THREE-DIMENSIONAL SUBSONIC FLOWS, AND ASYMPTOTIC ESTIMATES FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

During the past half century there has occurred a series of striking advances in the mathematical theory of compressible irrotational fluid flow in two dimensions. These developments have been concomitant with discoveries of general mathematical interest and to some extent with the growth of new mathematical disciplines. Among these we may cite the hodograph transformation of Chaplygin [3], the use of function-theoretical properties of quasi-conformal mappings [1, 6, 15], the theory of regular variational problems [19, 20], potential-theoretic investigations [7], the development of fixed point theorems in function space [11, 16], and the theory of pseudo-analytic functions [2].

Since most of these methods are by their nature limited in application to two-dimensional phenomena, the theory of three-dimensional flow has not fared so well, and there seems to be little literature of a precise mathematical character on the subject. There appear, in fact, to be serious difficulties in the way of a comprehensive discussion, for the study of such flows is equivalent to the study of a non-linear second order equation in three independent variables, and the problem of finding

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a priori estimates in a general case for the solutions of such equations remains open.(1)

In the present paper we discuss questions of existence, uniqueness, and asymptotic behavior of three-dimensional compressible flows. Our method of deriving asymptotic estimates is a modification of a procedure suggested by Nirenberg which will be published elsewhere. It differs from that of Nirenberg in important respects, and permits us to obtain a fairly precise estimate on asymptotic behavior of the velocity field (Theorem 1). This result is then applied to prove the uniqueness of a given subsonic flow past an obstacle B (Theorems 2, 3, 4). We remark that we do not require the flow to be uniformly subsonic (it may become sonic at B) and the uniqueness appears in a class of flows which may admit supersonic regions. We prove also that in a flow past B which is subsonic at infinity, no net force is exerted on B by the fluid (Theorem 5).(2)

Section 9 is devoted to a proof of existence of compressible flows past an obstacle B provided the maxium speed in the flow is sufficiently small (Theorems 6, 7). In principle such a result should follow by straightforward application of the method used by Frankl and Keldysh [7] in the case of two-dimensional flows, since this method is based on potential-theoretic considerations which do not respect dimension. We have been informed (oral communication) that an independent proof based on a variational method has been given by Berg and Nirenberg. The proof we present uses the fixed-point theorem of Leray and Schauder [11] and does not seem intrinsically simpler than the others; however, it leads to an explicit estimate of the maximum speed for which existence is demonstrated. In the case of polytropic flow(3) with $\gamma = 1.5$, it is sufficient that the Mach number does not exceed 0.53.(4)

Although this paper is directed toward the study of subsonic flows, the results can be interpreted as statements on the solutions of elliptic differential equations.

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(1) After the completion of this work our attention was called to a paper of H. O. Cordes [4] in which a priori estimates in a bounded region are obtained under remarkably weak hypotheses. It would be of interest to determine whether methods similar to those of Cordes will yield an improvement on the estimate at infinity which is essential for our results.

(2) This result has been announced by Ludford [13], but is not substantiated by the contents of his paper.

(3) For definitions see Section 2.

(4) *Added in proof:* Professors L. E. Payne and H. F. Weinberger have remarked that a modification of one of our estimates will lead to a significantly better result. Using this remark and also some improvements noted by the authors, it is possible to prove the existence of flows for which the Mach number does not exceed 0.7. However, it seems likely that a much stronger result can be proved. See the note "Added in proof II" at the end of this paper.

We adopt this point of view in Section 11, where we apply a modification of our method to a study of asymptotic behavior of solutions of elliptic equations in n-variables. The main result consists in the statement (Theorem 10) that if a solution $u(x)$ is restricted to a certain order of growth at infinity then it must tend to a limit with at least a prescribed rate.

Finally we present in Section 12 an extension of the classical maximum principle for solutions of elliptic equations to a statement which applies also to the point at infinity. Our method here is quite different from the argument on which the earlier sections are based, and we are able to announce a result (Theorem 11) which is true for a rather broad class of non-linear equations (which need not be uniformly elliptic), and which is nevertheless sharp in the special case of Laplace's equation. In conjunction with Theorem 1, this result implies the property of subsonic compressible flows (Theorem 8) that the speed cannot tend to a maximum at infinity.

We point out that, except for the existence theorem in Section 9, the discussion is entirely elementary, being based on formal integration by parts. The estimates we have obtained in this way suffice to establish the qualitative theorems of the present paper, but they are not best possible. In a later work we shall derive precise estimates of asymptotic behavior for solutions of elliptic equations in n-dimensions by an extension of the method of our earlier paper [6].

2. Notation and Definitions

To facilitate the exposition we shall use the summation convention,

$$a_i x_i = \sum_{i=1}^{n} a_i x_i.$$

The symbol x will denote the vector $(x_1, \ldots, x_n)$. A scalar solution[1] $\phi(x)$ of

$$\frac{\partial}{\partial x_i}\left[\varrho \frac{\partial \phi}{\partial x_i}\right] = 0, \qquad \varrho = \varrho(\phi_i^2) > 0, \quad i = 1, \ldots, 3, \tag{1}$$

will be called the *velocity potential* of an irrotational gas flow with *density* ϱ. The gradient $\nabla \phi$ of $\phi(x)$ then represents the *velocity vector* of the flow. We shall write

$$\nabla \phi = w = (u_1, u_2, u_3).$$

The *speed* q of the flow is defined as the magnitude of $\nabla \phi$,

$$q = (u_i^2)^{\frac{1}{2}}.$$

[1] We assume $\phi(x)$ has continuous partial derivatives with respect to all x_i up to the third order.

For simplicity, we shall occasionally restrict attention to equations of the form (1), although most of our results admit immediate extension to solutions of equations of the form

$$\frac{\partial}{\partial x_i}[\Theta_i(u_1, \ldots, u_n)] = 0, \qquad u_i = \frac{\partial \phi}{\partial x_i},$$

with suitable assumptions on the functions Θ_i. Our special concern is the case of *polytropic flow*, for which

$$\varrho = \left[1 - \frac{\gamma - 1}{2} q^2\right]^{1/(\gamma-1)}. \tag{2}$$

Here γ is the ratio of specific heats of the fluid, $\gamma > 1$.

If ϱ is any function of q^2, then (1) takes the form

$$a_{ij}\frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0, \tag{3}$$

with $$a_{ij} = \varrho\,\delta_{ij} + 2\varrho' u_i u_j, \qquad \varrho' = \frac{d\varrho}{dq^2}, \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

It is not hard to show that the quadratic form

$$a_{ij}\,\xi_i\,\xi_j$$

admits only two distinct eigenvalues, $\lambda_1 = \varrho + 2\varrho' q^2$, $\lambda_2 = \lambda_3 = \varrho$. Equation (3) is of elliptic type if all eigenvalues are positive, parabolic if at least one eigenvalue vanishes, otherwise hyperbolic.

If ϱ is defined by (2) there exists a *critical speed*

$$q_c = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2}}$$

and a *maximum speed* $$q_{\max} = \left(\frac{2}{\gamma - 1}\right)^{\frac{1}{2}}$$

such that (3) is elliptic, parabolic, or hyperbolic according as $0 \leq q < q_c$, $q = q_c$, $q_c < q \leq q_{\max}$, and ϱ is (in general) not defined for $q > q_{\max}$.

The speed with which infinitesimal disturbances are propagated in a flow is called the *speed of sound*,

$$a = \left(-\frac{\varrho}{2\varrho'}\right)^{\frac{1}{2}}.$$

The ratio q/a of speed in the flow to the speed of sound is called the *Mach number*,

$$M = \left(-\frac{2q^2}{\varrho}\varrho'\right)^{\frac{1}{2}}.$$

If ϱ is defined by (2), then

$$M^2 = 1 - \frac{1 - \frac{\gamma+1}{2} q^2}{1 - \frac{\gamma-1}{2} q^2}.$$

Accordingly we shall call a flow

subsonic, if $0 \leqslant q < q_c$;

sonic, if $q = q_c$;

supersonic, if $q_c < q \leqslant q_{\max}$.

A flow for which both subsonic and supersonic values of q appear will be called *mixed*.

Clearly the constants q_c and $q_{\max}$ are defined (perhaps infinite) for any density function $\varrho = \varrho(q^2)$ with $\varrho' < 0$. If this is not the case there will nevertheless be an interval $0 \leqslant q < q_c$ in which (3) is elliptic. We shall then consider only those flows for which q lies in this interval throughout the flow region, and we shall refer to such a flow as *subsonic*.

By an *obstacle* B we shall understand a closed surface which bounds a finite region in three-dimensional space. We assume that B possesses at each point a unit normal which varies continuously on the surface. Further restrictions on B will occasionally be imposed and are indicated in the context. By a *flow past* B we mean a velocity field derived from a solution of (1), which is defined throughout the exterior of B and which is directed tangentially on B. A flow is called *uniform at infinity* if the velocity vector tends to a limit as $x \to \infty$.

The symbols C and K will be used to represent constants, the value of which may change even within a given demonstration. Thus, from $\psi \leqslant C(R^2+1)^{\frac{1}{2}}$ we may conclude $\psi \leqslant CR$ for $R \geqslant 1$. We shall also use the notations $\psi = O(R)$, $\psi = o(R)$ to mean, respectively, $(1/R)|\psi| < C$, $(1/R)|\psi| \to 0$, as R tends to some limit R_0. Alternatively, the symbol $\varepsilon(R)$ will be used to denote a quantity which $\to 0$ as $R \to R_0$.

3. A Preliminary Estimate

We consider in this section a function $u(x)$ defined and continuously differentiable in a region including the exterior E of a sphere S in n-space, $n \geqslant 3$. We denote by $\Delta(\varrho)$ the Dirichlet integral of $u(x)$ extended over the exterior E_ϱ of a sphere S_ϱ about the origin of radius ϱ sufficiently large that $E_\varrho \subset E$. The existence of $\Delta(\varrho)$ is here assumed, later to be proved. We assume further that for any point P in E the

Dirichlet integral $D(r)$ of $u(x)$ in a sphere of radius r about P admits a suitable growth estimate as a function of r. We shall prove that $u(x)$ then tends to a limit u_0 at infinity and we shall estimate the rapidity with which $|u(x)-u_0|$ tends to zero. Our method of proof is essentially the same as that already used by Nirenberg in a similar connection.

