sigma(T)$ in its interior, we have $f^*(T) = \Sigma_0^\infty a_n(T-\lambda_0 I)^n$.*

Proof: Assertion (a) is obvious. We may prove (b) by an argument like that used in proving Theorem 4.3(b). Alternatively, (b) follows from Theorem 4.3(b) and Theorem 4.4, since there exist $\mathcal{G}_0(T)$ extensions of f and g. Assertion (c) is a slight extension of a theorem given by Dunford ([1], Theorem 2.8, p. 195). If $f(\lambda)$ is a polynomial the result (c) follows from Dunford's result; the proof depends essentially on the expansion (2.2). For the general case let D be the interior of a circle C lying inside the circle of convergence of $f(\lambda)$, and such that $\sigma(T)$ is inside C. Then

$$f^*(T) = \frac{1}{2\pi i}\int_C f(\lambda)R_\lambda(T)\,d\lambda = \sum_{n=0}^{\infty} a_n\left(\frac{1}{2\pi i}\int_C (\lambda-\lambda_0)^n R_\lambda(T)\,d\lambda\right).$$

But the integral under the summation sign is equal to $(T-\lambda_0 I)^n$, by the result for polynomials. This completes the proof.

This theorem furnished the basis for the operational calculus used by Dunford. The operator $f^*(T)$ was denoted by $f(T)$ in Dunford's work. We see from Theorem 4.4 and the remarks which follow it that when $T \in [X]$ the class of all operators $f^*(T)$, $f \in \mathfrak{F}(T)$, is the same as the class of all operators $g(T)$, $g \in \mathcal{G}(T)$ (or merely $g \in \mathcal{G}_0(T)$).

5. **The resolvent of a resolvent.** Let T be fixed, and let α be a fixed point of $\varrho(T)$. Define

$$A = (T-\alpha I)^{-1} = -R_\alpha(T)\ , \tag{5.1}$$

so that $A \in [X]$. The operator A defines a 1—1 mapping of X onto $\mathfrak{D}(T)$. We are going to study the relations between the resolvents of T and A, and between their spectra.

We readily verify that

$$TAx = \alpha Ax + x\ , \qquad x \in X\ , \tag{5.2}$$

$$ATx = \alpha Ax + x\ , \qquad x \in \mathfrak{D}(T)\ . \tag{5.3}$$

Lemma 1. *Suppose* $\mu \neq 0$, $x \in X$, $y = \mu x - Ax$. *Define* λ by $(\lambda-\alpha)\mu = 1$. *Then* $(T-\lambda I)(\mu x - y) = \mu^{-1} y$.

Proof: $Ax = \mu x - y$; therefore $\mu x - y \in \mathfrak{D}(T)$. By (5.2), $T(\mu x - y) = \alpha Ax + x$. Hence $T(\mu x - y) - \lambda(\mu x - y) = \alpha Ax + x - (\alpha\mu + 1)x + \lambda y$, as we see by noting that $\lambda\mu = \alpha\mu + 1$. Thus

$$(T-\lambda I)(\mu x - y) = \alpha(Ax - \mu x) + \lambda y = (\lambda - \alpha)y = \mu^{-1} y\ .$$

Lemma 2. *Suppose* $\lambda \neq \alpha$, $x \in \mathfrak{D}(T)$, $y = Tx - \lambda x$. *Define* μ *by* $(\lambda-\alpha)\mu = 1$. *Then* $(\mu I - A)x = \mu A y$.

Proof: By (5.3), $\alpha Ax + x = ATx = A(y + \lambda x)$. Hence $(\lambda-\alpha)Ax - x = -Ay$. Multiplication by μ gives $Ax - \mu x = -\mu A y$, which is equivalent to the result asserted in the lemma.

Lemma 3. *Suppose* $\lambda \neq \alpha$, $(\lambda-\alpha)\mu = 1$, *and that* $(T-\lambda I)^{-1}$ *exists. Then* $(\mu I - A)^{-1}$ *exists. The two inverses have the same domain of definition, and in this domain* $\mu^2(\mu I - A)^{-1} = \mu I + (T-\lambda I)^{-1}$.

Proof: Suppose $x \in X$ and $(\mu I - A)x = 0$. Now $\mu \neq 0$, and $\mu x \in \mathfrak{D}(T)$, since $Ax = \mu x$. Hence $x \in \mathfrak{D}(T)$. By Lemma 1 we have $(T-\lambda I)\mu x = 0$ the y of the lemma being 0 in this instance. Hence $(T-\lambda I)x = 0$ and $x = 0$ since $(T-\lambda I)^{-1}$ exists. Thus $(\mu I-A)^{-1}$ exists also.

Now the domains of $(\mu I-A)^{-1}$ and $(T-\lambda I)^{-1}$ are, respectively, the ranges of $\mu I-A$ and $T-\lambda I$. By lemma 1 we see that if $y \in \mathfrak{R}(\mu I-A)$, then $\mu^{-1}y$ and y are in $\mathfrak{R}(T-\lambda I)$. The Lemma also suggests that if $y \in \mathfrak{R}(T-\lambda I)$, then also $y \in \mathfrak{R}(\mu I-A)$, the solution of the equation $y = (\mu I-A)x$ being, conjecturally, $x = \mu^{-1}y+\mu^{-2}(T-\lambda I)^{-1}y$. If we justify this conjecture we shall have completed the proof of the theorem.

Suppose then that $y = (T-\lambda I)x_1$, for some x_1 in $\mathfrak{D}(T)$. Define $x = \mu^{-1}y+\mu^{-2}(T-\lambda I)^{-1}y$, that is, $x = \mu^{-1}y+\mu^{-2}x_1$, or $\mu^2 x-\mu y = x_1$. Now, by Lemma 2, $(\mu I-A)x_1 = \mu Ay$. Hence $\mu^2(\mu I-A)x-\mu(\mu I-A)y = \mu Ay$, or $\mu^2(\mu I-A)x = \mu^2 y$.

Thus $(\mu I-A)x = y$, and the proof is complete.

Lemma 4. *Suppose $\mu \neq 0$, $(\lambda-\alpha)\mu = 1$, and that $(\mu I-A)^{-1}$ exists. Then $(T-\lambda I)^{-1}$ exists, and in the common domain of the inverses we have $(T-\lambda I)^{-1} = \mu(\mu I-A)^{-1}A = \mu A(\mu I-A)^{-1}$.*

Proof: Suppose $x \in \mathfrak{D}(T)$ and $(T-\lambda I)x = 0$. Then $(\mu I-A)x = 0$ by Lemma 2; therefore $x = 0$, since $(\mu I-A)^{-1}$ exists. Hence $(T-\lambda I)^{-1}$ exists also. We know by Lemma 3 that the two inverses have the same domain. If y is in this domain the unique solution x of $(T-\lambda I)x = y$ satisfies the equation $(\mu I-A)x = \mu Ay$, by Lemma 2. Hence $x = \mu(\mu I-A)^{-1}y$. It is readily proved that A permutes with $(\mu I-A)^{-1}$ in the domain of the latter operator, and with that the proof is complete.

Theorem 5.1. *Let λ and μ satisfy $(\lambda-\alpha)\mu = 1$. Then λ belongs to one of the sets $\varrho(T)$, $p(T)$, $c(T)$, $r(T)$, if and only if μ belongs to the corresponding one of the sets $\varrho(A)$, $p(A)$, $c(A)$, $r(A)$. When $\lambda \in \varrho(T)$ we have*

$$R_\mu(A) = \mu^{-1}I-\mu^{-2}R_\lambda(T)\,, \tag{5.4}$$

$$R_\lambda(T) = -\mu R_\mu(A)A = -\mu AR_\mu(A)\,. \tag{5.5}$$

The theorem is an immediate consequence of the four preceding lemmas.

Theorem 5.2. *The point $\mu = 0$ is not in $p(A)$. It is in $\varrho(A)$, $c(A)$ or $r(A)$ according as $\mathfrak{D}(T)$ is all of X, a proper subset of X dense in X, or not dense in X.*

The theorem follows from the definitions of the resolvent set and the various parts of the spectrum.

Next we examine the operator $f(T)$, where $f \in \mathcal{G}(T)$, in relation to the operator

A. Under the transformation $\mu = (\lambda-\alpha)^{-1}$ let $\varphi(\mu) = f(\lambda)$. We take $\Delta(\varphi)$ as the map in the μ-plane of $\Delta(f)$ in the λ-plane. Since $\Delta(f)$ contains a neighborhood of $\lambda = \infty$, $\Delta(\varphi)$ contains a neighborhood of $\mu = 0$; we define $\varphi(0) = f(\infty)$, so that φ is regular at the origin. Now, by Theorem 5.1 and 5.2 $\sigma(A)$ consists of the image of $\sigma(T)$ and possibly the point $\mu = 0$. Hence $\sigma(A) \subset \Delta(\varphi)$, so that $\varphi \in \mathfrak{F}(A)$.

Theorem 5.3. *When $f \in \mathcal{G}(T)$, $\alpha \in \varrho(T)$, and when A and φ are defined as in the preceding work, we have $\varphi \in \mathfrak{F}(A)$ and $\varphi^*(A) = f(T)$.*

Proof: By Theorem 3.3 there exists an unbounded Cauchy domain D such that $\sigma(T) \subset D$, $\overline{D} \subset \Delta(f)$, $\alpha \in C(\overline{D})$. Let D_1 be the image of D in the μ-plane, with the point $\mu = 0$ added. Then D_1 is a bounded Cauchy domain, and $\sigma(A) \subset D_1$, $\overline{D}_1 \subset \Delta(\varphi)$. Using formula (5.4) we have

$$\begin{aligned} f(T)-f(\infty)I &= \frac{1}{2\pi i}\int_{+B(D)} f(\lambda)R_\lambda(T)d\lambda \\ &= \frac{1}{2\pi i}\int_{+B(D_1)} \varphi(\mu)[-\mu^{-1}I+R_\mu(A)]d\mu \\ &= -\varphi(0)I+\varphi^*(A)\ , \end{aligned}$$

which is equivalent to the required result.

We shall frequently have need to refer to this theorem. It is convenient to have a short way of referring to A and $\varphi(\mu)$ when T and $f(\lambda)$ are given. We shall call A the α-associate of T, and φ the α-associate of f, it being understood that $\alpha \in \varrho(T)$ and $(\lambda-\alpha)\mu = 1$.

6. **Polynomials in T.** Owing to the fact that $\mathfrak{D}(T)$ need not be the whole of X we must exercise care in dealing with powers of T.

Definition. *By $\mathfrak{D}_n(T)$ $(n \geqq 1)$ we mean the set of all $x \in X$ such that $x, Tx, \ldots, T^{n-1}x \in \mathfrak{D}(T)$. We write $\mathfrak{D}_0(T) = X$.*

Of course T^0 denotes the identity operator I. We see that $\mathfrak{D}_1(T) = \mathfrak{D}(T)$ and that $\mathfrak{D}_{n+1}(T) \subset \mathfrak{D}_n(T)$, $n = 0, 1, 2, \ldots$. Evidently $\mathfrak{D}_n(T)$ is a linear subspace of X.

Definition. *Let $P(\lambda) = \Sigma_0^n a_k\lambda^k (n \geqq 0,\ a_n \neq 0)$ be a polynomial in λ. We define an operator $P(T)$ with domain $\mathfrak{D}_n(T)$ by the formula $P(T)x = \Sigma_0^n a_k T^k x$, $x \in \mathfrak{D}_n(T)$.*

Remark. If $n = 0$, $P(T) = a_0 I$. Hence the notation in this case is in agreement with the meaning of $P(T)$ when we consider $P(\lambda)$ as a member of $\mathcal{G}(T)$. When $n > 0$, $P(\lambda)$ is not in $\mathcal{G}(T)$, so we are on fresh ground in defining $P(T)$. In case $T \in [X]$ we see by Theorem 4.5 that $P(T) = P^*(T)$.

