ON CERTAIN DISCONTINUITIES CONNECTED WITH PERIODIC ORBITS

BY

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In the final part of his work on *Celestial Mechanics* which has lately appeared M. POINCARÉ devotes some space to the consideration of the orbits discussed by Prof. DARWIN in his recent memoir on *Periodic Orbits.*[1] From considerations of analytical continuity M. POINCARÉ has been driven to the conclusion that Prof. DARWIN is in error in classifying together certain orbits of the form of a figure-of-8 and others which he designates as satellites of the class A. "Je conclus" says POINCARÉ "que les satellites A instables ne sont pas la continuation analytique des satellites A stables. Mais alors que sont devenus les satellites A stables?"

Besides the question here raised by POINCARÉ a second immediately presents itself. After explaining the disappearance of the stable orbits A it is necessary also to give a satisfactory account of the origin of the unstable orbits A. These questions had occupied my mind prior to the publication of M. POINCARÉ's work, and the present paper contains in substance the conclusions at which I had arrived in connection with them.

It will be seen that the difficulties which have occurred in following up the changes in form of DARWIN's orbits arise in some measure from the omission to take into account the orbits described in the present paper as 'retrograde', and the failure to recognize the analytical continuity between

[1] Acta Mathematica, vol. 21.

these orbits and the direct orbits. It had been my intention to defer publication of my conclusions until I had made an exhaustive examination of the retrograde orbits with something approaching the completeness devoted by DARWIN to the direct orbits, but as I see little prospect of obtaining the necessary leisure for so vast an undertaking in the immediate future I have thought it desirable to announce the results at which I have arrived, with some confidence that a closer investigation will prove them to be correct in their essential features though possibly subject to modification as regards details largely of a speculative character.

A summary of the contents of the paper and of the conclusions derived will be found in the last section.

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§ 1. *On the form of an orbit in the neighbourhood of a point of zero force.*

We shall throughout adopt the notation of Prof. DARWIN. Thus S will denote the Sun, J a planet Jove, ν the ratio of the mass of the Sun to that of Jove whose mass is unity, n the angular velocity of J about S.

Then the equations of motion of a satellite of infinitesimal mass referred to rectangular axes rotating with uniform angular velocity n about the centre of gravity of S and J, the origin being at the point S and the axis of x coinciding with the line SJ, are

$$(1) \qquad \left\{\begin{aligned} \frac{d^2x}{dt^2} - 2n\frac{dy}{dt} &= \frac{\partial\Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2n\frac{dx}{dt} &= \frac{\partial\Omega}{\partial y}, \end{aligned}\right.$$

where

$$(2) \qquad 2\Omega = \nu\left(r^2 + \frac{2}{r}\right) + \rho^2 + \frac{2}{\rho},$$

the length SJ being taken as unity, and r, ρ denoting the distances of the satellite from S, J respectively.

These equations admit of JACOBI's integral

$$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C, \tag{3}$$

where V denotes the velocity of the satellite relatively to the moving axes.

The points of zero force at which a satellite might remain in a position of relative equilibrium are determined by the equations

$$\frac{\partial \Omega}{\partial x} = 0, \qquad \frac{\partial \Omega}{\partial y} = 0. \tag{4}$$

The positions of these points has been examined by DARWIN who finds that there are three of them situated on the line SJ and two more at the vertices of the equilateral triangles described on the line SJ.

Now suppose that x_0, y_0 are the coordinates of one of these points and that ξ, η are the coordinates of the satellite referred to it as origin, so that

$$x = x_0 + \xi, \qquad y = y_0 + \eta.$$

Then if ξ, η be sufficiently small we may expand $\frac{\partial \Omega}{\partial x}$, $\frac{\partial \Omega}{\partial y}$ in ascending powers of ξ, η and by TAYLOR's theorem we shall obtain

$$\frac{\partial \Omega}{\partial x} = \xi \frac{\partial^2 \Omega}{\partial x_0^2} + \eta \frac{\partial^2 \Omega}{\partial x_0 \partial y_0} + \frac{1}{2}\left[\xi^2 \frac{\partial^3 \Omega}{\partial x_0^3} + 2\xi\eta \frac{\partial^3 \Omega}{\partial x_0^2 \partial y_0} + \eta^2 \frac{\partial^3 \Omega}{\partial x_0 \partial y_0^2}\right] + \ldots,$$

$$\frac{\partial \Omega}{\partial y} = \xi \frac{\partial^2 \Omega}{\partial x_0 \partial y_0} + \eta \frac{\partial^2 \Omega}{\partial y_2^2} + \frac{1}{2}\left[\xi^2 \frac{\partial^3 \Omega}{\partial x_0^2 \partial y_0} + 2\xi\eta \frac{\partial^3 \Omega}{\partial x_0 \partial y_0^2} + \eta^2 \frac{\partial^3 \Omega}{\partial y_0^3}\right] + \ldots.$$

Thus in the immediate neighbourhood of a point of zero force x_0, y_0 the motion of the satellite will be approximately determined by the equations

$$\begin{cases} \dfrac{d^2\xi}{dt^2} - 2n\dfrac{d\eta}{dt} = \xi \dfrac{\partial^2 \Omega}{\partial x_0^2} + \eta \dfrac{\partial^2 \Omega}{\partial x_0 \partial y_0}, \\ \dfrac{d^2\eta}{dt^2} + 2n\dfrac{d\xi}{dt} = \xi \dfrac{\partial^2 \Omega}{\partial x_0 \partial y_0} + \eta \dfrac{\partial^2 \Omega}{\partial y_0^2}, \end{cases} \tag{5}$$

where terms involving squares and products of ξ, η have been omitted. These equations being linear, it will be possible to obtain particular solutions of them by assuming for ξ, η the forms $ae^{\lambda t}$, $be^{\lambda t}$. On substituting these forms in the differential equations (5) we find

$$(6)\qquad \begin{aligned} a\left(\lambda^2-\frac{\partial^2\Omega}{\partial x_0^2}\right)-b\left(2n\lambda+\frac{\partial^2\Omega}{\partial x_0\partial y_0}\right)&=0,\\ b\left(\lambda^2-\frac{\partial^2\Omega}{\partial y_0^2}\right)+a\left(2n\lambda-\frac{\partial^2\Omega}{\partial x_0\partial y_0}\right)&=0,\end{aligned}$$

whence, on eliminating a, b, we obtain for the determination of λ the biquadratic equation

$$\left(\lambda^2-\frac{\partial^2\Omega}{\partial x_0^2}\right)\left(\lambda^2-\frac{\partial^2\Omega}{\partial y_0^2}\right)+4n^2\lambda^2-\left(\frac{\partial^2\Omega}{\partial x_0\partial y_0}\right)^2=0$$

or

$$(7)\qquad \lambda^4+\lambda^2\left\{4n^2-\frac{\partial^2\Omega}{\partial x_0^2}-\frac{\partial^2\Omega}{\partial y_0^2}\right\}+\left\{\frac{\partial^2\Omega}{\partial x_0^2}\frac{\partial^2\Omega}{\partial y_0^2}-\left(\frac{\partial^2\Omega}{\partial x_0\partial y_0}\right)^2\right\}=0.$$

The character of the motion indicated by the equations (5) will turn on the nature of the roots of this equation. Now the conditions implied by the equations (4) involve that the point x_0, y_0 is a singular point of the curve belonging to the family $\Omega=$ const., which contains it, while the nature of the singularity for the different points of zero force has been examined by DARWIN. For those points which lie on the axis of x he finds that there will be two real intersecting branches, and thus at these points

$$\frac{\partial^2\Omega}{\partial x_0^2}\cdot\frac{\partial^2\Omega}{\partial y_0^2}-\left(\frac{\partial^2\Omega}{\partial x_0\partial y_0}\right)^2$$

will be negative.