LEMMA 3.1. *On every sphere S_ϱ about the origin there is a point $\overline{P}$ such that on the extended radius from $\overline{P}$ to infinity $u(x)$ tends to a limit u_0^ϱ, and such that*

$$|u(\overline{P})-u_0^\varrho| \leqslant \left[\frac{\Delta(\varrho)}{\omega_n\cdot(n-2)}\right]^{\frac{1}{2}} \varrho^{1-\frac{1}{2}n},$$

where ω_n is the area of the surface of the unit sphere in n-space.

Proof: Evidently,
$$\int_{E_\varrho} u_r^2\, r^{n-1}\, dr\, d\omega \leqslant \Delta(\varrho).$$

Thus, for some ray through the center of S_ϱ,

$$\int_\varrho^\infty u_r^2\, r^{n-1}\, dr \leqslant \frac{1}{\omega_n}\Delta(\varrho).$$

But for $R>\varrho$,

$$|u(R)-u(\varrho)| = \left|\int_\varrho^R u_r\, dr\right| \leqslant \left|\int_\varrho^R |u_r|\, r^{\frac{1}{2}(n-1)}\, r^{\frac{1}{2}(1-n)}\, dr\right|$$

and by Schwarz' inequality,

$$|u(R)-u(\varrho)|^2 \leqslant \int_\varrho^R u_r^2\, r^{n-1}\, dr \int_\varrho^R r^{1-n}\, dr \leqslant \frac{\Delta(\varrho)}{\omega_n}\,\frac{\varrho^{2-n}}{n-2}$$

which proves the lemma.

LEMMA 3.2. *Let S_r^P denote a sphere of radius r about a point P such that the sphere S_{2r}^P lies in E. Let $\delta_P(r)$ denote the oscillation of $u(x)$ on the surface of S_r^P. If for all such points P and all $\varrho<2r$,*

$$D(\varrho) \leqslant C\frac{\varrho^{n-2+\alpha}}{r^{2(n-2+\alpha)}},$$

where $D(\varrho)$ denotes the Dirichlet integral of $u(x)$ extended ower the interior of S_ϱ^P, then for all $\varrho \leqslant r$,

$$\delta_P(\varrho) \leqslant K\varrho^{\frac{\alpha}{2}}\, r^{2-n-\alpha},$$

where we may choose
$$K = 2^{n+\frac{\alpha}{2}}\,\frac{4\sqrt{C}}{\alpha}\sqrt{\frac{n-2+\alpha}{\omega_{n-1}}}.$$

This lemma is essentially due to Morrey [15]. A simple proof in two dimensions has been given by Shiffman [19]; the extension to n-space requires only formal changes.

LEMMA 3.3. *Let R be chosen so large that $E_{R/2}\subset E$. Then under the hypothesis of Lemma 3.2,*

$$|u(P)-u(\overline{P})|\leqslant K R^{2-n-\frac{1}{2}\alpha}$$

for any two points P, $\overline{P}$ on a sphere S_R about the origin, where

$$K\leqslant 2^n \frac{28\sqrt{C}}{\alpha}\sqrt{\frac{n-2+\alpha}{\omega_{n-1}}}.$$

This result follows from Lemma 3.2 and the triangle inequality, using the fact that any two points on S_R can be joined by a chain of at most seven spheres of radius $R/4$.

LEMMA 3.4. *Under the hypothesis of Lemma 3.2, the function $u(x)$ tends to a limit u_0 at infinity. If $E_{R/2}\subset E$, and if P lies in E_R, then*

$$|u(P)-u_0|\leqslant K_1\sqrt{\Delta(R)}\,R^{1-\frac{1}{2}n}+K_2 R^{2-n-\frac{1}{2}\alpha},$$

where
$$K_1\leqslant[(n-2)\,\omega_{n-1}]^{-\frac{1}{2}},\qquad K_2\leqslant 2^n\frac{28\sqrt{C}}{\alpha}\sqrt{\frac{n-2+\alpha}{\omega_{n-1}}}.$$

Proof: Apply Lemmas 3.1 and 3.3.

LEMMA 3.5. *Suppose satisfied the hypothesis of Lemma 3.2 and also the condition $\Delta(\varrho)\leqslant C\varrho^{-(n-2+\alpha)}$ for some constant C. Then there is a constant K such that if P lies in E_R and $E_{R/2}\subset E$,*

$$|u(P)-u_0|\leqslant K R^{2-n-\frac{1}{2}\alpha}.$$

4. Growth of the Dirichlet Integral

We consider now a function $u(x)$ which satisfies in E an equation of the form

$$\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u}{\partial x_j}\right]=0,\qquad a_{ij}=a_{ji},\ i,\ j=1,\ldots,n. \tag{4}$$

We suppose that the $a_{ij}(x)$ are continuously differentiable functions which tend to limits a_{ij}^0 at infinity, and that the eigenvalues of the quadratic form

$$a_{ij}^0\,\xi_i\,\xi_j$$

are all positive.

LEMMA 4.1. *Suppose* $|u(x)| \leqslant C r^{\varkappa}$ *as* $x \to \infty$, $|x| = r$, *where* $\varkappa < \sqrt{n-1} - \frac{1}{2}(n-2)$, *and that the "outflow integral"*

$$\Omega = \int_{S_r} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i \, dS,$$

where the ν_i *are the direction cosines of the normal to* S_r *vanishes for some* r *for which* $\bar{E}_r \subset E$. *Then* (i) $\Omega = 0$ *for every such* r, (ii) *the Dirichlet integral*

$$\Delta(r) = \int_{E_r} |\nabla u|^2 \, dV$$

is finite, and (iii) *for any prescribed* $\varepsilon > 0$ *there is a constant* C *such that* $\Delta(r) \leqslant C r^{-\lambda}$, $\lambda = 2\sqrt{n-1} - \varepsilon$.

Proof: A suitable linear transformation of coordinates will carry (4) into an equation of the same form for which the coefficients tend to

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad \text{at infinity.}$$

For simplicity of notation we assume that this is already the case in (4). An integration by parts, using (4), shows that

$$\int_{S_r} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i \, dS = \int_{S_R} a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i \, dS$$

whenever $E_r \subset E$, $E_R \subset E$. This proves (i). A similar integration by parts yields the relation, for $r > R$,

$$\int_{E_R - E_r} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dV = A + \int_{S_r} u \, a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS, \tag{5}$$

where A represents a surface integral over S_R. Denoting the volume integral in (5) by $Q(r)$, we have by Schwarz' inequality(1)

$$(Q - A)^2 \leqslant \int_{S_r} u^2 \, dS \int_{S_r} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) (a_{ij} \nu_i \nu_j) \, dS$$

$$\leqslant C r^{2\varkappa + n - 1} \int_{S_r} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dS = C r^{2\varkappa + n - 1} \frac{dQ}{dr}$$

for a suitable constant C.

(1) cf. SHIFFMAN [18].

Suppose $\Delta(r)=\infty$. Since the smallest eigenvalue of the form $a_{ij}\xi_i\xi_j$ is bounded from zero in a neighborhood of infinity, $Q(r)$ is monotone in r and tends to infinity as $r\to\infty$. In particular there will be an $\bar r$ such that $Q(r)>A$ for $r\geqslant\bar r$. Thus,

$$\frac{1}{(Q-A)^2}\frac{dQ}{dr}\geqslant\frac{1}{C r^{n-1+2\varkappa}},$$

and for $r_1>r>\bar r$

$$\frac{1}{Q(r)-A}-\frac{1}{Q(r_1)-A}\geqslant\frac{1}{C(n-2)}\left[\frac{1}{r^{n-2+2\varkappa}}-\frac{1}{r_1^{n-2+2\varkappa}}\right].$$

Letting $r_1\to\infty$, we find
$$Q(r)\leqslant C r^{n-2+2\varkappa} \tag{6}$$
for a suitable C.

We shall now show that this estimate leads to a contradiction. If we introduce the average of $u(x)$ on S_r,

$$\bar u(r)=\frac{1}{r^{n-1}\omega_n}\int_{S_r}u(x)\,dS,$$

we may write (5) in the form

$$Q(r)-A=\int_{S_r}(u-\bar u)\,a_{ij}\frac{\partial u}{\partial x_j}\nu_i\,dS$$

since $\bar u$ is constant on S_r. By the assumed behavior of the coefficients a_{ij} near infinity, we may write on S_r,

$$a_{ij}\frac{\partial u}{\partial x_j}\nu_i=\frac{\partial u}{\partial n}+\varepsilon(x)\,|\nabla u|,$$

where $\varepsilon(x)\to 0$ as $r\to\infty$. Thus,

$$\begin{aligned}Q(r)-A&=\int_{S_r}(u-\bar u)\frac{\partial u}{\partial n}\,dS+\varepsilon(r)\int_{S_r}|u-\bar u|\;|\nabla u|\,dS\\&\leqslant\frac{\sqrt{n-1}}{2r}\int_{S_r}(u-\bar u)^2\,dS+\frac{r}{2\sqrt{n-1}}\int_{S_r}\left(\frac{\partial u}{\partial n}\right)^2 dS+\\&\qquad+\varepsilon(r)\left\{\int_{S_r}(u-\bar u)^2\,dS\int_{S_r}|\nabla u|^2\,dS\right\}^{\frac12},\end{aligned}$$

where we have used Schwarz' inequality and the inequality $ab\leqslant\frac12(a^2+b^2)$. By an inequality of Wirtinger(1)

(1) A proof in two dimensions appears in [9, p. 185]. In the more general case considered here a proof can be based on the fact that the minimum of $\int_S u_t^2\,dS$ under the auxiliary condition

$$\int_{S_r} (u - \bar{u})\, dS \leq \frac{r^2}{n-1} \int_{S_r} u_t^2\, dS, \tag{6 a}$$

where u_t denotes the magnitude of the projection of ∇u on the plane tangent to S_r. Thus, since

$$u_t^2 + \left(\frac{\partial u}{\partial n}\right)^2 = |\nabla u|^2,$$

$$Q(r) - A \leq \frac{r}{2\sqrt{n-1}} \int_{S_r} |\nabla u|^2\, dS + r\,\varepsilon(r) \int_{S_r} |\nabla u|^2\, dS.$$

Again using the behavior of the a_{ij} near infinity, we find

$$|\nabla u|^2 \leq (1 + \varepsilon(r))\, a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

for a suitable $\varepsilon(r)$ tending to zero. Thus,

$$Q(r) - A \leq \frac{r}{2\sqrt{n-1}} (1 + \varepsilon(r)) \frac{dQ}{dr}. \tag{7}$$

Integrating, we obtain, for $r > r_1$,

$$Q(r) - A \geq [Q(r_1) - A] \left(\frac{r}{r_1}\right)^{2\sqrt{n-1}/(1+\varepsilon)},$$

where ε is an upper bound for $\varepsilon(r)$ in E_{r_1}. If r_1 is chosen so large that

$$\frac{2\sqrt{n-1}}{(1+\varepsilon)} > n - 2 + 2\varkappa,$$

this contradicts the previous estimate (6). Thus (ii) is proved, and we have proved also that $Q(r) < A$ for all r. Therefore (7) can be replaced by

$$A - Q(r) \leq \frac{r}{2\sqrt{n-1}} (1 + \varepsilon(r)) \frac{dQ}{dr}.$$

Again integrating, we find

$$A - Q(r) \leq [A - Q(r_1)] \left(\frac{r_1}{r}\right)^{2\sqrt{n-1}/(1+\varepsilon)},$$

which we may write as

$$A - Q(r) \leq C r^{-\lambda}, \quad \lambda = 2\sqrt{n-1} - \varepsilon.$$

$\int_{S_1} (u - \bar{u})^2\, dS = 1$ is the smallest eigenvalue of $\bar{\Delta} u + \lambda u = 0$, where $\bar{\Delta}$ denotes Beltrami's operator (Laplacian) on the unit sphere. As is known, this smallest eigenvalue is equal to $n-1$ and is achieved by the function $u = x/r$. The use of this inequality is an essential feature of the method of Nirenberg.