Theorem 6.1. *The operator $P(T)$ is closed.*

Proof: We assume $n > 0$, since the assertion is obviously true for $n = 0$. Choose $\alpha \in \varrho(T)$, and let A be the α-associate of T. Then it is easily proved that A^n has the range $\mathfrak{D}_n(T)$ and that its inverse is $(T-\alpha I)^n$. Hence $(T-\alpha I)^n$ is closed, for A^n is closed and the inverse of a closed operator is closed. Now let us write $P(\lambda)$ in the form

$$P(\lambda) = \sum_{k=0}^{n} b_k(\lambda-\alpha)^{n-k} . \tag{6.1}$$

The α-associate of $P(\lambda)$ is $\mu^{-n}p(\mu)$, where

$$p(\mu) = \sum_{k=0}^{n} b_k\mu^k . \tag{6.2}$$

If $x \in \mathfrak{D}_n(T)$ we have $p^*(A)x \in \mathfrak{D}_n(T)$ and

$$P(T)x = (T-\alpha I)^n p^*(A)x = p^*(A)(T-\alpha I)^n x . \tag{6.3}$$

Suppose now that $x_\nu \in \mathfrak{D}_n(T)$ and that $x_\nu \to x$, $P(T)x_\nu \to y$. We have to show that $x \in \mathfrak{D}_n(T)$ and that $P(T)x = y$. Since $(T-\alpha I)^n$ is closed, the fact that $p^*(A)x_\nu \to p^*(A)x$ and $(T-\alpha I)^n p^*(A)x_\nu = P(T)x_\nu \to y$ implies that $p^*(A)x \in \mathfrak{D}_n(T)$ and $(T-\alpha I)^n p^*(A)x = y$. It remains to prove that $x \in \mathfrak{D}_n(T)$. The desired conclusion will then follow from (6.3). Now

$$p^*(A)x = b_0x+b_1Ax+\cdots+b_nA^nx ,$$

and $b_0 = a_n \neq 0$. Each term after the first on the right is in $\mathfrak{D}(T)$, since $\mathfrak{R}(A) = \mathfrak{D}(T)$. Also $p^*(A)x \in \mathfrak{D}_n(T)$, as we saw above. Hence $x \in \mathfrak{D}(T)$. Suppose we know that $x \in \mathfrak{D}_m(T)$, where $1 \leqq m < n$. Then

$$(T-\alpha I)^m p^*(A)x = b_0(T-\alpha I)^m x+b_1(T-\alpha I)^{m-1}x+\cdots+b_nA^{n-m}x ; \tag{6.4}$$

here the left member and each term after the first on the right are in $\mathfrak{D}(T)$. It follows that $(T-\alpha I)^m x \in \mathfrak{D}(T)$ and that $x \in \mathfrak{D}_{m+1}(T)$. Hence, by induction, $x \in \mathfrak{D}_n(T)$. This completes the proof of the theorem.

Definition. *Suppose $f \in \mathcal{G}(T)$ and $\alpha \in \varrho(T)$. Let φ be the α-associate of f. We say that f has a zero of order m at $\lambda = \infty$ if φ has a zero of order m at $\mu = 0$. We take $m = 0$ to mean $f(\infty) \neq 0$, and $m = \infty$ to mean that $f(\lambda)$ vanishes identically in a neighborhood of $\lambda = \infty$.*

Definition, *By $\mathfrak{D}_\infty(T)$ we mean the set of elements common to all the sets $\mathfrak{D}_k(T)$,* $k = 1, 2, \ldots .$

Theorem 6.2. (a) *Suppose* $x \in \mathfrak{D}_n(T)$ *(n finite),* $f \in \mathcal{G}(T)$, *and that* $P(\lambda)$ *is a polynomial of degree* n. *Then* $f(T)x \in \mathfrak{D}_n(T)$ *and* $f(T)P(T)x = P(T)f(T)x$.

(b) *Suppose* $x \in \mathfrak{D}_n(T)$, $0 \leqq n \leqq \infty$, *and suppose that* $f \in \mathcal{G}(T)$, f *having a zero of order* m $(0 \leqq m \leqq \infty)$ *at* $\lambda = \infty$. *Then* $f(T)x \in \mathfrak{D}_{m+n}(T)$ *(where* $m+n = \infty$ *if* $m = \infty$ *or* $n = \infty$*).*

Proof: Since $f(T) = \varphi^*(A)$, where A and φ are α-associates of T and f, respectively, it is clear that $f(T)$ permutes with any $h^*(A)$ where $h \in \mathcal{F}(A)$. We see from (6.3) that $P(T)A^n y = p^*(A)y$, for any y. Under the hypotheses of (a) we have $x = A^n y$, where $y = (T-\alpha I)^n x$. Then $f(T)x = f(T)A^n y = A^n f(T)y$, whence $f(T)x \in \mathfrak{D}_n(T)$. Also, $P(T)f(T)x = P(T)A^n f(T)y = p^*(A)f(T)y = f(T)p^*(A)y = f(T)p^*(A)(T-\alpha I)^n x = f(T)P(T)x$. This completes the proof of (a).

In proving (b), let q denote m if m is finite; if $m = \infty$ let q denote any positive integer. In $\Delta(\varphi)$ define $\psi(\mu) = \mu^{-q}\varphi(\mu)$ when $\mu \neq 0$, $\psi(0) = \lim_{\mu \to 0} \mu^{-q}\varphi(\mu)$. Then $\psi \in \mathcal{F}(A)$ and, by Theorems 4.5 and 5.3, $A^q\psi^*(A) = \varphi^*(A) = f(T)$. Thus $\mathfrak{R}\{f(T)\} \subset \mathfrak{R}(A^q) = \mathfrak{D}_q(T)$. It follows that $\mathfrak{R}\{f(T)\} \subset \mathfrak{D}_m(T)$. Now suppose that $x \in \mathfrak{D}_k(T)$ (k finite). Write $y = (T-\alpha I)^k x$, $x = A^k y$. Then $f(T)x = A^q\psi^*(A)A^k y = A^{q+k}\psi^*(A)y \in \mathfrak{D}_{q+k}(T)$. The conclusion of (b) now follows.

Theorem 6.3. *Suppose* $f \in \mathcal{G}(T)$, *with a zero of order* $m (0 \leqq m \leqq \infty)$ *at* $\lambda = \infty$. *Let* $P(\lambda)$ *be a polynomial of degree* n, $0 \leqq n \leqq m$. *Define* $g(\lambda) = P(\lambda)f(\lambda)$ *with* $\Delta(g) = \Delta(f)$. *Then* $g \in \mathcal{G}(T)$ *and* $P(T)f(T) = g(T)$. *[Note that* $\mathfrak{R}\{f(T)\} \subset \mathfrak{D}_m(T) \subset \mathfrak{D}_n(T)$, *by Theorem 6.2, and that* $g(\infty) = 0$ *if* $n < m$.*]*

Proof: Let $\gamma(\mu)$ be the α-associate of $g(\lambda)$ and let $p(\mu)$ be defined by (6.2). Then $\gamma(\mu) = \mu^{-n}p(\mu)\varphi(\mu)$. Clearly $\gamma(\mu)$ has a removable singularity at $\mu = 0$, and so $g \in \mathcal{G}(T)$, since it is regular at $\lambda = \infty$. Evidently $g(\infty) = 0$ if $n < m$. Now $A^n\gamma^*(A) = p^*(A)\varphi^*(A)$, whence $A^n g(T) = p^*(A)f(T)$. Operating on both sides with $(T-\alpha I)^n$ and using (6.3) we obtain $P(T)f(T) = g(T)$, as asserted.

Theorem 6.4. *Suppose that* $f \in \mathcal{G}(T)$ *and that* $P(\lambda)$ *is a polynomial of degree* $n (n \geqq 1)$. *Suppose that* $\alpha \in \varrho(T)$, *and let* D *be any unbounded Cauchy domain such that* $\sigma(T) \subset D$, $\overline{D} \subset \Delta(f)$, $\alpha \in C(\overline{D})$. *Then, if* $x \in \mathfrak{D}_n(T)$ *we have*

$$\{f(T)+P(T)\}x = \frac{1}{2\pi i}\int_{+B(D)} \frac{f(\lambda)+P(\lambda)}{(\lambda-\alpha)^{n+1}}(T-\alpha I)^{n+1}R_\lambda(T)x\,d\lambda\,. \tag{6.5}$$

Proof: We start from the formula

$$(T-\alpha I)^{n+1}R_\lambda(T)x = (\lambda-\alpha)^{n+1}R_\lambda(T)x - \sum_{k=0}^{n}(\lambda-\alpha)^{n-k}(T-\alpha I)^k x\,,$$

which is valid if $x \in \mathfrak{D}_n(T)$. It is easily established by induction, the case $n=0$ being an immediate consequence of the fact that $(T-\lambda I)^{-1} = -R_\lambda(T)$. The integral in (6.5) is now seen to be equal to

$$\frac{1}{2\pi i}\int_{+B(D)}[f(\lambda)+P(\lambda)]R_\lambda(T)x\,d\lambda - \sum_{k=0}^{n}\frac{1}{2\pi i}\int_{+B(D)}\frac{f(\lambda)+P(\lambda)}{(\lambda-\alpha)^{k+1}}(T-\alpha I)^k x\,d\lambda\,.$$

The first of these integrals is equal to $f(T)x - f(\infty)x$; the term involving $P(\lambda)$ yields zero when integrated, because it is equal to

$$-\frac{1}{2\pi i}\int_{+B(D_1)}P(\lambda)R_\lambda(T)x\,d\lambda\,,$$

where $D_1 = C(\overline{D})$ (the integrand is regular in $\overline{D}_1$, and D_1 is a bounded Cauchy domain). Now

$$\frac{1}{2\pi i}\int_{+B(D)}\frac{f(\lambda)}{(\lambda-\alpha)^{k+1}}d\lambda = \begin{cases} -f(\infty)\,, & k=0 \\ \quad 0\,, & k=1,\dots,n\,, \end{cases}$$

by an argument like that used at one stage of the proof of Theorem 4.3. Finally,

$$-\frac{1}{2\pi i}\int_{+B(D)}\frac{P(\lambda)}{(\lambda-\alpha)^{k+1}}d\lambda = \frac{1}{2\pi i}\int_{+B(D_1)}\frac{P(\lambda)}{(\lambda-\alpha)^{k+1}}d\lambda = \frac{P^{(k)}(\alpha)}{k!}\,.$$

Thus the right member of (6.5) is equal to

$$f(T)x + \sum_{k=0}^{n}\frac{P^{(k)}(\alpha)}{k!}(T-\alpha I)^k x = \{f(T)+P(T)\}x\,.$$

Theorems 6.2–6.4, and particularly Theorem 6.3, are useful supplements to the operational calculus based on Theorem 4.3. It will be seen later in the paper that many of Dunford's arguments which seem to depend upon Theorem 4.5 (c) (mainly for the case in which $f(\lambda)$ is a polynomial of degree 0 or 1) may be retrieved for general operators T with the help of Theorem 6.3.

7. **Inverse operators.** Our first theorem is implicit in Dunford's paper. It follows at once from Theorem 4.5.