Hence the values of λ^2 derivable from the equation (7) will be both real for these points, but they will be of opposite signs. Also at points on the axis of x

$$\frac{\partial^2\Omega}{\partial x_0\partial y_0}=0,$$

which introduces some simplification into the equations (6). We propose only to concern ourselves with the points of zero force which lie on the axis of x. Suppose that for one of these points the two values of λ^2 derivable from equation (7) are

$$\lambda^2=\alpha^2,\qquad \lambda^2=-\beta^2,$$

where α, β are real quantities. We then obtain the following particular solutions of (5)

$$\text{(i)} \qquad \xi = a_1 e^{\alpha t}, \qquad \eta = \quad a_1 \frac{\alpha^2 - \frac{\partial^2 \Omega}{\partial x_0^2}}{2n\alpha} e^{\alpha t},$$

$$\text{(ii)} \qquad \xi = a_2 e^{-\alpha t}, \qquad \eta = - a_2 \frac{\alpha^2 - \frac{\partial^2 \Omega}{\partial x_0^2}}{2n\alpha} e^{-\alpha t},$$

$$\text{(iii)} \qquad \xi = a_3 e^{\beta i t}, \qquad \eta = - a_3 \frac{\beta^2 + \frac{\partial^2 \Omega}{\partial x_0^2}}{2n\beta i} e^{\beta i t},$$

$$\text{(iv)} \qquad .\xi = a_4 e^{-\beta i t}, \qquad \eta = \quad a_4 \frac{\beta^2 + \frac{\partial^2 \Omega}{\partial x_0^2}}{2n\beta i} e^{-\beta i t},$$

and the general solution, involving the four arbitrary constants a_1, a_2, a_3, a_4 will be obtained by adding together these particular solutions.

If we put

$$a_1 + a_2 = h, \qquad a_1 - a_2 = k, \qquad a_3 + a_4 = H, \qquad a_3 - a_4 = - Ki,$$

and write for brevity γ^2 in place of $\frac{\partial^2 \Omega}{\partial x_0^2}$, we obtain as the general solution of (5) involving four arbitrary constants h, k, H, K

$$(8) \quad \left\{ \begin{aligned} \xi &= h \cosh \alpha t + k \sinh \alpha t + H \cos \beta t + K \sin \beta t, \\ \eta &= \frac{\alpha^2 - \gamma^2}{2n\alpha} (h \sinh \alpha t + k \cosh \alpha t) - \frac{\beta^2 + \gamma^2}{2n\beta} (H \sin \beta t - K \cos \beta t). \end{aligned} \right.$$

This solution being free from imaginary quantities is capable of a real hysical interpretation.

To avoid circumlocution we shall speak of the region surrounding a point of zero force L within which the equations (5) may be regarded as giving an approximation to the motion as the 'domain' of L. If then a satellite be initially in the domain of L it will be possible to determine four constants h, k, H, K so that the equations (8) will represent its motion at least for a finite time, but the terms involving the hyperbolic functions will rapidly increase with t so that the satellite will depart from the domain of L, and only a short length of its path will be sensibly represented by these equations.

In like manner whatever be the initial circumstances, if the satellite should at any instant enter the domain of L, it will be possible to determine four arbitrary constants h, k, H, K so that the equations (8) will represent its motion so long as it remains within this domain. Let us then examine the character of the motion represented by the equations (8).

First suppose that a satellite is describing a path within the domain of L such that $h = 0$, $k = 0$. Its motion will then be given by

$$(9) \qquad \begin{cases} \xi = H \cos \beta t + K \sin \beta t, \\ \eta = -\dfrac{\beta^2 + \gamma^2}{2n\beta}(H \sin \beta t - K \sin \beta t). \end{cases}$$

Without loss of generality we may put $K = 0$, since this evidently only involves a change in the epoch from which t is measured. We thus have

$$\xi = H \cos \beta t, \qquad \eta = -\frac{\beta^2 + \gamma^2}{2n\beta} H \sin \beta t.$$

The path of the satellite referred to the moving axes is therefore elliptic and the satellite will not tend to leave the domain of L.

Next suppose that $H = 0$, $K = 0$ so that the motion is given by

$$(10) \qquad \begin{cases} \xi = h \cosh \alpha t + k \sinh \alpha t, \\ \eta = \dfrac{\alpha^2 - \gamma^2}{2n\alpha}(h \sinh \alpha t + k \cosh \alpha t). \end{cases}$$

The path of the satellite is now a hyperbola whose centre is at L. The satellite will enter the domain of L along one branch of the hyperbola and after traversing the part of the curve which lies within the domain it will recede along another branch. Of course the path before entering and after leaving the domain of L may depart rapidly from the infinite branches of the hyperbola, but we are at present only concerned with the form of the orbit within the domain of L.

A special case of importance occurs when the path within the domain of L is such that $h^2 = k^2$. The hyperbola represented by (10) then degenerates into a pair of straight lines, coincident with the asymptotes.

If h, k have like signs the satellite will then be continually receding from L along a straight line, whereas if h, k have unlike signs it will be continually approaching L, but it will not reach L until $t = +\infty$. In fact the nearer it approaches to L the smaller does its velocity become, so that it will not be able to reach this point within a finite time.

If now the initial circumstances of projection undergo continuous change in such a manner that the satellite always enters the domain of L, and that its motion within the domain may be represented by the equations (10), the quantities h, k and therefore also $h^2 - k^2$ will vary continuously. It may happen that in the course of the change $h^2 - k^2$ will pass through the value zero and change sign. This will imply that the infinite branch of the hyperbola along which the satellite enters the domain will cross the asymptote, and that consequently immediately after the change in sign of $h^2 - k^2$ the satellite will recede along the second asymptote in the opposite direction to that in which it receded before the change. It follows that, if two satellites be projected simultaneously under initial circumstance which differ infinitesimally, but so that, when they enter the domain of L, the values of $h^2 - k^2$ for their two paths have infinitesimal values with opposite signs, the paths of the satellite though differing infinitesimally prior to entering the domain of L will have lost all similarity of character before they depart from this region. The nearer however the hyperbolic paths approach to the asymptotes the longer will the satellites take in passing round the vertices, and consequently the smaller the difference in the initial circumstances the longer will be the interval before the separation commences.

We have so far for simplicity supposed that the satellite moves within the domain of L either in an elliptic path (9) or in a hyperbolic path (10). In general however its path will be represented by (8) which indicates an elliptic path superposed upon a hyperbolic path. We may form a conception of this motion by supposing the satellite to move in an ellipse whose form is represented by (9) while the centre of this ellipse moves along the hyperbola represented by (10).

The character of the motion of the centre of the ellipse will then be to some extend shared by that of the satellite, but, when the major axis of the hyperbolic path becomes small, the time which the centre of the ellipse takes to move round the vertex of the hyperbola will increase and the elliptic element of the motion will commence to shew its independent existence by the formation of loops in the orbit of the satellite. The nearer the hyperbola approaches to its asymptote the greater will be the number of loops described by the satellite prior to leaving the domain of L, until when the hyperbola actually coincides with an asymptote the number of loops will become infinite. The orbit of the satellite will then approach closer and closer to the simple elliptic orbit represented by (9), and will in fact be asymptotic to this orbit in the sense in which the term is used by POINCARÉ.

Except in the case just considered the satellite will finally recede from the domain of L in one of two essentially different ways according as the centre of the ellipse recedes along a branch of the hyperbola which approximates to one or other of the infinite arms of the second asymptote.

The 'asymptotic' orbit just dealt with is the limiting orbit which separates those which leave in one way from those which leave in the other.

§ 2. *Application to the Orbits of Professor Darwin.*

The results proved in the last section rigorously apply only to the very limited region surrounding a point of zero force within which the motion can be sensibly represented by the approximate equations (5), but there can be no reasonable doubt that the general characteristics will be maintained over a far more extensive field. A good illustration is furnished by the orbits traced by DARWIN.

For example the orbits of the 'oscillating satellites' figured in Darwin's plates are closely analogous to the elliptic orbits represented by our equations (8). They are in fact the orbits at which we should arrive by the continuous deformation of the elliptic orbits which would result from diminution of the constant of relative energy C. By the time C has attained the values for which the figures have been drawn, these orbits have lost their symmetrical form but are still roughly elliptic in character.

