MULTIDIMENSIONAL PRINCIPAL INTEGRALS, BOUNDARY VALUE PROBLEMS AND INTEGRAL EQUATIONS.

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1. **Introduction.** The object of this work is to study principal integrals and kernels, extended over sufficiently smooth bounded surfaces S, possibly having 'edges', imbedded in the Euclidean 3-space; the edges are to be suitably 'smooth' (precise formulations are given in the sequel). On the basis of this study developments are given, relating to *boundary value problems of Hilbert-Riemann type,*

$$\Phi^{+}(t) = \Phi^{-}(t)A(t)+B(t) \qquad (t \text{ on } S) \tag{1.1}$$

[A, B are of a Hölder class on S; $A \neq 0$ on S]. Certain classes of solutions $\Phi(x)$ will be sought, regular in a suitable sense for x in $C(S)$ (complement of S), for which the boundary values $\Phi^{+}(t)$, $\Phi^{-}(t)$ on designated positive and negative sides of S satisfy (1.1). These boundary values will generally depend on the direction of approach. Further, on the basis of our theory, *we study singular integral equations*

$$a(t)u(t)+\int_S \frac{k(y, t)}{r^2(y, t)}\, u(t)d\sigma(y)+T(u|t) = f(t) \tag{1.2}$$

$$\big(r(y, t) = \text{distance between } y \text{ and } t\big)\,,$$

the kernel being of a principal type (section 3), the operator T of a suitably regular kind, $a(t), f(t)$ of a Hölder class on S (of a specified order of infinity near the edges). We shall actually give a process of regularizing (1.2), so that the resulting equation is a regular integral equation of the second kind.

There exist many developments along these directions in the complex plane E, with S denoting a finite number of open or closed, suitably smooth curves in E, the principal kernels being essentially of Cauchy type and the integrations being in the sense of Cauchy principal values; this field has been studied by a number of authors, of whom we shall mention N. E. Mushelishvili, whose book[1] contains an extensive bibliography, Vecoua, W. J. Trjitzinsky[2] (who considers the case of intersecting curves) and Michlin, *whose monograph*[3] *will be referred to as* [*M*]. The transition from the situation in the complex plane, as indicated above, to a greater number of dimensions presents substantial new difficulties. Instead of studying the more general problem, when S is a (suitably smooth) n-dimensional manifold ($n \geqq 2$), *with edges*, imbedded in m-space ($m > n$) — we are limiting ourselves, as stated at the beginning of this section. This is done for simplicity and is justified by the fact that the case actually treated in these pages embodies the essential difficulties of the case when S is a n-dimensional ($n > 3$) manifold. In this sense the subject of multi-dimensional principal integrals, and the related problems, have been implicitly treated in the present work.

Amongst the outstanding developments in the field of multidimensional integrals are those of Michlin [*M*], G. Giraud[4] (also see references to Giraud in [*M*] and Tricomi) (see [*M*]). *The essentially novel feature of our work is the possible presence of edges in S, a circumstance adding great new difficulties.* It is to be noted, however, that very special instances, when surfaces with edges are present, have been ingeniously treated by Tricomi. We did not find it possible to generalize Tricomi's methods to our more general case; thus our methods are unrelated to those of Tricomi. The work of Michlin [*M*] contains some valuable indications for the purposes at hand, especially with respect to regularizing (1.2). On the other

[1] N. E. Mushelishvili, *Granicnye Zadaci Teorii Funkcii i Nekotorye ih Prilozeniya k Matematiceskoi Fisike*, Moscow-Leningrad, 1946, pp. 1–448.

[2] W. J. Trjitzinsky, *Singular Integral Equations with Cauchy Kernels*, Trans. Amer. Math. Soc., vol. 60 (1946), pp. 167–214.

[3] S. G. Michlin, *Singular Integral Equations*, Uspehi Matem. Nauk (N. S.) 3, no. 3 (25), pp. 29–112 (1948). This work is referred to in the text as [*M*].

[4] G. Giraud, *Equations à intégrales principales*, Ann. Sc. de l'Ecole Norm. Sup., t. 51, 1934, pp. 251–372; also, in the same Journal: *Sur une classe d'équations intégrales où figurent des valeurs principales d'intégrales simples*, t. 56, 1939, pp. 119–172.

hand, it is Giraud's work that enables transition to n-dimensional ($n > 2$) manifolds.

S is to denote a finite number of bounded surfaces, some closed and some open (that is, having edges); these surfaces are to be without common points; for each a positive and a negative side can be assigned. In section 2 precise hypotheses satisfied by S are given; also a definition and investigation of so called 'completely regular' surfaces is presented; the latter are used just in a few connections.

Much of this work relates to integrals

$$\Psi(t) = \int_S \frac{k(y, t)}{r^2(y, t)} q(y) d\sigma(y) \qquad [q \text{ Hölder on } S;\ t \text{ on } S] , \tag{1.3}$$

where the kernel $k(y,t)r^{-2}(y,t)$ $\big((3.1), (3.1\text{a})\big)$ is a principal one in the sense of section 3. In Definition 3.19 classes $[\alpha|S]$, $[\alpha|C(S)]$ are defined. Most of the developments are under the conditions of Hypothesis 3.20 (supplemented by other assumptions, such as (3.27)). Theorem 3.25 presents conditions in order that the integral (1.3) should exist, for t on S, in the sense of principal values. The integral, related to (1.3),

$$\Psi(x) = \int_S \frac{k(y, x)}{r^2(y, x)} q(y) d\sigma(y) \qquad \big(x \text{ in } C(S)\big) , \tag{1.3a}$$

exists in the ordinary sense. In section 4 it is proved that for $\Psi(x)$ there exist analogues of the well known Plemelj formulas (for integrals with Cauchy kernels in the complex plane); thus Theorem 4.28 asserts that, when x $\big($in $C(S)\big)$ tends nontangentially to a point t on S, one has

$$\lim_{x \to t} \Psi(x) = q(t)K(t) + \Psi(t) ,$$

where $\Psi(t)$ is (1.3) (that is, an integral in the sense of principal values), while $K(t)$ is a function independent of q, but generally depending on the direction of approach; $K(t)$ is explicitly given by (4.22); this is obviously a very important function in all boundary value problems, relating to integrals of form (1.3), (1.3a); some of its properties are stated in Lemma 4.26.

Theorem 5.38 asserts, substantially, that $\Psi(x)$ is $[\alpha|C(S)]$ (if $\alpha > 0$), is $[0, \log|C(S)]$ (if $\alpha = 0$; cf. Definition 3.19), provided $q(y)$ is $[\alpha|S]$, with $0 \leqq \alpha < 1$. This result refers essentially to the order of infinity, near the edges of S, of $\Psi(x)$. In section 6 a study is made of the order of infinity of the principal integral $\Psi(t)$ [(1.3), t on S], for t near edges; theorem 6.36 amounts essentially to the assertion that $\Psi(t)$ is $[\lambda|S]$, if $q \subset [\alpha|S]$ $(\alpha+\beta < 1)$, where λ is a certain number depending on the various Hölder exponents and numbers, specifying orders of infinity (near edges) of $q(y)$ and of the kernel in (1.3). Theorem 6.38 supplements the above result in the case when S is completely regular (in the sense of section 2).

In theorem 7.18 is found the asymptotic form, near a point c on the edges β, of the curvilinear potential (7.1), whose density is along β and is of a Hölder class; (7.20) presents a solution of a certain related functional problem of use in treating (1.1).

Boundary value problems (1.1) of HILBERT-RIEMANN type are studied in section 8 [cf.: Notation 8.3; Definition 8.12; Lemmas 8.13, 8.14; Classes (A^*) (8.16), (B^*, A) (8.16a); Theorems 8.19, 8.25, 8.27, 8.29]; in these developments use is made of most of the preceding developments.

Now, in the complex plane (when S is a collection of curves and Cauchy kernels are involved) the situation is as follows. With the aid of Plemelj formulas singular integral equations are related to suitable Hilbert-Riemann boundary value problems; appropriate classes of solutions of the latter are found; then, using the fact that in $C(S)$ the integrals involved are analytic, one derives solutions of the integral equation from those for the boundary value problems. This idea was carried out first in a special case by T. CARLEMAN[1]; subsequently this idea of Carleman was combined with some other considerations, leading to a fairly complete theory of singular integral equations (with Cauchy kernels) in the complex plane, when S consists of a finite number of closed and open curves [Mushelishvili, Vecoua and many others]. It is natural therefore to attempt treatment of the singular integral equation (1.2) along similar lines. With the aid of Theorem 4.28 the equation (1.2) can be transformed into a Hilbert-Riemann boundary value problem (1.1); on the basis of section 8 one can find certain classes of solutions of the latter; however, it appears impossible to obtain solutions of the integral equation from those for the boundary problem, the reason for this being that nothing as simple (from our point of view) as the theory of analytic functions is now available. Thus, for the present, the indicated approach to integral equations will be not attempted. Instead we take the cue from the other method, largely due to MICHLIN and used by him in the complex plane as well as for equations with multidimensional principal integrals (cf. [M]). This method consists in forming an operator, whose application to the integral equation transforms the latter into a regular Fredholm equation of the second kind [provided the 'symbol' (Def. in [M]) does not vanish]. *Presence of edges in our case adds serious difficulties.* This approach is carried out in sections 9, 10, 11. Singular operators are studied in

[1] T. Carleman, *Sur la résolution de certaines équations intégrales*, Arkiv för Math. Astr. och Physik, t. 16, No. 26, 1922.

section 9. Theorem 10.32 presents a formula of composition of singular integrals (this involves Michlin's operators h_n, defined over the Euclidean plane E_2). Regular operators are specified in Definition 11.2 and, finally, the regularization is carried out in accord with Theorem 11.12.

The following notation will be used: points on S are denoted by $t = (t_1, t_2, t_3)$, $\tau, y, \eta, \ldots$; points in the complement $C(S)$ of S are designated by $x = (x_1, x_2, x_3), \ldots$; c^* is the generic designation for a positive constant; $l(t)$ is the distance from t to the 'edges' β of S; $r(x, y)$ = distance between x and y. The edges β are assumed to consist of a finite number of simple closed curves β_i, without common points and with continously turning tangents. Let c be any point of β_i and β_i' be the projection of β_i on the tangential plane P_c (at c) to S; Let the y_1-axis be along the tangent line at c and the y_2-axis extend from c in P_c; *we assume that near c the representation of β_i' in the (y_1, y_2) system is of form $y_2 = O(y_1^2)$.*

2. **Completely regular surfaces.** It will be assumed that in a neighborhood of every point τ, for which $l(\tau) > 0$ (that is, not on edges β) the following is true. On choosing the coordinate system (y_1, y_2, y_3) so that its origin O is at τ and that the y_1, y_2-plane is concident with the tangential plane at τ (*such a plane is assumed to exist for every τ*), the equation of the surface for y near O has the form

$$(2.1) \qquad y_3 = F(y_1, y_2) = a_{11}y_1^2 + 2a_{12}y_1y_2 + a_{22}y_2^2 + R(y_1, y_2)\,,$$

while

$$(2.1') \qquad \frac{\partial F}{\partial y_1} = 2a_{11}y_1 + 2a_{12}y_2 + R_1(y_1, y_2),\ \frac{\partial F}{\partial y_2} = 2a_{12}y_1 + 2a_{22}y_2 + R_2(y_1, y_2)\,,$$

$$(2.1\text{a}) \qquad \begin{gathered} \frac{\partial^2 F}{\partial y_i \partial y_j} = 2a_{ij} + R_{ij}(y_1, y_2)\ \ (i, j = 1, 2); \qquad |R(y_1, y_2)| \leqq c^* r^3(y)\,, \\ |R_i(y_1, y_2)| \leqq c^* r^2(y),\ \ |R_{ij}(y_1, y_2)| \leqq c^* r(y) \quad [r^2(y) = y_1^2 + y_2^2]\,. \end{gathered}$$

The following is assumed with respect to the nature of S near edges. If τ is a point on the edges and the y-system is chosen, as above, with its origin O at τ, the equation of S near this point being $y_3 = F(y_1, y_2)$, then the first and second order partial derivatives of $F(y_1, y_2)$ exist and are continuous and

$$(2.1'') \qquad \left| \frac{\partial^2 F(y_1, y_2)}{\partial y_i \partial y_j} \right| \leqq c^* \qquad (i, j = 1, 2)\,,$$

including a portion of β in the neighborhood of O (that is, of τ).

The above conditions will suffice for most of this work.

S will be said to be *completely regular* if the above holds as well as the following. On writing $\varrho^2 = (y_1-t_1)^2+(y_2-t_2)^2$ and forming the function

$$G_t(\varrho, \theta) = [F(t_1+\varrho\cos\theta, t_2+\varrho\sin\theta)-F(t_1, t_2)-F_{t_1}\cos\theta\,\varrho-F_{t_2}\sin\theta\,\varrho]\varrho^{-2} \tag{2.1b}$$
$$[F_{t_i} = \partial F/\partial t_i],$$

where ϱ, θ are polar coordinates (in the y_1, y_2-plane) with pole at (t_1, t_2), one has

$$|G_t(\varrho, \theta)-G_0(\varrho, \theta)| \leqq c^*(t_1^2+t_2^2)^{\frac{1}{2}}; \tag{2.1c}$$

$$\frac{\partial F}{\partial y_i} = \frac{\partial F}{\partial t_i}+G_{i,t}(\varrho, \theta)\varrho, \qquad |G_{i,t}(\varrho, \theta)-G_{i,0}(\varrho, \theta)| \leqq c^*(t_1^2+t_2^2)^{\frac{1}{2}}. \tag{2.1d}$$

S is completely regular if in (2.1) one has

$$R(y_1, y_2) = \sum_{i,j,k} b_{ijk}(y_1, y_2)y_iy_jy_k \tag{2.2}$$

$[b_{ijk} = b_{\alpha\beta\gamma}$ when (α, β, γ) is a permutation of $(i, j, k)]$, where

$$b_{ijk}^{\nu}(y_1, y_2) = \frac{\partial}{\partial y_\nu} b_{ijk}, \qquad b_{ijk}^{\nu\sigma}(y_1, y_2)y_k = y_k\frac{\partial^2}{\partial y_\nu \partial y_\sigma} b_{ijk} \tag{2.2a}$$

are $O(1)$ (that is, uniformly bounded in a vicinity of $y_1 = y_2 = 0$).

We now proceed to prove the above assertion. For R_1, R_2, R_{ij} in (2.1') we have

$$R_\nu(y_1, y_2) = \sum_{i,j} A_{ij}^{\nu}(y_1, y_2)y_iy_j, \qquad R_{\nu\sigma}(y_1, y_2) = \sum_k B_k^{\nu\sigma}(y_1, y_2)y_k, \tag{2.3}$$

$$A_{ij}^{\nu}(y_1, y_2) = 3b_{ij\nu}(y_1, y_2)+\sum_k b_{ijk}^{\nu}(y_1, y_2)y_k, \tag{2.3a}$$

$$B_k^{\nu\sigma}(y_1, y_2) = 6b_{k\nu\sigma}+3\sum_i [b_{ik\nu}^{\sigma}+b_{ik\sigma}^{\nu}]y_i+\sum_{i,j} b_{ijk}^{\nu\sigma}y_iy_j. \tag{2.3b}$$

By (2.2a) the coefficients in (2.2), (2.3) are $O(1)$; the inequalities (2.1a) thus ensue. In view of (2.2a)

$$b_{ijk}(y_1, y_2) \subset Lip\ 1, \qquad b_{ijk}^{\nu}(y_1, y_2)y_k \subset Lip\ 1; \tag{2.4}$$

$$A_{ij}^{\nu}(y_1, y_2) \subset Lip\ 1, \qquad B_k^{\nu\sigma}(y_1, y_2) \subset Lip\ 1. \tag{2.4a}$$

For the function $G_t(\varrho, \theta)$ (2.1b) we obtain

$$I = \varrho^2[G_t(\varrho, \theta)-G_0(\varrho, \theta)] = R(t_1+\varrho_1, t_2+\varrho_2)-R(t_1, t_2)-R(\varrho_1, \varrho_2) \tag{2.5}$$

$$-R_1(t_1, t_2)\varrho_1-R_2(t_1, t_2)\varrho_2 \qquad \left\{R_i(t_1, t_2) = \frac{\partial R_i}{\partial t_i};\ \varrho_1 = \varrho\cos\theta;\ \varrho_2 = \varrho\sin\theta\right\}$$

Substitution of (2.2), (2.3), (2.3a) yields

$$I = I'+I''; \qquad I' = \sum_{i,j,k}[b_{ijk}(t_1+\varrho_1,\, t_2+\varrho_2)-b_{ijk}(t_1,\, t_2)]t_i t_j t_k$$
$$-\sum_{i,j,\varkappa,\lambda} b^{\lambda}_{ij\varkappa}(t)t_i t_j t_\varkappa \varrho_\lambda; \qquad I'' = 3\sum_{i,j,k}[b_{ijk}(t_1+\varrho_1,\, t_2+\varrho_2)-b_{ijk}(t_1,\, t_2)]t_i t_j \varrho_k$$
$$+3\sum_{i,j,k} b_{ijk}(t_1+\varrho_1,\, t_2+\varrho_2)\varrho_i\varrho_j t_k+\sum_{i,j,k}[b_{ijk}(t_1+\varrho_1,\, t_2+\varrho_2)-b_{ijk}(\varrho_1,\, \varrho_2)]\varrho_i\varrho_j\varrho_k\,.$$

In view of the first property (2.4) one has

$$I'' = O\big(r^2(t)\varrho^2\big)+O\big(r(t)\varrho^2\big)+O\big(r(t)\varrho^3\big) = O\big(r(t)\varrho^2\big)\,, \tag{2.5a}$$

where $r(t) = (t_1^2+t_2^2)^{\frac{1}{2}}$. With the aid of a mean value theorem

$$I' = \sum_{i,j,\varkappa,\lambda}[b^{\lambda}_{ij\varkappa}(t_1+\vartheta\varrho_1,\, t_2+\vartheta\varrho_2)-b^{\lambda}_{ij\varkappa}(t_1,\, t_2)]t_i t_j t_\varkappa \varrho_\lambda \qquad (0 < \vartheta < 1)$$

and $I' = I_1+I_2$, where

$$I_1 = \sum_{i,j,\varkappa,\lambda}[b^{\lambda}_{ij\varkappa}(t_1+\vartheta\varrho_1,\, t_2+\vartheta\varrho_2)(t_\varkappa+\vartheta\varrho_\varkappa)-b^{\lambda}_{ij\varkappa}(t_1,\, t_2)t_\varkappa]t_i t_j \varrho_\lambda\,,$$
$$I_2 = -\vartheta\sum_{i,j,\varkappa,\lambda} b^{\lambda}_{ij\varkappa}(t_1+\vartheta\varrho_1,\, t_2+\vartheta\varrho_2)t_i t_j \varrho_\varkappa\varrho_\lambda\,.$$

By the second property (2.4)

$$I_1 = O\big(r^2(t)\varrho^2\big);$$

on the other hand, in view of the assumption regarding the first function displayed in (2.2a), we have $I_2 = O(r^2(t)\varrho^2)$. Thus I' is of the form $O(r^2(t)\varrho^2)$; together with (2.5a), this implies that

$$I = O(r(t)\varrho^2)\,. \tag{2.5b}$$

As a consequence of the first equality in (2.5) the above signifies that $G_t(\varrho,\, \theta)$ has the property (2.1c).

Turning to $G_{i,\,t}(\varrho,\, \theta)$, as defined by the equality in (2.1d), we obtain

$$I_\nu = \varrho[G_{\nu,\,t}(\varrho,\, \theta)-G_{\nu,\,0}(\varrho,\, \theta)] = R_\nu(t_1+\varrho_1,\, t_2+\varrho_2)-R_\nu(t_1,\, t_2)-R_\nu(\varrho_1,\, \varrho_2)\,.$$

Substituting (2.3) one deduces

$$I_\nu = \sum_{i,j}[A^{\nu}_{ij}(t_1+\varrho_1,\, t_2+\varrho_2)-A^{\nu}_{ij}(t_1,\, t_2)]t_i t_j$$
$$+2\sum_{i,j} A^{\nu}_{ij}(t_1+\varrho_1,\, t_2+\varrho_2)t_i\varrho_j+\sum_{i,j}[A^{\nu}_{ij}(t_1+\varrho_1,\, t_2+\varrho_2)-A^{\nu}_{ij}(\varrho_1,\, \varrho_2)]\varrho_i\varrho_j\,.$$

Since (by the remark subsequent (2.3b)) $A^{\nu}_{ij}(\ldots) = O(1)$, on taking note of the first property (2.4a), one obtains

$$I_\nu = O\big(r^2(t)\varrho\big)+O\big(r(t)\varrho\big)+O\big(r(t)\varrho^2\big) = O\big(r(t)\varrho\big);$$

accordingly

$$G_{\nu,t}(\varrho, \theta) - G_{\nu,0}(\varrho, \theta) = O(r(t)),$$

which establishes the inequality in (2.1d). *The italicized statement* (2.2), (2.2a) *is thus proved.* An analytic surface (that is, one for which F in (2.1) is a series in positive powers of y_1, y_2, convergent near $y_1 = y_2 = 0$) is completely regular.

3. **Principal kernels.** We look for conditions under which a series

$$\frac{k(y, x)}{r^2(y, x)} = \frac{1}{r^2(y, x)} [k_1(y, x) + k_2(y, x) + \cdots + k_m(y, x) + \cdots], \tag{3.1}$$

$$k_m(y, x) = \sum_{i_1, \ldots i_m = 1}^{3} \gamma_{i_1, i_2, \ldots i_m}(y) w_{i_1}(y, x) w_{i_2}(y, x) \ldots w_{i_m}(y, x) \tag{3.1a}$$

[$\gamma_{i_1 \ldots i_m} = \gamma_{j_1 \ldots j_m}$ when $(i_1 \ldots i_m)$ is a permutation of $(j_1 \ldots j_m)$], $w_i(y, x) = \dfrac{y_i - x_i}{r(y, x)}$, convergent for all y on S and all x, represents a principal kernel; the $\gamma_{i_1 \ldots i_m}(y)$ will be assumed to be of a Hölder class on S; more precise conditions in this regard will be given in the sequel. Let t be a fixed point on S (not on β); we write

$$k_m(y, x) = k'_m(t|y, x) + k''_m(t|y, x), \tag{3.2}$$

$$k'_m(t|y, x) = \sum_{i_1, \ldots i_m} \gamma_{i_1 \ldots i_m}(t) w_{i_1}(y, x) \ldots w_{i_m}(y, x),$$

$$k''_m(t|y, x) = \sum_{i_1, \ldots i_m} (\gamma_{i_1 \ldots i_m}(y) - \gamma_{i_1 \ldots i_m}(t)) w_{i_1}(y, x) \ldots w_{i_m}(y, x)$$

and

$$k(y, x) = k'(t|y, x) + k''(t|y, x), \tag{3.2a}$$

$$k'(t|y, x) = \sum_{m=1}^{\infty} k'_m(t|y, x), \qquad k''(t|y, x) = \sum_{m=1}^{\infty} k''_m(t|y, x).$$

Correspondingly for (1.3) one has

$$\Psi(t) = \Psi'(t) + \Psi''(t), \qquad \Psi'(t) = \int_S \frac{k'(t|y, t)}{r^2(y, t)} q(y) d\sigma(y), \tag{3.3}$$

$$\Psi''(t) = \int_S \frac{k''(t|y, t)}{r^2(y, t)} q(y) d\sigma(y).$$

Provided, as we assume it for the present without further detail, the $\gamma \ldots$ and q behave near β suitably, the integral for $\Psi''(t)$ will exist as an ordinary integral; the conditions for this will be inferred in the sequel by examining the inequality

$$\left| \frac{k''(t|y, t)}{r^2(y, t)} q(y) \right| \leq \frac{1}{r^2(y, t)} |q(y)| \sum_{m=1}^{\infty} \sum_{i_1, \ldots i_m} |\gamma_{i_1 \ldots i_m}(y) - \gamma_{i_1 \ldots i_m}(t)|. \tag{3.4}$$

Under such an assumption, *we are to show that* $k'(t|y, t)$ *is a principal kernel*, that is that the integral for $\Psi'(t)$ exists in the sense of principal values.

Introduce a Cartesian coordinate system (Y_1, Y_2, Y_3), whose origin O is at t and whose $+Y_3$-axis is coincident with the positive normal to S at t; the Y_1, Y_2-axes will be in the tangential plane to S at t. Capital letters will designate points in the new coordinate system. We thus have an orthogonal transformation

$$y_i = t_i + a_{i1}Y_1 + a_{i2}Y_2 + a_{i3}Y_3 \qquad (i = 1, 2, 3)\,, \tag{3.5}$$

where

$$\sum_i a_{ij}a_{ik} = \sum_i a_{ji}a_{ki} = \delta_{jk}\begin{cases} = 0 & (j \neq k) \\ = 1 & (j = k) \end{cases}; \qquad a_{i3} = n_i(t)\,, \tag{3.5a}$$

with the $n_i(t)$ denoting the direction cosines of the positive normal at t. One may, for instance, choose the a_{ij} as follows (when $n_2 \neq \pm 1$):

$$\begin{pmatrix} a_{11}, a_{12}, a_{13} \\ \\ a_{21}, a_{22}, a_{23} \\ \\ a_{31}, a_{32}, a_{33} \end{pmatrix} = \begin{pmatrix} \dfrac{n_3(t)}{\sqrt{1-n_2^2(t)}}, & \dfrac{-n_1 n_2}{\sqrt{1-n_2^2}}, & n_1 \\ 0, & \sqrt{1-n_2^2}, & n_2 \\ \dfrac{-n_1}{\sqrt{1-n_2^2}}, & \dfrac{-n_2 n_3}{\sqrt{1-n_2^2}}, & n_3 \end{pmatrix}. \tag{3.5b}$$

When $|n_2(t)|$ is near 1, a suitable modification of the above matrix is to be used. One accordingly has (cf. (3.2))

$$k'_m(t|y, t) = k'_m(Y, O) = \sum_{s_1, \ldots s_m = 1}^{3} \Gamma_{s_1 \ldots s_m}(t) W_{s_1}(Y, O) \ldots W_{s_m}(Y, O)\,, \tag{3.6}$$

where

$$W_s(Y, O) = r^{-1}(Y, O) Y_s\,, \tag{3.6a}$$

$$\Gamma_{s_1, s_2, \ldots s_m}(t) = \sum_{i_1, \ldots i_m = 1}^{3} \gamma_{i_1 \ldots i_m}(t) a_{i_1, s_1} a_{i_2, s_2} \ldots a_{i_m, s_m}\,.$$

Let $S_{t,a}$ ($a, > 0$, sufficiently small) be the portion of S, projecting orthogonally on the tangential plane at t into a circular region of center t and radius a. On writing

$$\Psi'(t) = \Psi'_a(t) + \Psi_a^{1,0}(t), \quad \Psi'_a(t) = \int_{S_{t,a}} \frac{k'(t|y, t)}{r^2(y, t)} q(y) d\sigma(y)\,, \tag{3.7}$$

$$\Psi_a^{1,0}(t) = \int \frac{k'(t|y, t)}{r^2(y, t)} q(y) d\sigma(y) \qquad (\text{over } S - S_{t,a})\,,$$

it is observed that the integral for $\Psi_a^{1,0}(t)$ exists in the ordinary sense, provided the kernel and $q(y)$ is of proper order of infinity for y near β. Accordingly *we are* *to secure existence of the integral for $\Psi'_a(t)$ in the sense of principal values.* In the (Y_1, Y_2, Y_3) coordinates one has

$$\Psi'_a(t) = \int_{S(O,a)} k'(Y,O) r^{-2}(Y,O) q(Y) d\sigma(Y)\,, \tag{3.8}$$

where $S(O,a) = S_{t,a}$ and

$$k'(Y,O) = \sum_{m=1}^{\infty} k'_m(Y,O) \qquad \text{(cf. (3.6))}\,. \tag{3.8a}$$

Introduce polar coordinates in the Y_1, Y_2-plane,

$$Y_1 = \varrho\cos\theta_1,\ Y_2 = \varrho\cos\theta_2,\ \theta_1 = \theta,\ \theta_2 = \frac{\pi}{2} - \theta\,. \tag{3.9}$$

Near O the equation of the surface will be of form

$$Y_3 = F(Y_1, Y_2) = O(\varrho^2)\,; \tag{3.10}$$

one will also have

$$\frac{\partial F}{\partial Y_i} = O(\varrho) \quad (i = 1,2), \qquad \left[1 + \left(\frac{\partial F}{\partial Y_1}\right)^2 + \left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}} = 1 + O(\varrho^2)\,,$$

$$d\sigma(Y) = \varrho\, d\varrho\, d\theta[1 + O(\varrho^2)]\,; \tag{1°}$$

furthermore

$$\frac{\varrho^2}{r^2(Y,O)} = 1 - \frac{Y_3^2}{Y_1^2 + Y_2^2 + Y_3^2}\,, \qquad \frac{Y_3^2}{Y_1^2 + Y_2^2 + Y_3^2} = \frac{O(\varrho^4)}{\varrho^2 + O(\varrho^4)} = O(\varrho^2)$$

so that

$$\frac{1}{r^2(Y,O)} = \frac{1}{\varrho^2}[1 + O(\varrho^2)]\,; \tag{2°}$$

whence, by (3.6a), (3.10) and (3.9),

$$W_3(Y,O) = O(\varrho^2) r^{-1}(Y,O) = O(\varrho^2)\frac{1}{\varrho}[1 + O(\varrho^2)]^{\frac{1}{2}} = O(\varrho)\,. \tag{3°}$$

$$W_s(Y,O) = [1 + O(\varrho^2)]\cos\theta_s \qquad (s = 1, 2)\,.$$

Thus, in view of (3.6)

$$\begin{aligned} k'_m(Y,O) &= \sum_{s_1,\dots s_m = 1}^{2} \Gamma_{s_1\dots s_m}(t) W_{s_1}(Y,O)\dots W_{s_m}(Y,O) + O(\varrho) \\ &= \sum_{s_1,\dots s_m = 1}^{2} \Gamma_{s_1\dots s_m}(t)\cos\theta_{s_1}\dots\cos\theta_{s_m} + O(\varrho) \end{aligned} \tag{3.11}$$

and, provided suitable conditions of convergence (to be specified in the sequel) are satisfied, (3.11a) $$k'(Y, O) = k^{1,*}(t, \theta) + k^{1,0}(\varrho, \theta); \qquad k^{1,0}(\varrho, \theta) = O(\varrho);$$

$$k^{1,*}(t, \theta) = \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{2} \Gamma_{s_1 \ldots s_m}(t) \cos \theta_{s_1} \ldots \cos \theta_{s_m}.$$

Accordingly as a consequence of (3.8), (1°), (2°) we may write formally

$$\Psi'_a(t) = \int_{\varrho=0}^{a} \int_{\theta=0}^{2\pi} [k^{1,*}(t, \theta) + k^{1,0}(\varrho, \theta)][1 + O(\varrho^2)] q(\varrho, \theta) \frac{d\varrho}{\varrho} d\theta,$$

where $q(\varrho, \theta) = q(Y)$. Since

$$[k^{1,*}(t, \theta) + k^{1,0}(\varrho, \theta)][1 + O(\varrho^2)] = k^{1,*}(t, \theta) + O(\varrho),$$

it follows that

(3.12) $$\Psi'_a(t) = \Psi_a^{1,*}(t) + \Psi_a^{1,0},$$

with the last term expressed by an ordinary integral and

(3.13) $$\Psi_a^{1,*}(t) = \int_{\varrho=0}^{a} \int_{\theta=0}^{2\pi} k^{1,*}(t, \theta) q(\varrho, \theta) \frac{d\varrho}{\varrho} d\theta$$

(formally). If in the Fourier expansion of $k^{1,*}(t, \theta)$,

(3.13a) $$k^{1,*}(t, \theta) = \tfrac{1}{2} f_0(t) + \sum_{n=1}^{\infty} [f_n(t) \cos n\theta + g_n(t) \sin n\theta]$$

one has $f_0(t) = 0$, it follows that

(3.14) $$\int_0^{2\pi} k^{1,*}(t, \theta) d\theta = 0 \qquad \text{[cf. (3.11a)]};$$

$k^{1,*}(t, \theta)$ is then the 'characteristic' (terminology of [M]) *of the kernel* (3.1) *at the point t (on S). When $f_0(t)$ in* (3.13a) *is zero, the integral* (3.13) *and, accordingly, the integral for $\Psi(t)$* (1.3) *will exist in the sense of principal values* (q is, of course, of a Hölder class). Since the condition securing the principal character of the kernel (3.1) is (3.14), we may also proceed as follows. One has

(3.15) $$\sum_{s_1, \ldots s_m = 1}^{2} \Gamma_{s_1 \ldots s_m}(t) \cos \theta_{s_1} \ldots \cos \theta_{s_m} = p_m(t) + p_m(t, \theta) [= k_m^{1,*}(t, \theta)],$$

$$p_m(t, \theta) = \sum_{\nu=1}^{m} [p'_{m,\nu}(t) \cos \nu\theta + p''_{m,\nu}(t) \sin \nu\theta], \qquad \int_0^{2\pi} p_m(t, \theta) d\theta = 0;$$

$p_m(t) + p_m(t, \theta)$ is the 'characteristic' of the kernel $k_m(y, x) r^{-2}(y, x)$ (cf. (3.1)); this kernel is, accordingly, a principal one if and only if $p_m(t) = 0$. In order that the kernel $k(y, x) r^{-2}(y, x)$ (3.1) be a principal kernel it is not necessary that all the kernels

$k_m(y, x)r^{-2}(y, x)$ $(m = 1, 2, \dots)$ be principal; $k(y, x)r^{-2}(y, x)$ will be a principal kernel if

$$p_1(t)+p_2(t)+\cdots = 0\,; \tag{3.16}$$

in fact, (3.16) will secure (3.14). One has

$$p_m(t) = 0 \qquad (m \text{ odd}) \tag{3.16a}$$

and

$$p_2(t) = \tfrac{1}{2}[\Gamma_{1,1}(t)+\Gamma_{2,2}(t)]\,, \tag{3.16b}$$

$$p_4(t) = \tfrac{3}{8}[\Gamma_{1,1,1,1}(t)+2\Gamma_{1,1,2,2}(t)+\Gamma_{2,2,2,2}(t)]\,,$$

$$p_6(t) = \tfrac{5}{16}[\Gamma_{1,1,1,1,1,1}(t)+3\Gamma_{1,1,1,1,2,2}(t)+3\Gamma_{1,1,2,2,2,2}(t)+\Gamma_{2,2,2,2,2,2}(t)], \dots .$$

To get the general expression for $p_{2\mu}(t)(\mu \geqq 1)$ we write

$$\sum_{s_1,\dots s_m=1}^{2} \Gamma_{s_1\dots s_m}(t) \cos\theta_{s_1}\dots\cos\theta_{s_m} = \sum_{k=0}^{2\mu} \Gamma^{2\mu,k}(t) \cos^{2\mu-k}\theta \sin^{k}\theta \tag{3.17}$$

$(m = 2\mu)$. The term free of θ in the Fourier expansion of $\cos^{2\mu-k}\theta\sin^{k}\theta$ is $A^{\mu,k}$, where

$$(-i)^k 2^{2\mu} A^{\mu,k} = \sum_{s=0}^{k} C^{2\mu-k}_{\mu-s} C^{k}_{s}(-1)^s \quad (\text{for } k \leqq \mu)\,,$$

$$= \sum_{s=k-\mu}^{\mu} C^{2\mu-k}_{\mu-s} C^{k}_{s}(-1)^s \quad (\text{for } k > \mu), \quad = 0 \ (\text{for } k \text{ odd})$$

($C^{\dots}_{\dots}$ are binomial coefficients). Thus

$$p_{2\mu}(t) = \sum_{k=0}^{\mu} \Gamma^{2\mu,2k}(t) A^{\mu,2k} \qquad (\mu \geqq 1)\,, \tag{1°}$$

$$A^{\mu,2k} = 2^{-2\mu} \sum_{s=0}^{2k} C^{2\mu-2k}_{\mu-s} C^{2k}_{s}(-1)^{s+k} \qquad \left(0 \leqq k \leqq \frac{\mu}{2}\right), \tag{3.17a}$$

$$A^{\mu,2k} = 2^{-2\mu} \sum_{s=2k-\mu}^{\mu} C^{2\mu-2k}_{\mu-s} C^{2k}_{s}(-1)^{s+k} \qquad \left(\frac{\mu}{2} < k \leqq \mu\right).$$

$\Gamma^{2\mu,k}(t)$ is the sum of $\Gamma_{s_1,\dots s_{2\mu}}(t)$ over sets $(s_1,\dots s_{2\mu})$ consisting of $2\mu-k$ numbers 'one' and k numbers 'two'; the number of such sets is $C^{2\mu}_{k}$; furthermore, $\Gamma_{s_1,\dots s_{2\mu}}(t)$ is unchanged when the subscripts are permuted; hence

$$\Gamma^{2\mu,k}(t) = C^{2\mu}_{k}\Gamma_{2\mu-k;k}(t)\,, \qquad \Gamma_{2\mu-k;k}(t) = \Gamma_{1\dots1,2\dots2} \tag{3.17b}$$

(1 repeated $2\mu-k$ times, 2 repeated k times).

In view of (3.16), (3.16a), (3.17b), (1°), (3.6a), *it is observed that* $k(y, x)r^{-2}(y, x)$ (3.1) *is a principal kernel* (on S), *provided*

$$p_2(t)+p_4(t)+\cdots+p_{2\mu}(t)+\cdots = 0\,, \tag{3.18}$$

where

$$p_{2\mu}(t) = \sum_{k=0}^{\mu} C_{2k}^{2\mu} A^{\mu,\,2k} \Gamma_{2\mu-2k;\,2k}(t) \tag{3.18a}$$

[$A^{\mu,\,2k}$ *from* (3.17a); *cf. notation* (3.17b) *for* $\Gamma_{\ldots,\ldots}$],

$$\Gamma_{2\mu-2k;\,2k}(t) = \sum_{i_1,\ldots i_{2\mu}=1}^{3} \gamma_{i_1\ldots i_{2\mu}}(t)[a_{i_1,1}a_{i_2,1}\ldots a_{i_\nu,1}]\,\cdot \tag{3.18b}$$

$$\cdot\,[a_{i_{\nu+1},2}\ldots a_{i_{2\mu},2}] \qquad (\nu = 2\mu-2k)\,;$$

here the $a_{ij} = a_{ij}(t)$ *may be defined as stated in connection with* (3.5b) (the a_{ij} satisfy (3.5a)); the above is asserted under the supposition that suitable conditions of convergence of the series involved are satisfied (this will be formulated in the sequel). With the aid of (3.5a) we obtain the explicit formulas

$$p_2(t) = \tfrac{1}{2}\Big\{\sum_i \gamma_{i,i} - \sum_{i_1,i_2} \gamma_{i_1,i_2} n_{i_1} n_{i_2}\Big\}, \tag{3.18c}$$

$$p_4(t) = \tfrac{3}{8}\Big\{\sum_{i,k}\gamma_{i,i,k,k} - 2\sum_{i_1,i_2,i}\gamma_{i_1,i_2,i,i}n_{i_1}n_{i_2} + \sum_{i_1,i_2,i_3,i_4}\gamma_{i_1,i_2,i_3,i_4}n_{i_1}n_{i_2}n_{i_3}n_{i_4}\Big\},$$

$$p_6(t) = \tfrac{5}{16}\Big\{\sum_{i,k,\nu}\gamma_{i,i,k,k,\nu,\nu} - 3\sum_{i_1,i_2,i,k}\gamma_{i_1,i_2,i,i,k,k}n_{i_1}n_{i_2}$$

$$+3\sum_{i_1,i_2,i_3,i_4,i}\gamma_{i_1,i_2,i_3,i_4,i,i}n_{i_2}n_{i_2}n_{i_3}n_{i_4} - \sum_{i_1,\ldots i_6}\gamma_{i_1\ldots i_6}n_{i_1}\ldots n_{i_6}\Big\}, \ldots$$

as remarked before, $k_{2\mu}(y,x)r^{-2}(y,x)$((3.1), (3.1a)) is a principal kernel (on S), if $p_{2\mu}(t) = 0$ (on S). Use will be made of the following.

Definition 3.19. *It will be said that* $q(y)$ *is of a Hölder class or, simply, is a Hölder function on* S, *if*

$$|q(y')-q(y'')| \leqq Qr^{\nu}(y',y'') \qquad (0 < \nu \leqq 1) \tag{3.19a}$$

for all y', y'' *on* S, *not on* β; *here* Q *is bounded for* y', y'' *at any positive distance from* β; Q *may become infinite as* y' *or* y'' *tends to* β; *a function satisfying* (3.19a) *will be termed of class* H *or, more specifically,* H_ν; *if* Q *can be selected as a constant, it will be said that* $q(y)$ *is uniformly* H *or* H_ν. *The class of functions* $q(y)$ *of class* H, *for which*

$$|q(y)| \leqq c^* l^{-\alpha}(y) \quad [y \text{ on } S \text{ near } \beta;\ l(y) \text{ from (1.11)};\ 0 \leqq \alpha]\,, \tag{3.19b}$$

will be designated by $[\alpha|S]$. *The number involved in the latter symbol will be always* $\geqq 0$. *If* $q(x)$ *is defined in* $C(S)$ *(complement of* S*) near* β *and*

$$|q(x)| \leqq c^* l^{-\alpha}(x) \tag{3.19c}$$

[x in $C(S)$ near β; tangential approaches to S or β excluded], its class will be designated by $[\alpha|C(S)]$. In the cases when (3.19 b), (3.19 c) *are replaced by*

$$\text{(3.19 b}^1\text{)} \qquad |q(y)| \leqq c^* l^{-\alpha}(y) \log \frac{c^*}{l(y)},$$

$$\text{(3.19 c}^1\text{)} \qquad |q(x)| \leqq c^* l^{-\alpha}(x) \log \frac{c^*}{l(y)},$$

respectively, the classes will correspondingly be designated by

$$[\alpha, \log |S], \qquad [\alpha, \log |C(S)].$$

Hypothesis 3.20. *We assume that the $\gamma_{i_1\ldots i_m}(y) \subset [0|S]$; specifically,*

$$\text{(3.20 a)} \qquad |\gamma_{i_1\ldots i_m}(y)| \leqq c_m;$$

furthermore, it will be supposed that

$$\text{(3.20 b)} \qquad c' = \sum_1^\infty c_m 3^m, \quad c^0 = \sum_1^\infty c_m 3^{2m} < \infty.$$

With regard to continuity of the $\gamma\ldots(y)$ it is assumed that

$$\text{(3.20 c)} \quad |\gamma_{i_1\ldots i_m}(y) - \gamma_{i_1\ldots i_m}(t)| \leqq \lambda_m \gamma(y, t) r^h(y, t) \quad (0 < h \leqq 1;\ h \text{ independent of } m),$$

where $\gamma(y, t)$ is bounded for $l(y) \geqq \delta$, $l(t) \geqq \delta (\delta > 0)$ and

$$\text{(3.20 d)} \qquad c'' = \sum_1^\infty m^2 \lambda_m 6^m < \infty.$$

Under the above hypothesis for the functions $k_m(y, x)$, $k(y, x)$ of (3.1), (3.1 a) one has

$$\text{(3.21)} \qquad |k_m(y, x)| \leqq \sum_{i_1,\ldots i_m=1}^{3} |\gamma_{i_1\ldots i_m}(y)| \leqq 3^m c_m, \qquad |k(y, x)| \leqq c';$$

moreover, the series for $k(y, x)$ converges absolutely and uniformly (with respect to x, y) when $l(y) \geqq \delta$ (any $\delta > 0$); for the functions of (3.2), (3.2 a) we have

$$\text{(3.21 a)} \qquad |k'_m(t|y, x)| \leqq \sum_{i_1,\ldots i_m} |\gamma_{i_1\ldots i_m}(t)| \leqq 3^m c_m,$$

$$|k''_m(t|y, x)| \leqq \sum_{i_1,\ldots i_m} |\gamma_{i_1\ldots i_m}(y) - \gamma_{i_1\ldots i_m}(t)| \leqq 3^m \lambda_m \gamma(y, t) r^h(y, t),$$

$$|k'(t|y, x)| \leqq c', \qquad |k''(t|y, x)| \leqq c'' \gamma(y, t) r^h(y, t);$$

the series for $k'(t|y, x)$ converges absolutely and uniformly; the series for $k''(t|y, x)$ converges in the same way when $l(y) \geqq \delta$, $l(t) \geqq \delta$.

We recall that the study of the integral $\Psi(t)$ (1.3) can be carried out on the basis of the decomposition $\Psi(t) = \Psi'(t)+\Psi''(t)$ (3.3) when the kernel $k(y, x)r^{-2}(y, x)$ is a principal kernel, *which is henceforth assumed*. Let t be a fixed point on S not on β. Let $S_{t,a}$ be a portion of S, as stated subsequent (3.6a). The integral for $\Psi'(t)$, extended over $S_{t,a}$, that is, $\Psi'_a(t)$ (3.7), exists in the sense of principal values; the integral for $\Psi'(t)$ over $S-S_{t,a} = S'$ has been designated as $\Psi_a^{1,0}(t)$ (3.7). One has

$$|\Psi_a^{1,0}(t)| \leq \int_{S'} \frac{k'(t|y, t)}{r^2(y, t)} |q(y)| d\sigma(y) \leq c^* \int_{S'} |k'(t|y, t)|\, |q(y)| d\sigma(y) .$$

Thus by (3.21a)

$$|\Psi_a^{1,0}(t)| \leq c^* \int_{S'} |q(y)| d\sigma(y) \leq c^* \int_{S} |q(y)| d\sigma(y) .$$

The integral last displayed, and hence the one for $\Psi_a^{1,0}(t)$, exists in the ordinary sense if

$$q(y) \subset [\alpha|S] \qquad (0 \leqq \alpha < 1) . \tag{3.22}$$

The truth of this assertion can be seen from the following considerations. It is sufficient to prove existence of the integral

$$\nu = \int_{s} |q(y)| d\sigma(y) , \tag{3.22a}$$

where s is a part of S consisting of a narrow strip, whose boundary is a 'curvilinear rectangle' one of whose sides is a small portion β' of β. In view of the 'smooth' character of the surfaces S and the curves β, it is sufficient to regard s as a true rectangular domain R, with β' as one of its rectilinear sides; choose the origin o of the y-coordinate system at an end point of β', so that β' lies on the $+y_1$-axis, while one of the other sides of R is on the $+y_2$-axis; $l(y)$ is then replaced by y_2. It is then observed that existence of the integral ν (3.22a) (under the condition (3.22)) is secured if the integral

$$\int_{y_2=0}^{c_2} \int_{y_1=0}^{c_1} y_2^{-\alpha} dy_1 dy_2 \qquad [c_1 > 0,\ c_2 > 0]$$

exists; this is the case since $\alpha < 1$. Our conclusion, then, is that *the integral* $\Psi'(t)$ (3.3) *exists* (in the sense of principal values), *provided Hypothesis* 3.20 *holds and* $q(y)$ is $[\alpha|S]$, *with* $\alpha < 1$. We turn now to $\Psi''(t)$ (3.3); as a consequence of (3.21a)

$$|\Psi''(t)| \leq c^* \int_S \gamma(y, t) |q(y)| r^{h-2}(y, t) d\sigma(y) \qquad \big(h, > 0, \text{ from } (3.20\text{c})\big) . \tag{3.22b}$$

The integrand above is bounded when $l(t) \geqq 2\delta (> 0)$ and y satisfies

$$l(y) \geqq \delta , \qquad r(y, t) \geqq \delta .$$

Hence to prove existence of $\Psi''(t)$ (fixed t, with $l(t) \geqq \delta > 0$) it is sufficient to prove that

$$(1°) \qquad \nu = \int_s \gamma(y, t)|q(y)|r^{h-2}(y, t)d\sigma(y) < \infty ,$$

where s is small neighborhood of β as in (3.22a), and that

$$(2°) \qquad \nu_1 = \int \gamma(y, t)|q(y)|r^{h-2}(y, t)d\sigma(y) < \infty \quad \left(\text{over } r(y, t) \leqq \frac{\delta}{2}\right).$$

In the latter integral

$$(3.23) \qquad \gamma(y, t)|q(y)| < a_\delta(t) < \infty \qquad \left(a_\delta(t) \text{ independent of } y\right),$$
$$r^{h-2}(y, t)d\sigma(y) = O(\varrho^{h-1}d\varrho d\theta) ,$$

where ϱ, θ are polar coordinates in the tangential plane P_t to S at t, with pole at t and ϱ being the length of the orthogonal projection of the radius vector (t, y) upon P_t; accordingly

$$\nu_1 \leqq c^* \int_{\varrho \leq \delta_1} a_\delta(t)\varrho^{h-1}d\varrho d\theta < \infty \qquad (\text{some } \delta_1 > 0) ,$$

inasmuch as $h > 0$; (2°) is thus established. As to ν (1°), it is observed that

$$(3°) \qquad \nu \leqq a_\delta(t)\nu^\circ , \qquad \nu^\circ = \int_s \gamma(y, t)|g(y)|d\sigma(y) ,$$

($a_\delta(t) < \infty$), provided the strip s is taken sufficiently narrow so that

$$r(y, t) \geqq \delta^\circ > 0 \qquad (\delta^\circ \text{ independent of } y;\ y \text{ in } s) .$$

The conclusion thus is that *the integral for* $\Psi''(t)$ (3.3) *exists if*

$$(3.24) \qquad \int_S \gamma(y, t)|q(y)|d\sigma(y) < \infty ;$$

in particular, if $\gamma(y, t) < c^*$, then $\Psi''(t)$ exists for all $q(y) \subset [\alpha|S]$ with $\alpha < 1$ (this follows in view of the remarks with respect to (3.22a)).

We sum the above as follws.

Theorem 3.25. *Suppose the* $\gamma_{i_1 \ldots i_m}(y)$ *are subject to Hypothesis* 3.20 (*cf. Definition* 3.19), *while*

$$(3.25a) \qquad p_2(t)+p_4(t)+\cdots+p_{2\mu}(t)+\cdots = 0 \qquad (\textit{all } t \textit{ on } S)$$

($p_{2\mu}(t)$ *given in* (3.18a, b, c)). *The kernel* $k(y, x)r^{-2}(y, x)$ (3.1) *is then a principal kernel for all integrals of the form*

(3.25 b) $$\Psi(t) = \int_S \frac{k(y, t)}{r^2(y, t)} q(y) d\sigma(y) ,$$

where $q(y), \subset [\alpha|S]$ *with* $0 \leqq \alpha < 1$, *is such that the integral*

(3.25 b¹) $$\int_S \gamma(y, t)|q(y)|d\sigma(y)$$

exists for t *on* S (*not on* β).

Note I. The condition with respect to (3.25 b¹) is deleted when $\gamma(y, t) < c^*$. If Hypothesis 3.20 and (3.25 a) are satisfied and

(3.26) $$\gamma(y, t) < a(t) l^{-\beta}(y) \quad (\text{for } 0 < l(y) \leqq \tfrac{1}{2} l(t);\ 0 \leqq \beta < 1) ,$$

the principal integral $\Psi(t)$ in the Theorem will exist for all $q(y)$ such that

(3.26 a) $$q(y) \subset [\alpha|S] \qquad (0 \leqq \alpha;\ \alpha+\beta < 1) .$$

To establish the above we note that, to start with,

$$\gamma(y, t) < \gamma_0(t) < \infty \text{ (some } \gamma_0(t) \text{ independent of } y)$$

for $l(t) > 0$ and $l(y) \geqq \frac{1}{2} l(t)$; this follows by the statement subsequent (3.20 c). Therefore for any $q(y)$ of class H (Definition 3.19) the integral

$$\gamma_t' = \int \gamma(y, t)|q(y)|d\sigma(y) ,$$

extended over the part of S for which $l(y) \geqq \frac{1}{2} l(t)$, exists. In order to establish existence of

$$\gamma_t'' = \int \gamma(y, t)|q(y)|d\sigma(y) \qquad \left(l(y) \leqq \tfrac{1}{2} l(t)\right)$$

we make use of (3.26), obtaining

$$\gamma_t'' < a(t) \int l^{-\beta}(y)|q(y)|d\sigma(y) \qquad \left(l(y) \leqq \tfrac{1}{2} l(t)\right);$$

under (3.26 a) the integrand above is $O\left(l^{-\alpha-\beta}(y)\right)$; with $\alpha+\beta < 1$, by the same reasons as previously applied to (3.22 a) it follows that the integral for γ_t'' exists. The integral in (3.25 b¹) is $\gamma_t'+\gamma_t''$ and, accordingly, it exists for all $q(y)$ satisfying (3.26 a). The statement (3.26), (3.26 a) ensues from the Theorem.