Theorem 7.1. *Suppose that $T \in (X)$, that $f \in \mathcal{F}(T)$, and that f has no zeros in $\sigma(T)$. Let Z be the set of points in $\Delta(f)$ at which $f(\lambda)$ vanishes. Let $\Delta(g) = \Delta(f) - Z$ and define $g(\lambda) = 1/f(\lambda)$ in $\Delta(g)$. Then $g \in \mathcal{F}(T)$ and $g^*(T)f^*(T) = f^*(T)g^*(T) = I$, so that $f^*(T)$ defines a $1-1$ mapping of X onto itself.*

Theorem 7.2. *Suppose that $f \in \mathcal{G}(T)$, that $f(\lambda)$ has no zeros in $\sigma(T)$, and that $f(\infty) \neq 0$. Then $f(T)$ defines a $1-1$ mapping of X onto itself. The inverse is given by $\big(f(T)\big)^{-1} = g(T)$, where $g(\lambda) = 1/f(\lambda)$.*

Proof: A proof may be based on Theorems 5.3 and 7.1. We give, instead, a proof independent of considerations about bounded operators. It is clear that $g \in \mathcal{G}(T)$ and $f(\lambda)g(\lambda) = 1$. Hence, by Theorems 4.2 and 4.3 (b), $f(T)g(T) = I$. The conclusion now follows.

Theorem 7.3. *Suppose that $f \in \mathcal{G}_0(T)$, that $f(\lambda)$ has no zeros in $\sigma(T)$, and that f has a zero of finite order n at $\lambda = \infty$. Then $\mathfrak{R}\{f(T)\} = \mathfrak{D}_n(T)$ and $\{f(T)\}^{-1}$ exists. There exists an unbounded Cauchy domain D such that $\sigma(T) \subset D, \overline{D} \subset \Delta(f)$, and $f(\lambda)$ does not vanish in D. If D is such a domain and α is a point of $\varrho(T)$ not in D, we have*

$$\{f(T)\}^{-1}x = \frac{1}{2\pi i}\int_{+B(D)} \{f(\lambda)(\lambda-\alpha)^{n+1}\}^{-1}(T-\alpha I)^{n+1}R_\lambda(T)x\,d\lambda \tag{7.1}$$

for each x in $\mathfrak{D}_n(T)$.

Proof: The existence of D follows readily from Theorem 3.3. With α and D as indicated let φ and A be the α-associates of f and T, respectively. The function φ has a zero of order n at $\mu = 0$. Hence the function $\psi(\mu) = \mu^{-n}\varphi(\mu)$, when properly defined at $\mu = 0$, belongs to $\mathcal{F}(A)$ and is never zero in $\sigma(A)$. Now $f(T) = \varphi^*(A) = A^n\psi^*(A)$, by Theorems 5.3 and 4.5. By Theorem 7.1 we know that $\psi^*(A)$ defines a $1-1$ mapping of X onto itself. We also know that A^n defines a $1-1$ mapping of X onto $\mathfrak{D}_n(T)$. Hence $f(T)$ is seen to define a $1-1$ mapping of X onto $\mathfrak{D}_n(T)$. The inverse operator $\{f(T)\}^{-1}$, with domain $\mathfrak{D}_n(T)$, is given by $\{f(T)\}^{-1} = \{\psi^*(A)\}^{-1}(T-\alpha I)^n$. Let D_1 be the domain in the μ-plane corresponding to D in the λ-plane (with $\mu = 0$ added to D_1). Then ψ is regular and never zero in D_1, so that

$$\{\psi^*(A)\}^{-1} = \frac{1}{2\pi i}\int_{+B(D_1)} \frac{1}{\psi(\mu)} R_\mu(A)d\mu\,.$$

Since $\mu\psi(\mu) = (\lambda-\alpha)^{n-1}f(\lambda)$ and $\mu R_\mu(A) = -(T-\alpha I)R_\lambda(T)$ (by formula (5.5)), we have, on changing the variable of integration from λ to μ,

$$\{\psi^*(A)\}^{-1}(T-\alpha I)^n x = \frac{1}{2\pi i}\int_{+B(D)} \{(\lambda-\alpha)^{n+1}f(\lambda)\}^{-1}(T-\alpha I)R_\lambda(T)(T-\alpha I)^n x\,d\lambda\,.$$

The result (7.1) now follows by an application of Theorem 6.2.

Theorem 7.4. *Let $P(\lambda)$ be a polynomial of degree n ($n \geqq 1$) all of whose zeros lie* *in $\varrho(T)$. Then $P(T)$ defines $1-1$ mapping of $\mathfrak{D}_n(T)$ onto X. The inverse operator belongs to $[X]$ and is given by*

$$\{P(T)\}^{-1} = \frac{1}{2\pi i}\int \frac{1}{P(\lambda)}(T-\lambda I)^{-1}d\lambda\,, \tag{7.2}$$

the integration being extended counterclockwise over a set of nonoverlapping circles, one around each zero of $P(\lambda)$, each circle and its interior lying in $\varrho(T)$.

Proof: Define $f(\lambda) = 1/P(\lambda)$, taking $\Delta(f)$ to be the entire plane with the exception of the zeros of $P(\lambda)$. Then $f \in \mathcal{G}_0(T)$, and f has a zero of order n at $\lambda = \infty$; also $f(\lambda)$ has no zeros in $\sigma(T)$. Hence $f(T)$ defines a $1-1$ mapping of X onto $\mathfrak{D}_n(T)$, with inverse given by (7.1). Upon comparing (7.1) with (6.5) we see that $\{f(T)\}^{-1}x = P(T)x$ when $x \in \mathfrak{D}_n(T)$. Thus $\{P(T)\}^{-1}$ exists and is equal to $f(T)$. To express $f(T)$ as an integral we may take an unbounded Cauchy domain D consisting of the region exterior to a set of circles as described in the theorem. These circles are oriented clockwise in $+B(D)$, and since $(T-\lambda I)^{-1} = -R_\lambda(T)$, the formula (7.2) now follows from the integral formula for $f(T)$.

8. **Spectral sets and projections.** In our definitions of the sets $\sigma(T)$, $\varrho(T)$ we were classifying the finite points of the complex plane. The question arises: if we consider the extended complex plane, how are we to classify the point $\lambda = \infty$? Our answer is as follows.

Definition. *We define the extended spectrum $\sigma_e(T)$ of T to be the set $\sigma(T)$ when $T \in [X]$, and otherwise the set $\sigma(T)$ together with the point $\lambda = \infty$. The extended resolvent set $\varrho_e(T)$ is defined as the complement of $\sigma_e(T)$ in the extended plane.*

Evidently $\sigma_e(T)$ is closed and $\varrho_e(T)$ is open in the extended plane. Also, $\sigma_e(T)$ is never empty.

The expansion (2.2) shows that $R_\lambda(T)$ is regular at $\lambda = \infty$ when $T \in [X]$. If $T \notin [X]$, $R_\lambda(T)$ is not regular at $\lambda = \infty$. This is clear if $\sigma(T)$ is unbounded; the assertion will be justified later in the case when $\sigma(T)$ is bounded (see Theorem 10.5). Hence $\varrho_e(T)$ is the set of points in the extended plane at which $R_\lambda(T)$ is regular.

Definition. *A set σ in the extended complex plane is called a spectral set of T if σ is a subset of $\sigma_e(T)$ which is both open and closed in $\sigma_e(T)$ in the topology of the extended plane. We use the notation $\sigma' = \sigma_e(T) - \sigma$.*

If σ is unbounded it must contain the point $\lambda = \infty$. The set σ' is also a spectral set of T. One of the sets σ, σ' must be bounded.

Definition. *Let σ be a bounded spectral set of T. Let D be a bounded Cauchy domain such that $\sigma \subset D$, $\sigma' \subset C(\overline{D})$. Define*

$$E_\sigma[T] = \frac{1}{2\pi i}\int_{+(B)D} R_\lambda(T) d\lambda . \tag{8.1}$$

If σ is an unbounded spectral set of T we define $E_\sigma[T]$ by the formula

$$E_\sigma[T] = I + \frac{1}{2\pi i}\int_{+B(D)} R_\lambda(T) d\lambda , \tag{8.2}$$

where in (8.2) *we take D as an unbounded Cauchy domain such that $\sigma \subset D$, $\sigma' \subset C(\overline{D})$.*

If σ is a spectral set, there exist nonnull open sets U, V in the extended plane such that $\sigma \subset U$, $\sigma' \subset V$, the closures of U and V are disjoint, and one of the sets U, V contains a neighborhood of $\lambda = \infty$ (the other set therefore being bounded). We define $f_\sigma(\lambda) = 1$ on U, $f_\sigma(\lambda) = 0$ on V, $f_{\sigma'}(\lambda) = 1 - f_\sigma(\lambda)$ on $U+V$. It is then easily seen that $E_\sigma[T] = f_\sigma(T)$, $E_{\sigma'}[T] = f_{\sigma'}(T)$. It follows from the operational calculus that $(E_\sigma[T])^2 = E_\sigma[T]$, so that $E_\sigma[T]$ is a projection. We say that it is the projection associated with σ. Projections defined by (8.1) have been considered by Lorch ([2], pp. 241–42). We observe the further properties

$$\begin{aligned} &E_\sigma[T] + E_{\sigma'}[T] = I , \\ &E_\sigma[T] E_{\sigma'}[T] = 0 , \end{aligned} \tag{8.3}$$

of the projections associated with complementary spectral sets.

Definition. *We write $X_\sigma[T] = \Re(E_\sigma[T])$. When there can be no ambiguity we drop the T and write simply E_σ, X_σ, etc.*

To clear the ground for the discussion of spectral sets and their associated projections we first consider two lemmas and a general theorem.

If A and B are linear subspaces of X with zero as their only common element, $A \oplus B$ denotes the set of all elements of the form $a+b$, $a \in A$, $b \in B$. The representation of an element of $A \oplus B$ in this fashion is unique. If A and B are linear subspaces of X, our use of the expression $A \oplus B$ will carry the implicit assumption that A and B have 0 as their only common element.

Lemma 1. *Suppose that A, B, M, and N are linear subspaces of X such that $A \subset M$, $B \subset N$, and $A \oplus B = M \oplus N$. Then $A = M$ and $B = N$.*

We omit the proof, which is exceedingly simple.

Lemma 2. *Suppose that A, B, M, N are linear subspaces of X, that $A \subset M$, $B \subset N$, and that M, N, and $M \oplus N$ are closed. Then $A \oplus B$ is dense in $M \oplus N$ if and only if A and B are dense in M and N, respectively.*

The proof, which we omit, is a simple consequence of well known properties of projection operators (see, for example, Lorch [1], p. 220, Theorem 2.2).

Theorem 8.1. *Let E_1 and E_2 be projections such that $E_1+E_2 = I$ (and hence $E_1E_2 = E_2E_1 = 0$). Let M_i be the range of E_i ($i = 1, 2$). For convenience write $\mathfrak{D} = \mathfrak{D}(T)$. Suppose that $E_i(\mathfrak{D}) \subset \mathfrak{D}$ and that $T(\mathfrak{D}M_i) \subset M_i$. When T is considered as an operator in the space M_i, with domain $\mathfrak{D}M_i$, denote its resolvent set, spectrum, point spectrum, etc. by $\varrho(T, M_i)$, $\sigma(T, M_i)$, etc. Also, denote by $\mathfrak{R}(\lambda I-T, M_i)$ the transform of $\mathfrak{D}M_i$ by $\lambda I-T$. Then*

(a) $E_iTx = TE_ix$ if $x \in \mathfrak{D}$;

(b) $\mathfrak{R}(\lambda I-T) = \mathfrak{R}(\lambda I-T, M_1) \oplus \mathfrak{R}(\lambda I-T, M_2)$;

(c) $p(T) = p(T, M_1)+p(T, M_2)$;

(d) $\sigma(T) = \sigma(T, M_1)+\sigma(T, M_2)$.

If we assume further that $\sigma(T, M_1)$ and $\sigma(T, M_2)$ have no points in common, then

(e) $c(T) = c(T, M_1)+c(T, M_2)$;

(f) $r(T) = r(T, M_1)+r(T, M_2)$.

