# SURFACE TRANSFORMATIONS AND THEIR DYNAMICAL APPLICATIONS.

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A state of motion in a dynamical system with two degrees of freedom depends on two space and two velocity coördinates, and thus may be represented by means of a point in space of four dimensions. When only those motions are considered which correspond to a given value of the energy constant, the points lie in a certain three-dimensional manifold. The motions are given as curves in this manifold. One such curve passes through each point.

Imagine these curves to be cut by a surface lying in the manifold. As the time increases, a moving point of the manifold describes a half-curve and meets the surface in successive points, $P, P', \ldots$. In this manner a particular transformation of the surface into itself — namely that which takes any point $P$ into the unique corresponding point $P'$ — is set up.

This fundamental reduction of the dynamical problem to a transformation problem was first effected by POINCARÉ and later, more generally, by myself.[1] In order to take further advantage of it I consider such transformations at length in the following paper, which appears here by the kind invitation of Professors MITTAG-LEFFLER and NÖRLUND. The dynamical applications are made briefly in conclusion. These bear on the difficult questions of integrability, stability, and the classification and interrelation of the various types of motions.

## Chapter I. Formal Theory of Invariant Points.

### § 1. Hypotheses.

For the present we shall confine attention to the consideration of a one-to-one, direct, analytic transformation $T$ in the vicinity of an invariant point of

[1] *Dynamical systems with two degrees of freedom. Transactions of the American Mathematical Society*, vol. 18, 1917.

the surface $S$ undergoing transformation. Hence, if $u, v$ be properly taken coördinates with the invariant point at $u = v = 0$, the transformation may be written

$$u_1 = au + bv + \cdots,$$
$$(1) \qquad v_1 = cu + dv + \cdots,$$

where the right-hand members are real power series in $u, v$ (i. e. with real coefficients), where $u_1, v_1$ are the coördinates of the transformed point, and where

$$(2) \qquad ad - bc > 0.$$

More generally, the notation $(u_k, v_k)$ or $P_k$ $(k = 0, \pm 1, \pm 2, \ldots)$ will stand for the point obtained by applying the $k$th iterate (power) of $T$ to $(u, v)$ or $P$.

Furthermore it will be assumed that there exists a real analytic function $Q(u, v)$, not zero for $u = v = 0$, such that the double integral

$$\int\int Q(u, v)\, du\, dv$$

has the same value when extended over any region as over its image under $T$. Following a dynamical analogy such a transformation will be called *conservative*. Also $Q$ will be termed a *quasi-invariant function of $T$*.

An explicit form for the condition that a quasi-invariant function must satisfy is well-known[1] and may be readily derived. If the double integral be expressed in terms of the new variables $u_1, v_1$, it takes the form

$$\int\int Q(u, v)\left[\frac{\partial u}{\partial u_1}\frac{\partial v}{\partial v_1} - \frac{\partial v}{\partial u_1}\frac{\partial u}{\partial v_1}\right] du_1\, dv_1,$$

where the integration extends over the image of the given region under $T$. Since the given region is arbitrary, and since by hypothesis the last written integral has the same value as $\int\int Q(u_1, v_1)\, du_1\, dv_1$ taken over the same region, we infer that the two integrands are equal. But the Jacobian of $u, v$ as to $u_1, v_1$ is the reciprocal of the Jacobian of $u_1, v_1$ as to $u, v$. Hence we obtain

$$(3) \qquad Q(u, v) = Q(u_1, v_1)\left[\frac{\partial u_1}{\partial u}\frac{\partial v_1}{\partial v} - \frac{\partial v_1}{\partial u}\frac{\partial u_1}{\partial v}\right].$$

[1] Cf. E. Goursat, *Sur les transformations ponctuelles qui conservent les volumes. Bulletin des Sciences Mathématiques,* vol. 52, 1917.

Conversely, if $Q(u, v)$ is a real analytic function, not zero for $u = v = 0$, and if (3) is true, it follows at once that $Q$ is a quasi-invariant function.

If there exists a second quasi-invariant function $Q'$ not a constant multiple of $Q$, it is clear that the ratio $\frac{Q'}{Q}$ is an analytic *invariant function of* $T$, not zero for $u = v = 0$. Moreover, if any quasi-invariant function be multiplied by such an invariant function, the product is clearly a quasi-invariant function.

When a conservative transformation $T$ has an analytic invariant function (not a constant), the transformation will be said to be *integrable*.[1]

A transformation $T$ remains conservative under a change of variables, say from $u, v$ to $\bar{u}, \bar{v}$. The quasi-invariant function $Q$ is thereby modified to a function $\bar{Q}$ obtained by multiplying $Q$ by the Jacobian of $u, v$ as to $\bar{u}, \bar{v}$.

## § 2. Preliminary Classification of Invariant Points.

We first make an evident and well-known preliminary classification of invariant points which is wholly based on the nature of the linear terms in the power series for $u_1, v_1$. Under real linear change of variables these first degree terms are transformed among themselves without reference to terms of higher degree. Consequently the theory of linear transformations applies to these terms. According to this theory the classification depends largely upon the nature of the roots of the quadratic equation in $\varrho$,

$$\varrho^2 - (a + d)\varrho + ad - bc = 0.$$

In the case at hand this equation is a reciprocal quadratic equation, i. e.

$$ad - bc = 1. \tag{4}$$

For, if $u = v = 0$, we have $Q = Q_1 \neq 0$ and also

$$\frac{\partial u_1}{\partial u} = a, \ \frac{\partial u_1}{\partial v} = b, \ \frac{\partial v_1}{\partial u} = c, \ \frac{\partial v_1}{\partial v} = d.$$

Thus from (3) the stated equation (4) follows. The roots of this reciprocal equation will be designated as $\varrho$ and $\frac{1}{\varrho}$.

[1] It should be observed that the definition refers to the vicinity of an invariant point.

There are the following three cases to consider. First, $\varrho$ may be real with a numerical value not unity; $T$ can then be taken in the normal form

$$\text{I.} \qquad \begin{cases} u_1 = \varrho u + \sum\limits_{m+n=2}^{\infty} \varphi_{mn} u^m v^n, & (\varrho \neq \pm 1), \\ v_1 = \dfrac{1}{\varrho} v + \sum\limits_{m+n=2}^{\infty} \psi_{mn} u^m v^n. \end{cases}$$

We subdivide this case according as $\varrho$ is positive (case I') or negative (case I''). Secondly, $\varrho$ may be complex and so of modulus 1. With this case we group that case $\varrho = \pm 1$ in which the two elementary divisors are distinct. Here $T$ may be taken in the normal form

$$\text{II.} \qquad \begin{cases} u_1 = u \cos\theta - v \sin\theta + \sum\limits_{m+n=2}^{\infty} \varphi_{mn} u^m v^n, & (\varrho = e^{\sqrt{-1}\theta}), \\ v_1 = u \sin\theta + v \cos\theta + \sum\limits_{m+n=2}^{\infty} \psi_{mn} u^m v^n. \end{cases}$$

It is convenient to subdivide case II into the *irrational* case II' when $\frac{\theta}{2\pi}$ is irrational, and the *rational* cases II'' when $\theta = 0$, and II''' when $\frac{\theta}{2\pi} = \frac{p}{q}$ with $\frac{p}{q}$ not an integer. Case II'' yields the case $\varrho = 1$; and II''', the case $\varrho = -1$. Thirdly, we have that case in which the two elementary divisors are not distinct; here $T$ may be taken in the normal form

$$\text{III.} \qquad \begin{cases} u_1 = \pm u + \sum\limits_{m+n=2}^{\infty} \varphi_{mn} u^m v^n, & (\varrho = \pm 1), \\ v_1 = \pm v + du + \sum\limits_{m+n=2}^{\infty} \psi_{mn} u^m v^n, & (d \neq 0). \end{cases}$$

We subdivide this case according as $\varrho = 1$ (case III') or $\varrho = -1$ (case III'').

If only linear terms are present in $u_1$, $v_1$ we obtain the linear transformations:

$$\text{I.} \qquad u_1 = \varrho u, \; v_1 = \frac{1}{\varrho} v, \qquad (\varrho \neq \pm 1),$$

$$\text{II.} \qquad u_1 = u \cos\theta - v \sin\theta, \; v_1 = u \sin\theta + v \cos\theta,$$

$$\text{III.} \qquad u_1 = \pm u, \; v_1 = \pm v + du, \qquad (d \neq 0).$$

These may be regarded as furnishing a first approximation to the corresponding general types. According to our definition all three linear transformations are conservative with $Q = 1$ a quasi-invariant function since areas are left invariant. Furthermore these cases are integrable with invariant functions $uv$, $u^2 + v^2$, $u^2$ respectively.

In the first case a point $P$ will move on a hyperbola $uv = \text{const.}$ upon successive application of $T$ or $T_{-1}$ ($u$, $v$ being taken as rectangular coördinates); in the third case $P$ will move along a pair of parallel lines $u^2 = \text{const.}$ Unless the point $P$ lies on the degenerate hyperbola $uv = 0$ in the first case, or on the pair of coincident straight lines $u^2 = 0$ in the third, $P$ will recede to infinity upon successive application of $T$ or $T_{-1}$. When $P$ lies on the degenerate hyperbola in the first case, it will approach the invariant point $(0, 0)$ upon successive application of $T$ or else of $T_{-1}$, and recede to infinity upon application of the inverse transformation. In the third case all points of the line $u = 0$ are invariant or are reflected into points of the same line on the other side of $(0, 0)$, according as the $+$ or $-$ sign is used.

On the other hand, in the second case the transformation is a rotation about $(0, 0)$ through an angle $\theta$, and every point $P$ remains at a fixed distance from $(0, 0)$ upon successive application of $T$ or $T_{-1}$.

The essence of the distinction here existing is brought out clearly by means of the following fundamental definition: if a neighborhood of an invariant point can be so taken that points arbitrarily near the invariant point leave this neighborhood upon successive application of $T$ (or of $T_{-1}$), the invariant point is *unstable;* in the contrary case the invariant point is *stable.*[1]

Thus the linear transformations I, III are unstable in this sense, while those of type II are stable.

## § 3. An auxiliary Lemma.

Before proceeding to the consideration of formal series for $u_k$, $v_k$ ($k = 0, \pm 1, \pm 2, \ldots$), we will establish the following obvious but useful lemma:

*Lemma.* The linear difference equation of the first order in $y(k)$,

$$y(k + 1) - \sigma y(k) = c\lambda^k k^\mu,$$

[1] See T. Levi-Civita, *Sopra alcuni criteri di instabilità. Annali di Matematica,* Ser. III vol. 5, 1901.

($\sigma$, $c$, $\lambda$ real, and $\mu$ a positive integer or zero) admits a solution

$$\lambda^k \text{(real polynomial in } k \text{ of degree } \mu)$$

if $\lambda \neq \sigma$, and otherwise a solution

$$\lambda^k \text{(real polynomial in } k \text{ of degree } \mu + 1).$$

Suppose first that $\lambda \neq \sigma$. Let us make the substitution $y = \lambda^k w$, when the difference equation takes the form

$$w(k+1) - \frac{\sigma}{\lambda} w(k) = \frac{c}{\lambda} k^\mu .$$

If we write

$$w = w^{(0)} k^\mu + w^{(1)} k^{\mu-1} + \cdots + w^{(\mu)},$$

we find that $w$ will be a solution if the following conditions are satisfied

$$\left(1 - \frac{\sigma}{\lambda}\right) w^{(0)} = \frac{c}{\lambda},$$

$$\mu w^{(0)} + \left(1 - \frac{\sigma}{\lambda}\right) w^{(1)} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$w^{(0)} + w^{(1)} + \cdots + \left(1 - \frac{\sigma}{\lambda}\right) w^{(\mu)} = 0.$$

On account of the assumption made, we see at once that these equations determine real quantities $w^{(0)}$, $w^{(1)}, \ldots,$ $w^{(\mu)}$ in succession, and lead to a solution of the kind specified.

If $\lambda = \sigma$ a slightly modified argument applies. Here we write $y = \lambda^k w$ as before, and then

$$w = w^{(0)} k^{\mu+1} + w^{(1)} k^\mu + \cdots + w^{(\mu+1)}.$$

The conditions on the coefficients take the form

$$(\mu+1) w^{(0)} = \frac{c}{\lambda},$$

$$\frac{(\mu+1)\mu}{1 \cdot 2} w^{(0)} + \mu w^{(1)} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$w^{(0)} + w^{(1)} + \cdots + w^{(\mu)} = 0.$$

These equations determine real quantities $w^{(0)}$, $w^{(1)}, \ldots,$ $w^{(\mu)}$ in succession but leave $w^{(\mu+1)}$ undetermined, although it is to be taken real.

## § 4. Formal Series for $u_k, v_k$. Case I.

By iteration one can obtain convergent series for $u_k$, $v_k$ in terms of $u$, $v$. In case I the linear terms of these series are evidently $\varrho^k u$, $\varrho^{-k} v$ respectively. This fact suggests that higher degree terms may be similarly given an explicit form in $k$, and we shall show this to be the fact.

*If $u_1$, $v_1$ are real series of the form I with $\varrho > 0$ (case I′), $u_k$, $v_k$ may be represented for all integral values of $k$ in the form*

$$
\mathrm{I}'_k. \qquad \begin{cases} u_k = \varrho^k u + \sum_{m+n=2}^{\infty} \varphi_{mn}^{(k)} u^m v^n, \\ v_k = \varrho^{-k} v + \sum_{m+n=2}^{\infty} \psi_{mn}^{(k)} u^m v^n, \end{cases}
$$

*where $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ are real polynomials in $\varrho^k$, $\varrho^{-k}$, $k$ of degree at most $m+n$ in these variables.*

Let us consider first the quadratic terms in the series for $u_k$, $v_k$.

If in $u_k$, $v_k$ we replace $u$, $v$ by $u_1$, $v_1$ respectively, we obtain $u_{k+1}$, $v_{k+1}$ by definition. By comparison of coefficients in $\mathrm{I}'_k$ above, this leads to the equations

$$
\varphi_{20}^{(k+1)} = \varrho^k \varphi_{20} + \varrho^2 \varphi_{20}^{(k)}, \quad \varphi_{11}^{(k+1)} = \varrho^k \varphi_{11} + \varphi_{11}^{(k)}, \quad \varphi_{02}^{(k+1)} = \varrho^k \varphi_{02} + \varrho^{-2} \varphi_{02}^{(k)},
$$

$$
\psi_{20}^{(k+1)} = \varrho^{-k} \psi_{20} + \varrho^2 \psi_{20}^{(k)}, \; \psi_{11}^{(k+1)} = \varrho^{-k} \psi_{11} + \psi_{11}^{(k)}, \; \psi_{02}^{(k+1)} = \varrho^{-k} \psi_{02} + \varrho^{-2} \psi_{02}^{(k)}.
$$

The first three of these equations are obtained by comparing the coefficients of $u^2$, $uv$, $v^2$ respectively in $u_{k+1}(u, v)$ and $u_k(u_1, v_1)$; the second three are found by a like comparison of $v_{k+1}(u, v)$ and $v_k(u_1, v_1)$.

By considering $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ with $m+n=2$ as undetermined functions of the index $k$, it is clear that these six equations constitute six difference equations of the type treated in the lemma of § 3.

Moreover these equations suffice to determine these six functions fully for all integral values of $k$ if their value is known for any particular $k$. In the case at hand we have of course $\varphi_{mn}^{(0)} = \psi_{mn}^{(0)} = 0$ for all $m$ and $n$, since $u_0 = u$, $v_0 = v$.

According to the lemma we can find explicit solutions of these difference equations of a very simple type, namely constant multiples of $\varrho^k$ for the first three equations, and of $\varrho^{-k}$ for the second three equations. Also the six reduced homogeneous equations obtained by removing the first term on the right in the six equations admit the following respective particular solutions:

$$\varrho^{2k},\ 1,\ \varrho^{-2k};\ \varrho^{2k},\ 1,\ \varrho^{-2k}.$$

By adding real constant multiples of these solutions to the respective solutions of the non-homogeneous equations, we find a new set of particular solutions vanishing for $k=0$ as desired.

In this way we obtain the explicit values of $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ for $m+n=2$:

$$(5)\qquad \begin{cases} \varphi_{20}^{(k)} = \dfrac{\varphi_{20}(\varrho^k-\varrho^{2k})}{\varrho-\varrho^2}, & \varphi_{11}^{(k)} = \dfrac{\varphi_{11}(\varrho^k-1)}{\varrho-1}, \quad \varphi_{02}^{(k)} = \dfrac{\varphi_{02}(\varrho^k-\varrho^{-2k})}{\varrho-\varrho^{-2}}, \\ \psi_{20}^{(k)} = \dfrac{\psi_{20}(\varrho^{-k}-\varrho^{2k})}{\varrho^{-1}-\varrho^2}, & \psi_{11}^{(k)} = \dfrac{\psi_{11}(\varrho^{-k}-1)}{\varrho^{-1}-1}, \quad \psi_{02}^{(k)} = \dfrac{\psi_{02}(\varrho^{-k}-\varrho^{-2k})}{\varrho^{-1}-\varrho^{-2}}. \end{cases}$$

We proceed to show that explicit expressions for $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ of the type stated exist also for $m+n=3$, $m+n=4, \ldots$ in succession.

To begin with, we write the equations obtained by a comparison of the coefficients of $u^m v^n$ in $u_{k+1}(u,v)$, $u_k(u_1,v_1)$ and $v_{k+1}(u,v)$, $v_k(u_1,v_1)$ in the respective abbreviated forms:

$$\varphi_{mn}^{(k+1)} = \varrho^k \varphi_{mn} + \varrho^{m-n}\varphi_{mn}^{(k)} + P_{mn},$$

$$\psi_{mn}^{(k+1)} = \varrho^{-k} \psi_{mn} + \varrho^{m-n}\psi_{mn}^{(k)} + Q_{mn}.$$

The expansions of $\varrho^k u_1$ and $\varrho^{-k} v_1$ in $u_k(u_1,v_1)$ and $v_k(u_1,v_1)$ respectively yield the first terms on the right in these equations. The second terms arise from the expansion of $\varphi_{mn}^{(k)} u_1^m v_1^n$ and $\psi_{mn}^{(k)} u_1^m v_1^n$ in the same functions. The last terms arise from the expansion of $\varphi_{\alpha\beta}^{(k)} u_1^\alpha v_1^\beta$ and $\psi_{\alpha\beta}^{(k)} u_1^\alpha v_1^\beta$ respectively, with $\alpha+\beta<m+n$; thus $P_{mn}$ and $Q_{mn}$ are linear and homogeneous in $\varphi_{\alpha\beta}^{(k)}$, $\psi_{\alpha\beta}^{(k)}$ respectively, with real coefficients, polynomial in $\varrho$, $\varrho^{-1}$, $\varphi_{\mu\nu}$, $\psi_{\mu\nu}$ $(\mu+\nu<\alpha+\beta)$.

Suppose now that we take $m+n=3$ and assume that the explicit expressions for $\varphi_{\alpha\beta}^{(k)}$ $(\alpha+\beta=2)$ are substituted in $P_{mn}$, $Q_{mn}$. The above equations become linear difference equations in $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$. Furthermore, it is clear that these equations, together with the fact that $\varphi_{mn}^{(0)}$, $\psi_{mn}^{(0)}$ vanish, determine these variables completely for all integral values of $k$.

By a similar process to that employed in the case $m+n=2$ we may arrive now at explicit expressions for $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ in the case $m+n=3$.

In this new case we have a non-homogeneous part composed of more than one term. But each term is of the form $c\lambda^k k^\mu$ occurring on the right-hand side of the equation of the lemma (§ 3), since the non-homogeneous part is a polynomial in $\varrho^k$, $\varrho^{-k}$ of degree at most 2.

If we add together the various particular solutions corresponding to each of these terms, as given by the lemma, we obtain a solution of each difference equation for $m+n=3$ in the form of a real polynomial in $\varrho^k$, $\varrho^{-k}$, $k$, of at most the third degree in these variables.

The corresponding homogeneous reduced equation has a solution $\varrho^{(m-n)k}$. If a suitable real constant multiple of this solution is added to the above particular solution of the non-homogeneous equation, a new particular solution is obtained which vanishes for $k=0$. Solutions of this type are real polynomials in $\varrho^k$, $\varrho^{-k}$, $k$ of degree at most 3 in these variables, and form the desired expressions.

Proceeding indefinitely in this way we establish the truth of the italicized statement for $m+n=3$, $m+n=4, \ldots$.

It is obvious that the coefficients in the polynomials $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ are themselves real polynomials in the coefficients of the series $u_1$, $v_1$, save for divisors of the form $\varrho^\alpha - \varrho^\beta$ where $\alpha$ and $\beta$ are unequal integers.

In the later discussion it is convenient to bring back the case I″ ($\varrho<0$) to the case I′ by means of the following remark:

*If $u_1$, $v_1$ are real series of the form I with $\varrho<0$ (case I″), then $u_2$, $v_2$ are of the form I′ treated above.*

## § 5. Formal series for $u_k$, $v_k$. Case II.

Next let us consider series of type II in the general case when $\theta$ is incommensurable with $2\pi$.

*If $u_1$, $v_1$ are real series of the form II with $\frac{\theta}{2\pi}$ irrational (case II′), $u_k$, $v_k$ may be represented for all integral values of $k$ in the form*

$$\mathrm{II}'_k. \qquad \begin{cases} u_k = u\cos k\theta - v\sin k\theta + \sum_{m+n=2}^{\infty} \varphi_{mn}^{(k)} u^m v^n, \\ v_k = u\sin k\theta + v\cos k\theta + \sum_{m+n=2}^{\infty} \psi_{mn}^{(k)} u^m v^n, \end{cases}$$

*where $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ are real polynomials in* $\cos k\theta$, $\sin k\theta$, $k$ *of degree at most $m+n$ in these variables.*

Let us introduce new variables $\bar{u}$, $\bar{v}$, namely

$$\bar{u} = u + \sqrt{-1}\, v, \quad \bar{v} = u - \sqrt{-1}\, v.$$

The equations II give series for $\bar{u}_1$, $\bar{v}_1$ in terms of $\bar{u}$, $\bar{v}$, which are of the form I with $\varrho = e^{\sqrt{-1}\,\theta}$.

Now the lemma of § 3 can evidently be extended to the case when $\sigma$, $c$, $\lambda$ are complex constants. Here of course the polynomial factors in the solutions are no longer real in general. Hence the same formal treatment of $\bar{u}_k$, $\bar{v}_k$ is posssible as was made in case I' for $u_k$, $v_k$; in fact for the case at hand none of the divisors $\varrho^\alpha - \varrho^\beta$ are o so that the solutions are precisely of the same form. Thus $\bar{u}_k$, $\bar{v}_k$ can be expressed as power series in $\bar{u}$, $\bar{v}$ with coefficients $\bar{\varphi}_{mn}^{(k)}$, $\bar{\psi}_{mn}^{(k)}$ of $\bar{u}^m \bar{v}^n$ respectively, polynomial in $\varrho^k$, $\varrho^{-k}$, $k$ of degree not more than $m+n$.

Recalling the simple relation between $u$, $v$ and $\bar{u}$, $\bar{v}$, and utilizing the trigonometric form of $\varrho^k$, $\varrho^{-k}$ we arrive at series $u_k$, $v_k$ of the desired type, save that the reality of the polynomials $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ is not established.

Although an inspection of the actual formulas employed would establish this reality, it suffices to note that, since $u_k$, $v_k$ are real power series, the real parts of $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ constitute real polynomials of the type required.

In the rational case II, $\theta = 0$, series of type II are also of type I with $\varrho = 1$. Consequently the method of § 4 leads at once to the conclusion:

*If $u_1$, $v_1$ are real series of the form II with $\theta = 0$ (case II''), $u_k$, $v_k$ may be represented for all integral values of $k$ in the form*

$$\mathrm{II}''_k. \qquad \begin{cases} u_k = u + \sum\limits_{m+n=2}^{\infty} \varphi_{mn}^{(k)} u^m v^n, \\ v_k = v + \sum\limits_{m+n=2}^{\infty} \psi_{mn}^{(k)} u^m v^n, \end{cases}$$

*where $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ are real polynomials in $k$ of degree at most $m+n-1$.*[1]

The rational case $\theta \neq 0$ can be brought back to the case $\theta = 0$:

*If $u_1$, $v_1$ are real series of the form II with $\frac{\theta}{2\pi} = \frac{p}{q}$ (case II'''), then $u_q$, $v_q$ are of the form II''.*

There are series similar to $\mathrm{II}'_k$ in the general rational case, but we do not need to use them.

[1] This fact has been noted by C. L. Bouton, *Bulletin of the American Mathematical Society*, vol. 23, 1916, p. 73. See also A. A. Bennett, *A case of iteration in several variables*, *Annals of Mathematics*, vol. 17, 1915—1916.

## § 6. Formal series for $u_k$, $v_k$. Case III.

Finally we have to consider case III:

*If $u_1$, $v_1$ are real series of the form III with $\varrho = 1$ (case III'), $u_k$, $v_k$ may be represented for all integral values of $k$ in the form*

$$\mathrm{III}'_k. \quad \begin{cases} u_k = u + \sum_{m+n=2}^{\infty} \varphi_{mn}^{(k)} u^m v^n, \\ v_k = v + kdu + \sum_{m+n=2}^{\infty} \psi_{mn}^{(k)} u^m v^n, \end{cases}$$

*where $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ are real polynomials in $k$ of degree at most $2m+n-1$.*

We propose to deal with this case by reducing it to the case II'' as follows. Write

$$u = \overline{u}\overline{v}, \quad v = \overline{v},$$

and let us make this change of variables in the given transformation. We obtain

$$\overline{u}_1 \overline{v}_1 = \overline{u}\overline{v} + \sum_{m+n=2}^{\infty} \varphi_{mn} \overline{u}^m \overline{v}^{m+n},$$

$$\overline{v}_1 = \overline{v} + d\overline{u}\overline{v} + \sum_{m+n=2}^{\infty} \psi_{mn} \overline{u}^m \overline{v}^{m+n}.$$

Now the right-hand member of each of these equations contains $v$ as a factor. Hence, dividing the first equation, member for member, by the second, we find the equivalent equations

$$\overline{u}_1 = \overline{u} + \sum_{m+n=2}^{\infty} \overline{\varphi}_{mn} \overline{u}^m \overline{v}^n,$$

$$\overline{v}_1 = \overline{v} + \sum_{m+n=2}^{\infty} \overline{\psi}_{mn} \overline{u}^m \overline{v}^n,$$

which is formally of the type II''. Hence by our result in § 5 we may write for all integral values of $k$

$$\bar{u}_k = \bar{u} + \sum_{m+n=2}^{\infty} \bar{\varphi}_{mn}^{(k)} \bar{u}^m \bar{v}^n,$$

$$\bar{v}_k = \bar{v} + \sum_{m+n=2}^{\infty} \bar{\psi}_{mn}^{(k)} \bar{u}^m \bar{v}^n,$$

where $\bar{\varphi}_{mn}^{(k)}$, $\bar{\psi}_{mn}^{(k)}$ are real polynomials in $k$ of degree at most $m+n-1$.