Note II. *The condition in the Theorem, stated in connection with* (3.25 b¹) *can be replaced by the following* (special case of (3.26), (3.26 a)):

(3.27) $$\gamma(y, t) < \begin{cases} c^* l^{-\beta}(y) & (\text{if } l(y) \leqq l(t)), \\ c^* l^{-\beta}(t) & (\text{if } l(t) \leqq l(y)) \end{cases} \qquad [\alpha+\beta < 1;\ 0 \leqq \beta] .$$

4. **Limits of $\Psi(x)$ (1.3a) as $x \to t$.** Assume Hypothesis 3.20 and (3.25a). Thus the kernel $k(y, x)r^{-2}(y, x)$ in the integral (1.3a) is a principal one for all $q(y)$ satisfying the conditions of Theorem 3.25. Let (λ_t) be a continuously varying direction at t; more precisely, let

$$(4.1) \qquad \lambda_j(t) \qquad (j = 1,2,3)$$

be the direction cosines of (λ_t); the $\lambda_j(t)$ are assumed to be of class H. Let x be a point on the line L_t, extending from t and having the direction (λ_t); $r(x, t) = h > 0$; L_t is not to lie in the tangential plane P_t to S (at t). Suppose for the present that

(4.2) $\vartheta(t) =$ *angle between the directions* $(+n_t)$, (λ_t) [$(+n_t)$ is the direction of the positive normal at t]

satisfies

$$(4.2\text{a}) \qquad 0 \leqq \vartheta(t) < \frac{\pi}{2}.$$

Let (Y) be the coordinate system (origin O at t), defined by (3.5), (3.5a) and achieving the situation as described preceding (3.5). Whenever $\vartheta(t) \neq 0$, the half plane extending from the normal (that is, from the Y_3-axis) through X intersects the tangential plane at t (the Y_1, Y_2-plane) in a certain ray extending from O; *let* $\varphi(t)$ *be the angle from the* $+Y_1$*-axis to this ray.* The angles $\vartheta(t)$, $\varphi(t)$ obviously define the direction L_t (when $\vartheta(t) = 0$, $\varphi(t)$ is undefined and is superfluous).

On taking note of the decomposition of $k(y, x)$, given by (3.2a), (3.2), we write

$$(4.3) \qquad \Psi(x) = \Psi_a'(x) + \Psi_a''(x) + \Psi_a^0(x),$$

where

$$(4.3\text{a}) \qquad \Psi_a'(x) = \int_{S_{t,a}} k'(t|y, x)r^{-2}(y, x)q(y)d\sigma(y),$$

$$\Psi_a''(x) = \int_{S_{t,a}} k''(t|y, x)r^{-2}(y, x)q(y)d\sigma(y), \qquad \Psi_a^0(x) = \int_{S'} \frac{k(y, x)}{r^2(y, x)} q(y)d\sigma(y);$$

here, with $a(>0)$ suitably small, $S_{t,a}$ is the portion of S which projects orthogonally upon the P_t plane as a circle of center t and radius a; $S' = S - S_{t,a}$. For x on L_t (h suitably small) and y on S' $r(y, x)$ is bounded below by a positive number independent of y, x; thus

$$(1°) \qquad \int_{S'} \left| \frac{k(y, x)}{r^2(y, x)} q(y)d\sigma(y) \right| \leqq a(t)\int_{S'} |k(y, x)||q(y)|d\sigma(y) \leqq a(t)I(x);$$

$$I(x) = \int_S |k(y, x)|\,|q(y)|d\sigma(y) \qquad [a(t), < \infty, \text{ independent of } x].$$

By (3.21)

$$(2^\circ)\qquad I(x) \leqq c' \int_S |q(y)| d\sigma(y) .$$

The latter integral exists for all $q(y) \subset [\alpha|S]$, with

$$(3^\circ)\qquad \alpha < 1$$

for reasons of the kind applied to (3.22a); Moreover, the integrand in (2°) is independent of x. Hence the integral for $\Psi_a^0(x)$ converges uniformly with respect to x (when x is on L_t and, in fact, also when x is on the prolongation of L_t to the negative side of S); accordingly $\Psi_a^0(x)$ is continuous in x at t, that is in h at $h = 0$. One has

$$(4.4)\qquad \Psi_a^0(t)^+ = \Psi_a^0(t)^- = \lim_{h \to 0} \Psi_a^0(x) = \int_{S'} \frac{k(y, t)}{r^2(y, t)} q(y) d\sigma(y) = \Psi_a^0(t) ,$$

independent of the direction of approach (provided $q(y) \subset [\alpha|S]$, $\alpha < 1$).

As a preliminary to the study of $\Psi_a''(x)$ (4.3a) we shall need to prove that

$$(4.5)\qquad \frac{r(y, t)}{r(y, x)} \leqq b(t) < \infty \qquad (b(t) \text{ independent of } y, x)$$

for x on L_t and for y on $S_{t.a}$ ($a = a_t$ sufficiently small). Introducing the (Y) system, as stated after (4.2a), we let X, Y be the designation for x, y in the new coordinates; O (the new origin) will designate t in the new coordinate system. We introduce polar coordinates in the Y_1, Y_2-plane (cf. (3.9)), so that

$$(4.6)\qquad Y_1 = \varrho \cos\theta,\ Y_2 = \varrho \sin\theta,\ \varrho^2 = r^2(O, Y') = Y_1^2 + Y_2^2 .$$

With X on L_t, the angle between $O, +Y_3$ and O, X being $\vartheta(t)$ ((4.2), (4.2a)), $r(O, X) = h$, and the point $Y' = (Y_1, Y_2, 0)$ in the Y_1, Y_2-plane, we find that

$$(4.7)\qquad r^2(X, Y') = h^2 + p^2 - 2h\varrho \cos(\theta - \varphi) \sin\vartheta(t) ,$$

$$\frac{r^2(X, Y')}{r^2(O, Y')} = \left(\frac{h}{\varrho}\right)^2 - 2B\left(\frac{h}{\varrho}\right) + 1 , \qquad B = \cos(\theta - \varphi) \sin\vartheta(t) .$$

Since $|B| \leqq \sin\vartheta(t) < 1$, it follows that

$$u^2 - 2Bu + 1 \geqq 1 - B^2 \geqq \cos^2\vartheta(t) > 0$$

(for all real u); whence

$$(4.7\,\text{a})\qquad r^2(X, Y') r^{-2}(O, Y') \geqq \cos^2\vartheta(t) .$$

In view of (3.10) $Y_3\varrho^{-1} = O(\varrho)$, $Y_3\varrho^{-2} = O(1)$; hence

$$(4.7\,\text{b})\qquad |\nu| = |(Y_3\varrho^{-1})^2 - 2h\cos\vartheta(t) Y_3\varrho^{-2}| \leqq \tfrac{1}{2}\cos^2\vartheta(t) \qquad (\text{for } h \leqq h_t,\ \varrho \leqq a) ,$$

where $h_t(>0)$, $a = a_t$ (>0) are chosen suitably small, independent of Y. Also

(4.7 c) $$X_1 = h \sin \vartheta(t) \cos \varphi(t),\ X_2 = h \sin \vartheta(t) \sin \varphi(t),\ X_3 = h \cos \vartheta(t)\,.$$

Accordingly, by (4.6), (4.7)

$$r^2(O,\ Y)r^{-2}(X,\ Y) = \frac{Y_1^2+Y_2^2+Y_3^2}{(X_1-Y_1)^2+(X_2-Y_2)^2+(X_3-Y_3)^2} =$$

$$= \frac{\varrho^2+Y_3^2}{h^2+\varrho^2+Y_3^2-2\varrho h \cos(\theta-\varphi)\cos\vartheta-2h\cos\vartheta Y_3} = \frac{\varrho^2+O(\varrho^4)}{r^2(X,\ Y')+Y_3^2-2h\cos\vartheta Y_3}\,.$$

Thus by virtue of (4.7a), (4.7b)

$$r^2(O,\ Y)r^{-2}(X,\ Y) = \frac{1+O(\varrho^2)}{r^2(X,\ Y')\varrho^{-2}+\nu} \leqq \frac{O(1)}{\cos^2\vartheta(t)-\frac{1}{2}\cos^2(t)\vartheta} \leqq b^2(t) < \infty\,;$$

since $r(O,\ Y) = r(t,\ y)$, $r(X,\ Y) = r(x,\ y)$, (4.5) has been established.

By (3.21 a) and (4.5) the absolute value of the integrand in the integral representing $\Psi_a''(x)$ (4.3a) satisfies

(1°) $$|k''(t|y,\ x)r^{-2}(y,\ x)q(y)| \leqq c''\gamma(y,\ t)r^{h-2}(y,\ t)\left[\frac{r(y,\ t)}{r(y,\ x)}\right]^2|q(y)|$$

$$\leqq c''b^2(t)\Lambda(y,\ t)\,,\qquad \Lambda(y,\ t) = \gamma(y,\ t)r^{h-2}(y,\ t)|q(y)|\,.$$

Now the integral

$$\int_S \Lambda(y,\ t)d\sigma(y)$$

is identical with that in (3.22 b); it accordingly exists, if (3.24) holds; the latter is the case in view of the assumed conditions of Theorem 3.25.

It is also observed that $b^2(t)\Lambda(y,\ t)$ in the third member in (1°) is independent of x. Hence one may pass to the limit under the sign of integration, obtaining

(4.8) $$\lim_{h\to 0}\Psi_a''(x) = \int_{S_{t,a}}\lim\ldots = \int_{S_{t,a}} k''(t|y,\ t)r^{-2}(y,\ t)q(y)d\sigma(y)$$

$$= \Psi_a''(t) = \Psi_a''^{+}(t) = \Psi_a''^{-}(t)\,;$$

here the integral exists in the ordinary sense and the limit is independent of the direction of approach.

We now come to the consideration of $\Psi_a'(x)$ (4.3 a). The transformation introduced subsequent (4.5) gives

$$x_i - t_i = a_{i,1}X_1+a_{i,2}X_2+a_{i,3}X_3\,,\qquad y_i-t_i = a_{i,1}Y_1+a_{i,2}Y_2+a_{i,3}Y_3$$

(here the a_{ij} are certain functions of t); thus for $w_i(y,\ x) = r^{-1}(y,\ x)(y_i-x_i)$ one has

(4.9) $$w_i(y, x) = w_i(Y, X) = r^{-1}(Y, X) \sum_{s=1}^{3}{}' a_{is}(Y_s - X_s) .$$

In the new coordinates (cf. (3.2))

$$k'_m(t|y, x) = k'_m(t|Y, X) = \sum_{i_1,\dots i_m=1}^{3} \gamma_{i_1\dots i_m}(t) w_{i_1}(Y, X)\dots w_{i_m}(Y, X) .$$

Thus

(4.10) $$k'_m(t|Y, X) = \sum_{s_1,\dots s_m=1}^{3} \Gamma_{s_1\dots s_m}(t) W_{s_1}(Y, X)\dots W_{s_m}(Y, X) ,$$

$$W_s(Y, X) = r^{-1}(Y, X)(Y_s - X_s) ;$$

here $\Gamma_{s_1\dots s_m}(t)$ is identical with the function so designated in (3.6a). The expression for $\Psi'_a(x)$ becomes

(4.11) $$\Psi'_a(x) = \Psi'_a(X) = \int_{S(O,a)} k'(t|Y, X) r^{-2}(Y, X) q(Y) d\sigma(Y) ;$$

$$S(O, a) = S_{t,a} , \quad q(Y) = q(y) , \qquad k'(t|Y, X) = \sum_{1}^{\infty} k'_m(t|Y, X) \qquad (4.10) ;$$

we further write

(4.11a) $$\Psi'_a(X) = q(t) A(X) + B(X) ,$$

$$A(X) = \int_{S(O,a)} k'(t|Y, X) r^{-2}(Y, X) d\sigma(Y) ,$$

$$B(X) = \int_{S(O,a)} k'(t|Y, X) r^{-2}(Y, X)[q(Y) - q(t)] d\sigma(Y) .$$

Since q is of class H, one has

$$|q(Y) - q(t)| = |q(Y) - q(O)| = O(r^{\nu}(O, Y))$$

(some $0 < \nu \leqq 1$), where $O(\dots)$ may depend on t; thus by (3.21a)

(1°) $$|k'(t|Y, X)\, r^{-2}(Y, X)(q(Y) - q(t))| \leqq b'(t) r^{-2}(Y, X) r^{\nu}(O, Y) ,$$

where $b'(t) (< \infty)$ is independent of Y, X; in view of (4.5)

(2°) first member in (1°) $\leqq b'(t) \dfrac{r^2(O, Y)}{r^2(Y, X)} r^{\nu-2}(O, Y) \leqq b'(t) b^2(t) r^{\nu-2}(O, Y)$.

Accordingly, the absolute value of the integrand in $B(X)$ is bounded by a function independent of X, whose integral, with respect to Y (over $S(O, a)$), exists (since in the last member in (2°) $\nu - 2 > -2$); whence one can pass to the limit under the integral sign, obtaining

$$(4.12)\quad B^+(t) = B^-(t) = \lim_{x\to t} B(X) = \int_{S(O,a)} k'(t|Y, O)r^{-2}(Y, O)[q(Y)-q(t)]d\sigma(Y)$$
$$= \int_{S_{t,a}} k'(t|y, t)r^{-2}(y, t)\big(q(y)-q(t)\big)d\sigma(y) = B(t)\,;$$

this limit is independent of the direction of nontangential approach.

As a preliminary to the study of $A(X)$ we establish the relations

$$(4.13)\qquad r^{-2}(Y, X) = r^{-2}(Y', X)+O\big(r^{-1}(Y', X)\big)\,,$$

$$(4.13\text{a})\qquad W_s(Y, X) = W_s(Y', X)+\nu_s\,,\qquad \nu_s = O(\varrho)\qquad (s = 1, 2, 3)\,.$$

(Here and throughout this section $O(\ldots)$ may depend on t). This will be proved, regarding the left members as functions of Y_3 alone, with the aid of the relation (valid in the present situation)

$$(\text{i})\qquad f(Y_3) = f(0)+f^{(1)}(Z_3)Y_3$$

and of the notation

$$(\text{ii})\qquad Z = (Y_1, Y_2, Z_3)\,,\quad \text{some } Z_3 \text{ between } 0 \text{ and } Y_3\,.$$

It is observed that by (4.5)

$$(4.14)\qquad r^{-1}(Y, X),\ r^{-1}(Y', X) \leqq b(t)r^{-1}(Y, O) \leqq b(t)\varrho^{-1} = O(\varrho^{-1})\,.$$

Let $f(Y_3) = r^{-1}(Y, X)$; then

$$f^{(1)}(Y_3) = -(Y_3-X_3)r^{-3}(Y, X)\,,\qquad |f^{(1)}(Y_3)| \leqq r^{-2}(Y, X) \leqq b^2(t)\varrho^{-2}\,;$$

the last member is independent of Y_3. In view of (i)

$$r^{-1}(Y, X) = r^{-1}(Y', X)+\nu\,,\qquad |\nu| = |f^{(1)}(Z_3)Y_3| \leqq b^2(t)\varrho^{-2}|Y_3|\,;$$

since $Y_3 = O(\varrho^2)$, one has $\nu = O(1)$; inasmuch as $r(Y', X) = O(1)$, we have

$$(\text{iii})\qquad O\big(r^{-1}(Y', X)\big)+O(1) = O\big(r^{-1}(Y', X)\big)\,;$$

as a consequence of this (4.13) follows.

To prove (4.13a; $s \leqq 2$) let $f(Y_3) = W_s(Y, X)$; now (by (4.14))

$$f^{(1)}(Y_3) = -(Y_s-X_s)(Y_3-X_3)r^{-3}(Y, X),\ |f^{(1)}(Y_3)| \leqq r^{-1}(Y, X) \leqq b(t)\varrho^{-1}\,;$$

whence $|f^{(1)}(Z_3)|$ (Z_3 as in (ii)) $\leqq b(t)\varrho^{-1}$; accordingly

$$W_s(Y, X) = W_s(Y', X)+\nu_s\,,\qquad \nu_s = f^{(1)}(Z_3)Y_3 = O(\varrho)\,.$$

When $s = 3$, we write $f(Y_3) = W_3(Y, X)$, obtaining

$$f^{(1)}(Y_3) = [(Y_1-X_1)^2+(Y_2-X_2)^2]r^{-3}(Y, X) \leqq r^{-1}(Y, X) \leqq b(t)\varrho^{-1}\,,$$
$$W_3(Y, X) = W_3(Y', X)+\nu_3\,,\qquad \nu_3 = f^{(1)}(Z_3)Y_3 = O(\varrho)\,.$$

The $|W_s(Y, X)|$, $|W_s(Y', X)|$ are bounded (≤ 1). Hence, as a consequence of (4.13a)

$$W_i(Y, X)W_j(Y, X) = W_i(Y', X)W_j(Y', X)+\nu_{ij}\,, \qquad \nu_{ij} = O(\varrho)\,.$$

Step by step one arrives at

$$W_{s_1}(Y, X)\ldots W_{s_m}(Y, X) = W_{s_1}(Y', X)\ldots W_{s_m}(Y', X)+\nu_{s_1\ldots s_m}\,, \tag{4.15}$$

where $\nu_{s_1\ldots s_m} = O(\varrho)$. By (4.10) and the above

$$k'(t|Y, X) = \sum_{m=1}^{\infty} k'_m(t|Y, X) = k(h, \varrho, \theta)+\nu'(Y, X)\,, \tag{4.16}$$

where

$$k(h, \varrho, \theta) = \sum_{m=1}^{\infty} \sum_{s_1,\ldots s_m=1}^{3} \Gamma_{s_1\ldots s_m}(t)W_{s_1}(Y', X)\ldots W_{s_m}(Y', X) \tag{4.16a}$$

and $\nu'(Y, X) = O(\varrho)$. In deriving the above use is made of the satisfied conditions involved in Theorem 3.25.

As a consequence of (4.13) and $\big(1°$; subsequent $(3.10)\big)$

$$\begin{aligned} r^{-2}(Y, X)d\sigma(Y) = \varrho d\varrho d\theta[&r^{-2}(Y', X)+O\big(r^{-1}(Y', X)\big) \\ &+O\big(\varrho\cdot\varrho r^{-1}(Y', X)\big)+O\big(\varrho^2 r^{-2}(Y', X)\big)\,; \end{aligned}$$

now, by (4.14), $\varrho r^{-1}(Y', X) = O(1)$; hence the last three terms in $[\ldots]$ above, combine into

$$O\big(r^{-1}(Y', X)\big)+O(\varrho)+O(1) = O\big(r^{-1}(Y', X)\big)+O(1) = O\big(r^{-1}(Y', X)\big)$$

(cf. (iii)); thus

$$r^{-2}(Y, X)d\sigma(Y) = \varrho d\varrho d\theta[r^{-2}(Y', X)+O\big(r^{-1}(Y', X)\big)]\,, \tag{4.17}$$

By virtue of the above and of (4.16)

$$\begin{aligned} \frac{1}{\varrho d\varrho d\theta}\, k'(t|Y, X)r^{-2}(Y, X)d\sigma(Y) &= [k(h, \varrho, \theta)r^{-2}(Y', X)+ \\ +\nu'(Y, X)r^{-2}(Y', X)+k(h, \varrho, \theta)&O\big(r^{-1}(Y', X)\big)+\nu'(Y, X)O\big(r^{-1}(Y', X)\big)]\,; \end{aligned}$$

since $k(h, \varrho, \theta) = O(1)$ and $\nu'(Y, X) = O(\varrho)$, one obtains

$$[\ldots] = k(h, \varrho, \theta)r^{-2}(Y', X)+O\big(\varrho r^{-2}(Y', X)\big)+O\big(r^{-1}(Y', X)\big)+O\big(\varrho r^{-1}(Y', X)\big)\,,$$

which (by (4.14)) equals

$$k(h, \varrho, \theta)r^{-2}(Y', X)+O\big(r^{-1}(Y', X)\big)+O(1)\,.$$

Thus, on taking note of (iii), it is deduced that

$$(4.18)\quad k'(t|Y, X)r^{-2}(Y, X)d\sigma(Y) = \varrho d\varrho d\theta[k(h, \varrho, \theta)r^{-2}(Y', X)+O(r^{-1}(Y', X))]$$
$$= k(h, \varrho, \theta)r^{-2}(Y', X)\varrho d\varrho d\theta+O(1)d\varrho d\theta .$$

From the above we obtain for $A(X)$ (4.11a) the decomposition

$$(4.19)\qquad A(X) = A_a^*(h)+A_a^0(X) ,$$

where

$$A_a^*(h) = \int_{\varrho=0}^{a}\int_{\theta=0}^{2\pi} K(h, \varrho, \theta)\varrho d\varrho d\theta , \qquad K(h, \varrho, \theta) = \frac{k(h, \varrho, \theta)}{r^2(Y', X)}$$

(cf. (4.16a), (4.7)) and

$$(4.19\text{a})\quad A_a^0(X) = \int_{\varrho=0}^{a}\int_{\theta=0}^{2\pi} O(1)d\varrho d\theta = O(a) \text{ (uniformly with respect to } h) .$$

We shall now proceed finding the limit of $A_a^*(h)$ for $h \to 0$, that is, for $X \to O$, which means for $x \to t$ along the direction (λ_t). Introduce quantities $\beta(t)$, $\beta_j(t)$, θ_j as follows:

$$(4.20)\qquad \beta_1(t) = \sin\vartheta(t)\cos\varphi(t),\ \beta_2(t) = \sin\vartheta(t)\sin\varphi(t),\ \beta_3(t) = \cos\vartheta(t);$$
$$\theta_1 = \theta,\ \theta_2 = \frac{\pi}{2}-\theta,\ \theta_3 = \frac{\pi}{2};\ \beta(t) = \sin\vartheta(t) .$$

By (4.7c) and (3.9) one then has

$$(4.20\text{a})\qquad X_j = \beta_j(t)h, \quad Y_j = \varrho\cos\theta_j \quad (j = 1, 2, 3) ,$$

when X is on the line L_t and when $Y_3 = 0$. In this connection we note that Y_3 in the expression for $K(h, \varrho, \theta)$ is zero. In view of (4.7) one may write

$$r^2(Y', X) = h^2-2\beta(t)h\varrho\cos(\theta-\varphi)+\varrho^2;$$

accordingly by (4.16a), (4.19), (4.10)

$$(4.21^\circ)\qquad K(h, \varrho, \theta) = \sum_{m=1}^{\infty}[h^2-2\beta(t)h\varrho\cos(\theta-\varphi)+\varrho^2]^{-\frac{m}{2}-1}\cdot$$
$$\sum_{s_1,\dots s_m=1}^{3}\Gamma_{s_1\dots s_m}(t)(\varrho\cos\theta_{s_1}-\beta_{s_1}(t)h)\dots(\varrho\cos\theta_{s_m}-\beta_{s_m}(t)h) .$$

On substituting $\varrho = \tau h$ and taking note of the absolute and uniform convergence of the series involved, we infer

$$(4.21)\qquad A_a^*(h) = \int_{\tau=0}^{ah^{-1}} K_t^*(\tau)d\tau ,$$

where

(4.21a)
$$K_t^*(\tau)=\int_{\theta=0}^{2\pi}\sum_{m=1}^{\infty}\tau[1-2\beta(t)\tau\cos(\theta-\varphi)+\tau^2]^{-\frac{m}{2}-1}\cdot$$
$$\sum_{s_1,\dots s_m=1}^{3}\Gamma_{s_1\dots s_m}(t)\big(\tau\cos\theta_{s_1}-\beta_{s_1}(t)\big)\dots\big(\tau\cos\theta_{s_m}-\beta_{s_m}(t)\big)d\theta$$

(cf. (3.6a), (4.20)); we observe the important fact that $K_t^*(\tau)$ is independent of h. The limit

(4.22)
$$\lim_{h\to0}A_a^*(h)=\int_{\tau=0}^{\infty}K_t^*(\tau)d\tau=K(t)\quad[\text{cf. (4.21a), (3.6a), (4.20)}]$$

exists if and only if the integral above converges. We shall prove that this integral exists. By (3.6a), (3.20a)

$$|\Gamma_{s_1\dots s_m}(t)|\leqq\sum_{i_1,\dots i_m=1}^{3}|\gamma_{i_1\dots i_m}(t)|\leqq3^mc_m\,,\quad\sum_{s_1,\dots s_m=1}^{3}|\Gamma_{s_1\dots s_m}(t)|\leqq3^{2m}c_m\,,$$

(1°)
$$\sum_{m=1}^{\infty}\sum_{s_1,\dots s_m=1}^{3}|\Gamma_{s_1\dots s_m}(t)|\leqq c^\circ\qquad\big(\text{cf. (3.20b)}\big).$$

Since the quantities

(4.22′) $\omega_s(\tau,\theta)=[1-2\beta(t)\tau\cos(\theta-\varphi)+\tau^2]^{-\frac{1}{2}}\big(\tau\cos\theta_s-\beta_s(t)\big)\qquad(s=1,2,3)$

are in the nature of direction cosines and are thus bounded in absolute values ($\leqq1$), we obtain

$$|\text{integrand for }K_t^*(\tau)|\leqq\frac{c^0\tau}{1-2\beta(t)\tau\cos(\theta-\varphi)+\tau^2}$$
$$=\frac{c^0u}{1-2\beta(t)u\cos(\theta-\varphi)+u^2}\qquad\left(u=\frac{1}{\tau}\right).$$

Using the fact that $1-2q\tau+\tau^2\geqq1-q^2$ (all real τ), we deduce

(2°) $1-2\beta\cos(\theta-\varphi)\tau+\tau^2,\ 1-2\beta\cos(\theta-\varphi)u+u^2\geqq1-\beta^2\cos^2(\theta-\varphi)\geqq1-\beta^2=$
$$\cos^2\vartheta(t)>0\,;$$

hence

(3°)
$$|\text{integrand for }K_t^*(\tau)|\leqq\begin{cases}c^0\,\tau\sec^2\vartheta(t)\,,\\ c^0\dfrac{1}{\tau}\sec^2\vartheta(t)\,.\end{cases}$$

Accordingly

(4.22a)
$$|K_t^*(\tau)|\leqq2\pi c^0\tau\sec^2\vartheta(t),\qquad2\pi c^0\frac{1}{\tau}\sec^2\vartheta(t).$$

In view of the above, existence of the integral

$$\int_{\tau=0}^{\tau_0} K_t^*(\tau)d\tau$$

is evident for all $0 < \tau_0 < \infty$. On the other hand, for τ large (and t fixed on S) the relation

$$K_t^*(\tau) = O\left(\frac{1}{\tau}\right)$$

is insufficient for the existence of the integral defining $K(t)$ (4.22). If one thinks of τ as a complex variable, it is observed that $K_t^*(\tau)$ is analytic in τ for $|\tau| \geqq \tau_0$ (any $\tau_0 > 1$) and that $K_t^*(\tau) = O(|\tau|^{-1})$ for $|\tau|$ large; we have an expansion

$$\text{(i)} \qquad K_t^*(\tau) = \frac{k_0}{\tau}+\frac{k_1}{\tau^2}+\cdots \qquad \text{(convergent for } |\tau| \geqq \tau_0\text{)}.$$

The integral (4.22) thus exists if $k_0 = 0$; one has

$$\text{(4.23)} \qquad k_0 = \int_{\theta=0}^{2\pi} k_0'(\theta)d\theta ,$$

where $k_0'(\theta)$ is from the expansion

$$\text{(ii)} \qquad \text{integrand for } K_t^*(\tau) = \sum_{j=0}^{\infty} k_j(\theta)\tau^{-j-i}$$

(the series here converges absolutely and uniformly with respect to θ, τ for $|\tau| \geqq \tau_0$); by (4.21a)

$$k_0'(\theta) = \operatorname*{limit}_{\tau\to\infty} [\tau \cdot (\text{integrand for } K_t^*(t))]$$

$$= \lim_{u\to 0} \sum_{m=1}^{\infty} \sum_{s_1,\ldots s_m=1}^{3} \Gamma_{s_1\ldots s_m}(t)[u^2-2\beta(t)u\cos(\theta-\varphi)+1]^{-\frac{m}{2}-1}(\cos\theta_{s_1}-\beta_{s_1}u)\ldots(\cos\theta_{s_m}-\beta_{s_m}u)$$

$$= \sum_{m=1}^{\infty} \sum_{s_1,\ldots s_m=1}^{3} \Gamma_{s_1\ldots s_m}(t)\cos\theta_{s_1}\ldots\cos\theta_{s_m} .$$

Since (by (4.20)) $\cos\theta_3 = 0$, in view of the remark with respect to (4.23) we conclude that *for the existence of the integral for* $K(t)$ (4.22) *it is necessary and sufficient that*

$$\text{(iii)} \qquad \int_{\theta=0}^{2\pi} \sum_{m=1}^{\infty} \sum_{s_1,\ldots s_m=1}^{2} \Gamma_{s_1\ldots s_m}(t)\cos\theta_{s_1}\ldots\cos\theta_{s_m}d\theta = 0 \qquad \left(\theta_1 = \theta,\ \theta_2 = \frac{\pi}{2}-\theta\right).$$

The integrand here is identical with $k^{1,*}(t, \theta)$ (3.11a); thus (iii) is precisely the condition securing vanishing of the term $f_0(t)$ in the Fourier expansion (3.13a); (iii) accordingly holds inasmuch as the kernel $k(y, x)r^{-2}(y, x)$ (3.1) has been assumed to be a principal one.

This completes the proof of the existence of the limit (4.22).

Let us study $K(t)$ (4.22) near edges β of S. Denoting the integrand in (4.21a) by $K_t^*(\tau, \theta)$, define $B_t^*(\tau, \theta)$ by the relation

$$K_t^*(\tau, \theta) = \frac{1}{\tau} k_0'(\theta) + \frac{1}{\tau^2} B_t^*(\tau, \theta); \tag{4.24}$$

inasmuch as k_0 (4.23) is zero, one then will have

$$K_t^*(\tau) = \frac{1}{\tau^2} \int_{\theta=0}^{2\pi} B_t^*(\tau, \theta) d\theta . \tag{4.24a}$$

$B_t^*(\tau, \theta)$ is expressible in the form

$$B_t^*(\tau, \theta) = \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{3} \Gamma_{s_1 \ldots s_m}(t) \Lambda^{s_1 \ldots s_m}, \tag{I$_1$}$$

where

$$\Lambda^{s_1 \ldots s_m} = \tau^3 [1 - 2\beta\tau \cos(\theta - \varphi) + \tau^2]^{-\frac{m}{2} - 1} (\tau \cos \theta_{s_1} - \beta_{s_1}) \ldots (\tau \cos \theta_{s_m} - \beta_{s_m})$$
$$- \tau \cos \theta_{s_1} \ldots \cos \theta_{s_m} = \Lambda' + \Lambda'' ,$$

with

$$\Lambda' = \tau [1 - 2\beta\tau \cos(\theta - \varphi) + \tau^2]^{-\frac{m}{2}} (\tau \cos \theta_{s_1} - \beta_{s_1}) \ldots (\tau \cos \theta_{s_m} - \beta_{s_m}) - \tau \cos \theta_{s_1} \ldots \cos \theta_{s_m},$$
$$\Lambda'' = [1 - 2\beta\tau \cos(\theta - \varphi) + \tau^2]^{-1} [2\beta\tau^2 \cos(\theta - \varphi) - \tau] \cdot \omega_{s_1}(\tau, \theta) \ldots \omega_{s_m}(\tau, \theta)$$

(cf. (4.22′)). One has

$$|\Lambda''| \leqq \left| \frac{2\beta\tau^2 \cos(\theta - \varphi) - \tau}{1 - 2\beta\tau \cos(\theta - \varphi) + \tau^2} \right| = \left| \frac{2\beta \cos(\theta - \varphi) - u}{1 - 2\beta u \cos(\theta - \varphi) + u^2} \right| \qquad \left(u = \frac{1}{\tau} \right);$$

whence in view of (2°)

$$|\Lambda''| \leqq (2 + u) \sec^2 \vartheta(t) \leqq 3 \sec^2 \vartheta(t) \qquad (\text{for } \tau \geqq 1) . \tag{I$_2$}$$

The set of integers $s_1, \ldots s_m$ consists of i_1, i_2 and i_3 numbers 1,2 and 3, respectively, with $i_1 + i_2 + i_3 = m$; accordingly, by virtue of (4.20)

$$\Lambda' = \tau [1 - 2\beta\tau \cos(\theta - \varphi) + \tau^2]^{-\frac{m}{2}} (\tau \cos \theta - \beta_1)^{i_1} (\tau \sin \theta - \beta_2)^{i_2} (-\beta_3)^{i_3}$$
$$- \tau \cos^{i_1} \theta \sin^{i_2} \theta 0^{i_3} \qquad (0^0 = 1)$$

and

$$u\Lambda' = p^{-m} (\cos \theta_1 - \beta_1 u)^{i_1} (\sin \theta - \beta_2 u)^{i_2} (-\beta_3 u)^{i_3} - \cos^{i_1} \theta \sin^{i_2} \theta 0^{i_3} , \tag{I$_3$}$$
$$p = p(u) = [1 - 2\beta(t) u \cos(\theta - \varphi) + u^2]^{\frac{1}{2}} .$$

Designate the function $u\Lambda'$ by $f(u)$; use will be made of the formula

$$f(u) = f(0) + f^{(1)}(v) u = f^{(1)}(v) u \qquad (\text{some } 0 < v < u) .$$

One has

$$f^{(1)}(u) = f_1 + \cdots + f_4 ,$$

where

$$f_1 = -mp^{-2}(u - \beta \cos(\theta - \varphi)) \left(\frac{\cos\theta - \beta_1 u}{p}\right)^{i_1} \left(\frac{\sin\theta - \beta_2 u}{p}\right)^{i_2} \left(\frac{-\beta_3 u}{p}\right)^{i_3},$$

$$f_2 = \frac{i_1}{p} \left(\frac{\cos\theta - \beta_1 u}{p}\right)^{i_1 - 1} (-\beta_1) \left(\frac{\sin\theta - \beta_2 u}{p}\right)^{i_2} \left(\frac{-\beta_3 u}{p}\right)^{i_3}$$

and

$$f_3 = \left(\frac{\cos\theta - \beta_1 u}{p}\right)^{i_1} \frac{i_2}{p} \left(\frac{\sin\theta - \beta_2 u}{p}\right)^{i_2 - 1} (-\beta_2) \left(\frac{-\beta_3 u}{p}\right)^{i_3},$$

$$f_4 = \left(\frac{\cos\theta - \beta_1 u}{p}\right)^{i_1} \left(\frac{\sin\theta - \beta_2 u}{p}\right)^{i_2} \frac{i_3}{p} \left(\frac{-\beta_3 u}{p}\right)^{i_3 - 1} (-\beta_3) .$$

It is observed that the functions

$$p^{-1}(\cos\theta - \beta_1 u) , \ p^{-1}(\sin\theta - \beta_2 u), \ -\beta_3 u p^{-1}$$

are in the nature of direction cosines and, thus, their absolute values are $\leqq 1$; moreover, as noted previously, $p^{-1} \leqq \sec \vartheta(t)$. Whence, for $0 \leqq u \leqq 1$,

$$|f^{(1)}(u)| \leqq 2mp^{-2} + i_1 p^{-1} + i_2 p^{-1} + i_3 p^{-1} \leqq 3m \sec^2 \vartheta(t) .$$

The same inequality is satisfied by $f^{(1)}(v)$. Hence

$$|u\Lambda'| = |f^{(1)}(v)u| \leqq 3m \sec^2 \vartheta(t) \cdot u \qquad (0 \leqq u \leqq 1)$$

and, by (I_2),

$$(\mathrm{I}_4) \qquad |\Lambda^{s_1 \cdots s_m}| \leqq |\Lambda'| + |\Lambda''| \leqq 3(m+1) \sec^2 \vartheta(t) \qquad (\tau \geqq 1) .$$

By (I_1) and the preceding, on noting a formula subsequent (4.22), we obtain the inequality

$$B_t^*(\tau, \theta) \leqq 3 \sec^2 \vartheta(t) \sum_{m=1}^{\infty} (m+1) 3^{2m} c_m \qquad (\tau \geqq 1) ,$$

useful only if the latter series converges (note the assumed convergence of the series c^0 (3.20b)). In view of (4.24a)

$$(4.25) \qquad |K_t^*(\tau)| \leqq \tau^{-2} 6\pi \sec^2 \vartheta(t) \sum (m+1) 3^{2m} c_m = \tau^{-2} B^*(t) \qquad (\text{for } \tau \geqq 1) .$$

By the first inequality (4.22a)

$$(4.25\text{a}) \qquad |K_t^*(\tau)| \leqq 2\pi c^0 \sec^2 \vartheta(t) \cdot \tau \qquad (\text{for } 0 \leqq \tau \leqq 1) .$$

Hence from (4.22) it is inferred that

$$|K(t)| \leqq \left(\int_{\tau=0}^{1} + \int_{\tau=1}^{\infty}\right) |K_t^*(\tau)| d\tau \leqq \pi c^0 \sec^2 \vartheta(t) + B^*(t) .$$

The following has been proved.

Lemma 4.26. *Assume the conditions of Theorem* 3.25. *Let t be on S (not on the edges of S). Suppose $x \to t$, nontangentially to S, along a direction (λ_t), as described in* (4.1)—(4.2a). *The limit $K(t)$* (4.22) *will then exist. If the series*

$$s^0 = \sum_{m=1}^{\infty} (m+1) 3^{2m} c_m \tag{4.26a}$$

converges, $K(t)$ satisfies

$$|K(t)| \leqq (c^0 + 6s^0)\pi \sec^2 \vartheta(t) \qquad \big(c^0 \textit{ from } (3.20\text{b})\big) . \tag{4.26b}$$

In view of (4.3), (4.4), (4.8)

$$\Psi(x) - \Psi_a'(x) \to \Psi_a^0(t) + \Psi_a''(t) \qquad (\text{as } h \to 0) .$$

By virtue of (4.11), (4.11a) we may substitute above $\Psi_a'(x) = q(t)A(X) + B(X)$; thus from (4.12) it is deduced that

$$\Psi(x) - q(t)A(X) \to B(t) + \Psi_a^0(t) + \Psi_a''(t) ;$$

as a consequence of (4.19) we may here let $A(X) = A_a^0(X) + A_a^*(h)$, obtaining (by (4.22))

$$\Psi(x) - q(t)A_a^0(X) \to q(t)K(t) + B(t) + \Psi_a^0(t) + \Psi_a''(t) = J_a(t) . \tag{4.27}$$

From (4.12), (4.4), (4.8) it is inferred that (with $S' = S - S_{t,a}$)

$$J_a(t) = q(t)K(t) + \int_{S'} k(y,t) r^{-2}(y,t) q(y) d\sigma(y)$$
$$+ \int_{S_{t,a}} k'(t|y,t) r^{-2}(y,t)\big(q(y) - q(t)\big) d\sigma(y) + \int_{S_{t,a}} k''(t|y,t) r^{-2}(y,t) q(y) d\sigma(y) .$$

As $a \to 0$, the first integral above tends to the principal integral $\Psi(t)$ (1.3) the second and third integrals are in the ordinary sense and tend to zero. Thus

$$J_a(t) = \nu(t) + \nu_a(t) , \qquad \nu(t) = q(t)K(t) + \Psi(t) , \tag{1_0}$$

where

$$\lim \nu_a(t) = 0 \qquad (\text{as } a \to 0) . \tag{2_0}$$

The meaning of (4.27) is that

$$\Psi(x) - q(t)A_a^0(X) = \nu(t) + \nu_a(t) + \nu_a(t,h) , \tag{3_0}$$

where $\nu_a(t,h)$ (as $h \to 0$). As stated in (4.19a), $A_a^0(X)$ is $O(a)$, uniformly with respect to h; hence (by (2_0))

$$(4_0) \qquad |\nu_0(t)+q(t)A_a^0(X)| < \frac{\varepsilon}{2}$$

for some sufficiently small $a(>0)$, independent of h. By (3_0) one has

$$|\Psi(x)-\nu(t)| \leqq |\nu_a(t)+q(t)A_a^0(X)|+|\nu_a(t, h)| < \frac{\varepsilon}{2}+|\nu(t, h)| .$$

Now, with a fixed so that (4_0) holds, choose h_ε (>0) so that

$$|\nu_a(t, h)| < \frac{\varepsilon}{2} \qquad (\text{for } 0 < h \leqq h_\varepsilon) ;$$

one thus has

$$|\Psi(x)-\nu(t)| < \varepsilon \qquad (\text{for } 0 < h \leqq h_\varepsilon) ;$$

that is, $\lim \Psi(x) = \nu(t)$ (for $h \to 0$).

On taking account of (1_0) the following is concluded.

Theorem 4.28. *Cosider the integral*

$$(4.28\text{a}) \qquad \Psi(x) = \int_S \frac{k(y, x)}{r^2(y, x)} q(y) d\sigma(y)$$

and assume that $k(y, x)$, $q(y)$ satisfy the conditions in Theorem 3.25. Suppose $x \to t$, nontangentially to S, along a direction (λ_t), as described in (4.1)—(4.2a). We then have

$$(4.28\text{b}) \qquad \lim_{x\to t} \Psi(x) = q(t)K(t)+\Psi(t) ;$$

here $K(t)$ is defined by (4.22), is independent of $q(t)$, but generally depends on (λ_t); the integral

$$(4.28\text{c}) \qquad \Psi(t) = \int_S \frac{k(y, t)}{r^2(y, t)} q(y) d\sigma(y)$$

is in the sense of principal values.

Let us distinguish between two distinct directions

$$(4.29) \qquad (\lambda_t') , \qquad (\lambda_t'') ,$$

the corresponding direction cosines $\lambda_j'(t)$, $\lambda_j''(t)$ being functions of the type of the $\lambda_j(t)$, as described at the beginning of this section; let $\vartheta'(t)$ be the angle between (λ_t') and $(+n_t)$ and assume $0 \leqq \vartheta'(t) < \frac{\pi}{2}$ (as in (4.2a)); for the angle $\vartheta''(t)$, between (λ_t'') and $(+n_t)$, we shall assume either

$$(4.29\text{a}) \qquad 0 \leqq \vartheta''(t) < \frac{\pi}{2} \qquad (\text{all } t \text{ on } S)$$

or

$$\frac{\pi}{2} < \vartheta''(t) \leqq \pi \qquad (\text{all } t \text{ on } S)\,. \tag{4.29b}$$

Designate by $\varphi'(t), \varphi''(t)$ the angles corresponding to the angle $\varphi(t)$, introduced subsequent (4.2a). In all cases tangential approaches to t are avoided. Let $K'(t)$ be the function $K(t)$ (4.22) for $(\lambda_t) = (\lambda_t')$ and let $K''(t)$ be the function $K(t)$ corresponding to $\lambda''(t)$; (4.28b) gives

$$\begin{aligned} \Psi'(t) &= \lim_x \Psi(x) = q(t)K'(t) + \Psi(t) \qquad \bigl(x \to t \text{ along } (\lambda_t')\bigr)\,, \\ \Psi''(t) &= \lim_x \Psi(x) = q(t)K''(t) + \Psi(t) \qquad \bigl(x \to t \text{ along } (\lambda_t'')\bigr)\,; \end{aligned} \tag{4.30}$$

at points t for which $K'(t) - K''(t) \neq 0$ we, accordingly, have

$$\begin{aligned} q(t) &= \alpha(t)[\Psi'(t) - \Psi''(t)]\,, \Psi(t) = \alpha_1(t)\Psi'(t) + \alpha_2(t)\Psi''(t)\,, \\ \alpha(t) &= [K'(t) - K''(t)]^{-1}, \ \alpha_1(t) = -K''(t)\alpha(t), \ \alpha_2(t) = K'(t)\alpha(t)\,. \end{aligned} \tag{4.31}$$

Suppose for the moment that (λ_t') is a direction opposite to (λ_t). To obtain the function $K_t^{*}{}'(\tau)$ (4.21a), corresponding to the direction (λ_t), we replace $\vartheta(t)$ by $\vartheta'(t) = \pi - \vartheta(t)$ and $\varphi(t)$ by $\varphi'(t) = \varphi(t) + \pi$; by (4.20)

$$\beta_j'(t) = -\beta_j(t) \ (j = 1, 2, 3); \ \beta'(t) = \beta(t)\,;$$

$$K_t^{*}{}'(\tau) = \int_{\theta=0}^{2\pi} d\theta \sum_{m=1}^{\infty} \tau[1 + 2\beta(t)\tau\cos\bigl(\theta - \varphi(t)\bigr) + \tau^2]^{-\frac{m}{2}-1} \,.$$

$$\cdot \sum_{s_1, \ldots s_m = 1}^{3} \Gamma_{s_1 \ldots s_m}(t)(\tau\cos\theta_{s_1} + \beta_{s_1}) \ldots (\tau\cos\theta_{s_m} + \beta_{s_m})\,.$$

Replacing θ by $\theta + \pi$, the $\cos\theta_s$ are replaced by $-\cos\theta_s$ $(s = 1, 2, 3)$, respectively, and one obtains

$$\begin{aligned} K_t^{*}{}'(\tau) &= \int_{\theta=0}^{2\pi} d\theta \sum_{m=1}^{\infty} \tau[1 - 2\beta(t)\tau\cos\bigl(\theta - \varphi(t)\bigr) + \tau^2]^{-\frac{m}{2}-1} \,. \\ &\cdot (-1)^m \sum_{s_1, \ldots s_m = 1}^{3} \Gamma_{s_2 \ldots s_m}(t)\bigl(\tau\cos\theta_{s_1} - \beta_{s_1}(t)\bigr) \ldots \bigl(\tau\cos\theta_{s_m} - \beta_{s_m}(t)\bigr)\,. \end{aligned} \tag{4.32}$$

The function (4.22), corresponding to (λ_t'), is

$$K'(t) = \int_{\tau=0}^{\infty} K_t^{*}{}'(\tau)\,d\tau\,.$$

By (4.21a), (4.32)

$$K(t)-K'(t) = \int_0^\infty d\tau \int_0^{2\pi} d\theta \sum_{\mu=0}^{\infty} 2\tau[1-2\beta(t)\tau\cos(\theta-\varphi(t))+\tau^2]^{-\mu-\frac{3}{2}}$$

(4.33)

$$\sum_{s_1,\dots s_{2\mu+1}} \Gamma_{s_1\dots s_{2\mu+1}}(t)(\tau\cos\theta_{s_1}-\beta_{s_1}(t))\dots(\tau\cos\theta_{s_{2\mu+1}}-\beta_{s_{2\mu+1}}(t))$$

(for opposite directions). If in the kernel $k(y, x)r^{-2}(y, x)$ (3.1) we have

(I) $$\gamma_{i_1\dots i_m}(y) = 0 \qquad (\text{for } m \text{ odd}),$$

so that $k_m(y, x)$ (3.1a) $= 0$ (for m odd), in view of (3.6a) the $\Gamma_{s_1\dots s_m}(t)$ will be zero for m odd and, by (4.33), we shall have

(4.33a) $$K(t)-K'(t) = 0 \qquad (\text{opposite directions}).$$

Consider the case when

(II) $$\gamma_{i_1\dots i_m}(y) = 0 \qquad (\text{for } m \text{ even});$$

then the $\Gamma_{s_1\dots s_m}(t)$ will be zero for m even; one then has

(4.33b) $$K(t)+K'(t) = 0 \qquad (\text{opposite directions}).$$

For the present we shall not examine the conditions under which positive lower bounds for $\alpha(t)$ (4.31), $|K'(t)|$ (or $|K''(t)|$) exist.

Consider the approach along the positive normal, $(\lambda_t) = (+n_t)$. We then have $\beta_1 = \beta_2 = 0$, $\beta_3 = 1$, $\beta(t) = 0$ and

$$\tau\cos\theta_s-\beta_s = \tau\cos\theta \ (s = 1), = \tau\sin\theta \ (s = 2), = -1 \ (s = 3).$$

In view of (4.21a)

(1₀) $$K_t^*(\tau) = \int_{\theta=0}^{2\pi} \left\{\sum_{m=1}^{\infty} \tau^{m+1}[1+\tau^2]^{-\frac{m}{2}-1} \sum_{s_1,\dots s_m=1}^{2} \Gamma_{s_1\dots s_m}(t)\cos\theta_{s_1}\dots\cos\theta_{s_m} + S_t(\tau,\theta)\right\} d\theta,$$

where

$$S_t(\tau,\theta) = \sum_{m=1}^{\infty} \sum_{s_1,\dots s_m}{}' \tau[1+\tau^2]^{-\frac{m}{2}-1} \Gamma_{s_1\dots s_m}(t)(\tau\cos\theta_{s_1}-\beta_{s_1}(t))\dots(\tau\cos\theta_{s_m}-\beta_{s_m}(t));$$

here the primed sum is over sets $(s_1,\dots s_m)$ containing one or more numbers 3. Let

(2_0) $$\Gamma_{i_1:i_2:i_3}(t) = \Gamma_{1,\dots 1,2,\dots 2,3,\dots 3}(t) \qquad (i_1+i_2+i_3 = m);$$

in the second member 1, 2, 3 are repeated i_1, i_2, i_3 times, respectively. By (3.6a) $\Gamma_{s_1\dots s_m}(t)$ is unchanged when the subscripts are permuted; the number of permutations of i_1 numbers 1, i_2 numbers 2, i_3 numbers 3 is $\frac{m!}{i_1!\,i_2!\,i_3!}$; hence

$$S_t(\tau, \theta) = \sum_{m=1}^{\infty} \sum_{i_1, i_2, i_3} \tau[1+\tau^2]^{-\frac{m}{2}-1} \frac{m!}{i_1!\, i_2!\, i_3!} \Gamma_{i_1:i_2:i_3}(\tau \cos \theta_1 - \beta_1(t))^{i_1} \cdot$$

$$\cdot (\tau \cos \theta_2 - \beta_2(t))^{i_2} (\tau \cos \theta_3 - \beta_3(t))^{i_3} = \sum_{m=1}^{\infty} \sum_{i_1, i_2, i_3} \tau[1+\tau^2]^{-\frac{m}{2}-1} \frac{m!}{i_1!\, i_2!\, i_3!} \cdot$$

$$\cdot \Gamma_{i_1:i_2:i_3}(t)(\tau \cos \theta)^{i_1} (\tau \sin \theta)^{i_2} (-1)^{i_3}.$$

The contribution to $K_t^*(\tau)$ arising from $S_t(\tau, \theta)$ is

$$(3_0) \qquad \sum_{m=1}^{\infty} \sum_{i_1, i_2, i_3} (-1)^{i_3} \frac{m!}{i_1!\, i_2!\, i_3!} \tau^{i_1+i_2+1} [1+\tau^2]^{-\frac{m}{2}-1} \Gamma_{i_1:i_2:i_3}(t) \int_0^{2\pi} \cos^{i_1}\theta \sin^{i_2}\theta d\theta$$

$(i_1+i_2+i_3 = m,\ i_3 > 0)$. We shall modify the function of τ, displayed after the summation symbol with respect to m in (1_0), subtracting from it τ^{-1}, when $\tau \geqq 1$, and leaving it unchanged for $0 \leqq \tau < 1$: this can be done in view of the satisfied condition (3.14). *Let* $\lambda(\tau)$ *be defined as* 0 *for* $\tau < 1$ *and as* 1 *for* $\tau \geqq 1$. The part of $K_t^*(\tau)$ (1_0) obtained by disregarding $S_t(\tau, \theta)$ will then be

$$(4_0) \qquad \sum_{m=1}^{\infty} \left[\tau^{m+1} [1+\tau^2]^{-\frac{m}{2}-1} - \frac{\lambda(\tau)}{\tau} \right] \sum_{s_1, \ldots s_m = 1}^{2} \Gamma_{s_1 \ldots s_m}(t) \int_0^{2\pi} \cos \theta_{s_1} \ldots \cos \theta_{s_m} d\theta ;$$

the function $[\ldots]$ above is bounded and is $O(\tau^{-2})$ for τ large. $K_t^*(\tau)$ is the sum of the functions (3_0), (4_0). Thus, by (4.22), *for the approach along* $(+n_t)$, *one has*

$$(4.34) \qquad K(t) = \sum_{m=1}^{\infty} \left\{ \sum_{s_1, \ldots s_m = 1}^{2} C_{s_1 \ldots s_m} \Gamma_{s_1 \ldots s_m}(t) + \sum_{i_1+i_2+i_3 = m} (i_3 > 0) C_{i_1:i_2:i_3} \Gamma_{i_1:i_2:i_3}(t) \right.$$

(*cf.* (2_0) *for* $\Gamma_{i_1:i_2:i_3}$), *where* $\left(\textit{with } \theta_1 = \theta,\ \theta_2 = \frac{\pi}{2} - \theta \right)$

$$(4.34\,\mathrm{a}) \qquad C_{s_1 \ldots s_m} = \int_0^{\infty} \left[\tau^{m+1} (1+\tau^2)^{-\frac{m}{2}-1} - \frac{\lambda(\tau)}{\tau} \right] d\tau \cdot \int_0^{2\pi} \cos \theta_{s_1} \ldots \cos \theta_{s_m} d\theta ,$$

$$C_{i_1:i_2:i_3} = \int_0^{\infty} (-1)^{i_3} \frac{m!}{i_1!\, i_2!\, i_3} \tau^{i_1+i_2+1} (1+\tau^2)^{-\frac{m}{2}-1} d\tau \cdot \int_0^{2\pi} \cos^{i_1}\theta \sin^{i_2}\theta d\theta$$

$(i_1+i_2+i_3 = m)$; the integral last displayed is zero except only when i_1 and i_2 are both even. In view of (3.6a) and since $a_{i,3} = n_i(t)$, the $\Gamma_{s_1 \ldots s_m}(t)$ $(s_1, \ldots s_m \leqq 2)$ are unchanged when the approach is changed to the negative normal; inasmuch as

$$\Gamma_{i_1:i_2:i_3}(t) = \sum_{j_1, \ldots j_m = 1}^{3} \gamma_{j_1 \ldots j_m}(t) [a_{j_1,1} \ldots a_{j_\nu,1}] [a_{j_{\nu+1},2} \ldots a_{j_k,2}] [n_{j_{k+1}} \ldots n_{j_m}]$$

$(\nu = i_1,\ k = i_1+i_2;\ m-k = i_3 > 0)$, it follows that $\Gamma_{i_1:i_3:i_3}(t)$ changes to $-\Gamma_{i_1:i_2:i_3}(t)$

for i_3 odd and is unchanged for i_3 even. With the aid of these remarks we come to an agreement with (4.33a), (4.33b), when the directions are normal.