Clearly, $$\Delta(r) \leqslant (1+\varepsilon(r))\,[A-Q(r)].$$

This proves (iii).

We shall need also an estimate for the growth of the Dirichlet integral in a neighborhood of a finite point. We use here the notation of Lemma 3.2.

LEMMA 4.2. *Under the hypotheses of Lemma* 4.1,

$$D(\varrho) \leqslant C\,\frac{\varrho^{\lambda}}{r^{2\lambda}} \quad \lambda = 2\sqrt{n-1} - \varepsilon,$$

where $\varepsilon \to 0$ *as* $r \to \infty$.

Proof: Repeating, for a sphere S_ϱ^P about P, the reasoning which led to (7), we find $A=0$ and

$$Q(\varrho) \leqslant \frac{\varrho}{2\sqrt{n-1}}\,(1+\varepsilon(r/2))\,\frac{dQ}{d\varrho},$$

where $Q(\varrho)$ denotes an integral over the interior of S_ϱ^P. As before, this leads to the inequality

$$Q(\varrho) \leqslant Q(r/2)\left(\frac{\varrho}{r/2}\right)^{\lambda}.$$

By Lemma 4.1, $Q(r/2) \leqslant C(r/2)^{-\lambda}$, q.e.d.

5. Asymptotic Behavior of the Velocity Field

Suppose $\phi(x)$ is a solution of an equation of the form

$$\frac{\partial}{\partial x_i}[\Theta_i(u_1, \ldots, u_n)] = 0, \tag{8}$$

$$u_i = \frac{\partial \phi}{\partial x_j}, \qquad \frac{\partial \Theta_i}{\partial u_j} = \frac{\partial \Theta_j}{\partial u_i} = a_{ij}.$$

The equation (1) for the velocity potential of the flow an ideal gas is a particular case of (8). Let $u = u_1$, and differentiate (8) with respect to x_1. We obtain, since $\partial u_j/\partial x_1 = \partial u_1/\partial x_j$,

$$\frac{\partial}{\partial x_i}\left[a_{ij}\,\frac{\partial u}{\partial x_j}\right] = 0.$$

Thus, each velocity component u_i is a solution of an equation of the form (4).

Suppose in addition, that all velocity components tend to limits at infinity. Then the coefficients a_{ij} tend to limits a_{ij}^0 at infinity.

If $\phi(x)$ represents the velocity potential of a flow past an obstacle B, and if S denotes a sphere enclosing B, then the "outflow integral" over S vanishes,

$$\int_S \Theta_i \nu_i \, dS = \int_S \varrho \frac{d\phi}{dn} dS = 0. \tag{9}$$

For

$$\int_S \varrho \frac{\partial \phi}{\partial n} dS = \int_B \varrho \frac{\partial \phi}{\partial n} dS$$

and the latter integral vanishes since $\partial\phi/\partial n = 0$ on B.

If $\phi(x) = \phi(x_1, \ldots, x_n)$ is a solution of (8), then $\phi^h(x) = \phi(x_1 + h, x_2, \ldots, x_n)$ is again a solution, since (8) does not contain x explicitly. If $\partial\phi/\partial n = 0$ on B then $\partial\phi^h/\partial n = 0$ on the translation B^h of B. If S is a sphere enclosing B and if h is sufficiently small, then S encloses B^h. Thus, the "outflow integral"

$$\int_S \Theta_i^h \nu_i \, dS, \qquad \Theta_i^h = \Theta\left(\frac{\partial \phi^h}{\partial x_1}, \ldots, \frac{\partial \phi^h}{\partial x_n}\right),$$

vanishes. Subtracting this integral from (9), dividing by h, and passing to the limit as $h \to 0$ shows that the "outflow integral"

$$\int_S a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS$$

vanishes for any flow past B. It it easily seen that this relation persists after a linear transformation of coordinates.

Finally, we suppose that $\phi(x)$ represents a flow past B which is subsonic in a neighborhood E of infinity. Then the quadratic form

$$a_{ij} \xi_i \xi_j$$

is definite (we may assume it positive definite) in E.

All hypotheses of Lemmas 4.1 and 4.2 are now satisfied by $u(x)$. We conclude the estimates of these lemmas on the Dirichlet integral of $u(x)$ near infinity. Inserting these estimates in the hypotheses of Lemmas 3.1 to 3.5 we obtain information on the behavior of the velocity components of the flow at infinity.

THEOREM 1. *Denote by $u(x)$ a velocity component of a flow past an obstacle B. If the velocity $w = (u_1, u_2, u_3)$ of the flow tends to a subsonic limit at infinity, and if $u_0 = \lim_{x \to \infty} u(x)$, then for any prescribed $\varepsilon > 0$ there is a constant C such that in E_r, $|u(x) - u_0| < C r^{-\gamma}$, $\gamma = \frac{1}{2} + \sqrt{2} - \varepsilon$.*

For an equation in n variables, $n \leqslant 6$, we find

$$\gamma = \frac{n-2}{2} + \sqrt{n-1} - \varepsilon.$$

6. Uniqueness of Subsonic Flows

Consider two subsonic flows with velocity potentials ϕ, $\bar{\phi}$, past an obstacle B, for which the velocities tend to equal subsonic limits at infinity. We may assume that this limiting velocity is directed parallel to the x_1-axis, and we denote its magnitude by u_0. We do not assume that the flows are uniformly subsonic up to B, but we require the velocity field to be continuous up to B, with $\partial\phi/\partial n = \partial\bar{\phi}/\partial n = 0$ on B. We shall prove that ϕ differs from $\bar{\phi}$ at most by a constant.

Both ϕ and $\bar{\phi}$ are solutions of an equation of the form (8). Let $\bar{\Theta}_i = \Theta_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $\bar{u}_i = \partial\bar{\phi}/\partial x_i$. For a region B_r bounded by B and by a sphere S_r containing B we consider the identity

$$\int_{B_r} (u_i - \bar{u}_i)(\Theta_i - \bar{\Theta}_i)\, dV = \int_{S_r} (\phi - \phi_i)(\Theta_i - \bar{\Theta}_i)\, \nu_i\, dS, \tag{10}$$

the integral over B vanishing because $\partial\phi/\partial n = \partial\bar{\phi}/\partial n = 0$ on B.

The velocity potential $\phi(x)$ can be obtained by an integration of its gradient. By Theorem 1, if $\phi(x)$ is changed by a suitable constant, $|\phi - u_0 x| \leqslant C r^{-\gamma+1}$. A similar estimate must hold for $\bar{\phi}$, hence $|\phi - \bar{\phi}| \leqslant C r^{-\gamma+1}$ in E_r. Further, by the mean value theorem, $\Theta_i - \bar{\Theta}_i = a_{ij} \cdot (u_j - \bar{u}_j)$, where the a_{ij} are to be evaluated at certain intermediate values of their arguments. Again by Theorem 1, $|\Theta_i - \bar{\Theta}_i| \leqslant C r^{-\gamma}$. Since $-2\gamma + 1 < -2$, the surface integral in (10) tends to zero as $r \to \infty$. Thus, the volume integral in (10), extended over the exterior of B, exists and is equal to zero. The integrand, however, is at each point x non-negative and zero only if $u_i(x) = \bar{u}_i(x)$.

Proof: Set
$$F(t) = (u_i - \bar{u}_i)[\Theta_i - \Theta_i^*],$$

where the arguments in Θ_i^* are $u_j + t(\bar{u}_j - u_j)$, $0 \leqslant t \leqslant 1$. Then, $F(0) = 0$, and $F(1)$ is the given integrand. But $F'(t) = a_{ij}(u_i - \bar{u}_i)(u_j - \bar{u}_j)$. Uniqueness follows directly.

THEOREM 2. *There is at most one subsonic flow past an ostacle B, for which the velocity tends to a prescribed subsonic limit at infinity.*[1]

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[1] It is possible to prove this result with less knowledge of the asymptotic behavior of the velocity than is used in the present proof. In fact, the proof as first obtained by us was based on the estimate $|u - u_0| < C r^{-1/(2-\varepsilon)}$ for some $\varepsilon > 0$. The stronger estimate of Theorem 1 has permitted a considerable simplification of our original proof.