Proof: The subspaces M_1 and M_2 are closed, and $M_1 \oplus M_2 = X$. We observe that $\mathfrak{R}(\lambda I-T, M_i) \subset M_i$. The assertions (a) and (b) are quite obvious. Evidently $p(T, M_i) \subset p(T)$. Suppose $\lambda \in p(T)$, so that we have $(\lambda I-T)x = 0$ for some $x \in \mathfrak{D}$, $x \neq 0$. Then $(\lambda I-T)E_1x = -(\lambda I-T)E_2x$. The left member of the equation belongs to M_1, the right member to M_2. Hence both members are zero. But E_1x and E_2x are not both zero, since $x \neq 0$. Hence λ belongs either to $p(T, M_1)$ or to $p(T, M_2)$. Thus (c) is proved. Now suppose that $\lambda \in \varrho(T)$, so that $\mathfrak{R}(\lambda I-T) = X$. By (b) and Lemma 1 we conclude that $\mathfrak{R}(\lambda I-T, M_i) = M_i$. Now λ cannot be in $p(T, M_i)$, by (c). Hence we conclude that $\lambda \in \varrho(T, M_i)$, $i = 1, 2$. Suppose conversely that $\lambda \in \varrho(T, M_i)$ for $i = 1$ and 2. Then $\mathfrak{R}(\lambda I-T, M_i) = M_i$, and hence $\mathfrak{R}(\lambda I-T) = X$, by (b). This means that λ is either in $\varrho(T)$ or $p(T)$. But it cannot be in $p(T)$, by (c). We have thus proved that $\varrho(T)$ is the intersection of $\varrho(T, M_1)$ and $\varrho(T, M_2)$, which is equivalent to (d).

We now assume that $\sigma(T, M_1)$ and $\sigma(T, M_2)$ are disjoint. Suppose $\lambda \in c(T)+r(T)$; then λ belongs to $\sigma(T, M_1)$ or $\sigma(T, M_2)$, by (d). Suppose it belongs to $\sigma(T, M_1)$, and hence to $\varrho(T, M_2)$. We see by (b) and the two lemmas that $\lambda \in c(T)$ implies $\lambda \in c(T, M_1)$ and that $\lambda \in r(T)$ implies $\lambda \in r(T, M_1)$. If on the other hand we assume that $\lambda \in c(T, M_1)$ (and hence that $\lambda \in \varrho(T, M_2)$), we have $\mathfrak{R}(\lambda I-T, M_2) = M_2$; from

(b) and the two lemmas we conclude that $\lambda \in c(T)$. The same argument shows that $r(T, M_1) \subset r(T)$. The proofs of (e) and (f) are now complete.

Theorem 8.2. *Let σ be a spectral set of T. Then*

(a) $E_\sigma(\mathfrak{D}(T)) \subset \mathfrak{D}(T)$;

(b) $T(\mathfrak{D}(T)X_\sigma) \subset X_\sigma$;

(c) $\sigma = \sigma_e(T, X_\sigma)$;

(d) $\sigma \cdot p(T) = p(T, X_\sigma)$;

(e) $\sigma \cdot c(T) = c(T, X_\sigma)$;

(f) $\sigma \cdot r(T) = r(T, X_\sigma)$.

If the spectral set σ is bounded we have the further conclusions

(g) $X_\sigma \subset \mathfrak{D}_\infty(T)$,

(h) *as an operator on X_σ, T is bounded.*

Proof: Assertion (a) follows from Theorem 6.2 (a) and the fact that $E_\sigma = f_\sigma(T)$ for a suitable $f_\sigma \in \mathcal{G}(T)$. If $x \in \mathfrak{D}(T)X_\sigma$ we have $x = E_\sigma x$. Hence $E_{\sigma'}Tx = TE_{\sigma'}x = TE_{\sigma'}E_\sigma x = 0$, by Theorem 6.2 (a) and (8.3). It follows that $Tx \in X_\sigma$; thus assertion (b) is true. If σ is bounded we can arrange the definitions of f_σ and $f_{\sigma'}$ so that f_σ has a zero of infinite order at $\lambda = \infty$. In view of this, assertion (g) follows from Theorem 6.2 (b). Since T is defined throughout X_σ and closed on X_σ, it is bounded there (Banach, [1], p. 41, Theorem 7); thus (h) is established.

To prove (c) we first prove that $\lambda \in \varrho(T, X_\sigma)$ if λ is a finite point not in σ. For, if λ is such a point we may suppose that the sets U, V mentioned above are so chosen that $\lambda \in V$. We then define $g(\xi) = (\lambda - \xi)^{-1}$ on U, $g(\xi) = 0$ on V. Thus $(\lambda - \xi)\, g(\xi) = f_\sigma(\xi)$ and $f_\sigma(\xi) g(\xi) = g(\xi)$, whence $(\lambda I - T)g(T) = E_\sigma$ and $E_\sigma g(T) = g(T)$. Then $g(T)$ and $\lambda I - T$ are inverse to each other in the space X_σ. Therefore $\lambda \in \varrho(T, X_\sigma)$. Now let σ_1 and σ_2 be the sets of finite points of σ and σ', respectively. We have $\sigma(T) = \sigma_1 + \sigma_2$, and, by Theorem 8.1 (d), $\sigma(T) = \sigma(T, X_\sigma) + \sigma(T, X_{\sigma'})$. We conclude that $\sigma_1 = \sigma(T, X_\sigma)$. This conclusion is equivalent to assertion (c) when σ is bounded, in view of (h). When σ is not bounded, it necessarily contains $\lambda = \infty$, and $T \notin [X]$. But in this case T is bounded on $X_{\sigma'}$ and hence cannot be defined throughout X_σ and bounded there. Thus $\sigma_e(T, X_\sigma)$ also contains $\lambda = \infty$. The proof of (c) is thus complete.

Assertion (d) now follows from Theorem 8.1 (c), since we know that $\sigma(T, X_\sigma)$

and $\sigma(T, X_{\sigma'})$ are disjoint. Assertions (e) and (f) follow in similar fashion, using Theorem 8.1 (e) and (f).

Theorem 8.3. *The spectral set σ is empty if and only if $E_\sigma = 0$.*

One half of this follows from Theorem 8.2 (c); the other half follows from the definition of E_σ, by Cauchy's theorem.

Theorem 8.4. (a) *If σ is a bounded spectral set of T and if $E_\sigma = I$, then $\sigma = \sigma(T)$ and $T \in [X]$.* (b) *If $T \in [X]$ and $\sigma = \sigma(T)$, then $E_\sigma = I$.*

Proof: Assertion (a) follows from Theorem 8.2 (c) and (h). Assertion (b) follows from Theorem 4.5 (c).

Theorem 8.5. *Let $\sigma(T)$ be bounded. We define the operator*

$$E[T] = \frac{1}{2\pi i}\int_C R_\lambda(T)d\lambda \tag{8.4}$$

where C is a circle, oriented counterclockwise, containing $\sigma(T)$ in its interior. Then $E[T]$ is a projection, and $E[T] = I$ if and only if $T \in [X]$.

Proof: We observe that $E[T]$ is the special case of an $E_\sigma[T]$ with $\sigma = \sigma(T)$. Hence $E[T]$ is a projection. The rest of the theorem follows from Theorem 8.4.

We can now state a theorem which completes the considerations begun in Theorems 7.2 and 7.3.

Theorem 8.6. *Let $\sigma(T)$ be bounded. Suppose that $f \in \mathcal{G}_0(T)$, that f does not vanish on $\sigma(T)$, and that f has a zero of infinite order at $\lambda = \infty$ (see one of the definitions preceding Theorem* 6.2). *Then $f(T)$ has the same range and null manifold as the operator $E[T]$ defined in* (8.4).

Proof: The null manifold of an operator $S \in [X]$ is the set of elements $x \in X$ such that $Sx = 0$. The theorem is trivial when $\sigma(T)$ is empty, since $f(T) = E = 0$ in that case. Hence we suppose $\sigma(T)$ not empty. Then $\Delta(f)$ consists of two parts: a neighborhood of $\lambda = \infty$ in which $f(\lambda) \equiv 0$, and the remainder of $\Delta(f)$ in which $f(\lambda)$ is not identically zero. Define $g(\lambda) = 1$ in the first mentioned part of $\Delta(f)$, and $g(\lambda) = 0$ in the second part. Define $h(\lambda) = f(\lambda)+g(\lambda)$. Then $g, h \in \mathcal{G}(T)$, $h(\infty) = 1$, and $h(\lambda)$ has no zeros in $\sigma(T)$. We observe that $f(\lambda)g(\lambda) \equiv 0$, and hence $f(T)g(T) = 0$. Also $g(T) = I-E$. Hence

$$f(T) = f(T)E = Ef(T)\,. \tag{8.5}$$

Next, $h(T) = f(T)+g(T) = f(T)+I-E$. Hence $h(T)E = f(T)E+E-E^2 = f(T)$. We know by Theorem 7.2 that $\{h(T)\}^{-1}$ exists and belongs to $[X]$. Hence

$$E = \{h(T)\}^{-1}f(T) = f(T)\{h(T)\}^{-1}. \tag{8.6}$$

The assertions of the theorem follow from (8.5) and (8.6).

9. The spectral mapping theorem. Composite functions. If G is a point set in the extended plane, and if $G \subset \Delta(f)$ where $f \in \mathcal{G}(T)$, we use $f(G)$ to denote the set of values assumed by f at points of G.

Theorem 9.1. *If $f \in \mathcal{G}(T)$ we have $f(\sigma_e(T)) = \sigma(f(T))$.*

The theorem asserts (a) $f(\sigma_e(T)) \subset \sigma(f(T))$ and (b) $\sigma(f(T)) \subset f(\sigma_e(T))$.

Proof of (a): Here and often later we use ξ as a complex variable when λ is used as a fixed number. Suppose $\lambda \in \sigma(T)$. Take $\Delta(g) = \Delta(f)$ and define $g(\xi) = (f(\lambda)-f(\xi))(\lambda-\xi)^{-1}$ if $\xi \neq \lambda$, $g(\lambda) = f'(\lambda)$. Observe that $(\lambda-\xi)g(\xi) = f(\lambda)-f(\xi)$. To obtain from this an equation involving operators we apply Theorems 6.3, 4.2, and 4.3 (a). The result is $(\lambda I-T)g(T) = f(\lambda)I-f(T)$. Now, if $\lambda \in p(T)$ there is an $x \in \mathfrak{D}(T)$, $x \neq 0$, such that $(\lambda I-T)x = 0$. Then $(f(\lambda)I-f(T))x = 0$, as we see by an application of Theorem 6.2 (a). Thus $f(\lambda) \in p(f(T))$. Also, if $\lambda \in c(T)+r(T)$ we see that $\mathfrak{R}\{f(\lambda)I-f(T)\}$ is a proper subset of X, so that $f(\lambda) \in \sigma(f(T))$. It remains to prove $f(\infty) \in \sigma(f(T))$ if $T \notin [X]$. If $f(\lambda) = f(\infty)$ at some point $\lambda \in \sigma(T)$, the situation is covered by what we have proved. Otherwise, $f(\infty)-f(\xi)$ belongs to $\mathcal{G}_0(T)$ and does not vanish on $\sigma(T)$. The range of $f(\infty)I-f(T)$ is therefore not all of X, either by Theorem 7.3 or by Theorems 8.5 and 8.6. Thus $f(\infty) \in \sigma(f(T))$.