5. **Order of infinity of $\Psi(x)$ (1.3a) near β.** Consider a point c on the 'edges' β of S. Let P_c be the tangential plane to S at c, T_c be the tangent line to β at c and n_c be the positive normal to S at c. Designate by β' the orthogonal projection of β on P_c; let $S(c, a)$ (small a, > 0) be the neighborhood of c, such that its orthogonal projection on P_c is a region $S'(c, a)$, bounded by a portion of β' and a portion σ' of the circumference of a circle of center c and radius a. Denote by H_c the half plane part of P_c, bounded by T_c and containing 'most' of $S'(c, a)$; that is, H_c contains the intersection of σ' with the perpendicular N_c to T_c at c (in P_c). We introduce

Definition 5.1. *With the above notation in view, let $N(c, \varepsilon)$ denote the neighborhoods of the tangent line T_c (to β) at c, consisting of two circular conical regions with common vertex at c and T_c as axis, the angle at c (for each cone) between the generating lines of the surfaces and T_c being ε. Designate by $W(c, \varepsilon)$ the neighborhood of the tangential half plane H_c, bounded by two half planes meeting along T_c and making angles ε with H_c (on the two sides of H_c); $W(c, \varepsilon)$ contains H_c.*

We consider $N(c, \varepsilon)$, $W(c, \varepsilon)$ as closed. The point of the above definition is that if x remains exterior $N(c, \varepsilon)+W(c, \varepsilon)$, x cannot tend to c tangentially either to the curve β nor to the surface S.

With ε (> 0) fixed, choose a (> 0) so small that the portions β, β' bounding $S(c, a)$, $S'(c, a)$, respectively, are in $N\left(c, \frac{\varepsilon}{2}\right)$, while $S(c, a)$ is in $N\left(c, \frac{\varepsilon}{2}\right)+W\left(c, \frac{\varepsilon}{2}\right)$. We shall proceed with x, near c, exterior $N(c, \varepsilon)+W(c, \varepsilon)$ and with $q(y)$ subject to conditions of Theorem 3.25.

We express $\Psi(x)$ (1.3a) as follows:

$$\Psi(x) = \Psi_a^*(x)+\Psi_a^0(x) \,, \tag{5.2}$$

$$\Psi_a^*(x) = \int_{S(c,a)} k(y, x)r^{-2}(y, x)q(y)d\sigma(y) \,, \tag{5.2a}$$

$$\Psi_a^0(x) = \int_{S'} k(y, x)r^{-2}(y, x)q(y)d\sigma(y) \qquad [S' = S-S(c, a)] \,.$$

For x at distance $\leqq 2^{-1}a$ from the perpendicular to P_c at c and for y on S' one has $r(y, x) \geqq 2^{-1}a$; in view of (3.21)

$$|\Psi_a^0(x)| \leqq \frac{4}{a^2}\int_{S'} |k(y, x)|\ |q(y)|d\sigma(y) \leqq \frac{4c'}{a^2}\int_{S'} |q(y)|d\sigma(y) \,.$$

Since $q(x) \subset [\alpha|S]$, the integrand here is $O(l^{-\alpha}(y))(\alpha < 1)$; thus by virtue of the remarks with respect to (3.22), (3.22a) one has

$$|\Psi_a^0(x)| \leqq c^* . \tag{5.3}$$

Some of the proofs in the sequel will be with *the coordinate axes* y_j *assumed so that the origin is at* c, *the* y_1*-axis coincides with the tangent line* T_c, *the* $+y_3$*-semiaxis falls along the positive normal* n_c *and the* $+y_2$*-axis lies in the* H_c *half plane*; let

$$y' = (y_1, y_2, 0) ; \tag{5.4}$$

for y on $S(o, a)$ one then has

$$y_3 = F(y_1, y_2) \qquad (F \text{ as in } (2.1)) ; \tag{5.4a}$$

$S'(o, a)$ is a subregion of the circular region $r^2(o, y') = y_1^2+y_2^2 \leqq a^2$, bounded by an arc σ' of the circle $r(o, y') = a$ and by a curvilinear arc β' (projection on the y_1, y_2-plane of a portion of β); β' is tangent to the y_1-axis at o and is given by an equation

$$y_2 = f(y_1) = O(y_1^2) ; \tag{5.4b}$$

the regions $N(o, \varepsilon)$ are given by the inequality

$$x_2^2+x_3^2 \leqq x_1^2 \operatorname{tg}^2 \varepsilon , \tag{5.4c}$$

while $W(o, \varepsilon)$ is represented by

$$|x_3| \leqq x_2 \operatorname{tg} \varepsilon , \qquad x_2 \geqq 0 . \tag{5.4d}$$

To say that x is exterior $N(c, \varepsilon)+W(c, \varepsilon)$ is equivalent to the relations

$$x_2^2+x_3^2 > x_1^2 \operatorname{tg}^2 \varepsilon; \qquad |x_3| > x_2 \operatorname{tg} \varepsilon \qquad (\text{if } x_2 \geqq 0) . \tag{5.5}$$

The following will be now proved.

Lemma 5.6. *When* y *is on* $S(c, a)$ $(a, > 0,$ *small*) *and* x *is* (*near* c) *exterior* $N(c, \varepsilon)+W(c, \varepsilon)$, *one has*

$$r^{-1}(x, y) \leqq k(\varepsilon) r^{-1}(c, x) , \tag{5.6a}$$

where $k(\varepsilon)$ (> 0) *is independent of* x, y *and is* $O(\varepsilon^{-2})$.

Choose coordinate axes as stated subsequent (5.3). It will suffice to give the proof when $x_3 > 0$; we then have

$$x_3 > x_2 \operatorname{tg} \varepsilon \ (\text{if } x_2 \geqq 0); \qquad x_2^2+x_3^2 > x_1^2 \operatorname{tg}^2 \varepsilon . \tag{1°}$$

Designate by $(W^+(o, \varepsilon)$ the 'top' part of the boundary of $W(o, \varepsilon)$; thus $(W^+(o, \varepsilon))$ consists of points x such that

$$x_3 = x_2 \operatorname{tg} \varepsilon \qquad (x_2 \geqq 0) .$$

Exterior $W(o, \varepsilon)$ $(x_3 > 0;\ x_2 \geqq 0)$ one has

$$(2^\circ) \qquad x_3[x_1^2+x_2^2]^{-\frac{1}{2}} > \sin\theta \operatorname{tg}\varepsilon \qquad \left(\theta = \operatorname{arc\,tg}\frac{x_2}{|x_1|}\right);$$

on $(W^+(o, \varepsilon)) > 0$ is here replaced by $=$; $(W^+(o, \varepsilon))$ intersects the surfaces of the cones $N(o, \varepsilon)$ along lines l^+, l^-, extending from o and expressible parametrically as follows:

$$(3^\circ) \qquad x_1 = \pm t,\ x_2 = t\sin\varepsilon,\ x_3 = t\sin\varepsilon\operatorname{tg}\varepsilon$$

$(+$ for l^+; $-$ for l^-; $t \geqq 0)$; on l^+, l^-

$$(5.7) \qquad x_3[x_1^2+x_2^2]^{-\frac{1}{2}} = \sin\varepsilon\operatorname{tg}\varepsilon[1+\sin^2\varepsilon]^{-\frac{1}{2}} = \operatorname{tg} 2\varepsilon_1;\ \frac{x_2}{|x_1|} = \operatorname{tg}\varepsilon_2 = \sin\varepsilon.$$

On the part of $(W^+(o, \varepsilon))$ between l^+ and l^-, by (2°),

$$(4^\circ) \qquad x_3[x_1^2+x_2^2]^{-\frac{1}{2}} \geqq \sin\varepsilon_2 \operatorname{tg} e = \operatorname{tg} 2\varepsilon_1.$$

The intersections (for $x_3 > 0$, $x_2 \geqq 0$) of the conical surfaces $N(o, \varepsilon)$ (5.4 c), $x_2^2+x_3^2 = x_1^2\operatorname{tg}^2\varepsilon$, with the planes

$$x_2 = |x_1|\operatorname{tg}\theta \qquad (0 \leqq \operatorname{tg}\theta \leqq \operatorname{tg}\varepsilon_2;\ x_2 \geqq 0)$$

are given parametrically by

$$x_1 = \pm t,\ x_2 = t\operatorname{tg}\theta,\ x_3 = t\sqrt{\operatorname{tg}^2\varepsilon - \operatorname{tg}^2\theta} \quad (\text{parameter } t \geqq 0);$$

along these intersections (with $x_2 \geqq 0$)

$$x_3[x_1^2+x_2^2]^{-\frac{1}{2}} = \sqrt{\operatorname{tg}^2\varepsilon-\operatorname{tg}^2\theta}\cos\theta \geqq \sqrt{\operatorname{tg}^2\varepsilon-\operatorname{tg}^2\varepsilon_2}\cos\varepsilon_2 = \operatorname{tg} 2\varepsilon_1.$$

Accordingly

$$(5^\circ) \qquad x_3[x_1^2+x_2^2]^{-\frac{1}{2}} \geqq \operatorname{tg} 2\varepsilon_1$$

for x (with $x_3 > 0$, $x_2 \geqq 0$) on the conical surfaces $N(o, \varepsilon)$, between the line $l^+[l^-]$ and the y_1, y_3-plane. By virtue of (4°), (5°) *it is seen that*

$$(5.7\,\text{a}) \qquad x_3[x_1^2+x_2^2]^{-\frac{1}{2}} > \operatorname{tg} 2\varepsilon_1 \qquad (\varepsilon_1 \text{ from } (5.7))$$

when x *(with* $x_3 > 0$, $x_2 \geqq 0$*) is anywhere exterior* $N(o, \varepsilon)+W(o, \varepsilon)$; *designate by* $K_{2\varepsilon_1}$ *the conical domain* (5.7 a) *and by* $(K^+_{2\varepsilon_1})$ *its surface.* Choose $a(>0)$ so small that *the portion* $S(o, a)$ *of the surface* S *lies (except for* o*) between the conical surfaces* $(K^+_{\varepsilon_1})$, $(K^-_{\varepsilon_1})$ $[K^-_{\varepsilon_1}$ being the symmetrical image of $(K^+_{\varepsilon_1})$ across the y_1, y_2-plane]; thus for y on $S(o, a)$

$$(5.8) \qquad |y_3|[y_1^2+y_2^2]^{-\frac{1}{2}} \leqq \operatorname{tg}\varepsilon_1.$$

Case I. *Let x be in $K_{2\varepsilon_1}$ and y satisfy* (5.8) (y not necessarily on the surface S). With x, y on the opposite sides of $(K_{\varepsilon_1}^{+})$, one has

$$(1_0) \qquad r(x, y) \geqq r(x, \eta)\,,$$

where $\eta = (\eta_1, \eta_2, \eta_3)$ with

$$\eta_1 = \sqrt{y_1^2+y_2^2}\cos\theta,\ \eta_2 = \sqrt{y_1^2+y_2^2}\sin\theta,\ \operatorname{tg}\theta = \frac{x_2}{x_1};$$

η satisfies (5.8) and the points x, η are on the opposite sides of $(K_{\varepsilon_1}^{+})$, but in the same half plane bounded by the y_3-axis. We have

$$r(x, \eta) \geqq r(x, y^0)\,,$$

where y^0 is the point of intersection of $(K_{\varepsilon_1}^{+})$ and of the line joining x and η. Let x^0 be the foot of the perpendicular from x to the line (o, y^0); clearly $r(x, y^0) \geqq r(x, x^0)$ and, by (1_0),

$$r(x, y) \geqq r(x, x^0)\,.$$

Now x_0, x are on the opposite sides of the conical surface $(K_{2\varepsilon_1}^{+})$; the angle at o between $(K_{\varepsilon_1}^{+})$ and $(K_{2\varepsilon_1}^{+})$ being ε_1, it is inferred that

$$r(x, x^0) \geqq r(o, x)\sin\varepsilon_1\,;$$

thus

$$(5.9) \qquad r^{-1}(x, y) \leqq \csc\varepsilon_1 r^{-1}(o, x) \ \big(\textit{in the Case } \mathrm{I};\ \varepsilon_1 \textit{ from } (5.7)\big).$$

Case II. *Let x be exterior $N(o, \varepsilon)$ and y be in $N\left(o, \frac{\varepsilon}{2}\right)$* ($y$ not necessarily on the surface S). $\Big[$It is observed that if y, with $y_2 \leqq 0$, is on $S(o, a)$ (a sutably small) then y is in $N\left(o, \frac{\varepsilon}{2}\right)\Big]$. We now have

$$(5.10') \qquad (x_2^2+x_3^2)^{\frac{1}{2}}|x_1|^{-1} > \operatorname{tg}\varepsilon; \qquad (y_2^2+y_3^2)^{\frac{1}{2}}|y_1|^{-1} \leqq \operatorname{tg}\frac{\varepsilon}{2}.$$

It will suffice to give the developments for $y_1 > 0$, $x_1 \geqq 0$. The plane $y_1 =$ const. intersects a cone $N\left(o, \frac{\varepsilon}{2}\right)$ in a circular region K_y of radius $y_1 \operatorname{tg}\frac{\varepsilon}{2}$. Let C_{y_1} be the surface of the right circular cylinder having K_{y_1} for a cross section. *Suppose first that x* (*subject to* (5.10^1)) *is exterior* C_{y_1}. The plane D_x, containing x and the y_1-axis, intersects K_{y_1} in one of its diameters; let η he the end point of this diameter nearest to x; we have $r(x, y) \geqq r(x, \eta)$. Designate by x_0 the foot of the perpendicular from x on the line (possibly extended) joining o, η; clearly $r(x, \eta) \geqq r(x, x_0)$. In the plane D_x we accordingly have a triangle with vertices o, η, x; the segment (o, η)

forms part of a generator of $N\left(o, \frac{\varepsilon}{2}\right)$, while a generator of $N(o, \varepsilon)$ extends from o, intersecting the side (η, x) internally; this is due to the hypothesis that x is exterior $N(o, \varepsilon)$. Thus in the triangle the angle at o exceeds $\varepsilon/2$; hence

$$r(x, x_0) \geqq r(o, x) \sin\frac{\varepsilon}{2};$$

that is

$$r(x, y) \geqq r(o, x) \sin\frac{\varepsilon}{2} \qquad (\text{Case II for } x \text{ exterior } C_{y_1}).$$

Suppose now x is on or interior the cylinder C_{y_1} and is still subject to (5.10′). The point $\eta_3 = (y_1, x_2, x_3)$ is in the circular region K_{y_1}. It is observed that $r(x, y) \geqq r(x, \eta^0)$. The segment (x, η^0) intersects a generator of $N\left(o, \frac{\varepsilon}{2}\right)$ in a point η^* (it may happen that $\eta^* = \eta^0$); one has

$$r(x, \eta^0) \geqq r(x, \eta^*) \geqq r(x, x^0),$$

where x^0 is the foot of the perpendicular from x on the segment (o, η^*). A generator of $N(o, \varepsilon)$ passes between (o, x) and (o, η^*); accordingly the angle between (o, x) and (o, η^*) exceeds $\varepsilon/2$ and we have

$$r(x, y) \geqq r(x, x^0) \geqq r(o, x) \sin\frac{\varepsilon}{2} \qquad (\text{Case II for } x \text{ on or interior } C_{y_1}).$$

From the above it is inferred that

$$r^{-1}(x, y) \leqq \csc\frac{\varepsilon}{2} r^{-1}(o, x) \qquad (\textit{in Case II}). \tag{5.10}$$

Case III. *x is exterior $N(o, \varepsilon)$ and $x_2 \leqq 0$; $y_2 \geqq 0$ and y lies in the region bounded above by $(K_{\varepsilon_1}^+)$ and below by $(K_{\varepsilon_1}^-)$* (that is, y satisfies (5.8), $y_2 \geqq 0$). For purposes of the discussion one may take $x_1, x_3 \geqq 0$ (note that the regions for x and y are each symmetric with respect to the y_2, y_3-plane, as well as with respect to the y_1, y_2-plane). Let $y' = (y_1, y_2, 0)$. If $y_3 < 0$, then $(x_3-y_3)^2 > x_3^2$ and, so,

$$\frac{r^2(x, y)}{r^2(x, y')} = \frac{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2} > 1;$$

with $r(x, y) > r(x, y')$, it will then suffice to obtain a suitable lower bound for $r(x, y')$. In view of this remark we shall proceed under the supposition that $y_3 \geqq 0$. We may therefore take

$$x_1 \geqq 0,\ x_2 \leqq 0,\ x_3 \geqq 0; \qquad y_2 \geqq 0, y_3 \geqq 0. \tag{5.11}$$

Let C be the semicircle consisting of points z such that

$$z_3 = y_3,\ z_1^2+z_2^2 = y_1^2+y_2^2,\ z_2 \geqq 0\,;$$

the point y is on C. As a consequence of elementary considerations, of all the points on C one of its end points, namely

$$\eta = (\sqrt{y_1^2+y_2^2},\, 0,\, y_3)$$

is nearest to x. Thus

$$(1_0) \qquad r(x, y) \geqq r(x, \eta)\,.$$

For the angle α between the $+y_1$-axis and the line (o, η) one has

$$(2_0) \qquad 0 \leqq \alpha \leqq \varepsilon_1 \ (< \varepsilon)\,.$$

Since $\varepsilon_1 < \varepsilon$, the segment (o, η) lies in $N(o, \varepsilon)$. Let x_0 be the foot of the perpendicular from x on the line (o, η) (the line extended, if necessary). Consider the triangle o, x, η; inasmuch as η is interior $N(o, \varepsilon)$ and x is exterior $N(o, \varepsilon)$, there is a generator g of the surface of $N(o, \varepsilon)$, extending from o and intersecting the segment (x, η) between x and η. Thus the angle at o in the right triangle o, x, x_0 exceeds the angle

$$(3_0) \qquad \beta = \text{angle between } g \text{ and the segment } (o, \eta)\,;$$

one has

$$(5.12) \qquad r(x, \eta) \geqq r(x, x_0) \geqq r(o, x) \sin \beta\,.$$

To get a lower bound for β introduce new coordinates (Y_1, Y_2, Y_3) so that the $+Y_2$-axis coincides with the $+y_2$-axis and the $+Y_1$-axis falls along the line joining o, η; the surface of $N(o, \varepsilon)$ (cf. (5.4c) with the equality sign) is representable by the equation

$$(4_0) \qquad a_3 Y_3^2 + Y_2^2 - a_1 Y_1^2 + a_{1,3} Y_1 Y_3 = 0 \quad [a_3 = \cos^2\alpha - \sin^2\alpha \operatorname{tg}^2\varepsilon\,;$$
$$a_1 = \cos^2\alpha \operatorname{tg}^2\varepsilon - \sin^2\alpha\,;\quad a_{1,3} = \sin 2\alpha \sec^2\varepsilon]\,.$$

The pencil of planes through (o, η) will be given by $Y_3 = \lambda Y_2$ (λ a real parameter); the intersection of one of these planes with (4_0) is a generator G (of which g is one) of the cone (4_0); along such a generator (with $Y_2 < 0,\ Y_1 > 0$)

$$(5_0) \qquad Y_1 : Y_2 : Y_3 = 1 : -h(\lambda) : -h(\lambda)\lambda\,;\ h(\lambda) = \frac{a_{1,3}\lambda + \sqrt{a^0\lambda^2 + 4a_1}}{2(1+a_3\lambda^2)}\,,$$

where $a^0 = a_{1,3}^2 + 4a_3a_1$. Let ω be generically a function of ε, tending to zero with ε. We have $\varepsilon_1 = \frac{1}{2}\varepsilon^2(1+\omega)$. Hence by (2_0), (4_0), (5_0)

$$(5.13) \qquad h(\lambda) = \frac{\varepsilon}{\sqrt{1+\lambda^2}}(1+\omega) \qquad (\text{for } |\lambda| \text{ bounded})\,.$$

Let B be the acute angle between $O, +Y_1$ and the generator G (5_0); by (5.13) one has

$$\cos^2 B = [1+h^2(\lambda)+h^2(\lambda)\lambda^2]^{-1} = [1+\varepsilon^2(1+\omega)^2]^{-1} = [1+\varepsilon^2+\varepsilon^2\omega]^{-1} .$$

Hence

$$\sin^2 B = \varepsilon^2(1+\omega)[1+\varepsilon^2(1+\omega)]^{-1}$$

and

$$\sin B \geqq c_0\varepsilon \qquad (\text{some } c_0 = c^*) , \tag{5.14}$$

provided ε is suitably small. Now β in (5.12) is a particular angle B, involved in (5.14); namely, the angle corresponding to the value of λ for which the plane of the pencil of planes through the line (o, η) passes through the point x. Accordingly (by (5.12), (5.14))

$$r(x, \eta) \geqq r(o, x) \sin \beta \geqq r(o, x)c_0\varepsilon ,$$

which together with (1_0) yields

$$r^{-1}(x, y) \leqq \frac{1}{c_0}\frac{1}{\varepsilon} r^{-1}(o, x) \qquad (\textit{Case III}) . \tag{5.15}$$

We now come to the proof of the Lemma 5.6. It will suffice to proceed with the coordinate axes chosen as done in the text after the formulation of the Lemma, with $x_3 \geqq 0$ and with the conditions of the Lemma satisfied.

By the remark preceding (5.8) points y on the surface $S(o, a)$ (a suitably small) are between the conical surfaces $(K^+_{\varepsilon_1})$, $(K^-_{\varepsilon_1})$ and, thus, satisfy (5.8). In view of the text in connection with (5.7a) points x, for which $x_2 \geqq 0$, are in $K_{2\varepsilon_1}$. Whence from (5.9) it is inferred that

(i) $\quad r^{-1}(x, y) \leqq \csc \varepsilon_1 r^{-1}(o, x)$ $\big(x$ exterior $N(o, \varepsilon)+W(o, \varepsilon)$, with $x_2 \leqq 0$;

all y on $S(o, a)\big)$.

Points y on $S(o, a)$, for which $y_2 \leqq 0$ (if any) will be in $N\left(o, \frac{\varepsilon}{2}\right)$; on the other hand, points x with $x_2 \leqq 0$ will be exterior $N(o, \varepsilon)$. Case II is then applicable; hence by (5.10)

(ii) $\quad r^{-1}(x, y) \leqq \csc \frac{\varepsilon}{2} r^{-1}(o, x)$ $[x$ exterior $N(o, \varepsilon)+W(o, \varepsilon)$, with $x_2 \leqq 0$;

y, with $y_2 \leqq 0$, on $S(o, a)]$.

The points y on $S(o, a)$, for which $y_2 \geqq 0$, lie between $(K^+_{\varepsilon_1})$, $(K^-_{\varepsilon_1})$ $\big($a fact stated previously for all y on $S(o, a)\big)$. The x, with $x_2 \leqq 0$, are exterior $N(o, \varepsilon)$. Case III now applies, yielding $\big($cf. (5.15)$\big)$

(iii) $$r^{-1}(x, y) \leqq \frac{1}{c_0}\frac{1}{\varepsilon} r^{-1}(o, x) \big(x \text{ exterior } N(o, \varepsilon)+W(o, \varepsilon), \text{ with } x_2 \leqq 0;$$
$$y, \text{ with } y_2 \geqq 0, \text{ on } S(o, a)\big).$$

Cases (i), (ii), (iii) embody all the possibilities envisaged in the Lemma. The inequality (5.6a) accordingly holds, with $k(\varepsilon)$ equal the greatest of the three quantities in the second members in (i)-(iii). Since $\varepsilon_1^{-1} = O(\varepsilon^{-2})$, it is inferred that $k(\varepsilon) = O(\varepsilon^{-2})$. *The Lemma is thus proved.*

We return now to the function $\Psi(x)$ (5.2). By (3.21) and since $q(y) \subset [\alpha|S]$

(5.16) $$|\Psi_a^*(x)| < c^* \int_{S(c,a)} l^{-\alpha}(y) r^{-2}(y, x) d\sigma(y).$$

Choose again the coordinates as in the text subsequent (5.3) and recall the definitions of $S'(o, a)$, β', σ' (the text from (5.3) to (5.4b)). *We proceed with x exterior $N(o, \varepsilon)+W(o, \varepsilon)$, near o.* With the surface $S(o, a)$ (of which $S'(o, a)$ is the orthogonal projection on the y_1, y_2-plane) and the 'edge' β in the vicinity of o suitably regular, the essential features (for the purposes of study of the order of infinity for x near o) of the integral above are embodied in the case when $S(o, a)$ is a semicircle (in the y_1, y_2-plane),

(5.17) $$S(o, a) = S'(o, a) = \{0 \leqq \varrho = \sqrt{y_1^2+y_2^2} \leqq a; \quad y_3 = 0; \; y_2 \geqq 0\},$$

while β is the rectilinear boundary of $S(o, a)$,

(5.17a) $$\beta = \beta' = \{-a \leqq y_1 \leqq a; \; y_2 = y_3 = 0\}.$$

Introduce polar coordinates (with pole at o),

(5.17b) $$\varrho = \sqrt{y_1^2+y_2^2}, \qquad \theta = \operatorname{arc\,tg}\left(\frac{y_2}{y_1}\right).$$

Use will be made of the decomposition

(5.17c) $$S(o, a) = \sigma_1+\sigma_2; \quad \sigma_1 = \text{part of } S(o, a) \text{ for which } \varrho < 2r(o, x);$$
$$\sigma_2 = \text{part of } S(o, a) \text{ for which } \varrho \geqq 2r(o, x).$$

For the case under consideration

(5.17d) $$l(y) = y_2 = \varrho \sin \theta.$$

For the component of the integral in (5.16), corresponding to σ_1, one has

(1_0) $$I_1(x) = \int_{\sigma_1} y_2^{-\alpha} r^{-2}(y, x) dy_1 dy_2.$$

As a consequence of Lemma 5.6,

$$I_1(x) \leqq k^2(\varepsilon) r^{-2}(o, x) \int_{\sigma_1} y_2^{-\alpha} dy_1 dy_2 ;$$

σ_1 lies in the rectangle

$$-2r(o, x) \leqq y_1 \leqq 2r(o, x), \qquad 0 \leqq y_2 \leqq 2r(o, x) ;$$

hence the integral above is bounded by

$$\int_{-2r(o,x)}^{2r(o,x)} dy_1 \int_0^{2r(o,x)} y_2^{-\alpha} dy_2 = c^* r^{2-\alpha}(o, x) ,$$

inasmuch as $\alpha < 1$; accordingly for x exterior $N(o, \varepsilon) + W(o, \varepsilon)$

$$(5.18) \qquad I_1(x) \leqq c^* k^2(\varepsilon) r^{-\alpha}(o, x) .$$

There is occasion to consider

$$(2_0) \qquad I_2(x) = \int_{\sigma_2} y_2^{-\alpha} r^{-2}(y, x) d\sigma(y)$$

only if $2r(o, x) < a$. Let K be the sphere of center o and radius $r(o, x)$; the plane $y_3 = x_3$ intersects K in a circle C_x, with center on the y_3-axis and radius $\sqrt{x_1^2 + x_2^2}$; x is on C_x. Designate by x^0 the point on C_x in the half plane

$$(3_0) \qquad \theta = \text{arc tg} \left(\frac{y_2}{y_1}\right) ;$$

clearly $r(x, y) \geqq r(x^0, y)$. Let η be the point of intersection of K, the half plane (3_0) and the y_1, y_2-plane. With y in $\sigma_2(a \geqq \varrho = \sqrt{y_1^2 + y_2^2} \geqq 2r(o, x);\ y_2 \geqq 0;\ y_3 = 0)$ and

$$\eta = (\eta_1, \eta_2, o), \qquad \sqrt{\eta_1^2 + \eta_2^2} = r(o, x), \qquad \frac{\eta_2}{\eta_1} = \frac{y_2}{y_1} \quad (\eta_2 \geqq 0) ,$$

it is observed that

$$r(x^0, y) \geqq r(\eta, y) ;$$

furthermore

$$r(\eta, y) \geqq \tfrac{1}{2} r(o, y) = \tfrac{1}{2} \sqrt{y_1^2 + y_2^2} = \tfrac{1}{2} \varrho ;$$

thus

$$(5.19) \qquad r(x, y) \geqq \tfrac{1}{2} \varrho \qquad \big(y = (y_1, y_2, o) \text{ on } \sigma_2\big) .$$

In (2_0) one may put $d\sigma(y) = \varrho d\varrho d\theta$. By the above inequality and (5.17d)

$$I_2(x) \leqq 4 \int_{\sigma_2} (\varrho^{-\alpha} \sin^{-\alpha} \theta) \varrho^{-2} \cdot (\varrho d\varrho d\theta) = 4 \int_0^{\pi} \sin^{-\alpha} \theta d\theta \int_{2r(o,x)}^{a} \varrho^{-1-\alpha} d\varrho ;$$

the integral with respect to θ of course exists, since $\alpha < 1$; thus

$$(5.20) \qquad I_2(x) \leqq c^* r^{-\alpha}(o, x) \qquad (\text{if } 0 < \alpha < 1) ,$$

(5.20a) $$I_2(x) \leqq c^* \log \frac{1}{r(o,x)} \qquad (\text{if } \alpha = 0)$$

(one may as well take $r(o,x) \leqq \frac{1}{2}$).

The second member in (5.16) equals $c^*(I_1+I_2)$; whence, by (5.18), (5.20), (5.20a), it is inferred that

(5.21) $$|\Psi_a^*(x)| < c^* k^2(\varepsilon) r^{-\alpha}(o,x) \qquad (\text{if } 0 < \alpha < 1),$$

$$|\Psi_a^*(x)| \leqq c^* k^2(\varepsilon) \log \frac{1}{r(o,x)} \qquad (\text{if } \alpha = 0),$$

for x near o exterior $N(o,\varepsilon)+W(o,\varepsilon)$ in the case (5.17), (5.17a).

Before considering the more general case, when

(5.22) $$S(o,a) = S'(o,a), \qquad (\beta = \beta' \text{ near } o),$$

with the 'edge' β not necessarily rectilinear near o, we shall need the following result.

Lemma 5.23. *Let $y = (y_1, y_2, 0)$, $\eta = (y_1, y_2+d, 0)$, $0 \leqq d \leqq |y_1| \operatorname{tg} \frac{\varepsilon}{2}$, with*

(5.23a) $$-\frac{\varepsilon}{2} \leqq \theta = \operatorname{arc\,tg} \frac{y_2}{y_1} \leqq \pi + \frac{\varepsilon}{2}, \qquad \varrho = \sqrt{y_1^2+y_2^2} \leqq a;$$

one then has

(5.23b) $$\frac{r(\eta,x)}{r(y,x)} < k_0(\varepsilon), \qquad \frac{r(y,x)}{r(\eta,x)} < k_0(\varepsilon)$$

for x exterior $N(o,\varepsilon)+W(o,\varepsilon)$; here $k_0(\varepsilon)(<\infty)$ is independent of y, η, x and is $O\left(\frac{1}{\varepsilon}\right)$.

It will suffice to proceed with $x_3 \geqq 0$. We note first that

(1°) $$\frac{r^2(\eta,x)}{r^2(y,x)} = 1+\omega(y,\eta,x), \qquad \omega(y,\eta,x) = \left(\frac{d}{r(y,x)}\right)^2 - 2\frac{x_2-y_2}{r(y,x)}\frac{d}{r(y,x)};$$

$$\frac{r^2(y,x)}{r^2(\eta,x)} = 1+q(y,\eta,x), \qquad q(y,\eta,x) = \left(\frac{d}{r(\eta,x)}\right)^2 + 2\frac{x_2-y_2-d}{r(\eta,x)}\frac{d}{r(\eta,x)}.$$

Case (i). *x is exterior $N(o,\varepsilon)+W(o,\varepsilon)$, while $x_2 \geqq 0$, $y_2 \geqq 0$.* By the remark with respect to (5.7a), x is then above the conical surface $(K_{2\varepsilon_1}^+)$; that is,

(2°) $$x_3^2 > \operatorname{tg}^2 2\varepsilon_1 (x_1^2+x_2^2) \qquad (\varepsilon_1 \text{ from } (5.7)).$$

Consider the semicircle consisting of points $z = (z_1, z_2, z_3)$, such that

(3°) $$z_3 = x_3;\ z_1^2+z_2^2 = x_1^2+x_2^2;\ z_2 \geqq 0;$$

x is on (3°); of all the points on this semicircle the point $z^0 = (z_1^0, z_2^0, x_3)$ in the plane

$z_2 = z_1 \operatorname{tg} \theta$ (containing y) is nearest to y; thus $r(y, x) \geqq r(y, z^0)$; z^0 is above $(K_{2\varepsilon}^+)$ and, so, satisfies (2°); clearly

$$r(y, z^0) > r(y, z'), \quad z' = (z_1^0, z_2^0, z_3) \; [z_3 = \operatorname{tg} 2\varepsilon_1 \sqrt{z_1^{0^2} + z_2^{0^2}}];$$

z' is on $(K_{2\varepsilon}^+)$. The angle at o in the triangle o, y, z' is $2\varepsilon_1$; let z^* be the foot of the perpendicular from y on the side (o, z') (extended, if necessary); we have

$$r(y, z') \geqq r(y, z^*) = r(o, y) \sin 2\varepsilon_1$$

and, finally,

$$(4°) \qquad r(y, x) \geqq r(o, y) \sin 2\varepsilon_1 .$$

In view of the inequality for d, given in the Lemma, one accordingly obtains (cf. (5.7) for ε_1)

$$\frac{d}{r(y, x)} \leqq \frac{|y_1|}{r(o, y)} \operatorname{tg} \frac{\varepsilon}{2} \csc 2\varepsilon_1 \leqq \operatorname{tg} \frac{\varepsilon}{2} \csc^2 \varepsilon \leqq c^* \varepsilon^{-1};$$

hence for ω in (1°) we have

$$(5.24) \qquad |\omega(y, \eta, x)| \leqq c^* \varepsilon^{-2} \qquad \text{(in Case (i))}.$$

In Case (i) one has $y_2 + d \geqq 0$. Repeating the developments from (2°) to (5.24), with η in place of y, *we find that* $|q(y, \eta, x)|$ *also satisfies* (5.24).

Case (ii). x is exterior $N(o, \varepsilon)$; $y_2 \leqq 0$. We now have

$$(1_0) \qquad -\frac{\varepsilon}{2} \leqq \theta = \operatorname{arc\,tg} \frac{y_2}{y_1} \leqq 0 \quad \text{or} \quad \pi \leqq \theta \leqq \pi + \frac{\varepsilon}{2};$$

$$-\frac{\varepsilon}{2} \leqq \theta' = \operatorname{arc\,tg} \frac{y_2 + d}{y_1} \leqq \frac{\varepsilon}{2} \quad \text{or} \quad \pi - \frac{\varepsilon}{2} \leqq \theta' \leqq \pi + \frac{\varepsilon}{2}.$$

Let $\tau = (\tau_1, \tau_2, 0)$ stand either for y or for η; thus

$$(2_0) \qquad -\frac{\varepsilon}{2} \leqq \alpha = \operatorname{arc\,tg} \frac{\tau_2}{\tau_1} \leqq \frac{\varepsilon}{2} \quad \text{or} \quad \pi - \frac{\varepsilon}{2} \leqq \alpha \leqq \pi + \frac{\varepsilon}{2}.$$

It will suffice to proceed with $\tau_1 \geqq 0$. Let C_{x_1} be the circle consisting of points $z = (z_1, z_2, z_3)$ for which

$$z_1 = x_1; \qquad z_2^2 + z_3^2 = x_2^2 + x_3^2;$$

x is on C_{x_1}; C_{x_1} intersects the y_1, y_2-plane in two points of which one, say $z^0 = (x_1, z_2^0, 0)$, is nearer to τ (the sign of z_2^0 is not opposite to that of τ_2). We have $r(\tau, x) \geqq r(\tau, z^0)$. A generator g of the conical surface $N(o, \varepsilon)$ extends from o between the segments (o, τ), (o, z^0); τ, z^0 being on the opposite sides of g, one has

$r(\tau, z^0) \geqq r(\tau, z^*)$, where z^* is the foot of the perpendicular from τ on g. In view of (2_0), the angle β at o in the triangle o, τ, z^* is $\geqq \varepsilon/2$; thus

$$r(\tau, z^*) = r(o, \tau) \sin \beta \geqq r(o, \tau) \sin \frac{\varepsilon}{2}$$

and

$$r(\tau, x) \geqq r(o, \tau) \sin \frac{\varepsilon}{2} \qquad (\tau = y \text{ or } \eta) .$$

Similar to the inequalities preceding (5.24) we now obtain

$$\frac{d}{r(y, x)} \leqq \frac{|y_1|}{r(o, y)} \operatorname{tg} \frac{\varepsilon}{2} \csc \frac{\varepsilon}{2} \leqq \sec \frac{\varepsilon}{2}, \qquad \frac{d}{r(\eta, x)} \leqq \frac{|y_1|}{r(o, \eta)} \operatorname{tg} \frac{\varepsilon}{2} \csc \frac{\varepsilon}{2} \leqq \sec \frac{\varepsilon}{2};$$

accordingly, as a consequence of (1°),

$$(5.25) \qquad |\omega(y, \eta, x)|, \ |q(y, \eta, x)| \leqq \sec^2 \frac{\varepsilon}{2} + 2 \sec \frac{\varepsilon}{2} \leqq c^* \qquad \big(\text{in Case (ii)}\big) .$$

Case (iii). *x is exterior $N(o, \varepsilon)$, while $x_2 \leqq 0$; $y_2 \geqq 0$.* Let $\tau = (\tau_1, \tau_2, 0)$ represent either y or η; in either case $\tau_2 \geqq 0$. Designate by $C(x_1)$ the semicircle, containing x, consisting of points z for which

$$z_1 = x_1, \ z_2^2 + z_3^2 = x_2^2 + x_3^2, \ z_2 \leqq 0 ;$$

τ and x are not on the same side of the y_1, y_3-plane; hence the points of $C(x_1)$ nearest to τ are its end points; consequently

$$r(\tau, x) \geqq r(\tau, z^0) , \qquad z^0 = (x_1, 0, \sqrt{x_2^2 + x_3^2}) ;$$

z^0 is exterior $N(o, \varepsilon)$. Consider the semicircle C_0, containing τ and consisting of points u for which

$$u_1^2 + u_2^2 = \tau_1^2 + \tau_2^2 , \qquad u_2 \geqq 0, \ u_3 = 0 ;$$

the end points of C_0 are points

$$\tau^0 = [\pm r(o, \tau), \ 0, \ 0] ;$$

we have $r(\tau, z^0) \geqq r(\tau^0, z^0)$, where τ^0 is given by the above with the sign chosen so that τ^0, z^0 are not on the opposite sides of the y_3-axis. In the y_1, y_3-plane there is on hand the triangle o, τ^0, z^0. Since z^0 is exterior $N(o, \varepsilon)$ there is a generator g of the conical surface $N(o, \varepsilon)$, extending from o between τ^0, z^0. The angle between g and the side (o, τ^0) is ε; hence the angle β, at o, exceeds ε; thus, designating by z^* the foot of the perpendicular from τ^0 on the side (o, z^0) (this side possibly extended), we obtain

$$r(\tau^0, z^0) \geqq r(\tau^0, z^*) = r(o, \tau^0) \sin \beta > r(o, \tau^0) \sin \varepsilon .$$

Since $r(o, \tau^0) = r(o, \tau)$, it is inferred that

$$r(\tau, x) \geqq r(\tau, z^0) \geqq r(\tau^0, z^0) > r(o, \tau) \sin \varepsilon \qquad (\tau = y \text{ or } \eta) .$$

Recalling that $d \leqq |y_1| \operatorname{tg}\frac{\varepsilon}{2}$, it is observed that

$$\frac{d}{r(y, x)} \leqq \frac{|y_1|}{r(o, y)} \frac{\operatorname{tg}\frac{\varepsilon}{2}}{\sin \varepsilon} \leqq \tfrac{1}{2} \sec^2 \frac{\varepsilon}{2}; \quad \frac{d}{r(\eta, x)} \leqq \frac{|y_1|}{r(o, \eta)} \tfrac{1}{2} \sec^2 \frac{\varepsilon}{2} \leqq \tfrac{1}{2} \sec^2 \frac{\varepsilon}{2} .$$

Whence, by virtue of (1°)

(5.26) $$|\omega(y, \eta, x)|, |q(y, \eta, x)| \leqq \tfrac{1}{4} \sec^4 \frac{\varepsilon}{2} + \sec^2 \frac{\varepsilon}{2} \leqq c^* \qquad \big(\text{in Case (iii)}\big) .$$

Cases (i), (ii), (iii) cover the situation envisaged in the Lemma. Therefore from (1°), (5.24), (5.25), (5.26) it follows that, under the conditions of the Lemma,

$$\frac{r^2(\eta, x)}{r^2(y, x)}, \frac{r^2(y, x)}{r^2(\eta, x)} = 1 + O(\varepsilon^{-2}) = O(\varepsilon^{-2}) ;$$

this leads to the desired result (5.23 b).

We are now in position to study $\Psi_a^*(x)$ (5.2 a) in the case (5.22). In view of (5.16) one now has

(5.27) $$|\Psi_a^*(x)| < c^* \int_{S'(o,a)} l^{-\alpha}(y) r^{-2}(y, x)\, d\sigma(y)$$

$\big(d\sigma(y) =$ element of plane area, at y, in $S'(o, a)\big)$. The portion β' of the boundary of $S'(o, a)$ consists of points $\eta = (\eta_1, \eta_2, 0)$ such that $\big(\text{cf. } (5.4\text{b})\big)$

(5.27 a) $$\eta_2 = f(\eta_1) = O(\eta_1^2) \qquad (-a' \leqq \eta_1 \leqq a''; \ 0 < a', a'' \leqq a) ,$$

the end points being

$$A\big(-a', f(-a')\big), \ B\big(a'', f(a'')\big) ;$$

furthermore

(5.27 b) $$|\eta_2| = |f(\eta_1)| \leqq |\eta_1| \operatorname{tg} \frac{\varepsilon}{2} \qquad (\text{on } \beta') ,$$

if a (> 0) is suitably small; $f^{(1)}(\eta_1) = O(\eta_1)$ *is assumed continuous for* $-a' \leqq \eta_1 \leqq a''$. With y in $S'(o, a)$ not on β' and $l(y)$ denoting the distance from y to β', one has

(5.27 c) $$l^{-1}(y) \leqq c^* \delta^{-1}(y), \ \delta(y) = y_2 - f(y_1) > 0 \qquad (-a' \leqq y_1 \leqq a'') .$$

In view of (5.27)

$$|\Psi_a^*(x)| \leqq c^* \Gamma(x), \quad \Gamma(x) = \int_{S'(o,a)} \delta^{-\alpha}(y) r^{-2}(y, x) d\sigma(y) \, . \tag{5.28}$$

Introduce a point transformation in the y_1, y_2-plane

$$Y_1 = y_1, \quad Y_2 = y_2 - f(y_1) \, ; \tag{1°}$$

its inverse is

$$y_1 = Y_1, \quad y_2 = Y_2 + f(Y_1) \, . \tag{2°}$$

The Jacobian here equals unity and $d\sigma(Y) = d\sigma(y)$; we have $\delta(y) = Y_2$. The boundary β' is transformed into the rectilinear segment β^*

$$\beta^*\{Y_2 = 0; \ -a' \leqq Y_1 \leqq a''\} \, ; \tag{3°}$$

The region $S'(o, a)$ is transformed into S^*; in S^* one has $Y_2 \geqq 0$; S^* is bounded by β^* and by a curve σ^* (the transform of the circular portion σ' of the boundary of $S'(o, a)$); for Y on σ^* one has

$$Y_1^2 + Y_2^2 = a^2 + \nu(y), \quad \nu(y) = f^2(y_1) - 2y_2 f(y_1) = O(a^4) + O(a^3) = O(a^3) \, ;$$

thus along σ^*

$$a(1-\varepsilon') < R = [Y_1^2 + Y_2^2]^{\frac{1}{2}} < a(1+\varepsilon'') = a^* \, , \tag{4°}$$

where $0 < \varepsilon', \varepsilon'' < 1$ and $\varepsilon', \varepsilon''$ can be made arbitrarily small by taking a (> 0) suitably small.

Let y, Y, in the preceding, play the role of y, η of Lemma 5.23 (not necessarily in the stated order) and let d of the Lemma be equal $|f(y_1)|$. This can be done in view of (5.27b) and the location of the points y, Y, provided a in the Lemma is replaced by a^* (4°). Accordingly

$$\frac{r(Y, x)}{r(y, x)} < k_0(\varepsilon) = O\left(\frac{1}{\varepsilon}\right) \qquad (x \text{ exterior } N(o, \varepsilon) + W(o, \varepsilon))$$

for all Y in S^* and all y in $S'(o, a)$. *Thus* $\Gamma(x)$ (5.28) *satisfies*

$$\Gamma(x) = \int_{S^*} Y_2^{-\alpha} r^{-2}(y, x) d\sigma(Y) \leqq \int_{C^*(a)} Y_2^{-\alpha} r^{-2}(Y, x) \frac{r^2(Y, x)}{r^2(y, x)} d\sigma(Y) \tag{5.29}$$

$$< k_0^2(\varepsilon) \int_{C^*(a)} Y_2^{-\alpha} r^{-2}(Y, x) d\sigma(Y)$$

for x exterior $N(o, \varepsilon) + W(o, \varepsilon)$; here $C^*(a)$ is the semicircular region (containing S^*) consisting of points Y such that

$$Y_1^2 + Y_2^2 \leqq a^{*2} \ (\text{cf. (4°) for } a^*); \qquad Y_2 \geqq 0 \, . \tag{5.29a}$$

We thus reduced the case (5.22) of a plane surface, whose edge in the vicinity of o is curvilinear, to the case when the edge near o is rectilinear—that is, to the case (5.17), (5.17a). Apply the result (5.21) to the last member in (5.29) and take note of (5.28). It is inferred that *in the case* (5.22)

$$(5.30) \qquad |\Psi_a^*(x)| < c^* k_1(\varepsilon) r^{-\alpha}(o, x) \qquad (\text{for } 0 < \alpha < 1),$$

$$|\Psi_a^*(x)| < c^* k_1(\varepsilon) \log \frac{1}{r(o, x)} \qquad (\text{for } \alpha = 0)$$

for x *exterior* $N(o, \varepsilon) + W(o, \varepsilon)$; *here*

$$(5.30\text{a}) \qquad k_1(\varepsilon) = k_0^2(\varepsilon) k^2(\varepsilon) = O(\varepsilon^{-6}) \qquad (k(\varepsilon) \text{ from Lemma 5.6}).$$

Before treating the general case we shall prove the following.

Lemma 5.31. *Suppose* $x = (x_1, x_2, x_3)$ *is interior the conical domain* $K_{2\varepsilon_1}$ *(thus* (5.7a) *is satisfied),* $x_3 \geqq 0$, *and is exterior the conical regions* $N(o, 2\varepsilon_1)$, *when* $x_2 \leqq 0$. *Suppose* y *is in the region bounded above and below by the surfaces* $(K_{\varepsilon_1}^+)$, $(K_{\varepsilon_1}^-)$, *respectively, when* $y_2 \geqq 0$ (*cf.* (5.8)), *and is in* $N(o, \varepsilon_1)$, *when* $y_2 \leqq 0$. *On letting* $y' = (y_1, y_2, 0)$, *one then has*

$$(5.31\text{a}) \qquad \frac{r(y', x)}{r(y, x)} \leqq 1 + \sec \varepsilon_1 \leqq c^*.$$

It is observed that

$$\frac{r^2(y', x)}{r^2(y, x)} = 1 + 2 \frac{x_3 - y_3}{r(y, x)} \frac{y_3}{r(y, x)} + \frac{y_3^2}{r^2(y, x)},$$

so that

$$\frac{r(y', x)}{r(y, x)} \leqq 1 + \frac{|y_3|}{r(y, x)}.$$

Now

$$|y_3| \leqq \sqrt{y_1^2 + y_2^2} \operatorname{tg} \varepsilon_1 \ (\text{for } y_2 \geqq 0), \ (y_3 \leqq) \sqrt{y_2^2 + y_3^2} \leqq |y_1| \operatorname{tg} \varepsilon_1 \ (\text{for } y_2 \leqq 0);$$

thus, in either case $|y_3| \leqq r(0, y') \operatorname{tg} \varepsilon_1$; whence

$$(5.32) \qquad \frac{r(y', x)}{r(y, x)} \leqq 1 + \operatorname{tg} \varepsilon_1 \frac{r(o, y')}{r(y, x)} \leqq 1 + \operatorname{tg} \varepsilon_1 \frac{r(0, y)}{r(y, x)}.$$

Case (i). $x_2 \geqq 0$. Let $C(x_3)$ be the circle consisting of points $z = (z_1, z_2, z_3)$ such that

$$z_3 = x_3; \ z_1^2 + z_2^2 = x_1^2 + x_2^2;$$

$C(x_3)$ contains x. Of the points on $C(x_3)$ the point $x^0 = (x_1^0, x_2^0, x_3)$, lying in the half plane extending from the y_3-axis through y, is nearest to y, thus $r(y, x) \geqq r(y, x^0)$.