Multiplying these two equations together, member for member, we get

$$\bar{u}_k \bar{v}_k = \bar{u}\bar{v} + \sum_{m+n=2}^{\infty} \bar{\chi}_{mn}^{(k)} \bar{u}^m \bar{v}^n,$$

where $\bar{\chi}_{mn}^{(k)}$ is a real polynomial in $k$ of degree at most $m+n-2$. Compare this equation with that for $u_k$ as a power series in $u$, $v$, and so in $\bar{u}$, $\bar{v}$. The two series must be identical so that the exponent of $\bar{v}$ must be at least as great as that of $\bar{u}$ in every term. Hence $\bar{\chi}_{mn}^{(k)}$ vanishes identically for $n < m$. Consequently, if we write

$$\varphi_{mn}^{(k)} = \bar{\chi}_{m,\,m+n}^{(k)},$$

we have $u_k$ expressed in the stated form.

Likewise, if we compare the series for $\bar{v}_k$ with that for $v_k$, we are led to see that $\bar{\psi}_{mn}^{(k)}$ vanishes identically for $n < m$ and to write

$$\psi_{mn}^{(k)} = \bar{\psi}_{m,\,m+n}^{(k)},$$

so that $v_k$ is of the stated form.

It may be observed that all of the series employed converge for $\bar{u}$, $\bar{v}$ sufficiently small in absolute value. This fact justifies the method of formal comparison employed.

The case III with $\varrho = -1$ is taken care of by the following remark:

*If $u_1$, $v_1$ are real series of the form III with $\varrho = -1$ (case III″), $u_2$, $v_2$ are of the form II″.*

## § 7. Uniqueness of series for $u_k$, $v_k$.

The following is easily proved:

*Lemma.* Unless $\varrho$ is a root of unity, a polynomial in $\varrho^k$, $\varrho^{-k}$, $k$, $\varrho^l$, $\varrho^{-l}$, $l, \ldots$ cannot vanish for all integral values of $k, l, \ldots,$ without vanishing identically.

If possible, suppose that the lemma is not true when there is a single variable $k$, i. e. suppose that there exists a polynomial in $\varrho^k$, $\varrho^{-k}$, $k$ which vanishes for all integral values of $k$ without vanishing identically, although $\varrho$ is not a root of unity.

In the first place we cannot have $|\varrho| > 1$. For in this case divide the hypothetical polynomial by the highest power of $\varrho^k$ which appears explicitly. Let $k$ take on larger and larger integral values. All of the terms of the modified polynomial tend to zero save the term formed by the coefficient of this highest power, inasmuch as $\varrho^k$ becomes infinite more rapidly than any power of $k$. This coefficient is itself a polynomial in $k$ which is not identically 0. Hence it cannot approach 0 as $k$ becomes positively infinite. But, since the hypothetical polynomial vanishes for all integral $k$, this is absurd.

The possibility $|\varrho| < 1$ is disposed of similarly by dividing through by the highest power of $\varrho^{-k}$ which appears.

Hence we have $|\varrho| = 1$ and may write $\varrho = e^{\sqrt{-1}\,\theta}$ where $\theta$ is real. Here we fix upon the coefficient of the highest power of $k$ which appears in the hypothetical polynomial. An argument like that made above shows that this coefficient must approach 0 as $k$ becomes infinite through integral values. However, this coefficient is a polynomial in $\cos k\theta$, $\sin k\theta$; and $\frac{\theta}{2\pi}$ is irrational since $\varrho$ is not a root of unity. Hence $k\theta$ can be made to differ from an integral multiple of $2\pi$ by nearly any assigned quantity $t$ for large integral $k$. Thus the coefficient polynomial must vanish when $k\theta$ is replaced by the arbitrary real variable $t$. This is impossible.

A similar proof disposes of the case when two or more variables enter.

An application of the lemma shows at once:

*The polynomials* $\varphi_{mn}^{(k)}$, $\psi_{mn}^{(k)}$ *of §§ 4, 5, 6 are unique.*

In fact it is clear that the difference of two such polynomials with the same subscripts $m, n$ vanishes for all integral $k$. But these polynomials are of the type dealt with in the lemma, and must therefore coincide.

## § 8. The formal group for $T$.

The various integral powers of the transformation $T$ combine according to the rule $T_k T_l = T_{k+l}$, where $k$ and $l$ are any integers whatever.

In the preceding sections we have been led to real *formal* series giving $T_k$ *for all integral values of* $k$ in the cases I', II', II'', III', to which all other cases were reduced.

*The formulas*

$$u_k(u_l, v_l) = u_{k+l}(u, v), \quad v_k(u_l, v_l) = v_{k+l}(u, v) \tag{6}$$

*hold for all real values of $k$ and $l$.*

The content of this statement is wholly formal of course.

In the cases II″, III′ its truth is at once obvious. The equations (6) stand for an infinite number of ordinary polynomial relations between the coefficients $\varphi^{(k)}_{mn}$, $\psi^{(k)}_{mn}$, $\varphi^{(l)}_{mn}$, $\psi^{(l)}_{mn}$, $\varphi^{(k+l)}_{mn}$, $\psi^{(k+l)}_{mn}$ which are known to hold for all integral values of $k$ and $l$. Since these coefficients are themselves ordinary polynomials in $k$, $l$, these relations hold identically. Similar reasoning, based on the lemma of § 7, shows that the statement is also true in cases I′, II′.

From the italicized statement thus established it appears that we have to deal with a one-parameter continuous group of formal transformations and that $k$ is an additive parameter for the group.[1] In treating of its properties we need a few of the general formal ideas for such groups.

We shall write formally

$$\delta u = \left.\frac{\partial u_k}{\partial k}\right|_{k=0}, \quad \delta v = \left.\frac{\partial v_k}{\partial k}\right|_{k=0}, \tag{7}$$

so that we have the following table:

$$\begin{cases} (\mathrm{I}'), & \delta u = u \log \varrho + \cdots, \quad \delta v = -v \log \varrho + \cdots, \\ (\mathrm{II}'), & \delta u = -\theta v + \cdots, \quad \delta v = \theta u + \cdots, \\ (\mathrm{II}''), & \delta u = \varphi_{20} u^2 + \cdots, \quad \delta v = \psi_{20} u^2 + \cdots, \\ (\mathrm{III}'), & \delta u = \left(\varphi_{20} - \frac{d}{2}\varphi_{11} + \frac{d^2}{6}\varphi_{02}\right) u^2 + \cdots, \quad \delta v = du + \cdots. \end{cases} \tag{8}$$

The series $\delta u$, $\delta v$ are real formal power series in $u$, $v$.

*The series $u_k$, $v_k$ satisfy the formal differential equations*

$$\frac{du_k}{dk} = \delta u(u_k, v_k), \quad \frac{dv_k}{dk} = \delta v(u_k, v_k), \tag{9}$$

*and the initial conditions $u_0 = u$, $v_0 = v$; conversely $u_k$, $v_k$ are formally determined by these equations and conditions.*

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[1] C. L. Bouton observed these facts in case II″, loc. cit.

To begin with, by differentiating the first equation (6) formally as to $k$ and noting symmetry, we find

$$\frac{d}{dk} u_k(u_l, v_l) = \frac{d}{dk} u_{k+l}(u, v) = \frac{d}{dl} u_l(u_k, v_k).$$

Putting $l = 0$ and recalling the definition of $\delta u$ we obtain the first of the differential equations (9). The second equation may be deduced in like manner.

The initial conditions $u_0 = u$, $v_0 = v$ are clearly satisfied.

Conversely, if we write $u_k$, $v_k$ as power series in $u$, $v$ without constant term and with coefficients which are undetermined functions of $k$, and substitute in the differential equations, we get at each step linear differential equations of the first order in these coefficients. When joined with the condition that all of these coefficients are 0 for $k = 0$, save the coefficients of $u$ in $u_0$ and of $v$ in $v_0$ which are 1, these equations successively determine the coefficients.

These facts explain the complete analogy between the classification of transformations $T$ near an invariant point and the classification of differential equations of type (9) at a point $\delta u = \delta v = 0$. This analogy was noted by POINCARÉ.[1]

## § 9. The invariant operator $L(w)$.

We shall now define the invariant operator $L(w)$:

$$L(w) = \delta u \frac{\partial w}{\partial u} + \delta v \frac{\partial w}{\partial v}.^{2} \tag{10}$$

It is clear the $L(w(u, v))$ is the formal derivative of $w(u_k, v_k)$ as to $k$ for $k = 0$. Consequently $L(w)$ is unaltered (formally) by a change of variables. The fundamental property of this operator is expressed in the following statement:

*The necessary and sufficient condition that a formal series $F$ be invariant under $T$ is that $L(F) = 0$.*

First, this condition is necessary. In fact, if $F$ is an invariant series we have $F(u_k, v_k) = F(u, v)$ for all integral values of $k$. Hence, by the lemma of

[1] *Sur les courbes définies par les équations différentielles, Journal de mathématiques*, ser. 3, vols. 7—8, 1881—1882 and ser. 4, vols. 1—2, 1885—1886. The analogy was explained partially by means of a limiting process by S. LATTÈS, *Sur les équations fonctionelles qui définissent une courbe ou une surface invariante par une transformation, Annali di Matematica*, ser. 3, vol. 13, 1907.

[2] This is the »symbol of the infinitesimal transformation» in the terminology of LIE.

§ 7, this relation holds for all values of $k$. Differentiating as to $k$ and taking $k=0$, we find $L(F)=0$.

Secondly, this condition is sufficient. For if $L(F)=0$ we find, using (9),

$$\frac{d}{dk}F(u_k, v_k) = \frac{\partial F(u_k, v_k)}{\partial u_k}\frac{du_k}{dk} + \frac{\partial F(u_k, v_k)}{\partial v_k}\frac{dv_k}{dk}$$
$$= L(F(u_k, v_k)) = 0.$$

Hence we infer that $F(u_k, v_k)$ is a power series with coefficients independent of $k$. Putting $k=0$ we get $F(u_k, v_k) = F(u, v)$, and in particular $F(u_1, v_1) = F(u, v)$. That is, $F$ is invariant under $T$.

## § 10. Existence of invariant series.

In §§ 2—9 the fact that $T$ was assumed conservative did not enter, save that we made use of the equation (4). We shall now prove the following:

*Any conservative transformation $T$ of the form $I'$, $II'$, $II''$ or $III'$ leaves invariant a real formal series $F^*$ defined by the equations*

$$\frac{\partial F^*}{\partial v} = Q\delta u, \quad \frac{\partial F^*}{\partial u} = -Q\delta v. \tag{11}$$

By multiplying together the equation (3) for $u, v$, for $u=u_1$, $v=v_1, \ldots$, for $u=u_{k-1}$, $v=v_{k-1}$, we obtain

$$Q(u, v) = Q(u_k, v_k)\begin{vmatrix} \frac{\partial u_k}{\partial u} & \frac{\partial u_k}{\partial v} \\ \frac{\partial v_k}{\partial u} & \frac{\partial v_k}{\partial v}\end{vmatrix} \tag{$3_k$}$$

for any positive integral value of $k$. We employ the familiar rule for the combination of Jacobians in obtaining this result. Likewise ($3_k$) holds for $k=0$ and also for negative integral values of $k$, as is easily seen.

Hence this relation ($3_k$) will hold identically when the formal series for $u_k, v_k$ are substituted. This follows from the lemma of § 7.

Differentiating with respect to $k$ and setting $k=0$, we find

$$0 = \frac{\partial}{\partial u}(Q\delta u) + \frac{\partial}{\partial v}(Q\delta v). \tag{12}$$

Here we have employed the definitions (7) of $\delta u$, $\delta v$ and we have made use of the fact that the Jacobian determinant reduces to

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

for $k=0$. Now consider the terms of a particular degree in $Q\delta u$ and $-Q\delta v$. These homogeneous polynomials $p$ and $q$ have the property

$$\frac{\partial p}{\partial u}=\frac{\partial q}{\partial v},$$

deduced from (12). Hence there exists a homogeneous polynomial $r$ of degree one higher such that

$$p=\frac{\partial r}{\partial v}, \quad q=\frac{\partial r}{\partial u}.$$

The sum of the polynomials $r$ of all degrees ($\geqq 2$) is the formal series $F^*$ required.

From the equations (11) we have immediately $L(F^*)=0$, so that by § 9 the series $F^*$ is formally invariant under $T$.

If a change of variables from $u, v$ to $U, V$ be made, the series $F^*$ for the new variables can be obtained by direct substitution. For, from the equations (11) we find

$$\frac{\partial F^*}{\partial U}\frac{\partial U}{\partial v}+\frac{\partial F^*}{\partial V}\frac{\partial V}{\partial v}= Q\left[\frac{\partial u}{\partial U}\delta U+\frac{\partial u}{\partial V}\delta V\right],$$

$$\frac{\partial F^*}{\partial U}\frac{\partial U}{\partial u}+\frac{\partial F^*}{\partial V}\frac{\partial V}{\partial u}=-Q\left[\frac{\partial v}{\partial U}\delta U+\frac{\partial v}{\partial V}\delta V\right].$$

Multiplying the first of these equations by $\frac{\partial v}{\partial V}$, and the second by $\frac{\partial u}{\partial V}$, and adding, we find

$$\frac{\partial F^*}{\partial V}=Q\left[\frac{\partial u}{\partial U}\frac{\partial v}{\partial V}-\frac{\partial u}{\partial V}\frac{\partial v}{\partial U}\right]\delta U.$$

But the quasi-invariant function for the new variables is the product of $Q(u, v)$ and the Jacobian of $u, v$ as to $U, V$ (§ 1). Hence the equation last written shows that $F^*(u, v)$, regarded as a formal series in $U, V$, satisfies the first equation (11) for the new variables. Similarly the second equation (11) is seen to hold in these variables.

From the equations (8) and (11) the explicit forms of the series $F^*$ are immediately evident:

$$(13)\quad \begin{cases} (\mathrm{I}'), & F^* = uv \log \varrho + \cdots, \\ (\mathrm{II}'), & F^* = -\frac{\theta}{2}(u^2 + v^2) + \cdots, \\ (\mathrm{II}''), & F^* = -\frac{1}{3}\psi_{20} u^3 + \cdots, \\ (\mathrm{III}'), & F^* = -\frac{d}{2} u^2 + \cdots. \end{cases}$$

It is apparent that any formal power series in $F^*$ furnishes an invariant series.

In order to determine to what extent the existence of formally invariant series for a transformation I′, II′, II″, III′ is characteristic of conservative transformations we need to make a digression.

## § 11. Factorization of formal series.[1]

We consider formal series without constant terms. Such a series will be called *prime* when it cannot be expressed as the product of two others. Since the lowest degree of any term in a product is the sum of the lowest degrees for any terms in the factors, any formal series can be decomposed into prime factors in at least one way, and the number of such factors cannot exceed the degree of the initial terms of that series.

Two factors, either of which can be obtained from the other by multiplication by a formal series with constant term, are regarded as essentially equivalent. Since products and quotients of formal series with constant terms yield series of the same type, the propriety of this convention is obvious.

By a linear change of variables any series $G(u, v)$ can be given the form $cv^n + \cdots, c \neq 0$, where the indicated terms are of degree at least $n$. Any possible factor of $G$ is readily seen to have the same prepared form. Also WEIERSTRASS's factorization theorem holds formally, i. e., we may write $G = EH$ where $E$ is a power series with constant term $c$ and $H$ is a power series, $v^n + \cdots$, in which $v$ does not occur with an exponent as large as $n$ after the first term.

Now let us determine the formal series $S(u^{\frac{1}{a}})$ in powers of $u^{\frac{1}{a}}$ which satisfy the equation $H = 0$, and let us proceed at each step of this determination pre-

[1] Cf. W. F. OSGOOD, *Factorization of analytic functions of several variables, Annals of Mathematics*, vol. 19, 1917—1918.

cisely as though $H(u, v)$ were a polynomial in $u, v$. The well-known method for doing so yields higher and higher terms of such series, with $\Sigma\alpha = n$.

At first sight it might seem conceivable that this process breaks down at some point so that it is not possible to proceed further. But, since the process used involves only a finite set of terms of $H$ at each stage, the same difficulty would necessarily arise if $H$ were broken off at some advanced term. This is absurd since then we are dealing with a polynomial. Thus we obtain a contradiction. Consequently we can obtain formal series of the stated type in $u^{\frac{1}{a}}$ which, when substituted for $v$, reduce $H$ to 0. The initial terms in these power series are at least of the first degree in $u$.

Let $\omega$ be any $\alpha$th root of 1 and consider

$$\prod_{a}\Big[\prod_{\omega}\big(v - S\big(\omega u^{\frac{1}{a}}\big)\big)\Big].$$

This product is precisely $H$, at least if $H$ is a polynomial in $u$ as well as in $v$. By breaking off $H$ at an advanced term and employing a limiting process, we infer that the same is always true.

The bracketed products involve only integral powers of $u$ as well as of $v$, and are prime factors of $G$. Indeed, if such a product $P$ is not prime, its component factors are of prepared form and may be decomposed as $G$ has been. But any new series $S$ so obtained must fail to reduce $P$ to 0 when we write $v = S$. This is absurd.

For a similar reason it appears that, if a prime series divides a product, the series must divide one of the factors.

It follows that, *as far as the fundamental theorems of decomposition are concerned, the situation for convergent series carries over directly to divergent series.*

## § 12. Condition for conservativeness.

We are now in a position to prove the following:

*A necessary and sufficient condition that a transformation $T$ given by real series $I'$, $II'$, $II''$, $III'$ (but otherwise unrestricted) be conservative is (1) that there exists a real invariant series $F$ of lowest terms one degree higher than those of $\delta u, \delta v$ and containing each common prime factor of $\delta u, \delta v$ to precisely one power higher than it appears as a common factor in $\delta u, \delta v$, and (2) that the formal power series given by the equal ratios*

$$\frac{\frac{\partial F}{\partial v}}{\delta u}, \quad -\frac{\frac{\partial F}{\partial u}}{\delta v}$$

*converges.*

Before entering upon the proof, it may be observed that an inspection of $\delta u, \delta v$ as given by (8) shows that, in the cases I', II', $\delta u$ and $\delta v$ have no common factor. In these cases the condition (1) reduces to the condition merely that there exists a formal series $F$ with lowest terms of the second degree. It will appear later that $\delta u$ and $\delta v$ admit of a common factor only in the extraordinarily special cases II'', III' when there exist curves through (0, 0) made up of invariant points.

We first prove the conditions *necessary*.

We take $F = F^*$. The equations (11) show that this invariant series has lowest terms of degree one higher than the terms in $\delta u, \delta v$ of least degree, inasmuch as $Q$ possesses a constant term.

From the equations (11) it follows also that the ratio series of the italicized statement converges to $Q$. It remains to show that $F^*$ contains the common prime factors of $\delta u, \delta v$ to a power one higher than these occur as common factors of $\delta u, \delta v$.

Let $P^k$ be the highest power of any such prime $P$ occurring in $\delta u, \delta v$. By (11) we have

$$\frac{\partial F^*}{\partial u} = P^k a, \quad \frac{\partial F^*}{\partial v} = P^k b$$

where either $a$ or $b$ is prime to $P$.

If $F^*$ contains $P$ to higher than the $(k+1)$th power, $\frac{\partial F^*}{\partial u}$ and $\frac{\partial F^*}{\partial v}$ will contain $P$ to higher than the $k$th power. This is in manifest contradiction with the equations last written.

If $F^*$ contains $P$ to a power $m$ with $0 < m < k+1$, and if we write $F^* = P^m G$, we find

$$mG\frac{\partial P}{\partial u} + P\frac{\partial G}{\partial u} = P^{k+1-m}a,$$

$$mG\frac{\partial P}{\partial v} + P\frac{\partial G}{\partial v} = P^{k+1-m}b.$$

Hence, since $G$ is prime to $P$, both $\frac{\partial P}{\partial u}$ and $\frac{\partial P}{\partial v}$ are divisible by $P$. At least one of these partial derivatives is possessed of initial terms of lower degree than $P$, so that this possibility is likewise excluded.

The statement under consideration is certainly true then unless, perchance, $F^*$ is not divisible by the prime factor $P$. We have merely to eliminate this possibility.

It was seen in the preceding section that we can write

$$P = E \prod_{\omega} \left(v - S\left(\omega u^{\frac{1}{n}}\right)\right),$$

when $E$ is a formal power series with constant term, where $S$ in an ascending power series in its argument, and where $\omega$ stands for any $n$th root of 1.

Now introduce the variable $t = u^{\frac{1}{n}}$ instead of $u$. We have

$$\frac{\partial F^*}{\partial t} = n t^{n-1} \frac{\partial F^*}{\partial u},$$

while $\frac{\partial F^*}{\partial v}$ is unaltered. Hence the partial derivatives $\frac{\partial F^*}{\partial t}$ and $\frac{\partial F^*}{\partial u}$ are divisible by $v - S(t)$. Let us effect a further change of variables from $v, t$ to $w, z$ where $w = v - S(t)$, $z = t$. Evidently one has

$$\frac{\partial F^*}{\partial w} = \frac{\partial F^*}{\partial v}, \quad \frac{\partial F^*}{\partial z} = \frac{\partial F^*}{\partial t} + \frac{\partial F^*}{\partial v}\frac{dS}{dt},$$

so that $\frac{\partial F^*}{\partial w}$ and $\frac{\partial F^*}{\partial z}$ are divisible by $w$.

The fact that $\frac{\partial F^*}{\partial z}$ is divisible by $w$ shows that $F^*$ contains no terms in $z$ alone and is divisible by $w$.

Passing back to the variables $v, t$, we infer that $F^*$ expressed as a power series in $v, t$ is divisible by $v - S(t)$. It follows that $F^*(u, v)$ is divisible by $v - S\left(u^{\frac{1}{n}}\right)$ and by $P$ of course. This completes the proof that the conditions stated are necessary.

It remains to prove them *sufficient*.

We may assume that an invariant series $F^*$ exists for which (11) holds in which $Q$ is a convergent power series with constant term 1. These equations follow at once from the second part of the italicized statement under consideration. Our aim is to show that $T$ is conservative.

By direct differentiation and use of the formal differential equations (9) we obtain

$$\frac{d}{dk}\left\{Q(u_k, v_k)\begin{vmatrix}\frac{\partial u_k}{\partial u} & \frac{\partial u_k}{\partial v}\\ \frac{\partial v_k}{\partial u} & \frac{\partial v_k}{\partial v}\end{vmatrix}\right\} =$$

$$= \left\{\frac{\partial Q(u_k, v_k)}{\partial u_k}\delta u(u_k, v_k) + \frac{\partial Q(u_k, v_k)}{\partial v_k}\delta v(u_k, v_k)\right\}\begin{vmatrix}\frac{\partial u_k}{\partial u} & \frac{\partial u_k}{\partial v}\\ \frac{\partial v_k}{\partial u} & \frac{\partial v_k}{\partial v}\end{vmatrix} +$$

$$+ Q(u_k, v_k)\left\{\begin{vmatrix}\frac{\partial\delta u(u_k, v_k)}{\partial u} & \frac{\partial\delta u(u_k, v_k)}{\partial v}\\ \frac{\partial v_k}{\partial u} & \frac{\partial v_k}{\partial v}\end{vmatrix} + \begin{vmatrix}\frac{\partial u_k}{\partial u} & \frac{\partial u_k}{\partial v}\\ \frac{\partial\delta v(u_k, v_k)}{\partial u} & \frac{\partial\delta v(u_k, v_k)}{\partial v}\end{vmatrix}\right\}.$$

But the first determinant in the final brace is the Jacobian of $\delta u(u_k, v_k), v_k$ with respect to $u, v$. This determinant may be broken up into the product of the Jacobian of $\delta u(u_k, v_k), v_k$ as to $u_k, v_k$ $\left(\text{which is } \frac{\partial\delta u(u_k, v_k)}{\partial u_k}\right)$ and the Jacobian of $u_k, v_k$ as to $u, v$. Likewise the second determinant in the same brace may be expressed as the product of $\frac{\partial\delta v(u_k, v_k)}{\partial v_k}$ and the Jacobian of $u_k, v_k$ as to $u, v$. Hence we find that the right-hand member of the above equations reduces to

$$\left\{\frac{\partial}{\partial u_k}[Q(u_k, v_k)\,\delta u_k] + \frac{\partial}{\partial v_k}[Q(u_k, v_k)\,\delta v_k]\right\}\begin{vmatrix}\frac{\partial u_k}{\partial u} & \frac{\partial u_k}{\partial v}\\ \frac{\partial v_k}{\partial u} & \frac{\partial v_k}{\partial v}\end{vmatrix}.$$

The first factor vanishes identically by (11). Hence the left-hand member of the above equation vanishes identically in $k$. Integrating formally we obtain $(3_k)$. For $k = 1$ this becomes (3), which is precisely the condition that $T$ be conservative with a quasi-invariant function $Q$.

It is natural to call a transformation $T$ of types I′, II′, II″, III′ *formally conservative* if there exists a formal series $F$ satisfying the conditions in part (1) of the italicized statement.

We may inquire precisely what condition the existence of formally invariant series lays upon transformations $T$ of these types. The ratio $Q$ of the italicized statement may or may not be convergent. If it is convergent, then $\iint Q(u, v)\,du\,dv$ is invariant under $T$. If the ratio is not convergent, the double integral is only formally invariant.

These considerations bring out the vitally close connection between conservativeness and formally invariant series.

## § 13. The formal vanishing of the Jacobian.

To complete our treatment of formally invariant series we need to establish the formal extension of a well-known property of Jacobians:

*The Jacobian of two formal series in $u, v$ without constant terms vanishes identically if and only if either can be expressed as a power series in the other or in fractional powers of the other.*[1]

It is immediately apparent that, if two functions $A, B$ are so expressible one in terms of the other, their Jacobian will vanish identically.