7. Uniqueness Among Mixed Flows

If the flow defined by $\phi(x)$ is everywhere subsonic it is unique among all flows $\bar{\phi}(x)$ with the same limiting velocity, even though the competing flows may be supersonic in finite regions. For set $\Phi(x)=\phi(x)-\bar{\phi}(x)$. If $\Phi(x)\not\equiv$ constant, then either the maximum or the minimum (we may suppose the maximum) of $\Phi(x)$ must be attained on B.(1) To prove this, we note that $\Phi(x)$ satisfies an elliptic equation(2)

$$a_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}+b_i\frac{\partial\Phi}{\partial x_i}=0$$

for suitable coefficients b_i, and hence admits no interior maximum or minimum [10]. Since $\Phi(x)$ has a limit at infinity, then either its maximum or minimum is achieved on B. Denote the maximum of $\Phi(x)$ by M. A point P on B at which M is achievd cannot be a supersonic point for the flow $\bar{\phi}$, since $\nabla\Phi\neq 0$ at such a point, while $\partial\Phi/\partial n=0$ on B and therefore $\nabla\Phi=0$ at P. For the same reason, if the flow ϕ is uniformly subsonic (speed bounded from the sonic speed), P must be a subsonic point of both flows. For a sufficiently small $\varepsilon>0$, the set G: $\Phi>M-\varepsilon$ is finite and contains no supersonic points. In the identity

$$\int\limits_G (u_i-\bar{u}_i)(\Theta_i-\overline{\Theta}_i)\,dV=\int\limits_\Gamma(\Phi-M+\varepsilon)(\Theta_i-\overline{\Theta}_i)\,\nu_i\,dS,$$

where Γ is the boundary of G, the surface integral over $\Gamma\cap B$ vanishes sine $\partial\Phi/\partial n=0$ on B, and the integral over the remaining part of Γ vanishes since $\Phi=M-\varepsilon$ on this set.(3) As in the proof of Theorem 2, the integrand of the volume integral can be expressed as a define form in $(u_i-\bar{u}_i)$, since both flows are subsonic in G. Therefore, $\Phi\equiv M$ in G, a contradiction.

In case the flow ϕ is not uniformly subsonic it is conceivable that P is a sonic point for both flows. In this case the use of a lemma of Gilbarg and Shiffman [8], under an additional hypothesis on the behavior of the velocity near P, leads to a contradiction with the assumption that $\partial\Phi/\partial n=0$ on B. The discussion that can be made quite rigorous, follows exactly that given in Section 6.1 of [6] and we therefore limit ourselves to a statement of the result.

(1) We shall prove in Section 12 that both extreme values are attained on B, but we do not need this fact for the present discussion.

(2) See, e.g., Courant-Hilbert, *Mathematische Physik* II, Julius Springer (1937), pp. 274–277. The coefficients a_{ij} are those of (3) and correspond to the (subsonic) solution $\varphi(x)$.

(3) The technical difficulties arising from possible irregularities of Γ are easily circumvented, cf. Lemma 4 in [5].

THEOREM 3. *A uniformly subsonic flow past B is unique among all (possibly mixed) flows with the same velocity at infinity.*

THEOREM 4. *In the class of flows past B for which the velocity vectors have bounded derivatives up to B, a subsonic flow is unique among all (possibly mixed) flows with the same velocity at infinity.*(1)

8. D'Alembert's Paradox

The classical expression for the x_j component of the force exerted by the fluid on a body B immersed in an irrotational flow is

$$X_j = -\int_S [p\nu_j + \varrho u_j (u_i \nu_i)]\,dS, \tag{11}$$

where ϱ is the density, S denotes a surface enclosing B, and p is the pressure in the fluid, $p = \text{const.} - \int \varrho q\,dq$. From the point of view of the formal development of this paper, we may take (11) as definition.

THEOREM 5. *The force exerted by the fluid on $\dot{B}$ vanishes for any flow past B with subsonic limiting velocity at infinity.*

We remark that this result is true even in the case that supersonic regions appear in the flow.

Proof of Theorem 5: We may assume that the limiting velocity vector w_0 has the form $w_0 = (u_0, 0, 0)$. By the mean value theorem, we have

$$\varrho = \varrho_0 + O(|w - w_0|),$$
$$p = \text{const.} - \varrho u (u - u_0) + O(|w - w_0|^2)$$

as $|w - w_0| \to 0$, where we have set $u = u_1$. If $j = 1$, we write (11) in the form

$$X_1 = \int_{S_r} [\varrho u (u - u_0)\nu_1 - \varrho (u - u_0)(u_i \nu_i) - \varrho u_0 (u_i \nu_i) + O(|w - w_0|^2)]\,dS$$

where we have chosen for S a sphere S_r of radius r. Thus,

$$X_1 = -u_0 \int_{S_r} \varrho u_i \nu_i\,dS + \int_{S_r} O(|w - w_0|^2)\,dS.$$

(1) We assume here that every point P of B can be contacted by a sphere S such that $S - P$ lies exterior to B.

The first integral vanishes since it equals $-u_0 \int_B \varrho\, u_i \nu_i\, dS$ and the normal component of velocity must vanish on B. By Theorem 1, $|w - w_0| \leqslant C r^{-\gamma}$, $\gamma > 1$. Hence the second integral tends to zero as r tends to infinity and we conclude $X_1 = 0$.

Similarly, for $j = 2$, we find

$$X_2 = \varrho_0 u_0 \int_{S_r} (u_1 \nu_2 - u_2 \nu_1)\, dS + O(r^{2-2\gamma}).$$

Since $\partial u_1/\partial x_2 = \partial u_2/\partial x_1$, a simple application of the divergence theorem shows that the first integral vanishes, and again the second integral tends to zero. Thus $X_2 = 0$, and a parallel argument shows $X_3 = 0$.

9. The Existence of Slow Flows

The methods of this paper, in conjunction with the fixed point theorem of Leray and Schauder [11] and classical results on linear elliptic equations, suffice to prove the existence of compressible flows past a prescribed obstacle provided the maximum speed in the flow is sufficiently small. As we have pointed out in the Introduction, a result of this type can also be obtained by other methods; however the proof we present leads to an explicit estimate of the permissible maximum speed.

THEOREM 6. *Let* $\delta = \varrho + 2\varrho' q^2$, $\varrho' = d\varrho/dq^2 < 0$, $\varrho(0) = 1$. *Let* q_m *denote a positive number so small that for* $0 \leqslant q \leqslant q_m$, $\delta > 3/2\,(1 + \sqrt{2})$. *Then for any given obstacle* B(1), *there is a unique flow past* B *which at infinity is uniform and has prescribed direction, and for which the maximum speed (achieved on* B) *is* q_m.

We remark that for a polytropic gas with $\gamma = 1.5$, Theorem 6 ensures the existence of flows for which the Mach number does not exceed 0.53.

Proof of Theorem 6: We shall show that the problem can be formulated in terms of a functional equation

$$w - \mathfrak{J}(w, k) = 0, \quad 0 \leqslant k \leqslant 1, \tag{12}$$

for which the hypotheses of Leray and Schauder [11] are satisfied.

As function space $\mathfrak{S}$ we choose the linear manifold of all vectors $w = (u_1, u_2, u_3)$ for which $u_2, u_3 \to 0$ at infinity(2) and such that w satisfies, in the closure $\bar{E}$ of the

(1) We shall here require B to be of class Bh. (For definition see LICHTENSTEIN [12].)

(2) Any prescribed direction at infinity can be changed to a direction $(u_1, 0, 0)$ by a rotation of coordinates which leaves the problem invariant.

exterior E of B, a suitable Hölder condition. Precisely, let α be chosen so that for $q \leqslant q_m$, $0 < \alpha < \min\,[1,\, 2\sqrt{2}\,\delta/(3-2\,\delta)-1]$; a vector function w will be said to belong to $\mathcal{S}$ if there is a constant C such that for any two points x', x'' of the closure $\bar{E}$ of E, the inequalities

$$|w(x') - w(x'')| \leqslant C\,R^{-1-\alpha}\,r^{\alpha/2}, \quad |w(x') - w(x'')| \leqslant C\,R^{-1-\alpha/2},$$

where $r = |x' - x''|$, $R = \min\,\{|x'|, |x''|\}$, are both satisfied.[1] As norm of w we choose

$$\|w\| = \inf C + \max_{\bar{E}} |w(x)|.$$

To define the transformation $\mathfrak{J}(w, k)$, we first regularize the differential equation in a manner suggested by M. Shiffman [19].

We observe first that (1) can be obtained as the Euler equation of a variational problem

$$\delta \int F(q^2)\, d\,V = 0,$$

where $F(q^2) = \int^{q^2} \varrho(q^2)\, d\,q^2$. For values of q larger than q_m, we may modify $F(q^2)$ so that the resulting Euler equation remains uniformly elliptic for all values of w, i.e. so that the eigenvalues of the quadratic form $a_{ij}\,\xi_i\,\xi_j$ are positive and bounded from zero. For details we refer the reader to Shiffman [19]. Since we shall deal only with solutions for which $q \leqslant q_m$, these solutions will appear also as solutions of the original equation. Accordingly, we shall refer to the coefficients a_{ij} of (1) without change of notation.

Consider now the *linear* elliptic equation, for prescribed $w(x)$,

$$a_{ij}(w)\frac{\partial^2 \Phi}{\partial x_i\,\partial x_j} = 0. \tag{13}$$

If w is in $\mathcal{S}$ there will be a unique solution $\Phi(x)$ of (13) such that (i) the vector $\nabla\Phi$ is directed parallel to the x_1-axis at infinity, (ii) $\partial\Phi/\partial n = 0$ on B, and (iii) $|\nabla\Phi|$ has a prescribed maximum in $\bar{E}$.[2] We define $\mathfrak{J}(w, k)$ to be the gradient of that

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(1) The oiigin of coordinates is assumed interior to B.

(2) Precisely: if B is of class $B\,h$ and if the coefficients of (13) satisfy both the conditions

$$|a_{ij}(w(x')) - a_{ij}(w(x''))| < C\,R^{-1-\alpha}\,r^{\alpha/2}, \qquad |a_{ij}(w(x')) - a_{ij}(w(x''))| < C\,R^{-1-\alpha/2}$$

in $\bar{E}$, then there is a unique solution $\Phi(x)$ of (13) satisfying conditions (i), (ii), (iii), such that the solution $\Phi(x)$ for which

$$\max_{\bar{E}} |\nabla \Phi| = k q_m.$$

1. $\mathfrak{F}(w, k)$ is completely continuous for each k. For $\nabla \Phi$ is in $\mathfrak{S}$, and $\|\nabla\Phi\|$ is bounded in terms of $\|w\|$. (We use here the uniform ellipticity of the modified equation (1).)

2. If $k=0$ there is only the single solution $w \equiv 0$ of (12). Since $\mathfrak{F}(w, 0) \equiv 0$, the index of this solution is obviously one.

3. For $0 \leqslant k \leqslant 1$, all solutions of (1) are bounded in $\mathfrak{S}$. This result requires an a priori estimate on the gradient of a solution of (1) which we proceed to obtain by the methods of this paper.