In this half plane we have the triangle o, y, x^0; let β be the angle at o; between the sides (o, η), (o, x^0) there extends a generator g_1 of $(K^+_{\varepsilon_1})$ (possibly coincident with (o, y) and a generator g_2 of $(K^+_{2\varepsilon_1})$; the angle between g_1 and g_2 is ε_1; hence $\beta > \varepsilon_1$; let x^* be the foot of the perpendicular from y on the side (o, x^0) (this side extended, if necessary); one has

$$r(y, x^0) \geqq r(y, x^*) = r(o, y) \sin \beta > r(o, y) \sin \varepsilon_1 ;$$

thus $r(y, x) > r(o, y) \sin \varepsilon_1$ and, by (5.32),

$$\frac{r(y', x)}{r(y, x)} \leqq 1+\operatorname{tg} \varepsilon_1 \csc \varepsilon_1 = 1+\sec \varepsilon_1 \ \big(\text{in Case (i)}\big) . \tag{5.33}$$

Case (ii). $y_2 < 0$. In this case we shall use the fact that y is in $N(o, \varepsilon_1)$ and that x is exterior $N(o, 2\varepsilon_1)$. Designate by $C(x_1)$ the circle, containing x,

$$z_1 = x_1 , \qquad z_2^2+z_3^2 = x_2^2+x_3^2 .$$

Let P be the half plane extending from the y_1-axis through y. The intersection $x^0 = (x_1, x_2^0, x_3^0)$ of $C(x_1)$ and P is the point of $C(x_1)$ nearest to y; thus $r(y, x) \geqq r(y, x^0)$. Consider the triangle o, y, x^0 (in P); the angle β at o exceeds ε_1, because from o and between y and x^0 there extends a generator g_1 of the conical surface $N(o, \varepsilon_1)$ and a generator g_2 of the conical surface $N(o, 2\varepsilon_1)$; the angle between g_1, g_2 is ε_1 $\big(g_1$ may coincide with the side $(o, y)\big)$. Denote by x^* the foot of the perpendicular from y on the side o, x^0 (extended if necessary). We have

$$r(y, x) \geqq r(y, x^0) \geqq r(y, x^*) = r(o, y) \sin \beta > r(o, y) \sin \varepsilon_1 .$$

Thus, by (5.32),

$$\frac{r(y', x)}{r(y, x)} < 1+\sec \varepsilon_1 \qquad \big(\text{in Case (ii)}\big) . \tag{5.34}$$

Case (iii). $x_2 < 0$; $y_2 \geqq 0$. We shall now use the fact that x is exterior $N(o, 2\varepsilon_1)$ and that y is in the region bounded above and below by $(K^+_{\varepsilon_1})$, $(K^-_{\varepsilon_1})$, respectively. For all z_2, satisfying $x_2 \leqq z_2 \leqq 0$, one has

$$(y_2-x_2)^2 \geqq (y_2-z_2)^2 ;$$

whence

$$r(y, x) \geqq r(y, z) \qquad [\text{for all } z = (x_1, z_2, x_3), \text{ with } x_2 \leqq z_2 \leqq 0] ;$$

the points z, referred to above, constitute a rectilinear segment L whose end points are x and $(x_1, 0, x_3)$. If $x_3 |x_1|^{-1} \geqq \operatorname{tg} 2\varepsilon_1$, that is if $(x_1, 0, x_3)$ is on or exterior the surface of $N(o, 2\varepsilon_1)$, we have

(1°) $$r(y, x) \geqq r(y, z^0) \qquad \big(z^0 = (x_1, 0, x_3)\big) .$$

When $x_3|x_1|^{-1} < \operatorname{tg} 2\varepsilon_1$, one has

(2°) $$r(y, x) \geqq r(y, z^0) \qquad [z^0 = (x_1, z_2^0, x_3);\ x_2 < z_2^0 < 0] ,$$

where z^0 is the intersection of L with the surface of $N(o, 2\varepsilon_1)$. Now if (1°) is on hand, the reasoning used in Case (i) (with $x_2 = 0$) applies; thus, corresponding to the inequalities preceding (5.33), we obtain

$$r(y, z^0) > r(o, y) \sin \varepsilon_1 \qquad \big(\text{Case } (1°)\big)$$

and $r(y, x) > r(o, y) \sin \varepsilon_1$, which $\big($by (5.32)$\big)$ yields

(5.35) $$\frac{r(y', x)}{r(y, x)} \leqq 1+\sec \varepsilon_1 \qquad \big(\text{in Case } (1°)\big) .$$

It remains to examine the case (2°). We designate by $C(y_3)$ the semicircle consisting of points z, such that $z_3 = y_3$, $z_1^2+z_2^2 = y_1^2+y_2^2$, $z_2 \geqq 0$; $C(y_3)$ contains y; its end points are $y^0 = \big(\pm\sqrt{y_1^2+y_2^2}, 0, y_3\big)$. When $x_1 > 0$, we use the plus sign; in the contrary case — the minus sign. With such a definition of y^0, it is observed that of all the points of $C(y_3)$ the point y^0 is nearest to $z^0 = (x_1, z_2^0, x_3)$. Thus, by (2°),

(3°) $$r(y, x) \geqq r(y, z^0) \geqq r(y^0, z^0) .$$

Suppose, for example, $x_1 > 0$ (we previously let $x_3 \geqq 0$). Then $y^0 = \big(\sqrt{y_1^2+y_2^2}, 0, y_3\big)$. Consider the semicircle $C(x_1)$ consisting of points $u = (u_1, u_2, u_3)$ such that

$$u_1 = x_1;\ u_2^2+u_3^2 = z_2^2+x_3^2;\ u_2 \leqq 0 ;$$

$C(x_1)$ contains z^0 $\big(C(x_1)$ lies in the surface of $N(o, 2\varepsilon_1)\big)$. The end points of $C(x_1)$ are $u^0 = (x_1, 0, \pm u_3^0)$, where

(4°) $$u_3^0 = \sqrt{z_2^2+x_3^2} = |x_1| \operatorname{tg} 2\varepsilon_1 ;$$

we note that the lines o, u^0 are generators of the conical surfaces $N(o, 2\varepsilon_1)$. Of all the points on $C(x_1)$ the end point u^0 is nearest to y^0, if we let:

$$u^0 = (x_1, 0, u_3^0) \qquad (\text{if } y_3 \geqq 0) ,$$

$$u^0 = (x_1, 0, -u_3^0) \qquad (\text{if } y_3 < 0) .$$

It will suffice to proceed under the first of the above alternatives. Accordingly, $r(y^0, z^0) \geqq r(y^0, u^0)$. Consider the triangle o, y^0, u^0 (in the y_1, y_3-plane); let the angle at o be β. The sides (o, u^0), (o, y^0) make angles $2\varepsilon_1$, $\operatorname{arc\,tg} y_3(y_1^2+y_2^2)^{-\frac{1}{2}}$ with the $+y_3$-axis, respectively; the latter angle is $\leqq \varepsilon_1$ $\big($because y^0 is on or under $(K_{\varepsilon_1}^+)\big)$. Whence $\beta \geqq \varepsilon_1$. On letting u^* denote the foot of the perpendicular from y^0 on the line (o, u^0), we obtain

$$r(y^0, z^0) \geqq r(y^0, u^0) \geqq r(y^0, u^*) = r(o, y^0) \sin\beta \geqq r(o, y^0) \sin\varepsilon_1 .$$

Now $r(o, y^0) = r(o, y)$; in view of (3°)

$$r(y, x) \geqq r(o, y) \sin\varepsilon_1 ;$$

accordingly by virtue of (5.32)

$$(5.36) \qquad \frac{r(y', x)}{r(y, x)} \leqq 1+\sec\varepsilon_1 \qquad \big(\text{in Case (iii)}\big) .$$

The truth of the Lemma follows by (5.33), (5.35), (5.36).

Now in the general case we have (cf. (5.16)), with suitable choice of coordinates,

$$(1_0) \qquad |\Psi_a^*(x)| < c^*\Lambda(x), \qquad \Lambda(x) = \int_{S(o,a)} l^{-\alpha}(y) r^{-2}(y, x) d\sigma(y) .$$

We keep x exterior $N(o, \varepsilon)+W(o, \varepsilon)$ and let a (> 0) be sufficiently small (to enable application of the various Lemmas). By Lemma 5.31 and since

$$d\sigma(y) < c^* d\sigma(y') \qquad [y' = (y_1, y_2, 0);\ d\sigma(y') = dy_1 dy_2] ,$$

it is inferred that

$$(2_0) \quad \Lambda(x) = \int_{S(o,a)} l^{-\alpha}(y) r^{-2}(y', x) \left[\frac{r(y', x)}{r(y, x)}\right]^2 d\sigma(y) < c^* \int_{S'(o,a)} l^{-\alpha}(y) r^{-2}(y', x) d\sigma(y') ;$$

here $S'(o, a)$ is a plane surface, being the orthogonal projection of $S(o, a)$ on the y_1, y_2-plane. In the above $l(y)$ is the distance from y (on $S(o, a)$) to the edges β; thus $l(y) = r(y, u(y))$, where

$$u(y) = \big(u_1(y), u_2(y), u_3(y)\big)$$

is a certain point on β; we observe that

$$u_3 = F(u_1, u_2) , \qquad u_2 = f(u_1) ,$$

where F, f are from the equations of the surface (near o) and β', respectively. It is to be recalled that the first order derivatives of F, f are continuous and

$$F(u_1, u_2) = O(\varrho^2),\ \frac{\partial F}{\partial u_i} = O(\varrho),\ f(u_1) = O(u_1^2) ,$$

$$f^{(1)}(u_1) = O(|u_1|) \qquad (\varrho = \sqrt{u_1^2+u_2^2}) .$$

It is observed that

$$\frac{r(y', u')}{l(y)} \leqq c^* \qquad [u' = u'(y) = \big(u_1(y), u_2(y), 0\big)]$$

and

$$\frac{l'(y')}{r(y', u')} \leqq c^* \qquad [l'(y') = \text{distance from } y' \text{ to } \beta'] ,$$

provided a is suitably small. To establish this we use essentially the fact that $S(o, a)$, β approximate (near o) $S'(o, a)$, β', respectively, while β' approximates a rectilinear segment. Thus $l'(y')l^{-1}(y) < c^*$ and

$$\int_{S'(o,a)} l^{-\alpha}(y)r^{-2}(y', x)d\sigma(y') = \int_{S'(o,a)} l'(y')^{-\alpha}\left[\frac{l'(y')}{l(y)}\right]^{\alpha} r^{-2}(y', x)d\sigma(y')$$
$$< c^* \int_{S'(o,a)} l'(y')^{-\alpha}r^{-2}(y', x)d\sigma(y') .$$

To the integral in the last member the result (5.30) (the case of surface planar near o) applies; whence, in view of (1_0), (2_0), *in the general case we have*

$$(5.37)\quad \begin{aligned} |\Psi_a^*(x)| &< c^*k_1(\varepsilon)r^{-\alpha}(o, x) \qquad (\text{if } 0 < \alpha < 1) , \\ |\Psi_a^*(x)| &< c^*k_1(\varepsilon) \log \frac{c^*}{r(o, x)} \qquad (\text{if } \alpha = 0) \end{aligned}$$

$\big(k_1(\varepsilon)$ from (5.30a)$\big)$ *for all x exterior $N(o, \varepsilon)+W(o, \varepsilon)$.* By virtue of (5.2), (5.3) the function $\Psi(x)$ (1.3a) will satisfy inequalities of form (5.37). We accordingly state (with the assumption after (5.3) easily deleted) the following.

Theorem 5.38. *Suppose that $q(y) \subset [\alpha|S]$ (Definition 3.19) and that Hypothesis 3.20 holds. Assume that $0 \leqq \alpha < 1$. The function $\Psi(x)$ (1.3a) will then satisfy*

$$(5.38)\quad |\Psi(x)| < c^*k_1(\varepsilon)r^{-\alpha}(c, x) \qquad (\textit{if } \alpha > 0) ,$$

$$|\Psi(x)| < c^*k_1(\varepsilon) \log \frac{c^*}{r(c, x)} \qquad (\textit{if } \alpha = 0) , \qquad k_1(\varepsilon) = O(\varepsilon^{-6}) ,$$

for x exterior $N(c, \varepsilon)+W(c, \varepsilon)$ (Definition 5.1) near any point c on the 'edges' β of S. In view of Definition 3.19 it can be also stated that

$$(5.38\text{a})\quad \Psi(x) \subset [\alpha|C(S)] \quad (\textit{if } \alpha > 0), \qquad \Psi(x) \subset [0, \log|C(S)] \quad (\textit{if } \alpha = 0) .$$

6. Order of infinity of principal integrals near β. *We now proceed under the conditions of Theorem 3.25, with $\gamma(y, t)$ satisfying (3.27). Consider the principal integral (3.25b),*

$$(6.1)\quad \Psi(t) = \int_S k(y, t)r^{-2}(y, t)q(y)d\sigma(y)$$

$\big(q(y) \subset [\alpha|S]$; $0 \leqq \alpha < 1$; $\alpha+\beta < 1$; $0 \leqq \beta\big)$. As follows from the works of G. Giraud and Michlin (cf. references to Giraud in [M]) the principal integral $\Psi(t)$ is certainly of a Hölder class for t on the surface S, at positive distance from the edge (that is, for $l(t) > 0$), provided $q(y)$ is of a Hölder class $\big($for $l(y) > 0\big)$ on S. We shall not examine any closer these aspects of $\Psi(t)$. Let c be a point on the 'edges' β. The problem now is to study the order of infinity of $\Psi(t)$ for t (on S) near c, avoiding approaches tangential to β (near c).

Proceeding with the notation of the beginning of section 5, *we let t be in the neighborhood of c, defined by the conditions*

$$(6.2) \qquad t \text{ in } S\left(c, \frac{a}{2}\right), \qquad t \text{ is exterior cones } N(c, \varepsilon)$$

(Definition 5.1). *We take a (> 0) so small that the portions of the curves β, β' bounding $S(c, a)$, $S(c', a)$, respectively, are in $N\left(c, \frac{\varepsilon}{2}\right)$.*

Now $r(y, t) \geqq \frac{a}{2}$ $\left(\text{for } t \text{ in } S\left(c, \frac{a}{2}\right) \text{ and } y \text{ in } S-S(c, a)\right)$; in view of (3.21) and since $q(y) \subset [\alpha|S]$ one has

$$(6.3) \qquad \left|\int k(y, t)r^{-2}(y, t)q(y)d\sigma(y)\right| \leqq c^* \int l^{-\alpha}(y)d\sigma(y) \leqq c^*$$

(integration over $S-S(c, a)$). Hence it will suffice to study the component of the integral $\Psi(t)$ (6.1) extended over $S(c, a)$,

$$(6.4) \qquad \Phi(t) = \int_{S(c, a)} k(y, t)r^{-2}(y, t)q(y)d\sigma(y) .$$

We recall the definitions of $k'(t|y, t)$, $k''(t|y, t)$ in (3.2a), (3.2) and write

$$(6.4\text{a}) \qquad \Phi(t) = \Phi'(t)+\Phi''(t) ,$$

where

$$(6.4\text{b}) \qquad \Phi'(t) = \int_{S(c, a)} k'(t|y, t)r^{-2}(y, t)q(y)d\sigma(y) ,$$

$$\Phi''(t) = \int_{S(c, a)} k''(t|y, t)r^{-2}(y, t)q(y)d\sigma(y) .$$

Here $\Phi''(t)$ is $\Psi''(t)$ in (3.3), with S replaced by $S(c, a)$; thus by (3.22b)

$$(6.4\text{c}) \qquad |\Phi''(t)| < c^*\Gamma''(t) , \qquad \Gamma''(t) = \int_{S(c, a)} \gamma(y, t)l^{-\alpha}(y)r^{h-2}(y, t)d\sigma(y)$$

$\left(h, \gamma(y, t) \text{ from } (3.20\text{c})\right)$, where the integral exists by Note II (end of section 3).

As a consequence of (3.27) for the integral $\Gamma''(t)$, above, we have

$$(6.5) \qquad \Gamma''(t) < c^*\left(\Gamma_1''(t)+\Gamma_2''(t)\right) ,$$

where

$$\Gamma_1''(t) = \int_{\omega_1} l^{-\alpha-\beta}(y)r^{h-2}(y, t)d\sigma(y) , \qquad \Gamma_2''(t) = l^{-\beta}(t)\int_{\omega_2} l^{-\alpha}(y)r^{h-2}(y, t)d\sigma(y) ,$$

with ω_1 and ω_2 denoting regions

$$\omega_1\{y \text{ in } S(c, a);\ l(y) \leqq l(t)\}\ , \qquad \omega_2\{y \text{ in } S(c, a);\ l(y) > l(t)\}\ .$$

Choose the (y_1, y_2, y_3) *coordinates with the origin o at c, as described subsequent* (5.3). Consider first the planar case when $S(o, a)$ is a semicircle (in the y_1, y_2-plane)

$$S(o, a) = S'(o, a) = \{0 \leqq \sqrt{y_1^2+y_2^2} \leqq a; \qquad y_3 = 0;\ y_2 \geqq 0\}\ , \tag{6.6}$$

β (near o) is the rectilinear boundary of $S(o, a)$,

$$\beta = \beta' = \{-a \leqq y_1 \leqq a; \qquad y_2 = y_3 = 0\}\ ; \tag{6.6a}$$

suppose for the present that $t = (t_1, t_2, 0)$ is on the normal to β at o,

$$t = (0, t_2, 0)\ ; \qquad 0 < t_2 \leqq \frac{a}{2}\ . \tag{6.6b}$$

We then have $l(y) = y_2$, $d\sigma(y) = dy_1 dy_2$ and

$$\omega_1 = \text{part of } S(o, a) \text{ with } y_2 \leqq t_2; \quad \omega_2 = \text{part of } S(o, a) \text{ with } y_2 > t_2\ . \tag{6.6b$'$}$$

Introduce the transformation, between sets of variables (y_1, y_2) and (l, r),

$$l = y_2, \qquad r = [y_1^2+(y_2-t_2)^2]^{\frac{1}{2}}\ ; \tag{6.7}$$

we have

$$\left| J\left(\frac{y_1, y_2}{l, r}\right) \right| = r[r^2-(l-t_2)^2]^{-\frac{1}{2}}\ , \qquad d\sigma(y) = \frac{r}{\sqrt{r^2-(l-t_2)^2}}\,|dldr|\ .$$

On taking account of the symmetry of $S(o, a)$ and of the integrands involved with respect to the y_2-axis, it is inferred that

$$\Gamma_1''(t) = 2\int_0^{t_2} l^{-\alpha-\beta}dl \int_{t_2-l}^{r'} r^{h-1}[r^2-(t_2-l)^2]^{-\frac{1}{2}}dr\ ,$$

where

$$r' = \sqrt{(t_2-l)^2+(a^2-l^2)} \leqq \sqrt{t_2^2+a^2} \leqq a' = a\frac{\sqrt{5}}{2}\ . \tag{1°}$$

Let $r = (t_2-l)\sec\theta$; one has

$$\Gamma_1''(t) \leqq 2\int_0^{t_2} l^{-\alpha-\beta}dl \int_0^{\theta'} (t_2-l)^{h-1}\sec^h\theta d\theta\ , \qquad \theta' = \operatorname{arc\,cos}\frac{t_2-l}{a'}$$

$\left(0 \leqq \theta' \leqq \frac{\pi}{2}\right)$; further, with $l = \tau t_2$,

$$\Gamma_1''(t) \leqq 2t_2^{h-\alpha-\beta}\int_0^1 \tau^{-\alpha-\beta}(1-\tau)^{h-1}d\tau \int_0^{\theta'(\tau)} \sec^h\theta d\theta;\ \cos\theta'(\tau) = \frac{t_2}{a'}(1-\tau)\ . \tag{2°}$$

It is noted that

$$\nu = \int_0^{\theta'} \sec^h \theta d\theta = \int_{\theta_0}^1 u^{-h}(1-u^2)^{-\frac{1}{2}} du \qquad \left[\theta_0 = \frac{t_2}{a'}(1-\tau);\ 0 \leqq \theta_0 < \tfrac{1}{2}\right];$$

when $h < 1$

$$\nu \leqq \int_0^1 u^{-h}(1-u^2)^{-\frac{1}{2}} du = c^*:$$

for $h = 1$ we have

$$\nu = \int_{\frac{1}{2}}^1 \cdots + \int_{\theta_0}^{\frac{1}{2}} \cdots = c^* + \int_{\theta_0}^{\frac{1}{2}} (1-u^2)^{-\frac{1}{2}} \frac{du}{u} \leqq c^* + \frac{2}{\sqrt{3}} \int_{\theta_0}^{\frac{1}{2}} \frac{du}{u} \leqq c^* \log \frac{a'}{t_2} + c^* \log \frac{1}{1-\tau}.$$

Whence, by (2°), the following is inferred. For $h < 1$

$$\Gamma_1''(t) \leqq c^* t_2^{h-\alpha-\beta} \int_0^1 \tau^{-\alpha-\beta}(1-\tau)^{h-1} d\tau \leqq c^* t_2^{h-\alpha-\beta}$$

(since $h-1 > -1$, $-\alpha-\beta > -1$); for $h = 1$

$$\Gamma_1''(t) \leqq c^* t_2^{1-\alpha-\beta} \log \frac{a'}{t_2} \int_0^1 \tau^{-\alpha-\beta} d\tau + c^* t_2^{1-\alpha-\beta} \int_0^1 \tau^{-\alpha-\beta} \log \frac{1}{1-\tau} d\tau$$

$$\leqq c^* t_2^{1-\alpha-\beta} \log \frac{a'}{t_2} + c^* t_2^{1-\alpha-\beta}.$$

Accordingly (in case (6.6)—(6.6b))

(3°) $\qquad \Gamma_1''(t) \leqq c^* t_2^{h-\alpha-\beta}$ (for $h < 1$), $\Gamma_1''(t) \leqq c^* t_2^{1-\alpha-\beta} \log \dfrac{a'}{t_2}$ $\qquad$ (for $h = 1$).

Continuing in the case (6.6)—(6.6b), we turn to $\Gamma_2''(t)$ of (6.5) and note that

$$\Gamma_2''(t) = 2t_2^{-\beta} \int_{t_2}^a l^{-\alpha} dl \int_{l-t_0}^{r'} r^{h-1}[r^2-(l-t_2)^2]^{-\frac{1}{2}} dr$$

(r' from (1°)). The substitution $r = (l-t_2)\sec\theta$ yields

$$\Gamma_2''(t) < 2t_2^{-\beta} \int_{t_2}^a l^{-\alpha} dl \int_0^{\theta'} (l-t_2)^{h-1} \sec^h \theta d\theta, \qquad \theta' = \operatorname{arc}\cos \frac{l-t_2}{a'},$$

where $0 \leqq \theta' \leqq 2^{-1}\pi$; we again let $l = t_2\tau$, obtaining

(4°) $$\Gamma_2''(t) < 2t_2^{h-\alpha-\beta} \int_1^{at_2^{-1}} \tau^{-\alpha}(\tau-1)^{h-1} \nu(\tau) d\tau,$$

with

$$\cos\theta' = \frac{t_2}{a'}(\tau-1), \qquad \nu(\tau) = \int_0^{\theta'} \sec^h \theta d\theta.$$

Since $a' = 2^{-1}\sqrt{5}a$ and $1 \leqq \tau \leqq at_2^{-1}$, we have

$$\nu(\tau) = \int_{\theta_0}^{1} u^{-h}(1-u^2)^{-\frac{1}{2}}du \qquad \left[\theta_0 = \frac{t_2}{a'}(\tau-1);\ 0 \leqq \theta_0 < \frac{2}{\sqrt{5}}\right];$$

thus

$$\nu(\tau) \leqq \int_0^1 \ldots < c^* \qquad (\text{when } h < 1)\,.$$

Whence by (4°) (for $h < 1$)

$$\Gamma_2''(t) < c^* t_2^{h-\alpha-\beta} I(t)\,, \qquad I(t) = \int_1^{at_2^{-1}} \tau^{-\alpha}(\tau-1)^{h-1}d\tau$$

($at_2^{-1} \geqq 2$), where

$$I(t) = \int_1^2 \cdots + \int_2^{at_2^{-1}} \cdots \leqq c^* + \int_2^{at_2^{-1}} \tau^{-\alpha+h-1}\left(1-\frac{1}{\tau}\right)^{h-1} d\tau\,;$$

thus

$$I(t) < c^* + c^* \int_2^{at_2^{-1}} \tau^{-\alpha+h-1} d\tau\,.$$

Accordingly

(5°) $$\Gamma_2''(t) < c^* t_2^{h-\alpha-\beta} \qquad (h < \alpha)\,,$$

$$\Gamma_2''(t) < c^* t_2^{-\beta} \log\frac{a}{t_2} \qquad (h = \alpha)\,, \qquad \Gamma_2''(t) < c^* t_2^{-\beta} \qquad (\alpha < h < 1)\,.$$

By virtue of (6.5), (3°), (5°), the following holds. *In the case* (6.6)—(6.6b) *one has*

(6.8) $$\Gamma''(t) < c^* t_2^{h-\alpha-\beta} \quad (\textit{for } h < \alpha)\,; \qquad \Gamma''(t) < c^* t_2^{-\beta} \quad (\textit{for } \alpha < h < 1)\,,$$

$$\Gamma''(t) < c^* t_2^{-\beta} \log\left(\frac{a}{t_2}\right) \qquad (\textit{for } h = \alpha)\,.$$

Consider now the case when $h = 1$. The inequalities (3.20 c) (Hypothesis 3.20) now yield (with $\gamma(y, t)$ subject to (3.27) and with y, t on the surface)

$$|\gamma_{i_1\ldots i_m}(y) - \gamma_{i_1\ldots i_m}(t)| \leqq \lambda_m \gamma(y, t) r(y, t) = \lambda_m \gamma(y, t) r^{\nu}(y, t) r^{1-\nu}(y, t)$$
$$< \lambda'' \lambda_m \gamma(y, t) r^{\nu}(y, t) \qquad (\lambda'', > 0, \text{ independent of } m)\,,$$

where we let ν be a fixed number such that $\alpha < \nu < 1$. Applying the second inequality (6.8), with $h = \nu$, *it is inferred that*

(6.8a) $$\Gamma''(t) < c^* t_2^{-\beta} \qquad \big(\textit{for } h = 1;\ \textit{in the case } (6.6)\text{—}(6.6\text{b})\big)\,.$$

It does not appear possible to get a sharper result for $h = 1$, using the integral for $\Gamma_2''(t)$ (6.5).

Generalize now the case (6.6)—(6.6b), *replacing* (6.6b) *by the requirement that*

(6.9) $\quad t = (t_1, t_2, 0)$ is on $S(o, a)$ (6.6), exterior $N(o, \varepsilon)$; $r(o, t) \leqq \frac{a}{2}$.

We have

$$|t_2| > |t_1| \operatorname{tg} \varepsilon , \qquad |t_1| < \frac{a}{2} \cos \varepsilon .$$

Introduce coordinates $y'_1 = y_1 - t_1$, $y'_2 = y_2$; in these, $S(o, a)$ (6.6) is a certain semicircular region (with center at $(-t_1, 0)$) S'; S' is contained in another semicircular region S^*, consisting of points (y'_1, y'_2) such that

$$S^*\{\sqrt{y_1'^2 + y_2'^2} \leqq \tfrac{3}{2} a ; \qquad y'_2 \geqq 0\} ;$$

furthermore, the point t' is on the $+y'_2$-axis, $t' = (0, t_2)$;

$$l(y) = l(y') = y_2 = y'_2,\ l(t) = l(t') = t_2,\ d\sigma(y) = d\sigma(y'),\ r(y, t) = r(y', t') .$$

For $\Gamma_1''(t)$, $\Gamma_2''(t)$ of (6.5) one accordingly has

$$(1_0) \qquad \Gamma_j''(t) < \Gamma_j^*(t) \ (j = 1{,}2), \quad \Gamma_1^*(t) = \int_{\omega'_1} l^{-\alpha-\beta}(y) r^{h-2}(y', t') d\sigma(y') ,$$

$$\Gamma_2^*(t) = l^{-\beta}(t') \int_{\omega'_2} l^{-\alpha}(y') r^{h-2}(y', t') d\sigma(y') ;$$

where

$$\omega'_1 = \text{part of } S^* \text{ with } y'_2 \leqq t'_2; \quad \omega'_2 = \text{part of } S^* \text{ with } y'_2 > t_2 .$$

The regions ω'_1, ω'_2 are formed precisely as ω_1, ω_2 were in (6.6b′), except that a is replaced by $\frac{3}{2} a$; moreover, $t_2 \leqq \frac{a}{2}$. The integrals for $\Gamma_j^*(t)$ $(j = 1{,}2)$ are identical in form with those for $\Gamma_j''(t)$ (6.5), considered in the case (6.6), (6.6a), (6.6b), with a replaced by $\frac{3}{2}a$. *Whence the results* (6.8), (6.8a) *continue to hold in the case* (6.6), (6.6a), (6.9).

We observe that, under (6.6), (6.6a), (6.9), $t_2 > r(o, t) \sin \varepsilon$ and there exists a positive constant b^0 so that

$$(6.9\text{a}) \qquad t_2^{-1} < b^0 \varepsilon^{-1} r^{-1}(o, t), \qquad \log \left(\frac{a}{t_2}\right) < \log \frac{b^0}{\varepsilon r(o, t)} .$$

By virtue of (6.4c), (6.8), (6.8a) and of the preceding italics *it follows that*

$$(6.10) \quad |\Phi''(t)| < c^*[\varepsilon r(o, t)]^{h-\alpha-\beta} (h < \alpha) ; \ |\Phi''(t)| < c^*[\varepsilon r(o, t)]^{-\beta} \log \left[\frac{b^0}{\varepsilon r(o, t)}\right]$$
$$(h \doteq \alpha) ; \qquad |\Phi''(t)| < c^*[\varepsilon r(o, t)]^{-\beta} \qquad (\alpha < h \leqq 1)$$

in the case (6.6), (6.6a), (6.9) (that is, when the surface is planar and the 'edge' is rectilinear near o). *Note that* by (6.5) $|\Phi''(t)| < c^*\Gamma''(t) < c^*(\Gamma_1''(t)+\Gamma_2''(t))$ *and that* Γ'', $\Gamma_1''+\Gamma_2''$ *satisfy* (6.10).

Continuing the study of $\Gamma''(t)$ (6.5) we go from the case (6.6), (6.6a), already studied, to *the case when* $S(o, a) = S'(o, a)$ *is still a plane surface, but the boundary* $\beta = \beta'$ *of* $S(o, a)$ *near* o *may be curvilinear*; $t = (t_1, t_2, 0)$ *will be kept in* $S'\left(o, \frac{a}{2}\right)$ *exterior* $N(o, \varepsilon)$. With (5.4b) in view, one may say that the part of β' involved in the boundary of $S'(o, a)$ consists of points $\eta = (\eta_1, \eta_2, 0)$ such that

$$(6.11) \qquad \eta_2 = f(\eta_1) = O(\eta_1^2) \qquad (-a' \leqq \eta_1 \leqq a''; \ 0 < a', a'' \leqq a);$$

the end points are $A(-a', f(-a'))$, $B(a'', f(a''))$. As remarked subsequent (6.2), the arc (6.11) is in $N\left(o, \frac{\varepsilon}{2}\right)$; thus on this arc

$$(6.11\,\text{a}) \qquad |\eta_0| = |f(\eta_1)| \leqq |\eta_1| \operatorname{tg}\frac{\varepsilon}{2}.$$

On letting $\delta(y) = y_2 - f(y_1)$, we have

$$(6.11\,\text{b}) \quad l^{-1}(y) \leqq c^*\delta^{-1}(y) \qquad [y \text{ on } S'(o, a), \text{ not on } \beta'; \ -a' \leqq y_1 \leqq a''].$$

Apply the transformation (1°) (given subsequent (5.28)) between sets of variables (y_1, y_2), (Y_1, Y_2); $S'(o, a)$ goes into a region S^*, in which $Y_2 \geqq 0$; the arc (6.11) is transformed into the rectilinear segment

$$(1°) \qquad \beta^*\{Y_2 = 0, \ -a' \leqq Y_1 \leqq a''\};$$

to the circular part σ' of the boundary of $S'(o, a)$ corresponds an arc σ^*, joining the points $(Y_1 = -a', Y_2 = 0)$, $(Y_1 = a'', Y_2 = 0)$. As subsequent (5.28), one now has

$$(2°) \qquad a(1-\varepsilon') < R = [Y_1^2 + Y_2^2]^{\frac{1}{2}} < a(1+\varepsilon'') = a^*$$

$(0 < \varepsilon', \varepsilon'' < 1; \ \varepsilon', \varepsilon'' \to 0$ with $a)$ for (Y_1, Y_2) on σ^*. Since $\delta(y) = Y_2$,

$$(3°) \qquad l^{-1}(y) < c^* Y_2^{-1} \qquad (\text{in } S^*).$$

S^* lies in *the semicircle* $C^*\{Y_1^2 + Y_2^2 \leqq a^{*2}; \ Y_2 \geqq 0\}$. Apply the same transformation to t as was applied to y; t then goes into T, where

$$(4°) \quad T = (T_1, T_2, 0); \qquad T_1 = t_1, \ T_2 = t_2 - f(t_1); \qquad |f(t_1)| \leqq |t_1| \operatorname{tg}\frac{\varepsilon}{2};$$

clearly $r(y, t) = r(Y, T)$. Since $|t_2 t_1^{-1}| > \operatorname{tg}\varepsilon$,

$$(5°) \qquad \frac{T_2}{|T_1|} \geqq \frac{t_2}{|t_1|} - \frac{|f(t_1)|}{|t_1|} > \operatorname{tg}\varepsilon - \operatorname{tg}\frac{\varepsilon}{2} > \operatorname{tg}\frac{\varepsilon}{2}.$$

Now

$$r(o, T) \leqq r(o, t)(1+\varepsilon^0), \qquad \varepsilon^0 = O(\varepsilon)\,;$$

hence

(6°) $$r(o, T) \leqq \tfrac{1}{2}a^0 \qquad [a^0 = (1+\varepsilon^0)a = a+O(\varepsilon)]\,,$$

when t is in $S'\left(o, \dfrac{a}{2}\right)$ exterior $N(o, \varepsilon)$. We shall need an inequality for $r(Y, T)$, $r(y, t)$. We note that

$$|f(y_1)-f(t_1)| \leqq c^*|y_1-t_1|\,;$$

moreover,

$$\frac{r^2(Y, T)}{r^2(y, t)} = 1+g\,, \qquad g = r^{-2}(y, t)\,[f(y_1)-f(t_1)]^2-2\frac{y_2-t_2}{r(y, t)}\frac{f(y_1)-f(t_1)}{r(y, t)}\,;$$

here

$$|g| \leqq \left[\frac{c^*|y_1-t_1|}{r(y, t)}\right]^2+2\left[\frac{c^*|y_1-t_1|}{r(y, t)}\right]\frac{|y_2-t_2|}{r(y, t)} \leqq c^*\,;$$

hence

(6.12) $$r^{-1}(y, t) < c^* r^{-1}(Y, T)\,.$$

Turning to $\Gamma_j''(t)$ (6.5) (with the y coordinates chosen as stated subsequent (6.5)), by (3°) and (6.12) we obtain

(6.13) $$\Gamma_1''(t) < c^*\int_{\omega_1^*} Y_2^{-\alpha-\beta} r^{h-2}(Y, T)\,d\sigma(Y)\,,$$

$$\Gamma_2''(t) < c^* T_2^{-\beta}\int_{\omega_2^*} Y_2^{-\alpha} r^{h-2}(Y, T)\,d\sigma(Y)\,,$$

where $d\sigma(Y)$ is element of area at Y and ω_1^*, ω_2^* are transforms of ω_1, ω_2 (in (6.5)), respectively. To describe ω_1, ω_2 consider the curve (α) (in the y_1, y_2-plane), within $S'(o, a)$ and consisting of points y such that $l(y) = l(t)$; let (α^*) be the transform of (α); (α^*) goes through T and joins two points on the part σ^* of the boundary of S^*; ω_1^* is the part of S^* bounded by the rectilinear segment β^*, by (α^*) and by two portions of σ^*; ω_2^* is the rest of S^*. The arc (α^*) is 'nearly' a rectilinear segment; it is 'nearly' parallel to the arc β' (of which β^* is the transform); in particular, on letting (t^*) denote the part within S^* of the parallel (in the Y_1, Y_2-plane) to β^* through T, we observe that the arc (α^*) is tangent to (t^*) at T; (α^*) lies in a region $R(\varepsilon')$, consisting of points Y such that

(7°) $$R(\varepsilon') = \{Y \text{ in } S^*;\qquad |Y_2-T| \leqq |Y_1-T_1|\varepsilon'\}\,,$$

where ε' (> 0) is small. By adding or subtracting from the integrals in (6.13) integrals (with integrands as in (6.13)) over suitable subregions of $R(\varepsilon')$, the integrals in (6.13) can be modified so that ω_1^* is replaced by the part of S^* for which $0 \leqq Y_2 \leqq T_2$, while ω_1^* is replaced by the portion of S^* for which $Y_2 \geqq T_2$. Let e_1, e_2 be any (measurable) subregions of $\omega_1^* R(\varepsilon')$, $\omega_2^* R(\varepsilon')$, respectively; it is observed that the functions

$$(8^\circ) \quad I_1(e_1; T) = \int_{e_1} Y_2^{-\alpha-\beta} r^{h-2}(Y, T) d\sigma(Y), \qquad I_2(e_2; T) = T_2^{-\beta} \int_{e_2} Y_2^{-\alpha} r^{h-2}(Y, T) d\sigma(Y)$$

are not of greater order of infinitude (in T, for T near o) than the corresponding functions in the second members of inequalities (6.10). The integrals in (6.13) are expressible in the form

$$(6.14) \qquad \Gamma_1^*(T) + I_1(e_1; T), \qquad \Gamma_2^*(T) + I_2(e_2; T)$$

(suitable sets e_1, e_2 as in (8°)), respectively, with

$$\Gamma_1^*(T) = \int Y_2^{-\alpha-\beta} r^{h-2}(Y, T) d\sigma(Y) \qquad (\text{over } S^*, \text{ with } Y_2 \leqq T_2),$$

$$\Gamma_2^*(T) = T_2^{-\beta} \int Y_2^{-\alpha} r^{h-2}(Y, T) d\sigma(T) \qquad (\text{over } S^*, \text{ with } Y_2 > T_2).$$

Recall the statement, subsequent (3°), with reference to S^* and the semicircle C^*; one has

$$(6.14\text{a}) \qquad \Gamma_1^*(T) \leqq \int_{\omega'} Y_2^{-\alpha-\beta} r^{h-2}(Y, T) d\sigma(Y),$$

$$\Gamma_2^*(T) \leqq T_2^{-\beta} \int_{\omega''} Y_2^{-\alpha} r^{h-2}(Y, T) d\sigma(Y),$$

where ω', ω'' are the parts of C^* for which $Y_2 \leqq T_2$, $Y_2 > T_2$, respectively. The integrals in (6.14a) are precisely of the form of the integrals for $\Gamma_1''(t)$, $\Gamma_2''(t)$ (6.5), respectively, in the case (6.6), (6.6a); now, however, they are modified in accord with (2°) (note a^*) and (6°); moreover, in view of (5°), ε is replaced by $\frac{\varepsilon}{2}$. At any rate, the results (6.10) apply to the sum of the second members in (6.14a). Thus, for T satisfying (5°), (6°), one has

$$(6.15) \qquad \Gamma_1^*(T) + \Gamma_2^*(T) < c^*[\varepsilon r(o, T)]^{h-\alpha-\beta} \qquad (\text{if } h < \alpha),$$

$$< c^*[\varepsilon r(o, T)]^{-\beta} \log\left[\frac{b^0}{\varepsilon r(o, T)}\right] \qquad (\text{if } h = \alpha), \quad < c^*[\varepsilon r(o, T)]^{-\beta} \qquad (\text{if } \alpha < h \leqq 1).$$

By virtue of the statement with respect to (8°), the sum of the functions (6.14), that is the sum of the integrals in (6.13), satisfies inequalities of the above form; *the same is true for* $\Gamma_1''(t) + \Gamma_2''(t)$ *and, accordingly, for* $\Gamma''(t)$ *of* (6.5). Since

$$|f(T_1)| \leqq |T_1| \operatorname{tg} \frac{\varepsilon}{2} \quad (4^\circ),$$

we have

$$\frac{r^2(o, t)}{r^2(o, T)} = 1 + 2 \frac{T_2}{r(o, T)} \left[\frac{f(T_1)}{r(o, T)}\right] + \left[\frac{f(T_1)}{r(o, T)}\right]^2 \leqq \left(1 + \operatorname{tg} \frac{\varepsilon}{2}\right)^2 < c^*;$$

thus

$$r^{-1}(o, T) < c^* r^{-1}(o, t) .$$

Whence, T *in the second members of* (6.15) *can be replaced by* t; t, as assumed preceding (6.11), is on $S\left(o, \frac{a}{2}\right) = S'\left(o, \frac{a}{2}\right)$, exterior $N(o, \varepsilon)$. In view of the above and of (6.4c) the following holds. *In the case described preceding* (6.11)

$$|\Phi''(t)| < c^*[\varepsilon r(o, t)]^{h-\alpha-\beta} (h < \alpha) ; \tag{6.16}$$

$$|\Phi''(t)| < c^*[\varepsilon r(o, t)]^{-\beta} \log \left[\frac{b^0}{\varepsilon r(o, t)}\right] \ (h = \alpha); \quad |\Phi''(t)| < c^*[\varepsilon r(o, t)]^{-\beta} \ (\alpha < h \leqq 1)$$

for t *exterior* $N(o, \varepsilon)$. *These inequalities are also satisfied by* $\Gamma_1''(t) + \Gamma_2''(t)$ (6.5).

When $S(o, a) = S'(o, a)$ is a plane surface, one may utilize the fact that $\delta(y) = y_2 - f(y_1)$ satisfies inequalities

$$b'l(y) > \delta(y) > b''l(y) \qquad (b' = c^*, \ b'' = c^*) , \tag{6.17}$$

of which (6.11b) is a part; (6.17) implies that $\delta(y)$ can be made to play the role of $l(y)$ in local considerations (near the point o, under consideration). Suppose in the developments, leading to (6.13), we replace $l(y)$, $l(t)$ by $\delta(y)$, $\delta(t)$ (absorbing b' and b'' in the generic designation c^* of positive constants). Then the curvilinear arc α^*, separating ω_1^*, ω_2^*, becomes a rectilinear segment t^*, through T and parallel to the rectilinear boundary β^* of S^*; consideration of integrals (8°) will be unnecessary; $I_1(e_1; T)$, $I_2(e_2; T)$ in (6.14) will be zero.

We now consider the general case (*with origin of* (y) *at* c) *when* $S(o, a)$ *is not necessarily a plane surface near* o. Let

$$y' = (y_1, y_2, 0), \ t' = (t_1, t_2, 0) , \qquad l'(y') = \text{distance from } y' \text{ to } \beta' .$$

On taking note of the developments preceding (5.37), we have $l^{-1}(y) < c^* l'^{-1}(y)$; moreover,

$$d\sigma(y) < c^* d\sigma(y') , \qquad r^{-1}(y, t) \leqq r^{-1}(y', t') ,$$

where $d\sigma(y)$ is element of area, at y', in $S'(o, a)$. Accordingly, for $\Gamma_1''(t)$, $\Gamma_2''(t)$ in (6.5) one has

(6.18)
$$\Gamma_1''(t) < c^* \int_{d_1} l'(y')^{-\alpha-\beta} r^{h-2}(y', t') d\sigma(y') ,$$
$$\Gamma_2''(t) < c^* l'(t')^{-\beta} \int_{d_2} l'(y')^{-\alpha} r^{h-2}(y', t') d\sigma(y') ,$$

with d_1, d_2 denoting orthogonal projections on the y_1, y_2-plane of the regions ω_1, ω_2 (involved in (6.5)). Repeating the developments, which led to (6.13), *we obtain for the* $\Gamma_j''(t)$ *inequalities of form* (6.13), where ω_1^*, ω_2^* are replaced by regions of the same type; that is, an argument of the kind used with respect to (8°) again applies, leading to inequalities of form (6.16) (with $r(o, t')$ in place of $r(o, t)$) for $\Gamma_1''(t)+\Gamma_2''(t)$. We have

$$|t_3| = |F(t_1, t_2)| \leqq c^* r^2(o, t') ;$$

thus

$$\frac{r^2(o, t)}{r^2(o, t')} = 1+t_3^2 r(o, t')^{-2} < c^* ; \qquad r^{-1}(o, t') < c^* r^{-1}(o, t) .$$

Whence in the inequalities of form (6.16) (with $r(o, t')$ for $r(o, t)$), referred to above, one may replace $r(o, t')$ in the second members by $r(o, t)$. Since $|\Phi''(t)| < c^*(\Gamma_1''(t)+\Gamma_2''(t))$, the following can be stated. In the general case (with the origin o of the coordinates y at c), *as formulated at the beginning of this section, the function* $\Phi''(t)$ *involved in* (6.4a) *satisfies*

(6.19)
$$|\Phi''(t)| < c^*[\varepsilon r(o, t)]^{h-\alpha-\beta} \quad (h < \alpha), \quad < c^*[\varepsilon r(o, t)]^{-\beta} \quad (\alpha < h \leqq 1),$$
$$< c^*[\varepsilon r(o, t)]^{-\beta} \log \left[\frac{b^0}{\varepsilon r(o, t)}\right] \quad (h = \alpha)$$

for t *in* $S\left(o, \frac{a}{2}\right)$, *exterior cones* $N(o, \varepsilon)$. *This is also satisfied by* $\Gamma_1''(t)$, $\Gamma_2''(t)$.

We now come to the study of $\Phi'(t)$ (6.4b),

(6.20)
$$\Phi'(t) = \int_{S(o,a)} k'(t|y, t) r^{-2}(y, t) q(y) d\sigma(y) \quad \text{(cf. (3.2a), (3.2))}$$

(origin of the y system at o); let t be in $S\left(o, \frac{a}{2}\right)$, not on β, exterior $N(o, \varepsilon)$. *Designate by* $S_{t,b}$ *the portion of* S, *whose orthogonal projection on the tangential plane,* P_t, *at* t, *is a circular region* $S'(t, b)$, *with center* t *and radius* b. *We shall take*

(1_0)
$$b = c_0 \varepsilon r(o, t) \qquad (\textit{small positive constant } c_0).$$

Use will be made of the decomposition

(6.21) $$\Phi'(t) = \Phi'_b(t)+\Phi_b^{1,0}(t)\,, \quad \Phi'_b(t) = \int_{S_{t,b}} \frac{k'(t|y,t)}{r^2(y,t)} q(y)d\sigma(y)\,,$$

$$\Phi_b^{1,0}(t) = \int_s k'(t|y,t)r^{-2}(y,t)q(y)d\sigma(y) \qquad [s = S(o,a)-S_{t,b}]\,.$$

Inasmuch as

$$q(y) \subset [\alpha|S]\,, \quad |k'(t|y,t)| \leqq c' \ (3.21\,\mathrm{a});\ r(y,t) \geqq b \quad (\text{for } y \text{ in } s)\,,$$

on letting $0 < \delta \leqq 1$ we obtain

$$|\Phi_b^{1,0}(t)| < c^* \int_s r^{-2}(y,t) l^{-\alpha}(y)d\sigma(y) = c^* \int_s r^{-\delta}(y,t)[l^{-\alpha}(y)r^{\delta-2}(y,t)d\sigma(y)]\,,$$

and

(2_0) $$|\Phi_b^{1,0}(t)| < c^* b^{-\delta} J(t)\,, \qquad J(t) = \int_{S(o,a)} l^{-\alpha}(y)r^{\delta-2}(y)d\sigma(y)\,;$$

we write

$$J(t) = J_1(t)+J_2(t) = \int_{\omega_1} \cdots + \int_{\omega_2} \cdots\,,$$

where ω_1, ω_2 are regions as in (6.5) (c at the origin o of the y system). It is observed that

$$J_1(t) = \Gamma_1''(t)\,, \qquad J_1(t) = \Gamma_2''(t)\,,$$

the $\Gamma_j''(t)$ ($j = 1, 2$) being defined by (6.5), with $\beta = 0$ and $h = \delta$. In view of the remark subsequent (6.19), J_1, J_2 satisfy (6.19). Thus, on letting $\beta = 0$, $h = \delta$ in the second members in (6.19), it is inferred that

$$J(t) < c^*[\varepsilon r(o,t)]^{\delta-\alpha} \ (\text{if } \delta < \alpha),\ < c^* \log\left[\frac{b^0}{\varepsilon r(o,t)}\right] \qquad (\text{if } \delta = \alpha)\,,$$

$$< c^* \qquad (\text{if } \alpha < \delta \leqq 1)\,;$$

whence by (2_0), (1_0)

$$|\Phi_b^{1,0}(t)| < c^*[\varepsilon r(o,t)]^{-\alpha} \ (\text{if } 0 < \delta < \alpha),\ < c^*[\varepsilon r(o,t)]^{-\alpha} \log\left[\frac{b^0}{\varepsilon r(o,t)}\right]$$

$$(\text{if } 0 < \delta = \alpha),\ < c^*[\varepsilon r(o,t)]^{-\delta} \quad (\text{if } \alpha < \delta \leqq 1)\,;$$

here δ ($0 < \delta \leqq 1$) is at our disposal, while α is fixed ($0 \leqq \alpha < 1$). Whence the above yields

(6.22) $$|\Phi_b^{1,0}(t)| < c^*[\varepsilon r(o,t)]^{-\alpha} \quad (\text{if } \alpha > 0)\,,$$

$$|\Phi_b^{1,0}(t)| < c^*[\varepsilon r(o,t)]^{-\delta} \qquad (\text{if } \alpha = 0)\,;$$

in the latter inequality $\delta(> 0)$ may be taken arbitrarily small; this inequality can be improved. Let y' be the orthogonal projection of y on the plane P_t and let $\varrho = r(y', t)$, θ be polar coordinates in P_t, with pole at t; one has

$$r(y,t) \geqq \varrho, \; d\sigma(y) < c^* \varrho d\varrho d\theta \,;$$

accordingly, when $\alpha = 0$,

$$|\Phi_b^{1,0}(t)| < c^* \int_s r^{-2}(y,t) d\sigma(y) < c^* \int_{\theta=0}^{2\pi} \int_{\varrho=b}^{L} \frac{d\varrho}{\varrho} d\theta \qquad (\text{some } L = c^*).$$

In view of (1_0), the second inequality (6.22) can be replaced by

$$(6.22\text{a}) \qquad |\Phi_b^{1,0}(t)| < c^* \log\left[\frac{b^0}{\varepsilon r(o,t)}\right] \qquad (\text{if } \alpha = 0)\,.$$

Introduce the orthogonal transformation (3.5)

$$y_i = t_i + \sum_j a_{ij} Y_j \qquad (a_{ij} \text{ from } (3.5\text{b}));$$

its inverse is

$$(6.23) \qquad Y_j = \sum_i a_{ij}(y_i - t_i)\,.$$

It is observed that when $r(o,t) \to 0$, the positive Y_j-axes tend to the corresponding y_i-axes. As before, let O be the origin of the Y system. The tangential plane P_t to S at t is the Y_1, Y_2-plane; $t = O$ and o will be designated by

$$Z = (Z_1, Z_2, Z_3)\,, \qquad Z_j = -\sum_i a_{ij} t_i$$

(capital letters are used for representation of points in the Y system). $\Phi_b'(t)$ can be represented as follows:

$$(6.24) \qquad \Phi_b'(t) = \int_{S(O,b)} k'(Y,O) r^{-2}(Y,O) q(Y) d\sigma(Y)\,,$$

where

$$q(Y) = q(y), \; k'(Y,O) = \sum_{m=1}^{\infty} \sum_{s_1,\dots s_m=1}^{3} \Gamma_{s_1\dots s_m}(t) W_{s_1}(Y,O)\dots W_{s_m}(Y,O)$$

(cf. (3.6a)); $S(O,b) = S_{t,b}$, that is $S(O,b)$ is the portion of S projecting orthogonally on P_t in a circular region, consisting of points Y for which

$$(1^\circ) \qquad Y_1^2 + Y_2^2 \leqq b^2, \; Y_3 = 0 \qquad \big(b = c_0 \varepsilon r(o,t)\big)\,.$$

We reintroduce the polar coordinates (3.9), $Y_i = \varrho \cos\theta_i$ $(i = 1, 2)$, $\theta_1 = \theta$, $\theta_2 = \frac{\pi}{2} - \theta$, and recall the formula (3.11a),

$$(2^\circ) \qquad k'(Y,O) = k^{1,*}(t,\theta) + k^{1,0}(\varrho,\theta)\,,$$

where $k^{1,*}(t,\theta)$ is the 'characteristic' of the original kernel. We let

(3°) $\qquad Y_3 = F(Y_1, Y_2) = O(Y_1^2 + Y_2^2)$ (as in the early part of section 2)

be the equation of the surface near O (that is, near $y = t$); the equation of the surface (near $y = o$) in the y coordinates will be written as

$$y_3 = F^0(y_1, y_2) = O(y_1^2+y_2^2)\,; \tag{4°}$$

$F(Y_1, Y_2)$ generally depends on t; $F^0(y_1, y_2)$ is independent of t. Introduce quantities ν_1, ν_2 as follows (with $Y_3 = F(Y_1, Y_2)$):

$$\frac{Y_1^2+Y_2^2}{Y_1^2+Y_2^2+Y_3^2}\left[1+\left(\frac{\partial F}{\partial Y_1}\right)^2+\left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}} = 1+\nu_1(\varrho, \theta)\,, \qquad q(Y)-q(O) = \nu_2(\varrho, \theta)\,;$$

the ν_i depend on t. We then have

$$r^{-2}(Y, O)d\sigma(Y) = [Y_1^2+Y_2^2+Y_3^2]^{-1}\left[1+\left(\frac{\partial F}{\partial Y_1}\right)^2+\left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}}\varrho d\varrho d\theta$$

$$= \big(1+\nu_1(\varrho, \theta)\big)\frac{d\varrho}{\varrho}d\theta\,; \qquad q(Y) = q(O)+\nu_2(\varrho, \theta).$$

In view of (2°), $\Phi_b'(t)$ (6.24) is expressible in the form of a principal integral

$$\int_{\varrho=0}^{b}\int_{\theta=0}^{2\pi}(k^{1,*}+k^{1,0})(1+\nu_1)(q(O)+\nu_2)\frac{d\varrho}{\varrho}d\theta\,; \tag{6.24'}$$

that is

$$\Phi_b'(t) = q(O)\int_{\varrho=0}^{b}\int_{\theta=0}^{2\pi}k^{1,*}(t, \theta)\frac{d\varrho}{\varrho}d\theta+\int_{\varrho=0}^{b}\int_{\theta=0}^{2\pi}\Lambda_t(\varrho, \theta)\frac{d\varrho}{\varrho}d\theta\,,$$

where the first integral displayed is in the sense of principal values and is zero in view of the satisfied condition (3.14). Thus

$$\Phi_b'(t) = \int_{\varrho=0}^{b}\int_{\theta=0}^{2\pi}\Lambda_t(\varrho, \theta)\frac{d\varrho}{\varrho}d\theta\,, \tag{6.25}$$

with

$$\Lambda_t(\varrho, \theta) = k'(Y, O)(1+\nu_1)\nu_2+q(O)k'(Y, O)\nu_1+q(O)k^{1,0}\,. \tag{6.25a}$$

The integral (6.25) exists in the ordinary sense.