Next consider the non-periodic orbits traced on p. 177 of Darwin's memoir, viz. those started at right angles to the line SJ on the side of J remote from S with C equal to 39·0. As x_0 (the abscissa of the point of projection) decreases and reaches a value in the neighbourhood of 1·095 the orbit begins to approach the region of the point of zero force L. It however recedes from this region towards the planet J after describing a loop. But when x_0 has the value 1·09375, or a smaller value, the orbit after passing near the orbit of the oscillating satellite no longer recedes towards the planet J, but towards the Sun S. As in the last section we may regard the motion of the satellite when in the neighbourhood of the point of zero force as consisting of two independent motions, (1) a motion in a closed periodic orbit similar to that of the oscillating satellite, (2) a bodily transference of this closed orbit in virtue of which each point of the orbit is carried along a curve analogous to the hyperbola of the last section. We may refer to these two parts of the motion briefly as the 'elliptic' and the 'hyperbolic' elements. Evidently the fate of a satellite after passing near L will turn on the character of the hyperbolic element of its motion, and the satellite will recede towards J or towards S according as the branch of the hyperbolic curve along which the elliptic orbit is carried recedes towards J or towards S.

The critical case which separates orbits receding towards J from those receding towards S, occurs when the hyperbolic element of the motion takes place in a curve which plays the part of the asymptotes in the last section. Each point of the periodic 'elliptic' orbit then tends towards a fixed limiting position, but it takes an infinite time for it to reach this position. Consequently the satellite will describe an infinite series of loops each of which approximates closer than the preceding to the orbit of the oscillating satellite a. If the hyperbolic element of the motion differs only very slightly from its asymptotic form a large but finite number of loops approximating to the orbit a will be described, but the satellite will ultimately recede either towards J or towards S according as the hyperbolic path lies on one side or the other of its critical form.

We are thus able to describe the manner in which the interval between the orbits $x_0 = 1·095$, $x_0 = 1·09375$ is to be filled up. As x_0 decreases from 1·095, loops, the first of which has already shewn its existence in the figure traced, will be formed in gradually increasing numbers and these will tend to approximate in figure to the orbit of the oscillating satellite; the path however will ultimately fall away in the direction of the planet J.

At length a stage will be reached when an infinite number of loops will be described before the satellite recedes. The orbit will then approach the orbit of the oscillating satellite asymptotically after the manner of the 'asymptotic orbits' treated of by POINCARÉ.[1]

As x_0 still further decreases the satellite will after describing at first a large number of loops recede towards S. The number of loops described will however rapidly diminish with x_0, until when $x_0 = 1{\cdot}09375$ all trace of them will have disappeared.

It will save circumlocution if we make use of the terms 'lunar' and 'planetary' to distinguish those of our orbits which, after passing near the orbit of the oscillating satellite, recede towards J from those which recede towards S. These terms are however at present only to be used to describe the character of the motion in the course of a single revolution round the primary. Thus an orbit which is 'lunar' so far as its first approach to L is concerned might become 'planetary' after two or more revolutions round the primary.

The lunar and planetary orbits will be separated from one another by orbits which are asymptotic to the orbit of the oscillating satellite. These orbits we shall speak of briefly as 'asymptotic' orbits.

With large values of C, DARWIN has shewn that any infinitesimal body moving in the plane of the orbit of J about S may be regarded either as a satellite, as an inferior planet, or as a superior planet, but that with smaller values of C it may be transferred from one category to another. The circumstances described in the present section are those which occur when an orbit is undergoing the change from that of a satellite to that of an inferior planet. It is clear that a similar sequence of events will occur when the orbit changes from that of a satellite or an inferior planet to that of a superior planet.

[1] In consideration of the fact that the orbit is symmetrical with respect to the line SJ it will be asymptotic to the orbit of the oscillating satellite for $t = -\infty$ as well as for $t = +\infty$, and will thus furnish an interesting illustration of one of POINCARÉ's 'doubly asymptotic orbits'.

§ 3. *On the relative orbit of a satellite which approaches indefinitely close to its primary.*

Imagine a satellite P to be moving subject solely to the attraction of its primary J. The orbit will then be a conic section.

Let us suppose that the initial circumstances are such that the orbit is an ellipse of large eccentricity. At perijove the satellite will then pass very near to its primary. Further suppose that the initial circumstances are varied continuously in such a manner that the distance of the satellite at perijove diminishes without limit, while the length and position of the major axis remain invariable. The elliptic orbit will become more and more flattened until it becomes sensibly a straight line except through very small portions of its length at perijove and apojove.

The same will be true in whichever direction the satellite is moving in its orbit and the two orbits which correspond to the two different directions of projection will approach the same limiting rectilinear form.

It is clear that if we suppose all the circumstances remote from the primary to undergo continuous variation the two forms of orbit may be regarded as continuations of one another, the rectilinear orbit forming the connecting link between the direct and the retrograde orbits, though physically only the part of this path between two successive perijove passages can be regarded as having a real existence, owing to the collision which would ensue between the satellite and primary at the instant which corresponds to the time of perijove passage.

There will be a similar connection between the direct and the retrograde orbits if we suppose that the initial circumstances are more general in character, admitting of change in the length and position of the major axis. If the initial conditions vary continuously the length and position of the major axis will likewise vary continuously, and, provided only that the length retains a finite value at the critical rectilinear stage, the direct and the retrograde orbits will merge into one another.

Let us next consider the figures of the relative orbit, when the motion is referred to axes which rotate uniformly. The motion may then be regarded as taking place in a moving ellipse, the line of apses of which revolves uniformly in a direction opposite to that of the rotation of the axes. We wish to consider the form of the relative orbit when this ellipse is very much flattened and approximates to the rectilinear form.

When the satellite is at perijove it is evident that the motion in the ellipse takes place very rapidly and therefore that the form of the path will resemble closely that which occurs when the axes are at rest. This results from the fact the axes can only be very slightly displaced during the passage of the satellite round Jove. The more eccentric the ellipse the more closely will the relative orbit correspond with the actual orbit. Thus the motion in the relative orbit will be direct or retrograde according as that in the actual orbit is direct or retrograde.[1]

On the other hand when the satellite is in apojove the motion in the actual orbit will be very slow and the apparent motion in the relative orbit will be chiefly that due to the rotation of the axes themselves. Thus the apparent motion will he retrograde whatever be the direction of motion in the true orbit. It is then evident that the path from apojove will be of the form indicated in the annexed diagrams (figs 1 and 2) according as the motion in the true orbit is direct or retrograde. In these figures the axes are supposed to be rotating in a counter-clockwise direction, and the direction of motion of the satellite is indicated by arrowheads.

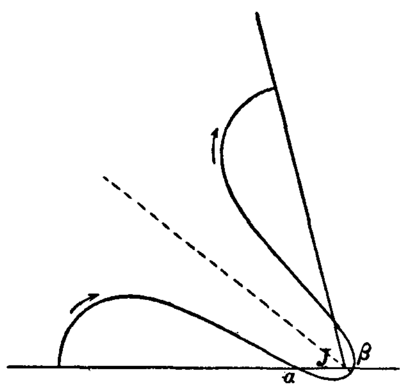

Fig. 1.

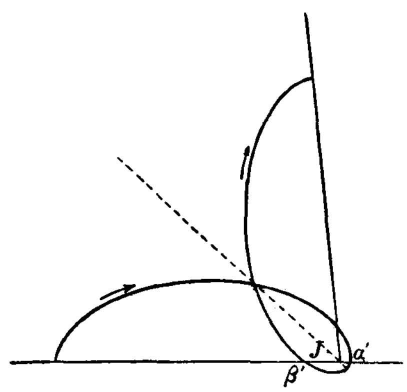

Fig. 2.

The critical form of orbit which corresponds with our previously rectilinear orbit, and which separates the orbits of the type represented in

[1] A 'direct' motion implies motion in the same sense as that of the rotation of the axes.

figure 1 from those of the type represented in figure 2, evidently possesses a cusp at J. Of course if a satellite moving in this orbit arrived at J a collision would occur and the physical continuity would be interrupted, but if we suppose that in the event of a collision the satellite rebounds without loss of energy in the direction opposite to that in which it fell into the primary the physical as well as the analytical continuity between the direct and retrograde orbits will be maintained.