Suppose, conversely, that $A$ and $B$ are power series in $u, v$ with vanishing Jacobian:

$$\frac{\partial A}{\partial u}\frac{\partial B}{\partial v}-\frac{\partial A}{\partial v}\frac{\partial B}{\partial u}=0.$$

Both $A$ and $B$ are exact powers of base series for which it suffices to establish the functional relation. But the Jacobian for the bases also vanishes. Consequently we may confine attention to the case in which neither $A$ nor $B$ is an exact power other than the first.

We begin by showing that $A$ and $B$ have the same prime factors.

If this is not the case, suppose that $A$ is divisible by a prime series $P$, while $B$ is not. After a suitable preliminary change of variables, $P$ is expressible as a product of series $v-S\left(u^{\frac{1}{n}}\right)$ (§ 11). Now take new variables

$$w=v-S\left(u^{\frac{1}{n}}\right),\quad t=u^{\frac{1}{n}}.$$

The series $A$ and $B$ are power series in these variables without constant terms, and their Jacobian as to $w, t$ is 0 by direct reckoning:

---

[1] The presence of fractional powers means that the root indicated is to be formally extracted.

$$\frac{\partial A}{\partial w}\frac{\partial B}{\partial t}-\frac{\partial A}{\partial t}\frac{\partial B}{\partial w}=0.$$

But $A$ is divisible by $w$, and $\frac{\partial A}{\partial t}$ is divisible by $w$ to a power at least as high. Also $\frac{\partial A}{\partial w}$ is divisible by $w$ to a power at least one lower than $A$. Hence $\frac{\partial B}{\partial t}$ is divisible by $w$. From this it follows that $B$ is divisible by $w$.

Proceeding to the original variables we infer that $B$ is divisible by the prime factor $P$, contrary to hypothesis.

Suppose that a prime factor $P$ is contained $p$ times in $A$ and $q$ times in $B$, and choose that factor for which $\frac{p}{q} \neq 0$ is as small as possible, and thus smaller than for some other factor unless $\frac{p}{q}$ is the same throughout. Except in this case, $\frac{A^q}{B^p}$ will yield a power series without constant term and not containing $P$. But the Jacobian of this series and $A$ is easily verified to be 0 also. This is not possible by the argument used above, since $\frac{A^q}{B^p}$ has not the prime factor $P$ which $A$ admits.

We are thus forced to the conclusion that the power series $\frac{A^q}{B^p}$ starts off with a constant term. But $A$ and $B$ are not exact powers so that we must have $p=q$. Consequently the prime factors of $A$ and $B$ occur with the same multiplicity in $A$ and $B$.

Now consider

$$A = B(c + C), \qquad (c \neq 0),$$

where $C$ is a power series without constant term. It is readily inferred that the Jacobian of $C, B$ is 0, and thence that, if $C$ is an exact $q$th power, $\frac{C^{\frac{1}{q}}}{B}$ is a power series with constant term. Hence we may write

$$C = B^q(d + D), \qquad (d \neq 0),$$

where $D$ is a power series without constant term. Proceeding in this way indefinitely we find

$$A = cB + dB^q + \cdots.$$

This establishes the statement.

## § 14. The totality of invariant series.

We may now prove the following:

*If $F^*$ is a qth power the most general invariant series is an arbitrary power series in $F^{*\frac{1}{q}}$. The integer $q$ is 1 unless all the prime factors of $F^*$ are common to $\delta u, \delta v$.*

The results of § 13 assure us that the most general invariant series can be represented as stated if the Jacobian of $F^*$ and any invariant series $F$ vanishes. But we have $L(F^*) = 0$, $L(F) = 0$, whence it appears that the Jacobian does vanish.

If $q \neq 1$ we may write $F^* = G^q$, and (11) gives

$$qG^{q-1}\frac{\partial G}{\partial v} = Q\delta u, \; qG^{q-1}\frac{\partial G}{\partial u} = -Q\delta v,$$

so that all of the factors of $G$ (and hence of $F^*$) are common to $\delta u$ and $\delta v$.

## § 15. Conditions for Formal Conservativeness.

At the very outset of the paper the condition (4) was obtained as a consequence of the fact that $T$ was assumed to be conservative. There exist an infinite set of similar conditions on the coefficients of higher degree terms in the power series $u_1$ and $v_1$. These conditions may be found by use of the existence of invariant formal series. We illustrate the method in case I'.

Since $F^*$ begins with a term $uv \log \varrho$ in this case, an invariant series $F$, also with first degree term $uv \log \varrho$, can be written down without any other terms having equal exponents in $u, v$:

$$F = uv \log \varrho + \sum_{m+n=3}^{\infty} F_{mn} u^m v^n, \qquad (m \neq n).$$

This series $F$ may be obtained by writing $F = F^* + cF^{*2} + \cdots$, and choosing the arbitrary coefficients so as to eliminate terms with equal exponents.

Moreover, it is easy to see that there is only one such series, since any invariant series can be expressed as in a power series in $F^*$ (§ 14).

Now, when coefficients of $u^m v^n$ are compared, the formal relation $F(u_1, v_1) = F(u, v)$ gives a series of equations

$$F_{mn}(\varrho^{m-n}-1)=P_{mn}, \qquad (m+n\geqq 3).$$

Here $P_{mn}$ is a linear expression in the quantities $F_{\alpha\beta}$ with $\alpha+\beta<m+n$. Thus we determine $F_{mn}$ for $m+n=3$, $m+n=4, \ldots$, as polynomials in the coefficients $\varphi_{mn}, \psi_{mn}$ of the series for $u_1, v_1$. For $m=n$ we have $P_{mn}=0$.

*In the case $I'$ the polynomials $P_{nn}$ in $\varphi_{\alpha\beta}, \psi_{\alpha\beta}$ $(\alpha+\beta<2n)$ vanish for $n=2, 3, \ldots$.*

Conversely, if these vanish we have a formally invariant series $F$, and formal conservativeness of $T$ in consequence.

*Similar conditions for formal conservativeness can be found in the other cases.*

## § 16. Invariant formal curves.

Let $f$ and $g$ be two formal power series in a parameter $t$, without constant terms and not both identically 0. Then we shall regard the equations

$$u=f(t), \quad v=g(t),$$

as furnishing a *formal curve through the point* $(0, 0)$. If the series $f, g$ converge for $|t|$ small we have an analytic curve.

Two curves of this sort will be regarded as identical if one can be obtained from the other by change of parameter $t=l(\tau)$ where $l$ is a formal power series in $\tau$ or a fractional power thereof.

A formal curve is regarded as real if the coefficients in $f$ and $g$ can be taken real.

By means of $T$ a formal curve of this sort is regarded as carried over into the formal curve

$$u=u_1(f(t), g(t)), \quad v=v_1(f(t), g(t)).$$

If this transformed curve is identical with the given curve $u=f(t)$, $v=g(t)$ then the given curve is said to be *formally invariant* under $T$.

The determination of the formally invariant curves is essential for our purpose. A fundamental division of types of invariant points will be made according as there do or do not exist curves of this sort given by real series. In cases I′, II′, II″, III′ the transformation $T$ will be called *hyperbolic* if real formally invariant curves exist, and *elliptic* in the contrary case. In cases II‴ or III″, $T$ is *hyperbolic* or *elliptic* according as $T_2$ or $T_2$ (of type II″) is one or the other.

If $t_1$ denotes the power series in $t$ or a fractional power thereof along the transformed invariant curve which relates its parameter and $t$, we have

$$f(t_1) = u_1(f(t), g(t)), \quad g(t_1) = v_1(f(t), g(t)).$$

In virtue of the fact that the determinant of the coefficients of the first degree terms in $u_1, v_1$ is not 0 (see (4)) we can show that the power series $t_1$ starts off with a first degree term in $t$. For suppose it commences with a term of higher degree. The initial term of one of the two right-hand members above will be $\alpha$, where $\alpha$ is the lowest degree of any term in $f$ or $g$. But the left-hand members will start off with higher degree terms, which is impossible. Similarly we may rule out the possibility that the initial term in $t$ is of lower degree than the first, by making use of the inverse equations

$$f(t) = u_{-1}(f(t_1), g(t_1)), \quad g(t) = v_{-1}(f(t_1), g(t_1)).$$

Hence $t_1$ is a power series in $t$ or a fractional power thereof beginning with a term of the first degree.

If $\alpha$ is the degree of the lowest term in $f$ or $g$ (say in $f$), then from the corresponding equation (the first) we obtain on the left a series in $t_1$, $at_1^\alpha + \cdots$, and on the right a similar series in $t$ commencing with a term of degree not less than $\alpha$ and therefore of degree precisely $\alpha$ by the above. Extracting $\alpha$th roots we conclude finally that $t_1$ can be expressed as an ordinary power series in $t$ with first degree term:

$$t_1 = \varrho^* t + \cdots.$$

Having this explicit form of $t$ in mind, let us compare anew the two members of each of the pair of equations first written. We write

$$f(t) = pt^\alpha + \cdots, \quad g(t) = qt^\alpha + \cdots,$$

so that $|p| + |q| \neq 0$, and obtain

$$p\varrho^{*\alpha} = ap + bq, \quad q\varrho^{*\alpha} = cp + dq.$$

It follows at once that $\varrho^{*\alpha}$ is a root of the characteristic equation, i. e. that $\varrho^{*\alpha} = \varrho$.

If (0, 0) is an 'ordinary point' of the formal curve we have $\alpha = 1$, $\varrho = \varrho^*$.

By successive transformation of the invariant curve by $T$, we obtain not only $t_1$ but parameters $t_2, t_3, \ldots$. Likewise by the inverse transformation we obtain parameters $t_{-1}, t_{-2}, \ldots$. These can all be obtained from the series for $t_1$ by iteration.

### § 17. The formal series for $t_k$ and the formal group.

Since the constant $\varrho^*$ is an $\alpha$th root of $\varrho$, it is clear that, if we write $\tau_1(\tau) = t_\alpha(\tau)$, then we have $\tau_1 = \varrho\tau + \cdots$. By iteration $\tau_k$ may be defined for all integral values of $k$. Moreover, the methods used in § 4 serve at once to show that

$$\tau_k = \varrho^k \tau + \sum_{m=2}^{\infty} \varphi_m^{(k)} \tau^m$$

where $\varphi_m^{(k)}$ is a polynomial in $\varrho^k$ of degree at most $m$ if $\varrho \neq 1$, and a polynomial in $k$ of degree at most $m-1$ if $\varrho = 1$.

For all integral values of $k$ and $l$ we have obviously

$$\tau_k(\tau_l) = \tau_{k+l}.$$

Therefore, by the lemma of § 7, this holds formally for all real values of $k$ and $l$.

We write

$$\delta\tau = \left.\frac{d\tau_k}{dk}\right|_{k=0},$$

and can then show (compare with § 8) that the formal differential equation

$$\frac{d\tau_k}{dk} = \delta\tau(\tau_k)$$

is satisfied, and, together with the initial condition $\tau_0 = \tau$, wholly determines the series for $\tau_k$.

### § 18. The invariant operator $L(u, v)$.

We shall define a second invariant differential operator:

$$L(u, v) = \delta u\, dv - \delta v\, du. \tag{14}$$

It can be immediately verified that, if the variables $u, v$ are changed to $\bar{u}, \bar{v}$, then $L(u, v)$ becomes $\bar{L}(\bar{u}, \bar{v})$ multiplied by the Jacobian of $\bar{u}, \bar{v}$ as to $u, v$. It is also obvious that, if $u = f(t)$, $v = g(t)$ is a formal curve, then $L(u, v)$ is independent of the particular parameter chosen for the curve.

*The necessary and sufficient condition for the invariance of a formal curve* $u = f(t)$, $v = g(t)$ *under* $T$ *is* $L(u, v) = 0$.

By definition of invariance we have for such an invariant curve

$$f(t_1) = u_1(f(t), g(t)), \quad g(t_1) = v_1(f(t), g(t)),$$

and thence for integral values of $k$

$$f(t_k) = u_k(f(t), g(t)), \quad g(t_k) = v_k(f(t), g(t)).$$

If we take $k$ as an integral multiple $k'\alpha$ of $\alpha$ (§ 17) and write $t = \tau$, $\tau_1 = t_\alpha(\tau)$ (§ 17), we have in particular

$$f(\tau_{k'}) = u_{k'\alpha}(f(\tau), g(\tau)), \quad g(\tau_{k'}) = v_{k'\alpha}(f(\tau), g(\tau)),$$

for integral values of $k'$.

Let the general series for $u_{k'\alpha}$, $v_{k'\alpha}$, $\tau_{k'}$, be substituted in the last equations. All the coefficients are either polynomials in $\varrho^{k'}$, $\varrho^{-k'}$, $k'$ (case I′), or in $\cos k'\theta$, $\sin k'\theta$, $k'$ (case II′), or in $k'$ (cases II″, III′). Hence, by the lemma of § 7, these equations are identically true from a formal standpoint.

Differentiating formally as to $k'$ and setting $k' = 0$, we get

$$\frac{df}{d\tau}\delta\tau = \alpha\,\delta u(f, g), \quad \frac{dg}{d\tau}\delta\tau = \alpha\,\delta v(f, g),$$

whence at once $L(u, v) = 0$.

Conversely, let us assume that $L(u, v)$ is $0$ for a formal curve $u = f(t)$, $v = g(t)$, and let us show that the curve is invariant under $T$.

In this case we have

$$\varkappa(t)\frac{df}{dt} = \delta u(f, g), \quad \varkappa(t)\frac{dg}{dt} = \delta v(f, g),$$

where $\varkappa$ is the sum of a polynomial in $\frac{1}{t}$ and a power series in $t$. Now, since $\delta u$, $\delta v$ begin with terms of the first degree or of higher degree, both right-hand members have initial terms of degree at least as high as $f$ or $g$. On the other hand $\frac{df}{dt}$ and $\frac{dg}{dt}$ are of degree one less than $f$ and $g$ respectively. Hence $\varkappa(t)$ cannot contain negative powers of $t$ or even a constant term. Thus $\varkappa(t)$ is an ordinary power series in $t$ without constant term.

Define $t_k$ by the differential equation

$$\frac{dt_k}{dk} = \varkappa(t_k),$$

and the initial condition $t_0 = t$. Thus $t_k$ is formally determined as a power series in $t$ with coefficients analytic in $k$.

For example in case I', $\delta u$ and $\delta v$ are given by (see (8))

$$\delta u = u \log \varrho + \cdots, \quad \delta v = -v \log \varrho + \cdots.$$

Hence an inspection of the above equations introducing $\varkappa(t)$ shows that this function possesses a first degree term in $t$, $\frac{\log \varrho}{\alpha} t$, $\alpha$ an integer.

Write then

$$t_k = \sum_{m=1}^{\infty} \varphi_m^{(k)} t^m, \quad \varkappa(t) = \frac{\log \varrho}{\alpha} t + \sum_{m=2}^{\infty} \varkappa^{(m)} t^m,$$

and the differential equation gives

$$\frac{d\varphi_1^{(k)}}{dk} = \frac{\log \varrho}{\alpha} \varphi_1^{(k)},$$

$$\frac{d\varphi_2^{(k)}}{dk} = \frac{\log \varrho}{\alpha} \varphi_2^{(k)} + \varkappa^{(2)}[\varphi_1^{(k)}]^2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

on comparison of terms in $t, t^2, \ldots$ Remembering the initial conditions $\varphi_1^{(0)} = 1$, $\varphi_2^{(0)} = 0, \ldots$, we find

$$\varphi_1^{(k)} = \varrho^{\frac{k}{\alpha}}, \quad \varphi_2^{(k)} = \frac{\varkappa^{(2)} \alpha}{\log \varrho} \left( \varrho^{\frac{2k}{\alpha}} - \varrho^{\frac{k}{\alpha}} \right), \ldots.$$

Thus the successive coefficients are polynomials of increasing degree in $\varrho^{\frac{k}{\alpha}}$.

Likewise in case II' these coefficients are polynomials of increasing degrees in $\cos \frac{k\theta}{\alpha}$, $\sin \frac{k\theta}{\alpha}$; and in cases II'', III', polynomials in $k$ only, since here $\varkappa(t)$ starts out with a term of the second degree or higher.

Consider now the formal series $f(t_k)$ and $g(t_k)$. Differentiating and using the definition of $t_k$, we find

$$\frac{df(t_k)}{dk} = \frac{df(t_k)}{dt_k}\varkappa(t_k) = \delta u(f(t_k),\, g(t_k)),$$

$$\frac{dg(t_k)}{dk} = \frac{dg(t_k)}{dt_k}\varkappa(t_k) = \delta v(f(t_k),\, g(t_k)),$$

These series $f(t_k)$ and $g(t_k)$ reduce to $f(t)$ and $g(t)$ for $k = 0$.

Consider next the formal series $u_k(f(t), g(t))$, $v_k(f(t), g(t))$. Differentiating and using (9) we find

$$\frac{d}{dk}[u_k(f(t),\, g(t))] = \delta u[u_k(f(t),\, g(t)),\quad v_k(f(t),\, g(t))],$$

$$\frac{d}{dk}[v_k(f(t),\, g(t))] = \delta v[u_k(f(t),\, g(t)),\quad v_k(f(t),\, g(t))].$$

Also these series reduce to $f(t)$, $g(t)$ for $k = 0$.

Hence, if either pair of series in $t$ be denoted by $p_k(t)$, $q_k(t)$, the differential equations

$$\frac{dp_k}{dk} = \delta u(p_k,\, q_k),\quad \frac{dq_k}{dk} = \delta v(p_k,\, q_k),$$

and the initial conditions $p_0 = f(t)$, $q_0 = g(t)$ will be satisfied.

But, just as in an analogous situation earlier, these equations and conditions uniquely determine the series. Hence the two solutions coincide:

$$f(t_k) = u_k(f(t),\, g(t)),\quad g(t_k) = v_k(f(t),\, g(t)).$$

Taking $k = 1$, we conclude that the given formal curve is invariant under $T$.

## § 19. Existence of invariant formal curves.

When a formal power series in $u$, $v$ without constant term is resolved into its prime factors in the sense of § 11, each such factor evidently corresponds to a formal curve $u = t^n$, $v = S(t)$ where $S$ is a power series in $t$. When the coördinates of this curve are substituted in the given formal series in $u$, $v$, it vanishes identically. Conversely, if the coördinates of a formal curve render such a series equal to 0, then it renders one and only one of the prime factors equal to 0, and this formal curve must be the one corresponding to the factor.

With these facts in mind we may prove:

*The totality of formally invariant curves for a conservative transformation $T$ is given by the equation $F = 0$, where $F$ is any invariant series under $T$.*

First, let us take any curve for which $F=0$. Now we have $F(u_k, v_k) = F(u, v)$, and thence, by formal differentiation as to $k$ and taking $k=0$,

$$\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v = 0.$$

But we have also

$$\frac{\partial F}{\partial u}du + \frac{\partial F}{\partial v}dv = 0.$$

Combining these equations we find $L(u, v) = 0$. By the preceding paragraph the formal curve is invariant.

Conversely, for any invariant formal curve $u = f(t)$, $v = g(t)$ we have

$$f(t_k) = u_k(f(t), g(t)), \quad g(t_k) = v_k(f(t), g(t)),$$

as we have seen. Hence it follows that

$$F(f(t_k), g(t_k)) = F[u_k(f(t), g(t)), \quad v_k(f(t), g(t))] = F(f(t), g(t)).$$

Now, taking $k = k'\alpha$, $t_\alpha = \tau$, we may regard this equation as holding for all $k'$ (§ 17). Differentiating as to $k'$ and taking $k'=0$, we find

$$\left[\frac{\partial F}{\partial u}\frac{df}{\partial \tau} + \frac{\partial F}{\partial v}\frac{dg}{d\tau}\right]\delta\tau = 0.$$

Unless $\delta\tau = 0$ we infer that $\dfrac{dF}{dt} = 0$. But $F(f(t), g(t))$ is a power series in $t$ without constant term. Hence except in this case we have $F(f(t), g(t)) = 0$, as we desire to prove.

However, if we take the equations which state that $u = f$, $v = g$, and its iterates under $T$ coincide (as written above), and differentiate as to $k'$ $(k = k'\alpha)$, we find for $k' = 0$

$$\frac{df}{d\tau}\delta\tau = \delta u(f, g), \quad \frac{dg}{d\tau}\delta\tau = \delta v(f, g).$$

Hence $\delta\tau$ vanishes formally if and only if $\delta u(f, g)$, $\delta v(f, g)$ vanish. In other words the given curve corresponds to a common factor of $\delta u$, $\delta v$. But it has been proved (§ 12) that such factors occur to a one higher power in $F$. Hence we have $F(f(t), g(t)) = 0$ in this case also.

Applying the above condition to $F^*$ (see (13)), we perceive that in case I′ we have two real formally invariant curves so that $T$ is hyperbolic, while in case II′ we have a pair of conjugate imaginary formally invariant curves so that $T$ is elliptic.

## § 20. Invariant point curves.

In an extremely special case the invariant point (0, 0) may not be isolated but may lie on one or more analytic curves of invariant points passing through (0, 0). These curves can be determined as the solutions of the ordinary equations

$$u_1(u, v) = u, \quad v_1(u, v) = v.$$

By iteration we get

$$u_k(u, v) = u, \quad v_k(u, v) = v,$$

which holds along these curves. Differentiating as to $k$, as we have often done, and setting $k = 0$, we find

$$\delta u = 0, \quad \delta v = 0,$$

along the invariant point curve. In other words the invariant point curves correspond to common factors of $\delta u$, $\delta v$. According to § 12 this means that the curve corresponds to a multiple factor of $F^*$.

Conversely, let us assume that $F^*$ has a multiple factor corresponding to a formal curve $u = f(t)$, $v = g(t)$, so that $\delta u = \delta v = 0$ along the curve. By formal integration we get $u_k(f, g) = f$, $v_k(f, g) = g$, and the formal curve is an invariant point curve.

*There exist formally invariant point curves if and only if $F^*$ has a multiple factor, and these curves are then analytic curves given by the equations $u_1 = u$, $v_1 = v$.*

## § 21. Normal form. Case I′.

Under a formal change of variables from $u, v$ to $U$, $V$ such as

$$U = u + \sum_{m+n=2}^{\infty} U_{mn} u^m v^n, \qquad V = v + \sum_{m+n=2}^{\infty} V_{mn} u^m v^n, \tag{15}$$

transformations of the type I′, II′, II″, III′ evidently maintain their type, and also remain formally conservative if they are so at the outset.

We propose to develop a normal form for the transformation $T$ in the cases I′, II′. In the other cases there appear to be an infinite number of invariants, and a similar normal form does not exist.

*By a formal change of variables (15), a formally conservative transformation of type I′ may be given either the normal form*

$$U_1 = \varrho\, U e^{c U^l V^l}, \quad V_1 = \frac{1}{\varrho} V e^{-c U^l V^l}, \qquad (c \neq 0), \tag{16}$$

*or the form*

$$U_1 = \varrho U, \quad V_1 = \frac{1}{\varrho} V. \tag{16′}$$

We propose first to choose $U$, $V$ so that

$$\delta U = U[1 + f(UV)] \log \varrho, \quad \delta V = -V[1 + g(UV)] \log \varrho,$$

where $f$ and $g$ are power series in their argument $UV$. More explicitly written, these equations take the form

$$\begin{aligned} &\frac{\partial U}{\partial u}\delta u + \frac{\partial U}{\partial v}\delta v = U[1 + f(UV)] \log \varrho, \\ &\frac{\partial V}{\partial u}\delta u + \frac{\partial V}{\partial v}\delta v = -V[1 + g(UV)] \log \varrho; \end{aligned} \tag{17}$$

recall the equations

$$\delta U = \left.\frac{\partial U_k}{\partial k}\right|_{k=0}, \quad U_k = U(u_k, v_k),$$

and similar equations in $V$.

By the first equation (8) the first degree terms on both sides of the above equations are the same.

Equating coefficient of $u^m v^n$ in these equations, we find

$$\begin{aligned} (m - n - 1)\, U_{mn} &= P_{mn}, && (m \neq n + 1), \\ (m - n + 1)\, V_{mn} &= Q_{mn}, && (n \neq m + 1), \\ 0 &= P_{n+1,n} + f_{2n+1}, \\ 0 &= Q_{n,n+1} + g_{2n+1}, \end{aligned}$$

where $P_{mn}, Q_{mn}$ are polynomials in $U_{\alpha\beta}, V_{\alpha\beta}, f_\gamma, g_\gamma$ with $\alpha+\beta<m+n, \gamma<m+n$, and where $f_{2k+1}, g_{2k+1}$ are the coefficients of $(UV)^k$ in $f$ and $g$ respectively. These equations are manifest as soon as the explicit series for $U$, $V$ are substituted in (17).

Let us compare second degree terms, so that $m+n=2$. The first two equations determine $U_{mn}, V_{mn}$ for $m+n=2$ uniquely.

Next let us compare third degree terms so that $m+n=3$. Here the quantities $U_{mn}, V_{mn}$, excepting $U_{21}, V_{12}$, are determined by the first two equations while $f_3, g_3$ are determined by the second two equations.

Continuing in this way we determine in succession $U_{mn}, V_{mn}, f_p, g_p$, save for $U_{21}, V_{12}, U_{32}, V_{23}, \ldots$ which can be taken arbitrarily.

Therefore it is possible to determine formal series $U$, $V$ so that (17) holds. In order to avoid complexity in our notation let us call these new variables $u, v$. It may be observed that the set of changes of variables (15) form a group. Accordingly, in accomplishing the desired normalization, we can compound any number of such changes of variables. With this understanding we may write

$$\delta u = u[1+f(uv)]\log \varrho, \quad \delta v = -v[1+g(uv)]\log \varrho.$$

If $Q$ denotes the formal quasi-invariant function, we have by (12)

$$\frac{\partial}{\partial u}[Qu(1+f)] = \frac{\partial}{\partial v}[Qv(1+g)]$$

on substituting in the above values of $\delta u, \delta v$. Here $f$ and $g$ are series in the product $uv$.

It follows from the equation just written that $Q$ must also be a series in the product $uv$. Suppose if possible that this is not the case, and let $du^m v^n$ be a term in $Q$ of minimum degree for which $m \neq n$. A term $(m+1)du^m v^n$ will then appear on the left of the equation written. But no other term of equal or lower degree in which the exponents of $u, v$ are unequal can occur on the left inasmuch as terms with unequal exponents are not present in $f$. A similar term $(n+1)du^m v^n$ will occur on the right. If then the above identity holds we must have $d=0$, contrary to hypothesis.