By (3.20a) and (3.6a) $|\Gamma_{s_1\ldots s_m}(t)| \leqq 3^m c_m$; whence (cf. (6.24))

$$|k'(Y, O)| \leqq c^0 \ (c^0, \text{ constant from } (3.20\text{b}))\,; \tag{6.26}$$

also, since $q \subset [\alpha|S]$,

$$|q(O)| = |q(t)| < c^* l^{-\alpha}(t)\,. \tag{6.26a}$$

By definition of $[\alpha|S]$ it follows that

$$|q(y)-q(t)| \leqq Q(y, t)r^{\nu}(y, t) \qquad (\text{some } \nu;\ 0 < \nu \leqq 1)\,, \tag{6.27}$$

where $Q(y, t)$ is bounded when $l(y)$, $l(t) \geqq \delta$ (> 0). It will be necessary to introduce some specific statement regarding the behaviour of $Q(y, t)$ for y and for t near edges;

this we shall do along the lines of the corresponding conditions (3.27) for $\gamma(y, t)$ (relating to the $\gamma_{i_1 \ldots i_m}$); *thus we assume that*

$$Q(y, t) < c^* l^{-\alpha_0}(y) \qquad \bigl(\text{if } l(y) \leqq l(t)\bigr) , \tag{6.27a}$$

$$< c^* l^{-\alpha_0}(t) \qquad \bigl(\text{if } l(y) \geqq l(t)\bigr) \qquad (\alpha \leqq \alpha_0;\ \alpha_0 - \nu < 1) .$$

The special case, important in applications, is when in (6.27), (6.27a) *one has*

$$\nu = 1,\ \alpha_0 = \alpha + 1 . \tag{6.27b}$$

As a consequence of the above, ν_2 of (5°) satisfies

$$|\nu_2(\varrho, \theta)| \leqq Q(y, t) r^\nu(y, t) = Q(y, t) r^\nu(Y, O) \leqq c^* Q(y, t) \varrho^\nu . \tag{1_0}$$

By methods of the type, previously used for similar purposes, we find that $l(t) \geqq c_0' \varepsilon r(o, t)$ $(c_0' > 0)$, that is

$$l^{-1}(t) < c^* [\varepsilon r(o, t)]^{-1} \qquad \bigl(t \text{ near } o, \text{ exterior } N(o, \varepsilon)\bigr); \tag{2_0}$$

in proving this use is made essentially of the fact that the curve β (near o) is in $N\left(o, \frac{\varepsilon}{2}\right)$. Furthermore, by the triangular relation

$$r(o, t) \leqq r(t, y) + r(o, y)$$

and on noting that for y in $S_{t,b}$ one has

$$r(t, y) = r(O, Y) \leqq k^0 r(O, Y') \leqq k^0 b = k^0 c_0 \varepsilon r(o, t) \ [Y' = (Y_1, Y_2, 0);\ k^0 = c^*] , \tag{3_0}$$

it is inferred that in $S_{t,b}$

$$r^{-1}(o, y) < c^* r^{-1}(o, t) , \tag{4_0}$$

provided c_0 in (1_0) is taken suitably small. In view of (2_0)

$$Q(y, t) < c^* \bigl(\varepsilon r(o, y)\bigr)^{-\alpha_0} \ \bigl(\text{if } l(y) \leqq l(t)\bigr), \qquad < c^* \bigl(\varepsilon r(o, t)\bigr)^{-\alpha_0} \ \bigl(\text{if } l(y) \geqq l(t)\bigr);$$

hence, by (4_0),

$$Q(y, t) < c^* \bigl(\varepsilon r(o, t)\bigr)^{-\alpha_0} \qquad \bigl(y \text{ in } S_{t,b};\ t \text{ exterior } N(o, \varepsilon)\bigr) .$$

Thus, as a consequence of (1_0),

$$|\nu_2(\varrho, \theta)| < c^* \bigl(\varepsilon r(o, t)\bigr)^{-\alpha_0} \varrho^\nu . \tag{6.28}$$

Before we study ν_1 (5°), $k^{1,0}$ (2°) it will be necessary to examine the first order derivatives of F (3°). The equation of the surface in the y system for y near o being $y_3 = F^0(y_1, y_2)$, consider the function

$$G(Y_1, Y_2, Y_3) \equiv y_3 - F^0(y_1, y_2) ,$$

where (cf. (3.5))

$$y_i = t_i + \sum_{j=1}^{3} a_{ij} Y_j \qquad (a_{ij} \text{ from (3.5b); } a_{ij} \text{ are functions of } t).$$

Regarding the Y_j as independent and letting

$$G_j = \frac{\partial}{\partial Y_j} G(Y_1, Y_2, Y_3)\,, \quad F_i^0(y_1, y_2) = \frac{\partial}{\partial y_i} F^0(y_1, y_2)\,,$$

one obtains

$$(\mathrm{I}_1) \qquad G_j = a_{3,j} - \sum_{i=1}^{2} F_i^0(y_1, y_2) a_{ij}\,.$$

$G(Y_1, Y_2, Y_3) = 0$ is the equation of the surface in the Y system; one has

$$(\mathrm{I}_2) \qquad \frac{\partial F}{\partial Y_j} = \frac{\partial Y_3}{\partial Y_j} = \frac{-G_j}{G_3} \quad (j = 1, 2)\,.$$

Now

$$|n_i(t)| \leqq |F_i^0(t_1, t_2)| \;\; (i = 1, 2),\;\; 0 < n_3(t) \leqq 1; \quad |F_i^0(t_1, t_2)| \leqq c^* r(o, t)\,;$$

hence the a_{ij} (3.5b) satisfy inequalities

$$(\mathrm{I}_3) \qquad |a_{ii}| \leqq 1,\; |a_{ij}| \leqq c^* r(o, t) \;\; (i \neq j;\; i, j = 1, 2, 3);\; a_{3,3} = n_3 \geqq n' = c^*\,.$$

Inasmuch as

$$\left| \sum_{i=1}^{2} F_i^0(y_1, y_2) a_{ij} \right| \leqq \sum_{i=1}^{2} |F_i^0(y_1, y_2)| \leqq c^* r(o, y)\,,$$

we have

$$(\mathrm{I}_4) \qquad |G_3| \geqq n' - c^* r(o, y) \geqq c^*\,,$$

provided $r(o, y)$ is sufficiently small (which is achieved by taking the number a, used in defining $S(o, a)$, suitably small). Write $-G_j$ (I_1) in the form

$$-G_j = \left[\sum_{i=1}^{2} F_i^0(t_1, t_2) a_{ij} - a_{3,j} \right] + \sum_{i=1}^{2} \left(F_i^0(y_1, y_2) - F_i^0(t_1, t_2) \right) a_{ij}\,;$$

[...] here is zero for $j = 1, 2$. On the other hand,

$$(\mathrm{I}_5) \qquad |F_i^0(y_1, y_2) - F_i^0(t_1, t_2)| \leqq c^* [(y_1 - t_1)^2 + (y_2 - t_2)^2]^{\frac{1}{2}} \leqq c^* r(y, t)$$

(as consequence of the assumed continuity and boundedness up to the edges of the second order partial derivatives of F^0). Thus

$$|G_j| \leqq \sum_{i=1}^{2} |F_i^0(y_1, y_2) - F_i^0(t_1, t_2)| \leqq c^* r(y, t) \quad (j = 1, 2)$$

and, by (I_2), (I_4),

$$(6.29) \qquad \left| \frac{\partial F}{\partial Y_j} \right| \leqq c^* r(y, t) = c^* r(O, Y) \leqq k_0 \varrho \quad (j = 1, 2;\; k_0 = c^*)\,;$$

it is essential to note that k_0 is independent of t.

A corollary of (6.29) is the relation

$$\left[1+\left(\frac{\partial F}{\partial Y_1}\right)^2+\left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}}=1+O(\varrho^2)$$

(here and in the sequel $O(\ldots)$ is independent of t). By a mean value theorem

$$Y_3=F(Y_1,\,Y_2)=\sum_{i=1}^{2}\frac{\partial}{\partial U_i}F(U_1,\,U_2)Y_i\ \ (Y_i=\varrho\cos\theta_i)\,,$$

where (U_1, U_2) is some point (in the Y_1, Y_2-plane) on the segment joining the points O, $(Y_1, Y_2, 0)$; hence, by (6.29), $|Y_3|\leqq c^*\varrho^2$ and

$$|Y_3\varrho^{-1}|\leqq c^*\varrho\,. \tag{6.29a}$$

Turning to ν_1 (5°) we note that

$$1+\nu_1=[1-Y_3^2r^{-2}(O,\,Y)]\left[1+\left(\frac{\partial F}{\partial Y_1}\right)^2+\left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}}.$$

Since $\varrho^2 = Y_1^2+Y_2^2$, in view of (6.29a) one has

$$\frac{Y_3^2}{r^2(O,\,Y)}=\frac{(Y_3\varrho^{-1})^2}{1+(Y_3\varrho^{-1})^2}\leqq(Y_3\varrho^{-1})^2\leqq c^*\varrho^2\,. \tag{6.29b}$$

Accordingly

$$1+\nu_1=[1+O(\varrho^2)]\,[1+O(\varrho^2)]$$

and, finally,

$$|\nu_1(\varrho,\,\theta)|\leqq c^*\varrho^2\,. \tag{6.30}$$

For the function $k^{1,0}$ $\big((2°), (3.11a)\big)$ the following holds

$$(\alpha_1)\qquad k^{1,0}(\varrho,\,\theta)=k'(Y,\,O)-k^{1,*}(t,\,\theta)=\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{3}\Gamma_{s_1\ldots s_m}(t)W_{s_1}(Y,\,O)\ldots W_{s_m}(Y,\,O)$$

$$-\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2}\Gamma_{s_1\ldots s_m}(t)\cos\theta_{s_1}\ldots\cos\theta_{s_m}=\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2}J_{s_1\ldots s_m}+J\,,$$

where

$$J_{s_1\ldots s_m}=\Gamma_{s_1\ldots s_m}(t)[W_{s_1}(Y,\,O)\ldots W_{s_m}(Y,\,O)-\cos\theta_{s_1}\ldots\ \cos\theta_{s_m}]\,,$$

$$J=\sum_{m=1}^{\infty}\ \sum_{s_1,\ldots s_m}{}'\ \Gamma_{s_1\ldots s_m}(t)W_{s_1}(Y,O)\ldots W_{s_m}(Y,\,O)\,;$$

the prime with the summation sign signifies summing over sets $(s_1,\ldots s_m)$ containing at least one element, say s', equal to 3. Now

$$(\alpha_2)\qquad W_s(Y,\,O)=Y_sr^{-1}(O,\,Y)=\varrho\cos\theta_s[\varrho^2+Y_3^2]^{-\frac{1}{2}}=\cos\theta_s[1+(Y_3\varrho^{-1})^2]^{-\frac{1}{2}}$$

for $s = 1, 2$; thus

(α_3) $$W_{s_1}(Y, O)\ldots W_{s_m}(Y, O) - \cos\theta_{s_1}\ldots\cos\theta_{s_m} = \cos\theta_{s_1}\ldots\cos\theta_{s_m} f_m \quad [s_1, \ldots, s_m \leqq 2]\,,$$

where (by (6.29a))

$$|f_m| = |-1 + [1+(Y_3\varrho^{-1})^2]^{-\frac{m}{2}}| \leqq \frac{m}{2}(Y_3\varrho^{-1})^2 \leqq mk'\varrho^2\,,$$

with $k' = c^*$ independent of m; $|\Gamma_{s_1\ldots s_m}(t)|$ (3.6a) is bounded by $3^m c_m$ (c_m from (3.20a)); hence by (α_3)

(α_4) $$\left|\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2} J_{s_1\ldots s_m}\right| \leqq \sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2} 3^m c_m |f_m| \leqq k'\sum_{m=1}^{\infty} m 2^m 3^m c_m \varrho^2 \leqq c^*\varrho^2$$

(the series last displayed converges by (3.20b)). We come to J. It is observed that, by (6.29b),

$$|W_3(Y, O)| = |Y_3| r^{-1}(O, Y) \leqq c^*\varrho\,;$$

using the fact that $|W_s(Y, O)| \leqq 1$ $(s = 1, 2, 3)$, one obtains

$$|J| \leqq \sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m}{}' \, 3^m c_m c^*\varrho \leqq c^*\varrho\,.$$

This, together with (α_4), (α_1), implies that

(6.31) $$|k^{1,0}(\varrho, \theta)| \leqq c^*\varrho\,.$$

As a consequense of (6.25a), (6.26), (6.26a), (2_0), (6.28), (6.30), (6.31) one has

(6.32) $$|\Lambda_t(\varrho, \theta)| < c^*(\varepsilon r(o, t))^{-\alpha_0}\varrho^\nu + c^*(\varepsilon r(o, t))^{-\alpha}\varrho\,.$$

Hence $|\Phi_b'(t)|$ (6.25) is bounded by an expression of the form

$$c^*(\varepsilon r(o, t))^{-\alpha_0} b^\nu + c^*(\varepsilon r(o, t))^{-\alpha} b\,.$$

Recalling that $b = c_0\varepsilon r(o, t)$, we finally obtain

(6.33) $$|\Phi_b'(t)| < c^*(\varepsilon r(o, t))^{\nu-\alpha_0} + c^*(\varepsilon r(o, t))^{1-\alpha} \leqq c^*(\varepsilon r(o, t))^{\nu-\alpha_0}\,;$$

in the case (6.27b) one has

(6.33a) $$|\Phi_b'(t) < c^*(\varepsilon r(o, t))^{-\alpha}\,.$$

Write, for short, $L(t) = (\varepsilon r(o, t))^{-1}$. By (6.4a), (6.21)

(6.34) $$\Phi(t) = \Phi''(t) + \Phi_b^{1,0}(t) + \Phi_b'(t)\,,$$

where the three terms in the second member satisfy (6.19), [(6.22), (6.22a)] and [(6.33), (6.33a)], respectively. There are following cases (valid for t exterior $N(o, \varepsilon)$, near o).

(6.34a) $$h < \alpha \quad (\text{then } 0 < h < \alpha < 1).$$

One has

$$\Phi(t) = O(L^{\lambda}(t)) \quad [\lambda = \max. (\alpha+\beta-h, \alpha, \alpha_0-\nu)] .$$

(6.34b) $$h = \alpha \text{ (then } 0 < h = \alpha < 1).$$

It is noted that

$$\Phi(t) = O(L^{\lambda}(t)) \quad [\text{if } \lambda = \max. (\beta, \alpha, \alpha_0-\nu) > \beta] ,$$

$$\Phi(t) = O(L^{\lambda}(t) \log L(t)) \quad (\text{if } \lambda = \beta) .$$

(6.34c) $$0 < \alpha < h \leqq 1 .$$

Then

$$\Phi(t) = O(L^{\lambda}(t)) \quad [\lambda = \max. (\beta, \alpha, \alpha_0-\nu)] .$$

(6.34d) $$0 = \alpha < h \leqq 1 .$$

It is observed that

$$\Phi(t) = O(L^{\lambda}(t)) \quad [\text{if } \lambda = \max. (\beta, \alpha_0-\nu) > 0] ,$$

$$\Phi(t) = O(\log L(t)) \quad (\text{if } \lambda = 0) .$$

We restate for convenience some of the previously made hypotheses. The $\gamma_{i_1 \dots i_m}(y) \subset [0|S]$ (cf. (3.20a)) and

$$|\gamma_{i_1 \dots i_m}(y) - \gamma_{i_1 \dots i_m}(t)| \leqq \lambda_m \gamma(y, t) r^h(y, t) \quad (0 < h \leqq 1) ;$$

$$\gamma(y, t) < c^* l^{-\beta}(y) \ (\text{for } l(y) \leqq l(t)) , \quad \gamma(y, t) < c^* l^{-\beta}(t) \quad (\text{for } l(y) \geqq l(t)) ;$$

$q(y) \subset [\alpha|S]$ and

$$|q(y) - q(t)| \leqq Q(y, t) r^{\nu}(y, t) \quad (0 < \nu \leqq 1) ;$$

$$Q(y, t) < c^* l^{-\alpha_0}(y) \ (\text{for } l(y) \leqq l(t)) , \quad Q(y, t) < c^* l^{-\alpha_0}(t) \quad (\text{for } l(y) \geqq l(t)) ;$$

$$0 \leqq \alpha ; \ 0 \leqq \beta ; \ \alpha+\beta < 1 ; \ \alpha \leqq \alpha_0 ; \ \alpha_0 - \nu < 1 .$$

On taking account of (6.34)—(6.34d) and of (6.3) we can formulate, independent of the choice of coordinates, the following.

Theorem 6.36. *Under Hypothesis* 3.20 *and, more specifically, under the conditions stated subsequent* (6.34d) *the principal integral* $\Psi(t)$,

$$\Psi(t) = \int_S k(y, t) r^{-2}(y, t) q(y) d\sigma(y)$$

satisfies inequalities

(6.36a) $$|\Psi(t)| < c^* L^{\lambda}(t) \quad [\textit{if } h < \alpha ; \ \lambda = \max. (\alpha+\beta-h, \alpha, \alpha_0-\nu)] ;$$

(6.36b) $$|\Psi(t)| < c^* L^{\lambda}(t) \quad [\textit{if } h = \alpha \textit{ and } \lambda = \max. (\beta, \alpha, \alpha_0-\nu) > \beta] ,$$

$$< c^* L^{\lambda}(t) \log L(t) \quad [\textit{if } h = \alpha \textit{ and } \lambda \text{ (above)} = \beta] ;$$

(6.36c) $|\Psi(t)| < c^* L^\lambda(t)$ [*if* $0 < \alpha < h \leqq 1$; $\lambda = \max. (\beta, \alpha, \alpha_0 - \nu)$];

(6.36d) $|\Psi(t)| < c^* L^\lambda(t)$ [*if* $0 = \alpha < h \leqq 1$ *and* $\lambda = \max. (\beta, \alpha_0 - \nu) > 0$],

$< c^* \log L(t)$ [*if* $0 = \alpha < h \leqq 1$ *and* λ (above) $= 0$].

Here $L(t) = [\varepsilon r(c, t)]^{-1}$ *and* c *is a point on the edges* β *of the surfaces* S. *The above is valid for* t *on* S, *near* c, *exclusive neighborhoods of* c *tangential to the curve* β *near* c; *specifically, for* t *in* $S(c, a^0)$ (a^0, > 0, *small*), *exterior* $N(c, \varepsilon)$ (*Def.* 5.1). *In all cases* $0 \leqq \lambda < 1$. *The above also implies that* $\Psi(t) \subset [\lambda|S]$, *or* $\subset [\lambda, \log |S]$, *or* $\subset [0, \log |S]$, *depending on the case.*

Note. In many cases one has $h = 1$, $\alpha_0 = \alpha + 1$, $\nu = 1$; the inequalities (6.36a)—(6.36d) can then be restated as follows:

(6.37) $|\Psi(t)| < c^* L^\lambda(t)$ [if $\alpha > 0$; $\lambda = \max. (\beta, \alpha)$];

(6.37a) $|\Psi(t)| < c^* L^\beta(t)$ [if $\alpha = 0$ and $\beta > 0$);

(6.37b) $|\Psi(t)| < c^* \log L(t)$ [if $\alpha = \beta = 0$].

Let S_δ be the part of the surface for which $0 \leqq l(t) \leqq \delta$. *Let* $c = c_t$ *be a continuous transformation of* S *on itself*; we arrange to have c_t of a Hölder class, edges included; furthermore, the choice of c_t is made so that neighborhoods of 'edges' are transformed into edges; more precisely, S_δ (for δ, > 0, small) is to transform into edges. We take δ, > 0, suitably small so that, for t in S_δ, c_t can be defined as a point on β such that the tangent to β at $c = c_t$ is perpendicular to the rectilinear segment (c_t, t).

Theorem 6.38 (*Supplement to Theorem* 6.36). *Suppose the surface* S *is completely regular* (*section* 2) *and the* $\gamma_{i_1 \ldots i_m}(y)$, $q(y)$ *are uniformly Lip.* 1 (*that is, of Hölder class* H_1, *edges included*); *one may then take* $\alpha = \alpha_0 = \beta = 0$, $h = \nu = 1$ *and the inequality* (6.37b) *will hold. This result can be improved replacing* (6.37b) *by*

$$\Psi(t) = \Psi^*(t) + q(c_t) v_0(t) \log \frac{c^*}{r(c_t, t)} \tag{6.38a}$$

(t *in* S_δ), *where* $\Psi^*(t)$, $v_0(t)$ *are uniformly of a Hölder class* (*edges included*).

The proof of the above result is not easy; it can be achieved by methods of type used in proving Theorem 6.36 and utilizing properties of completely regular surfaces. We shall omit the details.

$\Psi(t)$ *in Theorem* 6.36 *is a sum of three terms* $\Psi'(t)$ *such that*

(6.39) $|\Psi'(t) - \Psi'(t_0)| \leqq c^* l^{-\alpha_1}(t) r^{\nu_1}(t, t_0)$ $\big($*for* $l(t) \leqq l(t_0)\big)$, $\leqq c^* l^{-\alpha_1}(t_0) r^{\nu_1}(t, t_0)$ $\big($*for* $l(t_0) < l(t)\big)$;

if $\alpha+\beta, \alpha_0+\beta-\nu < 1$, *we may choose* $0 < \nu_1 \leqq 1, \alpha_1-\nu_1 < 1$. To prove this write $\Psi = \Psi_1+\Psi_2$, where Ψ_1, Ψ_2 are integrals over the parts of S for which $l(y) < \frac{1}{2}\min.\big(l(t), l(t_0)\big)$ and $l(y) \geqq \frac{1}{2}\min.\big(l(t), l(t_0)\big)$, respectively. To Ψ_1 we apply largely the methods of this section and to Ψ_2 those of GIRAUD, obtaining the stated result.

If $\alpha+\beta$, $\alpha_0+\beta-\nu < \frac{1}{2}$, *then* $\alpha_1-\nu_1 < \frac{1}{2}$.

7. Curvilinear potentials. We recall that $\beta = \beta_1+\beta_2+\cdots$ constitutes the edge (that is, the edges) of S, where $\beta_1, \beta_2, \ldots$ (finite in number) are regular (with continuously turning tangents) simple closed curves, without common points. In this section, we shall study the potential

$$K(x) = \int_\beta \frac{k(y)}{r(x, y)}\, ds(y)\,, \tag{7.1}$$

where $ds(y)$ is the element of length of β at y, $k(y)$ is real of a Hölder class on β and x is not on β. We shall determine the asymptotic form of $K(x)$ for x near c, exterior $N(c, 2\varepsilon)$ ($\varepsilon > 0$, suitably small). Write

$$K(x) = K_0(x)+K_1(x),\ \ K_0(x) = \int_{\beta_0} \frac{k(y)}{r(x, y)}\, ds(y)\,,\quad K_1(x) = \int_{\beta-\beta_0} \cdots\,, \tag{7.2}$$

where β_0 is the part of β, near c, for which

$$r(c, y) \leqq a \quad (\text{some } a > 0)\,; \tag{7.2a}$$

a is taken suitably small so that β_0 lies in $N(c, \varepsilon)$; x *(exterior* $N(c, 2\varepsilon)$*) is supposed to be near* c *so that*

$$r(c, x) \leqq a^0 \quad (\text{some } a^0 > 0;\ a^0 < a)\,. \tag{7.2b}$$

For y on $\beta-\beta_0$ one has $r(x, y) \geqq a-a^0 > 0$; hence

$$|K_1(x)| \leqq c^*\,. \tag{7.2c}$$

Turning to $K_0(x)$, we write

$$K_0(x) = k(c)\int_{\beta_0} \frac{ds(y)}{r(x, y)}+R(x)\,,\quad R(x) = \int_{\beta_0} \frac{k(y)-k(c)}{r(x, y)}\, ds(y)\,. \tag{7.3}$$

Now, c *separates* β_0 *into two parts* β_0', β_0''. Consider the integrals along β_0', for instance. Choose the y system so that c is at O and so that the positive y_1-axis coincides with the part of the tangent to β, at c, extending from c in the direction of β_0', while $x_2 > 0$, $x_3 = 0$. One has

$$dy_1 \leqq ds(y) \leqq s_0 dy\,, \tag{7.4}$$

where $s_0 \geqq 1$, is independent of y and tends to unity when $a \to 0$; moreover,

$$r^2(x, y) \geqq r^2(x, y^0) = (x_1-y_1)^2+(x_2-y_2)^2 \quad \big(y^0 = (y_1, y_2, 0)\big) \tag{7.4a}$$

and

(7.4b) $$x_2 \geqq |x_1| \operatorname{tg} 2\varepsilon; \quad |y_2| \leqq \sqrt{y_1^2+y_2^2} \leqq y_1 \operatorname{tg} 2\varepsilon .$$

Let y^+, y^- be the points of intersection of the line $y_1 = \text{const.}$ (> 0) with the traces in the y_1, y_2-plane of the conical surface $N(o, \varepsilon)$, i. e. with the lines $y_2 = \pm y_1 \operatorname{tg} \varepsilon$, respectively; thus

(7.4c) $$y^+ = (y_1, y_1 \operatorname{tg} \varepsilon, 0), \quad y^- = (y_1, -y_1 \operatorname{tg} \varepsilon, 0) .$$

For $x_2 \geqq y_1 \operatorname{tg} \varepsilon$ and $0 < x_2 \leqq y_1 \operatorname{tg} \varepsilon$ we have

(7.4d) $$r^2(x, y^0) \geqq r^2(x, y^+) = (x_1-y_1)^2+(x_2-y_1 \operatorname{tg} \varepsilon)^2, \quad r(x, y^0) \geqq y_1-x_1 \ (> 0) ,$$

respectively. By (7.4a), (7.4d)

(7.5) $$\frac{1}{r(x, y)} \leqq \frac{1}{r(x, y^+)} \qquad (\text{for } y_1 \leqq x_2 \operatorname{ctg} \varepsilon) ,$$

(7.5a) $$\frac{1}{r(x, y)} \leqq \frac{1}{y_1-x_1} \qquad (\text{for } y_1 \geqq x_2 \operatorname{ctg} \varepsilon) .$$

With $k(y)$, say of class H_h $(0 < h \leqq 1)$, with the aid of (7.4) we obtain

(7.6) $$\left| \int_{\beta_0'} \frac{k(y)-k(o)}{r(x, y)} \, ds(y) \right| \leqq c^* \int_0^{a'} \frac{r^h(o, y)}{r(x, y)} \, dy_1 ,$$

where

$$0 < a' \leqq a; \quad \frac{a'}{a} \to 1 \quad (\text{as } a \to 0)$$

and y_2, y_3 are thought of as functions of y_1 (the equations of β). Now y is some point in the cross section of $N(o, \varepsilon)$ by the plane $y_1 = \text{const.}$; clearly $r(o, y) \leqq r(o, y^*)$, where y^* is any point on the circumference of this cross section; thus

$$r(o, y) \leqq y_1 \sec \varepsilon .$$

Accordingly, in view of (7.5), (7.5a)

(7.7) $$\int_0^{a'} \frac{r^h(o, y)}{r(x, y)} \, dy_1 \leqq \sec^h \varepsilon \int_0^{x_2 \operatorname{ctg} \varepsilon} \frac{y_1^h dy_1}{r(x, y^+)} + \sec^h \varepsilon \int_{x_2 \operatorname{ctg} \varepsilon}^{a'} \frac{y_1^h dy_1}{y_1-x_1} ;$$

if $x_2 \operatorname{ctg} \varepsilon \geqq a'$, integration in the first term in the second member, above, is over $(0, a')$ and the last term is missing. When

$$x_2 \operatorname{ctg} \varepsilon \leqq y_1 \leqq a' ,$$

on noting (7.4b) we deduce that $y_1-x_1 \geqq r(x^*, y^+)$, where y^+ is from (7.4c) and x^* is the intersection of the lines

$$x_2 = y_1 \operatorname{tg} \varepsilon, \quad x_2 = x_1 \operatorname{tg} 2\varepsilon$$

in the y_1, y_2-plane; that is,

$$y_1 - x_1 \geqq y_1 - y_1 \operatorname{ctg} 2\varepsilon \operatorname{tg} \varepsilon \geqq y_1 c' ,$$

where c', > 0, is a constant. Hence

$$\int_{x_2 \operatorname{ctg} \varepsilon}^{a'} \frac{y_1^h dy_1}{y_1 - x_1} \leqq \frac{1}{c'} \int_{x_2 \operatorname{ctg} \varepsilon}^{a'} y_1^{h-1} dy_1 \leqq c^* , \tag{7.7a}$$

when there is occasion to consider the integral in the first member.

With y' denoting the foot of the perpendicular from y^+ (7.4c) upon the line $y_2 = y_1 \operatorname{tg} 2\varepsilon$, we observe that

$$r(x, y^+) \geqq r(y', y^+) = y_1 \operatorname{tg} \varepsilon ;$$

whence

$$\int_0^{x_1 \operatorname{ctg} \varepsilon} \frac{y_1^h dy_1}{r(x, y^+)} \leqq \operatorname{ctg} \varepsilon \int_0^{x_2 \operatorname{ctg} \varepsilon} y_1^{h-1} dy_1 \leqq c^* x_2^h \operatorname{ctg}^{h+1} \varepsilon \leqq c^* \varepsilon^{-h-1} x_2^h . \tag{7.7b}$$

In consequence of (7.6), (7.7)–(7.7b)

$$\left| \int_{\beta_0'} \frac{k(y) - k(o)}{r(x, y)} ds(y) \right| \leqq c^* \varepsilon^{-h-1} .$$

There is an inequality similar to the above for the integral over β_0'' (see the text subsequent (7.3)). Combining the two inequalities, we state the following result (independent of the choice of the y system)

$$|R(x)| \leqq c^* \varepsilon^{-h-1} \quad (x \text{ exterior } N(c, 2\varepsilon);\ r(c, x) \leqq a^0) . \tag{7.8}$$

We proceed on taking note of the text subsequent (7.3). By (7.4)

$$g(x) = \int_{\beta_0'} \frac{ds(y)}{r(x, y)} \leqq s_0 \int_{y_1=0}^{a'} \frac{dy_1}{r(x, y)} ; \tag{7.9}$$

utilizing (7.5), (7.5a) one obtains

$$\int_{y_1=0}^{a'} \frac{dy_1}{r(x, y)} = \int_0^{x_2 \operatorname{ctg} \varepsilon} [(x_1 - y_1)^2 + (x_2 - y_1 \operatorname{tg} \varepsilon)^2]^{-\frac{1}{2}} dy_1 + \int_{x_2 \operatorname{ctg} \varepsilon}^{a'} \frac{dy_1}{y_1 - x_1} ; \tag{7.9a}$$

if $x_2 \operatorname{ctg} \varepsilon > a'$, the last integral above is deleted and the first is between the limits $0, a'$.

Now $a^0 < a$, while a' ($\leqq a$) is arbitrarily near a for a suitably small; thus we may consider that $a^0 < a'$. On writing

$$\eta = \operatorname{arc} \operatorname{tg} \frac{x_2}{x_1},$$

we have

$$2\varepsilon \leqq \eta \leqq \pi - 2\varepsilon$$

and

$$0 < a'-a^0 \leqq a'-r \leqq a'-x_1 = a'-r\cos\eta < a'+a^0 \quad (r^2 = x_1^2+x_2^2)\,;$$

thus

$$(1^\circ) \qquad |\log(a-x_1)| \leqq c^* \quad (\text{for } x_2 \operatorname{ctg}\varepsilon \leqq y_1 \leqq a')\,;$$

on the other hand,

$$x_2 \operatorname{ctg}\varepsilon - x_1 = r\csc\varepsilon\sin(\eta-\varepsilon)\,;$$

since $\varepsilon \leqq \eta-\varepsilon \leqq \pi-3\varepsilon$, so that $\sin(\eta-\varepsilon) \geqq \sin\varepsilon$, one has

$$(2^\circ) \qquad r \leqq x_2 \operatorname{ctg}\varepsilon - x_1 \leqq r\csc\varepsilon.$$

In view of (1°), (2°)

$$(7.10) \qquad \int_{x_2 \operatorname{ctg}\varepsilon}^{a'} \frac{dy_1}{y_1-x_1} = \log\frac{1}{x_2\operatorname{ctg}\varepsilon - x_1} + \log(a'-x_1)$$
$$= \log\frac{1}{r} + \nu'(x)\,; \quad -c^* - \log\frac{1}{\varepsilon} \leqq \nu'(x) \leqq c^*\,.$$

It is observed that

$$\int_0^{x_2\operatorname{ctg}\varepsilon} [(x_1-y_1)^2+(x_2-y_1\operatorname{tg}\varepsilon)^2]^{-\frac{1}{2}} dy_1 = \cos\varepsilon\, I(x)\,,$$

$$I(x) = \int_0^{x_2\operatorname{ctg}\varepsilon} [(y_1-rp)^2+r^2q^2]^{-\frac{1}{2}} dy_1, \quad p = \cos(\eta-\varepsilon)\cos\varepsilon, \quad q = \sin(\eta-\varepsilon)\cos\varepsilon\,.$$

One has

$$I(x) = \log\left[\left((y_1-rp)^2+r^2q^2\right)^{\frac{1}{2}} + y_1 - rp\right]\Big|_0^{x_2\operatorname{ctg}\varepsilon}$$
$$= \log(\sigma_1^{\frac{1}{2}}+\sigma_2) - \log 2 + 2\log\csc\frac{\eta-\varepsilon}{2}\,,$$

where

$$\sigma_1 = \left(\sin\eta\csc\varepsilon - \cos(\eta-\varepsilon)\right)^2 + \sin^2(\eta-\varepsilon) = \csc^2\varepsilon[\sin^2\varepsilon + \sin\eta\sin(\eta-2\varepsilon)]\,,$$

$$\sigma_2 = \sin\eta\csc\varepsilon - \cos(\eta-\varepsilon) = \tfrac{1}{2}\csc\varepsilon[\sin\eta + \sin(\eta-2\varepsilon)], \quad \log\csc\frac{\eta-\varepsilon}{2} \leqq \log\csc\frac{\varepsilon}{2}.$$

Whence we infer that

$$1 \leqq \sigma_1^{\frac{1}{2}}+\sigma_2 \leqq (2^{\frac{1}{2}}+1)\csc\varepsilon; \quad 0 \leqq I(x) \leqq c^*\log\frac{1}{\varepsilon}\,;$$

hence

$$(7.11) \qquad \int_0^{x_2\operatorname{ctg}\varepsilon} \frac{dy_1}{\sqrt{(x_1-y_1)^2+(x_2-y_1\operatorname{tg}\varepsilon)^2}} \leqq c^*\log\frac{1}{\varepsilon}\,.$$

As a consequence of (7.9), (7.9a), (7.10), (7.11)

$$g(x) \leqq s_0\log\frac{1}{r(o,x)} + c^*\log\frac{1}{\varepsilon}\,.$$

A similar inequality holds for the integral

$$\int_{\beta_0''} \frac{ds(y)}{r(x, y)} \quad (\beta_0'' \text{ from the text after (7.3)}).$$

By virtue of these two inequalities we may assert that

$$\int_{\beta_0} \frac{ds(y)}{r(x, y)} \leqq 2s_0 \log \frac{1}{r(c, x)} + c^* \log \frac{1}{\varepsilon}$$

for x exterior $N(c, 2\varepsilon)$ $\big(r(c, x) \leqq a^0\big)$ (suitable s_0, $\geqq 1$, $\to 1$, as $a \to 0$).

In view of (7.2), (7.2c), (7.3), (7.8) and (7.12) one has

$$\text{(7.13)} \qquad K(x) = \int_{\beta} \frac{k(y)ds(y)}{r(x, y)} = k(c)\left[2s_0 \log \frac{1}{r(c, x)} + \zeta(c, x)\right] + \varrho(c, x);$$

$|\zeta(c, x)| \leqq c^* \log \frac{1}{\varepsilon}$, $|\varrho(c, x)| \leqq c^* \varepsilon^{-h-1}$ (for x exterior $N(c, 2\varepsilon)$; $r(c, x) \leqq a^0$); in the above

$$2s_0 \log \frac{1}{r(c, x)} + \zeta(c, x) = \int_{\beta_1} \frac{ds(y)}{r(x, y)} \quad (> 0).$$

Envisaging again the situation as set forth between (7.3) and (7.4), we proceed to obtain an upper bound for $r(x, y)$. Now $y = (y_1, y_2, y_3)$ is a point in the circular region $C(y_1; \varepsilon)$, at right angles with the y_1-axis, with center $(y_1, 0, 0)$ and radius $y_1 \operatorname{tg} \varepsilon$. For y_1, y_2 fixed

$$y_3^2 \leqq y_1^2 \operatorname{tg}^2 \varepsilon - y_2^2;$$

thus

$$r^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2 \leqq (x_1 - y_1)^2 + y_1^2 \operatorname{tg}^2 \varepsilon + x_2(x_2 - 2y_2);$$

since $|y_2| \leqq y_1 \operatorname{tg} \varepsilon$, so that

$$|x_2 - 2y_2| \leqq x_2 + 2y_1 \operatorname{tg} \varepsilon,$$

one has

$$r^2(x, y) \leqq (x_1 - y_1)^2 + y_1^2 \operatorname{tg}^2 \varepsilon + x_2^2 + 2x_2 y_1 \operatorname{tg} \varepsilon;$$

hence

$$r^2(x, y) \leqq \sec^2 \varepsilon [y_1^2 - 2(x_1 \cos \varepsilon - x_2 \sin \varepsilon) \cos \varepsilon \, y_1 + r^2 \cos^2 \varepsilon];$$

substitution of

$$x_1 = r \cos \eta, \; x_2 = r \sin \eta, \; r^2 = x_1^2 + x_2^2$$

yields

$$\text{(7.14)} \qquad r^2(x, y) \leqq \sec^2 \varepsilon [(y_1 - rp)^2 + r^2 q^2],$$

where

$$\text{(7.14a)} \qquad p = \cos(\eta + \varepsilon) \cos \varepsilon, \quad q = \sin(\eta + \varepsilon) \cos \varepsilon.$$

By (7.4) and (7.14)

$$\text{(7.15)} \qquad \int_{\beta_0'} \frac{ds(y)}{r(x, y)} \geqq \int_0^{a'} \frac{dy_1}{r(x, y)} \geqq I(x) \cos \varepsilon,$$

(7.15a) $$I(x) = \int_0^{a'} [(y_1 - rp)^2 + r^2q^2]^{-\frac{1}{2}} dy_1 .$$

Now

(1°) $$I(x) = \log [y_1 - rp + \sqrt{(y_1-rp)^2 + r^2q^2}] \Big|_{y_1=0}^{a'} = \log \nu_1 - \log \nu_2$$

where, in view of (7.14a),

(2°) $$\nu_1 = a' - rp + \sqrt{(a'-rp)^2 + r^2q^2}, \quad \nu_2 = 2r \cos \varepsilon \sin^2\left(\frac{\eta+\varepsilon}{2}\right).$$

Since

$$|rp| \leqq a^0 < a' \quad \text{(suitable choice of } a^0\text{)} ,$$

it follows without difficulty that

$$0 < 2(a'-a^0) \leqq \nu_1 \leqq c^* ;$$

thus

(3°) $$|\log \nu_1| \leqq c^* .$$

Inasmuch as

$$\frac{3\varepsilon}{2} \leqq \frac{\eta+\varepsilon}{2} \leqq \frac{\pi}{2} - \frac{\varepsilon}{2},$$

one has

$$1 \leqq \sec \varepsilon \leqq \sec \varepsilon \csc^2\left(\frac{\eta+\varepsilon}{2}\right) \leqq \sec \varepsilon \csc^2\left(\frac{3\varepsilon}{2}\right) \leqq c^* \varepsilon^{-2} ;$$

thus by (1°), (2°), (3°)

$$I(x) = \log \frac{1}{r(o, x)} + I_0(x) ,$$

where

$$|I_0(x)| = \left| \log \frac{\nu_1}{2} + \log \left((\sec \varepsilon \csc^2 \frac{\eta+\varepsilon}{2} \right) \right| \leqq c^* + 2 \log \frac{1}{\varepsilon}.$$

By (7.15), (7.15a) we accordingly obtain

(7.16) $$\int_{\beta_0'} \frac{ds(y)}{r(x, y)} \geqq \cos \varepsilon \log \frac{1}{r(o, x)} \left[1 + I_0(x) \left(\log \frac{1}{r(o, x)} \right)^{-1} \right] \geqq \sigma \cos \varepsilon \log \frac{1}{r(o, x)},$$

where σ *is arbitrarily near unity for* a^0 (> 0) *suitably small.* There is a similar inequality corresponding to β_0''. Independent of the choice of the coordinates (y_1, y_2, y_3) *we infer that*

(7.17) $$\int_{\beta_0} \frac{ds(y)}{r(x, y)} \geqq 2\sigma_1 \log \frac{1}{r(c, x)}$$

for x exterior $N(c, 2\varepsilon)$, with $r(c, x) \leqq a^0$, where σ_1 is arbitrarily near unity by suitable choice of a^0 (possibly depending on ε). From (7.12) it is inferred that

(7.17a) $$\int_{\beta_0} \frac{ds(y)}{r(x,y)} \leqq 2\sigma_0 \log \frac{1}{r(c,x)},$$

where σ_0 has the properties assigned to σ_1 in (7.17).

With the aid of (7.1), (7.2), (7.3), (7.8), (7.12), (7.17), (7.17a) the following is established.

Theorem 7.18. *With c any point on the 'edges' β, the curvilinear potential $K(x)$ [(7.1), with $k(y)$ of Hölder class H_h] satisfies*

(7.18a) $$2\sigma_1 k(c) \log \frac{1}{r(c,x)} - \varepsilon^{-h-1} c^* \leqq K(x) \leqq 2\sigma_0 k(c) \log \frac{1}{r(c,x)} + \varepsilon^{-h-1} c^*$$

for x exterior $N(c; 2\varepsilon)$ (Definition 5.1), with

$$r(c,x) \leqq a^0 \quad (a^0 > 0),$$

where a^0 is suitably small; in the above $0 < \sigma_1 \leqq \sigma_0$ $(when\ k(c) > 0)$ and $0 < \sigma_0 \leqq \sigma_1$ $(when\ k(c) < 0)$ and σ_1, σ_0 may be taken as near as desired (but not necessarily equal to) unity by choosing a^0 suitably small (possibly depending on ε).

Recall the transformation $c = c_t$ of S on itself, as described subsequent (6.37), and the definition of S_δ (set of points t such that $l(t) \leqq \delta$). For t in S_δ, $c = c_t$ is on β and the tangent to β at $c = c_t$ is perpendicular to the rectilinear segment (c_t, t). We extend this segment till it meets the boundary of S_δ other than β; *let $\lambda[c]$ denote this segment* (all the points t of $\lambda[c]$ are in S_δ and transform into the end point c). As a consequence of the theorem *one has*

(7.19) $$-c^0 + 2\sigma_1 k(c_t) \log r^{-1}(c_t, t) \leqq K(t) \leqq c^0 + 2\sigma_0 k(c_t) \log r^{-1}(c_t, t)$$

for t on S_δ (not on β), with σ_1, σ_0 as in the theorem and c^0 a positive constant.

Problem 7.20. *To construct a function $\gamma(x)$, real and harmonic for x everywhere not on β, $\gamma(\infty) = 0$, with the properties*

(7.20a) $$-\gamma(t) - f(t) \log \frac{1}{r(c_t, t)} \leqq c^*,$$

(7.20b) $$\gamma(t) + f(t) \log \frac{1}{r(c_t, t)} \leqq c^* + \nu \log \frac{1}{r(c_t, t)} \quad (0 \leqq \nu < 1)$$

for t on S_δ (not on β), $f(t)$ being an assigned real function of a Hölder class on S_δ, δ being suitably small.

Now (7.20a), (7.20b) are equivalent to

(7.21) $$-c^* \leqq \gamma(t)+f(t) \log \frac{1}{r(c_t,\, t)} \leqq c^*+\nu \log \frac{1}{r(c_t,\, t)}$$

(t on S_δ, not on β). Consider (7.19), where $\gamma(t) = K(t)$ is of the form (7.1) with $k(y)$ as yet undetermined; add to each member in (7.19) the term $f(t) \log \frac{1}{r}$; one has

(7.22) $$-c^0+\bigl(2\sigma_1 k(c_t)+f(t)\bigr) \log \frac{1}{r(c_t,\, t)} \leqq \gamma(t)+f(t) \log \frac{1}{r(c_t,\, t)}$$
$$\leqq c^0+\bigl(2\sigma_0 k(c_t)+f(t)\bigr) \log \frac{1}{r(c_t,\, t)}$$

(t on S_δ). Inequalities (7.21) will be secured as a consequence of (7.22) if (for t on S_δ) one has

(7.23) $$2\sigma_1 k(c_t)+f(t) \geqq 0\ ,$$

(7.23a) $$2\sigma_0 k(c_t)+f(t) \leqq \nu\ .$$

Now one may take

(7.24) $$\sigma_1 = 1-\xi,\ \sigma_0 = 1+\xi \quad \bigl(\text{if } k(c) > 0\bigr)\ ,$$
$$\sigma_1 = 1+\xi,\ \sigma_0 = 1-\xi \quad \bigl(\text{if } k(c) < 0\bigr)\ ,$$

where $1 > \xi > 0$, is a constant that may be taken as small as desired. (c^0 in (7.19), (7.22) is possibly increasing as $\xi \to 0$).

It will be shown that (7.23), (7.23a) are satisfied and Problem 7.20 is accordingly solved with $\gamma(t) = K(t)$ (7.1), provided one constructs $k(c)$ in accordance with the following succession of steps:

(I). Take *any* $0 < \nu < 1$.

(II). Let $\xi(> 0)$, δ be taken so small that

(7.25) $$H(\xi,\, \delta) = \xi B+(1+\xi)h(\delta) \leqq \frac{\nu}{2}\,(1-\xi)\ ,$$

where B is the upper bound of $|f(c)|$ on β and $h(\delta)$ is from the inequality

(7.25a) $$|f(c_t)-f(t)| \leqq h(\delta)$$

(t is on the segment $\lambda[c_t]$ 'crossing' S_δ; $h(\delta)$ is independent of t and $\to 0$ with δ).

(III). Let j be a constant such that

(7.26) $$\frac{H(\xi,\, \delta)}{1-\xi} \leqq j \leqq \frac{\nu-H(\xi,\delta)}{1+\xi}.$$

(IV). *In* $\gamma(t) = K(t)$, *as defined by* (7.1), *put*

(7.27) $$k(c) = -\tfrac{1}{2}f(c)+\frac{j}{2} \quad (c \text{ on } \beta)\ .$$

We observe that (7.25) makes the inequalities (7.26) consistent. In view of (7.27), it is observed that (7.23), (7.23a) hold if

$$\sigma_1(j-f(c_t))+f(t) \geqq 0, \quad \sigma_0(j-f(c_t))+f(t) \leqq \nu .$$

On writing $f(c_t) = f(t)-q(t)$ one has

$$(7.28) \qquad |q(t)| \leqq h(\delta) \quad (\text{on } S_\delta) ;$$

accordingly we are to secure

$$(7.29) \qquad \omega_1+\sigma_1 j \geqq 0, \quad \omega_2+\sigma_0 j \leqq \nu ,$$

where

$$\omega_1 = (1-\sigma_1)f(t)+\sigma_1 q(t), \quad \omega_2 = (1-\sigma_0)f(t)+\sigma_0 q(t) ;$$

by (7.24), (7.28) and (7.25)

$$|\omega_1|, |\omega_2| \leqq H(\xi, \delta) \quad (\text{on } S_\delta) .$$

We now note that

$$\omega_1+\sigma_1 j \geqq -H(\xi, \delta)+(1-\xi)j ;$$

hence the first inequality (7.29) holds by virtue of the first part of (7.26); on the other hand,

$$\omega_2+\sigma_0 j \leqq H(\xi, \delta)+(1+\xi)j ;$$

thus the second inequality (7.29) will be at hand as a consequence of the last part of (7.26). Accordingly, (IV) gives the required solution.

8. Boundary problems. Using the notation of section 4, let (λ_t'), (λ_t'') be asigned directions (nontangential to S) at t, defined by the direction cosines

$$(8.1) \qquad \lambda_j'(t), \ \lambda_j''(t) ,$$

respectively; these functions are to be of a Hölder class on S, edges included. The corresponding lines extending from t will be designated by L_t', L_t''; also we let

$$(8.1\text{a}) \qquad \vartheta'(t) = \text{angle between the directions } (+n_t), (\lambda_t')$$

$\big($similar definition for $\vartheta''(t)\big)$;

$$(8.2) \qquad 0 \leqq \vartheta'(t) < \frac{\pi}{2} ; \qquad \frac{\pi}{2} < \vartheta''(t) \leqq \pi .$$

Let $\varphi'(t)$, $\varphi''(t)$ be the angles corresponding to the angle $\varphi(t)$ $\big($cf. text after (4.2a)$\big)$. Designate by $K'(t)$, $K''(t)$ the functions $K(t)$ (4.22) corresponding to the directions (λ_t'), (λ_t''). Generally, the primes and double primes will relate to the directions (λ_t'), (λ_t'').

Notation 8.3. *Given any function* $A(x)$, *defined for* x *in* $C(S)$, *we write*

$$A^{(')}(t) = \lim A(x) \text{ (as } x, \text{ on } L'_t, \to t); \; A^{('')}(t) = \lim A(x) \text{ (as } x, \text{ on } L''_t, \to t),$$

provided of course the limits exist.

Use will be made of the formulas (4.30), (4.31), valid for $\Psi(x)$ of (4.28a) at points for which $K'(t) \neq K''(t)$,

$$\Psi^{(')}(t) = q(t)K'(t) + \Psi(t), \quad \Psi^{('')}(t) = q(t)K''(t) + \Psi(t); \tag{8.4}$$

$$q(t) = \alpha(t)[\Psi^{(')}(t) - \Psi^{('')}(t)], \; \alpha(t) = \frac{1}{K'(t) - K''(t)}; \tag{8.4a}$$

$$\Psi(t) = \alpha_1(t)\Psi^{(')}(t) + \alpha_2(t)\Psi^{('')}(t), \; \alpha_1 = -K''\alpha, \; \alpha_2 = K'\alpha. \tag{8.4b}$$

We shall now proceed to abtain classes of solutions of the *Hilbert-Riemann boundary problems*

$$\Phi^{(')}(t) = A(t)\Phi^{('')}(t), \tag{8.5}$$

$$\Phi^{(')}(t) = A(t)\Phi^{('')}(t) + B(t), \tag{8.6}$$

where $A(t) \neq 0$ on S, $B(t)$ are functions of Hölder class, assigned on S. Further hypotheses will be introduced in the sequel.

We shall first proceed heuristically. Let

$$\Phi_1(x) = \exp. V(x), \quad V(x) = \int_S \frac{k(y, x)}{r^2(y, x)} \mu(y) d\sigma(y), \tag{8.7}$$

where $\mu(y)$ is to be determined so that $\Phi_1(x)$ satisfies (8.5). One has

$$V^{(')} = \mu K' + V, \quad V^{('')} = \mu K'' + V \quad (\text{on } S);$$

thus

$$\Phi_1^{(')} = e^V \exp.(\mu K'), \quad \Phi_1^{('')} = e^V \exp.(\mu K'')$$

and we should have

$$A(t) = \exp.[(K' - K'')\mu];$$

that is,

$$\mu(t) = \alpha(t) \log A(t). \tag{8.7a}$$

The function

$$\Phi_0(x) = e^{\gamma(x)}\Phi_1(x), \tag{8.7b}$$

where $\gamma(x)$ is a curvilinear potential (as yet undefined) of a density distributed along β, as in (7.1), will also satisfy (8.5); we note that $\gamma^{(')}(t) = \gamma^{('')}(t)$. The non-homogeneous problem (8.6) can be solved on making the substitution

$$\Phi(x) = \Phi_0(x)\Psi(x). \tag{8.8}$$

We have

$$\Phi_0^{(')}\Psi^{(')} = A\Phi_0^{('')}\Psi^{('')}+B = \Phi_0^{(')}\Psi^{('')}+B\,;$$

thus

(8.8a) $$q(t) \equiv \alpha(t)B(t)[\Phi_0^{(')}(t)]^{-1} = \alpha(t)[\Psi^{(')}(t)-\Psi^{('')}(t)]\,;$$

in form this is identical with (8.4a). Whence a solution of (8.8a) is given by

(8.8b) $$\Psi(x) = \int_S \frac{k(y, x)}{r^2(y, x)} q(y)d\sigma(y)\,.$$

Since $\alpha K' = \alpha_2$, by (8.7b), (8.7a), we obtain

(8.8c) $$q(t) = \alpha(t)B(t)A^{-\alpha_2(t)}(t) \text{ exp. } [-\gamma(t)-V(t)]\,.$$

The above considerations indicate that it is desirable that $\alpha(t)$ be finite everywhere on S, except possibly at the edges; that is, *we should obtain conditions under which one can find a function* $k_0(t)$ *so that*

(8.9) $$|K'(t)-K''(t)| \geqq k_0(t) > 0 \quad \text{(edges possibly excluded)}\,.$$

Secondly, inasmuch as use is made of the principal value $V(t)$ of the integral $V(x)$ (8.7), *we are led to require that* A *be such that*

(8.10) $$\mu(y) = \alpha(y) \log A(y) \subset [\eta|S] \quad (\text{some } \eta;\ \eta+\beta < 1)\,.$$

Here β is from (3.27) (hypothesis (3.27) being assumed in place of the condition ($3.25b^1$) of the Theorem).