So far we have regarded the satellite as moving subject to the attraction of J alone, but if it be moving under the disturbing influence of the Sun, its path when in the neighbourhood of J will still be governed by the laws of elliptic motion. If then all the circumstances remote from J undergo continuous variation in such a manner that the orbit approaches J and in the course of the change actually falls into J, it is evident that so far as its form in the immediate neighbourhood of J is concerned a series of changes will occur similar to that already described. The orbit will pass through the cusped form and after the critical stage the direction of the motion round J will be reversed. If at first the satellite described an open loop round J as in fig. 1, it would afterwards describe a closed loop as in fig. 2 and vice-versâ.

As the orbit approaches the critical form from either direction its form in the neighbourhood of J will approximate closer and closer to that of a parabola. Suppose α, α' are the points in which the orbits (figures 1 and 2) cut any line through J as the satellite approaches J and β, β' the points in which they cut this line as the satellite recedes. Then the points $\alpha, \beta, \alpha', \beta'$ will all ultimately coincide with J, but the tangents to the orbits at α, β, and at α', β' being ultimately tangents at the extremities of a focal chord of a parabola will in their limiting position be at right angles. These limiting positions will be the two bisectors of the angles between the line $\alpha\beta$ and the limiting position of the axis of the parabola, i.e. the tangent at the cusp of the limiting orbit.

It follows from the figures that though we might regard the point α' as the analytical continuation of α since both coincide with J at the critical stage, if we wished to maintain continuity in the direction of the curve at the point where it crosses a given line, such as $\alpha\beta$, we must regard the point β' of fig. 2 as the continuation of α fig. 1 and α' as the continuation of β.

§ 4. *Application to the orbits of Professor Darwin.*

Let us deal with those orbits which start from points on the line SJ at right angles to this line, and confine our attention to those for which the starting point lies outside SJ on the side of J remote from S. With a given value of the constant of relative energy two such orbits may be regarded as originating from each point, distinguishable by the initial direction of projection. We may without ambiguity designate these orbits as direct or retrograde according as the initial direction of motion is direct or retrograde.

It is now clear that if the starting point moves up towards J the direct and the retrograde orbits will approach the same limiting (cusped) form, and that so far as the circumstances in the remote parts of the orbits are concerned each form of orbit can be regarded as the analytical continuation of the other.

Next suppose that the region to which the starting point is confined is limited by a branch of the curve of zero velocity as in the figure (5) on page 177 of DARWIN's paper, and let us further suppose that the starting point moves up to its extreme limit in the opposite direction viz: the point M where the curve of zero velocity cuts SJ. Now the form of an orbit in the neighbourhood of the curve of zero velocity has been dealt with by DARWIN, who has shewn that at such a point the orbit will possess very large curvature. The limiting form will be cusped while the figures before and after the passage through the cusped form will be similar in character to those presented in our figures 1 and 2 above.

Hence again we see that as the starting point approaches M the direct and retrograde orbits will approach the same limiting cusped form, the cusp being at M on the curve of zero velocity. Likewise also the direct and retrograde orbits may be regarded as continuations of one another.

As the starting point P moves along the line of syzygies the direct and retrograde orbits may then be regarded as forming a continuous cycle as P moves backwards and forwards between J and M. In this cycle when P arrives at J or M the orbit will assume the cusped form and an interchange from the direct to the retrograde or vice versâ will occur.

The tendency of the direct orbits to assume the cusped form is well indicated by the orbit $C = 39\cdot 0$, $x_0 = 1\cdot 001$ shewn by Professor DARWIN (fig. 5).

§ 5. *Conjectural Forms of Retrograde Orbits.*

The cusped orbit $C = 40\cdot 0$, $x_0 = 1$ has been computed in part by myself and independently by Prof. DARWIN. Its form is found to resemble that shown in fig. 3 below. Again when x_0 reaches its extreme limit in the opposite direction the form of the cusped orbit is that shown in fig. 4. We may pass from one of these forms to the other either by following the direct orbits or by following the retrograde orbits.

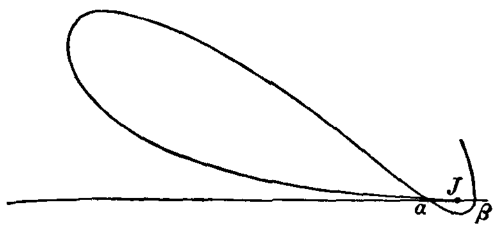

Fig. 3.

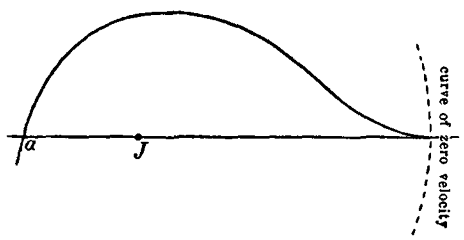

Fig. 4.

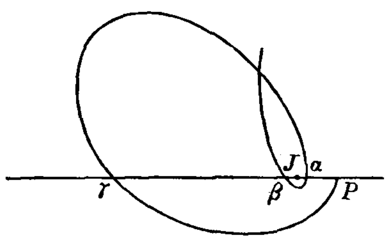

Fig. 5.

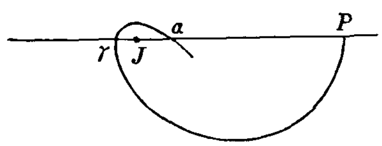

Fig. 6.

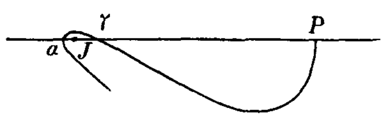

Fig. 7.

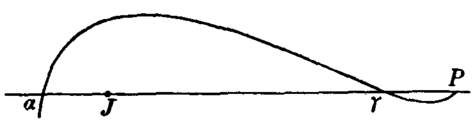

Fig. 8.

The sequence of changes in the forms of the orbits which occur as we follow the direct orbits may be inferred from the results already given by DARWIN, but as regards the retrograde orbits we have as yet no positive data available. It seems to me however that the probable sequence of changes is that indicated by figs. 5—8. In fig. 5 the starting point P has receded slightly from J and the direction of rotation round J at the next approach has been reversed by a passage through the cusped form.

In fig. 6 the large loop has diminished in size and the small loop (no longer represented in full) has increased, indicating another near approach of the satellite to the primary.

Fig. 7 represents the form of the orbit after this loop has passed through the cusped stage, while fig. 8 at once fills up the interval which remains between this orbit and that shewn in fig. 4.

§ 6. *φ-curves.*

The main object of DARWIN's research was to investigate the forms of the different varieties of *periodic* orbits, but in order to discover these periodic orbits he has incidentally determined the forms of a large number of non-periodic orbits. The method of procedure was to trace these latter orbits by a process of mechanical quadratures, and to examine the values of the angle (φ_1) at which the normal to the orbit was inclined to the axis-of-x at the instant when the orbit again crosses this axis. The periodic orbits are then selected by determining by interpolation the initial circumstances which allow of the orbit cutting the line of syzygies at right angles at the second crossing, i.e. those which make the value of φ_1 zero or a multiple of π.

Now the initial circumstances with which we are concerned involve two parameters viz: C, the constant of relative energy, and x_0, the abscissa of the starting point. If the value of C be assigned, φ_1 may be regarded as a function of the single quantity x_0, and the nature of this function may be represented graphically by a curve with x_0 as abscissa and φ_1 as ordinate. When the form of this curve is known we may at once recognize the existence of, and the initial circumstances associated with the periodic orbits by reading off the abscissae of the points of intersection of the curve with the lines $\varphi_1 = 0$, $\varphi_1 = \pm \pi$, $\varphi_1 = \pm 2\pi$, &c. As C varies, the curve, which we shall describe as a φ-curve, will undergo continuous deformation and the variations in its form will indicate the vicissitudes through which the different periodic orbits pass.

For large values of C the planet J will be surrounded by a closed branch of the curve of zero-velocity, and it therefore appears that two forms of discontinuity in the figures of the non-periodic orbits under discussion may present themselves; (1) where the satellite is instantaneously reduced to rest by attaining a point on the curve of zero-velocity and (2) where the satellite falls into the primary.

The former case has been frequently met with by DARWIN and it is clear from his figures that the crisis concerned involves no abrupt change at points on the orbit other than that where it occurs. Thus no discontinuity in the value of φ_1 will result.