Thus if we write $z=uv$, and use accents to denote differentiation with respect to $z$ we have easily

$$[Qz(1+f)]' = [Qz(1+g)]'$$

By formal integration we get then $f = g = h$, where $h$ a formal power series in $z$ without constant term. Consequently we have

$$\delta u = u[1 + h(uv)] \log \varrho, \quad \delta v = -v[1 + h(uv)] \log \varrho.$$

When use is made of this fact, the formal differential equations (9) for $u_k$, $v_k$ take the form

$$\frac{du_k}{dk} = u_k[1 + h(u_k v_k)] \log \varrho, \quad \frac{dv_k}{dk} = -v_k[1 + h(u_k v_k)] \log \varrho.$$

Hence, if we consider the product series $u_k v_k$, we have $\frac{d(u_k v_k)}{dk} = 0$. Noting that we have $u_0 = u$, $v_0 = v$, we conclude $u_k v_k = uv$.

If we substitute this value for $u_k v_k$ in the differential equations, these become

$$\frac{du_k}{dk} - u_k[1 + h(uv)] \log \varrho = 0, \quad \frac{dv_k}{dk} + v_k[1 + h(uv)] \log \varrho = 0.$$

If we multiply these two equations by

$$\varrho^{-(1+h(uv))k}, \quad \varrho^{(1+h(uv))k}$$

respectively, the left-hand members become exact formal derivatives. Integrating formally we find

$$\varrho^{-(1+h(uv))k}\, u_k = \text{const.}, \quad \varrho^{(1+h(uv))k}\, v_k = \text{const.},$$

where the constants are power series in $u, v$ with coefficients independent of $k$. Employing the initial conditions $u_0 = u$, $v_0 = v$, we get the following explicit formulas

$$u_k = \varrho^k u e^{h(uv)k}, \quad v_k = \varrho^{-k} v e^{-h(uv)k},$$

for the given transformation after the change of variables determined earlier.

If $h$ vanishes identically, a reduction to the normal form (16') has been effected.

In the contrary case we may write

$$h(uv) = cu^l v^l + \cdots.$$

Dividing by $c \neq 0$ and extracting $l$th root, we find

$$\sqrt[l]{\frac{h(uv)}{c}} = uvp(uv)$$

when $p(uv)$ is a power series in $uv$ with constant term 1. If we define a further change of variables (15)

$$U = u\sqrt{p(uv)}, \quad V = v\sqrt{p(uv)},$$

we obtain immediately the first normal form (16).

The normal forms are clearly of integrable type with $UV$ invariant.

It is apparent that, if $T$ is given by real series, the normalizing series $U, V$ can also be taken real.

### § 22. Generality of normal form. Case I'.

The normalizing series $U, V$ were not uniquely determined. The most general set $U^*, V^*$ of such series is related to any particular set $U, V$ as follows:

*The most general normalizing variables $U^*, V^*$ in case I' have the explicit form*

$$U^* = Ue^{\lambda(UV)}, \quad V^* = Ve^{-\lambda(UV)}, \tag{18}$$

*where $\lambda$ is an arbitrary power series in $UV$ without constant term, and $U, V$ are any particular set of normalizing series.*

Clearly we can pass directly from $U, V$ to $U^*, V^*$ by a change of variables (15). Since the invariant curves $U = 0$, $V = 0$ are carried into $U^* = 0$, $V^* = 0$, we infer further

$$U^* = U(1 + \cdots), \quad V^* = V(1 + \cdots).$$

Now the products $UV$, $U^*V^*$ are invariant under $T$. Hence (§ 14) $U^*V^*$ is given by a power series in $UV$, whose initial term is $UV$ of course. By the aid of this result we may conclude that in the series for $U^*, V^*$ only terms in $UV$ occur in the parentheses.

In fact, if we replace $U, V, U^*, V^*$ by $U_1, V_1, U_1^*, V_1^*$ respectively, the first of these equations gives

$$\varrho U^* e^{c^* U^{*l^*} V^{*l^*}} = \varrho U e^{c U^l V^l}(1 + \cdots).$$

On the left the exponential factor is a power series in $UV$ with constant term $1$ inasmuch as $U^* V^*$ is given by a power series in $UV$ without constant term. Suppose if possible that a term $d U^{m+1} V^n (m \neq n)$ occurs in the series for $U^*$ and let this term be of the minimum degree. On the left of the equation last written the corresponding term of this type is $\varrho d U^{m+1} V^n$, whereas on the right it is $\varrho^{m+1-n} d U^{m+1} V^n$. The two terms to be compared cannot be equal so that a contradiction results. In this way the parenthesis in the series for $U^*$, and likewise that in the series for $V^*$, are seen to only contain terms in $UV$.

We may now write

$$U^* = U(1 + \lambda'(UV)), \quad V^* = V(1 + \lambda''(UV)),$$

where $\lambda'$, $\lambda''$ are power series without constant terms.

Replacing $U, V, U^*, V^*$ by $U_1, V_1, U_1^*, V_1^*$ respectively here, we get

$$\varrho U^* e^{c^* U^{*l^*} V^{*l^*}} = \varrho U e^{c U^l V^l} [1 + \lambda'(UV)],$$

and also a companion equation. Bearing in mind the form of $U^*$, we conclude at once $c^* = c$, $l^* = l$ and then $U^* V^* = UV$. This yields the relation stated between $U^*, V^*$ and $U, V$, as well as the additional result:

*The integer $l$ and constant $c$ are independent of the normalizing series employed.*

Thus $l$ and $c$ are the only invariants. In the case of the normal form $(16')$ we write $l = \infty$, $c = 0$ for convenience.

Conversely, it is at once shown that any change of variables from $U, V$ to $U^*, V^*$ yields normalizing variables.

## § 23. Normal form. Case II′.

It has appeared earlier that cases I′ and II′ are of the same formal character in the complex domain. This is evident if variables

$$\bar{u} = u + \sqrt{-1}\, v, \quad \bar{v} = u - \sqrt{-1}\, v$$

are introduced in case II′, when we have

$$\bar{u}_1 = \varrho \bar{u} + \cdots, \quad \bar{v}_1 = \frac{1}{\varrho} \bar{v} + \cdots, \quad (\varrho = e^{\sqrt{-1}\,\theta}).$$

Moreover in case II′ we have $\varrho^k \neq 1$ for any integer $k \neq 0$. Consequently the same formal manipulation of the variables $\bar{u}, \bar{v}$ is possible as for $u, v$. Moreover changes of variables (15) of $u, v$ yield changes of variables (15) of $\bar{u}, \bar{v}$. Keeping these facts in mind, we deduce without difficulty the following important result:

*By a formal change of variables* (15), *a formally conservative transformation of type II′ may be given either the normal form*

$$(19)\qquad \begin{aligned} U_1 &= U \cos(\theta + c(U^2+V^2)^l) - V \sin(\theta + c(U^2+V^2)^l), \\ V_1 &= U \sin(\theta + c(U^2+V^2)^l) + V \cos(\theta + c(U^2+V^2)^l), \end{aligned}$$

*or the form*

$$(19')\qquad U_1 = U\cos\theta - V\sin\theta, \quad V_1 = U\sin\theta + V\cos\theta.$$

Also, on account of the possibility of preserving the conjugate relation of the series $\bar{u}, \bar{v}$ employed at every step of the formal work (so that $u, v$ are real series), we conclude that, if $T$ is given by real series, the normalizing series $U, V$ can also be taken real.

## § 24. Generality of normal form. Case II′.

Likewise in analogy with § 22 for case I′ we find:

*The most general normalizing variables $U^*, V^*$ in case II′ have the explicit form*

$$(20)\qquad \begin{aligned} U^* &= U\cos\lambda(U^2+V^2) - V\sin\lambda(U^2+V^2), \\ V^* &= U\sin\lambda(U^2+V^2) + V\cos\lambda(U^2+V^2), \end{aligned}$$

*where $\lambda$ is an arbitrary power series in $U^2+V^2$ without constant term, and $U, V$ are any particular set of normalizing series.*

## § 25. The integrable case.

The formal series $u_k, v_k$ used in the preceding part of the paper may converge. Suppose that these series converge uniformly for $|u|, |v|, |k|$ sufficiently small. By the definition of $\delta u, \delta v$ as derivatives of $u_k, v_k$ respectively as to $k$ for $k=0$, we see that in this case $\delta u, \delta v$ are given as convergent series. Consequently the formal differential equations (9) are of the ordinary type with $\delta u$, $\delta v$ analytic functions of $u, v$ vanishing for $u=v=0$. It follows that $u_k, v_k$ converge uniformly for $|k| \leq K$, an arbitrary positive quantity, if $|u|, |v|$ are sufficiently small. It is then that we speak of $u_k, v_k$ as convergent series.

*A necessary and sufficient condition for the convergence of the series $u_k$, $v_k$ (as specified) is that the corresponding conservative transformation $T$ be integrable.*

The fact that the integrability of $T$ is necessary is proved at once. In the convergent case the formal differential equations are of the ordinary type as noted above. Consequently the formally invariant function $F^*$ defined by means of the equations (11) is an actual invariant function. That is, $T$ is integrable.

To prove the sufficiency is not such an easy task. Let $F'$ be the given invariant analytic function. Every invariant series can be expressed as a power series in $F'$ or in fractional powers thereof (§ 14). In the latter case $F'$ is an exact formal $q$th power if the $q$th root is to be extracted. And furthermore this root is of course also given by a convergent invariant series. Hence, without loss of real generality, we may assume that the invariant formal series $F^*$ is a formal power series in $F'$ i. e. $F^* = \varphi(F')$.

Now write

$$Q(u'_k, v'_k)\frac{du'_k}{dk} = \frac{\partial F'(u'_k, v'_k)}{\partial v'_k}, \quad Q(u'_k, v'_k)\frac{dv'_k}{dk} = -\frac{\partial F'(u'_k, v'_k)}{\partial u'_k},$$

where $Q$ is a quasi-invariant function belonging to the conservative transformation $T$. The differential equations so defined, joined with the initial conditions $u'_0 = u'$, $v'_0 = v'$ determine convergent power series $u'_k$, $v'_k$ which converge uniformly for $|k| < K$ ($K$ arbitrary) if $|u'|$, $|v'|$ are sufficiently small. These functions define a conservative, integrable transformation $T'$.

Furthermore, $T'$ will be of the same type I′, II′, II″ or III′ as $T$, except possibly that when $T$ is of type II″, $T'$ may be of types I′, II′ or III′. For example, if $T$ is of type I′ then $F^*$ has an initial term $uv \log \varrho$ (see (8)) of the second degree. Hence $F'$ begins with terms of at most the second degree. But the initial terms cannot be of the first degree because of the relation $F^* = \varphi(F')$. Hence we have $F' = cuv + \cdots$, and, by introducing a constant factor in $F'$, we may take $c = \log \varrho$. An inspection of the initial terms of the transformation $T'$ shows then that $u'_1 = \varrho u' + \cdots$, $v'_1 = \frac{1}{\varrho} v' + \cdots$, as desired. An entirely similar argument holds in the cases II′, III′.

In all cases it is clear that either we can take the initial terms of $F^*$ to coincide exactly with those of $F'$, or these terms are of higher degree in $F^*$ than in $F'$. In the first case $\frac{d\varphi}{dF'} = 1$ for $F' = 0$, while in the second case $\frac{d\varphi}{dF'} = 0$ for $F' = 0$.

Consider the formal series

$$u'_{k'}(u', v'), \quad v'_{k'}(u', v'),$$

where $k' = k\frac{d\varphi}{dF'}$. It is necessary to elaborate further what is meant.

Take case I' for example. Here $u'_k$, $v'_k$ are power series in $u'$, $v'$ with coefficients polynomial in $\varrho^k$, $\varrho^{-k}$, $k$. Since

$$\frac{d\varphi}{dF'} = 1 + aF' + \cdots,$$

we have

$$\varrho^{k'} = \varrho^k(1 + aF'k \log \varrho + \cdots).$$

That is, $\varrho^{k'}$ can be written as $\varrho^k$ multiplied by a power series in $u'$, $v'$ with coefficients polynomial in $k$. A similar remark is true of $\varrho^{-k'}$ and $k'$. When these series are substituted in $u'_{k'}(u', v')$, $v'_{k'}(u', v')$, and the finite number of terms of any particular degree in $u'$, $v'$ are collected, new power series in $u'$, $v'$ with coefficients polynomial in $\varrho^k$, $\varrho^{-k}$, $k$ are formed. It is these series which we designate by $u'_{k'}(u', v')$, $v'_{k'}(u', v')$.

Similarly in all of the other cases the new series $u'_{k'}$, $v'_{k'}$ are of the same form as the series for $u_k$, $v_k$.

Now we have evidently

$$\frac{du'_{k'}}{dk} = \frac{du'_{k'}}{dk'}\frac{d\varphi}{dF'}$$

by a rule of formal differentiation which evidently applies to each constituent element of $u'_{k'}$ and thus to the entire series. A similar result holds for $v'_{k'}$. Making use of these results, and also of the defining differential equations for $u'_{k'}$, $v'_{k'}$ we find

$$Q\frac{du'_{k'}}{dk} = \frac{\partial F'}{\partial v'_{k'}}\frac{d\varphi}{dF'}, \quad Q\frac{dv'_{k'}}{dk} = -\frac{\partial F'}{\partial u'_{k'}}\frac{d\varphi}{dF'},$$

where the arguments in $Q$, $F'$ are understood to be $u'_{k'}$, $v'_{k'}$. But, from the relation $F^* = \varphi(F')$, it is clear that these differential equations for $u'_{k'}$, $v'_{k'}$ are the same as those for $u_k$, $v_k$. Also these two pairs of functions reduce to $u'$, $v'$ and $u$, $v$ respectively for $k = 0$.

Since such formal differential equations and conditions determine a unique power series in $u$, $v$ with coefficients functions of $k$ of the stated type, we obtain the formal identities

$$u'_{k'}(u, v) = u_k(u, v), \quad v'_{k'}(u, v) = v_k(u, v).$$

In particular, the above relation holds for $k = 1$ and gives

$$u'_{k'}(u, v) = u_1(u, v), \quad v'_{k'}(u, v) = v_1(u, v),$$

where now $k' = \frac{d\varphi}{dF'}$.

The noteworthy feature of these equations is that the only possible divergent element appearing is $k'$.

Now write $k' = \frac{d\varphi(0)}{dF'} + k''$. Then $u'_{k'}(u, v)$, $v'_{k'}(u, v)$ become convergent power series in $u, v, k''$ for sufficiently small values of these variables. A formal power series in $u, v$ without constant term satisfying the two equations above is $k'' = \frac{d\varphi(F')}{dF'} - \frac{d\varphi(0)}{dF'}$. Since these equations are of the ordinary analytic type, $k''$ is a convergent power series. Consequently $\frac{d\varphi(F')}{dF'}$ is a convergent series, and, since $F'$ is also, it follows that $\frac{d\varphi(z)}{dz}$ is a convergent power series in $z$. Finally then $\varphi$ is a convergent series.

It follows that $F^*$ is given by a convergent series in the integrable case and thus, by the differential equations (9), that the series $u_k, v_k$ are convergent.

The simplicity of the integrable case is sufficiently evident from the following fact:

*In the integrable case explicit formulas for $u_k, v_k$ are at hand, namely*

$$(21) \qquad F^*(u_k, v_k) = F^*(u, v), \quad k = \int_{u,v}^{u_k, v_k} \frac{Q\,du}{\frac{\partial F^*}{\partial v}} = -\int_{u,v}^{u_k, v_k} \frac{Q\,dv}{\frac{\partial F^*}{\partial u}},$$

*where the integrals are taken along the curve $F^* = const.$*

The normal forms (16), (16′) and (19), (19′), for cases I′ and II′ respectively, are integrable. If these normal forms can be obtained by means of a change of variables (15) in which the series $U, V$ are convergent, the given transformation $T$ is integrable.

Conversely, suppose $T$ to be integrable and of type I′. The series $F^*$ converges and by (8) can be written $UV \log \varrho$, where $U, V$ are convergent series of the form (15). If we introduce these new variables, which we call $u, v$ for brevity, then $uv$ is an invariant function.

For the integrable transformation $T$ in these variables, the convergent series $\delta u, \delta v$ must be of the forms $up \log \varrho$ and $-vp \log \varrho$, where $p$ is a convergent series with constant term 1. In fact $v\delta u + u\delta v$ vanishes and the initial terms of $\delta u$ and $\delta v$ are $u \log \varrho$ and $-v \log \varrho$ respectively.

If a further actual change of variables (15) can be made which gives $T$ the form

$$U_1 = \varrho U e^{h(UV)}, \quad V_1 = \frac{1}{\varrho} V e^{-h(UV)},$$

then an additional actual change of variables as in § 21 yields the desired normal form. But $U_1, V_1$ have this form if and only if

$$\delta U = U(\log \varrho + h(UV)), \quad \delta V = -V(\log \varrho + h(UV)),$$

i. e. if

$$\left(u\frac{\partial U}{\partial u} - v\frac{\partial U}{\partial v}\right) p \log \varrho = \quad U\ (\log \varrho + h(UV)),$$

$$\left(u\frac{\partial V}{\partial u} - v\frac{\partial V}{\partial v}\right) p \log \varrho = - V(\log \varrho + h(UV)).$$

We have then to find convergent series $U, V, h$ which satisfy this pair of equations, in order to establish the proposition under consideration.

It is sufficient to satisfy the equations

$$\left(u\frac{\partial U}{\partial u} - v\frac{\partial U}{\partial v}\right) p \log \varrho = \quad U\ (\log \varrho + \varphi(uv)),$$

$$\left(u\frac{\partial V}{\partial u} - v\frac{\partial V}{\partial v}\right) p \log \varrho = - V(\log \varrho + \varphi(uv)),$$

with convergent series $U, V, \varphi$, provided that $U$ and $V$ have initial terms $u$ and $v$ respectively. For, multiplying the first equation by $V$, the second by $U$, adding and integrating, we conclude that $UV$ is a function of the product $uv$ alone. Hence we have $UV = uv + \cdots$. Therefore $uv$ can be expressed inversely as a power series in $UV$, and $\varphi(uv) = h(UV)$ where $h$ is convergent.

But, by the same equations, $U$ and $V$ contain no terms in $v$ and $u$ alone respectively, since $p$ has a constant term 1. Consequently we may write

$$U = ue^M, \quad V = ve^{-N},$$

where $M$ and $N$ are convergent power series in $u, v$ without constant terms. The equations above take the form

$$u\frac{\partial M}{\partial u} - v\frac{\partial M}{\partial v} = \frac{1}{p}\left(1 + \frac{\varphi(uv)}{\log \varrho}\right) - 1,$$

$$u\frac{\partial N}{\partial u} - v\frac{\partial N}{\partial v} = \frac{1}{p}\left(1 + \frac{\varphi(uv)}{\log \varrho}\right) - 1.$$

If convergent power series solutions $M, N$ and $\varphi$ without constant terms can be found our proof will be complete.

We observe in the first place that there are no terms in $u, v$ with equal powers of $u$ and $v$ on the left. Hence, for any conceivable solution, the series development of

$$\frac{1}{p}\left(1 + \frac{\varphi(uv)}{\log \varrho}\right) - 1$$

contains no similar terms. But this property of a series is not modified if it is multiplied by a series in $uv$ only but having a constant term. Hence

$$\frac{1}{p} - \frac{\log \varrho}{\log \varrho + \varphi(uv)}$$

is a similar series. The second term must consist precisely of those terms $p'(uv)$ in $uv$ alone found in $\frac{1}{p}$, and we have

$$\varphi = \frac{1 - p'}{p'} \log \varrho.$$

Thus the only possible formal series $\varphi$ is convergent.

When this particular $\varphi$ is substituted, the right-hand members above become power series in $u, v$ without terms having equal exponents. Write then

$$M = N = \sum_{m+n=1}^{\infty} P_{mn} u^m v^n, \qquad (m \neq n).$$

The equations for the formal determination of the coefficients $P_{mn}$ show that these are uniquely determined and not greater numerically than the corresponding coefficients in the right-hand members. Thus the desired convergent solution $M, N, \varphi$ is obtained.

An entirely similar discussion can be made in case II'.

*In the integrable cases I', II', and then only, the normal forms* (16), (16') *and* (19), (19') *can be obtained by a change of variables* (15), *where $U$, $V$ are convergent series.*

### § 26. The non-integrable case and the integrable case.

Let two transformations $T$ and $T'$ be said to *osculate to the $\mu$th order if* $u_1 - u'_1$ and $v_1 - v'_1$ are given by series beginning with terms of at least the $(\mu + 1)$th degree; $T_k$ and $T'_k$ will also osculate to the $\mu$th order for any integral value of $k$. It is clear that the formal series for $\delta u, \delta v$ and $\delta u', \delta v'$ agree to terms of the $(\mu + 1)$th degree. Conversely, if $\delta u, \delta v$ and $\delta u', \delta v'$ agree out to terms of the $(\mu + 1)$th degree, then $T$ and $T'$ osculate to the $\mu$th order.

*Let $T$ be a given conservative transformation of types I', II', II'' or III' with a quasi-invariant function $Q$ and a formally invariant series $F^*$. If $\bar{Q}$ and $\bar{F}^*$ are convergent series agreeing with $Q$ and $F^*$ to terms of the $(\mu + 1)$th and $(\mu + 2)$th degrees respectively, then there will exist a corresponding integrable transformation $\bar{T}$ with a quasi-invariant function $\bar{Q}$ and an invariant function $\bar{F}^*$, which $T$ osculates to the $\mu$th order.*

The transformation $\bar{T}$ is evidently that defined by the equations

$$\bar{Q}\frac{du_k}{dk} = \frac{\partial \bar{F}^*}{\partial v_k}, \quad \bar{Q}\frac{dv_k}{dk} = -\frac{\partial \bar{F}^*}{\partial u_k},$$

with initial conditions $u_0 = u$, $v_0 = v$.

## Chapter II. Hyperbolic invariant points.

### § 27. The analytic invariant curves in case I'.

In the non-integrable as well as in the integrable case I' the two real formally given invariant curves correspond to actual curves. A proof of this fact was first given by Poincaré (loc. cit.) and later by Hadamard.[1] Our proof will be of a different character, and involves the hypothesis that $T$ is conservative. A similar method will be used later by us in treating more general cases.

[1] *Sur l'itération et les solutions asymptotiques des équations différentielles, Bulletin de la Société Mathématique de France*, vol. 29, 1901.

The two formally invariant curves in case I' may either be obtained from the equation $F^* = 0$ or from the equations $U = 0$, $V = 0$ where $U$, $V$ are the normalizing variables of § 21. In fact, when the transformation is in the normal form, these equations yield the formally invariant curves.

*In case I' the two real formally invariant curves give two analytic invariant curves through the invariant point.*

We commence our proof by choosing variables which osculate the normalizing variables $U$, $V$ to the $\mu$th order ($\mu \geq 2$). According to § 21 we have then

$$u_1 = \varrho u e^{cu^l v^l} + \omega(u, v), \quad v_1 = \frac{1}{\varrho} v e^{-cu^l v^l} + \eta(u, v),$$

where $\omega$, $\eta$ are convergent power series beginning with terms of the $(\mu + 1)$th degree or of higher degree.

Our proof will consist of the following three steps:

(1) the limits $\lim_{k=\infty} u_k(\varrho^{-k}t, 0)$, $\lim_{k=\infty} v_k(\varrho^{-k}t, 0)$ exist as formal power series $u^*(t)$, $v^*(t)$, and yield a formally invariant curve;

(2) $u_k(\varrho^{-k}t, 0)$, $v_k(\varrho^{-k}t, 0)$ are dominated by fixed convergent power series in $t$ for all $k$;

(3) and hence these series converge uniformly to limiting functions of $t$ for $|t|$ sufficiently small, namely to $u^*$, $v^*$ respectively, which are thus the coordinates of the invariant curve $V = 0$.

A similar treatment of the invariant curve $U = 0$ can be based on the inverse transformation $T_{-1}$.

*Proof of* (1).

We have directly

$$u_k = \varrho^k u e^{ku^l v^l} + \omega(u, v, k), \quad v_k = \varrho^{-k} v e^{-ku^l v^l} + \eta(u, v, k),$$

where $\omega(u, v, k)$, $\eta(u, v, k)$ are convergent power series beginning with terms of the $(\mu + 1)$th or higher degree. Furthermore, the first term on the right-hand side of either equation has evidently the property that when expanded in power series in $u, v$ the coefficient of a term of the $m$th degree is linear in $\varrho^k$ or $\varrho^{-k}$ and polynomial in $k$ of degree less than the degree of the term. But $u_k$, $v_k$ have the property that the coefficient of $u^m v^n$ is a polynomial in $\varrho^k$, $\varrho^{-k}$, $k$ of degree at most $m + n$ (§ 4). Therefore the same is true of $\omega(u, v, k)$, $\eta(u, v, k)$.

Thus we obtain at once

$$u_k(\varrho^{-k}t, 0) = t + \sum_{n=\mu+1}^{\infty} p^{(n)}(\varrho^{-k}, k) t^n, \quad v_k(\varrho^{-k}t, 0) = \sum_{n=\mu+1}^{\infty} q^{(n)}(\varrho^{-k}, k) t^n,$$

where $p^{(n)}, q^{(n)}$ are polynomials in $\varrho^{-k}$, $k$, and where every term involving $k$ is affected with a multiplier $\varrho^{-k}$ raised to a positive power.

It is thus seen that, inasmuch as $\varrho > 1$, each coefficient of the series for $u_k(\varrho^{-k}t, 0)$ and $v_k(\varrho^{-k}t, 0)$ approaches a limiting value as $k$ becomes infinite. In other words there exist limiting formal series $u^*$, $v^*$. Clearly $u^*$ is given by a series with first term $t$ and following terms of degree at least $\mu + 1$, while $v^*$ begins with a term of degree not less than $\mu + 1$.

Now we have the formal identities symbolized by

$$T(u_k(\varrho^{-k}t, 0), v_k(\varrho^{-k}t, 0)) = (u_{k+1}(\varrho^{-k-1}t', 0), v_{k+1}(\varrho^{-k-1}t', 0)),$$

where $t'$ stands for $\varrho t$. By allowing $k$ to become infinite we obtain the identities symbolized by

$$T(u^*(t), v^*(t)) = (u^*(\varrho t), v^*(\varrho t)).$$

This is precisely the condition that $u = u^*(t)$, $v = v^*(t)$ be a formally invariant curve under $T$. The parameter $t$ on the curve goes over into $\varrho t$.