Thirdly, since some of the above considerations indicate that the principal value $\Psi(t)$ of $\Psi(x)$ (8.8b) should exist, *we should have*

(8.11) $$q(y)\ (8.8c) \subset [\alpha|S] \quad (\text{some } \alpha;\ \alpha+\beta < 1)\,.$$

Definition 8.12. *Suppose* $\alpha(t)$ (8.4a) *is finite on* S (*edges possibly excluded*). *We shall designate by* (A^*) *the class of functions* A (*nonvanishing on* S) *such that* (8.10) *holds. Given a particular* $A(t) \subset (A^*)$, *let* (B^*, A) *denote the class of functions* $B(t)$ *such that*

(8.12a) $$q(t)\{= \alpha(t)B(t)A^{-\alpha_2(t)}(t) \text{ exp. } [-\gamma(t)-V(t)]\} \subset [\alpha|S]$$

for some α *such that* $\alpha+\beta < 1$. *Here* γ *is a fixed potential of form* (7.1).

With $K'-K'' \neq 0$, it is fairly easy to determine whether a function $A(t) \subset (A^*)$. With $A(t)$ denoting any particular function $\subset (A^*)$, the determination of whether $B(t) \subset (B^*, A)$ is more involved, but can be carried out (for instance with $\gamma(t) = 0$) by ascertaining with the aid af Theorems 6.36, 6.38, the behaviour of the principal integral $V(t)$ near the edges and by examining the expression for $q(t)$ in (8.12a).

When (8.9) *holds and* $A(t) \subset (A^*)$ *and* $B(t) \subset (B^*, A)$, *the heuristic process described from* (8.7) *to* (8.8b) *is rendered rigorous and we have on hand a class of solutions of the Hilbert-Riemann boundary problems* (8.5), (8.6). The behaviour of these solutions, that is the possible orders of infinity of these solutions for x (in $C(S)$) near the edges of S, can be ascertained with the aid of Theorem 5.38.

Relating to the question of (8.9) we have the following.

Lemma 8.13. *Suppose the* $\gamma_{i_1\dots i_m}(t)$, *for* $m = 1$, *satisfy*

(8.13a) $$\sum_1^3 \gamma_i(t) n_i(t) \geqq a_0 > b_0 = \tfrac{1}{2} \sum_{m=2}^{\infty} 3^{2m}(6m+7)c_m \quad (\textit{all } t \textit{ on } S)$$

or

(8.13b) $$\sum_1^3 \gamma_i(t) n_i(t) \leqq -a_0 < -b_0 \quad (\textit{all } t \textit{ on } S)\,.$$

Let $K'(t)$ *be the function* $K(t)$ (4.22), *corresponding to the approach along the positive normal. Then*

(8.13c) $$K'(t) \leqq -2\pi(a_0 - b_0) < 0 \quad (\textit{case } (8.13\text{a}));$$
$$K'(t) \geqq 2\pi(a_0 - b_0) > 0 \quad (\textit{case } (8.13\text{b}))\,.$$

For the purposes of the proof the prime will be deleted. We have

(1°) $$K(t) = K_1(t) + K_2(t)\,,$$

where $K_1(t)$ is the part of $K(t)$ arising from the $\gamma_i(y)$, while $K_2(t)$ is the part arising from the $\gamma_{i_1\dots i_m}$ $(m > 1)$. In view of (4.34), (4.34a)

$$K_1(t) = \sum_{s=1}^{2} C_s \Gamma_s(t) + C_{0:0:1}\Gamma_{0:0:1}(t)\,;$$

here $\Gamma_{0:0:1} = \Gamma_3$ and

$$C_1 = \int_0^\infty \left[\tau^2(1+\tau^2)^{-\frac{3}{2}} - \frac{\lambda(\tau)}{\tau}\right] d\tau \int_0^{2\pi} \cos\theta\, d\theta = 0\,;$$

$$C_2 = \int_0^\infty [\text{as above}]\; d\tau \int_0^{2\pi} \sin\theta\, d\theta = 0\,;$$

$$C_{0:0:1} = \int_0^\infty -\tau(1+\tau^2)^{-\frac{3}{2}} d\tau \int_0^{2\pi} d\theta = -2\pi\,.$$

Recalling (3.6a), we obtain

(2°) $$K_1(t) = -2\pi \sum_1^3 \gamma_i(t) n_i(t) \leqq -2\pi a_0 \quad \text{or} \geqq 2\pi a_0\,.$$

To $K_2(t)$ Lemma (4.26) can be applied, with $\vartheta(t) = 0$ and the $\gamma_{i_1\dots i_m}(y)$ for $m = 1$

omitted. Thus, replacing c_1 (in (3.20a)) by zero one obtains

$$(3^\circ) \qquad |K_2(t)| \leqq \pi \sum_{m=2}^{\infty} 3^{2m}(6m+7)c_m = 2\pi b_0\,.$$

The Lemma ensues by (1°), (2°), (3°).

Lemma 8.14. *Let $K'(t)$, $K''(t)$ be the functions $K(t)$* (4.22), *corresponding to the approaches along the positive and negative normals, respectively. If* (8.13a) *or* (8.13b) *holds, one has*

$$(8.14\text{a}) \qquad K'(t)-K''(t) \leqq -4\pi(a_0-b_0) < 0 \quad \big(\textit{case } (8.13\text{a})\big)\,;$$

$$K'(t)-K''(t) \geqq 4\pi(a_0-b_0) > 0 \ \big(\textit{case } (8.13\text{b})\big);\ |\alpha(t)| \leqq c^* \ (\textit{all } t \textit{ on } S)\,.$$

This is established by noting that, by (4.33) (with $\vartheta(t) = 0$), $K'(t)-K''(t)$ is double the expression for $K'(t)$, with the $\gamma_{i_1\ldots i_m}(y)$ for m even deleted, and by utilizing Lemma 8.13. *The above result can be generalized to fairly general situations, still obtaining $|\alpha(t)| < c^*$, as follows:*

I. When $\gamma_{i_1\ldots i_m}(y) = 0$ for $m = 1, 2, \ldots, 2\mu$, but not all the $\gamma_{i_1\ldots i_m}(y)$ for $m = 2\mu+1$ are zero, assume conditions analogous to (8.13a), (8.13b) for the $\gamma_{i_1\ldots i_m}(y)$ with $m = 2\mu+1$.

II. After making an extension of Lemma 8.14 on the basis of I, allow approaches to t not along the positive and negative normals, respectively, but require these approaches to be suitably near to approaches along opposite normals; more precisely, in this extension, assume that $\vartheta'(t)$ it near 0, while $\vartheta''(t)$ is near π and $\varphi''(t)$ is near $\varphi'(t)+\pi$ (cf. the text after (4.31)).

We shall omit the details of such extensions.

In the rest of this section it will be assumed, on the basis of Lemmas 8.13, 8.14 *and extensions* (I), (II), *that there is a following situation on hand:*

(8.15) *$\alpha(t)$ maintains sign on S; $|\alpha(t)| \leqq c^*$; $\alpha_1(t)$, $\alpha_2(t)$, $K'(t)$, $K''(t)$*

maintain signs. With the $\gamma_{i_1\ldots i_m}(y)$, the $n_j(t)$, $\vartheta'(t)$, $\varphi'(t)$, $\vartheta''(t)$, $\varphi''(t)$ uniformly of a Hölder class on S, edges included, we shall have

$$(8.15\text{a}) \qquad \alpha(t),\ K'(t),\ \alpha_1(t),\ \alpha_2(t)$$

uniformly of a Hölder class, edges included. The assumption with respect to the $\gamma_{i_1\ldots i_m}(y)$ means that β of (3.27) is 0.

Turning now to the italics subsequent (8.12a), we are now able to replace the definitions of classes (A^*), (B^*, A) by simpler ones as follows.

Class (A^*). This is the class of functions $A(t)$ such that

$$\log A(t) \subset [\eta|S] \quad (\text{some } \eta;\ 0 \leqq \eta < 1) \tag{8.16}$$

and (8.16b) holds.

Class (B^*, A). Let A be a particular function of class (A^*); (B^*, A) is the class of functions $B(t)$ such that

$$B(t)A^{-\alpha_2(t)}(t) \exp.[-\gamma(t)-V(t)] \subset [\alpha|S] \quad \big(\text{cf. } (8.7),\ (8.7\text{a})\big) \tag{8.16a}$$

for some $\alpha < 1$ $\big(\gamma(t)$ is a curvilinear potential at our disposal$\big)$.

Is is observed that $\mu(y) = \alpha(y) \log A(y)$ will be $[\eta|S]$; furthermore (with some ν),

$$|\mu(y)-\mu(t)| \leqq \mu(y, t)r^{\nu}(y, t) \ (0 < \nu \leqq 1); \quad \mu(y, t) < c^* l^{-\alpha_0}(y) \ \big(l(y) \leqq l(t)\big); \tag{8.16b}$$
$$\mu(y, t) < c^* l^{-\alpha_0}(t) \ \big(l(y) \geqq l(t)\big); \textit{ some } \alpha_0 \ (\geqq \eta) \textit{ such that } \alpha_0-\nu < 1.$$

Applying Theorem 6.36 with $\mu(y)$ in place of $q(y)$ and α, β replaced by η, 0, respectively, we obtain

$$\begin{aligned} |V(t)| &< c^* L^{\lambda}(t) \quad \big(\text{if } h < \eta;\ \lambda = \max.(\eta-h, \eta, \alpha_0-\nu)\big), \\ &< c^* L^{\lambda}(t) \quad \big(\text{if } h = \eta \text{ and } \lambda = \max.(0, \eta, \alpha_0;\nu) > 0\big), \\ &< c^* L^{\lambda}(t) \log L(t) \quad \big(\text{if } h = \eta \text{ and } \lambda \text{ (above)} = 0\big), \\ &< c^* L^{\lambda}(t) \quad \big(\text{if } 0 < \eta < h \leqq 1;\ \lambda = \max.(0, \eta, \alpha_0-\nu)\big), \\ &< c^* L^{\lambda}(t) \quad \big(\text{if } 0 = \eta < h \leqq 1 \text{ and } \lambda = \max.(0, \alpha_0-\nu) > 0\big), \\ &< c^* \log L(t) \ \big(\text{if } 0 = \eta < h \leqq 1 \text{ and } \lambda \text{ (above)} = 0\big). \end{aligned} \tag{8.17}$$

In the above h is the Hölder exponent for the $\gamma_{i_1 \ldots i_m}(y)$; since $h > 0$ and the third inequality cannot occur unless $\eta = 0$, this inequality could not possibly take place, as stated.

With $A(t)$ denoting some particular function $\subset (A^*)$ (8.16), the corresponding principal integral $V(t)$ [(8.7), (8.7a)] is of form

$$V(t) = v(t)\varrho\big(L(t)\big), \tag{8.18}$$

where $\varrho\big(L(t)\big)$ is one of the functions of $L(t)$ (depending on the case) in (8.17) and $|v(t)| \leqq c^*$, while $v(t)$ is of a Hölder class for $l(t) > 0$. However, in general, there is no assurance that $V(t)$ is uniformly of a Hölder class, edges included. *Under these circumstances the problem of determination of whether $B(t) \subset (B^*, A)$ $\big($with $\gamma(t) = 0\big)$ is that of finding whether (near edges)*

$$B(t)A^{-\alpha_2(t)}(t) \exp.[-v(t)\varrho\big(L(t)\big)] \subset [\alpha|S] \tag{8.18a}$$

for some $\alpha < 1$ $\big(\alpha_2(t)$ satisfies (8.15), (8.15a)$\big)$; this can be carried out without much diffuculty. We therefore may state the following.

Theorem 8.19. *Suppose* (8.15), (8.15a) *have been secured. Then the heuristic process, from* (8.7) *to* (8.8b) *is rendered rigorous, when* $\log A(t)$ *is* $[\eta|S]$ (*some* $\eta < 1$), *for all* $B(t)$ *such that* (8.18a) *holds* (*with* $\gamma = 0$) *for some* $\alpha < 1$; (8.16b) *assumed.*

Let us consider the following case.

(I). S is completely regular (section 2);

(II). The $\gamma_{i_1\ldots i_m}(t)$, $\vartheta'(t)$, $\varphi'(t)$, $\vartheta''(t)$, $\varphi''(t) \subset (u)$ Lip 1 (*that is, are uniformly of class* H_1, *edges included*);

(III). $$A(t) \subset (u.)\ \mathrm{Lip}\ 1. \quad \big(A(t) \neq 0 \text{ on } S\big).$$

In view of (II), $\alpha(t)$ will be $(u.)$ Lip 1; whence, as a consequence of (III), we shall have

$$\mu(t) = \alpha(t) \log A(t) \subset (u.)\ \mathrm{Lip}\ 1 \quad \big(\mu(t) \subset [0|S]\big);$$

in (8.16) one will have $\eta = 0$ and in (8.16b): $\nu = 1$, $\alpha_0 = 0$. Accordingly Theorem 6.38 will apply to $\mu(y)$; we have

$$V(t) = V^*(t) + \mu(c_t) v_0(t) \log \frac{1}{r(c_t, t)} \quad \text{(near edges)}, \tag{8.20}$$

where $V^*(t)$, $v_0(t)$ *are uniformly of a Hölder class, say* H_p $(0 < p \leqq 1)$, *edges included.* Furthermore, since $K'(t)$ is $(u.)$ Lip 1,

$$A^{-\alpha_2(t)}(t) = \exp.\,[-K'(t)\mu(t)] \subset (u.)\ \mathrm{Lip}\ 1\,. \tag{8.20a}$$

Since $\mu(y)$ may be complex valued (when $A(y)$ assumes negative values); $v_0(t)$ is independent of μ and is real; write

$$\mu(c_t) v_0(t) = v_1(t) + v_2(t)\sqrt{-1} \quad \big(v_1(t),\ v_2(t) \text{ real}\big)\,. \tag{8.21}$$

Construct a function $\gamma(x)$, real and harmonic for x not on edges, zero at infinity, with the properties:

$$(1^\circ)\quad -\gamma(t) + v_1(t) \log r(c_t, t) \leqq c^*\,, \tag{8.22}$$

$$\gamma(t) - v_1(t) \log r(c_t, t) \leqq c^* + \sigma \log \left(\frac{1}{r(c_t, t)}\right) \tag{2°}$$

$(0 \leqq \sigma < 1)$ for t on S_δ (that is, for $\delta \geqq l(t) > 0$, with $\delta (> 0)$ small).

Such a function $\gamma(x)$ can be actually obtained in the form of a curvilinear potential (7.1) ($K(x)$ of section 7 not to be confused with $K(x)$ of Theorem 4.28), extended over edges β in accordance with the scheme used in solving the Problem 7.20 (on the basis of Theorem 7.18); we just replace $f(t)$ of Problem 7.20 by $v_1(t)$ and ν by σ. In all cases $\gamma(x)$ can be so chosen so that σ (if not $= 0$) *is as small as desired.*

When $|A(y)| = 1$ *we have* $v_1(t) = 0$; *one then may take* $\gamma(x) = 0$, $\sigma = 0$.

As a consequence of (8.20), (8.21) and (8.22)

$$|\exp. [-\gamma(t) - V(t)]| \leqq c^* \exp. [-\gamma(t) + v_1(t) \log r(c_t, t)] \leqq c^* , \tag{8.23}$$

$|\exp. [\gamma(t) + V(t)]| \leqq c^* r^{-\alpha}(c_t, t)$; $\exp. [-\gamma(t) - V(t)]$ is of a Hölder class for $l(t) > 0$.

By (8.20a) and the above

$$A^{-\alpha_2(t)}(t) \exp. [-\gamma(t) - V(t)] \subset [0|S] .$$

By virtue of the statement with respect to (8.16a) it is observed that all functions

$$B(t) \subset [\alpha|S] \quad (\text{with } \alpha < 1) \tag{8.24}$$

belong to the class (B^*, A), for all $A(t) \subset (u.)$ Lip 1 (the same is true under certain more general conditions). The following has thus been established.

Theorem 8.25. *Assume the situation as described in connection with* (8.15), (8.15a); *suppose* S *is completely regular and that*

$$\gamma_{i_1 \ldots i_m}(t),\ \vartheta'(t),\ \varphi'(t),\ \vartheta''(t),\ \varphi''(t) \subset (u.) \text{ Lip } 1 ; \tag{8.25a}$$

then the heuristic process, from (8.7) *to* (8.8b), *for solving the Hilbert-Riemann boundary problems* (8.5), (8.6) *is rendered rigorous for all* $A(t) \subset (u.)$ Lip 1 $\big(A(t) \neq 0$ *on* $S\big)$ *and all* $B(t) \subset [\alpha|S]$ (*with* $\alpha < 1$), *with* $\gamma(t)$ *chosen in accordance with* (8.21), (8.22).

Suppose we obtained solutions in accordance with the theorem 8.19. The homogeneous problem is solved by

$$\Phi_0(x) = \Phi_1(x) = \exp. \left[\int_S \frac{k(y, x)}{r^2(y, x)} \mu(y) d\sigma(y) \right] \quad \big(\mu(y) = \alpha(y) \log A(y)\big) . \tag{8.26}$$

To study this solution for x (not on S) near edges β of S we apply Theorem 5.38. In view of the hypotheses involved in theorem 8.19

$$\alpha(y) \log A(y) \subset [\eta|S] \quad (0 \leqq \eta < 1) .$$

It is inferred that, with c denoting any point on β, one has

$$\left| \int_S \frac{k(y, x)}{r^2(y, x)} \mu(y) d\sigma(y) \right| < c^* k_1(\varepsilon) r^{-\eta}(c, x) \ (\text{if } \eta > 0), < c^* k_1(\varepsilon) \log \frac{1}{r(c, x)} \quad (\text{if } \eta = 0)$$

$\big(k_1(\varepsilon)$ from (5.38)$\big)$ for x near c, exterior $N(c, \varepsilon) + W(c, \varepsilon)$ (Definition 5.1). The integral in (8.26) can therefore be expressed in the form

$$v(c, x) r^{-\eta}(c, x) \quad (\text{if } \eta > 0),\ v(c, x) \log \frac{1}{r(c, x)} \quad (\text{if } \eta = 0) ,$$

where $|v(c, x)| < c^* k_1(\varepsilon)$; thus

(8.26a) $$\Phi_0(x) = \exp.\,[v(c, x) r^{-\eta}(c, x)] \quad (\text{if } \eta > 0)\,,$$

$$\Phi_0(x) = r(c, x)^{-v(c,x)} \quad \big(\text{if } \eta = 0;\ x \text{ near } c, \text{ exterior } N(c, \varepsilon) + W(c, \varepsilon)\big)\,.$$

With $B(t)$ such that (8.18a) holds, the function $q(t)$ [(8.12a), with $\gamma(t) = 0$] will be in $[\alpha|S]$ ($\alpha < 1$); by Theorem 5.38

$$|\Psi(x)| = \left| \int_S \frac{k(y, x)}{r^2(y, x)} q(y) d\sigma(y) \right| < c^* k_1(\varepsilon) r^{-\alpha}(c, x) \quad (\text{if } \alpha > 0)\,,$$

$$< c^* k_1(\varepsilon) \log \frac{1}{r(c, x)} \quad (\text{if } \alpha = 0);\ x \text{ exterior } N(c, \varepsilon) + W(c, \varepsilon)\,;$$

hence

(8.26b) $$\Psi(x) = u(c, x) r^{-\alpha}(c, x) \quad (\text{if } \alpha > 0), = u(c, x) \log \frac{1}{r(c, x)} \quad (\text{if } \alpha = 0)\,,$$

where $|u(c, x)| < c^* k_1(\varepsilon)$ $\big($exterior $N(c, \varepsilon) + W(c, \varepsilon)\big)$. On taking note of (8.8) and of the preceding, the following is concluded.

Theorem 8.27. *When solutions of* (8.5), (8.6) *are obtained in accordance with theorem* 8.19, *the solution* $\Phi(x) = \Phi_0(x) \Psi(x)$ (8.8) *of the nonhomogeneous problem has the forms*:

(8.27a) $$u(c, x) r^{-\alpha}(c, x) \exp.\,[v(c, x) r^{-\eta}(c, x)] \quad (\textit{if } \alpha > 0,\ \eta > 0)\,;$$

(8.27b) $$u(c, x) r^{-\alpha}(c, x) r(c, x)^{-v(c,x)} \quad (\textit{if } \alpha > 0,\ \eta = 0)\,;$$

(8.27c) $$u(c, x) \log \frac{1}{r(c, x)} \exp.\,[v(c, x) r^{-\eta}(c, x)] \quad (\textit{if } \alpha = 0,\ \eta > 0)\,;$$

(8.27d) $$u(c, x) \log \frac{1}{r(c, x)} r(c, x)^{-v(c,x)} \quad (\textit{if } \alpha = 0, \eta = 0)\,;$$

the above is asserted for x near any 'edge' point c, exterior $N(c, \varepsilon) + W(c, \varepsilon)$; the functions $u(c, x)$, $v(c, x)$ have bounded absolute values (the bounds may depend on ε).

In any actual case, in applying the above result, supplementary more precise information can be obtained by determining the numerical sign of the real part of $v(c, x)$ $\big(v(c, x)$ is defined by μ and may therefore be complex valued$\big)$. We will not go any further into this.

Proceeding on the basis of theorem 8.25, it is noted that a solution of (8.5) is given by

(8.28) $$\Phi_0(x) = e^{\gamma(x)} \exp.\, V(x) \quad \big(\gamma(x) \text{ as in } (8.22)\big)\,.$$

As noted preceding (8.20), $\mu(t) \subset (u.)$ Lip 1 and is $[0|S]$; hence in view of Theorem 5.38 $\big($where $q(y)$, α are replaced by $\mu(y), 0\big)$

$$V(x) = v(c, x) \log \frac{1}{r(c, x)}, \quad |v(c, x)| < c^* k_1(\varepsilon) .$$

Now $\gamma(x)$ is defined by a potential (7.1) (so that (8.22) holds). By Theorem 7.18

$$2\sigma_1 k(c) \log \frac{1}{r(c, x)} - \varepsilon^{-h-1} c^* \leqq \gamma(x) \leqq 2\sigma_0 k(c) \log \frac{1}{r(c, x)} + \varepsilon^{-h-1} c^* \quad (\text{exterior } N(c; \varepsilon))$$

(h here is the Hölder exponent of $k(y)$ of (7.1)), where σ_1, σ_0 are certain positive numbers, as stated in the theorem. Hence, near c,

$$\Phi_0(x) = r(c, x)^{-v_2(c,x)}; \quad |v_2(c, x)| < c^* k_2(\varepsilon) , \tag{8.28a}$$

where $k_2(\varepsilon)$ is a certain function of ε which may tend to ∞, as $\varepsilon \to 0$; these inequalities can be made sharper, utilizing the special construction of $\gamma(x)$. We shall not linger on this point. With $B(t) \subset [\alpha|S]$ $(\alpha < 1)$, $\alpha(t) \subset (u.)$ Lip 1 (cf. the text preceding (8.20)), $A^{-\alpha_2(t)}(t) \subset (u.)$ Lip 1 (8.20a), in view of (8.23) we infer

$$q(t) = \alpha(t) B(t) A^{-\alpha_2(t)}(t) \exp. [-\gamma - V] \subset [\alpha|S] ;$$

hence (8.26b) holds again for $\Psi(x)$. Now $\Phi(x) = \Phi_0(x)\Psi(x)$ is a solution of (8.6). Therefore the following can be stated.

Theorem 8.29. *When solutions of* (8.5), (8.6) *are obtained in accordance with theorem* 8.25, *the solution* $\Phi(x)$ (8.8) (*with* γ *defined as in* (8.22)) *of the nonhomogeneous problem has the forms*

$$u(c, x) r(c, x)^{-v_1(c,x)-\alpha} \quad (\text{if } \alpha > 0) , \tag{8.29a}$$

$$u(c, x) r(c, x)^{-v_2(c,x)} \log \frac{1}{r(c, x)} \quad (\text{if } \alpha = 0) \tag{8.29b}$$

for x *near any 'edge' point* c, *exterior* $N(c, \varepsilon) + W(c, \varepsilon)$; $|u(c, x)|$, $|v_2(c, x)|$ *have bounds finite for* $\varepsilon > 0$.

9. Singular operators. In the remaining sections we shall study integral equations, involving operators of type

$$a(t)u(t) + \int_S \frac{k(y, t)}{r^2(y, t)} u(y) d\sigma(y) [= A_t(u)] ; \tag{9.1}$$

here $k(y, t) r^{-2}(y, t)$ (3.1) is a principal kernel as described in section 3, while $a(t)$ is of a Hölder class on S (for $l(t) > 0$), $a(t) \neq 0$ (for $l(t) > 0$). *As remarked before, the essentially novel feature (and one involving substantial new difficulties) of our present developments, in so far as integral equations are concerned, consists in the possible* *presence of edges β in the manifold (surfaces) S.* The latter fact necessitates care regarding orders of infinity near β.

We shall proceed under the conditions of Theorem 3.25, with $\gamma(y, t)$ satisfying (3.27). In order that the integral in (9.1) should exist in the sense of principal values, in view of the considerations of section 6 we are led to require that

$$(9.2) \qquad u(t) \subset [\alpha|S] \quad \big(0 \leqq \alpha < 1;\ \alpha+\beta < 1;\ \beta \text{ from } (3.27)\big)\,.$$

Use will be made of a number of formulas of section 6, with $q(y)$ replaced by $u(y)$. Let c be a point on the edges β. Suppose the y system has its origin at c; thus $c = 0$. This hypothesis is not essential. We have

$$(9.3) \qquad \Psi(t) \equiv \int_S k(y, t) r^{-2}(y, t) u(y) d\sigma(y) = \int_{\sigma'} k(y, t) r^{-2}(y, t) u(y) d\sigma(y) + \Phi(t)$$

$[\sigma' = S - S(o, a);\ a, > 0, \text{ small}]$, where $\big(\text{by } (6.3)\big)$

$$(9.3\text{a}) \qquad \left| \int_{\sigma'} k(y, t) r^{-2}(y, t) u(y) d\sigma(y) \right| < c^*$$

and

$$\Phi(t) = \int_{S(o,a)} k(y, t) r^{-2}(y, t) u(y) d\sigma(y) = \Phi''(t) + \Phi'(t)$$

$\big($cf. (6.4), (6.4a), (6.4b)$\big)$; here

$$(9.3\text{b}) \qquad |\Phi''(t)| = \left| \int_{S(o,a)} k''(t|y, t) r^{-2}(y, t) u(y) d\sigma(y) \right| < c^* L^{\beta+\alpha-h}(t) \quad (h < \alpha)\,,$$

$$< c^* L^{\beta}(t) \ (\alpha < h \leqq 1), \quad < c^* L^{\beta}(t) \log L(t) \ (h = \alpha)\,; \quad \big(\text{cf. } (6.19)\big)\,;$$

$$[L(t) = \big(\varepsilon r(o, t)\big)^{-1}; \quad t \text{ in } S\left(o, \frac{a}{2}\right) \text{ exterior cones } N(o, \varepsilon)]\,;$$

further $\big($by (6.20), (6.21)$\big)$,

$$(9.3\text{c}) \qquad \Phi'(t) = \int_{S(o,a)} k'(t|y, t) r^{-2}(y, t) u(y) d\sigma(y) = \Phi_b^{1,0}(t) + \Phi_b'(t)\,;$$

here $\big($by (6.22), (6.22a)$\big)$

$$(9.3\text{d}) \qquad |\Phi_b^{1,0}(t)| = \left| \int_s k'(t|y, t) r^{-2}(y, t) u(y) d\sigma(y) \right| < c^* L^{\alpha}(t) \quad (\text{if } \alpha > 0)\,,$$

$$< c^* \log L(t) \quad (\text{if } \alpha = 0) \quad [s = S(o, a) - S_{t,b}]\,,$$

where $b = c_0 \varepsilon r(o, t)$ with $c_0, > 0$, suitably small (independent of t), in accordance with the text subsequent (6.20). At this stage *introduce the orthogonal transformation* (3.5), *going from the y system to the Y system, the origin O of the latter being at t, as described preceding* (6.24). We then have

(9.4) $$\Phi_b'(t) = \int_{S(O,\, b)} \dot{k}'(Y, O) r^{-2}(Y, O) u(Y) d\sigma(Y) ,$$

where $u(Y) = u(y)$ and $k'(Y, O)$ is as in (6.24). Introduce now polar coordinates

$$Y_i = \varrho \cos\theta_i \quad (i = 1, 2), \quad \theta_1 = \theta,\ \theta_2 = \frac{\pi}{2} - \theta .$$

One then has (cf. (2°) after (6.24))

(9.4a) $$k'(Y, O) = f(t, \theta) + k_t^{1,0}(\varrho, \theta) ,$$

where $f(t, \theta)$ is written for $k^{1,*}(t, \theta)$ of the preceding sections;

(9.4b) $$f(t, \theta) = \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{2} \Gamma_{s_1 \ldots s_m}(t) \cos\theta_{s_1} \ldots \cos\theta_{s_m} \quad \text{(cf. (3.11a))} .$$

The function $f(t, \theta)$ *is the characteristic of the kernel in* (9.1).

It will be necessary to modify the procedure that led from (6.24) to (6.25). In the expression for $\Phi_b'(t)$ (6.24′) replace $u(O) + \nu_2$ by $u(Y)$; thus

$$\Phi_b'(t) = \iint \big(f(t, \theta) + k_t^{1,0}(\varrho, \theta)\big)\big(1 + \nu_1(\varrho, \theta)\big) u(Y) \frac{d\varrho}{\varrho} d\theta$$

($0 \leqq \varrho \leqq b$; $0 \leqq \theta \leqq \pi$). One has

(9.5) $$\Phi_b'(t) = \Psi^*(t) + \Psi(t) ;$$

here

(9.5a) $$\Psi^*(t) = \int_0^b \int_0^{2\pi} f(t, \theta) u(Y) \frac{d\varrho}{\varrho} d\theta = \int_{S(O,\, b)} \frac{f(t, \theta)}{r^2(O, Y)} u(Y) dY_1 dY_2$$

$\big(Y = (Y_1, Y_2, 0)\big)$ is a principal integral and

(9.5b) $$\Psi_1(t) = \int_{S(O,\, b)} \int [k'(Y, O)\nu_1(\varrho, \theta) + k_t^{1,0}(\varrho, \theta)] u(Y) \frac{d\varrho}{\varrho} d\theta .$$

The integral $\Psi^*(t)$ (9.5a) *will be termed the characteristic part of the principal integral* $\Psi(t)$ (9.3); $\Psi^*(t)$ is defined for $l(t) > 0$; when $l(t) \leqq \delta_0$ (small fixed δ_0, > 0), b in $S(O, b)$ is taken as $c_0 \varepsilon r(o, t)$ (c_0, > 0, small), as stated before; when $l(t) > \delta_0$, b (> 0) can be defined as a fixed suitably small constant.

Inasmuch as (9.5)–(9.5b) differs from (6.25) merely in the grouping of the various terms, from the text leading from (6.25) to (6.32a) it is easily seen that $\Psi^*(t)$, $\Psi_1(t)$ satisfy inequalities of the same form as $\Phi_b'(t)$; thus

(9.6) $$|\Psi^*(t)|,\ |\Psi_1(t)| < c^* L(t)^{\alpha_0 - \nu} ;$$

(9.6a) $$|\Psi^*(t)|,\ |\Psi_1(t)| < c^* L^{\alpha}(t) \quad \text{(in the case (6.27b), for } u(y)\text{)}$$

for t (on S) near the edge-point o, exterior $N(o, \varepsilon)$; here α_0, ν are from the inequalities

$$|u(y)-u(t)| \leqq u(y, t)r^{\nu}(y, t) \quad (\text{some } \nu;\ 0 < \nu \leqq 1)\,, \tag{9.7}$$

$$u(y, t) < c^* l^{-\alpha_0}(y) \ \bigl(\text{for } l(y) \leqq l(t)\bigr), \quad < c^* l^{-\alpha_0}(t) \quad \bigl(\text{for } l(y) \geqq l(t)\bigr)\,; \tag{9.7a}$$
$$\alpha \leqq \alpha_0;\ \alpha_0 - \nu < 1\,.$$

Lemma 9.8. *Let $u(t)$ belong to the class of functions satisfying* (9.2) (*so that* (9.7), (9.7a) *hold*). *The operator $A_t(u)$* (9.1) *is representable near an 'edge' point c (in an y system, in which $c = o$) in the form*

$$A_t(u) = A_t^*(u) + A_t^0(u)\,, \tag{9.8 }$$

where

$$A_t^*(u) = a(t)u(t) + \Psi^*(t) \quad \bigl(\Psi^*(t) \text{ from } (9.5\text{a})\bigr) \tag{9.8b}$$

is the characteristic part of $A_t(u)$ and

$$A_t^0(u) = \Psi^0(t) = \int_{\sigma'} k(y, t)r^{-2}(y, t)u(y)d\sigma(y) \tag{9.8c}$$
$$+ \int_{S(o,a)} k''(t|y, t)r^{-2}(y, t)u(y)d\sigma(y) + \int_s k'(t|y, t)r^{-2}(y, t)u(y)d\sigma(y) + \Psi_1(t)\,,$$

$$[\Psi_1(t) \text{ is from (9.5b) and } \sigma' = S - S(o, a), \quad s = S(o, a) - S_{t,b}]$$

is the regular part of $A_t(u)$. We term $A_t^*(u)$, $A_t^0(u)$, briefly, characteristic operator and regular operator, respectively; $\Psi^*(t)$, $A_t^0(u)$ satisfy inequalities near edges of the same form as hold for $\Psi(t)$ in Theorem 6.36 (*obvious changes for $c \neq o$*).

Write the Fourier expansion of the characteristic of the kernel in $A_t(u)$ in the form

$$f(t, \theta) = \sum_{n=-\infty}^{\infty}{}' f_n(t)e^{in\theta}\,, \tag{9.9}$$

where the prime signifies omission of the term for $n = 0$; we recall that $f_0(t) = 0$, as a consequence of (3.14) (where $k^{1,*}(t, \theta) = f(t, \theta)$).

Use will be made of the following result in the theory of Fourier series. Let $F(\theta)$ be continuous, of period 2π, and let

$$\omega(\delta) = \max.\,|F(\theta + d) - F(\theta)| \quad (\text{for } |d| \leqq \delta)$$

be its modulus of continuity; then the complex Fourier coefficients of $F(\theta)$ satisfy

$$|F_n| \leqq \tfrac{1}{2}\omega\left(\frac{\pi}{n}\right) \quad (n = \pm 1, \pm 2, \ldots) \tag{9.10}$$

[Cf. A. Zygmund, Trig. Series, Warszawa–Lwow, 1935; p. 18].

Let a prime in parentheses denote the partial derivative with respect to θ; thus

$$f^{(\prime)}(t, \theta) = \frac{\partial}{\partial \theta} f(t, \theta) . \tag{9.11}$$

Since

$$\left| \frac{d}{d\theta} (\cos \theta_{s_1} \ldots \cos \theta_{s_m}) \right| \leqq m$$

and $|\Gamma_{s_1 \ldots s_m}(t)| \leqq 3c$ (3.20a), by (9.4b) one has

$$|f^{(\prime)}(t, \theta)| \leqq \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{2} m |\Gamma_{s_1 \ldots s_m}(t)| \leqq \sum_{1}^{\infty} m c_m 6^m < c^0 \tag{9.11a}$$

$\big($$c^0$ from (3.20b)$\big)$; *existence and continuity, in* θ, *of* $f^{(\prime)}(t, \theta)$ *is evident.*

The surface is regular; thus the $a_{ij} = a_{ij}(t)$ in the transformation (3.5) can be selected (u.) Lip 1, that is so that

$$|a_{ij}(y) - a_{ij}(t)| \leqq h_0 r(y, t) \quad (h_0 = c^*) ;$$

since $|a_{ij}(t)| \leqq 1$, it follows by induction that

$$|a_{i_1, s_1}(y) \ldots a_{i_m, s_m}(y) - a_{i_1, s_1}(t) \ldots a_{i_m, s_m}(t)| \leqq m h_0 r(y, t) ;$$

in view of (3.6a), the $\Gamma_{s_1 \ldots s_m}(y, t) = \Gamma_{s_1 \ldots s_m}(y) - \Gamma_{s_1 \ldots s_m}(t)$ are bounded in absolute value by

$$\sum_{i_1, \ldots i_m} |\gamma_{i_1 \ldots i_m}(y) [a_{i_1, s_1}(y) \ldots a_{i_m, s_m}(y) - a_{i_1, s_1}(t) \ldots a_{i_m, s_m}(t)]$$
$$+ \sum_{i_1, \ldots i_m} |\gamma_{i_1 \ldots i_m}(y) - \gamma_{i_1 \ldots i_m}(t)| |a_{i_1, s_1}(t) \ldots a_{i_m, s_m}(t)| ;$$

thus, as a consequence of (3.20a), (3.20c), (3.27),

$$\begin{aligned} |\Gamma_{s_1 \ldots s_m}(y, t)| &\leqq m c_m 3^m h_0 r(y, t) + 3^m \lambda_m \gamma(y, t) r^h(y, t) \\ &\leqq g_0 \lambda'_m 3^m r^h(y, t) l^{-\beta}(\eta) \; [\lambda'_m = \max. (m c_m, \lambda_m) ; \; g_0 = c^*] , \end{aligned} \tag{9.12}$$

where

$$\eta = y \; \big(\text{when } l(y) < l(t)\big); \; \eta = t \; \big(\text{when } l(t) \leqq l(y)\big) . \tag{9.12a}$$

We have

$$F(y, t, \theta) = f(y, \theta) - f(t, \theta) = \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{2} \Gamma_{s_1 \ldots s_m}(y, t) \cos \theta_{s_1} \ldots \cos \theta_{s_m} . \tag{9.13}$$

Inasmuch as

$$|\cos (\theta + d) - \cos \theta|, \; |\sin (\theta + d) - \sin \theta| \leqq |d| \quad \left(\theta_1 = \theta, \; \theta_2 = \frac{\pi}{2} - \theta\right),$$

one obtains by induction

$$|\cos \theta_{s_1} \ldots \cos \theta_{s_m} \text{ (for } \theta + d) - \cos \theta_{s_1} \ldots \cos \theta_{s_m} \text{ (for } \theta)| \leqq m |d| . \tag{1_0}$$

Now $\frac{d}{d\theta}(\cos\theta_{s_1}\ldots\cos\theta_{s_m})$ is a sum of m products of form

$$\pm\cos\theta_{j_1}\ldots\cos\theta_{j_m};$$

hence, in view of (1_0),

$$(2_0)\qquad \left|\frac{d}{d\theta}(\cos\theta_{s_1}\ldots\cos\theta_{s_m})(\text{for } \theta+d)-\frac{d}{d\theta}(\cos\theta_{s_1}\ldots\cos\theta_{s_m})\ (\text{for } \theta)\right|\leqq m^2|d|.$$

By virtue of (9.13), (9.12) and of the above

$$(9.13\text{a})\qquad |F^{(\prime)}(y, t, \theta+d)-F^{(\prime)}(y, t, \theta)|\leqq\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2}|\Gamma_{s_1\ldots s_m}(y, t)|m^2|d|$$
$$\leqq g'r^h(y, t)l^{-\beta}(\eta)|d|\quad \left[g'=g_0\sum_1^{\infty}m^2 6^m\lambda'_m=c^*\right],$$

where η is as in (9.12a) and the series for g' converges, since the series c^0 (3.20b), c'' (3.20d) converge. Similarly, by (9.4b) and (2_0)

$$(9.13\text{b})\qquad |f^{(\prime)}(t, \theta+d)-f^{(\prime)}(t, \theta)|\leqq\sum_{m=1}^{\infty}\sum_{s_1,\ldots s_m=1}^{2}|\Gamma_{s_1\ldots s_m}(t)|m^2|d|$$
$$\leqq h'|d|\quad\left[h'=\sum_1^{\infty}m^2c_m 6^m=c^*,\ \text{convergent by (3.20b)}\right].$$

We have

$$(9.14)\qquad f_n(t)=\frac{1}{in}f'_n(t);\ f_n(y)-f_n(t)=\frac{1}{in}F'_n(y, t)\quad (n\neq 0),$$

where $f'_n(t)$, $F'_n(y, t)$ are complex Fourier coefficients of

$$(9.14\text{a})\qquad f^{(\prime)}(t, \theta),\ F^{(\prime)}(y, t, \theta)=f^{(\prime)}(y, \theta)-f^{(\prime)}(t, \theta),$$

respectively; $f'_0(t)=F'_0(y, t)=0$. The third members in (9.13b), (9.13a) give upper bounds for the moduli of continuity (with respect to θ) of the functions $f^{(\prime)}(t, \theta)$, $F^{(\prime)}(y, t, \theta)$, respectively. Hence, as a consequence of (9.10), $|f'_n(t)|$ and $|F'_n(y, t)|$ are bounded by

$$\frac{\pi}{2}h'\frac{1}{n},\qquad \frac{\pi}{2}g'r^h(y, t)l^{-\beta}(\eta)\frac{1}{n};$$

whence (with η from (9.12a) and $n\neq 0$) by (9.14)

$$(9.14\text{b})\qquad |f_n(t)|\leqq\frac{\pi}{2}h'\frac{1}{n^2};\quad |f_n(y)-f_n(t)|\leqq\frac{\pi}{2}g'r^h(y, t)l^{-\beta}(\eta)\frac{1}{n^2}.$$

The above inequalities give information regarding the behaviour near the edges, as well as continuity properties of the coefficients in the expansion (9.9) *of the characteristic of the kernel in* $A_t(u)$.

Definition 9.15. *The function*

$$a(t, \varphi) = \sum_{n=-\infty}^{\infty} a_n(t) e^{in\varphi} \quad (0 \leqq \varphi \leqq 2\pi)\,, \tag{9.15a}$$

where $a_0(t) = a(t)$ *and*

$$a_n(t) = \frac{2\pi}{n} f_n(t)\,, \quad a_{-n}(t) = \frac{2\pi}{n}(-1)^n f_{-n}(t) \quad (n > 0) \tag{9.15b}$$

will be termed symbol of the operator $A_t(u)$ (9.1).

The definition of the symbol is in accord with [M; p. 92]. A simple condition for the nonvanishing of the symbol $a(t, \varphi)$ is that

$$|a(t)| \geqq a^0 > 2\pi^2 h' \sum_{1}^{\infty} |n|^{-3} \ (= a')\,; \tag{9.16}$$

this follows from the inequality (ensuing by (9.14b))

$$\left| \sum_{-\infty}^{\infty}{}' a_n(t) e^{in\varphi} \right| \leqq \sum_{-\infty}^{\infty}{}' \pi^2 h' |n|^{-3}\,.$$

If S has no edges, then, as can be seen from [M], the following is true. If the symbol $a(t, \varphi)$ of $A_t(u)$ does not vanish, the operator $A_t(u)$ can be regularized in the sense that there exists an operator $B(w)$ (whose symbol is $a^{-1}(t, \varphi)$) so that $BA(u) = u + T(u)$, where $T(u)$ is a completely continuous operator. Without further consideration, this cannot be asserted when edges are present.

Let $B_t(w) = B_t^*(w)$ be the characteristic operator (Definition in Lemma 9.8), defined by the symbol

$$b(t, \varphi) = \frac{1}{a(t, \varphi)} = \sum_{n=-\infty}^{\infty} b_n(t) e^{in\varphi} \quad (0 \leqq \varphi \leqq 2\pi)\,. \tag{9.17}$$

Whether as a consequence of (9.16) *or in any other way, we forthwith assume that*

$$|b(t, \varphi)| \{ = |a^{-1}(t, \varphi)| \} \leqq b^0 \quad (b^0 = c^*)\,. \tag{9.18}$$

A corollary to a theorem of N. Wiener asserts that, if the Fourier series $S(f)$ of $f(\theta)$ converges absolutely and $f(\theta) \neq 0$, then $S\left(\frac{1}{f}\right)$ also converges absolutely [cf. Zygmund, p. 143]. Now the series (9.15a) for $a(t, \varphi)$ converges absolutely; hence the series (9.17) for the symbol $b(t, \varphi)$ converges absolutely for $0 \leqq \varphi \leqq 2\pi$, for every t for which $a(t, \varphi) \neq 0$; in view of (9.18) such convergence is assured for all t on S. From (9.17) the characteristic of the kernel in the operator $B_t(w)$ is reconstructed in accord with (9.15b), (9.9); thus

(9.19)
$$g(t,\,\theta) = \sum_{n=-\infty}^{\infty}{}' g_n(t)e^{in\theta}\,,$$

$$g_n(t) = \frac{n}{2\pi}\,b_n(t)\,,\quad g_{-n}(t) = \frac{n}{2\pi}\,(-1)^n b_{-n}(t)\quad (n>0)\,,$$

will be the characteristic for $B_t(w)$. Convergence of the series for $g(t,\theta)$ is a corollary of the absolute convergence of (9.17) [M]. The operator $B_t(w)$, itself, has the structure of (9.8b), (9.5a); that is

(9.19a)
$$B_t(w) = b(t)w(t) + \Psi_1^*(t)\,;\quad b(t) = b_0(t)\,;$$

$$\Psi_1^*(t) = \int_0^b\int_0^{2\pi} g(t,\,\theta)w(Y)\frac{d\varrho}{\varrho}\,d\theta = \int_{S(O,\,b)}\frac{g(t,\,\theta)}{r^2(O,\,Y)}\,w(Y)dY_1dY_2$$

[the Y system has its origin O at t, as in (9.5a); in the above $Y = (Y_1,\,Y_2,\,0)$].

The following formula due to Michlin is found in [M; p. 93]:

(I)
$$a(t,\,\varphi) = -\int_{-\pi}^{\pi}\log\,[2i\sin\,(\theta-\varphi)]f(t,\,\theta)d\theta + a(t)$$

(in the present notation). This we put in the form

(I′)
$$a(t,\,\varphi) = -\int_0^{2\pi}\log\,(2i\sin\,\theta)f(t,\,\varphi+\theta)d\theta + a(t)\,.$$

By (9.15b), (9.14b) $a^{(')}(t,\varphi)$ can be obtained deriving (9.15a) term by term; $a^{(')}(t,\varphi)$ is continuous in φ and one has

(9.20)
$$|a^{(')}(t,\,\varphi)| \leqq \pi^2 h' 2\sum_1^{\infty} r^{-2} = h_0'(=c^*)\,.$$

Also, in view of the same formulas

(9.20a)
$$|a^{(')}(y,\,\varphi) - a^{(')}(t,\,\varphi)| \leqq h_0'' r^h(y,\,t)l^{-\beta}(\eta)\quad (h_0''=c^*)\,.$$

Further, by (9.18) and (9.20)

(9.20b)
$$|b^{(')}(t,\,\varphi)| \leqq b_0'\quad (=h_0'(b^0)^2 = c^*)\,.$$

It is observed that

(1°)
$$B^{(')}(y,\,t,\,\varphi) \equiv b^{(')}(y,\,\varphi) - b^{(')}(t,\,\varphi) = H(y,\,t,\,\varphi)b^2(t,\,\varphi)b^2(y,\,\varphi)\,,$$

where

(2°)
$$H(y,\,t,\,\varphi) = a^{(')}(t,\,\varphi)a^2(y,\,\varphi) - a^{(')}(y,\,\varphi)a^2(t,\,\varphi)$$

$$= a^{(')}(t,\,\varphi)\big(a(y,\,\varphi)-a(t,\,\varphi)\big)\big(a(y,\,\varphi)+a(t,\,\varphi)\big) + \big(a^{(')}(t,\,\varphi)-a^{(')}(y,\,\varphi)\big)a^2(t,\,\varphi)\,.$$

Now, by virtue of (9.4b), (3.20a), (3.20b)

$$(9.20a') \qquad |f(t, \theta)| \leqq g_1 = \sum_1^{\infty} c_m 6^m = c^* ;$$

also, from (9.13), (9.12) one derives

$$(9.20a'') \qquad |f(y, \theta) - f(t, \theta)| \leqq \sum_{m=1}^{\infty} \sum_{s_1, \ldots s_m = 1}^{2} |\Gamma_{s_1 \ldots s_m}(y, t)|$$

$$\leqq g_0' r^h(y, t) l^{-\beta}(\eta) \qquad \left[g_0' = g_0 \sum_1^{\infty} \lambda_m' 6^m = c^* \right];$$

hence by (I′) (with η from (9.12a))

$$(9.21) \qquad |a(t, \varphi)| \leqq g_1 \int_0^{2\pi} |\log (2i \sin \theta)| d\theta + |a(t)| = g_1 i' + |a(t)| = T_0(t) ;$$

$$|a(y, \varphi) - a(t, \varphi)| \leqq g_0' i' r^h(y, t) l^{-\beta}(\eta) + |a(y) - a(t)| = T(y, t) .$$

With the aid of (2°), (9.20), (9.21), (9.20a) one infers

$$(9.21a) \quad |H(y, t, \varphi)| \leqq h_0' T(y, t)[T_0(y) + T_0(t)] + h_0'' r^h(y, t) l^{-\beta}(\eta) T_0^2(t) = T_1(y, t) .$$

Whence by (1°) and (9.18)

$$(9.22) \qquad |B^{(\prime)}(y, t, \varphi)| \leqq (b^0)^4 T_1(y, t) .$$

Let the n-th Fourier coefficient of $b^{(\prime)}(t, \varphi)$ be $b_n'(t)$; by (9.20b) and since $b_n(t) = (in)^{-1} b_n'(t)$ $(n \neq 0)$,

$$(9.23) \qquad |b_n(t)| = |b_n'(t)| \cdot |n|^{-1} \leqq b_0' |n|^{-1}; \quad b_0' = c^*; \; n \neq 0 ;$$

$$b_n'(t) \to 0, \quad \text{as } n \to \pm\infty .$$

Since

$$B_n(y, t) = b_n(y) - b_n(t) = \frac{1}{ni} B_n'(y, t) \quad (n \neq 0) ,$$

where $B_n'(y, t)$ is the n-th Fourier coefficient of $B^{(\prime)}(y, t, \varphi)$, from (9.22) it follows that

$$(9.23a) \qquad |b_n(y) - b_n(t)| = |B_n'(y, t)| \, |n|^{-1} \leqq (b^0)^4 T_1(y, t) |n|^{-1}; \quad n \neq 0 ;$$

$$B_n'(y, t) \to 0, \text{ as } n \to \pm\infty \quad [T_1(y, t) \text{ from (9.21a), (9.21)}] .$$

The formulas (9.23), (9.23a) *are important because they furnish information regarding the behaviour of the coefficients* $g_n(t)$ *in the expansion* (9.19) *of the characteristic* $g(t, \theta)$ *for the operator* $B_t(w)$.

The following can be proved. *Suppose* (9.18) *holds and*

$$(9.23b^\circ) \qquad |a(t)| \leqq c^*, \; |a(y) - a(t)| \leqq c^* r^h(y, t) l^{-\beta}(\eta), \; \sum_1^{\infty} m^5 6^m \lambda_m < \infty ;$$

then

(9.23b)
$$|f_n(t)| \leqq c^*|n|^{-5}, \quad |f_n(y)-f_n(t)| \leqq c^* r^h(y, t) l^{-\beta}(\eta)|n|^{-5},$$
$$|g_n(t)| \leqq c^*|n|^{-3}, \quad |g_n(y)-g_n(t)| \leqq c^* r^h(y, t) l^{-\beta}(\eta)|n|^{-3},$$

where $\eta = t$ $\big($*for* $l(t) \leqq l(y)\big)$, $= y$ $\big($*for* $l(y) \leqq l(t)\big)$. *Furthermore,* $n^3 g_n(t) \to 0$, $n^3|g_n(y)-g_n(t)| \to 0$, *as* $|n| \to \infty$.

The proof of this will be omitted; we shall only remark that, under the above conditions, the fourth order partial derivatives, with respect to θ, of $f(t, \theta)$, $b(t, \theta)$ exist and that a suitable elaboration of the methods used in proving (9.23a) will lead to (9.23b).