Proof of (b): Consider first a point $\beta \in \sigma(f(T))$ such that $\beta \neq f(\infty)$. Suppose, contrary to what is to be proved, that $f(\xi)-\beta$ has no zeros in $\sigma(T)$. Let B be the set of points in $\Delta(f)$ at which $f(\xi) = \beta$. Then define $h(\xi) = (\beta-f(\xi))^{-1}$ with $\Delta(h) = \Delta(f)-B$. It is clear that $h \in \mathcal{G}(T)$ and $h(\infty) \neq 0$. It follows from Theorem 7.2 that $h(T)$ defines a $1-1$ mapping of X onto itself, the inverse operator being $\beta I-f(T)$. This contradicts the fact that $\beta \in \sigma(f(T))$. If $\beta = f(\infty)$ and $T \notin [X]$, we have $\beta \in f(\sigma_e(T))$, since $\lambda = \infty$ belongs to $\sigma_e(T)$. It remains only to consider the case $\beta = f(\infty) \in \sigma(f(T))$, $T \in [X]$. In this case there exists a $g \in \mathcal{G}(T)$ such that f and g coincide on an open set containing $\sigma(T)$, $f(T) = g(T)$, and $g(\infty) \neq \beta$ (see Theorem 4.4). We then conclude by the earlier argument that $\beta \in f(\sigma(T))$. Thus the proof is complete.

Remark: We cannot replace $\sigma_e(T)$ by $\sigma(T)$ in the (b) assertion of Theorem 9.1 unless $T \in [X]$. For, if $\mathfrak{D}(T)$ is a proper subset of X and $f \in \mathcal{G}_0(T)$, with a zero of the first order at $\lambda = \infty$ and no zeros in $\sigma(T)$, then we see by Theorem 7.3 that the point $\beta = 0$ belongs to $\sigma(f(T))$.

From Theorem 9.1 we obtain a criterion for T to belong to $[X]$.

Theorem 9.2. *Suppose that* $f \in \mathcal{G}(T)$ *and that* $\sigma(f(T))$ *does not contain the point* $\lambda = f(\infty)$. *Then* $T \in [X]$.

This follows from Theorem 9.1 and the definition of $\sigma_e(T)$.

Theorem 9.1 is a generalization of the spectral mapping theorem for bounded operators T, which was proved by Dunford ([1], p. 195, Theorem 2.9). Dunford's theorem is a corollary of Theorem 9.1 and 4.4.

Theorem 9.3. *Let* $P(\lambda)$ *be a polynomial. Then* $P(\sigma(T)) = \sigma(P(T))$.

Proof: We may assume that $P(\lambda)$ is of at least the first degree, for the assertion if evidently true for a polynomial of degree zero. In the special case that $T \in [X]$ this theorem is included in Dunford's spectral mapping theorem. To prove that $P(\sigma(T)) \subset \sigma(P(T))$ we assume $\lambda \in \sigma(T)$ and define a polynomial $Q(\xi)$ such that $P(\lambda) - P(\xi) = (\lambda - \xi)Q(\xi)$. The argument is then similar to that used in proving Theorem 9.1 (a). The proof that $\sigma(P(T)) \subset P(\sigma(T))$ is similar to that of Theorem 9.1 (b). If λ is not in $P(\sigma(T))$ let $h(\xi) = (\lambda - P(\xi))^{-1}$. Then Theorem 7.4 shows that $\lambda I - P(T)$ defines a $1-1$ mapping of $\mathfrak{D}_n(T)$ (where n is the degree of $P(\lambda)$) onto X, with inverse $h(T)$, so that $\lambda \in \varrho(P(T))$. This completes the proof.

We now come to the theorem on composite functions.

Theorem 9.4. *Suppose* $f \in \mathcal{G}(T)$, $S = f(T)$, $g \in \mathcal{F}(S)$, *and suppose further that* $\Delta(g)$ *contains the point* $f(\infty)$. *Let* $\Delta(F)$ *be the set of those points* λ *in* $\Delta(f)$ *such that* $f(\lambda) \in \Delta(g)$, *and define* $F(\lambda) = g(f(\lambda))$. *Then* $F \in \mathcal{G}(T)$ *and* $F(T) = g^*(S)$.

Proof: It is readily seen that $\Delta(F)$ is an open set which contains a neighborhood of the point $\lambda = \infty$. We have $\sigma(T) \subset \Delta(F)$, by Theorem 9.1. Thus $F \in \mathcal{G}(T)$; we see that $F(\infty) = g(f(\infty))$. Now let D be a bounded Cauchy domain such that $\sigma(S) \subset D$, $\overline{D} \subset \Delta(g)$, and further such that the point $f(\infty)$ lies in D. Next let D_1 be an unbounded Cauchy domain such that $\sigma(T) \subset D_1$, $\overline{D}_1 \subset \Delta(f)$, and $f(\overline{D}_1) \subset D$. The existence of D_1 follows with the help of Theorem 3.3. Now

$$F(T) - F(\infty)I = \frac{1}{2\pi i}\int_{+B(D_1)} g(f(\xi)) R_\xi(T)\, d\xi\,;$$

since $f(\xi) \in D$ when $\xi \in B(D_1)$ we have

$$g(f(\xi)) = \frac{1}{2\pi i}\int_{+B(D)} \frac{g(\lambda)}{\lambda - f(\xi)}\, d\lambda.$$

Therefore, setting $h(\xi) = (\lambda - f(\xi))^{-1}$, we have

$$F(T)-F(\infty)I = \frac{1}{2\pi i}\int_{+B(D)} g(\lambda)d\lambda \frac{1}{2\pi i}\int_{+B(D)} h(\xi)R_\xi(T)d\xi \,.$$

The inner integral here is equal to $h(T)-h(\infty)I$. By Theorem 7.2 we see that $h(T) = \big(\lambda I - f(T)\big)^{-1} = R_\lambda(S)$. Hence

$$F(T)-F(\infty)I = \frac{1}{2\pi i}\int_{+B(D)} g(\lambda)\left\{R_\lambda(S) - \frac{I}{\lambda - f(\infty)}\right\} d\lambda$$
$$= g^*(S) - g\big(f(\infty)\big)I \,,$$

whence $F(T) = g^*(S)$, as asserted.

10. **Isolated points of the spectrum.** If λ is an isolated point of $\sigma(T)$, the set σ consisting of λ alone (which we denote by $\sigma = (\lambda)$) is a bounded spectral set of T. We shall write E_λ for E_σ and X_λ for X_σ in such a situation. As in the classical theory of analytic functions, $R_\xi(T)$ possesses a Laurent expansion in the neighborhood of $\xi = \lambda$. A number of relations exist between the operator coefficients in the expansion, as we shall see in the following theorem. Some of these relations have been stated before (Dunford [1], p. 198; Taylor [2], pp. 659—660).

Theorem 10.1. *Let λ be an isolated point of $\sigma(T)$, and let the Laurent expansion of $R_\xi(T)$ in the neighborhood of $\xi = \lambda$ be*

$$R_\xi(T) = \sum_{n=0}^{\infty} (\xi-\lambda)^n A_n + \sum_{n=1}^{\infty} (\xi-\lambda)^{-n} B_n \,.$$

Then

(a) $B_1 = E_\lambda$;
(b) $A_m A_n = -A_{m+n+1}$, $m, n = 0, 1, 2, \ldots$;
(c) $B_m B_n = B_{m+n-1}$, $m, n = 1, 2, \ldots$;
(d) $A_m B_n = 0$, $m = 0, 1, \ldots$, $n = 1, 2, \ldots$;
(e) $A_n = (-1)^n A_0^{n+1}$, $n = 0, 1, 2, \ldots$;
(f) $\mathfrak{R}(A_n) \subset \mathfrak{D}_{n+1}(T)$, $n = 0, 1, \ldots$, and $\mathfrak{R}(B_n) \subset \mathfrak{D}_\infty(T)$, $n = 1, 2, \ldots$;
(g) $(T-\lambda I)A_0 = B_1 - I$;
(h) $(T-\lambda I)A_{n+1} = A_n$, $n = 0, 1, \ldots$;
(i) $B_{n+1} = (T-\lambda I)B_n = (T-\lambda I)^n E_\lambda$.

Proof: Let r be chosen positive and such that all of $\sigma(T)$ except $\xi = \lambda$ lies outside the circle of radius $2r$ about $\xi = \lambda$. We define functions $f_n(\xi)$ as follows:

$$n \geqq 0 \begin{cases} f_n(\xi) = 0, & |\xi-\lambda| < r\,, \\ f_n(\xi) = (\xi-\lambda)^{-(n+1)}, & |\xi-\lambda| > 2r\,, \end{cases}$$

$$n < 0 \begin{cases} f_n(\xi) = (\xi-\lambda)^{-(n+1)}, & |\xi-\lambda| < r, \\ f_n(\xi) = 0, & |\xi-\lambda| > 2r. \end{cases}$$

Observe that $f_n \in \mathcal{G}_0(T)$ for every n. Also observe that $f_m(\xi)f_n(\xi) \equiv 0$ if $m < 0$ and $n \geqq 0$, while $f_m(\xi)f_n(\xi) \equiv f_{m+n+1}(\xi)$ if m and n are either both nonnegative, or both negative. As in the standard development of the Laurent series we know that

$$A_n = \frac{1}{2\pi i}\int_C (\xi-\lambda)^{-(n+1)} R_\xi(T)d\xi,\ n \geqq 0,$$

$$B_n = \frac{1}{2\pi i}\int_C (\xi-\lambda)^{n-1} R_\xi(T)d\xi,\ n \geqq 1,$$

where C is any circle with center at $\xi = \lambda$ such that all of $\sigma(T)$ except the point λ is outside C, the integration being in the counterclockwise sense. A brief consideration of the definitions shows that $B_n = f_{-n}(T)$, $A_n = -f_n(T)$ for the appropriate ranges of the subscripts, and further that $B_1 = E_\lambda$. Assertions (b)–(d) follow at once by the operational calculus; (e) follows from (b) by induction. Assertions (f)–(i) follow from Theorems 6.2 and 6.3 (the last part of (i) coming from the first part by induction).

As in classical function theory we say that λ is a pole of $R_\xi(T)$ if the coefficients B_n are all zero from a certain n onward. Otherwise we say that λ is an essential singularity. We observe that $B_1 = E_\lambda \neq 0$, by Theorem 8.3.

Theorem 10.2. *If $B_n = 0$ for some n, then λ is a pole of $R_\xi(T)$ of order less than n. In this case λ belongs to $p(T)$.*

The first statement is a consequence of Theorem 10.1 (i). If m is the order of the pole we have $B_m \neq 0$, $B_{m+1} = 0$. Then $B_m x \neq 0$ for some x, and we have $(T-\lambda I)B_m x = B_{m+1}x = 0$. Hence $\lambda \in p(T)$.

The next theorem supplements Theorem 8.2.

Theorem 10.3. *Let σ be a spectral set of T, and let λ be an isolated finite point of σ. Then λ is a pole of order m of $R_\xi(T)$ if and only if it is a pole of order m of the resolvent of T when T is considered as an operator on the space $X_\sigma[T]$.*

Proof: When $|\xi-\lambda|$ is positive and sufficiently small, the operator $R_\xi(T)$, considered as an operator on X_σ, is the resolvent of T when T is considered as an operator on X_σ. It follows that the operator coefficients in the Laurent expansion of the last mentioned resolvent are the operators A_n, B_n, considered as operators on X_σ. Now $B_n E_\sigma = B_n$, as we see by the operational calculus. Hence $B_n x = 0$ for all $x \in X_\sigma$ if and only if $B_n = 0$. The theorem now follows at once.

We now consider the relation between isolated singularities of the resolvents of T and A, where $\alpha \in \varrho(T)$ and A is the α-associate of T, as defined in § 5.

Theorem 10.4. *Suppose that λ is an isolated point of $\sigma(T)$, and let $\mu = (\lambda-\alpha)^{-1}$. Then μ is an isolated point of $\sigma(A)$. We have*

$$(10.1) \qquad (\mu I-A)^n E_\mu[A] = \mu^n A^n (T-\lambda I)^n E_\lambda[T],\ n \geqq 0\,.$$

Thus λ is a pole or an essential singularity of $R_\xi(T)$ if and only if μ is a singularity of the corresponding type of $R_\nu(A)$.