*Proof of* (2).

To establish (2) we begin by observing that, inasmuch as $u_k(u, v)$, $v_k(u, v)$ are computed by successive substitutions, every coefficient in these series will certainly become positive, and as large numerically as it is originally, if $u_1$, $v_1$ are replaced by any series in $u$, $v$ with each coefficient positive or zero and as large numerically as the like coefficient in $u_1$, $v_1$. Such series can be taken of the form

$$\varrho u + \frac{K(u+v)^2}{1 - L(u+v)}, \quad \varrho v + \frac{K(u+v)^2}{1 - L(u+v)},$$

provided that $K$ and $L$ are sufficiently large positive constants.[1]

Hence when we compute $u_k(u, v)$, $v_k(u, v)$ for $u = \varrho^{-k}t$, $v = 0$, taking $u_1$, $v_1$ to be these modified series, we obtain power series in $t$ which have positive coefficients greater than originally and so dominate the earlier series. Again these series will certainly be dominated by the sum

$$u_k(\varrho^{-k}t, 0) + v_k(\varrho^{-k}t, 0),$$

where $u_1$, $v_1$ are the dominating series exhibited above.

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[1] The linear terms taken are clearly large enough. The coefficients of $u^m v^n$ $(m + n \geq 2)$ in either series is at least as large as $KL^{m+n-2}$, which evidently exceeds numerically the coefficient of $u^m v_n$ in $u_1$ or $v_1$ if $K, L$ be chosen sufficiently large to begin with.

The sum $\sigma_k = u_k + v_k$ obeys the law of formation

$$\sigma_{k+1} = \varrho\sigma_k + \frac{K\sigma_k^2}{1 - L\sigma_k}$$

with $\sigma_0 = \sigma = u + v$.

But the sequence $\sigma_k$ is itself obtained by the method of successive substitution. Therefore the dominating series $\sigma_k$ is increased if we take

$$\sigma_1 = \frac{\varrho\sigma}{1 - M\sigma},$$

provided we take $M$ to be positive and as large as $\frac{K}{\varrho}$ and $L$.

Under these circumstances we get the following general formula for $\sigma_k$:

$$\sigma_k = \frac{\varrho^k\sigma}{1 - M\frac{\varrho^k - 1}{\varrho - 1}\sigma}.$$

The corresponding series in $t$ is then

$$\frac{t}{1 - M\frac{1 - \varrho^{-k}}{\varrho - 1}t},$$

which is dominated for all $k$ by the convergent series

$$\frac{t}{1 - \frac{Mt}{\varrho - 1}}.$$

Hence this same series dominates the original series $u_k(\varrho^{-k}t, 0)$, $v_k(\varrho^{-k}t, 0)$ for all positive integral values of $k$. Q. E. D.

*Proof of* (3).

With the aid of (1) and (2), established above, we can at once show that the power series $u_k(\varrho^{-k}t, 0)$, $v_k(\varrho^{-k}t, 0)$ must be approaching a limiting pair of functions uniformly for $|t|$ sufficiently small.

To this end we choose $k$ so large that all of the coefficients up to the $m$th in both series ($m$ arbitrary) differ by less than an arbitrarily assigned positive $\varepsilon$ from their limiting values, which exist by our result (1). The sum of these $m$ terms never varies for greater $k$ by more than a fixed $\varepsilon'$ if $|t|$ be restricted.

But the remaining terms cannot exceed the sum of the corresponding terms of the fixed dominating series with $t$ replaced by $|t|$. Hence the sum of these terms is arbitrarily small if $m$ is sufficiently large.

The fact of uniform convergence is evident.

Thus $u^*(t)$, $v^*(t)$ are not only formal series but these series converge to actual analytic functions which we denote by $u^*(t)$, $v^*(t)$. Consequently we have an analytic invariant curve $u = u^*(t)$, $v = v^*(t)$. Since $u^*(t)$ begins with a term $t$ while $v^*(t)$ begins with a term of degree at least $\mu + 1$, this invariant curve has contact of order $\mu$ at least with the $u$-axis at the invariant point, and corresponds to the invariant curve $V = 0$. The parameter change under $T$ along the curve is $t_1 = \varrho t$.

Evidently the existence of these two analytic invariant curves $U = 0$, $V = 0$ establishes the fact that the invariant point is unstable in case I'.

## § 28. A general property in case I'.

Introduce new variables of the type (15)

$$U = u - \varphi(v), \quad V = v - \psi(u),$$

where $u = \varphi(v)$ and $v = \psi(u)$ are the analytic invariant curves of the preceding section. In the $UV$-plane the invariant curves are the axes. For the sake of brevity of notation we will let $u, v$ denote any set of variables which make the axes and the invariant curves coincide.

When such variables have been selected it is clear that $u_1 = 0$ if $u = 0$, and that $v_1 = 0$ if $v = 0$. Hence we have

$$u_1 = u(\varrho + \cdots), \quad v_1 = v\left(\frac{1}{\varrho} + \cdots\right).$$

From this form of $T$ we infer at once

$$\varrho - \varepsilon < \frac{u_1}{u} < \varrho + \varepsilon, \quad \frac{1}{\varrho} - \varepsilon < \frac{v_1}{v} < \frac{1}{\varrho} + \varepsilon$$

for points near $(0, 0)$. Here $\varepsilon$ is an arbitrarily small positive quantity. Thus $u$ increases numerically and $v$ decreases numerically upon iteration of $T$, in such wise that the following result is obvious.

*If the invariant curves are taken as the axes in case I' by means of a preliminary choice of variables* (15), *every point of the region* $u^2 + v^2 \leqq \delta^2$ *with* $u \neq 0$ *is* *carried out of the region by iteration of $T$, while every point $v \neq 0$ is carried out of the region by iteration of $T_{-1}$. The excluded points of the axes approach* (0, 0) *under the same conditions.*

These considerations show that there can exist no further invariant curves through (0, 0) besides the two analytic curves above obtained.

## § 29. **On the invariant series in case I′.**

The treatment of invariant points in case I′ as given above is sufficient for the later parts of the paper. Nevertheless, there remains open the question of the actual existence of divergent formal series $F^*$ in case I′. Unfortunately I have not been able to answer this question. In the present paragraph upper limits for the coefficients of a particular invariant formal series are obtained.

*If the invariant curves are taken as the axes in case I′ by means of a preliminary choice of variables* (15) *and if $F$ denotes the invariant formal series having no terms with equal exponents in $u, v$ save $uv \log \varrho$, then the coefficient of $u^m v^n$ in $F$ does not numerically exceed $C_{mn}\varrho^{2mn}$, where $C_{mn} > 0$ is the coefficient of $u^m v^n$ in a convergent power series.*

Before entering upon the proof of this statement, we note that *in a series $F$ with coefficients so restricted the terms in any power of $u$ or of $v$ form a convergent series.*

There exists such an invariant series $F$, for, by forming $\varphi(F^*)$ as a power series in $F^*$ beginning with a term $F^*$, we can eliminate the terms with equal exponents in $F$. Since the most general invariant series is a power series in $F^*$ in case I′, it follows that this particular series $F$ is uniquely determined by the given condition.

In order to effect a proof of the italicized statement we first write the equation $F(u_1, v_1) = F(u, v)$ in the form

$$F\left(u_1\left(\frac{u}{\varrho}, \varrho v\right), \ v_1\left(\frac{u}{\varrho}, \varrho v\right)\right) = F\left(\frac{u}{\varrho}, \varrho v\right),$$

obtained by replacing $u, v$ by $\frac{u}{\varrho}$, $\varrho v$ respectively. We have

$$u_1\left(\frac{u}{\varrho}, \varrho v\right) = up, \quad v_1\left(\frac{u}{\varrho}, \varrho v\right) = vq,$$

where $p$ and $q$ are convergent power series in $u$ and $v$ with constant term 1. The equation above may be written

$$F\left(\frac{u}{\varrho}, \varrho v\right) = F(up, vq).$$

Likewise from the equation $F(u_{-1}, v_{-1}) = F(u, v)$ we obtain an equation

$$F\left(\varrho u, \frac{v}{\varrho}\right) = F(ur, vs),$$

where $r$ and $s$ are convergent power series in $u, v$ with constant term 1.

If $F_{mn}$ denotes the coefficient of $u^m v^n$ in $F$ so that $F_{11} = \log \varrho$, $F_{20} = F_{02} = 0$, there results, by a comparison of coefficients in these two equations,

$$F_{mn}(\varrho^{n-m} - 1) = P_{mn}, \; F_{mn}(\varrho^{m-n} - 1) = Q_{mn},$$

where $P_{mn}, Q_{mn}$ are linear homogeneous expressions in $F_{\alpha\beta}$ with $\alpha \leqq m$, $\beta \leqq n$, $\alpha + \beta < m + n$. The coefficients of $F_{\alpha\beta}$ in $P_{mn}$ are polynomials in the coefficients of the series $p, q$ with positive integral coefficients, while the coefficients of $F_{\alpha\beta}$ in $Q_{mn}$ are similar polynomials in the coefficients of the series $r, s$. Combining the above equations we obtain

$$F_{mn}(\varrho^{n-m} + \varrho^{m-n} - 2) = P_{mn} + Q_{mn}.$$

For $m \neq n$ the coefficient of $F_{mn}$ is positive.

Suppose that $p, q, r, s$ are replaced by a single dominating series, say

$$\frac{1}{1 - A(u+v)}.$$

Then $P_{mn}, Q_{mn}$ takes a common form $R_{mn}$, and the modified equations

$$F_{mn}(\varrho^{n-m} + \varrho^{m-n} - 2) = 2R_{mn}, \qquad (m \neq n),$$

define new positive quantities $F_{mn}$ for $m \neq n$, at least as large as before in absolute magnitude.

Along with these equations we consider the equations

$$G_{mn}(\varrho^{m+n} - 1) = R_{mn}, \qquad (m + n \geqq 3),$$

in which the arguments $F$ in $R_{mn}$ are replaced by $G$. These equations determine $G_{mn}$ for $m+n=3$, $m+n=4, \ldots$, in succession, provided that we take $G_{11}=\log \varrho$, $G_{20}=G_{02}=0$. These differ from the equations determining the modified values $F_{mn}$ only in that the divisors $\frac{1}{2}(\varrho^{n-m}+\varrho^{m-n}-2)$ are replaced by the larger divisors $\varrho^{m+n}-1$. Consequently we have

$$\frac{F_{mn}}{G_{mn}} < \prod \frac{\varrho^{m'+n'}-1}{\frac{1}{2}(\varrho^{n'-m'}+\varrho^{m'-n'}-2)},$$

where the values $m', n'$ written are for all the divisors explicitly entering into some one term of the complete expression for $F_{mn}$.

Now take these divisors in order beginning with $m'=m$, $n'=n$. The next divisor has $m' \leqq m$, $n' \leqq n$, $m'+n' < m+n$, and in general $m', n'$ do not increase while $m'+n'$ decreases by at least unity at each stage.

For $m' > n'$ we have

$$\frac{\varrho^{m'+n'}-1}{\varrho^{n'-m'}+\varrho^{m'-n'}-2} = \frac{\varrho^{m'+n'}-1}{\varrho^{m'-n'}(1-\varrho^{m'-n'})^2} < \frac{\varrho^{2n'}}{\left(1-\frac{1}{\varrho}\right)^2},$$

and there is a symmetrical inequality which holds for $n' > m'$. Let us replace the factors above by these larger factors. If $m > n$ a superior limit for the product $\Pi$ is therefore obtained by making $m'$ diminish by unity successively and keeping $n'=n$ until we have $m'=n+1$, and thereafter decreasing $n'$ and $m'$ alternately by $1$. Hence the product of the factors is less than

$$2^{m+n} \frac{\varrho^{2n(m-n)}\varrho^{2(n-1)}\varrho^{2(n-2)}\cdots 1}{\left(1-\frac{1}{\varrho}\right)^{2(m+n)}}$$

and thus less than

$$\frac{2^{m+n}\varrho^{2nm}}{\left(1-\frac{1}{\varrho}\right)^{2(m+n)}}.$$

Thus we have

$$|F_{mn}| < \frac{2^{m+n}G_{mn}}{\left(1-\frac{1}{\varrho}\right)^{2(m+n)}}\varrho^{2nm}$$

for $m > n$, with the same inequality holding for $n > m$.

It is clear therefore that $|F_{mn}|$ is restricted by an inequality of the type stated if the series

$$G = \Sigma\, G_{mn}\, u^m v^n,$$

converges.

The coefficients $G_{mn}$ and $G_{nm}$ are equal so that $G_{mn} + R_{mn}$ is the coefficient of $u^m v^n$ in either

$$G\left(\frac{u}{1 - A(u+v)}, \frac{v}{1 - A(u+v)}\right) \text{ or } G\left(\frac{v}{1 - A(u+v)}, \frac{u}{1 - A(u+v)}\right).$$

Moreover, it follows also from the equations of definition of $G_{mn}$ that the difference

$$G(\varrho u, \varrho v) - \frac{1}{2}\left[G\left(\frac{u}{1 - A(u+v)}, \frac{v}{1 - A(u+v)}\right) + G\left(\frac{v}{1 - A(u+v)}, \frac{u}{1 - A(u+v)}\right)\right],$$

considered as a formal series, has no terms in $u^m v^n$ for $m + n \geqq 3$, and so reduces to

$$(\varrho^2 - 1)\, uv \log \varrho.$$

Furthermore, $G$ is determined formally by this property.

If we replace this last difference by

$$\frac{1}{2}(\varrho^2 - 1)(u + v)^2,$$

which dominates it, a modified $G$ series is obtained, satisfying the equation

$$G(\varrho u, \varrho v) = \frac{1}{2}\left[G\left(\frac{u}{1 - A(u+v)}, \frac{v}{1 - A(u+v)}\right) + G\left(\frac{v}{1 - A(u+v)}, \frac{u}{1 - A(u+v)}\right)\right] + $$
$$+ \frac{1}{2}(\varrho^2 - 1)(u + v)^2,$$

and certainly dominating the former $G$ series. This functional equation wholly determines the new series.

But the functional equation

$$f(\varrho z) = f\left(\frac{z}{1 - Az}\right) + \frac{1}{2}(\varrho^2 - 1)z^2$$

admits of an analytic solution, namely

$$\frac{\varrho^2-1}{2\varrho^2}z^2\sum_{n=0}^{\infty}\frac{\varrho^{-2n}}{\left(1-\frac{1-\varrho^{-n}}{\varrho-1}Az\right)^2}.$$

It follows that $G(u, v) = f(u+v)$ gives the solution of the modified equation for $G$, and thus that the original $G$ series converges. Consequently the proposition under consideration is fully established.

## § 30. The case I''.

This case is easily disposed of inasmuch as $T_2$ is of the type I' treated above.

If we choose formal normalizing variables, $T_2$ becomes precisely

$$u_2 = \varrho^2 u e^{cu^l v^l}, \quad v_2 = \frac{1}{\varrho^2} v e^{-cu^l v^l}.$$

This is possible by § 21.

We have $u_2 v_2 = uv$. In case $u_1 v_1 \neq uv$, write

$$u_1 v_1 = uv + \varphi(u, v) + \cdots$$

where $\varphi$ is a homogeneous polynomial in $u, v$ of least the third degree in $u, v$. This gives

$$u_2 v_2 = u_1 v_1 + \varphi(u_1, v_1) + \cdots = uv + \varphi(u, v) + \varphi\left(\varrho u, \frac{1}{\varrho}v\right) + \cdots.$$

Hence $\varphi(u, v) + \varphi\left(\varrho u, \frac{1}{\varrho}v\right)$ vanishes identically. This is impossible for any polynomial not identically 0 since $\varrho \neq -1$. Thus we conclude that $u_1 v_1 = uv$, and accordingly that $u_1$ and $v_1$ are divisible by $u$ and $v$ respectively.

We may now write

$$u_1 = \varrho u g(u, v), \quad v_1 = \frac{v}{\varrho g(u, v)}.$$

Here $g(u, v)$ is a power series in $u, v$ with initial term 1.

The first of these equations gives $u_2 = \varrho^2 u g(u, v)\, g(u_1, v_1)$ whence, by comparison with the normal form,

$$g(u, v)\, g(u_1, v_1) = e^{c u^l v^l}.$$

Replacing $u, v$ by $u_1, v_1$ in this equation, we have

$$g(u_1, v_1)\, g(u_2, v_2) = e^{c u^l v^l}$$

so that, by a comparison, $g(u_2, v_2) = g(u, v)$, i. e. $g(u, v)$ is an invariant function under $T_2$, and must be a function of the product $uv$ only (§ 14), namely $e^{\frac{c}{2} u^l v^l}$.

The form of the transformation $T$ is now fully determined.

*In the case I″ by the aid of a formal change of variables* (15) *the transformation $T$ may be reduced to the form*

$$U_1 = \varrho U e^{\frac{1}{2} c u^l v^l}, \quad V_1 = \frac{1}{\varrho} V e^{-\frac{1}{2} c u^l v^l},$$

*where we may have $c = 0$, $l = \infty$ as in case* I′.

This same reduction shows that the formally invariant curves under $T$ and $T_2$ coincide.

*In case I″ there is a formally invariant function $F^*$ and two analytic invariant curves through the invariant point, these being the same as for $T_2$.*

We can at once infer that *the same property holds in case I″ as is given in case I′ by the italicized statement of § 28.* Hence there are no further invariant curves through (0, 0).

In the integrable case these normal forms can be obtained by means of ordinary changes of variables (§ 25).

## § 31. An example in the hyperbolic case II″.

There are of course no real invariant formal curves in case II′ inasmuch as $F^*$ is of the form $-\frac{1}{2}\theta(u^2 + v^2)$ by (13). Thus the case I′ treated above may be regarded as the general hyperbolic case, while II′ is the general elliptic case. The cases II″, III′ may be either hyperbolic or elliptic.

In the hyperbolic case II″ we can set up an example showing that the formal series $F^*$ may diverge and also illustrating other significant features.

The transformation $T$ is the following

$$u_1 = \frac{u}{1+u}, \quad v_1 = (1+u)^2(v+u^2).$$

This is evidently of type II″ at the invariant point (0, 0). Moreover, since the Jacobian of $u_1, v_1$ as to $u, v$ is 1, areas are preserved, and $T$ is conservative with quasi-invariant function $Q = 1$.

By direct iteration we find

$$u_k = \frac{u}{1+ku}, \quad v_k = (1+ku)^2\left[v + u^2\left(1 + \frac{1}{(1+u)^4} + \cdots + \frac{1}{(1+(k-1)u)^4}\right)\right]$$

for all integral values of $k$. The expression for $u_k$ is of the type given in § 5. To express $v_k$ in such a form we introduce the well-known function $\psi(z) = \frac{d \log \Gamma(z)}{dz}$.[1]

We have, by means of the functional equation for $\Gamma(z)$

$$\psi(z+1) = \frac{1}{z} + \psi(z),$$

so that

$$\psi'''(z+1) = \frac{-6}{z^4} + \psi'''(z),$$

where $\psi'''$ stands for the third derivative of $\psi$ as to $z$. Hence we have

$$\psi'''\left(\frac{1}{u}+1\right) = -6u^4 + \psi'''\left(\frac{1}{u}\right), \quad \psi'''\left(\frac{1}{u}+2\right) = \frac{-6u^4}{(1+u)^4} + \psi'''\left(\frac{1}{u}+1\right), \ldots,$$

whence, by addition,

$$\psi'''\left(\frac{1+ku}{u}\right) - \psi'''\left(\frac{1}{u}\right) = -6\sum_{m=0}^{k-1}\frac{u^4}{(1+mu)^4}.$$

Thus we may write for positive integral values of $k$, and likewise for $k$ a negative integer or 0,

$$v_k = (1+ku)^2\left[v - \frac{1}{6u^2}\left(\psi'''\left(\frac{1+ku}{u}\right) - \psi'''\left(\frac{1}{u}\right)\right)\right].$$

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[1] For a simple development of the properties of $\psi(z)$ used here, see K. P. Williams, *The asymptotic form of the function $\psi(x)$*. *Bulletin of the American Mathematical Society*, vol. 19, 1912—1913.

Now $\psi(z)$ is given asymptotically by the series

$$\log z - \frac{1}{2z} + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{2n z^{2n}}$$

in the right half of the complex $z$-plane, in which $B_n$ denotes the $n$th Bernoullii number. By differentiating three times[1] we infer that $\psi'''(z)$ is given by an asymptotic series,

$$\frac{2}{z^3} + \frac{3}{z^4} - \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n+2) B_n}{z^{2n+3}}$$

in the right half of the complex $z$-plane. This series satisfies formally the functional equation for $\psi'''(z)$ given above, although the series diverges of course.

If replace $\psi'''$ by this divergent series in the expression for $v_k$ found above, $v_k$ becomes a power series in $u, v$ with coefficients polynomial in $k$. Moreover, for $k = 1, 2, 3, \ldots$ the series involved must converge. In fact $v_k$ is then a function analytic at $u = 0$, and therefore its asymptotic series is its power series expansion. Hence we have here the unique formal series for $v_k$ of the type considered in § 5.

By direct formal differentiation of these series for $u_k$, $v_k$ and setting $k = 0$, we obtain

$$\delta u = -u^2, \quad \delta v = 2uv - \frac{1}{6u^2} \psi^{(4)}\left(\frac{1}{u}\right).$$

Moreover, since $Q = 1$, we have for the invariant series $F^*$ by formula (11)

$$\frac{\partial F^*}{\partial v} = -u^2, \quad \frac{\partial F^*}{\partial u} = -2uv + \frac{1}{6u^2} \psi^{(4)}\left(\frac{1}{u}\right),$$

whence we get immediately

$$F^* = -u^2 v - \frac{1}{6} \psi'''\left(\frac{1}{u}\right).$$

This of course is a *divergent* formal power series in $u, v$. But it was seen in § 25 that the series for $F^*$ converges in the integrable case.

[1] See J. F. Ritt, *On the differentiability of asymptotic series. Bulletin of the American Mathematical Society*, vol. 24, 1917—1918, for a discussion of such differentiation.

*Thus $T$ is of non-integrable type.*

We note the very significant fact that if $\psi'''$ be regarded as a function and not as a formal series, we have here an actual invariant function, real and analytic for $u > 0$, and asymptotically given by the formal series for $F^*$ for $u$ small, so that this function is continuous together with all of its derivatives for $u \geqq 0$.

At first sight this seems to leave available no similar function for $u \leqq 0$. Such a function is readily furnished as follows. The function $\psi(1-z)$ satisfies the same functional equation as $\psi(z)$, is analytic in the left half of the complex $z$-plane, and is given asymptotically by the same divergent series as $\psi$. Hence a similar invariant function $F^*$ can be obtained by replacing the function $\psi'''(z)$ by the function $-\psi'''(1-z)$.

If the invariant functions for the two halves of the $uv$-plane are united we obtain a real invariant function, analytic save for $u = 0$, continuous together with its partial derivatives, and given asymptotically by $F^*$ at $(0, 0)$.

It is this general type of invariant function which probably exists in all cases.

In the case I′, I can prove the existence of a real invariant function, continuous with all of its derivatives and asymptotically given by $F^*$ at $(0, 0)$. But this discussion is omitted since the existence of an *analytic* invariant function is highly probable in the general hyperbolic case.

We come now to the question of formally invariant curves. These are obtained by factorization of $F^*$ (§ 19). Since $\psi'''\left(\frac{1}{u}\right)$ has an initial term $2u^3$, the curves are at once seen to be the curve $u = 0$ taken doubly, and the formal curve $v = -\frac{1}{6u^2}\,\psi'''\left(\frac{1}{u}\right)$. The latter curve also gives an invariant curve, analytic for $u \neq 0$ such that $v$ is continuous together with its derivatives of all orders in $u$ at $u = 0$. Half of this curve ($u \geqq 0$) is that arising when by $\psi(z)$ we understand the function $\psi(z)$ introduced above. For $u \leqq 0$ we get of course the other half of the invariant curve $v = \frac{1}{6u^2}\,\psi'''\left(1-\frac{1}{u}\right)$.

The analytic invariant curve $u^2 = 0$ corresponds to a multiple factor of $F^*$, and accordingly is an invariant point curve (§ 20). This is readily verified from the explicit formulas for $u_1, v_1$.

It is also worthy of note that, for any point $(u, v)$ not on either invariant curve and with $u > 0$, $u_k$ increases for $k = -1, -2, \ldots$. Also for $k = 1, 2, \ldots$, we note that

$$F^* = -u_k^2\left[v_k + \frac{1}{6u_k^2}\psi'''\left(\frac{1}{u_k}\right)\right] = c \neq 0.$$

Hence as $u_k$ diminishes the expression in brackets increases. In other words any such point $(u, v)$ leaves the vicinity of $(0, 0)$, both upon indefinite iteration of $T_{-1}$ and of $T$. A similar argument shows that the same is true for a point $(u, v)$ with $u < 0$.

## § 32. Preliminary normalization in the hyperbolic case II″.

We will consider first what may be termed the non-specialized case II″. The characteristic feature of the case II″ is that the invariant formal series for $F^*$ begins with terms of higher than the second degree. In general then we have

$$F^* = a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + \cdots,$$

where the roots of the cubic in $s$,

$$a_{30} + a_{21}s + a_{12}s^2 + a_{03}s^3 = 0,$$

are distinct and at least one of these roots is real. To each such real root $s$ corresponds a real formally invariant curve $v = su + \cdots$. In general therefore case II″ is of hyperbolic type. We consider any such real formally invariant curve $C$.

By a linear change of variables $C$ can be taken tangent to a new $u$-axis. That is, we can make $s = 0$ in this way. The equation of $C$ is now of the form $v = \varphi(u)$, where $\varphi(u) = au^2 + \cdots$. By a further formal change of variables (15), $U = u, V = v - \varphi(u)$, the formal curve $C$ can be taken into the $U$-axis. It is to be observed that any such linear change of variables as well as a change of variables (15) leaves $T$ of the same type II″. Since $V = 0$ is the equation of the invariant curve, both $V_1$ and $F^*$ are divisible by $V$.

If we do not make the above formal change of variables but make an actual change to variables which are the same to arbitrarily high degree $\mu + 1$, $u_1, v_1$ are unaltered to terms of degree $\mu + 1$, $Q$ to terms of degree $\mu$, and thus, by (11), $F^*$ is unaltered to terms of degree $\mu + 1$. Consequently we may write

$$v_1 = v[1 + \cdots] + \omega(u, v),$$

$$F^*(u, v) = v[a_{21}u^2 + a_{12}uv + a_{03}v^2 + \cdots] + \eta(u, v),$$

where the brackets are polynomials in $u, v$ of degree at most $\mu - 1$ and where $\omega, \eta$ are power series with initial terms of degree at least $\mu + 1$.