In so far as the coefficient of $w(t)$ in $B_t(w)$ (9.19a) is concerned, one has $\big($by (9.17), (9.18)$\big)$

(9.24)
$$|b(t)| = |b_0(t)| \leqq b^0 = c^*;$$

in view of (9.18), (9.21)
$$|b(y, \varphi)-b(t, \varphi)| \leqq (b^0)^2 T(y, t);$$

thus

(9.24a)
$$|b(y)-b(t)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |b(y, \varphi)-b(t, \varphi)| d\varphi = (b^0)^2 T(y, t).$$

The $a_n'(t)$, $b_n'(t)$ are the Fourier coefficients of $a^{(\prime)}(t, \varphi)$, $b^{(\prime)}(t, \varphi)$ (the partials with respect to φ); we have $\big($cf. (9.15a), (9.15b), (9.17), (9.19)$\big)$
$$|a_n'(t)| = 2\pi|f_n(t)|; \ |b_n'(t)| = 2\pi|g_n(t)|; \ a_0'(t) = b_0'(t) = 0.$$

It has been noted in [*M*; p. 101] that Parseval's identity leads to relations between the integrals of the squares of absolute values of derivatives of symbols and of characteristics; thus, in our case:
$$4\pi^2\int_0^{2\pi} |g(t, \varphi)|^2 d\varphi = \int_0^{2\pi} |b^{(\prime)}(t, \varphi)|^2 d\varphi;$$

(*i*)
$$\int_0^{2\pi} |a^{(\prime)}(t, \varphi)|^2 d\varphi = 4\pi^2\int_0^{2\pi} |f(t, \varphi)|^2 d\varphi.$$

By (9.18) $|b^{(\prime)}(t, \varphi)| \leqq (b^0)^2|a^{(\prime)}(t, \varphi)|$. Hence

(9.25)
$$\int_0^{2\pi} |g(t, \varphi)|^2 d\varphi \leqq (b^0)^4\int_0^{2\pi} |f(t, \varphi)|^2 d\varphi \leqq 2\pi g_1^2 (b^0)^4$$

$\big(g_1, = c^*$, is from (9.20a′)$\big)$. Inasmuch as
$$|a_n'(y)-a_n'(t)| = 2\pi|f_n(y)-f_n(t)|; \quad |b_n'(y)-b_n'(t)| = 2\pi|g_n(y)-g_n(t)|,$$

we have further relations

$$(ii) \qquad 4\pi^2 \int_0^{2\pi} |g(y,\varphi)-g(t,\varphi)|^2 d\varphi = \int_0^{2\pi} |b^{(\,)}(y,\varphi)-b^{(\,)}(t,\varphi)|^2 d\varphi\,;$$

$$(iii) \qquad \int_0^{2\pi} |a^{(\,)}(y,\varphi)-a^{(\,)}(t,\varphi)|^2 d\varphi = 4\pi^2 \int_0^{2\pi} |f(y,\varphi)-f(t,\varphi)|^2 d\varphi\,.$$

Now by (9.18) and the inequality subsequent (9.24)

$$|b^{(\,)}(y,\varphi)-b^{(\,)}(t,\varphi)| = |b^2(t,\varphi)\big(a^{(\,)}(t,\varphi)-a^{(\,)}(y,\varphi)\big)+a^{(\,)}(y,\varphi)\big(b(t,\varphi)+b(y,\varphi)\big)\cdot$$
$$\cdot\big(b(t,\varphi)-b(y,\varphi)\big)| \leqq (b^0)^2|a^{(\,)}(t,\varphi)-a^{(\,)}(y,\varphi)|+2(b^0)^3|a^{(\,)}(y,\varphi)|T(y,t)\,.$$

Hence

$$\left[\int_0^{2\pi} |b^{(\,)}(y,\varphi)-b^{(\,)}(t,\varphi)|^2 d\varphi\right]^{\frac12} \leqq (b^0)^2\left[\int_0^{2\pi} |a^{(\,)}(t,\varphi)-a^{(\,)}(y,\varphi)|^2 d\varphi\right]^{\frac12}$$
$$+2(b^0)^3T(y,t)\left[\int_0^{2\pi} |a^{(\,)}(y,\varphi)|^2 d\varphi\right]^{\frac12};$$

by (iii) and (i) the second member is bounded by

$$(b^0)^2 2\pi\left[\int_0^{2\pi} |f(y,\varphi)-f(t,\varphi)|^2 d\varphi\right]^{\frac12}+2(b^0)^3T(y,t)2\pi\left[\int_0^{2\pi} |f(y,\varphi)|^2 d\varphi\right]^{\frac12}.$$

Accordingly by (ii) and (9.20a′)

$$(9.25a) \qquad \left[\int_0^{2\pi} |g(y,\varphi)-g(t,\varphi)|^2 d\varphi\right]^{\frac12} \leqq (b^0)^2\left[\int_0^{2\pi} |f(y,\varphi)-f(t,\varphi)|^2 d\varphi\right]^{\frac12}$$
$$+2(b^0)^3T(y,t)\left[\int_0^{2\pi} |f(y,\varphi)|^2 d\varphi\right]^{\frac12} \leqq (b^0)^2\left[\int_0^{2\pi} |f(y,\varphi)-f(t,\varphi)|^2 d\varphi\right]^{\frac12}$$
$$+2\sqrt{2\pi}g_1(b^0)^3T(y,t) \quad \big(T(y,t) \text{ from } (9.21)\big)\,.$$

The above formula gives properties of mean square continuity (with respect to y) of the characteristic $g(y,\varphi)$ in the operator B; these properties are related to similar properties of the characteristic $f(y,\varphi)$ in the original operator A. In view of (9.20a″) *we have the corollary:*

$$(9.25b) \qquad \left[\int_0^{2\pi} |g(y,\varphi)-g(t,\varphi)|^2 d\varphi\right]^{\frac12} \leqq \sqrt{2\pi}(b^0)^2g_0'r^h(y,t)l^{-\beta}(\eta)$$
$$+2\sqrt{2\pi}g_1(b^0)^3T(y,t) \quad \big(\eta \textit{ as in } (9.12a)\big)\,.$$

10. Composition of singular integrals. It will be necessary to study in some detail the result of application of the operator B_t (9.19a) to the operator A_t^* (9.8b). For this purpose we introduce the notation: (10.1) t, t', t'' are points on the surface S

(near one another); $P_t, P_{t'}$ are tangential planes to S at t, t', respectively; τ' is the orthogonal projection of t' on P_t; τ'' is the orthogonal projection of t'' upon $P_{t'}$; $\varrho = r(t, \tau')$, $\varrho' = r(t', \tau'')$; ϱ, Ψ are polar coordinates of τ' (in P_t, with pole at t); ϱ', θ are polar coordinates of τ'' (in $P_{t'}$, with pole at t'); furthermore,

$$\tau' = (\tau_1', \tau_2'); \quad \tau_1' = \varrho \cos \Psi, \tau_2' = \varrho \sin \Psi; \quad d\tau' = d\tau_1' d\tau_2'; \tag{10.1a}$$

$$\tau'' = (\tau_1'', \tau_2''); \ \tau_1'' = \varrho' \cos \theta, \tau_2'' = \varrho' \sin \theta; \quad d\tau'' = d\tau_1'' d\tau_2''.$$

Let c be a point on edges, near which the operator product BA^*u will be studied. *Assume* $c = (0, 0, 0) = o$ *in the* $y = (y_1, y_2, y_3)$ *system*; *the* (y_1, y_2) *plane tangent at* c. Designate by $Y = (Y_1, Y_2, Y_3)$ a variable coordinate system with origin O at t, the $+Y_3$-axis coincident with $+n_t$ (the positive normal to S at t), the Y_1, Y_2-axes in P_t; we arrange so that the point τ' (10.1a) is representable in the Y system by

$$Y_1 = \tau_1', \ Y_2 = \tau_2', \ Y_3 = 0 .$$

We have (cf. (3.5))

$$y_i = t_i + \sum_{k=1}^{3} a_{ik}(t) Y_k , \quad Y_k = \sum_{i=1}^{3} a_{ik}(t)(y_i - t_i) . \tag{10.1b}$$

The system (Y), corresponding to t' will be designated by $Y' = (Y_1', Y_2', Y_3')$;

$$y_i = t_i' + \sum_{k=1}^{3} a_{ik}(t') Y_k' , \quad Y_k' = \sum_{i=1}^{3} a_{ik}(t')(y_i - t_i') ; \tag{10.1c}$$

the origin O' of the Y' system is at t', the Y_1', Y_2'-plane is identical with $P_{t'}$. *Choose the* $+Y_1'$*-axis in the plane* $Y_2 = \tau_2'$ (in the general direction of the $+Y_1$-axis). The angle θ (10.1a) will be measured from the $+Y_1'$-axis. The orthogonal projection of $O', +Y_1'$ on P_t is the ray extending from τ' parallel to $O, +Y_1$. The point τ'' (10.1a) is representable in the Y' system by

$$Y_1' = \tau_1'', \ Y_2' = \tau_2'', \ Y_3' = 0 .$$

Near $y = o$ (the edge point) the surface is representable in the form

$$y_3 = F^0(y_1, y_2) = O(y_1^2 + y_2^2) . \tag{10.2}$$

In the Y, Y' systems the equations are

$$Y_3 = F(Y_1, Y_2) = F(t|Y_1, Y_2) = O(Y_1^2 + Y_2^2) , \tag{10.2a}$$

$$Y_3' = F'(Y_1', Y_2') = F(t'|Y_1', Y_2') = O(Y_1'^2 + Y_2'^2) ;$$

in (10.2a) the symbols $O(\ldots)$ depend on t, t', respectively. As remarked preceding (6.29)

(10.2b) $|F_i^0(y_1, y_2) - F_i^0(t_1, t_2)| \leqq c^* r(y, t) \quad \left[F_i^0(y_1, y_2) = \frac{\partial}{\partial y_i} F^0(y_1, y_2);\ i = 1, 2\right]$,

since the second order partials of F^0 are continuous, bounded up to edges. Furthermore, in view of (6.29), (6.29a)

$$\text{(10.2c)} \qquad \left|\frac{\partial F}{\partial Y_j}\right| \leqq c^*[Y_1^2 + Y_2^2]^{\frac{1}{2}}, \quad \left|\frac{\partial F'}{\partial Y'_j}\right| \leqq c^*[Y_1'^2 + Y_2'^2]^{\frac{1}{2}} \quad (j = 1, 2);$$

$$|Y_3| \leqq c^*(Y_1^2 + Y_2^2), \ |Y_3'| \leqq c^*(Y_1'^2 + Y_2'^2); \ \left[1 + \left(\frac{\partial F}{\partial Y_1}\right)^2 + \left(\frac{\partial F}{\partial Y_2}\right)^2\right]^{\frac{1}{2}} = 1 + O(Y_1^2 + Y_2^2),$$

$$\left[1 + \left(\frac{\partial F'}{\partial Y_1'}\right)^2 + \left(\frac{\partial F'}{\partial Y_2'}\right)^2\right]^{\frac{1}{2}} = 1 + O(Y_1'^2 + Y_2'^2) \quad [O(\ldots) \text{ independent of } t].$$

The function $b(t)$ $\big(= b$ in (9.5a), (9.19a)$\big)$ can be defined as follows. *Assign* $\delta > 0$, $1 > b_0 > 0$ *suitably small and define* $b(t)$ *for all points of* S *by the relations*

$$\text{(10.3)} \qquad b(t) = \delta \left(\text{for } l(t) \geqq \frac{\delta}{b_0}\right), \quad b(t) = b_0 l(t) \quad \left(\text{for } 0 \leqq l(t) \leqq \frac{\delta}{b_0}\right);$$

$b(t)$ *is uniformly* Lip. 1 *(edges included) and vanishes on edges.*

The function $u(y)$ on which various operators will be applied, should satisfy conditions of the type imposed on $q(y)$ in section 6. Thus, assume $u(y) \subset [\alpha|S]$, that is

$$\text{(10.4)} \qquad |u(t)| < c^* l^{-\alpha}(t) \quad \big(0 \leqq \alpha < 1;\ \alpha + \beta < 1;\ \beta \text{ from } (3.27)\big);$$

also $\big($cf. (6.27), (6.27a)$\big)$

$$\text{(10.4a)} \qquad |u(y) - u(t)| \leqq c^* l^{-\alpha_0}(\eta) r^\nu(y, t) \quad [0 < \nu \leqq 1;\ \alpha \leqq \alpha_0;\ \alpha_0 - \nu < 1;$$

η is y or t, depending on whether $l(y)$ or $l(t)$ is smaller].

Let k, n $(\neq 0)$ be integers, possibly negative, and form

$$\text{(10.5)} \qquad A_n(u|t') \equiv \int_{\varrho' \leqq b(t')} u[t', \tau''] \frac{e^{in\theta}}{\varrho'^2} d\tau'' = v(t'),$$

where

$$\text{(10.5a)} \qquad u[t', \tau''] = u(t'')$$

[t'' is the point on S, whose orthogonal projection on $P_{t'}$ is τ''];

similarly

$$\text{(10.5b)} \qquad A_k(v|t) \equiv \int_{\varrho \leqq b(t)} v[t_0, \tau'] \frac{e^{ik\psi}}{\varrho^2} d\tau' \ [= A_k A_n(u|t)],$$

with

(10.5c) $$v[t, \tau'] = v(t')$$

[t' is the point on S, projecting orthogonally on P_t in τ'].

Designate by x'' the orthogonal projection of τ'' on P_t; write $\sigma' = r(\tau', x'')$ and denote by γ the projection of the angle θ; γ is the angle between the direction $(O, +Y_1)$ and the radius vector (τ', x''). Let

(10.6) $$x'' = (x_1'', x_2'', 0) \quad \text{(in the } Y \text{ system)}.$$

Now t' is (t_1', t_2', t_3') in the y system and is $(\tau_1', \tau_2', \tau_3')$ in the Y system; by (10.1b)

(10.7) $$t_i' = t_i + \sum_{k=1}^{3} a_{ik}(t)\tau_k' = t_i'(t, \tau'), \quad \tau_3' = F(\tau_1', \tau_2') = F(t|\tau_1', \tau_2').$$

Moreover, in view of (10.1c)

$$t_i'' = t_i' + \sum_{k=1}^{3} a_{ik}(t')\tau_k'', \quad \tau_3'' = F'(\tau_1'', \tau_2'') = F(t'|\tau_1'', \tau_2''),$$

$(\tau_1'', \tau_2'', \tau_3'')$ being the representation of t'' in the Y' system. One has

(10.7') $$t_i'' = t_i'(t, \tau') + \sum_{k=1}^{2} a_{ik}\big(t'(t, \tau')\big)\tau_k'' + a_{i3}\big(t'(t, \tau')\big)F\big(t'(t, \tau')|\tau_1'', \tau_2''\big),$$

where

(10.7a) $$t'(t, \tau') = \big(t_1'(t, \tau'), t_2'(t, \tau'), t_3'(t, \tau')\big) \quad \big(\text{cf. } (10.7)\big).$$

We shall need to express the τ_k'' $(k = 1, 2)$ *in terms of the* τ_i', x_i'' $(i = 1, 2)$; this will be done in the course of investigating the difference

(10.8) $$\omega = \varrho'^{-2}e^{in\theta}d\tau'' - \sigma'^{-2}e^{in\gamma}dx'' \quad (dx'' = dx_1''dx_2'').$$

One has

(10.8a) $$d\tau'' = dx''[1+F_1^2+F_2^2]^{\frac{1}{2}}; \quad \frac{1}{\varrho'^2} = \frac{1}{\sigma'^2}[1+(F_1^2+F_2^2)\cos^2(\gamma-\varphi(\tau'))]^{-1},$$

where

(10.8a') $$F_i = \frac{\partial}{\partial \tau_i} F(\tau_1', \tau_2') = \frac{\partial}{\partial \tau_i} F(t|\tau_1', \tau_2') \quad (i = 1, 2); \ \operatorname{tg} \varphi(\tau') = \frac{F_2}{F_1}$$

[unless $F_1 = F_2 = 0$, when $\varrho' = \sigma'$].

In fact, on letting ϑ be the angle between $(+n_t)$, $(+n_{t'})$, we obtain

$$\cos \vartheta = [1+F_1^2+F_2^2]^{-\frac{1}{2}};$$

thus, *the first relation* (10.8a) *is obtained* on noting that dx' is the projection on P_t of the areal element $d\tau''$ (in $P_{t'}$). The equation of $P_{t'}$ in the Y system is

(1°) $$(Y_1-\tau_1')F_1+(Y_2-\tau_2')F_2 = Y_3-\tau_3' \quad \big(\tau_3' = F(\tau_1', \tau')\big).$$

Now τ'' is the point in $P_{t'}$, projecting into x'' $(x_1'', x_2'', 0)$ (in P_t); hence in the Y system the coordinates of τ'' are

$$x_1'', x_2'', x_3''; \; x_3'' = Y_3 \text{ from } (1^\circ) \text{ (when } Y_i = x_i''; \; i = 1, 2);$$

thus

(2°)
$$x_3'' - \tau_3' = \sum_{i=1}^{2} (x_i'' - \tau_i') F_i \, .$$

Accordingly

$$\varrho'^2 = r^2(t', \tau'') = \sum_{i=1}^{2} (x_i'' - \tau_i')^2 + \left[\sum_{i=1}^{2} (x_i'' - \tau_i') F_i\right]^2 .$$

Substituting

(3°)
$$x_1'' - \tau_1' = \sigma' \cos\gamma, \quad x_2'' - \tau_2' = \sigma' \sin\gamma \, ,$$

we obtain

$$\varrho'^2 = \sigma'^2 [1 + (F_1 \cos\gamma + F_2 \sin\gamma)^2] \, ,$$

which leads to the second relation (10.8a). In view of (2°), (3°) the direction cosines of the vector t', τ'' (in $P_{t'}$) are

(4°)
$$\frac{\sigma'}{\varrho'} \cos\gamma, \; \frac{\sigma'}{\varrho'} \sin\gamma, \; \frac{\sigma'}{\varrho'} [F_1 \cos\gamma + F_2 \sin\gamma] \quad \text{(with respect to the } Y \text{ system)}$$

for any value of the polar angle γ (in P_t, with pole at $\tau' = (\tau_1', \tau_2', 0)$). As a consequence of the choice of the $+Y_2'$-axis (in $P_{t'}$; cf. the text subsequent (10.1c)) the direction cosines of the $+Y_2'$-axis, with respect to the Y system, are obtained replacing γ in (4°) by 0; these cosines are

$$[1 + F_1^2]^{-\frac{1}{2}}, \; 0, \; [1 + F_1^2]^{-\frac{1}{2}} F_1 \, .$$

Hence for the polar angle θ with the aid of (10.8a) we obtain

$$\cos\theta = \frac{1}{\sqrt{1 + \lambda^2(\gamma)} \sqrt{1 + F_1^2}} [\cos\gamma + \lambda(\gamma) F_1], \quad \lambda^2(\gamma) = (F_1^2 + F_2^2) \cos^2\big(\gamma - \varphi(\tau')\big) \, .$$

From this it follows that

$$\sin\theta = \frac{1}{\sqrt{1 + \lambda^2(\gamma)} \sqrt{1 + F_1^2}} \sin\gamma [1 + F_1^2 + F_2^2]^{\frac{1}{2}} \, .$$

In view of the first inequality (10.2c)

(4_0)
$$|F_i| \leqq c^* \varrho, \quad |\lambda(\gamma)| \leqq c^* \varrho \, ;$$

whence the preceding relations yield

(5°)
$$\cos\theta = \cos\gamma + \alpha, \; \sin\theta = \sin\gamma + \beta \, ;$$

$$|\alpha|, |\beta| \leqq c_0 \varrho^2; \quad c_0 = c^* \, .$$

Thus

$$(6^\circ) \qquad e^{i\theta} = e^{i\gamma} + (\alpha + i\beta); \quad |\alpha + i\beta| \leqq \sqrt{2c_0}\,\varrho^2 .$$

Now, for n a positive integer,

$$e^{in\theta} - e^{in\gamma} = (e^{i\theta} - e^{i\gamma})(e^{(n-1)i\theta} + \cdots + e^{(n-1)i\gamma}) ,$$

$$e^{-in\theta} - e^{-in\gamma} = (e^{i\gamma} - e^{i\theta}) e^{-i(\gamma+\theta)} (e^{-(n-1)i\theta} + \cdots + e^{-(n-1)i\gamma}) .$$

Hence by (6°)

$$|e^{\pm in\theta} - e^{\pm in\gamma}| \leqq c^* n \varrho^2 \quad (n > 0) .$$

Thus

$$(10.8\text{b}) \qquad e^{in\theta} = e^{in\gamma} + \nu_n, \quad |\nu_n| \leqq c_2 |n| \varrho^2 , \; c_2 = c^*$$

for integers $n = \pm 1, \pm 2, \ldots$.

We deduce

$$(\mathrm{I}_1) \qquad [1+\lambda^2(\gamma)]^{-1} \sqrt{1+F_1^2+F_2^2} = 1 + J; \quad |J| \leqq |\sqrt{1+F_1^2+F_2^2} - 1 - \lambda^2(\gamma)|$$

$$= \left| \frac{F_1^2+F_2^2}{1+\sqrt{1+F_1^2+F_2^2}} - \lambda^2(\gamma) \right| \leqq \tfrac{1}{2}(F_1^2+F_2^2) + \lambda^2(\gamma) ;$$

in view of (4_0)

$$(\mathrm{I}_2) \qquad |J| \leqq c^* \varrho^2 .$$

As a consequence of (10.8a), (10.8b) and (I_1)

$$(\mathrm{I}_3) \qquad \varrho'^{-2} e^{in\theta} d\tau'' = \frac{1}{\sigma'^2} (e^{in\gamma} + \nu_n) \frac{\sqrt{1+F_1^2+F_2^2}}{1+\lambda^2(\gamma)} dx''$$

$$= \frac{1}{\sigma'^2} (e^{in\gamma} + \nu_n)(1+J) dx'' = \frac{dx''}{\sigma'^2} (e^{in\gamma} + J_1) ;$$

$$(\mathrm{I}_4) \qquad |J_1| = |J e^{in\gamma} + \nu_n (1+J)| \leqq c^* \varrho^2 + c^* |\nu_n| .$$

Accordingly for ω (10.8) *one has*

$$(10.9) \qquad \omega = J_0 dx'', \; J_0 = J_1 \sigma'^{-2} ;$$

$$(10.9\text{a}) \qquad |J_1| \leqq c^* |n| \varrho^2 ;$$

furthermore, in view of (I_4), (I_1), (10.8b), J_1 *is a function of* t, τ', γ *(of period* 2π *in* γ*) and is independent of* σ', *while*

$$(10.9\text{b}) \qquad \int_0^{2\pi} J_1 d\gamma = 0 ;$$

(10.9b) also follows indirectly; in fact if (10.9b) did not hold, the integral $P_n''(\tau')$ (10.22a) would not exist in the principal sense (cf. text from (10.22b) to (10.22c)), which would contradict the existence of the principal integral $A_n(u|t')$ (10.5), (10.21).

We turn to the τ_k'' $(k = 1, 2)$, involved in (10.7′), obtaining

(10.10) $$\tau_k'' = x_k'' - \tau_k' + \lambda_k, \quad |\lambda_k| \leqq c^* \sigma' \varrho^2 \quad (k = 1, 2).$$

In fact, by (10.1a), (10.8a), (5°), (4_0)

$$\tau_1'' = \sigma' \sqrt{1+\lambda^2(\gamma)}\ (\cos\gamma + \alpha) = \sigma'[1+O(\varrho^2)][\cos\gamma + O(\varrho^2)]\,,$$

where $O(\ldots)$ is independent of t; the formula for τ_1'' follows on noting that $\sigma' \cos\gamma = x_1'' - \tau_1'$; the proof is similar for τ_2''.

By (10.7′) and (10.10)

(10.11) $$t_i'' = t_i''(t; \tau'; x'') = t_i''(t; \tau_1', \tau_2'; x_1'', x_2'') = t_i'(t, \tau') +$$
$$+ \sum_{k=1}^{2} a_{ik}(t'(t, \tau))(x_k'' - \tau_k' + \lambda_k) + a_{i3}(t'(t, \tau'))F\big(t'(t, \tau')|x_1'' - \tau_1' + \lambda_1, x_2'' - \tau_2' + \lambda_2\big)\,.$$

The function $u[t', \tau'']$ in (10.5) can be represented in the form

(10.12) $$u[t', \tau''] = u(t'') = u(t_1'', t_2'', t_3'') = u\big(t_1'', t_2'', F^0(t_1'', t_2'')\big) = u(t_1'', t_2'')\,;$$

thus, by (10.11),

(10.12a) $$u[t', \tau''] = u\big(t_1''(t; \tau'; x''), t_2''(t; \tau'; x'')\big) = U(\tau', x'') = U(\tau_1', \tau_2'; x_1'', x_2'')\,;$$

in the notation $U(\tau', x'')$, t is not displayed. In the above τ', x'' are regarded as points in the plane P_t, which are given in the (Y_1, Y_2) system by (τ_1', τ_2'), (x_1'', x_2''), respectively. We take note of the following configuration in the plane P_t (that is, the Y_1, Y_2-plane). Points τ', x'' are in the plane, $r(O, \tau') = \varrho\,(=\sqrt{\tau_1^2+\tau_2^2})$, $r(\tau', x'') = \sigma'$; the angle between the radius vector O, τ' and $O, +Y_1$ is Ψ; the angle between the radius vector τ', x'' and $O, +Y_1$ is γ; we write $\varrho'' = r(O, x'') = \sqrt{x_1''^2 + x_2''^2}$.

By (10.11), (10.7) and since $\lambda_k = 0$ $(k = 1, 2)$ for $(\tau_1', \tau_2') = (x_1'', x_2'')$,

(10.13) $$t_i'' = t_i'(t, x'') = t_i + \sum_{k=1}^{3} a_{ik}(t) x_k'', \quad x_3'' = F(x_1'', x_2'') = F(t|x_1'', x_2'')$$

when $(\tau_1', \tau_2') = (x_1'', x_2'')$. Thus

(10.13a) $$U(x'', x'') = u\Big(t_1 + \sum_{k=1}^{3} a_{1k}(t) x_k'', \ t_2 + \sum_{k=1}^{3} a_{2k}(t) x_k''\Big) \quad (\text{cf. } (10.12))\,.$$

It can be verified that

(10.13b) $$U(o, x'') = U(x'', x'')\,.$$

We write

(10.14b) $$u[t, \tau''] = U(\tau', x'') = U(x'', x'') + V(\tau', x'')\,.$$

One has

(10.14c) $$V(\tau', x'') = U(\tau', x'') - U(x'', x'') = u(t_1'', t_2'') - u\big(t_1''(x''), t_2''(x'')\big)\,,$$

where the t_i'' $(i = 1, 2)$ are from (10.11), (10.7), that is,

$$t_i'' = t_i + \sum_{k=1}^{3} a_{ik}(t)\tau_k' + \sum_{k=1}^{2} a_{ik}\big(t'(t, \tau')\big)(x_k'' - \tau_k' + \lambda_k) \tag{10.14c'}$$
$$+ a_{i3}\big(t'(t, \tau')\big)F\big(t'(t, \tau')|x_1'' - \tau_1' - \lambda_1, x_2'' - \tau_2' - \lambda_2\big),$$

and

$$t_i''(x'') = t_i + \sum_{k=1}^{3} a_{ik}(t)x_k'', \quad \tau_3' = F(\tau_1', \tau_2'), \quad x_3'' = F(x_1'', x_2''); \tag{10.14c''}$$

accordingly

$$t_i'' - t_i''(x'') = \sum_{k=1}^{2} \big(a_{ik}(t') - a_{ik}(t)\big)(x_k'' - \tau_k') + a_{i3}(t)(\tau_3' - x_3'') \tag{10.14d}$$
$$+ \sum_{k=1}^{2} a_{ik}(t')\lambda_k + a_{i3}(t')F(t'|x_1'' - \tau_1' - \lambda_1, x_2'' - \tau_2' - \lambda_2).$$

In any case the $a_{i3}(t)$ are defined as in (3.5b), while the $a_{ik}(k = 1, 2)$ might be possibly suitable modifications of the corresponding expressions in (3.5a). One has

$$|n_i(t)| = [1 + F_1^0(t_1, t_2)^2 + F_2^0(t_1, t_2)^2]^{-\frac{1}{2}} |F_i^0(t_1, t_2)| \leqq |F_i^0(t_1, t_2)|$$

(notation of (10.2b); $i = 1, 2$); thus by (2.1'), (2.1a)

$$|a_{i3}(t)| = |n_i(t)| \leqq c^* r(o, t), \quad |a_{i3}(t')| \leqq c^* r(o, t') \quad (i = 1, 2); \tag{1°}$$

one may replace $r(o, t)$ here by $\sqrt{t_1^2 + t_2^2}$. We have

$$|\tau_3' - x_3''| = |F(\tau_1', \tau_2') - F(x_1'', x_2'')| = \left| \sum_{1}^{2} \frac{\partial}{\partial u_i} F(u_1, u_2)(\tau_i' - x_i'') \right|,$$

where $u = (u_1, u_2)$ is some point on the segment (τ', x''); by (10.2c) and a triangular inequality the above is bounded by

$$c^* r(O, u)\sigma' \leqq c^*[r(O, \tau') + r(\tau', u)]\sigma';$$

$r(O, \tau') = \varrho$, $r(\tau', u) \leqq \sigma'$; hence

$$|\tau_3' - x_3''| \leqq c^*(\varrho + \sigma')\sigma' \quad (\leqq c^*\sigma'). \tag{2°}$$

In view of the inequality for Y_3' in (10.2c) and (10.10)

$$|F(t'|x_1'' - \tau_1' - \lambda_1, x_2'' - \tau_2' - \lambda_2)| \leqq c^*[(x_1'' - \tau_1' - \lambda_1)^2 + (x_2'' - \tau_2' - \lambda_2)^2] \tag{3°}$$
$$\leqq c^*\sigma'^2(1 + \varrho^2)^2 \leqq c^*\sigma'^2 \quad (\text{for } \varrho \leqq c^*).$$

With $u, u' \geqq 1$ one has

$$|(1 + u')^{-\frac{1}{2}} - (1 + u)^{-\frac{1}{2}}| \leqq c^*|u - u'|;$$

inasmuch as (by (10.2b)) $|F_i^0(t_1', t_2') - F_i^0(t_1, t_2)| \leqq c^*\varrho$ $(i = 1, 2)$ and since $|F_i^0(t_1, t_2)| \leqq c^* r(o, t)$, we have

(4°) $\quad |n_3(t')-n_3(t)| \leqq c^* \left| \sum_{i=1}^{2} [F_i^0(t_1', t_2')^2 - F_i^0(t_1, t_2)^2] \right| \leqq c^* \varrho \cdot \bigl(r(o, t)+r(o, t')\bigr)$.

Since $n_i(t) = -F_i^0(t) n_3(t)$ $(i = 1, 2)$, as a consequence of (4°) one has

(5°) $$|n_i(t')-n_i(t)| \leqq c^* \varrho \quad (i = 1, 2) .$$

Also, in view of (1°), (5°)

(6°) $$|(1-n_2^2(t'))^{\pm\frac{1}{2}} - (1-n_2^2(t))^{\pm\frac{1}{2}}| \leqq c^* \varrho \cdot \bigl(r(o, t)+r(o, t')\bigr) ;$$

further, by (4°), (6°), the modulus of continuity for $n_3(t)(1-n_2^2(t))^{-\frac{1}{2}}$ is bounded by an expression of the form of the second member above; by (1°), (6°), (5°) the modulus of continuity of $n_1(t)(1-n_2^2(t))^{-\frac{1}{2}}$ is bounded by $c^* \varrho$. The moduli of continuity of

$$-n_1(t) n_2(t)(1-n_2^2(t))^{-\frac{1}{2}}, \qquad -n_2(t) n_3(t)(1-n_2^2(t))^{-\frac{1}{2}}$$

are bounded by

$$c^* \varrho \cdot \bigl(r(o, t)+r(o, t')\bigr), \quad c^* \varrho ,$$

respectively. If the $a_{ik}(t)$ are defined as in (3.5a), the above shows that

(10.14e) $$|a_{ik}(t')-a_{ik}(t)| \leqq c^* \varrho .$$

The $+Y_1'$-axis has been chosen as indicated subsequent (10.1c). This entails a modification of the expressions given for the a_{ik} in (3.5b); however, (10.14e) can be shown to continue to hold in the present situation. On the other hand, the a_{ik} could have been left as in (3.5b), which would require a modification of the definition of the angle γ, but would have no effect on the conclusions of this section. As a consequence of (10.14d), (10.14e), (1°), (2°), (3°), since $\varrho \leqq c^*\bigl(r(o, t)+r(o, t')\bigr)$ and

$$\left| \sum_{k=1}^{2} a_{ik}(t') \lambda_k \right| \leqq \sum_{k=1}^{2} |\lambda_k| \leqq c^* \sigma' \varrho^2 \quad \text{(cf. (10.10))} ,$$

one obtains

(10.15) $$|t_i''-t_i''(x'')| \leqq c^*\bigl(r(o, t)+r(o, t')\bigr)\sigma' \quad (i = 1, 2) .$$

Accordingly [by (10.2b) and with $u = (u_1, u_2)$ on the segment (t_1'', t_2''), $\bigl(t_1''(x''), t_2''(x'')\bigr)$ in the y_1, y_2-plane]

(10.15a) $$|t_3''-t_3''(x'')| = |F^0(t_1'', t_2'')-F^0\bigl(t_1''(x''), t_2''(x'')\bigr)|$$

$$= \left| \sum_{i=1}^{2} \frac{\partial}{\partial u_i} F^0(u_1, u_2)\bigl(t_i''-t_i''(x'')\bigr) \right| \leqq c^* r(o, u)\bigl(r(o, t)+r(o, t')\bigr)\sigma' \leqq c^*\bigl(r(o, t)+r(o, t')\bigr)\sigma' ;$$

a more precise inequality can be obtained on noting that

$$r(o, u) \leqq \sqrt{t_1''^2+t_2''^2} + \sqrt{t_1''(x'')^2+t_2''(x'')^2} .$$

From the above it is inferred that

(10.15b) $$r(t'', t''(x'')) \leqq c^*(r(o,t)+r(o,t'))\sigma' \quad \text{(cf. (10.14c}', \text{c}''\text{))}.$$

Whence, as a consequence of (10.4a), (10.12) the function $V(\tau', x'')$ of (10.14c) satisfies

(10.16) $$|V(\tau', x'')| \leqq c^* l^{-\alpha_0}(\eta) r^\nu(t'', t''(x'')) \leqq c^* l^{-\alpha_0}(\eta)[r(o,t)+r(o,t')]^\nu \sigma'^\nu,$$

where η is t'' (10.14c′) or $t''(x'')$ (10.14c″), depending on whether $l(t'')$ or $l(t''(x''))$ is smaller.

The relation (10.13b) indicates that $V(\tau', x'') \to 0$, as $\varrho \to 0$. In fact, *it can be shown that*

(10.16′) $$|V(\tau', x'')| \leqq c^* l^{-\alpha_0}(\eta)[r(o,t)+r(o,t')]^\nu \varrho^\nu \quad (\eta \text{ as in } (10.16)).$$

To establish this rewrite (10.14d) in the form

(I_1) $$t_i''-t_i''(x'') = \sum_{k=1}^{3}(a_{ik}(t')-a_{ik}(t))(x_k''-\tau_k')+\sum_{k=1}^{2} a_{ik}(t')\lambda_k$$

$$+a_{i3}(t')\tau_3'+a_{i3}(t')(\tau_3''-x_3''); \; \tau_3'' = F(t'|\tau_1'', \tau_2''), \; \tau_k'' = x_k''-\tau_k'-\lambda_k;$$

$$\tau_3''-x_3'' = \nu_1+\nu_2; \; \nu_1 = F(t'|\tau_1'', \tau_2'')-F(t|\tau_1'', \tau_2''); \; \nu_2 = F(t|\tau_1'', \tau_2'')-F(t|x_1'', x_2'').$$

By the methods employed in proving (2°), with the aid of (10.10) we obtain

(I_2) $$|\nu_2| \leqq c^*(\varrho+\sigma')\varrho.$$

By (10.14e), (2°), (10.10), (1°), (10.2c) (for Y_3) and (I_2) and since $|\nu_1| \leqq c^*\varrho$,

$$|t_i''-t_i''(x'')| \leqq c^*\sigma'\varrho+c^*r(o,t')\varrho^2+c^*r(o,t')|\nu_1| \leqq c^*\sigma'\varrho+c^*r(o,t')\varrho.$$

Whence

(I_3) $$|t_i''-t_i''(x'')| \leqq c^*(r(o,t)+r(o,t'))\varrho \quad (i = 1, 2, 3),$$

which corresponds to (10.15), (10.15a); thus $r(t'', t''(x''))$ is $O(r(o,t)+r(o,t'))\varrho$; (10.16′) *follows by* (10.14c), (10.12), (10.4a).

Consider the triangle o, t, t'; a triangular inequality gives

$$r(o,t') \leqq r(o,t)+r(t,t').$$

Now by the third inequality (10.2c)

$$r^2(t,t') = \varrho^2+F^2(\tau_1', \tau_2') \leqq \varrho^2+c^*\varrho^4 \leqq c^*\varrho^2;$$

also, $\varrho \leqq b(t) \leqq c^* l(t) \leqq c^* r(o,t)$; thus

(1_0) $$r(o,t') \leqq c^* r(o,t).$$

Let $S_{t', b(t')}$ denote the portion of S projecting orthogonally on the plane $P_{t'}$ in the circular region

$$\varrho' = r(t', \tau'') \leqq b(t') \quad (\tau'' = (\tau_1'', \tau_2'', 0));$$

as stated before, τ' is the orthogonal projection of t' on P_t; *designate by* $S(t)$ *the sum of the (nondenumerably infinite) collection of portions* $S_{t', b(t')}$, *corresponding to all* τ' *belonging to the circular region*

$$\varrho = r(t, \tau') \leqq b(t) \quad (\tau' = (\tau_1', \tau_2', 0)) .$$

The points t'', $t''(x'')$, referred to in (10.14c$'$, c$''$) are in $S(t)$. Consider for a moment the special case when the 'edge' β is rectilinear near o and the surface is plane near o (thus lying in the y_1, y_2-plane). For t near o, by (10.3), one may take $b(t) = b_0 l(t)$ $(0 < b_0 = c^* < 1)$. In the plane case $S(t)$ will consist of a region at distance $(1-b_0)^2 l(t)$ from the edge, so that in (10.16) one has

$$l(\eta) \geqq c^* l(t) , \tag{2_0}$$

where $c^* = (1-b_0)^2$. In the general case (2_0) continues to hold, with a possibly different value of c^*. Such a positive constant can be shown to exist, provided $b_0 (> 0)$ is taken sufficiently small. The proof of this assertion is based essentially on the regular character of the 'edges' β near o (β has a continuously turning tangent) and on the regular character of the surface; the indicated circumstances imply sufficient closeness of the general case to the plane case; we shall omit the details. In view of (1_0), (2_0), one may write (10.16), (10.16$'$) in the form

$$|V(\tau', x'')| \leqq c^* l^{-\alpha_0}(t) r(o, t)^\nu \sigma'^\nu ; \tag{10.16a}$$

$$|V(\tau', x'')| \leqq c^* l^{-\alpha_0}(t) r(o, t)^\nu \varrho^\nu . \tag{10.16a$'$}$$

We now turn to the component $U(x'', x'')$ of $U(\tau', x'')$ (10.14b)

$$U(x'', x'') = u(t_1''(x''), t_2''(x'')),\ t_i''(x'') = t_i + \sum_{k=1}^{3} a_{ik}(t) x_k'' \tag{10.17}$$

$(x_3'' = F(x_1'', x_2''))$. The $t_i''(x'')$ $(i = 1, 2, 3)$ are coordinates in the y system of a point $t''(x'')$ on S; the representation of this point in the Y system is (x_1'', x_2'', x_3''). As a consequence of the remark in connection with (2°)

$$l(t''(x'')) \geqq c^* l(t) . \tag{10.17a}$$

By (10.12)

$$U(x'', x'') = u(t_1''(x''), t_2''(x''), t_3''(x'')) ; \tag{10.17a$'$}$$

whence in view of (10.4), (10.17a)

$$|U(x'', x'')| < c^* l^{-\alpha}(t''(x'')) < c^* l^{-\alpha}(t) . \tag{10.17b}$$

One has

$$|U(x'', x'') - U(x_0'', x_0'')| = |u(t_1''(x''), t_2''(x''), t_3''(x'')) - u(t_1''(x_0''), t_2''(x_0''), t_3''(x_0''))| ,$$

where $x_0'' = (x_{01}'', x_{02}'', x_{03}'')$, with $x_{03}'' = F(x_{01}'', x_{02}'')$. By virtue of (10.4a) the above is bounded by

$$(\mathrm{I}_1) \qquad c^* l^{-\alpha_0}(\eta) r^{\nu}(x'', x_0'') ,$$

where one may define $r(x'', x_0'')$ as a distance in the plane P_t, that is

$$(\mathrm{I}_2) \qquad r^2(x'', x_0'') = (x_1''-x_{01}'')^2+(x_2''-x_{02}'')^2 .$$

In (I_1) η is that one of the points x'', x_0'' on S which is nearer to the edge. For reasons essentially of the type that led to (2_0) (preceding (10.16a)) we have

$$(\mathrm{I}_3) \qquad l(\eta) \geqq c^* l(t) .$$

From (I_1), (I_3) it is inferred that

$$(10.17\mathrm{c}) \quad |U(x'', x'')-U(x_0'', x_0'')| < c^* l^{-\alpha_0}(t) r^{\nu}(x'', x_0'') \quad \big(r(x'', x_0'') \text{ from } (\mathrm{I}_2)\big) .$$

As noted before, t'' is on $S_{t', b(t')}$; the orthogonal projection of $S_{t', b(t')}$ on $P_{t'}$ is the circular region $\varrho' \leqq b(t')$; the orthogonal projection of the latter region on P_t is an elliptic region $E(t, \tau')$:

$$(10.18) \qquad 0 \leqq \sigma' \leqq \sigma'(\gamma),\ 0 \leqq \gamma \leqq 2\pi;\ \sigma'(\gamma) \leqq b(t') ;$$

the function $\sigma'(\gamma)$ can be determined with the aid of (10.8a). *Designate by $E(t)$ the sum of the (nondenumerably infinite) collection of regions $E(t, \tau')$, corresponding to all points $\tau' = (\tau_1', \tau_2', 0)$ (in P_t) for which $\varrho = r(t, \tau') \leqq b(t)$. In* (10.17b), (10.17c) *the point $(x_1'', x_2'', 0)$ is supposed to be in $E(t)$.*

We extend the function $U(x'', x'')$ over the whole Euclidean plane

$$(10.19) \qquad E_2 = P_t ,$$

define $U(x'', x'')$ for all x'' $(x_1'', x_2'', 0)$ of E_2 so that in $E(t)$ $U(x'', x'')$ coincides with the function (10.17), that the inequalities (10.17b), (10.17c) continue to hold in all of E_2 and that

$$(10.19\mathrm{a}) \qquad U(x'', x'') = 0 \quad (\text{for } \varrho'' = \sqrt{x_1''^2+x_2''^2} \geqq a = c^*) ,$$

where a is suitably great. To obtain such an extension $U^*(x'', x'')$ define $V(x'')$ by

$$V(x'') = U(x'', x'') l^{\alpha_0}(t) \quad \big(x'' \text{ in } E(t)\big) .$$

Then (10.17b), (10.17c) will yield

$$(10.20) \qquad |V(x'')| < c^* l^{\alpha_0-\alpha}(t), \quad |V(x'')-V(x_0'')| < c^* r^{\nu}(x'', x_0'')$$

(in $E(t)$). Designate by $V^*(x'')$ the continuous extension of $V(x'')$ so that

$$V^*(x'') = V(x'') \ \big(\text{in } E(t)\big);\ V^*(x'') = 0 \quad (\text{for } \varrho'' \geqq a) ;$$

$$|V^*(x'')| < c^* l^{\alpha_0-\alpha}(t),\ |V^*(x'')-V^*(x_0'')| < c^* r^{\nu}(x'', x_0'') \quad (\text{in all of } E_2) .$$

The function

(10.20a) $$U^*(x'', x'') = V^*(x'')l^{-\alpha_0}(t)$$

will be an extension of $U(x'', x'')$ with all the required properties. We drop the superscript with U^* and designate the extension of U by the same symbol.

With the aid of (10.8), (10.9), (10.14b) the operator $A_n(u|t')$ (10.5) is expressible in the form

(10.21) $$A_n(u|t') = \int_{E(t,\tau')} u[t', \tau''][\sigma'^{-2}e^{in\gamma}+J_0]dx''$$
$$= \int_{E(t,\tau')} U(x'', x'')[e^{in\gamma}+J_1]\frac{dx''}{\sigma'^2} + \int_{E(t,\tau')} V(\tau', x'')[e^{in\gamma}+J_1]\frac{dx''}{\sigma'^2};$$

here $E(t, \tau')$ is the elliptic region (in P_t) (10.18) and

(10.21a) $$J_1 = J_1(\tau', \gamma) \qquad \left\{ |J_1| \leqq c^*|n|\varrho^2; \int_0^{2\pi} J_1 d\gamma = 0 \right\}$$

depends on t, but is *independent of* σ'. The first integral in the last member of (10.21) is a principal one; the last term is an ordinary integral, by virtue of the presence of the factor σ'^{ν} $(0 < \nu \leqq 1)$ in (10.16a). Inasmuch as $E(t, \tau')$ is the projection of the region $\varrho' \leqq b(t')$ (in $P_{t'}$), by (10.8a) we find $E(t, \tau')$ defined by

(10.21b) $$0 \leqq \sigma' \leqq \sigma'(\gamma) = b(t')[1+\lambda^2(\gamma)]^{-\frac{1}{2}}, \ \lambda(\gamma) = F_1 \cos\gamma + F_2 \sin\gamma,$$
$$F_i = \frac{\partial}{\partial \tau_i'} F(\tau_1', \tau_2') \quad (i = 1, 2).$$

As a consequence of (10.16a) and of the inequality for J_1 in (10.21a),

$$|V(\tau', x'')[e^{in\gamma}+J_1]| \leqq c^*|n|l^{-\alpha_0}(t)r^{\nu}(o, t)\sigma'^{\nu}.$$

Since $\sigma'(\gamma) \leqq b(t') \leqq c^*b(t) \leqq c^*l(t)$, we have

(10.21c) $$\left|\int_{E(t,\tau')} V(\tau', x'')[e^{in\gamma}+J_1]\frac{dx''}{\sigma'^2}\right| \leqq c^*|n|l^{-\alpha_0}(t)r^{\nu}(o, t)\int_0^{2\pi} d\gamma \int_0^{c^*l(t)} \sigma'^{\nu-1}d\sigma'$$
$$\leqq c^*|n|l^{\nu-\alpha_0}(t)r^{\nu}(o, t) \quad (\nu-\alpha_0 > -1);$$

here $c^*|n|$ *can be replaced by* c^*, inasmuch as by (10.8a) and (I_3) (subsequent (10.8b))

(10.21c′) $$|\sigma'^{-2}(e^{in\gamma}+J_1)dx''| \leqq \varrho'^{-2}d\tau'' \leqq c^*\sigma'^{-2}dx''.$$

The integral

(10.22) $$P_n(\tau') = \int_{E(t,\tau')} U(x'', x'')J_1(\tau', \gamma)\frac{dx''}{\sigma'^2}$$

is a principal one. In order to express this in terms of ordinary integrals write

$$\text{(10.22a)}\quad P_n(\tau') = P_n'(\tau') + U(\tau', \tau')P_n''(\tau'), \quad P_n''(\tau') = \int_{E(t,\tau')} J_1(\tau', \gamma)\frac{dx''}{\sigma'^2}$$

$$P_n'(\tau') = \int_{E(t,\tau')} [U(x'', x'') - U(\tau', \tau')]J_1(\tau', \gamma)\frac{dx''}{\sigma'^2}.$$

$P_n'(\tau')$ is an ordinary integral; by (10.17c), (10.9a) and since $\sigma'(\gamma) \leqq c^* l(t)$,

$$\text{(10.22b)}\qquad |P_n'(\tau')| \leqq c^*|n|\varrho^2 l^{-\alpha_0}(t)\iint \sigma'^{\nu-1}d\sigma' d\gamma \leqq c^*|n| l^{\nu-\alpha_0}(t)\varrho^2 .$$

The principal integral $P_n''(\tau')$ is the limit, as $\varepsilon \to 0$, of

$$P_n^\varepsilon(\tau') = \int_{\gamma=0}^{2\pi} d\gamma \int_{\sigma'=\varepsilon}^{\sigma'(\gamma)} J_1(\tau', \gamma)\frac{d\sigma'}{\sigma'} ;$$

by (10.21a) (since J_1 is independent of σ')

$$P_n^\varepsilon(\tau') = \int_{\gamma=0}^{2\pi} J_1(\tau', \gamma) \log \sigma'(\gamma) d\gamma ;$$

thus

$$\text{(10.22c)}\qquad P_n''(\tau') = \int_{\gamma=0}^{2\pi} J_1(\tau', \gamma) \log \sigma'(\gamma) d\gamma ,$$

the integral being ordinary. Essentially as a consequence of (2_0) (preceding (10.16a)) one has $l(t') \geqq c^* l(t)$ (for $\varrho \leqq b(t)$); thus by (10.21b) (since $\lambda^2(\gamma) \leqq c^*$) and (10.3)

$$(1^\circ)\qquad \sigma'(\gamma) \geqq c^* b(t') \geqq c^* l(t') \geqq \sigma_0 l(t) \quad (\text{for } \varrho \leqq b(t))$$

($\sigma_0 = c^*$, suitably small); accordingly

$$(2^\circ)\qquad |\log \sigma'(\gamma)| < c^* \log\left(\frac{k'}{l(t)}\right) \quad (k' = c^*) .$$

From (10.22c), (10.21a), (2°) it is inferred that

$$\text{(10.22d)}\qquad |P_n''(\tau')| \leqq c^*|n|\varrho^2 \log\left(\frac{k'}{l(t)}\right);$$

In view of (10.17b), for $x'' = \tau'$,

$$\text{(10.22e)}\qquad |U(\tau', \tau')P_n''(\tau')| \leqq c^*|n|\varrho^2 l^{-\alpha}(t) \log\left(\frac{k'}{l(t)}\right).$$

Define the operator

$$\text{(10.23)}\qquad h_n(U) = h_n(U|t') = h_n(U; \tau') = \frac{1}{2\pi}\int_{E_2} U(x'', x'')\frac{e^{in\gamma}}{\sigma'^2} dx'' ,$$

where $U(x'', x'')$ is a continuous extension to the whole Euclidean plane E_2 of the function designated so originally, as stated in connection with (10.19), (10.19a).

This operator is identical with the operator h_n in [M; p. 90]. In view of (10.19a) there exists $a_0 = c^*$ so that, for $\varrho \leqq b(t)$,

$$(1_0) \qquad U(x'', x'') = 0 \quad (\text{for } \sigma' = \sqrt{(x_1''-\tau_1')^2+(x_2''-\tau_2')^2} \geqq a_0)\,.$$

On letting $e_0 = E_2 - E(t, \tau')$, by (10.17b), (1°), (1_0)

$$(10.23\text{a}) \quad \left|\int_{e_0} U(x'', x'')\frac{e^{in\gamma}}{\sigma'^2}dx''\right| \leqq \int_{e_0} |U(x'', x'')|\frac{d\sigma'}{\sigma'}d\gamma = \int_0^{2\pi} d\gamma \int_{\sigma'=\sigma'(\gamma)}^{\infty} |U(x'', x'')|\frac{d\sigma'}{\sigma'}$$

$$\leqq c^* l^{-\alpha}(t) \int_{\sigma_0 l(t)}^{a_0} \frac{d\sigma'}{\sigma'} \leqq c^* l^{-\alpha}(t) \log\left(\frac{a'}{l(t)}\right) \quad \left(a' = c^*;\ \varrho \leqq b(t)\right);$$

an inequality of the same form will hold when $\varrho \leqq c^*$.