Proof: Define $g_n(\xi) = (\xi-\alpha)^{-n}$, $n \geqq 1$. Then $(\xi-\alpha)^n g_n(\xi) = 1$, and so $(T-\alpha I)^n g_n(T) = I$, by Theorem 6.3. Therefore $g_n(T) = A^n$. Define $h_0(\xi) = f_{-1}(\xi)$, $h_n(\xi) = f_{-(n+1)}(\xi) g_n(\xi)$ if $n \geqq 1$, using the functions defined in the proof of Theorem 10.1. Then $h_n(T) = B_{n+1}A^n$, $n \geqq 0$. Let $\psi_n(\nu)$ be the α-associate of $h_n(\xi)$, $\nu = (\xi-\alpha)^{-1}$. The $\psi_n(\nu) = \mu^{-n}(\mu-\nu)^n$ in the neighborhood of $\nu = \mu$, while $\psi_n(\nu) = 0$ in the neighborhood of $\nu = 0$ and the rest of $\sigma(A)$. Hence, by Theorems 10.1 (i) (applied to A and the point μ) and 4.4, we have $\psi_n^*(A) = \mu^{-n}(\mu I-A)^n E_\mu[A]$. On the other hand, $\psi_n^*(A) = h_n(T) = B_{n+1}A^n = A^n(T-\lambda I)^n E_\lambda[T]$. Thus (10.1) is established. The final assertion of the theorem is an immediate consequence.

Next let us consider the behavior of $R_\lambda(T)$ at $\lambda = \infty$. If $T \in [X]$ we know from (2.2) that $R_\lambda(T)$ is regular in the neighborhood of $\lambda = \infty$ and that $R_\lambda(T) \to 0$ as $\lambda \to \infty$. The next theorem shows that, for operators with bounded spectrum, the point at infinity can never be a pole of the resolvent.

Theorem 10.5. *Suppose that $\sigma(T)$ is bounded and that T does not belong to $[X]$. Then the point $\lambda = \infty$ is an essential singularity of the resolvent.*

Proof: We again use the α-associate of T. By (5.4) we have $R_\lambda(T) = \mu I - \mu^2 R_\mu(A)$. The point $\mu = 0$ is an isolated point of $\sigma(A)$ but does not belong to $p(A)$, by Theorems 5.1 and 5.2. Hence, by Theorem 10.2, the point $\mu = 0$ is not a pole of $R_\mu(A)$. Thus the Laurent expansion of $R_\lambda(T)$ about the point $\mu = 0$ has nonzero coefficients for all the negative powers of μ.

We next give a useful characterization of the subspace X_λ, where λ is an isolated point of $\sigma(T)$. The corresponding characterization, for the case $T \in [X]$, was given by Dunford ([1], p. 199, Theorem 2.18).

Theorem 10.6. *Let λ be an isolated point of $\sigma(T)$, and let δ be the distance from λ to the rest of $\sigma(T)$ (let $\delta = \infty$ if $\sigma(T)$ is the single point λ).* (a) *Then, if $x \in X_\lambda$ and*

$\varepsilon > 0$, we assert that $\lim_{n\to\infty} (T-\lambda I)^n \varepsilon^{-n} x = 0$. (b) *Also, if for some fixed ε, $0 < \varepsilon < \delta$, we have $x \in \mathfrak{D}_\infty(T)$ and* $\lim_{n\to\infty} (T-\lambda I)^n \varepsilon^{-n} x = 0$, *then $x \in X_\lambda$.*

Proof: Observe that $X_\lambda \subset \mathfrak{D}_\infty(T)$, by Theorem 10.1 (a) and (f). We see from the proof of Theorem 10.1 that

$$(T-\lambda I)^n E_\lambda = \frac{1}{2\pi i}\int_C (\xi-\lambda)^n R_\xi(T) d\xi ,$$

where C is any sufficiently small circle with center λ. If $\varepsilon > 0$ is given, we can choose the radius r of C so that $r < \varepsilon$. Then, if $x \in X_\lambda$ we have

$$(T-\lambda I)^n \varepsilon^{-n} x = (T-\lambda I)^n E_\lambda \varepsilon^{-n} x = \frac{1}{2\pi i}\int_C (\xi-\lambda)^n R_\xi(T) \varepsilon^{-n} x d\xi ,$$

and this expression evidently tends to zero as $n \to \infty$.

To prove (b), choose circles C_1, C_2, of radii r_1, r_2, with centers at λ, such that $0 < r_1 < \varepsilon < r_2 < \delta$. Let

$$h_n(\xi) = [1-(\xi-\lambda)^n \varepsilon^{-n}]^{-1} .$$

Then $h_n \in \mathcal{G}_0(T)$. By Theorem 7.4 we see that $h_n(T)$ defines a $1-1$ mapping of X onto $\mathfrak{D}_n(T)$, the inverse operator being $I-\varepsilon^{-n}(T-\lambda I)^n$. We have

$$h_n(T)-E_\lambda = \frac{1}{2\pi i}\int_{C_1} \big(h_n(\xi)-1\big) R_\xi(T) d\xi - \frac{1}{2\pi i}\int_{C_2} h_n(\xi) R_\xi(T) d\xi ,$$

and hence $h_n(T) \to E_\lambda$ as $n \to \infty$. If now x is given as in (b) we have $x = h_n(T)[I-\varepsilon^{-n}(T-\lambda I)^n]x$; $x - E_\lambda x = [h_n(T)-E_\lambda]x - h_n(T)(T-\lambda I)^n \varepsilon^{-n} x \to 0$. Thus $x \in E_\lambda x$.

This argument is entirely similar to that of Dunford, but we must use Theorem 7.4 where Dunford could use Theorem 7.1.

Before coming to the last theorems of this section we give some definitions.

Definition. *For any λ, and $n \geqq 1$, let $\mathfrak{M}^n(\lambda)$ be the set of all elements $x \in \mathfrak{D}_n(T)$ such that $(\lambda I - T)^n x = 0$. Let $\mathfrak{R}^n(\lambda)$ be the range of the operator $(\lambda I - T)^n$ $\big($with the domain $\mathfrak{D}_n(T)\big)$.*

Theorem 10.7. *In the notation of the foregoing definition we have*

(a) $\mathfrak{R}^{n+1}(\lambda) \subset \mathfrak{R}^n(\lambda)$, $n = 1, 2, \ldots$;

(b) *if* $\mathfrak{R}^{k+1}(\lambda) = \mathfrak{R}^k(\lambda)$ *then* $\mathfrak{R}^n(\lambda) = \mathfrak{R}^k(\lambda)$ *when* $n \geqq k$;

(c) $\mathfrak{M}^n(\lambda) \subset \mathfrak{M}^{n+1}(\lambda)$, $n = 1, 2, \ldots$;

(d) *if* $\mathfrak{M}^{k+1}(\lambda) = \mathfrak{M}^k(\lambda)$ *then* $\mathfrak{M}^n(\lambda) = \mathfrak{M}^k(\lambda)$ *when* $n \geqq k$.

The assertions of this theorem are well known, at least for operators defined throughout X, and we omit the proofs, which are easy.

The next theorem was given by Dunford ([1], p. 198, Theorem 2.17).

Theorem 10.8. *Let λ be a pole of $R_\xi(T)$ of order m, and let $\sigma = (\lambda)$. Then*

(a) $X_\lambda = \mathfrak{M}^m(\lambda)$, $X_{\sigma'} = \mathfrak{N}^m(\lambda)$ $\big($*and* $X = \mathfrak{M}^m(\lambda) \oplus \mathfrak{N}^m(\lambda)\big)$.

(b) $\mathfrak{M}^n(\lambda)$ *is a proper subset of* $\mathfrak{M}^{n+1}(\lambda)$ *if* $1 \leqq n < m$, *but* $\mathfrak{M}^n(\lambda) = \mathfrak{M}^m(\lambda)$ *if* $n \geqq m$.

(c) $\mathfrak{N}^{n+1}(\lambda)$ *is a proper subset of* $\mathfrak{N}^n(\lambda)$ *if* $1 \leqq n < m$, *but* $\mathfrak{N}^n(\lambda) = \mathfrak{N}^m(\lambda)$ *if* $n \geqq m$.

Proof: Dunford, in treating the case $T \in [X]$, uses an argument that depends upon the expansion (2.2). This part of Dunford's argument is not available in the general case; we use Theorem 10.1 (g) and (h) instead.

We have $X_\lambda \subset \mathfrak{M}^n(\lambda)$ if $n \geqq m$, for $X_\lambda \subset \mathfrak{D}_\infty(T)$, and if $x \in X_\lambda$ then $(T-\lambda I)^n x = (T-\lambda I)^n E_\lambda x = B_{n+1} x = 0$, by Theorem 10.1 (i). Next, we have $\mathfrak{M}^n(\lambda) \subset X_\lambda$ if $n \geqq 1$. For it follows by induction from Theorem 10.1 (f) and (h) that $(T-\lambda I)^n A_{n-1} = (T-\lambda I) A_0$ if $n \geqq 1$. Also, if $x \in \mathfrak{D}_n(T)$, $(T-\lambda I)^n A_{n-1} x = A_{n-1}(T-\lambda I)^n x$, by Theorem 6.2 (a). Hence, if $x \in \mathfrak{M}^n(\lambda)$, $0 = A_{n-1}(T-\lambda I)^n x = (T-\lambda I) A_0 x = E_\lambda x - x$, by Theorem 10.1 (g). Therefore $\mathfrak{M}^n(\lambda) \subset X_\lambda$. It now follows that $X_\lambda = \mathfrak{M}^n(\lambda)$ if $n \geqq m$. Since $B_{m+1} = 0$ and $B_m \neq 0$ we easily see from Theorem 10.1 (i) that $\mathfrak{M}^{m-1}(\lambda)$ is a proper subset of $\mathfrak{M}^m(\lambda)$. Using Theorem 10.7 (d) we obtain the proof of (b) in the present theorem.

We next observe that $X_{\sigma'} \subset \mathfrak{N}^n(\lambda)$ if $n \geqq 1$. This follows by an induction argument, starting from the fact that $\lambda \in \varrho(T, X_{\sigma'})$ $\big($by Theorem 8.2 (c)$\big)$. Now, if $n \geqq m$, the only element common to $\mathfrak{M}^n(\lambda)$ and $\mathfrak{N}^n(\lambda)$ is 0; this follows from (b). Since $X = X_\lambda \oplus X_{\sigma'}$ it follows by Lemma 1 of § 8 that $\mathfrak{N}^n(\lambda) = X_{\sigma'}$ if $n \geqq m$. It remains to prove the first part of (c). Suppose $x \in \mathfrak{M}^m(\lambda) - \mathfrak{M}^{m-1}(\lambda)$ and let $y = (\lambda I - T)^{m-1} x$. Then $y \neq 0$ and $y \in \mathfrak{M}^m(\lambda)$, whence $y \in \mathfrak{N}^{m-1}(\lambda) - \mathfrak{N}^m(\lambda)$. Appealing to Theorem 10.7 (b) we complete the proof of Theorem 10.8.

11. **The Sylvester theorem.** Our first theorem is the generalization, for a general T, of what Dunford calls the Sylvester theorem (Dunford [1], p. 204, Theorem 2.24; [2], pp. 645–646). As usual we write $f^{-1}(\tau)$ for the set of points $\lambda \in \Delta(f)$ such that $f(\lambda) \in \tau$, τ being a given set.