Under our hypotheses $a_{21}$ is not 0, for in that case $s = 0$ would be a double root of the equation in $s$.

By direct iteration of the formal series in their original form (§ 2) we find

$$u_k = u + k(\varphi_{20} u^2 + \varphi_{11} uv + \varphi_{02} v^2) + \cdots,$$

$$v_k = v + k(\psi_{20} u^2 + \psi_{11} uv + \psi_{02} v^2) + \cdots.$$

Hence we find, on differentiating as to $k$ and taking $k = 0$,

$$\delta u = \varphi_{20} u^2 + \varphi_{11} uv + \varphi_{02} v^2 + \cdots,$$

$$\delta v = \psi_{20} u^2 + \psi_{11} uv + \psi_{02} v^2 + \cdots.$$

Using the equations (11) and bearing in mind the fact that $Q$ commences with a constant term 1, we see that

$$\varphi_{20} = a_{21},\ \varphi_{11} = 2a_{12},\ \varphi_{02} = 3a_{03},$$

$$\psi_{20} = 0,\ \psi_{11} = -2a_{21},\ \psi_{02} = -a_{12},$$

These results may be summarized as follows:

*Let $T$ be a conservative transformation of type $II''$ for which there are three formally invariant curves with ordinary points and distinct tangents at* (0, 0). *If variables $u, v$ are properly chosen, any real formal curve of this sort can be made to osculate the $v$-axis to any order $\mu > 2$. Under these circumstances we have*

$$u_1 = u + a_{21} u^2 + 2a_{12} uv + \cdots, \qquad (a_{21} \neq 0),$$

$$v_1 = v[1 - 2a_{21} u - a_{12} v + \cdots] + \omega(u, v),$$

$$F^*(u, v) = v[a_{21} u^2 + a_{12} uv + a_{03} v^2 + \cdots] + \eta(u, v),$$

*where the bracketed expressions are polynomials of degree at most $\mu - 1$ and $\omega, \eta$ are power series with initial terms of degree at least $\mu + 1$.*

## § 33. Some inequalities in the hyperbolic case II''.

Let us take variables $u, v$ as above. We will take $\mu > 5$ and also $a_{21} > 0$. Furthermore $u, v$ are taken to be complex, of small moduli, and such that

(22) $$\Re(u) > 0, \quad |v| < |u|^2,$$

where $\Re(u)$ designates the 'real part of $u$'.

The series for $u_1$ gives us at once

(23) $$|u_1 - u - a_{21} u^2| < E^{(1)} |u|^3,$$

where $E^{(1)}$ is a definite positive constant. If we introduce a new variable $z = \frac{1}{u}$, (23) can be given the essentially equivalent but more convenient form

(23') $$|z_1 - z + a_{21}| < E^{(2)} |z|^{-1}.$$

Suppose now that we take $v = 0$, $\Re(z) > R$, $R$ a large positive quantity; in this case the inequalities (22) are satisfied. By iteration of $T$ we obtain $(z_1, v_1)$, $(z_2, v_2), \ldots$. Let us assume for the present that $\Re(z_l) > R$, $|v_l| < |z_l|^{-2}$ for $l = 0, 1, \ldots, n-1$ with $n > 0$.

From the inequalities (23') for $z, z_1, z_2, \ldots, z_{n-1}$ we infer

$$|z_l - z_{l-1} + a_{21}| < E^{(2)} R^{-1}, \qquad (l = 1, 2, \ldots, n).$$

These inequalities show that the real part of $z_l$ diminishes by approximately $a_{21}$ as $l$ increases by 1, while the imaginary component varies slowly. By combination we obtain

$$|z_l - z_j + (l-j) a_{21}| < (l-j) E^{(2)} R^{-1}, \qquad (0 \leqq j < l),$$

and thence

$$|z_j| > |z_l + (l-j) a_{21}| - (l-j) E^{(2)} R^{-1}.$$

But, since $z_l$ has a positive real part, we have

$$|z_l + (l-j) a_{21}| > (l-j) a_{21},$$

whence

$$l - j < \frac{1}{a_{21}} |z_l + (l-j) a_{21}|.$$

Replacing $l - j > 0$ by this greater value in the negative term of the inequality for $|z_j|$ we find

$$|z_j| > \left(1 - \frac{E^{(2)} R^{-1}}{a_{21}}\right) |z_l + (l-j) a_{21}|.$$

The polynomial of degree $\mu$, $\bar{F} = F^* - \eta$, has the same terms as those of the formal series $F^*$ out to terms of degree $\mu + 1$. Consequently if (22) holds we have

$$|\bar{F}_1 - \bar{F}| < E^{(3)} |z|^{-\mu-1}.$$

Thus, under the above hypotheses, we have

$$|\bar{F}_l - \bar{F}_{l-1}| < E^{(3)} |z_{l-1}|^{-\mu-1}, \qquad (l = 1, 2, \ldots, n).$$

Moreover $\bar{F} = 0$ since $\bar{F}$ is divisible by $v$. By combination we therefore obtain

$$|\bar{F}_l| < E^{(3)} \sum_{j=0}^{l-1} |z_j|^{-\mu-1}.$$

Using the preceding inequality for $|z_j|$ we find

$$|\bar{F}_l| < E^{(4)} \sum_{j=0}^{l-1} |z_l + (l-j) a_{21}|^{-\mu-1}$$

$$< E^{(4)} \sum_{j=1}^{\infty} |z_l + j a_{21}|^{-\mu-1}.$$

But this sum is less than

$$\int_0^\infty \frac{dn}{|z_l + n a_{21}|^{\mu+1}} = \frac{1}{|z_l|^\mu} \int_0^\infty \frac{dt}{\left|\frac{z_l}{|z_l|} + t a_{21}\right|^{\mu+1}},$$

where we have written $n = |z_l| t$. The final integral which appears has evidently as greatest value

$$\int_0^\infty \frac{dt}{|\sqrt{-1} + t a_{21}|^{\mu+1}}$$

inasmuch as $z_l$ has a positive real part. Thus finally we obtain

$$(24) \qquad |\bar{F}_l| < E^{(5)} |z_l|^{-\mu}, \qquad (l = 1, 2, \ldots, n),$$

where $E^{(5)}$ is a definite positive constant which does not increase as $R$ increases.

Furthermore, from the explicit form of $\bar{F}$ we have

$$|\bar{F}_1| \, |z_1|^2 > E^{(6)} |v_1|$$

so long as (22) holds for $(z, v)$. Thus we obtain

$$|v_l| < \frac{|\bar{F}_l|\,|z_l|^2}{E^{(6)}}, \qquad (l = 1, 2, \ldots, n).$$

Combining this inequality with (24) there results

$$|v_l| < \frac{E^{(5)}}{E^{(6)}} |z_l|^{-\mu+2}, \qquad (l = 1, 2, \ldots, n).$$

Since $\mu - 2 > 3$ the second inequality (22) continues to hold until $\Re(z_l) \leqq R$.

Our main result may be formulated as follows:

*If* $\mu > 5$, $a_{21} > 0$, $\Re\left(\frac{1}{u}\right) > R > 0$, *and if* $v = 0$, *then we have*

$$|v_l| < E^{(7)} |u_l|^{\mu-2} < |u_l|^2 \tag{25}$$

*for* $l = 1, 2, \ldots,$ *until* $\Re\left(\frac{1}{u_l}\right) \leqq R$.

It is evident that $\Re\left(\frac{1}{u_l}\right)$ ultimately becomes less than $R$.

## § 34. Further inequalities in the hyperbolic case II″.

The inequalities of § 33 are not sufficient for our purposes. It is necessary to evaluate $u_l$ more precisely than we have done.

To this end we write

$$w = \frac{1}{u} + \alpha \log u + \beta u + \cdots + \varkappa u^k,$$

where $k$ is arbitrarily large. Also let $\bar{u}$ stand for the series formed by the terms in $u_1$ which involve $u$ only. We propose to determine real quantities $\alpha, \beta, \ldots, \varkappa$ so that

$$|w(\bar{u}) - w(u) + a_{21}| < K |u|^{k+1}.$$

This condition will be met if

$$\left[\frac{1}{u}\left(\frac{u}{\bar{u}} - 1\right) + a_{21}\right] + \alpha \log \frac{\bar{u}}{u} + \beta(\bar{u} - u) + \cdots + \varkappa(\bar{u}^k - u^k)$$

is of the $(k+1)$th order in $u$. The term in brackets is a convergent power series in $u$ without constant term. The following term is a similar series beginning with a linear term $\alpha a_{21} u$; hence $\alpha$ can be so chosen that the first two terms form a power series without constant or first degree term. The third term is a similar series with leading term $2\beta a_{21} u^2$; hence $\beta$ can be so chosen that the first three terms form a power series beginning with terms of the third degree or higher. Continuing in this way we arrive at a determination of $\alpha, \beta, \ldots, \varkappa$ which yields an expression $w$ with the desired property.

Suppose now that we introduce the variable $w$ instead of the similar variable $z = \frac{1}{u}$ (§ 33), taking $\Re\left(\frac{1}{u}\right) > R$ and choosing the principal value of $\log u$ in the expression $w$. It is clear that $\Re(w)$ is large when $\Re(z)$ is large and that the region $\Re(z) > R$ corresponds to a region of similar character in the $w$-plane, with nearly vertical tangent throughout and crossing the $w$-axis far to the right of the origin in the $w$-plane. Hence, by DARBOUX's well-known theorem, the correspondence between these regions in the $w$-plane and $z$-plane is one-to-one and conformal. Moreover, in this part of the $w$-plane $\frac{w}{z}$ is nearly 1.

From the definition of $\bar{u}$ it appears that

$$|u_1 - \bar{u}| < E^{(8)} |v|,$$

and thence from the explicit expression for $\bar{F}$,

$$|u_1 - \bar{u}| < E^{(9)} |\bar{F}| \, |u|^{-2}$$

when (22) holds. Further, we have

$$|u_1 - \bar{u}| < E^{(7)} E^{(8)} |u|^{\mu-2},$$

when (25) holds, from which

$$|w(u_1) - w(\bar{u})| < E^{(10)} |u|^{\mu-4}.$$

If we recall the defining property of $w(u)$ and take $k \leqq \mu - 5$, we get finally

$$|w(u_1) - w(u) + a_{21}| < E^{(11)} |u|^{k+1}.$$

Applying this inequality successively for the sequence of values $(u, v)$ of § 34 (when (22), (25) hold), we obtain

$$|w_1 - w + a_{21}| < E^{(12)} |w|^{-k-1},$$

$$|w_2 - w_1 + a_{21}| < E^{(12)} |w_1|^{-k-1},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$|w_n - w_{n-1} + a_{21}| < E^{(12)} |w_{n-1}|^{-k-1}.$$

Thus in the complex $w$-plane each point $w, w_1, \ldots, w_n$ in the region $R(w) > R'$ falls approximately at a distance $a_{21}$ to the left of its predecessor. By the method used in § 33 it is apparent that the sum

$$\frac{1}{|w_l|^{k+1}} + \frac{1}{|w_{l-1}|^{k+1}} + \cdots + \frac{1}{|w|^{k+1}}$$

and $|w_l - w + l a_{21}|$ are of the order $|w_l|^{-k}$. Hence we have:

*Assume* $\mu > 5$, $k < \mu - 5$, $a_{21} > 0$, $\Re\left(\frac{1}{u}\right) > R$, *and* $v = 0$. *Write*

$$w = \frac{1}{u} + \alpha \log u + \beta u + \cdots + \varkappa u^k,$$

*where* $\alpha, \beta, \ldots, \varkappa$ *are suitably determined constants. Then we have*

$$|w_l - w + l a_{21}| < E^{(13)} |w - l a_{21}|^{-k} \tag{26}$$

*for* $l = 1, 2, \ldots,$ *until* $\Re\left(\frac{1}{u_l}\right) \leqq R$.

## § 35. The invariant curves in the hyperbolic case II″.

With the facts deduced in §§ 33, 34 in mind, we can readily prove the existence of an invariant curve.

Let us take $w, v$ as our variables where $w$ is restricted to the region of the $w$-plane which corresponds to $\Re\left(\frac{1}{u}\right) > R$.

Consider the two sequences of functions of $w$:

$$w,\ w_1(w + a_{21}, 0),\ w_2(w + 2a_{21}, 0), \ldots,$$

$$0,\ v_1(w + a_{21}, 0),\ v_2(w + 2a_{21}, 0), \ldots.$$

According to the inequalities (26), (25), we have

$$|w_l(w + la_{21}, 0) - w| < E^{(13)} |w|^{-k},$$
$$|v_l(w + la_{21}, 0)| \qquad < E^{(14)} |w|^{-\mu+2},$$

inasmuch as $\frac{w}{z}$ approaches 1.

Thus, for $\Re(w)$ sufficiently large and positive, the sequences $w_l$, $v_l$ remain bounded and define a closed set $\Sigma$ of limiting functions $w^*(w)$, $v^*(w)$ analytic within the same $w$ domain,[1] and restricted by the inequalities

$$|w^*(w) - w| < E^{(13)} |w|^{-k}, \quad |v^*(w)| < E^{(14)} |w|^{-\mu+2}.$$

Now we have

$$T(w_l(w + la_{21}, 0), v_l(w + la_{21}, 0)) = (w_{l+1}(w + la_{21}, 0), v_{l+1}(w + la_{21}, 0)).$$

Thus the transformed sequences of functions have as limiting functions

$$w^*(w - a_{21}), \ v^*(w - a_{21}).$$

In other words the totality $\Sigma$ is carried over itself by $T$, the change of parameter being $w_1 = w - a_{21}$.

Now let us suppose $w$ and $v$ to be real, with $w$ sufficiently large and positive. The transformation from $(w, v)$ to $(w_1, v_1)$ is then a real analytic transformation, and the totality of curves specified above are analytic with the positive $w$-axis as asymptotes. In fact these curves $\Sigma$ have contact of order $\mu - 2$ at least with the $w$-axis at $\infty$.

Let us return now to the prepared real $uv$-plane of § 32. The relation between $w$ and $u$ shows that in the real $uv$-plane the curves $\Sigma$ are defined for $u$ sufficiently small and positive, and are analytic curves with contact of order $\mu - 2$ at least with the $u$-axis at $(0, 0)$. On account of the mode of definition of the curves in the complex domain the inequality $|v| < E^{(15)} |u|^{\mu-2}$ holds uniformly for all of these curves.

It follows that the totality $\Sigma$ consists of only one curve.

In fact, consider the region $u > 0$ bounded by these curves and the line $u = d$. If there were more than a single such curve, such a region would necessarily arise and lie within the region

$$0 < u < d, \ |v| < E^{(15)} |u|^{\mu-2}.$$

[1] Cf. W. F. Osgood, *On the uniformisation of algebraic functions*, *Annals of Mathematics*, vol. 14, 1912—1913, pp. 152—154.

By the transformation $T$ we have

$$u_1 = u + a_{21}u^2 + \cdots.$$

We see at once that this region goes into another which includes it. For, the upper and lower boundaries of the region bounded by the $\Sigma$ curves are carried into themselves (the totality $\Sigma$ being invariant), while the line $u = d$ is moved to the right. This is impossible since $\iint Q\,du\,dv$ is invariant under $T$.

Passing back to the original $uv$-plane, we infer the existence of an analytic invariant curve ending at the invariant point and having contact of order of $\mu - 2$ with the corresponding formally invariant curve. When the curve is represented in the form $v = \varphi(u)$ say, $\varphi$ is continuous together with its derivatives of the first $\mu - 2$ orders for $u = 0$.

Now $\mu$ is an arbitrarily large integer. By increasing $\mu$ we cannot obtain further invariant curves, as is seen at once by a repetition of the above argument as applied to the region between such curves. Therefore, the invariant curve when represented in the form $v = \varphi(u)$ say yields a function $\varphi$ analytic for $u \neq 0$, continuous together with its derivatives of all orders for $u = 0$, and formally coinciding with the formally invariant curve.

All of the above only applies if $a_{21} > 0$. But if $a_{21} < 0$ then the analogous quantity for $T_{-1}$ is $-a_{21}$. Hence we can arrive at the same conclusion if $a_{21} < 0$ by considering $T_{-1}$ instead of $T$.

Clearly we can deal with the case $u < 0$ by merely rotating the axes in the prepared $uv$-plane through the angle $\pi$.

Let us call a real function $f(t)$ of a real variable $t$ *hypercontinuous* for $t = t_0$ if $f(t)$ is analytic for $t \neq t_0$, $|t - t_0| \leqq \delta > 0$, and continuous together with all of its derivatives for $t = t_0$. Similarly a curve is *hypercontinuous* at a point if its coördinates can be expressed as hypercontinuous functions of a parameter $t$. With these definitions we can summarize our results as follows:

*In the case $II''$ when there are three formally invariant curves with ordinary points and distinct tangents, one or all three of these will be real. To each such real formal curve corresponds a unique hypercontinuous curve through the invariant point which is invariant under $T$ and has the corresponding asymptotic representation at the invariant point.*

It is clear that the method above is not essentially limited to the discussion of real invariant curves but these are all we need to consider.

### § 36. Extension to the general hyperbolic case II'', II'''.

It is easy to see that the above work admits of an extension to the most general case II''.

Suppose first we fix attention on any real formally invariant curve $C$ in the case II'' which has an ordinary point at (0, 0).

We can begin as before (§ 32) by taking a prepared $uv$-plane in which this curve osculates the $u$-axis to order $\mu$.

The series for $u_1$ can be taken to contain a term $cu^p$ of least degree $p > 1$, where $p$ does not increase indefinitely with $\mu$. Otherwise, when the $u$-axis is made the invariant curve by a formal change of variables, it will be an invariant point curve, and such a curve has previously been observed to be analytic.

The series in $v_1$ is divisible by $v$ out to terms of degree $\mu + 1$ as before.

The formal series $F^*$ consists of a polynomial of degree at most $\mu$ divisible by $v$ with a leading term $cvu^p$ and a formal series with initial terms of degree at least $\mu + 1$.

Let us assume $c > 0$ and take

$$\Re(u) > 0, \ |v| < |u|^p.$$

Further let us introduce the variable $z = u^{-p+1}$. We find easily that, for $\Re(z) > R$,

$$|z_1 - z + (p-1)c| < E|z|^{-\frac{1}{p-1}},$$

where $E$ is a suitable positive constant, and we can carry through a discussion analogous to that contained in §§ 33, 34.

Introducing next a variable $w$,

$$w = \frac{1}{u^{p-1}} + \frac{\alpha}{u^{p-2}} + \cdots + \lambda \log u + \cdots + \sigma u^{k+1-p},$$

we can determine $\alpha, \beta, \ldots, \sigma$ so that

$$|w(\bar{u}) - w(u) + (p-1)c| < E'|u|^{k+1}$$

as in § 34, and can generalize the results there obtained.

The existence of a unique invariant curve can then be proved as in § 35.

When the real formally invariant curve has a 'cusp' at (0, 0), this can be reduced to an 'ordinary point' by a succession of changes of variables of the type $u = \bar{u}\bar{v}$, $v = \bar{v}$, and then an argument may be made like that carried through in §§ 33—35.

We will not stop to enter into details, but merely state the conclusion:

*In any case $II''$ to every real formally invariant curve corresponds a unique hypercontinuous invariant curve with the corresponding asymptotic representation.*

In the hyperbolic case $\mathrm{II}'''$, $T_q$ is of type $\mathrm{II}''$. Hence we infer:

*In the hyperbolic case $II'''$, $\theta = \frac{2p\pi}{q}$, the invariant curves under $T_q$ are of type $II''$ and their images are invariant as a set under $T$.*

## § 37. A general property in case II''.

*In case $II''$ under the restrictions of § 35 every point of the region $u^2 + v^2 \leqq \delta^2$ not on one of the real invariant curves is carried out of the region by iteration of $T$ or $T_{-1}$, while every point on one of these curves approaches the invariant point $(0, 0)$ by iteration of $T$ and is carried out of the region by iteration of $T_{-1}$, or vice versa.*[1]

There may be either three real invariant curves, or a single such curve.

Let us consider the first of these subcases. Here the neighborhood of $(0, 0)$ in the plane is divided into six parts, bounded by arcs of the invariant analytic curves. These six regions evidently go over into themselves under $T$ or $T_{-1}$. Let us consider a particular one of these regions, and let us first take tangents to the corresponding arcs of the two invariant curves at $(0, 0)$ as axes. The hypercontinuous invariant curves have equations $v = \varphi(u)$, $u = \psi(v)$ referred to these axes.

Make the further change of variables

$$U = u - \psi(v), \quad V = v - \varphi(u).$$

The right-hand members of these equations are continuous together with their partial derivatives of all orders in $u, v$, analytic except for $u = 0$ or $v = 0$. In the new variables the invariant curves appear as the $U$- and $V$-axis, while the region under consideration becomes the first quadrant in the $UV$-plane.

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[1] It is apparent from this result that no other invariant curves through the invariant point can exist.

LEVI-CIVITA (loc. cit.) proved that certain nearly points are carried away from the invariant point in this and other hyperbolic cases, showing that the point is unstable.

See also A. R. CIGALA, *Sopra un criterio di instabilità, Annali di Matematica*, ser. 3, vol. 11, 1905.

This further change of variables is formally of the type (15) so that we have (see § 32)

$$U_1 = U[1 + a_{21}\,U + 2a_{12}\,V + \cdots], \quad V_1 = V[1 - 2a_{21}\,U - a_{12}\,V + \cdots].$$

The factors in brackets are analytic for $U > 0$, $V > 0$ of course. We can readily show that these factors are continuous together with all of their partial derivatives for $U \geqq 0$, $V \geqq 0$.

In fact consider

$$\lim \frac{U_1}{U},$$

as a point $(u, v)$ approaches the invariant curve $U = 0$. By the ordinary rule for the evaluation of an indeterminate form the limit will be given by

$$\lim \left(\frac{\partial u_1}{\partial u} - \frac{d\psi(v_1)}{d v_1}\,\frac{\partial v_1}{\partial u}\right)$$

at the point in question. Hence the first bracket, and likewise the second bracket, are continuous functions for $U \geqq 0$, $V \geqq 0$. By successive steps of like nature all of the partial derivatives of the brackets may be shown continuous.

In the subcase under consideration there is a third real invariant curve in the second and fourth quadrants obtained by factoring formally

$$F^* = U\,V[a_{21}\,U + a_{12}\,V + \cdots].$$

We see that $a_{21}$ and $a_{12}$ are of the same sign (say positive), for this third invariant curve is given by

$$a_{21}\,U + a_{12}\,V + \cdots = 0.$$

Returning to the explicit form of $U_1$, $V_1$ above, we infer that $U(V)$ increases and $V(U)$ diminishes under iteration of $T(T_{-1})$ for any point in the first quadrant. If $(U, V)$ approaches a definite point $(\bar{U}, \bar{V})$ with $\bar{U} > 0\,(\bar{V} > 0)$ within the region $U^2 + V^2 \leqq \delta^2$, this point is necessarily invariant under $T$. But, inasmuch as there are no multiple factors of $F^*$ and thus no invariant point curves in the case at hand, there will be no invariant point in this region except (0, 0). Thus the first part of the italicized statement holds in this case.

The part of the statement which deals with the behavior of points on the invariant curves is obviously true in all cases. If it was not we should have isolated invariant points on these invariant curves lying arbitrarily near to (0, 0), and this is impossible.

We have next to discuss the subcase where there is a single real formally invariant curve. Let us take this curve into the $U$-axis by a transformation like that made above. We have then

$$U_1 = U + a_{21} U^2 + 2a_{12} U V + 3a_{03} V^2 + \cdots,$$
$$V_1 = V[1 - 2a_{21} U - a_{12} V + \cdots],$$

where the brackets stand for a type of functions similar to those in brackets above.

Here one has

$$F^* = V[a_{21} U^2 + a_{12} U V + a_{03} V^2 + \cdots].$$

The quadratic form in brackets is definite since there are a pair of conjugate formally invariant curves with distinct conjugate directions.

We find

$$U_1 V - V_1 U = V[3(a_{21} U^2 + a_{12} U V + a_{03} V^2) + \cdots].$$

This equation renders it apparent that $\tau = \tan^{-1} \frac{V}{U}$ varies continually in one sense under indefinite iteration of $T$ or of $T_{-1}$ as long as a point and its iterates remain near (0, 0). If $\lim \tau = 0$ or $\pi$, $\left|\frac{V}{U}\right|$ approaches 0, and the formulas for $U_1$, $V_1$ show that $|U|$ and $|V|$ vary in opposite senses. In this case $|U|$ increases and the point cannot remain near (0, 0).

Moreover the point cannot remain near (0, 0) in the contrary case. If $\lim \tau = \bar{\tau} \neq 0, \pi$ the above equation shows that $V$ approaches 0; in fact the variation in $\tau$ is of the first order in $V$. The variable $U$ must likewise tend to 0. But the geometry of the figure in the plane makes it clear that $\frac{V_1 - V}{U_1 - U}$ must approach $\tan \bar{\tau}$ indefinitely often, at the same time. If we recall that $\frac{V}{U}$ approaches this value also and employ the formulas for $U_1$, $V_1$, we find readily

$$\frac{-2a_{21} \tan \bar{\tau} - a_{12} \tan^2 \bar{\tau}}{a_{21} + 2a_{12} \tan \bar{\tau} + 3a_{03} \tan^2 \bar{\tau}} = \tan \bar{\tau}$$

whence

$$3 \tan \bar{\tau} (a_{21} + a_{12} \tan \bar{\tau} + a_{03} \tan^2 \bar{\tau}) = 0,$$

which is impossible.

### § 38. Extension to a more general case II''.

The same property holds in the most general hyperbolic case.

The kernel of the method of proof employed in § 37 depends on the use of a function which increases or decreases upon iteration of $T$. This method can be applied to a somewhat more general case than has been treated above, namely that in which all the real directions of formally invariant curves at (0, 0) are distinct. It is this case which we treat first. In dealing with the most general case (§ 39), however, we are obliged to employ less direct means.

Suppose that the property fails to hold, so that there are real points not on an invariant arc which remain in an arbitrarily small neighborhood of (0, 0) under indefinite iteration of $T$ (or of $T_{-1}$, if not of $T$).