Consideration of (10.21), (10.22), (10.23) yields *the decomposition*

$$(10.24) \qquad A_n(u|t') = 2\pi h_n(U|t') + \varrho_n(u|t')\,,$$

where

$$(10.24\text{a}) \quad \varrho_n(u|t') = -\int_{e_0} U(x'', x'')\frac{e^{in\gamma}}{\sigma'^2}dx'' + \int_{E(t,\tau')} V(\tau', x'')[e^{in\gamma}+J_1]\frac{dx''}{\sigma'^2} + P_n(\tau')\,.$$

We recall that $\alpha < 1$, $\alpha_0 - \nu < 1$; thus, by (10.23a), (10.21c, c'), (10.22a, b, e) and since $\varrho \leqq c^* l(t)$ $\left(\text{for } \varrho \leqq b(t)\right)$, one has

$$(10.24\text{b}) \qquad |\varrho_n(u|t')| < c^*|n| l^{-\alpha}(t) \log\left(\frac{c^*}{l(t)}\right) \quad \left(\text{if } \alpha \geqq \alpha_0 - \nu;\ \varrho \leqq b(t)\right),$$

$$\leqq c^*|n| l^{-(\alpha_0-\nu)}(t) \quad (\text{if } \alpha_0 - \nu > \alpha)\,.$$

In agreement with (10.23) we write

$$h_k(v) = h_k(v|t) = \frac{1}{2\pi}\int_{E_2} v(t, \tau')\frac{e^{ik\psi}}{\varrho^2}d\tau' \quad (d\tau' = d\tau_1' d\tau_2')\,;$$

this is a principal integral extended over the total plane P_t. With (10.5b) in view, consider the operational product $\left(\text{cf. } (10.24)\right)$

$$(10.25) \qquad A_k A_n(u|t) = \int_{\varrho \leqq b(t)} [2\pi h_n(U|t') + \varrho_n(u|t')]\frac{e^{ik\psi}}{\varrho^2}d\tau'$$

$$= 4\pi^2 h_k h_n(U|t) + \Gamma_{kn};\ \Gamma_{kn} = \int_{\varrho \leqq b(t)} \varrho_n(u|t')\frac{e^{in\psi}}{\varrho^2}d\tau' - \int_{\varrho > b(t)} 2\pi h_n(U|t')\frac{e^{ik\psi}}{\varrho^2}d\tau'\,.$$

It is observed that here $h_k h_n(U|t)$ is the product of two principal operators, extended over all of E_2; the formulas of MICHLIN hold for this product; thus $\left(\text{cf. } (M;\ \text{p. } 90)\right)$, on writing $h = h_1$,

$$(10.25a)\qquad h_n(U|t) = \frac{1}{n} h^n(U|t) \quad (n > 0);$$

$$h_{-n}(U|t) = \frac{(-1)^n}{n} h^{-n}(U|t) \ (n > 0);\ h_{-1}(U|t) = -h^{-1}(U|t);$$

$$h^{-1}h(U|t) = hh^{-1}(U|t) = h^0(U|t) = U(o,o) = u(t);$$

the last relation follows by (10.13a), (10.12), on noting that $t_i''(o) = t_i$ $(i = 1, 2, 3)$. From (10.25a) it is inferred that, for $k > 0$, $n > 0$, one has

$$(10.25b)\qquad h_k h_n = \frac{1}{kn} h^{k+n}, \quad h_k h_{-n} = \frac{(-1)^n}{kn} h^{k-n},$$

$$h_{-k} h_n = \frac{(-1)^k}{kn} h^{-k+n}, \quad h_{-k} h_{-n} = \frac{(-1)^{k+n}}{kn} h^{-k-n}.$$

It will be shown that the second integral in the expression for Γ_{kn} (10.25) *satisfies*

$$(10.26)\qquad \left| \int_{\varrho > b(t)} 2\pi h_n(U|t') \frac{e^{ik\psi}}{\varrho^2} d\tau' \right| \leqq c^* l^{-\alpha}(t) \log^2\left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha \geqq \alpha_0 - \nu),$$

$$\leqq c^* l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha_0 - \nu > \alpha).$$

Inasmuch as

$$\int \frac{e^{in\gamma}}{\sigma'^2} dx'' = 0 \quad \left(\text{integration over } \sigma' \leqq l(t)\right),$$

by (10.23) we have

$$(1^\circ)\qquad 2\pi h_n(U|t') = \int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' + \int_{e_1} [U(x'', x'') - U(\tau', \tau')] \frac{e^{in\gamma}}{\sigma'^2} dx'',$$

where e_1 is the circular region $\sigma' \leqq l(t)$ and $e' = E_2 - e_1$. In view of (10.17c) (valid for the extension function $U(x'', x'')$)

$$(2^\circ)\qquad \left| \int_{e_1} [U(x'', x'') - U(\tau', \tau')] \frac{e^{in\gamma}}{\sigma'^2} dx'' \right| \leqq c^* l^{-\alpha_0}(t) \int_{e_1} \sigma'^{\nu} \frac{dx''}{\sigma'^2} \leqq c^* l^{\nu-\alpha_0}(t).$$

The number a in (10.19a) may be taken $> l(t)$; when $\varrho \geqq 2a$ one has

$$U(x'', x'') = U(\tau', \tau') = 0 \quad \left(\text{for } \sigma' \leqq l(t)\right)$$

(since then $\varrho'' > a$); thus

$$(3^\circ)\qquad \int_{e_1} [U(x'', x'') - U(\tau', \tau')] \frac{e^{in\gamma}}{\sigma'^2} dx'' = 0 \quad (\text{for } \varrho \geqq 2a).$$

Suppose $\varrho \leqq 2a$; if $\sigma' \geqq 3a$, in view of the triangular inequality

$$\sigma' \leqq \varrho + \varrho'' \leqq 2a + \varrho'',$$

we shall have $\varrho'' \geqq a$ and (by (10.19a)) $U(x'', x'') = 0$; hence

$$\int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' = \int_{e0} \dots \quad (\text{for } \varrho \leqq 2a) ,$$

where e^0 is the region $l(t) \leqq \sigma' \leqq 3a$; accordingly by (10.17b) (for the extension)

$$(4^\circ) \qquad \left| \int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' \right| \leqq c^* l^{-\alpha}(t) \int_0^{2\pi} d\gamma \int_{l(t)}^{3a} \frac{d\sigma'}{\sigma'} \leqq c^* l^{-\alpha}(t) \log \left(\frac{c^*}{l(t)} \right)$$

when $\varrho \leqq 2a$. Consider the case $\varrho > 2a$; then the circular regions

$$\varrho'' \leqq a, \quad \sigma' \leqq l(t)$$

are exterior each other; the first of these lies in the regions s' consisting of points x'' such that

$$(5_0) \qquad \varrho - a \leqq \sigma' \leqq \varrho + a, \qquad -\lambda_0 \leqq \lambda \leqq \lambda_0 \qquad \left(\lambda_0 = \text{arc sin} \frac{a}{\varrho} \right);$$

here σ' and angle λ are thought of as polar coordinates of x'', with pole at τ' and the polar axis extending from τ' through O; the region (5_0) is bounded by portions of the tangents from τ' to the circle $\varrho'' = a$ and by circular arcs with center at τ' and radii $\varrho \pm a$; since $U(x'', x'') = 0$ for $\varrho'' \geqq a$, we have

$$\int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' = \int_{s'} \dots ;$$

hence by (10.17b) (for the extension)

$$(5^\circ) \qquad \left| \int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' \right| = \left| \int_{s'} \dots \right| \leqq c^* l^{-\alpha}(t) \int_{s'} \frac{dx''}{\sigma'^2} \quad (\text{for } \varrho > 2a);$$

here

$$\int_{s'} \frac{dx''}{\sigma'^2} = \int_{-\lambda_0}^{\lambda_0} d\lambda \int_{\varrho-a}^{\varrho+a} \frac{d\sigma'}{\sigma'} = 2\lambda_0 \log \left[1 + \frac{2a}{\varrho - a} \right] < 2\lambda_0 \frac{2a}{\varrho - a} < 8a\lambda_0 \varrho^{-1};$$

since $a\varrho^{-1} < 2^{-1}$ and, for $0 < u < \frac{1}{2}$,

$$\text{arc sin } u = (1 - v^2)^{-\frac{1}{2}} u \quad (\text{some } 0 < v < u) < \frac{2}{\sqrt{3}} u ,$$

one has

$$\lambda_0 < \frac{2}{\sqrt{3}} \frac{a}{\varrho};$$

hence the integral in the last member in (5°) is bounded by $c^* \varrho^{-2}$; thus

$$(6^\circ) \qquad \left| \int_{e'} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'' \right| \leqq c^* l^{-\alpha}(t) \varrho^{-2} \quad (\text{for } \varrho > 2a) .$$

In view of (1°), (4°), (2°), when $\varrho \leqq 2a$, and by (1°), (6°), (3°), for $\varrho > 2a$, one has

$$(7°) \qquad |h_n(U|t')| \leqq c^* l^{-\alpha}(t) \log\left(\frac{c^*}{l(t)}\right) + c^* l^{\nu-\alpha_0}(t) = C(t) \quad (\varrho \leqq 2a);$$

$$|h_n(U|t')| \leqq c^* l^{-\alpha}(t) \varrho^{-2} \quad (\varrho > 2a).$$

The integral in (10.26) *is bounded by* $\Gamma' + \Gamma''$, *where*

$$\Gamma' = \int' 2\pi |h_n(U|t')| \frac{d\tau'}{\varrho^2} \qquad (\text{integration over } b(t) \leqq \varrho \leqq 2a),$$

$$\Gamma'' = \int'' 2\pi |h_n(U|t')| \frac{d\tau'}{\varrho^2} \qquad (\text{integration over } 2a \leqq \varrho < \infty).$$

Since $b(t) > c^* l(t)$, by (7°) one has

$$\Gamma' \leqq 2\pi C(t) \int_0^{2\pi} d\psi \int_{b(t)}^{2a} \frac{d\varrho}{\varrho} \leqq c^* C(t) \log \frac{c^*}{l(t)},$$

$$\Gamma'' \leqq c^* l^{-\alpha}(t) \int_0^{2\pi} d\psi \int_{2a}^{\infty} \varrho^{-3} d\varrho \leqq c^* l^{-\alpha}(t).$$

From the above (10.26) *follows.*

Turning to the first integral in the expression for Γ_{kn} (10.25), we note that in view of (10.24b)

$$(10.27) \qquad \int_{\varrho \leqq b(t)} \varrho_n(u|t') \frac{e^{ik\psi}}{\varrho^2} d\tau' = \int_{\varrho \leqq \alpha l(t)} \varrho_n(u|t') \frac{e^{ik\psi}}{\varrho^2} d\tau' + \Lambda^{k,n},$$

where (with α not to be confused with the same letter in $[\alpha|S]$)

$$(10.27a) \qquad \alpha = c^* < 1 \quad (\alpha \text{ as small as desired; } \alpha l(t) \leqq b(t)),$$

$$(10.27b) \qquad |\Lambda^{k,n}| \leqq c^* |n| l^{-\alpha}(t) \log\left(\frac{c^*}{l(t)}\right), \quad \text{or} < c^* |n| l^{\nu-\alpha_0}(t).$$

By (10.24a) (with $e_0 = E_2 - E(t, \tau')$)

$$(10.27c) \qquad \int_{\varrho \leqq \alpha l(t)} \varrho_n(u|t') \frac{e^{ik\psi}}{\varrho^2} d\tau' = \sum_{i=1}^{3} \int_{\varrho \leqq \alpha l(t)} \varrho_{n,i}(\tau') \frac{e^{ik\psi}}{\varrho^2} d\tau',$$

$$\varrho_{n,1}(\tau') = -\int_{e_0} U(x'', x'') \frac{e^{in\gamma}}{\sigma'^2} dx'', \quad \varrho_{n,2}(\tau') = \int_{E(t,\tau)} V(\tau', x'')(e^{in\gamma} + J_1) \frac{dx''}{\sigma'^2},$$

$$\varrho_{n,3}(\tau') = P_n(\tau').$$

It will be first shown that

$$(10.28) \qquad \left| \int_{\varrho \leqq \alpha l(t)} \varrho_{n,1}(\tau') \frac{e^{ik\psi}}{\varrho^2} d\tau' \right| \leqq c^* l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right).$$

Let $\sigma_0'(=\varrho'')$ be the distance from O to x''. Since $U(x'', x'') = 0$ (for $\sigma_0' > a$), one may express $\varrho_{n,1}(\tau')$ (10.27c) in the form

$$(1^*)\qquad \varrho_{n,1}(\tau') = -\int_{\bar{e}_0} U(x'', x'')\frac{e^{in\gamma}}{\sigma'^2}\,dx'' ,$$

where $\bar{e}_0$ is the part of e_0 for which $\sigma_0' \leqq a$, that is, *$\bar{e}_0$ is the set of points x'' such that simultaneously*

$$(1^\circ)\qquad \sigma' > \sigma'(\gamma)\ \ (0 \leqq \gamma \leqq 2\pi),\quad \sigma_0' \leqq a .$$

The outer boundary of $\bar{e}_0$ is independent of ϱ; the inner boundary of $\bar{e}_0$ depends on τ' and, thus, on ϱ. One has

$$(2^\circ)\qquad -\varrho_{n,1}(\tau') = \nu_1(\tau')+\nu_2(\tau'),\ \ \nu_1(\tau') = U(\tau', \tau')\int_{\bar{e}_0}\frac{e^{in\gamma}}{\sigma'^2}\,dx'' ,$$

$$\nu_2(\tau') = \int_{\bar{e}_0}[U(x'', x'')-U(\tau', \tau')]\frac{e^{in\gamma}}{\sigma'^2}\,dx'' .$$

We note that

$$(2')\qquad \sigma' = r(\varrho, \sigma_0', \omega) = -\varrho\cos\omega+\sqrt{\sigma_0'^2-\varrho^2\sin^2\omega},\quad \omega = \gamma-\psi ,$$

and $\big($for $\varrho \leqq \alpha l(t)\big)$

$$(2'')\qquad c^* \leqq a-\alpha l(t) \leqq a-\varrho \leqq r(\varrho, a, \omega) \leqq a+\varrho \leqq a+\alpha l(t) \leqq a^0 = c^* .$$

The set $\bar{e}_0$ consists of points x'' such that $\sigma'(\gamma) < \sigma' \leqq r(\varrho, a, \gamma-\psi)$. Thus, by (10.21b)

$$\int_{\bar{e}_0}\frac{e^{in\gamma}}{\sigma'^2}\,dx'' = \int_0^{2\pi}e^{in\gamma}d\gamma\int_{\sigma'(\gamma)}^{r(\varrho,a,\omega)}\frac{d\sigma'}{\sigma'} = \int_0^{2\pi}\log\left[\frac{r(\varrho, a, \omega)}{a}\sqrt{1+\lambda^2(\gamma)}\right]e^{in\gamma}d\gamma .$$

Now, $\lambda(\gamma)$ is uniformly $O(\varrho)$ and

$$\left|\log\frac{r(\varrho, a, \omega)}{a}\right| = \left|\log\left\{\sqrt{1-\left(\frac{\varrho}{a}\right)^2\sin^2\omega}-\frac{\varrho}{a}\cos\omega\right\}\right| \leqq c^*\varrho\quad \big(\text{for } \varrho \leqq \alpha l(t)\big) .$$

Hence

$$\left|\int_{\bar{e}_0}\sigma'^{-2}e^{in\gamma}dx''\right| \leqq c^*\varrho+c^*\varrho^2 \leqq c^*\varrho ;$$

thus, in view of (10.17b),

$$(3^\circ)\qquad \left|\int_{\varrho\leqq\alpha l(t)}\frac{e^{ik\psi}}{\varrho^2}\,\nu_1(\tau')d\tau'\right| \leqq c^*l^{1-\alpha}(t) .$$

We have

$$(4^\circ)\quad \nu_1(\tau') = \alpha_1(\tau')-\alpha_2(\tau')+\alpha_3(\tau'),\ \ \alpha_j(\tau') = \int_{e_j}[U(x'', x'')-U(\tau', \tau')]\frac{e^{in\gamma}}{\sigma'^2}\,dx'' ;$$

here e_1 is the set of points x'' such that $b(t) \leqq \sigma_0' \leqq a$; e_2 is $E(t, \tau')-E(t, o)$ and e_3 is

$E(t, o)-E(t, \tau')$. Now τ' is interior the region $\varrho < b(t)$ (since $\varrho \leqq \alpha l(t) < b(t)$ for α sufficiently small). The least distance from τ' to the circumference $\varrho = b(t)$ is $b(t)-\varrho \geqq c^*l(t)$. When x'' is in e_2, one has $b(t)-\varrho \leqq \sigma' \leqq \sigma'(\gamma)$. Thus by (10.17c) (for the extension)

$$(4_0) \qquad |\alpha_2(\tau')| \leqq c^*l^{-\alpha_0}(t)\int_{e_2} \sigma'^{\nu-2}dx'' \leqq c^*l^{-\alpha_0}(t)\int' d\gamma \int_{b(t)-\varrho}^{\varrho'(\gamma)} \sigma'^{\nu-1}d\sigma'$$

$$\leqq c^*l^{-\alpha_0}(t)\int' [\sigma'(\gamma)^\nu-(b(t)-\varrho)^\nu]d\gamma$$

(primed integration is over γ for which $\sigma_0' > b(t)$). Observing that $f(x) \geqq \mathrm{H} > 0$, $f(x) \subset \mathrm{Lip}\ 1$ implies

$$|f^\nu(x')-f^\nu(x)| \leqq c^*\mathrm{H}^{\nu-1}|x'-x|$$

and noting that $|b(t')-b(t)| \leqq c^*\varrho$, while $b(t') \geqq c^*l(t)$ (for $\varrho \leqq \alpha l(t)$), we infer

$$|b^\nu(t')-b^\nu(t)| \leqq c^*l^{\nu-1}(t)\varrho .$$

On the other hand (with some $0 < p < \varrho$),

$$|(b(t)-\varrho)^\nu-b^\nu(t)| = |\nu(b(t)-p)^{\nu-1}\varrho| \leqq c^*l^{\nu-1}(t)\varrho .$$

Hence (since $\lambda(\gamma) = O(\varrho)$)

$$|\sigma'^\nu(\gamma)-(b(t)-\varrho)^\nu| \leqq |\sigma'^\nu(\gamma)-b^\nu(t)|+|b^\nu(t)-(b(t)-\varrho)^\nu| \leqq c^*l^{\nu-1}(t)\varrho+$$

$$+|(b^\nu(t')-b^\nu(t))(1+\lambda^2(\gamma))^{-\frac{\nu}{2}}+b^\nu(t)[(1+\lambda^2(\gamma))^{-\frac{\nu}{2}}-1]| \leqq c^*l^{\nu-1}(t)\varrho+c^*l^{\nu-1}(t)\varrho+c^*l^\nu(t)\varrho^2.$$

The last member here is $O(l^{\nu-1}(t)\varrho)$. Accordingly, by (4_0),

$$(4') \qquad |\alpha_2(\tau')| \leqq c^*l^{-\alpha_0+\nu-1}(t)\varrho \quad (\text{same for } \alpha_3(\tau'))$$

and

$$(5^\circ) \qquad \left|\int_{\varrho\leqq\alpha l(t)} \frac{e^{ik\psi}}{\varrho^2}[-\alpha_2(\tau')+\alpha_3(\tau')]d\tau'\right| \leqq c^*l^{\nu-\alpha_0}(t) .$$

Inasmuch as for $\sigma_0' > b(t)$ and $\varrho \leqq \alpha l(t)$ one has

$$\sigma'^{-2} = \sigma_0'^{-2}[1+O(\varrho\sigma_0'^{-1})] = \sigma_0'^{-2}+O(\varrho\sigma_0'^{-3}) ,$$

by (4°) it is inferred that

$$(6^\circ) \qquad \int_{\varrho\leqq\alpha l(t)} \varrho^{-2}e^{ik\psi}\alpha_1(\tau')d\tau' = \beta_1(\tau')+\beta_2(\tau') ,$$

$$\beta_1(\tau') = \int_{\varrho\leqq\alpha l(t)} \frac{e^{ik\psi}}{\varrho^2}d\tau'\int_{e_1} [U(x'', x'') - U(\tau', \tau')]\frac{e^{in\gamma}}{\sigma_0'^2}dx'' ,$$

$$\beta_2(\tau') = \int_{\varrho\leqq\alpha l(t)} \frac{e^{ik\psi}}{\varrho^2}d\tau'\int_{e_1} [U(x'', x'')-U(\tau', \tau')]O\left(\frac{\varrho}{\sigma_0'^3}\right)dx'' .$$

Within the ranges of integration $\sigma' \leqq c^*\sigma_0'$; thus by (10.17c) (for the extension) $U(x'', x'') - U(\tau', \tau')$ is $O(l^{-\alpha_0}(t))\sigma'^{\nu} = O(l^{-\alpha_0}(t)\sigma_0'^{\nu})$; whence

$$(7^\circ) \qquad |\beta_2(\tau')| \leqq c^* l^{-\alpha_0}(t) \int_{\varrho \leqq \alpha l(t)} \frac{d\tau'}{\varrho^2} \int_{e_1} \sigma_0'^{\nu} \frac{\varrho}{\sigma_0'^3}\, dx'' \leqq c^* l^{-\alpha_0}(t) \int_0^{\alpha l(t)} d\varrho \cdot$$

$$\cdot \int_{b(t)}^{a} \sigma_0'^{\nu-2} d\sigma_0' \leqq c^* l^{\nu-\alpha_0}(t) \quad (\text{if } \nu < 1), \leqq c^* l^{1-\alpha_0} \log\left(\frac{c^*}{l(t)}\right) \quad (\text{if } \nu = 1)\,.$$

As a consequence of GIRAUD's work, a singular integral applied to an ordinary integral can be expressed with the order of integration changed; thus (e_1 being independent of τ')

$$\beta_1(\tau') = \int_{e_1} \frac{e^{in\gamma}}{\sigma_0'^2}\, dx'' \int_{\varrho \leqq \alpha l(t)} [U(x'', x'') - U(\tau', \tau')] \frac{e^{ik\psi}}{\varrho^2}\, d\tau'\,;$$

on writing

$$U(x'', x'') - U(\tau', \tau') = \big(U(x'', x'') - U(o, o)\big) - \big(U(\tau', \tau') - U(o, o)\big)$$

and noting that (with $dx'' = \sigma_0' d\sigma_0' d\gamma$)

$$\int_{e_1} \frac{e^{in\gamma}}{\sigma_0'^2}\, dx'' = \int_0^{2\pi} e^{in\gamma} d\gamma \int_{b(t)}^{a} \frac{d\sigma_0'}{\sigma_0'} = 0\,,$$

we obtain

$$|\beta_1(\tau')| = \left| \int_{e_1} \frac{e^{in\gamma}}{\sigma_0'^2}\, dx'' \int_{\varrho \leqq \alpha l(t)} [U(\tau', \tau') - U(o, o)] \frac{e^{ik\psi}}{\varrho^2}\, d\tau' \right|$$

$$\leqq c^* l^{-\alpha_0}(t) \int_{e_1} \frac{dx''}{\sigma_0'^2} \int_{\varrho \leqq \alpha l(t)} \varrho^{\nu-2} d\tau' \leqq c^* l^{-\alpha_0}(t) \int_{b(t)}^{a} \frac{d\sigma_0'}{\sigma_0'} \int_0^{\alpha l(t)} \varrho^{\nu-1} d\varrho\,;$$

thus

$$(8^\circ) \qquad |\beta_1(\tau')| \leqq c^* l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right).$$

The conclusion (10.28) *ensues by* (2°)–(8°).

We show next that

$$(10.29) \qquad \left| \int_{\varrho \leqq \alpha l(t)} \varrho_{n,2}(\tau') \frac{e^{ik\psi}}{\varrho^2}\, d\tau' \right| \leqq c^* |n| l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right) r^{\nu}(o, t)\,.$$

By (10.27c)

$$(\mathrm{I}_1) \qquad \varrho_{n,2}(\tau') = \sigma_1(\tau') + \sigma_2(\tau') - \sigma_3(\tau')\,; \ \ \sigma_j(\tau') = \int_{e_j} V(\tau', x'')(e^{in\gamma} + J_1) \frac{dx''}{\sigma'^2}$$

$$(j = 2, 3)\,; \ \ \sigma_1(\tau') = \int_{E(t,o)} V(\tau', x'')(e^{in\gamma} + J_1) \frac{dx''}{\sigma'^2}\,,$$

where the e_j are as in (4°). In view of (10.16a), (10.21c′)

$$|\sigma_2(\tau')| \leqq c^* l^{-\alpha_0}(t) r(o, t)^\nu \int_{e_2} \sigma'^{\nu-2} dx'' ;$$

except for the factor $r(o, t)^\nu$ the last member here is identical with the second term in (4_0); hence corresponding to (4′) we now have

$$|\sigma_j(\tau')| \leqq c^* l^{-\alpha_0+\nu-1}(t) r(o, t)^\nu \varrho \quad (j = 2, 3) .$$

Thus

$$(\mathrm{I}_2) \qquad \left| \int_{\varrho \leq \alpha l(t)} |[\sigma_2(\tau') - \sigma_3(\tau')] \frac{e^{ik\psi}}{\varrho^2} d\tau' \right| \leqq c^* l^{\nu-\alpha_0}(t) r(o, t)^\nu .$$

On the other hand,

$$(\mathrm{I}_3) \qquad \int_{\varrho \leq \alpha l(t)} \sigma_1(\tau') \frac{e^{ik\psi}}{\varrho^2} d\tau' = \Lambda_0 + \Lambda_1 ,$$

$$\Lambda_0 = \int_{\varrho \leq \alpha l(t)} \frac{e^{ik\psi}}{\varrho^2} d\tau' \int_{E(t,o)} V(\tau', x'') e^{in\gamma} \frac{dx''}{\sigma'^2} ,$$

$$\Lambda_1 = \int_{\varrho \leq \alpha l(t)} \frac{e^{ik\psi}}{\varrho^2} d\tau' \int_{E(t,o)} V(\tau', x'') J_1 \frac{dx''}{\sigma'^2} .$$

By (10.16a), (10.9a)

$$(\mathrm{I}_4) \qquad |\Lambda_1| \leqq c^* |n| l^{-\alpha_0}(t) r(o, t)^\nu \int_{\varrho \leq \alpha l(t)} d\tau' \int_{\sigma_0' \leq b(t)} \sigma'^{\nu-2} dx'' .$$

We let $dx'' = \sigma' d\sigma' d\omega$ and, on noting (2′) (subsequent (10.28)), obtain

$$\int_{\sigma_0' \leq b(t)} \sigma'^{\nu-2} dx'' = \int_0^{2\pi} d\omega \int_{\sigma_0'=0}^{b(t)} \sigma'^{\nu-1} d\sigma' \leqq c^* \int \left| \left(\sqrt{b^2(t) - \varrho^2 \sin^2 \omega} - \varrho \cos \omega \right)^\nu - \varrho^\nu \right| d\omega$$
$$\leqq c^* b^\nu(t) \leqq c^* l^\nu(t) ,$$

inasmuch as $\varrho \leqq \alpha l(t) \leqq \frac{1}{2} b(t)$ (for $\alpha = c^*$ sufficiently small); hence

$$(\mathrm{I}_5) \qquad |\Lambda_1(t)| \leqq c^* |n| l^{\nu-\alpha_0+2}(t) r(o, t)^\nu .$$

The inequalities (10.16a), (10.16a′) may be written in the form

$$(10.30) \qquad |V(\tau', x'')| \leqq c^* l^{-\alpha_0}(t) r(o, t)^\nu g(\sigma', \varrho) ;$$

$$(10.30\mathrm{a}) \qquad g(\sigma', \varrho) = \sigma'^\nu \text{ (for } \sigma' \leqq \varrho), = \varrho^\nu \text{ (for } \varrho \leqq \sigma') .$$

For Λ_0 of (I_3) one then has

$$|\Lambda_0| \leqq c^* l^{-\alpha_0}(t) r^\nu(o, t) \Gamma_0, \quad \Gamma_0 = \int_0^{\alpha l(t)} \frac{d\varrho}{\varrho} \int_{E(t, o)} g(\varrho, \sigma') \frac{dx''}{\sigma'^2} ;$$

here, by ((2′) after (10.28)) and on writing $\sigma'(\varrho, \omega) = \sqrt{b^2(t) - \varrho^2 \cos^2 \omega} - \varrho \cos \omega$, one has

$$\Gamma_0 = \int_0^{\alpha l(t)} \frac{d\varrho}{\varrho} \int_0^{2\pi} d\omega \int_0^{\sigma'(\varrho,\omega)} g(\varrho, \sigma') \frac{d\sigma'}{\sigma'} \leqq c^* \int_0^{\alpha l(t)} \frac{d\varrho}{\varrho} \int_0^{\varrho+b(t)} g(\varrho, \sigma') \frac{d\sigma'}{\sigma'};$$

$$\Gamma_0 \leqq c^* \int_0^{\alpha l(t)} \frac{d\varrho}{\varrho} \left[\int_0^{\varrho} \sigma'^{\nu} \frac{d\sigma'}{\sigma'} + \int_{\varrho}^{\varrho+b(t)} \varrho^{\nu} \frac{d\sigma'}{\sigma'} \right];$$

a direct calculation gives $\Gamma_0 = O\left(l^{\nu}(t) \log \frac{c^*}{l(t)}\right)$; thus

$$(\mathrm{I}_6) \qquad \Lambda_0 \leqq c^* l^{\nu-\alpha_0}(t) \log \left(\frac{c^*}{l(t)}\right) r^{\nu}(o, t) .$$

The result (10.29) *ensues by* (I_1)–(I_6) (actually a sharper inequality can be stated).

For $\varrho_{n,3}(\tau')$ (10.27c) one has (cf. (10.22), (10.22a))

$$\varrho_{n,3}(\tau') = P_n'(\tau') + U(\tau', \tau') P_n''(\tau');$$

in view of (10.22b), (10.22e)

$$(10.31) \qquad \begin{aligned} &\left| \int_{\varrho \leqq \alpha l(t)} \frac{e^{ik\psi}}{\varrho^2} P_n'(\tau') d\tau' \right| \leqq c^* |n| l^{\nu-\alpha_0}(t) l^2(t) ; \\ &\left| \int_{\varrho \leqq \alpha l(t)} \frac{e^{ik\psi}}{\varrho^2} U(\tau', \tau') P_n''(\tau') d\tau' \right| \leqq c^* |n| l^{-\alpha}(t) \log \left(\frac{c^*}{l(t)}\right) l^2(t) ; \\ &\left| \int_{\varrho \leqq \alpha l(t)} \varrho_{n,3}(\tau') \frac{e^{ik\psi}}{\varrho^2} d\tau' \right| \leqq c^* |n| l^{-\alpha+2}(t) \log \left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha \geqq \alpha_0 - \nu) , \\ &\qquad \leqq c^* |n| l^{\nu-\alpha_0+2}(t) \log \left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha < \alpha_0 - \nu) . \end{aligned}$$

By (10.27c, 28, 29, 31) the first term in the second member of (10.27) is $O\left(|n| \log \left(\frac{c^*}{l(t)}\right) \cdot l^{\nu-\alpha_0}(t)\right)$; hence by (10.27b) the integral in the first member of (10.27) is $O\left(|n| \log \left(\frac{c^*}{l(t)}\right) \cdot l^{-\alpha'}(t)\right)$, where α' is max. $(\alpha, \alpha_0 - \nu)$. Deletion of the assumption that $c = 0$ leaves the conclusions intact. By (10.25, 26) we infer.

Theorem 10.32. *Let* $u(y) \subset [\alpha | S]$ (*cf.* (10.4), (10.4a)); *consider the singular integrals* A_n, A_k ((10.5), (10.5b); $n, k \neq 0$). *For the operational product one has*

$$(10.32\mathrm{a}) \qquad A_k A_n(u|t) = 4\pi^2 h_k h_n (U|t) + \Gamma_{kn} ,$$

where h_k, h_n *are principal operators extended over the plane* E_2 *and are identical with the operators so designated in* [M; p. 90] (*cf.* 10.25a, b), *while*

$$(10.32\text{b}) \qquad |\Gamma_{kn}| \leqq c^*|n| l^{-\alpha}(t) \log^2\left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha \geqq \alpha_0 - \nu)\,,$$

$$\leqq c^*|n| l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right) \quad (\text{if } \alpha < \alpha_0 - \nu)\,.$$

11. Integral equations. Consider the singular integral equation

$$(11.1) \qquad a(t)u(t) + \int_S \frac{k(y,\,t)}{r^2(y,\,t)} u(y) d\sigma(y) + T(u|t) = f(t)\,;$$

here $k(y, t)r^{-2}(y, t)$ (3.1) is a principal kernel as described in section 3; $f(t)$ is given, $\subset [\text{H}|S]$ ($\text{H} < \frac{1}{2}$); $a(t)$ is of a Hölder class on S (for $l(t) > 0$) and $|a(t)| \geqq a^0 = c^*$ (as in (9.16)). T is an operator regular in the sense of

Definition 11.2. *An operator T^*, consisting (for example) of a number of ordinary integrals of type as in* (9.8c), *will be said to be regular, if for every $u(t) \subset [\alpha|S]$, satisfying* (10.4), (10.4a) *with $\alpha < \frac{1}{2}$, $\alpha_0 - \nu < \frac{1}{2}$, one has $T^*(u|t) \subset [\tau|S]$ (some $\tau < \frac{1}{2}$), while $u(t) + T^*(u|t) = F(t)$ ($F \subset L_2$ on S) is a regular Fredholm equation.*

The operator

$$(11.3) \qquad a(t)u(t) + \int_S \frac{k(y,\,t)}{r^2(y,\,t)} u(y) d\sigma(y) = A_t(u) = A(u|t)$$

is the one studied in section 9. By Lemma 9.8

$$(11.3\text{a}) \qquad A_t(u) = A_t^*(u) + A_t^0(u)\,,$$

where $A_t^0(u)$ (9.8c) is a regular operator (as a consequence of Theorem 6.36), while $A_t^*(u)$ is the singular operator (9.8b). Accordingly *the equation* (11.1) *may be written in the form*

$$(11.4) \qquad a(t)u(t) + \int_{S(O,\,b)} \frac{f(t,\,\theta)}{r^2(O,\,Y)} u(Y) dY_1 dY_2 + T'(u|t) = f(t)\,;$$

notation as in (9.4), (9.5), ... ; $f(t, \theta)$ is the characteristic of the kernel $k(y, t)r^{-2}(y, t)$ (cf. (9.4b), (9.9)); $T' = A_t^0(u) + T(u|t)$ is a regular operator. The operator

$$B_t(w) = b(t)w(t) + \int_{S(O,\,b)} \frac{g(t,\,\theta)}{r^2(O,\,Y)} w(Y) dY_1 dY_2 \quad [= B(w|t)]$$

(cf. (9.19a), (9.19), (9.17)) would certainly be a regularizing operator, as a consequence of $[M]$, if S had no edges. In the present case application of B to (11.4) yields

$$BA^*(u|t) + BT'(u|t') = B(f|t) \equiv g(t)\,,$$

which can be expressed in the form

(11.5) $$u(t)+T_t''(u)+BT'(u|t) = B(f|t) \equiv g(t)$$

This is a regular integral equation of the second kind, provided

(11.5a) $$T_t''(u),\ BT'(u|t)$$

are regular operators,

(11.5b) $$B(f|t) \subset [\mathrm{H}'|S] \quad (\mathrm{H}' < \tfrac{1}{2}) .$$

Thus the problem of regularizing the integral equation (11.1) *is solved when* (11.5a), (11.5b) *are proved.*

The statement with respect to (11.5b) is taken care of by

Lemma 11.6. *Suppose* $f(t) \subset [\mathrm{H}|S]$, *with* $\mathrm{H} < \frac{1}{2}$, *while the Hölder condition is of form*

(11.6a) $$|f(t')-f(t)| \leqq c^* l^{-\mathrm{H}_0}(\eta) r(t', t)^{\nu_0} \ [0 < \nu_0 \leqq 1;\ \mathrm{H}_0 \geqq \mathrm{H};\ \mathrm{H}_0 - \nu_0 < \tfrac{1}{2}]$$

(η *is* t *or* t', *whichever is nearer to edges*). *Then* $B(f|t) \subset [\mathrm{H}'|S]$, *where* $\mathrm{H}' =$ max. $(\mathrm{H}_0-\nu_0, \mathrm{H}) < \frac{1}{2}$.

Note. In many applications $\mathrm{H}_0 = \mathrm{H}+1$, $\nu_0 = 1$ and, so, the conditions of the Lemma are satisfied.

To prove the above, reverting to the notation of section 10 (cf. (10.5b)), we have

$$B(f|t) = b(t)f(t)+\Gamma(t);\ \Gamma(t) = \int_{\varrho \leq b(t)} \frac{g(t, \psi)}{\varrho^2} f(t')d\tau' = \int_{\varrho \leq b(t)} \frac{g(t, \psi)}{\varrho^2}(f(t')-f(t))d\tau'.$$

By (9.24) $|b(t)| \leqq c^*$; thus

(1°) $$b(t)f(t) \subset [\mathrm{H}|S] .$$

In view of (11.6a), and since $r(t', t) \leqq c^* \varrho$, $l^{-1}(\eta) \leqq c^* l^{-1}(t)$ (for $\varrho \leqq b(t)$),

$$|f(t')-f(t)| \leqq c^* l^{-\mathrm{H}_0}(t) \varrho^{\nu_0} .$$

Thus (since $b(t) \leqq c^* l(t)$)

$$|\Gamma(t)| \leqq c^* l^{-\mathrm{H}_0}(t) \int_0^{2\pi}\int_0^{b(t)} |g(t, \psi)| \varrho^{\nu_0-1} d\varrho d\psi \leqq c^* l^{\nu_0-\mathrm{H}_0}(t) \int_0^{2\pi} |g(t, \psi)| d\psi .$$

Accordingly, in view of (9.25), $\Gamma(t)$ is $O(l^{\nu_0-\mathrm{H}_0}(t))$; *the Lemma follows by* (1°).

Using the notation involved in (10.5)–(10.5c), one has

$$BA^*(u|t) = b(t)\left[a(t)u(t) + \int_{\varrho \leq b(t)} \frac{f(t, \psi)}{\varrho^2} u(t')d\tau'\right]$$
$$+\int_{\varrho \leq b(t)} \frac{g(t, \psi)}{\varrho^2}\left[a(t')u(t') + \int_{\varrho' \leq b(t')} \frac{f(t', \theta)}{\varrho'^2} u(t'')d\tau''\right] d\tau' .$$

Substitution of (9.9), (9.19) yields

$$BA^*(u|t) = b(t)a(t)u(t)+b(t)\sum_n{}' f_n(t)\int_{\varrho\leq b(t)} u(t')\frac{e^{in\psi}}{\varrho^2}d\tau'$$

$$+\sum_j{}' g_j(t)\int_{\varrho\leq b(t)} a(t')u(t')\frac{e^{ij\psi}}{\varrho^2}d\tau'+\sum_j{}'\sum_n{}' g_j(t)\int_{\varrho\leq b(t)} f_n(t')\cdot$$

$$\cdot\frac{1}{\varrho^2}e^{ij\psi}\left[\int_{\varrho'\leq b(t')} u(t'')\frac{e^{in\theta}}{\varrho'^2}d\tau''\right]d\tau'.$$

We replace $f_n(t')$, $a(t')$ by

$$f_n(t)+(f_n(t')-f_n(t)),\ a(t)+(a(t')-a(t)),$$

respectively, obtaining

$$BA^*(u|t) = b(t)a(t)u(t)+\sum_n{}' b(t)f_n(t)A_n(u|t)+$$

$$+\sum_j{}' g_j(t)a(t)A_j(u|t)+\sum_j{}'\sum_n{}' g_j(t)f_n(t)A_jA_n(u|t)+\Lambda;$$

$$(1_0)\qquad \Lambda = \sum_j{}' g_j(t)\int_{\varrho\leq b(t)} u(t')(a(t')-a(t))\frac{e^{ij\psi}}{\varrho^2}d\tau'+$$

$$+\sum_j{}'\sum_n{}' g_j(t)\int_{\varrho\leq b(t)}(f_n(t')-f_n(t))\frac{e^{ij\psi}}{\varrho^2}\left[\int_{\varrho'\leq b(t')} u(t'')\frac{e^{in\theta}}{\varrho'^2}d\tau''\right]d\tau'$$

We have

$$A_n(u|t) = 2\pi h_n(u)+\Gamma_n,\ A_jA_n(u|t) = 4\pi^2h_jh_n(u)+\Gamma_{jn},$$

where the h_j, Γ_{jn} are operators of Theorem 10.32; Γ_n is $\varrho_n(u|t)$ from (10.24) and satisfies (10.24b). Thus

$$(2_0)\qquad BA^*(u|t) = b(t)a(t)u(t)+2\pi\sum_n{}' b(t)f_n(t)h_n(u)+2\pi\sum_j{}' g_j(t)a(t)h_j(u)$$

$$+4\pi^2\sum_j{}'\sum_n{}' g_j(t)f_n(t)h_jh_n(u)+\Lambda^0+\Lambda;$$

$$(3_0)\qquad \Lambda^0 = \sum_n{}' b(t)f_n(t)\Gamma_n+\sum_j{}' g_j(t)a(t)\Gamma_j+\sum_j{}'\sum_n{}' g_j(t)f_n(t)\Gamma_{jn}.$$

In view of (10.25a), (10.25b), the terms in (2_0), apart from $\Lambda^0+\Lambda$, are expressible in the form

$$(4_0)\qquad b(t)a(t)h^0(u)+\sum_n{}' b(t)a_n(t)h^n(u)+\sum_j{}' b_j(t)a(t)h^j(u)+\sum_j{}'\sum_n{}' b_j(t)a_n(t)h^{j+n}(u);$$

here the $a_n(t)$ are from (9.15b) (cf. (9.9)) and the $b_n(t)$ ($b_0(t) = b(t)$) are from (9.17), (9.19); $a_0(t) = a(t)$. By (9.17) and since the series for $a(t,\varphi)$, $b(t,\varphi)$ are absolutely convergent one has

$$\sum_{j=-\infty}^{\infty} b_j(t)a_{m-j}(t) = 0\ (\text{for } m\neq 0),\ = 1\ (\text{for } m = 0);$$

hence the terms (4_0) combine into

$$\sum_{j=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' b_j(t)a_n(t)h^{j+n}(u) = h^0(u) = u(t) .$$

Accordingly

$$BA^*(u|t) = u(t)+T_t''(u),\ T_t''(u) = \Lambda^0+\Lambda\ \bigl(\text{cf. } (1_0),\ (3_0)\bigr) , \tag{11.7}$$

a fact of type already utilized, in [*M*], in MICHLIN's treatment of equations with integrations over E_2 or over 'edgeless' manifolds. With respect to $a(t)$ $\bigl(= a_0(t)\bigr)$ we have previously assumed $|a(t)| \geqq a^0$ and that $a(t)$ is of a Hölder class for $l(t) > 0$; we shall now specify the behaviour near edges and Hölder conditions by the assumption

$$|a(t)| \leqq c^*;\ |a(t')-a(t)| \leqq c^* l^{-\beta}(t) r^h(t', t) \quad \bigl(\text{for } \varrho \leqq b(t)\bigr) , \tag{11.8}$$

(here $0 < h \leqq 1$; $\beta \geqq 0$; $\alpha+\beta < 1$, as stated previously). These conditions are in conformity with the inequalities (9.14b) for $f_n(t), f_n(t')-f_n(t)$ (one may use a modification of (11.8)).

By (9.25a), (9.21), (11.8), (9.14b)

$$\left[\int_0^{2\pi} [|g(t', \varphi)-g(t,\varphi)|^2 d\varphi\right]^{\frac{1}{2}} \leqq c^* \left[\int_0^{2\pi} |f(t', \varphi)-f(t, \varphi)|^2 d\varphi\right]^{\frac{1}{2}} + \tag{I$_1$}$$

$$+c^* l^{-\beta}(t)\varrho^h \leqq c^* l^{-\beta}(t)\varrho^h \quad \bigl(\text{for } \varrho \leqq b(t)\bigr) .$$

Utilizing in succession the Bessel inequalities for series and integrals, by (9.25), (10.4), (11.8) we infer that the absolute value of the simple series in Λ (1_0) is bounded by

$$\left\{\sum_j{}' |g_j(t)|^2 \cdot \sum_j{}' \left|\int_0^{2\pi} e^{ij\psi} d\psi \int_0^{b(t)} u(t')\bigl(a(t')-a(t)\bigr)\frac{d\varrho}{\varrho}\right|^2\right\}^{\frac{1}{2}} \tag{I$_2$}$$

$$\leqq c^* \left\{\int_0^{2\pi} \left|\int_0^{b(t)} u(t')\bigl(a(t')-a(t)\bigr)\frac{d\varrho}{\varrho}\right|^2 d\psi\right\}^{\frac{1}{2}} \leqq c^* l^{-\alpha-\beta+h}(t) .$$

Denoting the double series in Λ by s'' and letting

$$F_n(\psi) = \int_0^{b(t)} \bigl(f_n(t')-f_n(t)\bigr)\frac{d\varrho}{\varrho}\int_{\varrho' \leqq b(t')} \bigl(u(t'')-u(t')\bigr)\frac{e^{in\theta}}{\varrho'^2}d\tau'' ,$$

in view of (9.25) one has

$$|s''|^2 = \left|\sum_j{}' g_j(t)\int_0^{2\pi} \sum_n{}' F_n(\psi)e^{ij\psi}d\psi\right|^2 \leqq \sum_j{}' |g_j^2| \cdot$$

$$\sum_j{}' \left|\int_0^{2\pi} \sum_n{}' F_n(\psi)e^{ij\psi}d\psi\right|^2 \leqq c^* \int_0^{2\pi} \left|\sum_n{}' F_n(\psi)\right|^2 d\psi .$$

Now by (9.14b), (10.4a) and since $b(t') \leqq c^* l(t)$, we have $F_n(\psi) = O\left(l^{h-\beta+\nu-\alpha_0}(t) \cdot \dfrac{1}{n^2}\right)$. Hence $s'' = O(l^{h-\beta+\nu-\alpha_0}(t)$; together with (I_2), *this implies that*

$$|A| \leqq c^* l^{-\alpha}(t) l^{h-\beta}(t) + c^* l^{\nu-\alpha_0}(t) l^{h-\beta}(t) \leqq c^* l^{-\alpha'}(t) \tag{11.7a}$$

$(\alpha' = \max. (\alpha+\beta-h, \alpha_0-\nu+\beta-h))$. *It is noted that with* $\alpha < \frac{1}{2}$, $\alpha_0-\nu < \frac{1}{2}$, *to start with (which is a property of* u*), we shall have*

$$\alpha' < \tfrac{1}{2}\,, \tag{11.7b}$$

provided $(0 \leqq)\ \beta \leqq h$ *(the latter being a property of the kernel).*

With the aid of Theorem 10.32, (3_0), (10.24b) and of certain other cosiderations of section 9, we obtain

$$\begin{aligned} |A^0| &\leqq c^* l^{-\alpha}(t) \log^2\left(\frac{c^*}{l(t)}\right) \quad (\text{for } \alpha \geqq \alpha_0-\nu)\,, \\ &\leqq c^* l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right) \quad (\text{for } \alpha_0-\nu > \alpha) \end{aligned} \tag{11.7c}$$

at least under the conditions involved in the result (9.23b). *These are forthwith assumed.*

In view of (10.7)–(10.7c) one has the result

Lemma 11.8. *Suppose* $u \subset [\alpha|S]$, *satisfying* (10.4), (10.4a) *with* $\alpha < \frac{1}{2}$, $\alpha_0-\nu < \frac{1}{2}$; *assume that* $\beta \leqq h$ $(\alpha+\beta < 1)$ *(a condition relating to the kernel). Then* $T_t''(u)$ *(in* (11.5), (11.5a)*) satisfies*

$$\begin{aligned} |T_t''(u)| &\leqq c^* l^{-\alpha}(t) \log^2\left(\frac{c^*}{l(t)}\right) \quad (\textit{if } \alpha \geqq \alpha_0-\nu), \\ &\leqq c^* l^{\nu-\alpha_0}(t) \log\left(\frac{c^*}{l(t)}\right) \quad (\textit{if } \alpha_0-\nu > \alpha)\,. \end{aligned} \tag{11.8a}$$

As a consequence of the above $T_t''(u)$ *is a regular operator when* $\beta = 0$.

Turning to T' in (11.4), one has

$$BT'(u|t) = B(A_t^0(u)) + BT(u|t)\,. \tag{11.9}$$

$A_t^0(u)$ is given by four terms in the third member of (9.8c); these terms satisfy inequalities (9.3a), (9.3b), (9.3d), (9.6), respectively; in the latter one may put $L(t) = c^* l^{-1}(t)$. *With* $\alpha < \frac{1}{2}$, $\alpha_0-\nu < \frac{1}{2}$, $\beta \leqq h$, $\beta < \frac{1}{2}$, *we certainly have*

$$|A_t^0(u)| \leqq c^* l^{-\tau^0}(t) \quad (\text{some } \tau^0 < \tfrac{1}{2})\,; \tag{11.9a}$$

if $\alpha+\beta < \frac{1}{2}$, $\alpha_0+\beta-\nu < \frac{1}{2}$, then by (6.39) A^0 is a sum of three terms A' for which

(1°) $$|A'_{t'}(u)-A'_t(u)| \leqq c^* l^{-\mathrm{H}_0}(t) r(t',t)^{\nu^0} \leqq c^* l^{-\mathrm{H}^0}(t) \varrho^{\nu^0} \quad (\text{for } \varrho \leqq b(t)),$$

where H^0, ν^0 are some numbers such that $0 < \nu^0 \leqq 1$, $\mathrm{H}^0 \geqq \tau^0$, $\mathrm{H}^0 - \nu^0 < \frac{1}{2}$ (in a wide variety of cases $\mathrm{H}^0 = \tau^0 + 1$, $\nu^0 = 1$). We have

(2°) $$B(A_t^0(u)) = b(t)A_t^0(u) + \int_{\varrho \leqq b(t)} \frac{g(t,\psi)}{\varrho^2} A_{t'}^0(u) d\tau' .$$

Since $|b(t)| \leqq c^*$, by (11.9a) one has

(3°) $$|b(t)A_t^0(u)| \leqq c^* l^{-\tau^0}(t) .$$

In the integral, above, $A_{t'}^0(u)$ may be replaced by $A_{t'}^0(u) - A_t^0(u)$; by (1°), (9.25) this integral is a sum of three terms, each of modulus bounded by

$$\int_0^{2\pi} |g(t,\psi)| \left[\int_0^{b(t)} |A'_{t'}(u) - A'_t(u)| \frac{d\varrho}{\varrho} \right] d\psi \leqq c^* l^{-\mathrm{H}^0}(t) \int_0^{2\pi} |g(t,\psi)| d\psi \cdot$$
$$\cdot \int_0^{b(t)} \varrho^{\nu^0 - 1} d\varrho \leqq c^* l^{\nu^0 - \mathrm{H}^0}(t) \int_0^{2\pi} |g(t,\psi)| d\psi \leqq c^* l^{\nu^0 - \mathrm{H}^0}(t) .$$

Thus by virtue of (2°), (3°)

(11.9b) $$|BA_t^0(u)| \leqq c^* l^{-\mathrm{H}'}(t) \quad \left(\mathrm{H}' = \max.\,(\tau^0, \mathrm{H}^0 - \nu^0) < \tfrac{1}{2}\right).$$

It is concluded that BA^0 is a regular operator (at least when $\beta = 0$). We shall proceed under

Hypothesis 11.10. *Let the operator $T(u|t)$ in (11.1) be of the form*

(11.10a) $$T(u|t) = \int_S H(y,t) u(y) d\sigma(y) ,$$

where

(11.10b) $$H(t) = \left[\int_S |H(y,t)|^2 d\sigma(y) \right]^{\frac{1}{2}} \leqq c^* l^{-s^0}(t) \quad (0 \leqq s^0 < \tfrac{1}{2}) ;$$
$$H^*(t',t) = \left\{ \int_S |H(y,t') - H(y,t)|^2 d\sigma(y) \right\}^{\frac{1}{2}} \leqq c^* l^{-h_0}(\tau) r^s(t',t)$$

$[0 < s \leqq 1;\ h_0 - s < \frac{1}{2};\ \tau = t'$ $\left(\textit{for } l(t') \leqq l(t)\right)$, $= t$ $\left(\textit{for } l(t) < l(t')\right)]$.

With u as in Lemma 11.8

(1_0) $$|T(u|t)| \leqq c^* H(t) \left[\int_S |u(y)|^2 d\sigma(y) \right]^{\frac{1}{2}} \leqq c^* l^{-s_0}(t) .$$

On the other hand, since the integral of $|u|^2$ over S exists, one gets

(2_0) $$|T(u|t') - T(u|t)| \leqq H^*(t',t) \left[\int_S |u(y)|^2 d\sigma(y) \right]^{\frac{1}{2}}$$
$$\leqq c^* l^{-h_0}(\tau) r^s(t',t) \leqq c^* l^{-h_0}(t) r^s(t',t) \quad \left(\text{for } r(t',t) \leqq c^0 l(t)\right),$$

when $c^0 = c^*$ is suitably small (say, as in $b(t) = c^0 l(t)$). Now

$$(3_0) \qquad BT(u|t) = b(t)T(u|t) + \int_{\varrho \leq b(t)} \frac{g(t, \psi)}{\varrho^2} T(u|t')d\tau'$$

$$= b(t)T(u|t) + \int_{\varrho \leq b(t)} \frac{g(t, \psi)}{\varrho^2} [T(u|t') - T(u|t)]d\tau' .$$

The latter integral is bounded in absolute value by (cf. (9.25))

$$(4_0) \qquad \int_0^{2\pi} |g(t, \psi)| \left[\int_0^{b(t)} |T(u|t') - T(u|t)| \frac{d\varrho}{\varrho} \right] d\psi \leq c^* l^{-h_0}(t) \int_0^{2\pi} |g(t, \psi)| d\psi \cdot$$

$$\cdot \int_0^{b(t)} \varrho^{s-1} d\varrho \leq c^* l^{s-h_0}(t) \left[\int_0^{2\pi} |g^2(t, \psi)| \, d\psi \right]^{\frac{1}{2}} \leq c^* l^{s-h_0}(t) .$$

By (3_0), (1_0), (4_0) one has

$$(11.11) \qquad |BT(u|t)| \leq c^* l^{-\mathrm{H}''}(t) \quad \left(\mathrm{H}'' = \max. (s^0, h_0 - s) < \tfrac{1}{2}\right) .$$

Hence BT (as well as T) is a regular operator.

Theorem 11.12. *The problem of regularizing the singular integral equation* (11.1) (*cf. the text up to* (11.5)) *is possible when the conditions of Lemmas* 11.6, 11.8 *and of Hypothesis* 11.10 *are satisfied, at least when* $\beta = 0$ *and* (9.23b°) *holds.*

We shall terminate this work with a few remarks. Of the remaining questions outstanding is the problem of equivalence (handled in [*M*] in the cases therein considered). This and other matters will be relegated to a later work. The developments given in these pages, in so far as integral equations are concerned, can be extended along following lines.

I. Systems of integral equations.

II. Hilbert space theory.

III. Equations involving principal integrals extended over (sufficiently 'smooth') m (> 2)-dimensional manifolds, with sufficiently 'smooth' edges, imbedded in a Euclidean space of n ($> m$) dimensions.

The developments of (I) present no essentially new difficulties. Only part of the work can be extended along the line of (II). Extension to (III) would involve use of expansions into spherical harmonics (instead of Fourier series) and is possible as a consequence of a very important formula of Giraud, found in [*M*; p. 94]. The various 'regular operators' in the texte are actually equivalent to regular Fredholm integral operators (when $\beta = 0$).