3 a. A priori estimation on the boundary

It is sufficient to show that a suitable Hölder estimate can be obtained in a (sufficiently small) neighborhood in E of a given point $\bar{x}$ of B, since an open set containing B can be covered by a finite number of such neighborhoods. Let ε be a prescribed positive number. Then there is an $\eta(\varepsilon) > 0$, a sphere S_η of radius η about $\bar{x}$, and a $1-1$ transformation $\xi = T(x)$ defined in S_η such that (i) the image $\bar{B}$ of $B \cap S_\eta$ is a plane, (ii) on B the transformation is conformal, and (iii) at $\bar{x}$ the Jacobian matrix $J(x)$ of the transformation is the identity I; throughout S_η, $|J(x) - I| < \varepsilon$. To see this, we may choose as parametric lines on B the image of a suitable conformal net. We may then introduce as a third set of coordinate lines the set of normals to the surface, metrized so that the differential of arc length at each point on B is equal to that induced by the conformal net. A scale transformation ensures that $J(\bar{x}) = I$, and by continuity of $J(x)$, $|J(x) - I| < \varepsilon$ if η is sufficiently small.

In the original space, equation (1) appears as Euler equation for the variational problem $\delta \int F(q^2)\, dx = 0$, $F(q^2) = \int^{q^2} \varrho\, d q^2$. This problem transforms to $\delta \int F(\alpha_{ij} \bar{u}_i \bar{u}_j) \times J\, d\xi = 0$, where $\bar{u}_i = \partial \phi / \partial \xi_i$ and $\alpha_{ii} = \delta_{jj}$ on the image $\bar{B}$ of B. For simplicity of notation we have written J instead of det $|J|$. We may assume that $\bar{B}$ is a plane through the origin and orthogonal to the ξ_3 axis. Equation (1) thus transforms to

$$\frac{\partial}{\partial \xi_j} (J \cdot F'(\alpha_{ij} \bar{u}_i \bar{u}_j) \cdot \alpha_{ij} \bar{u}_i) = 0$$

---

vector $\nabla \Phi$ is in $\mathfrak{S}$, and such that $\|\nabla \Phi\|$ depends only on C, α and on the ellipticity constant of (13). The second derivatives of Φ remain bounded up to B. A proof of this result will appear in a forthcoming work by these authors.

with $\partial\phi/\partial\xi_3=\bar{u}_3=0$ on $\bar{B}$. Differentiating with respect to ξ_l, we obtain

$$\frac{\partial}{\partial\xi_i}\left(\bar{a}_{ij}\frac{\partial\bar{u}}{\partial\xi_j}+b_i\right)=0, \tag{14}$$

where

$$\bar{a}_{ij}=2JF''\cdot\alpha_{ik}\alpha_{jm}\bar{u}_k\bar{u}_m+JF'\alpha_{ij},$$
$$\bar{u}=\frac{\partial\phi}{\partial\xi_l},$$
$$b_i=\frac{\partial}{\partial\xi_l}(J\alpha_{ij})\cdot F'\bar{u}_j+F''J\alpha_{ij}\frac{\partial\alpha_{km}}{\partial\xi_l}\bar{u}_j\bar{u}_k\bar{u}_m.$$

Note that $\bar{a}_{3i}=\bar{a}_{i3}=0$ on $\bar{B}$ if $i\neq 3$, and that $b_3=0$ on $\bar{B}$ if $l\neq 3$.

We now extend $\bar{u}$ to a solution of (14) on both sides of $\bar{B}$. Specifically, we set

$$\bar{u}(\xi_1,\xi_2,-\xi_3)=-\bar{\delta}_{3l}\,\bar{u}(\xi_1,\xi_2,\xi_3),$$
$$\bar{a}_{ij}(\xi_1,\xi_2,-\xi_3)=\bar{\delta}_{3i}\bar{\delta}_{3j}\bar{a}_{ij}(\xi_1,\xi_2,\xi_3),$$
$$b_i(\xi_1,\xi_2,-\xi_3)=\bar{\delta}_{3l}\bar{\delta}_{3i}\,b_i(\xi_1,\xi_2,\xi_3),$$

where

$$\bar{\delta}_{ij}=2\left(\delta_{ij}-\tfrac{1}{2}\right)=\begin{cases}1, & i=j\\ -1, & i\neq j.\end{cases}$$

In analogy with the procedure of Section 3, we form the identity

$$\int\left(\bar{a}_{ij}\frac{\partial\bar{u}}{\partial\xi_i}\frac{\partial\bar{u}}{\partial\xi_j}+b_i\frac{\partial\bar{u}}{\partial\xi_i}\right)dV=\int_{S_r}\bar{u}\left(\bar{a}_{ij}\frac{\partial\bar{u}}{\partial\xi_j}+b_i\right)\nu_i\,dS \tag{15}$$

for a sphere S_r of radius r about the origin. (The integrals over the surface $\bar{B}$ are easily seen to vanish.)

By the choice of the transformation, $\bar{a}_{ij}=a_{ij}+\varepsilon_{ij}$, where the $|\varepsilon_{ij}|$ can be made arbitrarily small if $|\eta|$ is chosen sufficiently small. For an equation of the form (1), we have $a_{ij}(\bar{u}_1,\bar{u}_2,\bar{u}_3)=\delta_{ij}\varrho+2\varrho'\bar{u}_i\bar{u}_j$. For this choice of a_{ij}, it is not hard to prove that the quadratic form $a_{ij}\xi_i\xi_j$ has only two eigenvalues, of which the larger is ϱ, the smaller is $\varrho+2\varrho' q^2$. Thus,

$$(\varrho+2\varrho' q^2-\varepsilon)\left(\frac{\partial\bar{u}}{\partial\xi_i}\right)^2\leqslant\bar{a}_{ij}\frac{\partial\bar{u}}{\partial\xi_i}\frac{\partial\bar{u}}{\partial\xi_j}\leqslant(\varrho+\varepsilon)\left(\frac{\partial\bar{u}}{\partial\xi_i}\right)^2$$

for any prescribed $\varepsilon>0$ in a neighborhood of each point of $\bar{B}$. Further,

$$\left|b_i\frac{\partial\bar{u}}{\partial\xi_i}\right|\leqslant\frac{1}{2}kb_i^2+\frac{1}{2k}\left(\frac{\partial\bar{u}}{\partial\xi_i}\right)^2$$

for any positive k. If k is chosen sufficiently large, we may write, for a suitable constant C and sufficiently small r,

$$C D(r) = C \int \left(\frac{\partial \bar{u}}{\partial \xi_i}\right)^2 dV$$

$$\leqslant \frac{k}{2} \int b_i^2 \, dV + \left\{ \int_{S_r} \bar{u}^2 \, dS \int_{S_r} (\bar{a}_{ij} \nu_i \nu_j) \left(\bar{a}_{ij} \frac{\partial \bar{u}}{\partial \xi_i} \frac{\partial \bar{u}}{\partial \xi_j}\right) dS \right\}^{\frac{1}{2}} + \int_{S_r} \bar{u} \, b_i \, \nu_i \, dS$$

where we have used Schwarz's inquality on the right side of (15). Thus there is a constant C such that

$$D(r) - C r^2 \leqslant C r \left(\frac{dD}{dr}\right)^{\frac{1}{2}}.$$

We may clearly suppose $r < 1$. If for $r = r_0$, $D(r) \geqslant C$ then by monotonicity $D(r) \geqslant C$ for all $r \geqslant r_0$. Thus,

$$[D(r) - C]^2 \leqslant C^2 r^2 D'(r),$$

the integration of which yields a bound on $D(r)$ which depends only on the prescribed bound for the speed and on r. A suitable covering of B by a finite number of spheres with center on B suffices to show the boundedness of the Dirichlet integral of u in a neighborhood of B, the bound depending only on B and on q_m.

Using the inequality $2ab \leqslant a^2 + b^2$, we obtain from (15)

$$(\delta_m - \varepsilon) D(r) \leqslant \frac{\sqrt{2}}{2r} \int_{S_r} \bar{u}^2 \, dS + \frac{r}{2\sqrt{2}} \int_{S_r} \left(\frac{\partial \bar{u}}{\partial n}\right)^2 dS +$$

$$+ \int_{S_r} \bar{u} (\bar{a}_{ij} - \delta_{ij}) \frac{\partial \bar{u}}{\partial \xi_i} \nu_j \, dS + \int_{S_r} \bar{u} \, b_i \, \nu_i \, dS + \frac{k}{2} \int b_i^2 \, dV, \quad (16)$$

where $\delta_m = \min(\varrho + 2\varrho' q^2)$. If λ is an eigenvalue of the quadratic form $\bar{a}_{ij} \xi_i \xi_i$, then $1 - \lambda$ is an eigenvalue of the form $(\delta_{ij} - \bar{a}_{ij}) \xi_i \xi_j$. If $\varrho = 1$ at the origin then in a suitable neighborhood of the origin, $|\delta_{ij} - \bar{a}_{ij}| < \varepsilon$ for any prescribed $\varepsilon > 0$, thus

$$\left[(\delta_{ij} - \bar{a}_{ij}) \frac{\partial \bar{u}}{\partial \xi_i} \nu_j\right]^2 \leqslant 9 \, \varepsilon^2 \left(\frac{\partial \bar{u}}{\partial \xi_i}\right)^2.$$

If $\varrho \neq 1$ at the origin then there is a neighborhood in which the eigenvalues of $\bar{a}_{ij} \xi_i \xi_j$ do not exceed unity, hence in which $(\delta_{ij} - \bar{a}_{ij}) \xi_i \xi_j$ is non-negative, and we find

$$\left[(\bar{a}_{ij} - \delta_{ij}) \frac{\partial \bar{u}}{\partial \xi_i} \nu_j\right]^2 \leqslant \left[(\delta_{ij} - \bar{a}_{ij}) \frac{\partial \bar{u}}{\partial \xi_i} \frac{\partial \bar{u}}{\partial \xi_j}\right] [(\delta_{ij} - \bar{a}_{ij}) \nu_i \nu_j].$$