There will then exist such points in some one of the regions into which the invariant arcs divide the vicinity of (0, 0), and it is upon such a region that we fix attention. For the present we assume there is more than a single real invariant curve.

By a change of variables $U = u - \psi(v)$, $V = v - \varphi(u)$ (§ 37), the region between the invariant boundary arcs may be taken into the first quadrant, in such wise that the invariant arcs become the $U$- and $V$-axes. The variables $U, V$ are analytic in $u, v$, save for $u = 0$ or $v = 0$ when $U, V$ are continuous together with all of their partial derivatives. Furthermore we have

$$U_1 = U\left[1 + \left(H + V\frac{\partial H}{\partial V}\right) + \cdots\right], \quad V_1 = V\left[1 - \left(H + U\frac{\partial H}{\partial U}\right) + \cdots\right],$$

where $UVH$ is the homogeneous polynomial of lowest degree $m \geqq 3$ in $F^*$, and where the brackets stand for functions analytic for $U \neq 0$, $V \neq 0$, and continuous together with all of their partial derivatives. The factors $U, V$ in $UVH$ correspond to the invariant axes. The factors of $H$ are either real linear factors $\alpha U + \beta V\,(\alpha\beta > 0)$ or complex linear factors, since there are no real invariant curves in the first quadrant of the $UV$-plane. Hence $H$ is of one sign, say positive, near (0, 0) and of the order $m - 2$ in $\sqrt{U^2 + V^2}$.

From the above equations and the facts stated we have

$$U_1 V - V_1 U = UV[mH + \cdots] > 0;$$

in consequence $\tau = \tan^{-1}\dfrac{V}{U}$ varies continually in one sense upon iteration of $T$ or of $T_{-1}$, and must approach a limit.

This limit must be o or $\frac{\pi}{2}$. In fact since there are no invariant points near (o, o) (invariant point curves correspond to multiple factors of $F^*$), the corresponding point would necessarily approach (o, o) in the contrary case. If $r > 0$ denotes the corresponding $\lim \frac{V}{U}$ the fraction

$$\frac{V_1 - V}{U_1 - U} = \frac{V}{U}\left(\frac{-\left(H + V\frac{\partial H}{\partial V}\right) + \cdots}{H + U\frac{\partial H}{\partial U} + \cdots}\right)$$

can be made nearly equal to $r$ with negative denominator and numerator; this is easily seen geometrically. Hence we have (compare § 37)

$$\lim \frac{H + V\frac{\partial H}{\partial V}}{H + U\frac{\partial H}{\partial U}} = -1$$

along this direction, whence $H = 0$ for $\frac{V}{U} = r$. This direction will correspond to a real formally invariant curve, which is absurd.

Also this limit is not o, for the formulas for $U_1$, $V_1$ show then that $|U_1|$, $|V_1|$ will vary in opposite senses and the point will recede from (o, o) along the $U$-axis. Similarly the limit is not $\frac{\pi}{2}$.

This completes the discussion when there is more than one real invariant curve. The argument is easily modified to meet the case of a single such curve (compare § 37).

*The property of § 37 holds therefore if the real tangent directions of the formally invariant curves are all distinct.*

## § 39. Extension to the general case II″, II‴.

We propose to deal in outline with the general case II″. As before, we assume the property not to hold, and show that a contradiction results.

The region under consideration is bounded by two invariant arcs which may or may not have the same tangent direction. An argument like that used above may be partially applied. If we form the difference $u_1 v - v_1 u$, it is given by a series beginning with a constant multiple of the homogeneous polynomial of lowest degree in $F^*$. But for directions within the region making this polynomial vanish there must be an even number of equal factors $\alpha u + \beta v$, since if there were an odd number there would be at least one corresponding real formally invariant curve and thus an invariant curve within the region, contrary to hypothesis. Hence $u_1 v - v_1 u$ preserves a constant sign save near these critical directions.

Moreover, if, under iteration of $T$, a point moves away from the vicinity of such a critical direction, it rotates in a constant sense about (o, o) to the vicinity of the next following critical direction (compare § 38).

Since there are only a finite number of such critical directions, there will then be points remaining in the indefinitely small vicinity of one such direction under indefinite iteration. It is upon such a critical direction and its neighborhood within the region under consideration that we now fix attention.

Let us take this critical direction along the positive $u$-axis, and make the change of variables.

$$u = \bar{u}, \quad v = \bar{u}\,\bar{v};$$

in the new variables the transformation $T$ then is readily found to have the form II''

$$\bar{u}_1 = \bar{u} + \cdots, \quad \bar{v}_1 = \bar{v} + \cdots.$$

The series $F^*(\bar{u}, \bar{u}\,\bar{v})$ is of course formally invariant, and in general the same methods of formal reckoning apply as earlier.

The first distinction to be noted is that the invariant integral $\iint Q\,du\,dv$ becomes $\iint \bar{u}\,Q\,d\bar{u}\,d\bar{v}$; the new quasi-invariant function $\bar{u}\,Q$ is analytic but vanishes at (o, o). The second distinction is that the line $\bar{u} = \text{o}$ in the $\bar{u}\bar{v}$-plane is evidently an invariant point curve corresponding to the invariant point (o, o) in the $uv$-plane.

Also there are infinitely many points $\bar{u} > \text{o}$ in the $\bar{u}\bar{v}$-plane which remain near (o, o) under indefinite iteration of $T$, and yet do not lie on an image of an invariant curve in the $uv$-plane.

The formal differential equations (9) in $\bar{u}_k, \bar{v}_k$ are clearly

$$\bar{u}_k Q_k \frac{d\bar{v}_k}{dk} = -\frac{\partial F^*}{\partial \bar{v}_k}, \quad \bar{u}_k Q_k \frac{d\bar{u}_k}{dk} = \frac{\partial F^*}{\partial \bar{u}_k},$$

where by $F^*$ is meant the series $F^*(\bar{u}, \bar{u}\,\bar{v})$.

In the $\bar{u}\bar{v}$-plane there are also certain critical directions, finite in number, along which the points above referred to cluster. In fact $\bar{u}_1\bar{v} - \bar{v}_1\bar{u}$ has the same initial terms as

$$\frac{1}{\bar{u}}\left[\bar{u}\frac{\partial F^*}{\partial \bar{u}} + \bar{v}\frac{\partial F^*}{\partial \bar{v}}\right].$$

As before, the lowest terms here form a homogeneous polynomial in $\bar{u}, \bar{v}$ of one sign or zero for $\bar{u} > 0$.

Repetition of the reasoning and further like changes of variables can now be made. It may be observed that the invariant point curve $\bar{u} = 0$ introduced at any stage is either eliminated by a further change of variable, or corresponds to the new $\bar{u}$-axis. Consequently the extraneous invariant point curves are either $\bar{u} = 0$ or $\bar{u} = 0$ and $\bar{v} = 0$.

Since there are only a finite number of formally invariant curves and the changes of variables used lower their order of contact, a stage must finally be reached at which either (1) there is no formally invariant curve not of extraneous type or (2) there is only one such curve. In case (2) it is clear that we may assume this curve to have an 'ordinary point' at (0, 0) with tangent direction distinct from that of an extraneous invariant curve; the changes of variables employed separate formally distinct curves and eliminate a 'cusp'. One further change of the same type will then make $\bar{u} = 0$ the only extraneous invariant curve.

Let us begin with case (1) when $\bar{u} = 0$ and $\bar{v} = 0$ are extraneous.

Since $\int\int \bar{Q}(\bar{u}, \bar{v})\, d\bar{u}\, d\bar{v}$ is an invariant integral it is clear that points $\bar{Q}(\bar{u}, \bar{v}) = 0$ go into points $\bar{Q}(\bar{u}, \bar{v}) = 0$. Thus $\bar{Q} = 0$ gives a set of real analytic curves invariant under $T$. Such curves are necessarily individually invariant inasmuch as $\bar{u} = 0$ is invariant. But there are no such curves save $\bar{u} = 0$ and $\bar{v} = 0$. Hence we have

$$\bar{Q}(\bar{u}, \bar{v}) = \bar{u}^l \bar{v}^m R(\bar{u}, \bar{v}), \quad (l > 0,\ m > 0),$$

where $R(0, 0) \neq 0$.

Also $F^* = 0$ yields formally invariant curves so that

$$F^* = \bar{u}^p \bar{v}^q G(\bar{u}, \bar{v}), \quad (p > l + 1,\ q > m + 1),$$

where $G$ is a formal power series with constant term.

Now by the formal differential equations (9) for this case we have

$$\bar{u}_k^l \bar{v}_k^m R \frac{d\bar{u}_k}{dk} = \frac{\partial}{\partial \bar{v}_k}[\bar{u}_k^p \bar{v}_k^q G], \quad \bar{u}_k^l \bar{v}_k^m R \frac{d\bar{v}_k}{dk} = -\frac{\partial}{\partial \bar{u}_k}[\bar{u}_k^p \bar{v}_k^q G].$$

Thus the series for $\bar{u}_1, \bar{v}_1$ have the form

$$\bar{u} + \bar{u}^{p-l}\bar{v}^{q-m-1}[qc + A], \quad \bar{v} + \bar{u}^{p-l-1}\bar{v}^{q-m}[-pc + B]$$

respectively, where $A$ and $B$ are power series without constant terms. Under iteration of $T$ or $T_{-1}$ either $|\bar{u}|$ increases and $|\bar{v}|$ decreases, or *vice versa*. Consequently a point which remains in the vicinity of (o, o) will approach a limiting point on the $\bar{u}$- or $\bar{v}$-axis, distinct from (o, o).

For definiteness suppose the point to lie in the first quadrant with $c > 0$. Such a point will then approach a point of the positive $\bar{u}$-axis near (o, o) under iteration of $T$. But the series above show that for such a point

$$\frac{\Delta \bar{v}}{\Delta \bar{u}} > -2K\bar{v}, \qquad (K > 0).$$

Thus $\bar{v}$ decreases less rapidly than if

$$\frac{d\bar{v}}{d\bar{u}} = -K\bar{v},$$

when by integration we find $\bar{v} = ce^{-K\bar{u}}$. Hence $\bar{u}$ cannot approach a limit as $\bar{v}$ approaches o but must increase indefinitely.

The case when only $\bar{u} = 0$ is extraneous admits of similar discussion.

Case (1) is now disposed of. Let us consider case (2).

By a formal change of variables of the type employed in § 37 we may take $\bar{u} = 0$ as the invariant point curve and $\bar{v} = 0$ as the other invariant curve. Formally then we are essentially in case (1), above disposed of. Indeed if the invariant curve is analytic no modification is required.

If the hypercontinuous invariant curve is not analytic there can be no corresponding factor of $\bar{Q}$, i. e. after the new change of variables we have

$$\bar{Q}(\bar{u}, \bar{v}) = \bar{u}^l R(\bar{u}, \bar{v}), \qquad (l > 0),$$

where $R(0, 0) \neq 0$.

Moreover, after this change of variables, $\bar{v}$ only occurs once as a factor of $F^*$. For a multiple factor gives an analytic invariant point curve (§ 20). Thus we have

$$F^* = \bar{u}^p \bar{v}\, G(\bar{u}, \bar{v}), \quad (p > l + 1),$$

where $G(0, 0) \neq 0$.

Consequently we have here

$$\bar{u}_1 = \bar{u}[1 + c\bar{u}^{p-l-1} + \cdots], \quad \bar{v}_1 = \bar{v}[1 - cp\bar{u}^{p-l-1} + \cdots],$$

where $c \neq 0$. The brackets stand for functions continuous together with all of their partial derivatives (see § 37), and with the asymptotic representation indicated at (0, 0).

But the points remaining near (0, 0) under indefinite iteration of $T$ or $T_{-1}$ lie approximately in the direction of the $\bar{u}$-axis; otherwise, before the above non-analytic change of variables was made, we might have removed the invariant curve by another change of variables, and thus have arrived at case (1).

As a result $|u|$ increases and $|v|$ decreases. The above formulas demonstrate this fact. This possibility is excluded since there are no invariant points near (0, 0) not on $\bar{u} = 0$.

Thus case (2) is also disposed of.

Since in case II''', $T_q$ is of type II'' we may state:

*The property of § 37 holds in the most general hyperbolic cases II'', II'''.*

## § 40. The hyperbolic case III', III''.

The non-specialized case III' is of hyperbolic type as appears from an inspection of (13). If we assume that the coefficient of $v^3$ in $F^*$ is not zero, we obtain a real formally invariant curve with cusp at (0, 0).

Now by the change of variables (see § 6)

$$u = \bar{u}\bar{v}, \quad v = \bar{v},$$

$T$ takes the form II''. By the use of the methods of § 32—39 we can infer.

*In the hyperbolic case III' to each real formally invariant curve corresponds a unique hypercontinuous curve which is invariant under $T$ and has the corresponding asymptotic representation at the invariant point.*

*In the hyperbolic case III'' the invariant curves under $T_2$ are of type II' and their images are invariant as a set under $T$.*

*The property of § 37 holds in the hyperbolic case III.*

## § 41. Invariant curves and the hyperbolic case.

We aim finally to show that a certain kind of converse to the above can be found:

*If $T$ is a conservative transformation $I'$, $II'$, $II''$, $III'$ for which* (o, o) *is an invariant point, and if there exists an invariant continuous arc ending at* (o, o) *for which* $\tan^{-1}\frac{v}{u}$ *remains finite, then the invariant point is hyperbolic and the invariant arc is an arc of a hypercontinuous invariant curve obtained above.*

If the invariant point can be proved hyperbolic the remainder of the statement can be demonstrated at once. In fact all points not on one of these hypercontinuous arcs leave a definite vicinity of (o, o) under both $T$ and $T_{-1}$, according to the general property developed above. But the invariant arc is carried into part of itself either by $T$ or $T_{-1}$. Therefore it must consist of points on one of the hypercontinuous arcs.

Let us take first the general case when $T$ is of type I$'$ or II$'$ at the invariant point (o, o) and let us suppose if possible that $T$ is elliptic at that point. Let $\alpha$, $\beta$ be the upper and lower limits of $\tan^{-1}\frac{v}{u}$ along the curve. These are invariant under $T$ of course. Hence the lines through (o, o) in these directions are carried into curves tangent to these respective lines at (o, o). Thus we have the phenomenon of invariant directions, which is absurd in case II$'$. Hence $T$ is of type I$'$, and (o, o) is a hyperbolic point.

If $T$ is of type II$''$ every direction through the invariant point is invariant. It is necessary here to have recourse to a more elaborate argument to show that $T$ is hyperbolic.

For definiteness we assume that $T$ carries the invariant arc into part of itself. Define $\alpha$ and $\beta$ as above. If $\alpha \neq \beta$ we can find a line $v = cu$ which intersects the invariant arc infinitely often near (o, o). But the image of this line lies on one side or the other of the line near o, at least near (o, o), since $T$ is analytic. Thus it is apparent that the total area between the line and invariant arc on the same side of the line is carried over into part of itself by $T$, which is absurd. Hence there is only a single limiting direction, i. e. $\alpha = \beta$, and the invariant arc does not meet the corresponding line $v = cu$ near the invariant point.

This direction corresponds to the real tangent direction of a formally invariant curve. Indeed the arguments employed in § 38 show that points not approximately in such a real invariant direction from the invariant point are rotated into such a direction under iteration of $T$, provided that the point remains near (o, o) as is the case for a point of the invariant arc.

This formally invariant curve which has a real tangent direction will correspond to a real formally invariant curve in general so that we have the hyperbolic case II$''$.

There remains the possibility, however, that we have an even number of formally invariant curves with real tangent directions but not with all coefficients real. Here further consideration is required.

Take the straight line from (0, 0) in the limiting tangent direction as the $u$-axis and write

$$u = \bar{u}, \quad v = \bar{u}\bar{v}$$

as in § 39. The $\bar{v}$-axis in the $\bar{u}\bar{v}$-plane is a line of invariant points under $T$, and the invariant arc approaches (0, 0) in this new plane. But this invariant arc does not cross the line of invariant points of course.

Repeating the argument above we infer that this arc approaches (0, 0) in a definite limiting direction in the $\bar{u}\bar{v}$-plane. But it was established in § 39 that such a limiting direction can only be along a real tangent direction to a formally invariant curve. Hence again we argue that the invariant arc has the direction tangent to $\bar{u} = 0$ or to a formally invariant curve, when another change of variables as above is in order.

At each stage these changes of variables diminish the number of real coefficients in the series for the formally invariant curve, until at last the first coefficient is not real and there is no invariant direction. This is impossible by our argument for case (1), § 39.

Similarly the case III' is disposed of.

## Chapter III. Elliptic invariant points. Stable case.

### § 42. Existence of closed invariant curves in the stable case.

In the integrable elliptic case there is a family of closed analytic curves $F^* = $ const. about the invariant point, each invariant under $T$ but not of the type above considered since these curves do not pass through the invariant point. Such an invariant point is stable of course.

A somewhat analogous property can be established in the non-integrable stable case. Let us understand by a *closed curve* the boundary of a simply connected open continuum in the finite plane, while regarding that plane as completed by the adjunction of a 'point at infinity'.

*In the stable case there exist an infinite number of invariant closed curves surrounding the invariant point and lying within any prescribed neighborhood of it.*[1]

[1] Compare the method of proof with a proof given by H. Poincaré, *Les methodes nouvelles de la mécanique céleste*, vol. 3, Paris, 1899, pp. 149—151.

Choose any arbitrarily small neighborhood of the invariant point. It is then possible to find a second neighborhood $r \leqq \delta < d$ such that any point of this latter neighborhood remains within the first under indefinite iteration of $T$. This is the direct statement of the property of stability.

Now the open region $r < \delta$ and all of its images under $T$ include $(0, 0)$ as an inner point and overlap. Let us speak of a point $P$ as *occluded* by this set of regions if it is possible to draw a regular closed curve lying entirely within the set and enclosing $P$ and $(0, 0)$. The set of occluded points $\Sigma$ is evidently an open simply connected continuum containing all of the set of regions.

The image continuum $\Sigma_1$ is also made up of points $\Sigma$; the curve enclosing $P$ and $(0, 0)$ is carried into a curve enclosing $P_1$ and $(0, 0)$, lying within the set of regions, and so $P_1$ is occluded, i. e. is a point of $\Sigma$.

Now $\Sigma$ cannot contain points not in $\Sigma_1$ since then $\iint Q\,du\,dv$ would be larger over $\Sigma$ then over $\Sigma_1$. Hence $\Sigma_1$ coincides with $\Sigma$. The boundary of $\Sigma$ is therefore an invariant curve lying in the arbitrary neighborhood $\delta \leqq r \leqq d$ and surrounding $(0, 0)$. Since $d$ is arbitrary there is clearly an infinitude of such curves, invariant under both $T$ and $T_{-1}$.

*Conversely, if there is an infinitude of such invariant curves about* $(0, 0)$, *that invariant point is clearly stable.*

## § 43. Some fundamental properties in the case II′, $l = 1$.

The cases I′, II′, $l = 1$, may be regarded as constituting the non-specialized case of an invariant point. In the second of these cases we have the first possibility of stability. The discussion of this elliptic case II′, $l = 1$ which we shall make (both in the stable and unstable case) will be based on certain properties established in the present paragraph.

Let us choose variables $u, v$ which osculate the normalizing variables $U, V$ of § 22, formula (19), to the order $\mu\,(\mu > 2)$. We will then have

$$
\begin{aligned}
u_1 &= u \cos[\theta + c(u^2 + v^2)] - v \sin[\theta + c(u^2 + v^2)] + P(u, v), \\
v_1 &= u \sin[\theta + c(u^2 + v^2)] + v \cos[\theta + c(u^2 + v^2)] + Q(u, v),
\end{aligned}
\tag{27}
$$

in which $P, Q$ are given by convergent power series which begin with terms of the $(\mu + 1)$th or higher degree.

We shall assume $c > 0$ for definiteness. It is clear that in the contrary case

$T_{-1}$ will be of this same form with $-c$ replacing $c$, so that our assumption is no essential restriction.

The particularly simple integrable case $P = Q = 0$ affords a clear insight as to the character of $T$. Circles with $(0, 0)$ as center are rotated into themselves through an angle $\theta + cr^2$, increasing with or decreasing with the radial distance $r$ according as $c > 0$ or $c < 0$.

This special case shows clearly the *vortical* nature of the transformation $T$ in the neighborhood of the invariant point.

It will be convenient for us to introduce polar coördinates $r, \varphi$. In these variables the equations above take an equivalent form

$$r_1 = r + R(r, \varphi), \quad \varphi_1 = \varphi + \theta + cr^2 + S(r, \varphi), \tag{28}$$

where

$$\begin{cases} R = \sqrt{r^2 + 2r(P \cos(\varphi + \theta + cr^2) + Q \sin(\varphi + \theta + cr^2)) + P^2 + Q^2} - r, \\ S = \tan^{-1} \dfrac{-P \sin(\varphi + \theta + cr^2) + Q \cos(\varphi + \theta + cr^2)}{r + P \cos(\varphi + \theta + cr^2) + Q \sin(\varphi + \theta + cr^2)}. \end{cases} \tag{29}$$

Since $P, Q$ are analytic power series in $r$ beginning with terms of degree $\mu + 1$ or higher, and with coefficients analytic in $\varphi$ with period $2\pi$, it is apparent that $R, S$ are continuous functions of $r, \varphi$ for $r \geq 0$, expansible as power series in $r$ with coefficients analytic in $\varphi$ of period $2\pi$. The first of these will begin with terms of at least the $(\mu + 1)$th degree in $r$, while the second will begin with terms of at least the $\mu$th degree.

These same considerations show that $R, S$ admit continuous partial derivatives in $r, \varphi$ of all orders.

The coördinates $r, \varphi$ will be regarded constantly as rectangular coördinates in an $r\varphi$-plane, so that the $r$-axis corresponds to the invariant point, and the half plane $r > 0$ to the $uv$-plane. Two points for which the coördinates $r$ are the same, but for which the coördinates $\varphi$ differ by a multiple of $2\pi$, are called *congruent*. Two congruent points represent the same point of the $uv$-plane.

Suppose now that we have a point in the $r\varphi$-plane and a direction at the point given by $\dfrac{d\varphi}{dr}$, which is the reciprocal of the slope. The corresponding reciprocal slope at the transformed point under $T$ is then given by $\dfrac{d\varphi_1}{dr_1}$. This quantity may be computed by means of (28) and has the value

$$\frac{d\varphi_1}{dr_1} = \frac{\dfrac{d\varphi}{dr} + 2cr + \dfrac{dS}{dr}}{1 + \dfrac{dR}{dr}},$$

where the indicated differentiation is directional in character.

From this equation there results

$$\frac{d\varphi_1}{dr_1} - \frac{d\varphi}{dr} = 2cr + \frac{\dfrac{dS}{dr} - \left(2cr + \dfrac{d\varphi}{dr}\right)\dfrac{dR}{dr}}{1 + \dfrac{dR}{dr}}.$$

If we evaluate the directional derivatives on the right in terms of the partial derivatives of $R$, $S$, we perceive at once that

$$\left|\frac{dR}{dr}\right| = \left|\frac{\partial R}{\partial r} + \frac{\partial R}{\partial \varphi}\frac{d\varphi}{dr}\right| < hr^{\mu}\left\{1 + \left|\frac{d\varphi}{dr}\right|\right\},$$

$$\left|\frac{dS}{dr}\right| < hr^{\mu-1}\left\{1 + \left|\frac{d\varphi}{dr}\right|\right\},$$

and thence

$$\left|\frac{d\varphi_1}{dr_1} - \frac{d\varphi}{dr} - 2cr\right| < h'r^{\mu-1}\left\{1 + \left(\frac{d\varphi}{dr}\right)^2\right\},$$

as long as $\left|\dfrac{d\varphi}{dr}\right| < \varepsilon r^{1-\mu}$ say.

That is, we may write

$$\frac{d\varphi_1}{dr_1} - \frac{d\varphi}{dr} = 2cr + \chi r^{\mu-1}\left\{1 + \left(\frac{d\varphi}{dr}\right)^2\right\}, \tag{30}$$

where $|\chi| < h'$ as long as $\left|\dfrac{d\varphi}{dr}\right|$ is restricted as stated.

Let us term the small sheaf of slopes $\dfrac{dr}{d\varphi}$ at any point in the $r\varphi$-plane for which

$$\left|\frac{dr}{d\varphi}\right| \leqq \frac{1}{\varepsilon} r^{\frac{\mu}{2}-1} \tag{31}$$

the *barred angle*. When (31) is not satisfied, the left-hand side of (30) is positive. Our conclusion may be formulated as follows:

*In the $r\varphi$-plane the transformation $T$ leaves $r$ unaltered to terms of order $\mu + 1$ in $r$, increases $\varphi$ by $\theta + cr^2$ to terms of order $\mu$, and rotates any direction not in the barred angle in a negative sense.*

An entirely analogous discussion of $T_{-1}$ shows that, if $\varepsilon$ be taken small enough in defining the barred angle, we have

*In the $r\varphi$-plane $T_{-1}$ rotates any direction not in the barred angle in a positive sense.*

A quantitative discussion of the amount of rotation of directions can be based on (30) and a similar equation for $\dfrac{d\varphi_{-1}}{dr_{-1}}$.

In particular we note that

*Under iteration of $T$ ($T_{-1}$) any direction at a point is ultimately rotated into or past a barred angle negatively (positively) if $r$ remains small.*

If possible assume this statement not to be true.

In the first place we must have $\lim r = 0$. For in the contrary case there is a rotation of definite negative amount occurring indefinitely often, and the statement must hold.

If we let $\varphi' = \dfrac{d\varphi}{dr}$ the inequality (30) gives

$$\Delta\varphi' > 2ar_1, \qquad (a > 0),$$

as long as $r$ remains small and $\varphi'$ does not lie in the barred angle. At the same time the formula (28) for $r_1$ shows that

$$|\Delta r| < br_1^{\mu+1}, \qquad (b > 0).$$

Consequently we have

$$\left|\frac{\Delta r}{\Delta\varphi'}\right| < \frac{b}{2a} r^{\mu}.$$

Hence $r$ diminishes as $\varphi'$ increases not more rapidly than if

$$\frac{dr}{d\varphi'} = -\frac{b}{2a} r^{\mu}.$$

But a direct solution of this differential equation establishes the fact that $\varphi'$ must increase indefinitely and be of order at least $r^{1-\mu}$ if $r$ is to approach zero. This is in contradiction with the hypothesis that the direction is not rotated into or past the barred angle.