No instance of the occurrence of the second event has been found by DARWIN prior to the satellite first crossing the line of syzygies. A case has however been found ($C = 39{\cdot}0$, $x_0 = 1{\cdot}095$) where the satellite passes very close to its primary after twice crossing the axis-of-x, while similar instances appear to occur among the retrograde orbits in the critical forms which separate orbits of the characters represented in figs. 5-6-7.

Now the angle φ made by the normal with the axis-of-x at the points where the orbit cuts this axis must be regarded analytically as a multiple-valued function of x_0 having an infinite number of determinations, since the orbit will evidently cut the axis-of-x an infinite number of times. The complete φ-curve will then consist of an infinite number of branches each one of which corresponds with a particular crossing. In dealing with the direct orbits we define φ_1 as that particular determination which corresponds to the first crossing of the axis after a semi-revolution round the primary, the values which correspond to subsequent crossings being denoted by different suffixes (φ_2, φ_3, &c.).

In so far as we can pass from the direct orbits to the retrograde orbits, by continuous deformation through the cusped form, the particular crossing, to which the angle φ_1 belongs in the case of the retrograde orbits, may be defined as the geometrical continuation of that previously defined so long as the point under consideration does not fall into the planet J. This definition will however lead to ambiguity when the orbit falls into J. To remove this ambiguity we will suppose that before and after the passage through the cusped form, the crossings corresponding to the approach of the satellite to the primary are continuations of one another, as also are those which correspond to the recession of the satellite from the primary. The angle φ_1 will then be defined without ambiguity, and it is evident that if the sequence of changes indicated in fig. 3—8 be the correct sequence through which the orbits pass, the points marked with similar letters on these figures will correspond with one another, the determination φ_1 being that which belongs to the point α throughout.

From the result proved at the end of § 3 it follows that the angle φ_1 will no longer be a continuous function of x_0, but when the crossing to which φ_1 belongs falls into J the ordinate of the φ-curve will change abruptly by $\frac{1}{2}\pi$. The points selected as the continuations of one another are, in fact not the true analytical continuations of one another so that when the orbit passes through the critical form we transfer our attention from one branch to another of the φ-curve.

The advantage gained by defining φ_1 in this manner, rather than by following the continuous changes in the angle φ, is that as we follow the cycle of orbits discussed in § 5, φ_1 will go through a series of cyclical changes, whereas if we maintained strict analytical continuity, on each passage through a cycle we should have to fix our attention on a different determination of the angle φ.

We are now in a position to examine that branch of the φ-curve which corresponds to the determination φ_1. Since the direct and retrograde orbits merge into one another when x_0 has its extreme values (1 and m) the curve will touch the lines $x_0 = 1$, $x_0 = m$.

For large values of C, DARWIN finds only a single periodic orbit for which $\varphi_1 = \pi$ among the direct orbits in question, while the angle φ_1 decreases with increasing values of x_0. Assuming that the forms of the retrograde orbits are similar to those represented in § 5 the φ_1-curve will then be as below, where two passages through the cusped form are indicated by abrupt diminutions by $\frac{1}{2}\pi$ as x_0 increases.

In this figure the point A corresponds with DARWIN's periodic orbit, the points J and M with orbits which start from a cusp while the points

P , P', Q , Q' correspond with orbits which have a cusp at J, at some time after starting.

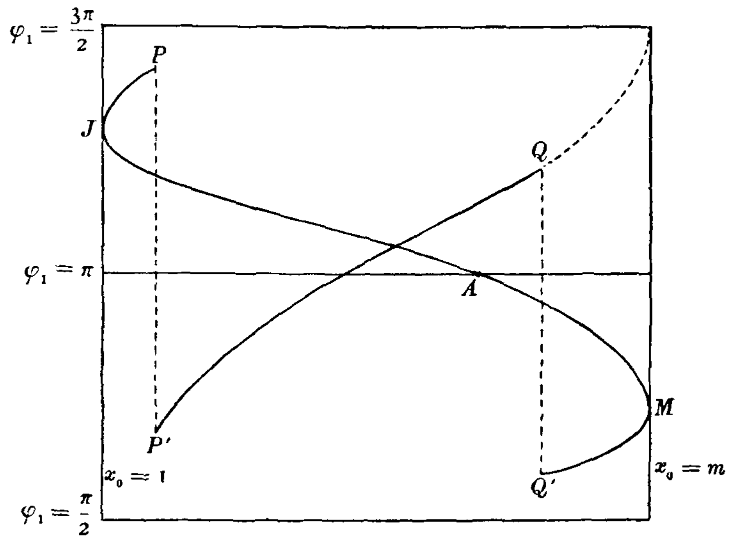

Fig. 9.

If we follow the retrograde orbits up to their limiting form when cusped at M it is evident that the normal at the crossing denoted by γ in figs. 7, 8, which may be regarded as the true analytical continuation of α in figs. 5 and 6, will approach nearer and nearer to a direction at right angles to the line of syzygies. At the final stage the crossing will disappear by coalescence with a second which occurs before the passage through the starting point. We thus conclude that the continuation of the branch $P'Q$ will touch the line $x_0 = m$ at the point where $\varphi_1 = \frac{3}{2}\pi$, as indicated by the dotted part of the curve in the above diagram.

Further since the points P , Q' necessarily lie between the lines $\varphi_1 = \frac{3\pi}{2}$ and $\varphi_1 = \frac{\pi}{2}$, it may be readily seen that the branch of our curve which corresponds with the retrograde orbits must necessarily cut the line $\varphi_1 = \pi$, indicating the existence of a retrograde periodic.

If as in the above figure P, Q' lie on opposite sides of the line $\varphi_1 = \pi$, P', Q will also lie on opposite sides of this line and the retrograde periodic will then be of the character indicated by figs. 5 and 6 in which the crossing α becomes rectangular, i.e. it will be a 'doubly' periodic orbit.

On the other hand if P, Q' lie on the same side of $\varphi_1 = \pi$ one of the branches JP, MQ' must have bent round so as to cross this line, and the form of the retrograde orbit will be modified in a manner which it is easy to trace.

§ 7. *First deformation of the φ-curve accounting for the disappearence of the orbit A.*

As C varies the figure of the φ-curve given in the last section will undergo continuous deformation, and we may follow the fate of the orbit A by fixing our attention on the point A of this figure. Now from the figures given by DARWIN we see that as C decreases the point A will approach the line $x_0 = 1$. Meanwhile the point J will move along the line $x_0 = 1$, and it is evident that the points A and J may at some stage coincide. When this occurs the cusped orbit J itself cuts the axis at right angles at the next crossing and may be regarded as periodic. Prof. DARWIN who is a present examining the forms of some of these cusped orbits informs me that the orbits in question appear to become periodic for a value of C about 39·5, but the actual numerical value has not yet been determined.

After the critical stage the point J will cross the line $\varphi_1 = \pi$ and the point A will no longer be found in that part of the curve which corresponds with the direct orbits, but in the part which corresponds with the retrograde orbits. Subsequently the curve will bend up so as to again cut the line $\varphi_1 = \pi$, indicating the growth of two new periodic orbits, the orbits B and C of Prof. DARWIN's paper. The form of the φ-curve, so far at least as regards that part of it with which we are concerned will now be as below (fig. 10).

It appears then that the starting point of the orbit A will move up to the planet J, that the orbit will at first become cusped and that afterwards it will have a loop round J which is described by the satellite in a retrograde direction. The critical stage occurs when C is in the neighbourhood of 39·5. This explains why Prof. DARWIN who confined his attention to the direct orbits alone, failed to find any trace of this orbit when $C = 39{\cdot}0$.

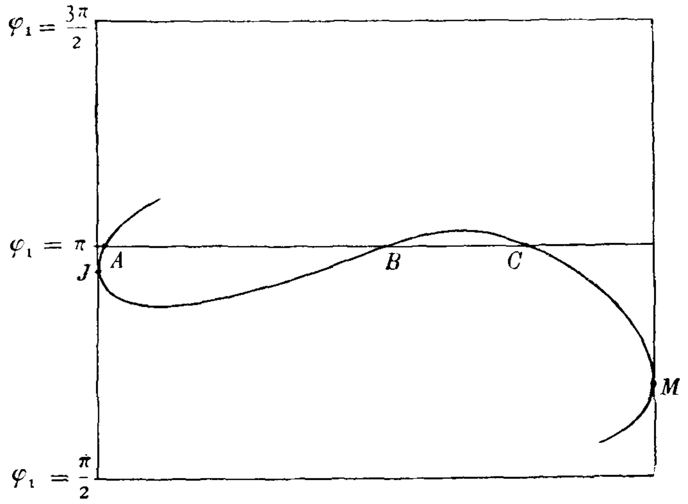

Fig. 10.