Thus, in any case we will have

$$\left[(\tilde{a}_{ij}-\delta_{ij})\frac{\partial \tilde{u}}{\partial \xi_i}\nu_j\right]^2 \leqslant (1-\delta_m+\varepsilon)^2\left(\frac{\partial \tilde{u}}{\partial \xi_j}\right)^2$$

for some $\varepsilon > 0$ which $\to 0$ with r. Hence,

$$\left|\int_{S_r} \tilde{u}\,(\tilde{a}_{ij}-\delta_{ij})\frac{\partial \tilde{u}}{\partial \xi_i}\nu_j\,dS\right| \leqslant \frac{(1-\delta_m+\varepsilon)\sqrt{2}}{2r}\int_{S_r}\tilde{u}^2\,dS + \frac{(1-\delta_m+\varepsilon)\,r}{2\sqrt{2}}\int_{S_r}\left(\frac{\partial \tilde{u}}{\partial \xi_i}\right)^2 dS.$$

We may assume, as in Section 4, that $\int_{S_r} \tilde{u}\,dS = 0$. Applying the inequality (6a), we obtain from (16),

$$(\delta_m-\varepsilon)\,D(r) \leqslant \frac{r}{2\sqrt{2}}\int_{S_r}|\nabla \tilde{u}|^2\,dS + \frac{(1-\delta_m+\varepsilon)\,r}{\sqrt{2}}\int_{S_r}|\nabla \tilde{u}|^2\,dS + \int_{S_r}\tilde{u}\,b_i\,\nu_i\,dS + \frac{k}{2}\int b_i^2\,dV$$

$$\leqslant \frac{(3-2\delta_m+\varepsilon)\,r}{2\sqrt{2}}D'(r) + C\,r^2$$

for some constant C. Thus, for suitable C and ε,

$$D'(r) - \frac{a}{r}D(r) + C\,r \geqslant 0,$$

where
$$a = \frac{2\sqrt{2}\,(\delta_m-\varepsilon)}{3-2\delta_m}.$$

Thus,
$$[r^{-a}D(r)]' + C\,r^{1-a} \geqslant 0,$$

$$D(r) \leqslant [r_1^{-a}D(r_1)]\,r^a + C\,r^a\int_r^{r_1} r^{1-a}\,dr, \quad r < r_1.$$

Case 1: $1 \leqslant a \leqslant 2$. Set $r^{1-a} = r^{-1}r^{2-a}$. Then $\int_r^{r_1} r^{1-a}\,dr \leqslant r_1^{2-a}\log r_1/r$, and using the known boundness of $D(r_1)$, we find $D(r) < C\,r^a \log r_1/r$ for a suitable constant C.

Case 2: $a > 2$. In this case $\int_r^{r_1} r^{1-a}\,dr \leqslant r^{2-a}\log(r_1/r)$, from which follows $D(r) < C\,r^2\log(r_1/r)$.

Thus we have found an estimate for the growth of the Dirichlet integral of u at the boundary. The same method evidently leads to a corresponding estimate at points in a sufficiently small neighborhood of B. Applying Lemma 3.2, we find that the solution $w=(u_1, u_2, u_3)$ satisfies in such a neighborhood a Hölder condition with exponent larger than $\frac{1}{2}\alpha$ for any $\alpha < \min\,[1,\, 2\sqrt{2}\,\delta_m/(3-2\,\delta_m)-1]$, with a constant depending only on q_m and on B. This completes the a priori estimate of w at the boundary.

3 b. A priori estimation at infinity

As in Section 4, we use the identity

$$Q(r) = \int\limits_{E_R - E_r} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dV = A + \int\limits_{S_r} u\, a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS \tag{5}$$

where we shall now permit S_R to denote an arbitrary closed surface containing B. The reasoning of Section 4 shows directly that $Q(r) \leqslant C r$ as $r \to \infty$. Using the estimates of part 3 a on the quadratic form $a_{ij}\xi_i\xi_j$ we find that if $Q(r)$ exceeds A, then

$$Q - A \leqslant \frac{3-2\,\delta_m}{2\sqrt{2}\,\delta_m}\, r \frac{dQ}{dr}$$

from which follows $Q \geqslant C r^{\gamma}$, $\gamma = 2\sqrt{2}\,\delta_m/(3-2\,\delta_m)$. By hypothesis, $\gamma > 1$, hence $Q(r)$ tends to a finite limit. As in Section 4, we then find

$$A - Q \leqslant (A - Q_1)\, r_1^{\gamma} r^{-\gamma}, \quad r > r_1.$$

In order to use this result we shall need an a priori bound for A. We have

$$A = \int\limits_{S_R} u\, a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS,$$

$$A^2 \leqslant C \int\limits_{S_R} \left(\frac{\partial u}{\partial x_j}\right)^2 dS$$

for some constant C, by Schwarz's inequality. In 3 a we have proved the boundedness of the Dirichlet integral of u in a neighbourhood of B, the size of the neighbourhood depending only on B. Hence there is at least one surface S_R lying in this neighborhood for which $\int\limits_{S_R} \left(\frac{\partial u}{\partial x_i}\right)^2 dS$ is bounded. This observation provides the needed estimate.

3 c. A priori estimation at interior points

We find, as above, if $Q(r)$ denotes now an integral over a solid sphere S_r^P of radius r about a point P of distance $2R$ from the origin,

$$Q(r) \leqslant \frac{Q(R)}{R^\gamma} r^\gamma.$$

By the estimate of 3 b, $Q(R) \leqslant C R^{-\gamma}$. Hence

$$Q(r) \leqslant C R^{-2\gamma} r^\gamma$$

from which we find by Lemma 3.2,

$$|u(x') - u(x'')| \leqslant C R^{-1-(\gamma-1)} r^{\frac{1}{2}(\gamma-1)}$$

with $r = |x' - x''|$, $R = \min \{|x'|, |x''|\}$; by Lemmas 3.4 and 3.5, there is a u_0 such that $|u(x) - u_0| < C R^{-1-\frac{1}{2}(\gamma-1)}$, hence $|u(x') - u(x'')| < 2 C R^{-1-\frac{1}{2}(\gamma-1)}$.

This completes the necessary a priori estimate on the solutions of (1). All hypotheses of the Leray-Schauder theorem [11] are now satisfied, and Theorem 6 follows.

THEOREM 7. *For any prescribed obstacle B there is a positive number q_M such that if $0 < q_0 < q_M$ there is a unique flow past B for which the velocity has prescribed direction and magnitude q_0 at infinity.*

Proof: The theorem of Leray-Schauder establishes the existence of a continuum of solutions corresponding to the segment $0 \leqslant k \leqslant 1$. The speed at infinity is a continuous function of k on any branch of the continuum. It is zero for $k = 0$ and nonzero for $k = 1$, since the solution for given velocity at infinity is unique (Theorem 2). Let q_M be the upper bound of limiting speeds for $0 \leqslant k \leqslant 1$ and all permissible values of q_m (Theorem 6). Evidently q_M has the property required by Theorem 7.

10. A maximum Principle for the Speed

It is an immediate consequence of the maximum principle for the solutions of second order elliptic equations that the speed of a subsonic compressible flow admits no maximum interior to the flow region. This result has been extended to include the point at infinity in a two-dimensional flow, first by Bers [1], later with a simpler proof under less restrictive hypotheses by the authors [6]. We present in this section two results of this type which are valid for three-dimensional flows.

THEOREM 8: *If a subsonic flow is defined in a region E which includes the exterior of a sphere, and if the velocity of the flow tends to a subsonic limit at infinity, then the speed admits no maximum interior to the region or at infinity.*

From Theorem 1 we conclude that the velocity vector (and hence the speed) tends to its limit as $o(1/r)$. It is thus sufficient to show that this estimate implies the validity of the maximum principle at infinity. Such a result is best formulated as a general theorem on elliptic equations; we present it in this context as Theorem 12 in Section 12.

THEOREM 9: *If a flow is defined in a neighborhood E of infinity and if the speed in E is everywhere so small that $(\delta-\varepsilon)/(3-2\delta)>1/2\sqrt{2}$, $\delta=\varrho+2\varrho' q^2$, $\varepsilon>0$, then the velocity of the flow tends to a limit at infinity (and hence Theorem 8 applies).*

By the discussion in Section 9 we see that the Dirichlet integral of the velocity vector is finite in a neighborhood of infinity and tends to its limit as $r^{-\gamma}$, $\gamma=2\sqrt{2}(\delta_m-\varepsilon)/(3-2\delta_m)>1$, $\delta_m=$ lower bound for δ in E. A corresponding estimate holds in a neighborhood of each finite point. Theorem 9 then follows from Lemma 3.4.

11. Asymptotic Behavior of Solutions of Elliptic Equations

The considerations of this section differ from those of Section 4 in that the outflow integral is not assumed to vanish. We consider again a function $u(x)$ which satisfies in a region including the closure of a neighbourhood E of infinity an equation of the form

$$\left.\begin{aligned}\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u}{\partial x_j}\right]=0, \quad & a_{ij}=a_{ji},\\ & i,j=1,\ldots,n.\end{aligned}\right\}\tag{4}$$

We suppose that the $a_{ij}(x)$ are continuously differentiable functions which tend to limits a_{ij}^0 at infinity, and that the eigenvalues of the quadratic form $a_{ij}^0\xi_i\xi_j$ are all positive.

LEMMA 11.1. *Let $\varkappa<\sqrt{n-1}-\frac{1}{2}(n-2)$, $n>2$. If $u(x)$ is a solution of (4) in E and if $|u(x)|\leqslant C r^{\varkappa}$ as $r\to\infty$, then the Dirichlet integral $D(\varrho)=\int\limits_{E-E_\varrho}|\nabla u|^2\,dV$ is bounded in ϱ.* Here E_ϱ denotes the exterior of a sphere S_ϱ of radius ϱ about the origin.

Proof. Suppose $D(\varrho)\to\infty$. The reasoning of Section 4 leads to the inequality

$$(Q-A)^2\leqslant C\frac{dQ}{dr}\int\limits_{S_r}u^2\,dS,$$

where A denotes a fixed contour integral and $Q(r) = \int\limits_{E-E_r} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dV$. Using the hypotheses on $u(x)$ and on the $a_{ij}(x)$, we find, as in Section 4,

$$Q(r) \leq C r^{n+2\varkappa-2}. \tag{6}$$

On the other hand, we may write (5) in the form

$$Q(\varrho) - A = \int\limits_{S_\varrho} (u - \bar{u}) a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS + C \bar{u}(\varrho), \tag{17}$$

where
$$C = \int\limits_{S_\varrho} a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS$$

is independent of ϱ and $\bar{u}(\varrho)$ is the average of $u(x)$ on S_ϱ. We may assume that a suitable linear transformation has been made, so that

$$a_{ij} \frac{\partial u}{\partial x_j} \nu_i = \frac{\partial u}{\partial n} + \varepsilon(\varrho) |\nabla u|,$$

where $\varepsilon(\varrho) \to 0$ as $\varrho \to \infty$. Thus, setting $A + C\bar{u}(\varrho) = B(\varrho)$ and applying the estimate introduced in Section 4, we find

$$Q(\varrho) - B(\varrho) \leq \frac{\varrho}{2\sqrt{n-1}} (1 + \varepsilon(\varrho)) \frac{dQ}{d\varrho}. \tag{18}$$

Suppose first that $u(x)$ is bounded. Then $\bar{u}(\varrho)$ is bounded, hence $B(\varrho) \leq B < \infty$ for all $\varrho > R$. This implies

$$Q(\varrho) - B \geq [Q(\varrho_1) - B] \left(\frac{\varrho}{\varrho_1}\right)^{2\sqrt{n-1}/(1+\varepsilon)}, \quad \varrho > \varrho_1,$$

where ε is an upper bound for $\varepsilon(\varrho)$ in E_ϱ. This contradicts (6).