Theorem 11.1. *Suppose that $f \in \mathcal{G}(T)$, and that τ is a spectral set of $f(T)$. Let $\sigma = \sigma_e(T) \cdot f^{-1}(\tau)$. Then σ is a spectral set of T, and $E_\tau[f(T)] = E_\sigma[T]$.*

Proof: It is readily seen that σ is closed, for τ and $\sigma_e(T)$ are closed and f is continuous on $\sigma_e(T)$. The set σ is bounded if $f(\infty)$ is not in τ or if $T \in [X]$. The point $\lambda = \infty$ is in σ if and only if $f(\infty) \in \tau$ and T is not in $[X]$. From Theorem 9.1 and the fact that τ and τ' are disjoint we see that $\sigma_e(T)-\sigma = \sigma_e(T) \cdot f^{-1}(\tau')$. If follows that σ and $\sigma_e(T)-\sigma$ are disjoint closed sets; hence σ is a spectral set of T. We see by (8.3) and remarks made earlier in this proof that it suffices to prove the last assertion of the theorem on the assumption that σ is bounded and that $f(\infty)$ is not in τ. Let U_1 and U_2 be open sets with the following properties: U_1 is bounded, U_2 contains a neighborhood of $\lambda = \infty$, $\overline{U}_2 \subset C(\overline{U}_1)$, $\tau \subset U_1$, τ' and $f(\infty)$ are in U_2. Let U_3 and U_4 be open sets with the following properties: U_3 is bounded, U_4 contains a neighborhood of $\lambda = \infty$, $\overline{U}_4 \subset C(\overline{U}_3)$, $\sigma \subset U_3$, $\sigma' \subset U_4$, $\overline{U}_3$ and $\overline{U}_4$ are in $\Delta(f)$, $f(\overline{U}_3) \subset U_1$, $f(\overline{U}_4) \subset U_2$. Such open sets exist. Now define the function f_τ to have the value 1 on U_1 and 0 on U_2, and define f_σ to have the value 1 on U_3 and 0 on U_4. Then $f_\tau(f(\lambda)) = f_\sigma(\lambda)$ on the union of U_3 and U_4, while $f_\tau(f(T)) = E_\tau[f(T)]$, $f_\sigma(T) = E_\sigma[T]$. The proof is now completed by appealing to Theorems 9.4 and 4.4.

Theorem 11.2. *Suppose that $f \in \mathcal{G}(T)$, that $\beta \in \sigma(f(T))$, and that $f(\lambda)-\beta$ is not identically zero on any component of $\Delta(f)$. Then $f(\lambda)-\beta$ has at most a finite number of zeros in $\sigma(T)$ and at least one zero in $\sigma_e(T)$. There is at least one such zero in $\sigma(T)$ if $\beta \neq f(\infty)$ or if $T \in [X]$, but not necessarily so otherwise. If there are such zeros, say $\lambda_1, \ldots, \lambda_k$, of multiplicities $m_1, \ldots, m_k$, and if β is a pole of $R_\xi(f(T))$ of order m, then λ_i $(i = 1, \ldots, k)$ is a pole of $R_\xi(T)$ of order not exceeding mm_i.*

Proof: The zeros of $f(\lambda)-\beta$ cannot have a limit point in $\sigma(T)$, nor can they form an unbounded set, for either of these alternatives would imply that $f(\lambda)-\beta$ vanishes identically in some component of $\Delta(f)$. Hence the set of zeros of $f(\lambda)-\beta$ in $\sigma(T)$ is not infinite. If $\beta \neq f(\infty)$ or if $T \in [X]$, there is at least one such zero, by Theorem 9.1 (b), but not necessarily otherwise, as we remarked in connection with the latter theorem.

Now define $P(\lambda) = (\lambda-\lambda_1)^{m_1} \ldots (\lambda-\lambda_k)^{m_k}$, and define $h(\lambda) \in \mathcal{G}_0(T)$ with $\Delta(h) = \Delta(f)$, in such a way that $P(\lambda)h(\lambda) = f(\lambda)-\beta$; then $h(\lambda)$ has a zero of finite order at $\lambda = \infty$, and no zeros in $\sigma(T)$. By Theorem 6.3 we have $P(T)h(T) = f(T)-\beta I$.

Thus far we have not assumed that β is a pole of $R_\xi(f(T))$. Making this assumption, we can choose $\tau = (\beta)$ as the spectral set of $f(T)$ in Theorem 11.1. The corresponding σ is the finite set $(\lambda_1, \ldots, \lambda_k)$, together with $\lambda = \infty$ in case $f(\infty) = \beta$ and $T \notin [X]$; then $E_\beta[f(T)] = E_\sigma[T] = E_{\lambda_1}[T] + \cdots + E_{\lambda_k}[T]$. Now $(f(T)-\beta I)^m E_\beta[f(T)] = 0$, by Theorem 10.1 (i). Thus after multiplying by $E_{\lambda_i}[T]$, we obtain

$$\{P(T)h(T)\}^m E_{\lambda_i}[T] = 0 \,. \tag{11.1}$$

We can permute the factors here to obtain

$$\{h(T)\}^m\{Q_i(T)\}^m(T-\lambda_i I)^{mm_i}E_{\lambda_i}[T] = 0 ,$$

where $Q_i(T)$ is the product of all factors composing $P(T)$ with the exception of $(T-\lambda_i I)^{m_i}$. Now $h(T)x = 0$ implies $x = 0$, by Theorem 7.3; also, $Q_i(T)x = 0$ and $x \in X_{\lambda_i}$ imply $x = 0$, by Theorem 8.2 (c). Hence $(T-\lambda_i I)^{mm_i}E_{\lambda_i}[T] = 0$; this means that λ_i is a pole of order not exceeding mm_i of $R_\lambda(T)$, for λ_i is isolated in $\sigma(T)$.

Theorem 11.3. *Suppose that $f \in \mathcal{G}(T)$, that $\beta \in (f(T))$, that $f(\lambda)-\beta$ has zeros of multiplicities $m_1,\dots, m_k$ at the points $\lambda_1,\dots,\lambda_k$ in $\sigma(T)$ and at no other points in $\sigma(T)$. If $\beta = f(\infty)$ we make the further assumption that $T \in [X]$. Suppose finally that the points $\lambda_1,\dots, \lambda_k$ are poles of orders $\nu_1,\dots, \nu_k$ of $R_\lambda(T)$. Then β is a pole of $R_\lambda(f(T))$ of order not exceeding the least integer n such that $m_i n \geqq \nu_i$, $i = 1,\dots, k$.*

Proof: We first prove that β is isolated in $\sigma(f(T))$. For suppose it is not. Then there must exist a sequence $\{\beta_n\}$ of numbers, distinct from each other, from $f(\infty)$ and β, such that $\beta_n \in \sigma(f(T))$ and $\beta_n \to \beta$. By Theorem 9.1 (b) there exist numbers $\alpha_n \in \sigma(T)$ such that $f(\alpha_n) = \beta_n$. The sequence $\{\alpha_n\}$ is bounded, for if not we conclude $\beta = f(\infty)$. But in that case we are assuming as well that $T \in [X]$, and this implies that $\sigma(T)$ is bounded. Hence $\{\alpha_n\}$ contains a subsequence converging to a limit $\alpha \in \sigma(T)$, so that $f(\alpha) = \beta$. Therefore α is one of the points λ_i. But since the sequence $\{\alpha_n\}$ and the set $(\lambda_1,\dots, \lambda_k)$ have no points in common, we are led to the conclusion that λ_i is not isolated in $\sigma(T)$, contrary to the hypothesis.

Now let n be an integer such that $m_i n \geqq \nu_i$, $i = 1,\dots, k$. We have $(T-\lambda_i I)^{\nu_i}E_{\lambda_i}[T] = 0$. Hence, if $\sigma = (\lambda_1,\dots, \lambda_k)$, we have $\{P(T)\}^n E_\sigma[T] = 0$, where $P(\lambda)$ is defined as in the proof of Theorem 11.2. As in that proof, $(f(T)-\beta I)^n E_\beta[f(T)] = \{P(T)h(T)\}^n E_\sigma[T] = 0$, so that β is a pole of $R_\lambda(f(T))$ of order not exceeding n.

The exceptional case $\beta = f(\infty)$ is covered in the following theorem.

Theorem 11.4. *Suppose that $f \in \mathcal{G}(T)$, that $f(\lambda)-f(\infty)$ is not identically zero on any component of $\Delta(f)$, and that the point $\lambda = f(\infty)$ is a pole of $R_\lambda(f(T))$. Then $T \in [X]$.*

Proof: Consider the α-associates of T and f, respectively. We have $f(T) = \varphi^*(A)$, by Theorem 5.3. Let $\psi(\mu)$ be a $\mathcal{G}(A)$ extension of $\varphi(\mu)$ (see the definition following the proof of Theorem 4.4) such that $\psi(\mu)-f(\infty)$ does not vanish identically on any component of $\Delta(\psi)$. Then $\psi(A) = \varphi^*(A)$, and the point $f(\infty)$ is a pole of $R_\lambda(\psi(A))$. Now $\psi(0) = f(\infty)$. If the point $\mu = 0$ is in $\sigma(A)$ we conclude by Theorem 11.2 that this point is a pole of $R_\mu(A)$, and hence, by Theorem 10.2, that the point $\mu = 0$ is in $p(A)$. But this contradicts Theorem 5.2. Hence the point $\mu = 0$ is not in $\sigma(A)$, and this implies $T \in [X]$, again by Theorem 5.2.

Theorem 11.5. *Suppose that $f \in \mathcal{G}_0(T)$, and let $f(T)$ be completely continuous. Suppose that $\lambda \in \sigma(T)$ and $f(\lambda) \neq 0$. Then λ is isolated in $\sigma(T)$, $X_\lambda(T)$ is finite dimensional, and λ is a pole of $R_\xi(T)$ of order not exceeding the dimension of $X_\lambda[T]$.*

This is a direct generalization of a theorem given by Dunford ([1], p. 207, Theorem 2.32). Dunford's proof applies without formal modification, since we have established, in Theorems 9.1, 11.1, and 8.2 (c), the validity of his arguments for the case of a T not necessarily in $[X]$. Hence we omit a detailed proof.

12. **The minimal equation theorem.** We are going to examine the conditions under which it may happen that $f(T) = 0$, where $f \in \mathcal{G}(T)$.

Theorem 12.1. *Suppose that $f \in \mathcal{G}(T)$. Then $f(T) = 0$ if and only if the following three conditions are fulfilled:*

(a) *$f(\xi)$ vanishes identically in any component of $\Delta(f)$ which contains an infinite number of points of $\sigma(T)$;*

b) *$f(\xi)$ vanishes identically in any component of $\Delta(f)$ which contains an isolated essential singularity of $R_\xi(T)$. (In accordance with Theorem 10.5 this condition is to be interpreted as meaning that $f(\xi)$ vanishes identically in a neighborhood of $\xi = \infty$ if $\sigma(T)$ is bounded and T does not belong to $[X]$.)*

(c) *If λ is a pole of $R_\xi(T)$ of order m, then $f^{(j)}(\lambda) = 0$, $j = 0, \ldots, m-1$* (i. e. *f has a zero of at least m th order at $\xi = \lambda$*).

Proof: This theorem generalizes one of Dunford's ([1], p. 200, Theorem 2.19). Observe that we must have $f(\xi) \equiv 0$ in a neighborhood of $\xi = \infty$ if T does not belong to $[X]$ (by (a) if $\sigma(T)$ is unbounded, otherwise by (b)).