§ 8. *Form of the φ-curve in the neighbourhood of an asymptotic orbit.*

So long as the curve of zero velocity possesses a closed branch round J, the only forms of discontinuity which the φ-curve can present are those dealt with in the preceding sections, but when the curve of zero-velocity has assumed an hour-glass form more complex figures, resulting from the existence of asymptotic orbits, may occur. The nature of the singularity which appears in the neigbourhood of an asymptotic orbit may however be inferred from the curves given by DARWIN for the case $C = 39{\cdot}0$.

Hitherto by defining φ_1 as a particular determination of a multiple-valued function we have been able to regard φ_1 as a single-valued function of the quantity x_0, but we must now attach a slightly extended meaning to the symbol φ_1, which will no longer permit us to regard it as single-valued.

Suppose that the crossing on which our attention is fixed and to which the determination φ_1 belongs approaches the point of zero-force. We have seen (§ 1) that the orbit may then acquire loops, and may therefore cut the axis of x in several points before it recedes again either towards the planet or the Sun. When such loops exist, we define φ_1 as the angle made by the normal with the axis at any crossing prior to its again receding from the point of zero force. The number of real determinations of φ_1 will then depend on the number of loops which cut the line of syzygies.

Now on reference to DARWIN's figures it will be seen that as x_0 decreases from the value $x_0 = m$, $\varphi_1 - \pi$ is initially negative and twice vanishes and changes sign. The vanishing points determine two periodic orbits B and C. The corresponding portion of the φ-curve will then resemble that of the curves previously given.

If we confine our attention to the value of φ_1 at the first crossing it is evident that after passing the point B the value of φ_1 diminishes, until when x_0 is rather less than 1·06 it attains the value $\frac{\pi}{2}$. For values of x_0 beyond this one, the corresponding crossing becomes imaginary by coalescence with a second. Hence the branch of the φ_1 curve, to which we limit ourselves in directing our attention only to the first crossing, will intersect the line $\varphi_1 = \frac{\pi}{2}$ at right angles and will at this stage identify itself with a second branch. This second branch applies to the values of φ_1 at the second crossing, and from the figures it is clear that it first appears when x_0 is rather larger than 1·095. The second value of φ_1 comes into existence simultaneously with a third and its initial value is $-\frac{\pi}{2}$. The value of φ_1 as we pass along the second branch ranges between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, and consequently vanishes at some stage. The vanishing stage is that which corresponds to the figure-of-8 orbit of DARWIN's paper. It is clear that the second branch may be continued into a third, the third into a fourth and so on, that these various branches are in reality different parts of one and the same continuous curve, which possesses an infinite branch consisting of a series of waves of the form shewn in fig. 11.

This figure gives a concise summary of the results implied in DARWIN's figures 3 and 5, and we may interpret all the features of it in connection with the orbits represented in those figures.

Thus consider the intersections with the curve of an ordinate which moves from right to left.

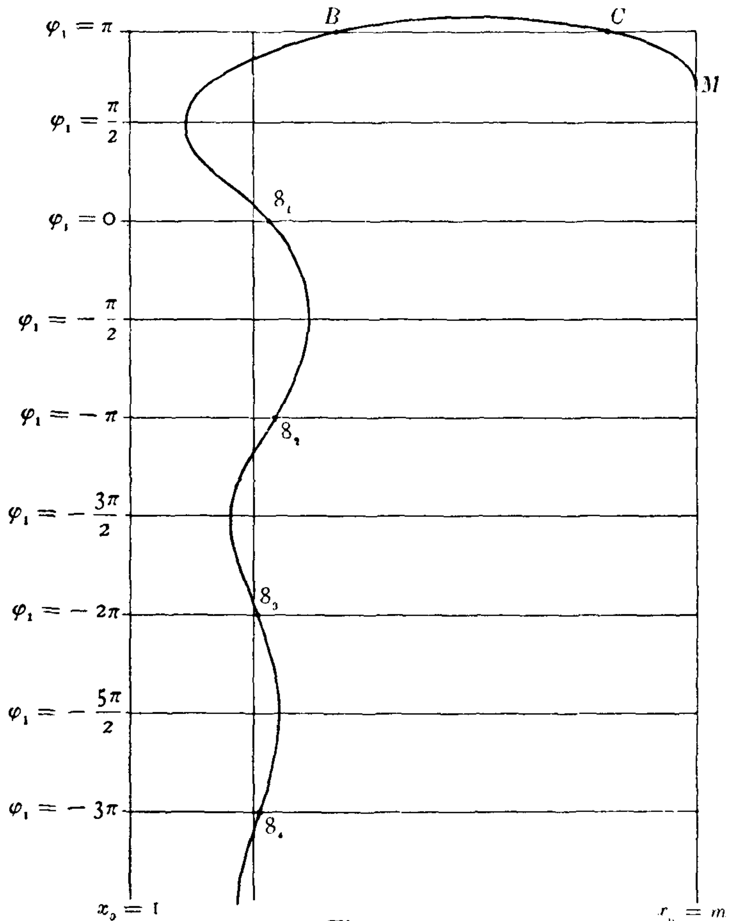

Fig. 11.

The first critical stage which will occur after its foot has passed B will be when the ordinate just touches the crest of the first wave. The ordinate will then intersect the curve in two new coincident points indicating that the orbit will intersect the axis of x in two new coincident points. This results from the fact that the orbit has acquired a single loop which just comes into contact with the axis of x at the critical stage. Evidently the initial values of φ_1 will be each equal to $-\frac{\pi}{2}$ and consequently the crest of the first wave lies on the line $\varphi_1 = -\frac{\pi}{2}$.

The new values of φ_1 will then separate and the next critical stage will occur when the larger of them attains the value zero, i. e. when the ordinate passes through the point marked 8_1 in the diagram. The orbit will then be periodic and of the form of a figure-of-8. It is the figure-of-8 orbit which has been found by DARWIN.

For further decrease in x_0 the ordinate will touch the crest of a second wave which indicates that a second loop will have been formed in the orbit and that this second loop bends upwards so as to touch and afterwards cut the axis-of-x. Subsequently when the ordinate passes through the point 8_2 the orbit with two loops will have become periodic. It will then be one of the more complex figure-of-8 orbits whose existence has been foreshadowed by Prof. DARWIN (p. 189).

Evidently the number of waves intersected by the ordinate will go on rapidly increasing in number, and with them the number of loops of the orbit intersecting the axis of x, until a critical stage will be reached when the ordinate attains a position shewn in the figure about which all the waves oscillate. When the ordinate attains this position the orbit will be the asymptotic orbit described in § 2 which possesses an infinite number of loops.

For further decrease in x_0 we see from the results of § 2 that the orbit will no longer be of the 'lunar' type but of the 'planetary' type. The number of real interesections of the ordinate with the φ-curve will rapidly diminish which implies that the orbit will shed its loops. Finally the ordinate will cease to intersect the curve in real points and the orbit will have attained the form shewn by DARWIN for the cases $x_0 = 1{\cdot}04$, $1{\cdot}02$ and $1{\cdot}001$, where there are no real intersections with the axis of x prior to the recession towards the Sun.

The intersections of the infinite branch with each of the lines $\varphi_1 = 0$, $\varphi_1 = -\pi$, $\varphi_1 = -2\pi$ &c. will indicate the existence of periodic orbits having 1, 2, 3 &c. loops. Since these orbits are all necessarily of the lunar type the points on the figure, marked 8_1, 8_2, 8_3 &c., which correspond to these periodic orbits, all lie to the right of the critical ordinate which separates the lunar orbits from the planetary.

We see then that the existence of an asymptotic orbit implies also the existence of an infinite number of complex figure-of-8 orbits, the first of which is that which has been found by DARWIN.