Suppose $u(x)$ is unbounded. We are given that $|B(\varrho)| < c\varrho^\varkappa$ as $\varrho \to \infty$. Set $\nu = [2\sqrt{n-1}/(1+\varepsilon)][(\bar{c} - c)/\bar{c}]$ and choose $\bar{c}$ and ϱ_1 so large that $\nu > \varkappa$. If for some $\varrho_0 > \varrho_1$, $Q(\varrho_0) > \bar{c}\varrho_0^\varkappa$ then there is an interval I of values $\varrho > \varrho_0$ in which this inequality is satisfied. We assert that I extends to infinity. For if not, let $\bar{\varrho}$ denote the upper end point of I. We will have $Q(\bar{\varrho}) = \bar{c}\bar{\varrho}^\varkappa$. But by (18) and the choice of $\bar{c}$,

$$Q(\varrho) \frac{\bar{c} - c}{\bar{c}} < Q(\varrho) - B(\varrho) \leq \frac{\varrho}{2\sqrt{n-1}} (1 + \varepsilon) \frac{dQ}{d\varrho}$$

or
$$\frac{1}{Q(\varrho)}\frac{dQ}{d\varrho}>\frac{\nu}{\varrho}.$$

Integrating from ϱ_0 to $\bar{\varrho}$,
$$\frac{Q(\bar{\varrho})}{Q(\varrho_0)}>\frac{\bar{\varrho}^{\nu}}{\varrho_0^{\nu}},$$

$$Q(\bar{\varrho})>\bar{c}\,\varrho_0^{\varkappa-\nu}\,\bar{\varrho}^{\nu}=\bar{c}\,\frac{\varrho_0^{\varkappa-\nu}}{\bar{\varrho}^{\varkappa-\nu}}\,\bar{\varrho}^{\varkappa},$$

hence
$$Q(\bar{\varrho})>\bar{c}\,\bar{\varrho}^{\varkappa},$$

a contradiction

Thus, either $Q(\varrho)>\bar{c}\,\varrho^{\varkappa}$ for all $\varrho>\varrho_1$ or else $Q(\varrho)\leqslant\bar{c}\,\varrho^{\varkappa}$ for all $\varrho>\varrho_1$. The former case contradicts (6), hence we conclude

$$Q(\varrho)\leqslant c\,\varrho^{\varkappa},\qquad D(\varrho)\leqslant c\,\varrho^{\varkappa}. \tag{19}$$

A simple modification of Lemma 3.1 shows the existence of a radial line joining S_r to S_{2r} on which

$$|u(2r)-u(r)|^2\leqslant\frac{D(2r)}{\omega_n}\,\frac{r^{2-n}}{n-2}\leqslant C\,r^{\varkappa+2-n}.$$

For a sphere S^p lying interior to E we find by (17) that

$$\int_{S^p} a_{ij}\frac{\partial u}{\partial x_j}\nu_i\,dS=0,$$

since in (17) the inner sphere can be contracted to a point in E. Repeating the reasoning which led to (18) we find that in this case $B(\varrho)\equiv 0$ and hence

$$\frac{Q(r/2)}{Q(\varrho)}\geqslant\left(\frac{r/2}{\varrho}\right)^{2\sqrt{n-1}/(1+\varepsilon)}.$$

But $Q(r/2)\leqslant c\,r^{\varkappa}$ by (19). Thus. in the notation of Lemma 3.2,

$$D(\varrho)\leqslant c\,r^{(\varkappa-2\sqrt{n-1}+2\varepsilon)}\,\varrho^{2\sqrt{n-1}}.$$

By Lemma 3.3, the oscillation of $u(x)$ on S_r is bounded by $C\,r^{\frac{1}{2}(\varkappa-n+2-\varepsilon)}$. Now $\varkappa-n+2<\frac{1}{2}[\sqrt{n-1}-\frac{3}{2}(n-2)]<0$, hence if r is sufficiently large, ε can be chosen small enough that $\frac{1}{2}(\varkappa-n+2)+\varepsilon<-\alpha<0$. We have therefore proved the existence of a constant C such that if r is sufficiently large, no value of $u(x)$ on S_r differs from any value of $u(x)$ on S_{2r} by more than $Cr^{-\alpha}$. The corresponding estimate for the spheres S_r and $S_{2^N r}$ becomes, by repeated use of the triangle inequality, $Cr^{-\alpha}[1+2^{-\alpha}+\cdots+2^{-N\alpha}]$. Thus, on the sphere $S_{2^N r}$, $u(x)$ remains uniformly bounded for all N. But a solution of an equation (4) admits no interior maximum or minimum [10]. Hence $u(x)$ is bounded, and by the reasoning we have already presented for this case, $u(x)$ has a finite Dirichlet integral.

THEOREM 10. *Let* $\varkappa < \sqrt{n-1} - \frac{1}{2}(n-2)$, $n>2$. *If* $u(x)$ *is a solution of* (4) *in* E *and if* $|u(x)| \leqslant Cr^{\varkappa}$ *as* $r \to \infty$, *then* $u(x)$ *tends to a limit* u_0 *at infinity. Further,* $|u-u_0| < Cr^{-\lambda+\varepsilon}$ *for any prescribed* $\varepsilon > 0$, *where* $\lambda = \min[(n-2), 2\sqrt{n-1}]$.(1)

Proof We have already shown the existence of a limit u_0 (for $n>6$ this is obvious), and we have proved $|u(x)-u_0| < Cr^{-\alpha}$ for some $\alpha > 0$. We may therefore write (17) in the form

$$Q(\varrho) - A = \int_{S_\varrho} (u-\bar{u})\, a_{ij} \frac{\partial u}{\partial x_j} \nu_i \, dS + C u_0 + C(\bar{u} - u_0),$$

from which
$$B - Q(\varrho) - C\varrho^{-\alpha} \leqslant \frac{\varrho}{2\sqrt{n-1}}(1+\varepsilon)\frac{dQ}{d\varrho} = \frac{\varrho}{\nu} Q'(\varrho),$$

where
$$B = A + C u_0,\ \nu = \frac{2\sqrt{n-1}}{1+\varepsilon}.$$

Thus
$$[\varrho^{\nu}(Q-B)]' + \nu C \varrho^{\nu-\alpha-1} \geqslant 0.$$

Suppose $\nu > \alpha$. Then, since by Lemma (11.1) Q is bounded,

$$0 < B - Q \leqslant C\varrho^{-\alpha},$$

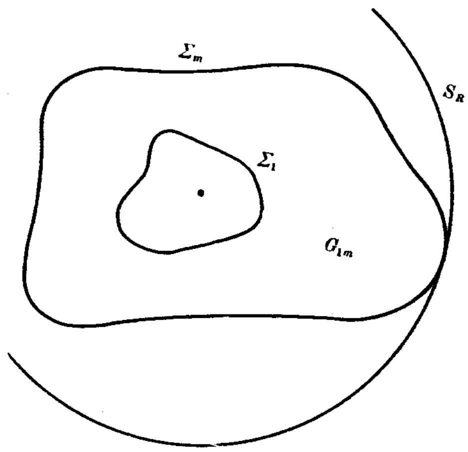
from which we conclude that the Dirichlet integral tends to its limit as $\varrho^{-\alpha}$. Similarly, for a neighborhood of a finite point

$$Q(\varrho) \leqslant \frac{Q(\varrho_1)}{\varrho_1}\varrho^{\nu} \leqslant c\,\varrho_1^{-\nu-\alpha}\varrho^{\nu}, \quad \varrho < \varrho_1.$$

Applying Lemma 3.1, 3.2, 3.3, we obtain

$$|u(x) - u_0| \leqslant C r^{1-\frac{1}{2}(n+\alpha)} = C r^{-\frac{1}{2}(n-2+\alpha)}.$$

(1) The hypothesis that the $a_{ij}(x)$ tend to limits at infinity can be weakened, cf. Theorem 9. Theorem 10 overlaps a similar limit theorem, first proved by L. Nirenberg (to appear).



If $n > 2$ this is a better estimate than the one we started with, and implies the correspondingly better relation

$$B - Q \leqslant C r^{-\frac{1}{2}(n-2+\alpha)}$$

provided $(n-2+\alpha)/2 < \nu$. This in turn implies

$$|u - u_0| \leqslant C r^{-\frac{1}{2}(n-2)-\frac{1}{4}(n-2+\alpha)}.$$

Thus, after a finite number of iterations, we find $|u - u_0| \leqslant C r^{-(n-2-\varepsilon)}$ for any prescribed $\varepsilon > 0$, or else $|u - u_0| \leqslant C r^{-\nu}$, q.e.d.

12. An Extension of the Maximum Principle

Let $u(x)$ be a solution of

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = 0, \quad a_{ij} = a_{ji}, \tag{4}$$

in a neighborhood E of infinity. It is known that if all eigenvalues of the form $a_{ij}\xi_i\xi_j$ are positive, $u(x)$ admits no maximum or minimum interior to its domain of definition [10]. In this section we present an extension of this result to include the point at infinity.

THEOREM 12: *Let $u(x)$ tend to a limit u_0 as $x \to \infty$ and suppose $|u(x) - u_0| = o(r^{2-n})$, $n \geqslant 3$. Suppose further that $a_{ij}\xi_i\xi_j \geqslant \lambda(x)\xi_i^2$, $\lambda > 0$, and that $|a_{ii}| < M < \infty$ in-*

dependent of x. *Then in every neighbourhood of infinity there are points where* $u(x) > u_0$ *and points where* $u(x) < u_0$.