Our arguments do not differ much from those used by Dunford, but it seems best to give the proof in full. Let us first consider an isolated point $\lambda \in \sigma(T)$. Suppose that $f \in \mathcal{G}(T)$, that $f(\xi)$ is not identically zero in the neighborhood of $\xi = \lambda$, and that k is the smallest integer such that $f^{(k)}(\lambda) \neq 0$. We do not yet assume that $f(T) = 0$. Let r be a positive number such that all of $\sigma(T)$ except λ lies outside the circle $|\xi - \lambda| = 2r$, and such that f is regular and $f(\xi) \neq 0$ when $0 < |\xi - \lambda| < r$. Let us define $g_k(\xi)$ so that it is regular at $\xi = \lambda$ and

$$g_k(\xi) = (\xi - \lambda)^{-k} f(\xi), \quad 0 < |\xi - \lambda| < r,$$
$$g_k(\xi) = (\xi - \lambda)^{-(k+1)}, \quad |\xi - \lambda| > 2r.$$

Then g_k does not vanish in $\sigma(T)$ and it has a zero of order $k+1$ at $\xi = \infty$. Hence, by Theorem 7.3, $g_k(T)$ defines a $1-1$ mapping of X onto $\mathfrak{D}_{k+1}(T)$. Following the notation used in Theorem 10.1 and its proof, we have $(\xi-\lambda)^k g_k(\xi)f_{-1}(\xi)=f(\xi)f_{-1}(\xi)$. Consequently, by the operational calculus, $(T-\lambda I)^k g_k(T)E_\lambda[T]=f(T)E_\lambda[T]$. The order of the factors may be rearranged, giving

$$g_k(T)(T-\lambda I)^k E_\lambda[T]=f(T)E_\lambda[T]\,. \tag{12.1}$$

If we now assume $f(T)=0$, we can conclude that $(T-\lambda I)^k E_\lambda[T]=0$, since $g_k(T)x=0$ implies $x=0$. It follows that λ is a pole of $R_\xi(T)$ of order not exceeding k. The necessity of condition (c) is now evident. The necessity of condition (b), insofar as it pertains to essential singularities in the finite plane, is likewise clear from the above argument.

Next we prove that $f(T)=0$ implies that condition (a) is satisfied. For, when $f(T)=0$, $\sigma(f(T))$ consists of the single point $\xi=0$ and so $f(\xi)$ vanishes identically on $\sigma(T)$, by Theorem 9. 1. Condition (a) now follows. For an unbounded component of $\Delta(f)$ it should be observed that $f(\xi)$ is either identically zero in the component or has an isolated zero at $\xi=\infty$, and the latter possibility is ruled out if there are an infinite number of points of $\sigma(T)$ in the component.

Now suppose that $f(T)=0$, that $\sigma(T)$ is bounded, and that T is not in $[X]$. We consider the α-associates of T and f. We have seen in the proof of Theorem 10.5 that $\mu=0$ is an essential singularity of $R_\mu(A)$. Now $f(T)=\varphi^*(A)=\psi(A)$, where $\psi(\mu)$ is a $\mathcal{G}_0(A)$ extension of $\varphi(\mu)$. Hence, by part (a), ψ (and also φ) vanishes identically in the neighborhood of $\mu=0$. This means that f vanishes identically in the neighborhood of $\xi=\infty$. We have now proved the necessity of all the conditions (a)–(c).

We now consider the converse proof. We may suppose that $f(\xi)$ is not identically zero in at least one component of $\Delta(f)$, for otherwise $f(T)=0$ by definition. There are at most a finite number of points $\lambda_1,\dots,\lambda_p$ of $\sigma(T)$, all of them poles of $R_\xi(T)$, in the components of $\Delta(f)$ where f does not vanish identically. It is easy to see that the integral formula defining $f(T)$ can, under these conditions, be given the form

$$f(T)=f(\infty)I+\sum_{\nu=1}^{p}\frac{1}{2\pi i}\int_{C_\nu} f(\xi)R_\xi(T)\,d\xi-\frac{1}{2\pi i}\int_C f(\xi)R_\xi(T)\,d\xi$$

where $C_1,\dots,C_p$ are small nonoverlapping circles about the points $\lambda_1,\dots,\lambda_p$, and C is a very large circle. All the circles are sensed counterclockwise. The term $f(\infty)I$ and the integral over the large circle actually occur only in case $T\in[X]$ (since otherwise f vanishes identically in the unbounded component of $\Delta(f)$). These terms cancel each other when $T\in[X]$, however, as is easily seen from formula (2.2). Now

$$\frac{1}{2\pi i}\int_{C_\nu} f(\xi)R_\xi(T)d\xi = f(T)E_{\lambda_\nu}[T]$$

by the operational calculus. Hence

$$f(T) = \sum_{\nu=1}^{p} f(T)E_{\lambda_\nu}[T]\,. \tag{12.2}$$

Now let m_ν be the order of λ_ν as a pole of $R_\xi(T)$, and k_ν its order as a zero of $f(\lambda)$. Then $k_\nu \geqq m_\nu$, by condition (c). Hence $(T-\lambda_\nu I)^{k_\nu}E_{\lambda_\nu}[T] = 0$. It now follows by (12.1) and (12.2) that $f(T) = 0$.

The foregoing theorem is applicable to $P(T)$ when $T \in [X]$, by taking $f(\lambda)$ as some $\mathcal{G}(T)$ extension of the polynomial $P(\lambda)$. The situation when T is not in $[X]$ is covered in the following theorem.

Theorem 12.2. *Suppose that T is not in $[X]$. Then there exists no polynomial $P(\lambda)$ of degree n ($n \geqq 1$), not identically zero, such that $P(T)x = 0$ for every x in $\mathfrak{D}_n(T)$.*

Proof: Suppose such a polynomial does exist. Take $\alpha \in \varrho(T)$, $g(\lambda) = (\lambda-\alpha)^{-(n+1)}$, $f(\lambda) = P(\lambda)g(\lambda)$. Then $f(T) = P(T)g(T) = 0$, by Theorem 6.3. Therefore, by Theorem 12.1, f must vanish identically in a neighborhood of $\lambda = \infty$. Since f does not in fact so vanish, the proof is complete.

We conclude with a theorem converse to Theorem 10.8.

Theorem 12.3. *Suppose, for fixed values λ and m ($m \geqq 1$), that $\mathfrak{R}^m(\lambda)$ is closed and that $X = \mathfrak{M}^m(\lambda) \oplus \mathfrak{R}^m(\lambda)$. Then, if $\mathfrak{M}^m(\lambda)$ contains nonzero elements, we conclude that λ is a pole of $R_\xi(T)$ of order not greater than m. Otherwise $\lambda \in \varrho(T)$.*

Proof: For the case $T \in [X]$ this theorem is included in a theorem given by Dunford ([1], p. 201, Theorem 2.23). For convenience let us write $M_1 = \mathfrak{M}^m(\lambda)$, $M_2 = \mathfrak{R}^m(\lambda)$. It is readily proved by induction that $M_1 \subset \mathfrak{D}_\infty(T)$. We have $T(M_1) \subset M_1$, since T and $(\lambda I - T)^m$ permute on the domain $\mathfrak{D}_{m+1}(T)$. Also $T\big(M_2\mathfrak{D}(T)\big) \subset M_2$, for if $x = (\lambda I - T)^m y$, where $y \in \mathfrak{D}_m(T)$ and $x \in M_2\mathfrak{D}(T)$, it follows that $y \in \mathfrak{D}_{m+1}(T)$ and $Tx = (\lambda I - T)^m Ty$. In the notations of Theorem 8.1 it is clear that $E_i\big(\mathfrak{D}(T)\big) \subset \mathfrak{D}(T)$, since $E_1[X] = M_1 \subset \mathfrak{D}_\infty(T)$. Hence we may apply Theorem 8.1 to the present situation.

We prove first that λ is a pole of the resolvent of T when T is considered as an operator on M_1, provided that M_1 contains nonzero elements. Consider the function $f(\xi) = (\lambda-\xi)^m g_{m+1}(\xi)$, where g_{m+1} is defined as in the proof of Theorem 10.4. Then $f(T) = (\lambda I - T)^m A^{m+1}$, by Theorem 6.3. If $x \in M_1$ we have $f(T)x = A^{m+1}(\lambda I - T)^m x = 0$. We may apply Theorem 12.1, with M_1 as the whole space instead of X (this change in point of view does not affect the definition of $f(T)$ as applied to elements of M_1). The conclusion is that $\sigma(T, M_1)$ is a finite set composed entirely of poles of the resolvent. Since T is bounded on M_1, the set $\sigma(T, M_1)$ is not empty. Now $f(\xi)$ has just one zero (at $\xi = \lambda$), and that of order m. We conclude further, by Theorem 12.1, that $\sigma(T, M_1)$ consists of the single point λ, which is a pole of order m_1, $m_1 \leqq m$, of the resolvent of T as an operator on M_1.

We next prove that λ is an isolated point of $\sigma(T)$. By Theorem 8.1 (d) it is enough to prove that $\lambda \in \varrho(T, M_2)$. For convenience let us write $P(\xi) = (\lambda - \xi)^m$. We prove that $P(T)$ defines a $1-1$ mapping of $M_2\mathfrak{D}_m(T)$ onto M_2. If $x \in M_2\mathfrak{D}_m(T)$ and $P(T)x = 0$ we have $x \in M_1M_2$, and hence $x = 0$. If $y \in M_2$ there exists an $x \in \mathfrak{D}_m(T)$ such that $P(T)x = y$, by the definition of M_2. Then $E_2x \in M_2\mathfrak{D}_m(T)$ and $P(T)E_2x = P(T)(x - E_1x) = y$, since $E_1x \in M_1$. It follows that the origin belongs to $\varrho(P(T), M_2)$. But $P(\sigma(T, M_2)) = \sigma(P(T), M_2)$ by Theorem 9.3, and hence $P(\xi)$ does not vanish on $\sigma(T, M_2)$. Thus $\lambda \in \varrho(T, M_2)$, and λ is isolated in $\sigma(T)$.

The next stage in the argument is the proof that $M_1 = X_\lambda$. Theorem 10.6 shows that $M_1 \subset X_\lambda$, since $M_1 \subset \mathfrak{D}_\infty(T)$. On the other hand, $X_\lambda \subset D_\infty(T)$ by Theorem 10.1 (a) and (f). Let H be the operator defined on the space M_2 as the inverse of $\lambda I - T$; H exists and is bounded, since $\lambda \in \varrho(T, M_2)$. Let ε be chosen positive and such that $\varepsilon \|H\| < 1$. Now suppose that $x \in X_\lambda$. Then $(\lambda I - T)^m E_1 x = 0$. By Theorem 10.6 we know that $\lim\limits_{n\to\infty} (\lambda I - T)^n \varepsilon^{-n} x = 0$. Hence $\lim\limits_{n\to\infty} (\lambda I - T)^n \varepsilon^{-n} E_2 x = 0$. Let $y_n = (\lambda I - T)^n \varepsilon^{-n} E_2 x$. By induction we may prove readily that $y_n \in M_2$ and $H^n y_n = \varepsilon^{-n} E_2 x$. Hence $\|E_2 x\| = \|\varepsilon^n H^n y_n\| \leqq (\varepsilon \|H\|)^n \|y_n\| \leqq \|y_n\|$. We conclude that $E_2 x = 0$, whence $x \in M_1$. The proof that $M_1 = X_\lambda$ is now complete.

It now follows by Theorem 10.3 that λ is a pole of order m_1 of $R_\xi(T)$. The proof of our theorem is now complete except for the consideration of the case in which $\mathfrak{N}^m(\lambda)$ contains the zero element alone. In that case $M_2 = X$ and $P(T) = (\lambda I - T)^m$ defines a $1-1$ mapping of $\mathfrak{D}_m(T)$ onto all of X, so that the origin belongs to $\varrho(P(T))$. By Theorem 9.3, $P(\xi)$ cannot vanish on $\lambda(T)$. Therefore $\lambda \in \varrho(T)$.

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