§ 9. *Completion of the φ-curve.*

The part of the curve shewn in our last figure is that which corresponds to the direct orbits. DARWIN's investigations enable us to figure this part of the curve with certainty, but no attempt has been made to draw it to scale in order that the essential features may be exhibited in a somewhat exaggerated form. Thus the amplitudes of the successive waves will be very much smaller than they are represented, all the critical features being included between $x_0 = 1.095$ and $x_0 = 1.06$ (approximately). As regards the remaining part of the curve we however have no such data available, and we have to fall back entirely on considerations of continuity to supply them.

Now DARWIN finds that when $x_0 = 1.02$ and even when $x_0 = 1.001$ the orbits are planetary and that they do not intersect the axis-of-x prior to their recession towards the Sun. It seems probable that they will retain this same character until x_0 reaches its limiting value (unity) and the orbit has a cusp at J. On the other hand when $x_0 = 1.3$ (see fig. 3) the orbits are lunar, and it is probable that they will retain their lunar character until the starting point reaches the curve of zero-velocity ($x_0 = m$).

If this be the case, as we pass from the orbit $x_0 = m$ to the orbit $x_0 = 1$ by continuous deformation, following the retrograde orbits instead of the direct orbits, we must pass through a stage where the orbits change from the lunar to the planetary form. Thus there must be a second asymptotic orbit among the retrograde orbits.

Whether or not the above assumptions as to the form of the cusped orbits be correct, it is clear than in such a cycle of orbits as that under consideration asymptotic orbits must occur in even numbers. Thus the existence of a single asymptotic orbit in the cycle necessarily involves the existence of a second.

Now it might appear at first sight that an asymptotic orbit could pass out of our cycle when the orbit falls into the planet and φ_1 undergoes an abrupt change by $\frac{1}{2}\pi$. Such however cannot be the case. To prove this let us suppose that the crossing on which our attention is fixed moves up to the planet J. A second crossing will reach J simultaneously with it. Now if our crossing corresponds with the approach to J, since a second crossing will occur after it on the opposite side of J, the orbit (so far as this crossing is concerned) will necessarily be of the lunar type both before and after the change. On the other hand if the crossing with which we are concerned belongs to the branch of the curve along which the satellite recedes from J, since the discontinuity in the geometrical form of the orbit is confined to the critical point and does not extend to remote parts of the curve, the orbit will be of the same character before and after its passage through the critical form. Thus such a discontinuity, as that represented by the passage from the point P to P' or from Q to Q' in fig. 9, can never involve a transition from the lunar to the planetary form or vice-versâ.

It appears then that asymptotic orbits can only disappear from our cycle by coalescence and that the development of them will always occur in pairs.

§ 10. *Second deformation of the φ-curve.*

Having recognized the existence of the asymptotic orbits and the form which the φ-curve assumes in their neighbourhood we next proceed to examine the transitional forms through which the curve will pass when two such orbits coalesce. A reversal of the order of the events considered will then indicate the state of affairs prior and subsequent to the development of a pair of asymptotic orbits.

First let us consider the forms of the orbits which possess a cusp at J as C increases. When $C = 39{\cdot}0$ these belong to the planetary class, but when $C = 40{\cdot}0$ they are found to be no longer planetary but lunar. The cusped orbit must therefore, for intermediate values of C, acquire loops, pass through the asymptotic form and finally shed these loops again. This will occur when one of the two ordinates which correspond to the asymptotic orbits in our cycle moves up to the line $x_0 = 1$. The corresponding asymptotic orbit will then undergo a change from the direct to the retrograde form or vice-versâ, and subsequently both asymptotic orbits will be direct or both retrograde. It seems probable that it is the retrograde orbit which passes through the critical form and becomes direct after the crisis. This assumption is however only made for the purpose of giving greater definiteness to our statements and is not essential to the arguments. After the passage of the asymptotic orbit through the cusped form the figure-of-8 orbits which accompany it will each in turn undergo a like change. For simplicity we will suppose that all these changes occur before the next critical stage is reached and that consequently the form of the φ-curve, so far as it applies to the direct orbits, is now as below, having two infinite branches.

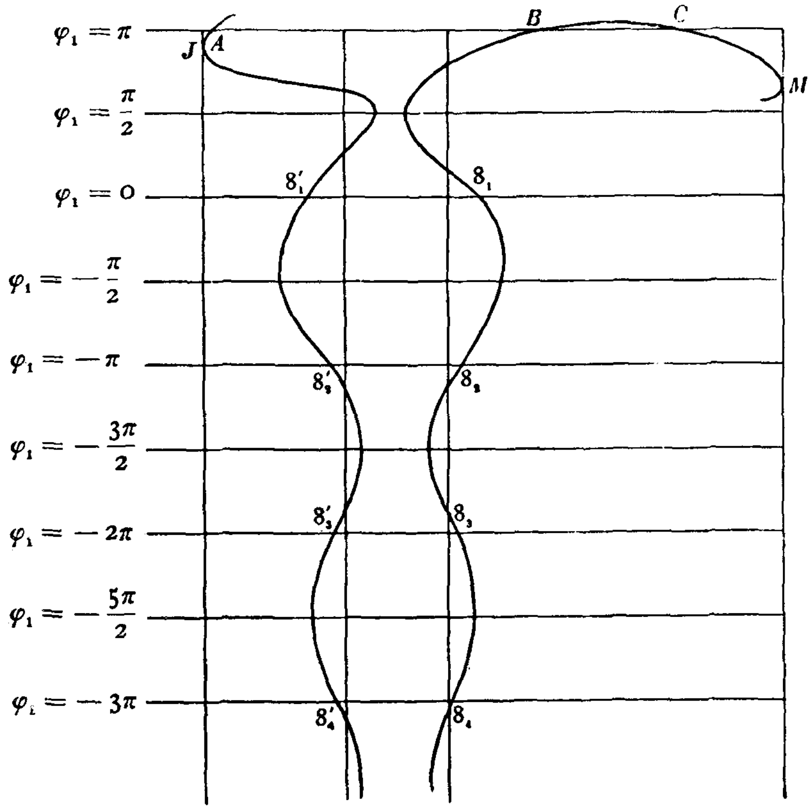

Fig. 12.

There will now be a second figure-of-8 orbit of a character similar to Prof. DARWIN'S, which corresponds to the point $8'_1$ of the figure, while there will be two infinite series of more complex figures-of-8 corresponding to the two series of points $8_2, 8_3, \ldots, 8'_2, 8'_3, \ldots$. It should be noticed that none of these points can be between the two critical ordinates which correspond with the asymptotic curves, since the periodic orbits all belong essentially to the lunar category. As C decreases further the next event of importance will be the coalescence of the crests of the two first waves in the above figure. After this coalescence the infinite branches of the curve will be severed from the remaining part, which will form a curve of the type shewn in figs. 9 and 10.

The significance of this change is that for larger values of C orbits of the type represented by $C = 39{\cdot}0$, $x_0 = 1{\cdot}04$, which do not intersect the axis of x before their recession, can no longer exist.

Subsequently as the two critical orbits approach one another the successive wave crests of the two branches will coalesce in turn and a series of isolated ovals will be formed. The curve will then consist of a branch similar to those of §§ 6 and 7, a finite number of isolated ovals and two infinite branches as below, where the curve is drawn on a reduced vertical scale with two ovals (fig. 13).

We arrive next at the case where the two critical ordinates move up to coincidence. The asymptotic orbits will then coalesce and disappear. The infinite branches of the φ-curve will have degenerated into an infinite series of isolated ovals. The two infinite series of figure-of-8 periodic orbits will however still have a real existence.

For further increase in C these ovals will shrink in size, reduce to points and finally disappear. It is clear that each of the points $8_1, 8_2, 8_3, \ldots$ will disappear by coalescence with the corresponding point of the series $8'_1, 8'_2, 8'_3, \ldots$ indicating that each of the periodic orbits disappears by coalescence with a second. Also it is evident that the more distant ovals will be the first to vanish indicating that the periodic orbits with a large number of turns round the orbit a will vanish before those with a smaller number of turns and that the last orbits to vanish will be the simple figures-of-8 of Prof. DARWIN's paper.