Remark 1. It is not assumed that (4) is uniformly elliptic in a neighborhood of infinity, i.e., there may be sequences $\{x\}$ of points along which $\lambda(x) \to 0$.

Remark 2. Theorem 12 is sharp, even for Laplace's equation $\Delta u = \partial^2 u / \partial x_i^2 = 0$. For this equation admits the solution $u(x) = 1 - r^{2-n}$ with $u_0 = 1$, $u(x) < u_0$ for all $x \neq 0$.

Remark 3. In conjunction with Theorem 1, Theorem 12 implies that the maximum speed in a subsonic flow past an obstacle B, with subsonic limiting velocity, occurs on B.

Remark 4. If $n \leqslant 2$ a much stronger result is true, cf. [6].

Proof of Theorem 12: Suppose $u(x) \leqslant u_0$ in E. The maximum principle of E. Hopf [10] shows that $u(x) < u_0$ in E. Let Σ denote a sphere about the origin whose surface lies in E and let $\bar{u}$ be the maximum of u on the surface Σ. Corresponding to a sequence of numbers $\bar{u} < u_1 < u_2 < \cdots < u_i < \cdots < u_0$, $u_i \to u_0$, there is a sequence of regions G_{1i} in E, defined by the inequalities $u_1 < u(x) < u_i$, and bounded by level surfaces Σ_1 and Σ_i on which $u = u_1$, $u = u_i$ respectively. We have[1]

$$\int_{\Sigma_m} a_{ij} \frac{\partial u}{\partial x_i} \nu_j \, dS = \int_{\Sigma_1} a_{ij} \frac{\partial u}{\partial x_i} \nu_j \, dS = K_1$$

Also,

$$Q_{1m} = \int_{G_{1m}} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dV = \int_{\Sigma_m} u_m \, a_{ij} \frac{\partial u}{\partial x_i} \nu_j \, dS - \int_{\Sigma_1} u_1 \, a_{ij} \frac{\partial u}{\partial x_i} \nu_j \, dS = (u_m - u_1) K_1.$$

Let $L = (u_0 - u_1) K_1$. Then

$$Q_{1m} - L = (u_m - u_0) K_1. \tag{20}$$

Let R be the smallest number such that the sphere S_R of radius R about the origin contains Σ_m. By (20) and by the definition of Q_{1m} we see that as $m \to \infty$, $Q_{1m} \to L$ in increasing. Hence the left side of (20) is decreased in magnitude if the region of integration is enlarged. Thus, if

$$Q(R) = \int_{G_R} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dV,$$

where G_R is the region bounded by Σ_1 and S_R,

---

(1) See footnote (3), p. 278.

$$L - Q(R) \leqslant |(u_0 - u_m) K_1|$$

But since S_R contacts Σ_m we have by hypothesis $|u_0 - u_m| = \varepsilon(R) R^{2-n}$ where $\varepsilon(R) \to 0$ as $R \to \infty$. Thus,

$$L - Q(R) = o(R^{2-n}), \tag{21}$$

that is, the quadratic functional $Q(R)$ is bounded and tends to a limit as fast as does $u(x)$.

On the other hand, we may write

$$Q(R) = \int_{S_R} (u - u_1)\, a_{ij} \frac{\partial u}{\partial x_i} \nu_j \, dS,$$

$$Q^2(R) \leqslant \int_{S_R} (u - u_1)^2 \, dS \int_{S_R} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) (a_{ij} \nu_i \nu_j) \, dS.$$

The eigenvalues of $a_{ij} \nu_i \nu_j$ are by assumption positive and $\nu_i^2 = 1$. Thus $|a_{ij} \nu_j \nu_i|$ is certainly bounded by the sum of its eigenvalues. An orthogonal transformation which carries this form into diagonal form with the elements on the principle diagonal equal to the eigenvalues leaves invariant the trace. Thus $|a_{ij} \nu_i \nu_j| \leqslant a_{ii} \leqslant M$, and we obtain

$$Q^2(R) \leqslant C R^{n-1} \frac{dQ}{dR}$$

for some constant C. Integrating, we have

$$\frac{1}{Q(R)} - \frac{1}{Q(R_1)} \geqslant C [R^{2-n} - R_1^{2-n}], \quad R_1 > R.$$

Letting $R_1 \to \infty$,

$$\frac{1}{Q(R)} - \frac{1}{L} \geqslant C R^{2-n}.$$

Thus,

$$L - Q(R) \geqslant C L Q(R) \cdot R^{2-n}$$

which contradicts (21) since $Q(R)$ is an increasing function of R. A similar discussion disposes of the case $u(x) > u_0$ in E and completes the proof of Theorem 12.

Added in proof I: The limit theorem of section 11 can be improved to the essentially sharp result, $|u - u_0| \leqslant C r^{2-n+\varepsilon}$, by means of the following inequality recently proved by Payne and Weinberger (see the next article, [21]): Let u be any function having a bounded Dirichlet integral in E. Then the average value of u over concentric spheres of radius r has a limit $\bar{u}$ as $r \to \infty$, and

$$\Delta(r) = \int_{E_r} |\nabla u|^2 dV \geqslant \frac{n-2}{r} \int_{S_r} (u-\bar{u})^2 dS.$$

where E_r is the region exterior to the sphere S_r. This can now be applied to a result of Nirenberg(1) which states that if u is a bounded solution of (4) in E, then the Dirichlet integral of u is bounded in E and the hypotheses of Lemma 3.2 are satisfied. It follows simply that for any $\varepsilon > 0$, $\Delta(r) \leqslant K r^{2-n+2\varepsilon}$, and hence, by Lemma 3.4, one concludes that $|u - u_0| \leqslant C r^{2-n+\varepsilon}$. This result remains valid, for fixed ε, if the coefficients in (4) are allowed some oscillation at infinity (depending on ε).

Added in proof II: In a note, "Parabolic Equations", *Proc. Nat. Acad. Sci., U.S.A.*, vol. 43 (1957), pp. 754–758, J. Nash states that if $u(x)$ is a solution of an elliptic equation (4) in a region R and if $|u| < M$, then

$$|u(x) - u(\bar{x})| < CM \left[\frac{|x - \bar{x}|}{\min\{d(x), d(\bar{x})\}}\right]^{\alpha} \qquad (*)$$

where $d(x)$ denotes distance to the boundary of R, and the constants C and α depend only on n and on upper and lower bounds for the eigenvalues of the associated quadratic form. No assumption is made on the eigenvalues except that they should be positive and bounded from zero and from infinity. The authors believe that this result, in conjunction with the methods of the present paper, will lead to a proof of the existence of a subsonic flow past an obstacle B for any prescribed maximum speed q_m smaller than the sonic speed q_c. New estimates on linear equations, to be published elsewhere by these authors, show that if the space S of section 9 is suitably enlarged, (*) is sufficient to provide an a-priori estimate at interior points and at infinity. The only remaining difficulty is a discussion of the boundary behavior of a solution. This may require a slight extension of the result of Nash to include equations of more general structure.

References

[1]. Bers, L., Existence and uniqueness of a subsonic flow past a given profile. *Comm. Pure Appl. Math.*, 7 (1954), 441–504.

[2]. ——, *Theory of Pseudo-analytic Functions*. Lecture notes (mimeographed). New York University, 1953.

(1) To appear in the *Communications on Pure and Applied Mathematics*.

[3]. CHAPLYGIN, S. A., On gas jets. *Sci. Mem. Imp. Univ. Moscow, Math. Phys. Sec.*, 21 (1902), 1–121.

[4]. CORDES, H. O., Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. *Math. Ann.* 130 (1956), 278–312.

[5]. FINN, R., Isolated singularities of solutions of non-linear partial differential equations. *Trans. Amer. Math. Soc.*, 75 (1953), 385–404.

[6]. FINN, R. & GILBARG, D., Asymptotic behavior and uniqueness of plane subsonic flows. *Comm. Pure Appl. Math.* (to appear).

[7]. FRANKL, F. I. & KELDYSH, M., Die äussere Neumann'sche Aufgabe für nichtlineare elliptische Differentialgleichungen mit Anwendung auf die Theorie der Flügel im kompressiblen Gas. *Bull. Acad. Sci., URSS*, 12 (1934), 561–607.

[8]. GILBARG, D. & SHIFFMAN, M., On bodies achieving extreme values of the Critical Mach number I. *J. Rational Mech. Anal.*, 3 (1954), 209–230.

[9]. HARDY, G. H., LITTLEWOOD, J. E. & POLYA, G., *Inequalities*. Cambridge Univ. Press, (1934).

[10]. HOPF, E., Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. *Sitzungsber. Berl. Akad. Wiss.*, 19 (1927), 147–152.

[11]. LERAY, J. & SCHAUDER, J., Topologie et équations fonctionnelles. *Ann. Ecole Norm.*, 51 (1934), 45–78.

[12]. LICHTENSTEIN, L., Neuere Entwicklung der Potentialtheorie. *Enc. Math. Wiss.*, Band II, 3. Teil, 1. Hälfte, Leipzig, 1909–1921, pp. 181 ff.

[13]. LUDFORD, G. S. S., D'Alembert's paradox and the moment formula in three-dimensional subsonic flow. *Rev. Fac. Sci. Univ. Istanbul, Ser. A*, 18 (1953), 1–13.

[14]. MIRANDA, C., *Equazioni alle derivate parziali di tipo ellitico*, p. 171. Springer-Verlag, 1955.

[15]. MORREY, C. B., On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 43 (1938), 126–166.

[16]. SCHAUDER, J., Der Fixpunktsatz in Funktionalräumen. *Studia Math.*, 2 (1930), 171–130.

[17]. ——, Equations du type elliptique, problèmes linéares. *L'Enseign. Math.*, 35 (1936), 126–139, esp. 138.

[18]. SHIFFMAN, M., Differentiability and analyticity of solutions of double integral variational problems. *Ann. of Math.*, 48 (1947), 274–284.

[19]. ——, On the existence of subsonic flows of a compressible fluid. *J. Rational Mech. Anal.*, 1 (1952), 605–652.

[20]. TONELLI, L., L'estremo assoluto degli integrali doppi. *Ann. Scuola Norm. Super. Pisa*, 2 (1933), 89–130.

[21]. PAYNE, L. E. & WEINBERGER, H. F., Note on a lemma of Finn and Gilbarg, *Acta Math.*, 98 (1957), 297—299.