There will be an interval between the disappearance of the points 8_1, $8'_1$ &c. and the final evanescence of the ovals. During this interval non-periodic orbits may occur having loops, but it will never be possible for the loops to adjust themselves so as to cut the line of syzygies at right angles.

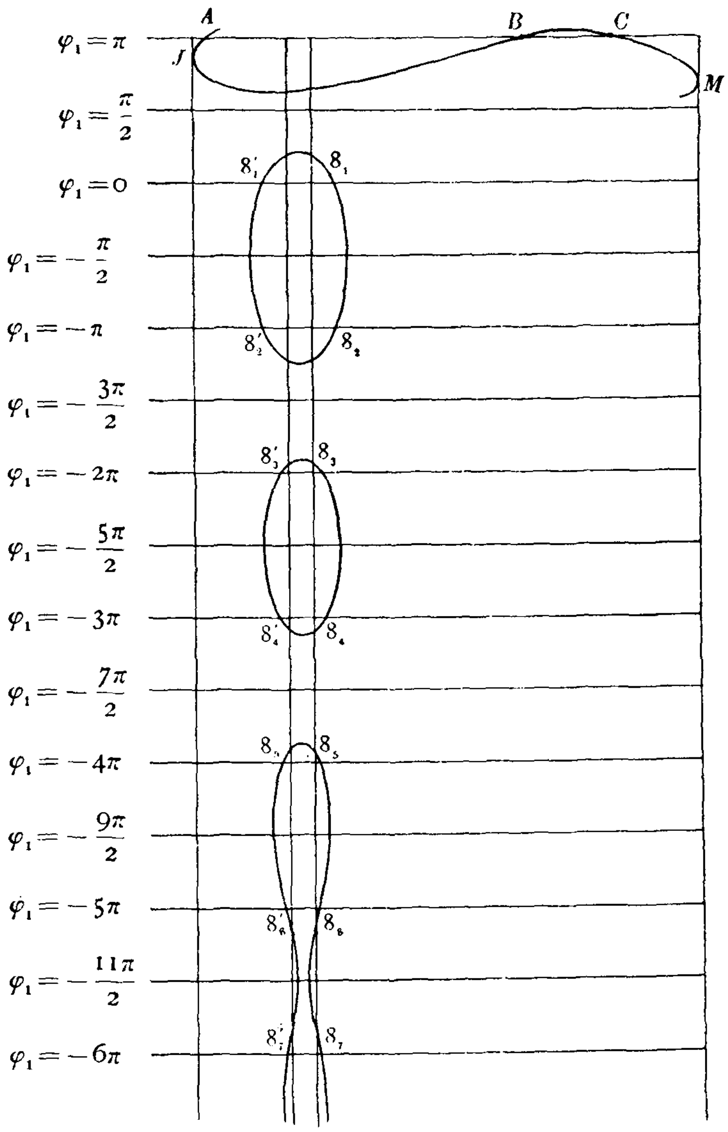

Fig. 13.

§ 11. *Summary and Conclusion.*

The problems dealt with arise as the result of the study of an apparently remarkable change in the forms of certain periodic orbits which have been examined by Prof. DARWIN, whereby an orbit originally in the form of a simple closed oval seems to have developed into the form of a figure-of-8. This change is of so surprizing a character that M. POINCARÉ and others have concluded that Prof. DARWIN was misled in regarding the simple oval orbit and the figure-of-8 orbit as members of the same family. But if we regard them as members of two different families questions at once arise as to the earlier history of the second family and the later history of the former. On these points DARWIN's results throw very little light, but by supplementing the numerical results obtained by him by arguments based on the consideration of geometrical continuity I have shewn that the history of either family may be traced in part, and have verified M. POINCARÉ's conclusion as to the independence of these two families. The results explain fully how the first family has been lost sight of by DARWIN and how the second family comes into existence. It appears to have been largely a matter of chance that with the actual numerical data adopted the appearance of the second family coincided exactly with the appearance of the first, a fact which naturally led DARWIN to the conclusion that the two forms of orbit really belonged to the same group.

The earlier sections of the paper (§§ 1—4) deal with two critical cases which may occur in connection with the forms of non-periodic orbits and lead to two classifications of these orbits. They are first classified as 'lunar' or 'planetary' according to the fate of the satellite after describing a semi-revolution round the primary. If it recedes towards the primary and proceeds to describe further revolutions round it, it is described as 'lunar'. If on the other hand it passes away towards the Sun it is described as 'planetary'. The critical orbits which separate the lunar from the planetary describe an infinite number of loops round the point of zero-force, approximating at each turn closer and closer to the orbit of the oscillating satellite. Adopting POINCARÉ's term we describe the critical orbit as an 'asymptotic' orbit. We next classify the orbits as 'direct' or 'retrograde' according as the initial direction of motion is direct or retrograde, the critical orbits which separate the one class from the other being described as 'cusped' on account of the forms which they assume. It is then shewn that by the inclusion of the retrograde orbits as well as the direct, we may arrange the orbits under consideration into a perfect self-contained cycle which may however involve certain abrupt discontinuities. The disappearance of the orbit A is accounted for by the fact that at the stage at which it was sought by DARWIN it had passed through the critical cusped form and thus was no longer to be found in the part of the cycle examined by him viz: the part which includes only the direct orbits.

We proceed to shew how all the results indicated in the figures of the non-periodic orbits traced by DARWIN may be represented on a single curve, and how a study in the variations in the form of this curve will enable us to trace the history of the different families of periodic orbits. The method employed by DARWIN to discover the periodic orbits in fact is equivalent to the examination of the form of this curve in regions where there were *à priori* grounds for suspecting the existence of a periodic orbit.

The chief point of interest in connection with these curves is the form which they assume in the neighbourhood of an asymptotic orbit i.e. one of the critical orbits implied in our first classification. The form indicates that such an asymptotic orbit is accompanied not only by a simple figure-of-8 orbit of the kind found by DARWIN, but also by an infinite series of complex figures-of-8 whose exact forms have not yet been examined but whose existence has been predicted by DARWIN.

We next shew that in a complete cycle of orbits, such as that which we have found to exist, asymptotic orbits must occur in even numbers and consequently must appear or disappear in pairs. As DARWIN's results indicate only one of such orbits among the direct orbits for the case $C = 39{\cdot}0$ we conclude that a second one must exist among the retrograde orbits. The two asymptotic orbits must have had a common origin and at the instant of their first appearance must have been both direct or both retrograde. For purposes of illustration it has been assumed that both are initially direct though this is not essential to the arguments employed.

The existence of two asymptotic orbits implies also the existence of two simple figure-of-8 orbits of the form found by DARWIN, which must likewise have had a common origin. The converse is however not necessarily true. For if we trace back the changes described in the last section we see that the first indication of the development of a pair of asymptotic orbits will be the growth of loops, as in the orbit $x = 1{\cdot}095$ of DARWIN's fig. 5. When these loops first appear it will not be possible for them to arrange themselves so as to cut the line of syzygies at right angles and render the orbit periodic, but with smaller values of C pairs of periodic looped orbits will appear which gradually separate from one another. The first to appear will be those with the smaller number of loops, the final ones being the asymptotic orbits which may be regarded as the limiting form of periodic looped orbits when the number of loops becomes infinitely great.

The passage of the second asymptotic orbit from the direct to the retrograde form will be preceded by a similar passage of the whole series of periodic looped orbits which accompany it, including the simple figure-of-8 orbit which at its origin coincided with that of Prof. DARWIN. Here again the failure to find any trace of this orbit is to be explained by the fact that the search was confined to the direct orbits alone, whereas at the stage under investigation ($C = 39\cdot 0$) this orbit had already passed into the retrograde form.

In conclusion I have to thank Prof. DARWIN not only for the unfailing courtesy with which be has placed at my disposal all the details of the prodigious amount of numerical work which formed the basis of his published memoir on this subject, but also for the readiness with which be has communicated to me his further results still under investigation and for valuable criticism which has saved me from numerous errors. Even should the present results be found to require modification on further investigation, and it is admitted that the detail is to be regarded as conjectural rather than proven, I shall feel that the paper will have served a useful purpose if it succeeds in enticing other investigators into the vast and hitherto unexplored field which seems to be opened up with each new development of this highly interesting subject.