Thus we see that the stated property holds for $T$. In the same way it may be proved for $T_{-1}$.

The following property is also useful:

*Given an arbitrary positive* $K$, *then for any point* $(r, \varphi)$ *with* $0 < r < \delta$ ($\delta$ *sufficiently small*) *we have*

$$\varphi_n > \varphi + n\theta + K$$

*for* $n \geqq N$ *until* $r_n > d$.

From the equations (28) we get the inequalities

$$|r_1 - r| < c' r^{\mu}, \quad \varphi_1 - \varphi > \theta + c'' r^2, \qquad (c' > 0,\ c'' > 0).$$

From the second of these inequalities there results

$$\varphi_n > \varphi + n\theta + c'' \sum_{j=0}^{n-1} r_j^2$$

as long as $r, r_1, \ldots, r_{n-1}$ are less than $d$. It suffices to prove that the sum on the right exceeds $\frac{K}{c''}$ before $r_n > d$, provided that $r$ is sufficiently small. Now from the first inequality we deduce

$$|r_p - r_q| < c' \sum_{j=q}^{p-1} r_j^{\mu}.$$

If $r_p$ and $r_q$ are the maximum $M$ and minimum $m$ of $r_j$, this yields for $\mu > 3$

$$M - m < c' M^2 \sum_{j=q}^{p-1} r_j^2,$$

whence

$$\sum_{j=q}^{p-1} r_j^2 > \frac{1}{c' M}\left(1 - \frac{m}{M}\right).$$

Consequently, if $r$ is sufficiently small and varies to a relatively much larger (but still small) value or to a relatively much smaller value, the corresponding sum $\sum_{j=q}^{p-1} r_j^2$ is very large and exceeds $\frac{K}{c''}$.

## § 44. Nature of the invariant curves.

Let us define a *regular neighborhood* of an elliptic invariant point (o, o) as a neighborhood such that any radial direction in the $r\varphi$-plane is rotated through a negative angle by $T$, and through a positive angle by $T_{-1}$.

Probably a hyperbolic point cannot lie in a neighborhood of this kind.

The reasoning of § 43 shows a regular neighborhood of this type to exist in case II′, $l = 1$.

The elliptic case does not arise in the general case II″ or III′. But a direct computation shows that, in case II′, $l$ finite, and in what may be termed the general elliptic subcases II″ and III′, a regular neighborhood exists.

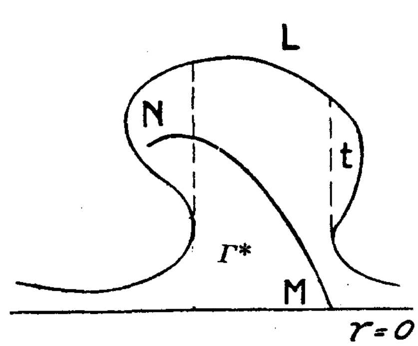
Throughout such a neighborhood we can evidently construct a barred angle through each point of the neighborhood such that directions outside the barred angle are rotated negatively by $T$ and positively by $T_{-1}$, and ultimately are rotated into or past the barred angle.

*In a regular neighborhood of an elliptic invariant point of type II′, $l = 1$, any invariant curve enclosing the invariant point meets every radius vector through the invariant point in only one point. If the barred angle in the plane be drawn at the corresponding point the curve lies entirely within it on either side in the vicinity of the point.*

In order to demonstrate this fundamental property of the invariant curves we make use of the $r\varphi$-plane employed above.

Let us suppose first that the invariant curve $L$ under consideration is defined by means of an inner simply connected open continuum $\Gamma$ containing (o, o) in the $uv$-plane.

If the first italicized statement is not true consider the continuum of points accessible from $r = 0$ along a perpendicular line $\varphi = \text{const.}$ in the $r\varphi$-plane without passing a point of the invariant curve itself. This open continuum $\Gamma^*$ forms all or part of the open continuum $\Gamma$ bounded by the invariant curve $L$ and $r = 0$ (see figure). The boundary of $\Gamma^*$ is evidently a closed curve made up of points of $L$ and open segments of lines $\varphi = \text{const.}$




If $\Gamma$ and $\Gamma^*$ coincide then either the first property is true or there exist one or more boundary segments $\varphi = \text{const.}$ of $\Gamma$ and

$\Gamma^*$. Now either $\Gamma^*$ lies to the right or to the left of such a segment. In the first case the tangent to the boundary makes an angle $\frac{\pi}{2}$ with the $\varphi$-axis. An application of $T_{-1}$ will carry this segment into another with tangent argument greater than $\frac{\pi}{2}$. But, on account of the form of the boundary, any tangent argument must be intermediate between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so that this is not possible. In the same way it may be concluded that $\Gamma$, $\Gamma^*$ cannot lie to the left of such a segment $\varphi = \text{const.}$ Hence there is no such segment, if $\Gamma$ and $\Gamma^*$ coincide.

In this case of coincidence every radial line must meet $\Gamma$ only once, as we wished to prove.

Let us now turn to the case when the two continua $\Gamma$ and $\Gamma^*$ are distinct.

Consider the part $\overline{\Gamma}$ of $\Gamma$ accessible from $r = 0$ along a regular simple curve in $\Gamma$ (such as $MN$ in figure above) with tangent argument never less than $\frac{\pi}{2}$. This part of $\Gamma$ evidently includes $\Gamma^*$, but can only coincide with $\Gamma^*$ if there are no bounding segments $\varphi = \text{const.}$ of $\Gamma^*$ which have a part (see region $t$ of figure above) of $\Gamma$ on the right. By the transformation $T_{-1}$ which increases every tangent argument which is equal to $\frac{\pi}{2}$ and does not diminish to $\frac{\pi}{2}$ any greater argument, the points of $\overline{\Gamma}$ are carried into points of $\Gamma$ which are still accessible from $r = 0$ along an auxiliary regular simple curve in $\Gamma$ with tangent argument greater than $\frac{\pi}{2}$, namely along the image of the auxiliary curve. Thus the continuum $\overline{\Gamma}_1$ forms part of $\overline{\Gamma}$. Hence $\overline{\Gamma}_1$ coincides with $\overline{\Gamma}$, since $T$ is conservative.

Consequently $\Gamma^*$ has no boundary segments $\varphi = \text{const.}$ with part of $\Gamma$ on the right. Similarly we can exclude the possibility of boundary segments $\varphi = \text{const.}$ of $\Gamma^*$ with part of $\Gamma$ to the left. Hence $\Gamma^*$ coincides with $\Gamma$. The first italicized statement has previously been demonstrated in this case.

It is now easy to show that the invariant curve lies within the barred angle in the vicinity of any one of its points.

Suppose for example it lies partially above the upper right arm of this angle. By sufficient iteration of $T_{-1}$ the direction of this arm rotates positively past the vertical, and it is intuitively clear that the radial line $\varphi = \text{const.}$ through this point will meet the invariant curve more than once, contrary to what has been proved above.

If the invariant curve is defined by means of an outer continuum, essentially the same argument leads us to the same conclusion.

## § 45. Rotation numbers.

Consider any closed set of points defined by an angular coördinate $\tau$ of period $2\pi$. Let us suppose a transformation given which takes each point of the set into a definite point $\tau_1$, in such wise that if $P$ precedes $Q$ (i. e. the $\tau$ of $P$ is less than the $\tau$ of $Q$) then $P_1$ precedes or coincides with $Q_1$, and also such that $\tau_1$ varies continuously with $\tau$. In particular if $P$ and $Q$ are the same point, represented by angular coördinates $\tau, \tau'$ differing by $2l\pi$, then $\tau_1, \tau_1'$ differ by $2l\pi$ also.

Consider now the difference $\tau_k - \tau$ for all points $P$. If $\tau$ increases through all the values of the set by $2\pi$ so does $\tau_k$. It follows that we have

$$a^{(k)} \leqq \tau_k - \tau \leqq b^{(k)}, \qquad (b^{(k)} < a^{(k)} + 2\pi)$$

In fact suppose $\tau_k - \tau$ is a minimum for $\tau = \tau^*$ and let $\tau$ vary by $2\pi$ from this minimum. Since $\tau_k$ increases but only by $2\pi$ altogether we have at the maximum of $\tau_k - \tau$

$$\tau_k - \tau < \tau_k^* - \tau^* + 2\pi,$$

which establishes the statement.

There is a number $\alpha$ such that for every $k$

$$\frac{a^{(k)}}{k} \leqq \alpha \leqq \frac{b^{(k)}}{k}.$$

For, if this is not the case, two intervals $\left(\frac{a^{(k)}}{k}, \frac{b^{(k)}}{k}\right)$, $\left(\frac{a^{(l)}}{l}, \frac{b^{(l)}}{l}\right)$ will fail to have a common point so that for instance

$$\frac{b^{(k)}}{k} < \frac{a^{(l)}}{l},$$

whence

$$l\, b^{(k)} < k\, a^{(l)}.$$

But, since $\tau_k - \tau \leqq b^{(k)}$ for any $\tau$, we have successively

$$\tau_k - \tau \leqq b^{(k)}, \quad \tau_{2k} - \tau_k \leqq b^{(k)}, \ldots \quad \tau_{lk} - \tau_{(l-1)k} \leqq b^{(k)},$$

whence by addition

$$\tau_{lk} - \tau \leqq l\, b^{(k)}.$$

Also since $\tau_l - \tau \geqq a^{(l)}$ for any $\tau$ we get similarly

$$\tau_{lk} - \tau \geqq k\, a^{(l)}.$$

These two inequalities and the inequality written above are contradictory.

The number $\alpha$ will be called the *rotation number* of the transformation $\tau_1 = f(\tau)$.[1] Evidently $\alpha$ measures what may be regarded as the mean angular advance of the points under this transformation, inasmuch as we have for any $\tau$ and $k$

$$|\tau_k - \tau - k\alpha| < 2\pi.$$

Since $k\alpha$ lies on the interval $(a^{(k)}, b^{(k)})$ some points advance more than $k\alpha$, and some by less than $k\alpha$.

When $\tau_1 = f(\tau)$ is a *one-to-one* transformation, then its inverse has evidently the negative rotation number $-\alpha$.

If $\frac{\alpha}{2\pi}$ is rational, say $\frac{\alpha}{2\pi} = \frac{p}{q}$, $p, q$ relatively prime integers, then we have

$$\frac{a^{(q)}}{q} \leqq \frac{2p\pi}{q} \leqq \frac{b^{(q)}}{q}$$

and hence

$$a^{(q)} \leqq 2p\pi \leqq b^{(q)}.$$

Consequently $\tau_q - \tau$ is exactly equal to $2p\pi$ for some $\tau$. It follows that, if $\frac{\alpha}{2\pi} = \frac{p}{q}$, there is at least one point $P$ for which $\tau$ increases by precisely $2p\pi$ upon $q$ iterations of the transformation.

## § 46. Rotation numbers along invariant curves.

Returning now to the invariant curves about an invariant point in the elliptic case II′, $l = 1$, it appears that for each such curve there is a definite rotation number, for $T$ yields a one-to-one, continuous transformation of each such curve into itself which preserves order.

*If such an invariant curve has a rotation number commensurable with $2\pi$, say $\frac{2p\pi}{q}$, it is made up of a finite number of analytic arcs ending at hypercontinuous points, invariant under $T_q$.*

---

[1] Introduced by Poincaré (loc. cit. § 8).

For mark off the invariant points under $T_q$ on this curve about (o, o). There exists at least one such point by the remark proved in § 45. The invariant curve near these invariant points forms then invariant curves in the sense of § 41. These points are thus hyperbolic and the invariant curves are hypercontinuous at the invariant points. But, by indefinite iteration of $T_q$ or $T_{-q}$, the part of the arc is carried into all of itself, since there are no invariant points on the arc save at its end points.

There are only a finite number of isolated invariant points on the invariant curve under $T_q$ or else the limit invariant point would have a non-analytic invariant curve of the type excluded in § 41.

Thus the statement is proved.

*If two such invariant curves have one or more points in common, the rotation numbers of the two curves are the same and of the form* $\frac{2p\pi}{q}$. *These common points and arcs are finite in number and invariant under* $T_q$.

If two invariant curves have points in common, but nevertheless are not coincident, there will be one or more continua included between them. Since the invariant curves are each cut only once by a radial line $\varphi = \text{const.}$ in the $r\varphi$-plane, these continua are of the form

$$f(\varphi) < r < g(\varphi), \qquad \varphi' \leqq \varphi \leqq \varphi''$$

where $f, g$ are continuous functions of $\varphi$ with $f < g$ for $\varphi' < \varphi < \varphi''$ and $f = g$ for $\varphi = \varphi'$ or $\varphi = \varphi''$.

Evidently the transformation $T$ carries any continuum of this type into another of the same type, included between the same two invariant curves. By further repetition of $T$ this continuum is carried into others which cannot all be distinct inasmuch as $\int\int Q\,du\,dv$ has the same value for any of them, and the total value of this integral taken over the complete neighborhood of the invariant point is finite. Thus after $q$ iterations the original continuum is carried into itself, and its two boundary arcs are carried into themselves. The end points of these arcs are therefore invariant under $T_q$.

If these invariant points are rotated $p$ times around the invariant point by $T_q$, clearly the rotation number belonging to either invariant curve is $\frac{2p\pi}{q}$. This demonstrates the first part of our statement.

Moreover it has been seen above that on such an arc there are a finite number of invariant points and invariant point arcs of the specified type under

$T_q$. Any point of an invariant arc terminated by invariant points will approach one end or the other under indefinite iteration of $T_q$ or of $T_{-q}$.

*If two such invariant curves are entirely distinct from one another, the rotation number of the one further removed from the invariant point is the greater, and both rotation numbers exceed $\theta$.*[1]

Consider the two curves in the $r\varphi$-plane. Since $T$ carries a line $\varphi = \text{const.}$ in a regular neighborhood into a curve cut by any line $\varphi = \text{const.}$ at most once we see that $\varphi_1$ for the outer curve exceeds $\varphi_1$ for the inner curve, and both are greater than the initial $\varphi$ by more than $\theta$. Hence we can affirm that $\varphi_1$ along the outer curve exceeds $\varphi_1$ along the inner curve by a definite amount $\delta$, and that this $\varphi_1$, in turn, exceeds the initial $\varphi$ by at least $\theta + \delta$.

This fact shows at once that the rotation nnmber of the outer curve is at least as great as that of the inner curve. For, points initially with the same $\varphi$ on the two curves are taken into points such that the $\varphi$ of the outer curve exceeds that of the inner curve by at least $\delta$ under indefinite iteration of $T$.

To establish our statement that the outer curve has the greater rotation number it is thus only needful to exclude the possibility that their rotation numbers are the same.

If the two rotation numbers are the same and rationally related to $2\pi$, then for some $q$ the transformation $T_q$ will have a rotation number $2p\pi$ ($p$ an integer), so that there will exist points on both curves which are carried into themselves by this transformation, $\varphi$ being increased by precisely $2p\pi$.

Suppose now that we follow the transformation $T_q$ by a rotation in the $uv$-plane through $2p$ complete negative revolutions. The resultant transformation will evidently be conservative with the same invariant area integral as before, and the two curves will appear as invariant curves with the rotation number o. It is also evident that by this resultant transformation points on the outer curve have their coördinate $\varphi$ increased by at least $\delta$ more than the increase in the like coördinate of the corresponding point on the inner curve.

Construct a curvilinear quadrilateral in the $r\varphi$-plane as follows. One vertex will be an invariant point of the inner curve under the resultant transformation and a second vertex the corresponding point on the outer curve. A third vertex will be the first invariant point on the outer curve with greater $\varphi$, and a fourth the corresponding point of the inner curve. The quadrilateral will then consist of the two radial segments $\varphi = \text{const.}$ through the two pairs of corresponding points, and the two arcs of the invariant curves included between them.

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[1] The assumption $c > 0$ is still made. This entails no specialization of course.

Consider the image of this quadrilateral under the resultant transformation. The invariant points remain fixed, but the two sides $\varphi = \text{const.}$ are carried into curves with tangent argument which is everywhere less than $\frac{\pi}{2}$, the argument before the transformation. In the image quadrilateral then, the curvilinear side through the invariant point on the inner curve lies to the right of the point, while the opposite side through the other invariant point on the outer curve lies to the left of this second invariant point. Consequently the quadrilateral has been taken into part of itself, the two sides formed of arcs of the invariant curves being carried over into part of themselves. This is impossible of course with a conservative transformation.

It is still easier to dispose of the case when both rotation numbers $\theta'$ are assumed to be equal but not rationally related to $2\pi$. Here again if a point on the outer invariant curve has a coördinate $\varphi$ not less than that of a point on the inner invariant curve, then, under indefinite iteration of $T$, it will always be true that the coördinate of any image of the first point will be greater by at least $\delta$ than the coördinate $\varphi$ of the like image of the second point.

Choose now a positive integer $q$ such that $q\theta'$ is less than some integral multiple of $2\pi$, say $2p\pi$, by a quantity less than $\delta$. It is always possible to do this precisely because $\theta'$ is incommensurable with $2\pi$. Every point on the inner invariant curve will then be advanced by less than $2p\pi$ under $T_q$. On the other hand, since the transformation $T_q$ has a rotation number $q\theta'$, it is always possible to choose a point of the inner curve which has its $\varphi$ coördinate increased by at least $q\theta'$ under $T_q$. The corresponding point of the outer curve then has its $\varphi$ coördinate increased by at least as much as $q\theta' + \delta$ i.e. by more than $2p\pi$. Hence the rotation number of the outer curve under $T_q$ is at least as great as $\frac{2p\pi}{q}$. This is impossible.

## § 47. On rings of instability.

If $C_1$ and $C_2$ are invariant curves in a regular neighborhood of an invariant point in the stable case II′, there may either be further invariant curves on the ring $C_1C_2$ or not. If there are, the ring $C_1C_2$ may be subdivided further into similar rings. Thus the neighborhood of the invariant point is divided into an infinite succession of *rings of instability* (reducing to single invariant curves in the integrable case). Each ring of this sort is bounded by two invariant curves

$C' C''$ and has no invariant curve upon it other than $C'$ and $C''$. We shall only prove:

*Let $C'$, $C''$ be entirely distinct invariant curves forming the boundary curves of a ring of instability. Then for any $\varepsilon > 0$ an integer $N$ can be assigned such that a point $P'$ exists within a distance $\varepsilon$ of any point $P$ of $C'$ (or $C''$) which goes into a point $Q'$ within distance $\varepsilon$ of any point $Q$ of $C''$ (or $C'$) in $n < N$ iteration of $T$ (or $T_{-1}$).*

In the contrary case points $P$ and $Q$ exist for which no point $P'$ can be found for some $Q'$ and any $N$. Consider a small circle with $P$ as center and of radius $\varepsilon$ in this case. By iteration of $T$ this region is carried into others, all lying partly within the ring, but not extending to $C''$. Consider the open continuum lying outside of $C'$ and occluded by all of these regions. This continuum is carried into all or part of itself by $T$. But it cannot be carried into part of itself. Hence the boundary curve is invariant under $T$. But this curve is distinct from $C''$ as well as $C'$, since it does not approach within distance $\varepsilon$ of the point $L$. Such an invariant curve does not exist in $C' C''$ by hypothesis.

## § 48. The other stable elliptic cases.

In the case II′, $l \neq 1$ but finite, the fundamental equalities (28) may be replaced by

$$r_1 = r + R(r, \varphi), \quad \varphi_1 = \varphi + \theta + cr^{2l} + S(r, \varphi),$$

where $R, S$ have the same character as before. Hence we see that a regular neighborhood of (0, 0) exists in this case. Here the arguments made above for the case II′, $l = 1$, apply without modification.

This is also true in the general elliptic subcases II″, III′ (see § 45).

*In the case II′, $l$ finite, in the general elliptic subcases II″, III′, and, more generally, whenever there exists a regular neighborhood of an elliptic invariant point, all of the properties of invariant closed curves established in case II′, $l = 1$, continue to hold.*

It is highly probable that an integer analogous to $l$ in the case II′ can be defined in all elliptic cases and that, if the notion of regular neighborhood be generalized appropriately, such a neighborhood exists when $l$ is finite. When $l = \infty$ it appears possible that an invariant linear family of series $G + cH$ exists.

The formal and analytical questions to be answered here are extremely important and interesting.

## §49. Invariant curves and the function $F^*$.

In case II', $l = 1$, the invariant closed curves investigated above are closely like the curves $F =$ const. where $F$ is a polynomial in $u, v$ obtained by breaking off $F^*$ at terms of high degree $\mu$.

To see this let us employ the variables $u, v$ osculating the normalizing variables of § 22 to high order. The formulas (28) and (31) show that the tangent directions along the invariant curve in the $r\varphi$-plane have a slope less than $\frac{1}{\varepsilon} r^{\frac{\mu}{2}-1}$. If the slope exceeds this magnitude the invariant curve will not lie within the barred angle at the corresponding point of the invariant curve.

On the other hand $F^*$ is given by $-\frac{1}{2}\theta(u^2+v^2)$ out to terms of degree $\mu+1$.

Combining these results and observing that $\mu$ is arbitrary, we find:

*In the stable case II', $l = 1$, if $F$ stands for the polynomial in $u, v$ formed by the terms of degree less than $\mu$ in $F^*$, then $|F' - F''| < k|F'|^{\frac{\mu}{2}}$ at any points $P'$, $P''$ of an invariant curve.*

*Evidently similar results hold in any case II', II'', III', III'' when a regular neighborhood exists.*

## § 50. Remark on the integrable elliptic case.

In the integrable elliptic case there is a family of closed analytic invariant curves $F^* =$ const. about the invariant point.

Since an area integral $\int\int Q(u, v)\,du\,dv$ is invariant under $T$, an integral of the form

$$\int\int P(\sigma, \tau)\,d\sigma\,d\tau$$

remains invariant, where $\sigma$ is a parameter varying from curve to curve, and $\tau$ an angular parameter varying from 0 to $2\pi$ as each curve is described. But $T$ has the form

$$\sigma_1 = \sigma, \quad \tau_1 = f(\tau, \sigma)$$

so that

$$P(\sigma_1, \tau_1)\frac{\partial f}{\partial \tau} = P(\sigma, \tau).$$

Consequently along any particular curve

$$\int_A^B P(\sigma, \tau)\, d\tau = \int_{A_1}^{B_1} P(\sigma_1, \tau_1)\, d\tau_1 .$$

Thus, if $\int P(\sigma, \tau)\, d\tau$ be taken as proportional to a modified parameter, the equations for $T$ take the form

$$\sigma_1 = \sigma, \quad \tau_1 = \tau + g(\sigma),$$

where $g$ is an analytic function of $\sigma$ for $\sigma \neq 0$.

A noteworthy special feature of this case now appears:

*In the integrable elliptic case if any invariant closed curve of the analytic family* $F^* = const.$ *has a rotation number* $\frac{2p\pi}{q}$, *then* $T_q$ *leaves every point of the curve invariant.*

It is obvious that the integrable case is not the general case, inasmuch as such an invariant point curve will not exist in general.

It would be a vital advance to be able to determine the distribution of the invariant curves in the non-integrable case by analytic tests. This appears to be possible only in the case of a rotation number commensurable with $2\pi$, when the invariant curve is hypercontinuous.

## Chapter IV. Elliptic invariant points. Unstable case.

### § 51. Existence of $\alpha$ and $\omega$ points. General case.

In the unstable case a neighborhood of the invariant point (0, 0), of the form $r \leq d$ say, can be so taken that, under indefinite iteration of $T$ or of $T_{-1}$, points arbitrarily near (0, 0) leave this neighborhood. It has previously been pointed out that this property holds for both $T$ and $T_{-1}$ if it holds for either (§ 42). We restrict attention to such a neighborhood $D$.

Let us fix attention upon $\omega$ *points* which remain in $D$ under indefinite iteration of $T$ and upon $\alpha$ *points* which remain in $D$ under indefinite iteration of $T_{-1}$. The two sets of points are clearly closed sets.

An $\omega$ point is evidently carried into an $\omega$ point by $T$, and also by $T_{-1}$ if its image under $T_{-1}$ lies in $D$. Similar results hold for $\alpha$ points.

*For an unstable elliptic point the point set of $\alpha(\omega)$ points has a connected subset $A(\Omega)$ extending from $r = 0$ to the boundary of $D$.*[1]

Take a very small neighborhood $r \leqq \delta$ of $(0, 0)$. Under iteration of $T_{-1}$ $N$ times ($N$ large), some point of the image extends out to $r = d$, by virtue of the instability. Within this image we can draw a curve from $r = 0$ to $r = d$ which remains within $D$ under $N$ iterations of $T$ of course. This curve cuts any closed curve about $(0, 0)$ in at least one point. By a limiting process, in which $N$ becomes larger and larger, we see that at least one $\omega$ point lies on any such curve. Similarly an $\alpha$ point lies upon it.

It is then intuitively evident that the italicized statement holds, inasmuch as a point $\alpha$ lies upon every such closed curve in $D$ which encloses the invariant point.

The following is evident:

*The connected sets $A(\Omega)$ are carried into parts of themselves by $T_{-1}$ $(T)$ and into all of themselves together with a part outside of $D$ by $T$ $(T_{-1})$.*

Let us term an unstable invariant point *regular* if there do not exist closed invariant curves in $D$ of which it is a boundary point.

The regular case embraces the general unstable elliptic case for $l$ finite and indeed any case in which the invariant point is surrounded by a regular neighborhood. Consider for simplicity the general case II$'$, $l = 1$. Here $r = 0$ functions as an invariant curve in the $r\varphi$-plane of polar coördinates. If another invariant curve has a point in common with this invariant curve (i. e. with $r = 0$) the rotation number is $\theta$ of course and commensurable with $2\pi$ (§ 46), and this is impossible in case II$'$ by definition.

*In the regular case the set $A(\Omega)$ connected with $r = 0$ tends uniformly to $r = 0$ under iteration of $T_{-1}$ $(T)$.*

Suppose if possible that this does not hold for the set $\Omega$. There exists then a quantity $\delta > 0$ such that for $n$ indefinitely large the set $T_n(\Omega)$ does not lie entirely within $r \leqq \delta$. Now any point of $T_n(\Omega)$ is an $\omega$ point which remains in $D$ under $n$ iterations of $T_{-1}$ also. Therefore, recalling that $\Omega$ and its images are connected with $r = 0$, we see that any curve within $r \leqq \delta$ and surrounding $r = 0$ has on it at least one point $P$ which remains in $D$ under indefinite iteration of $T$ and $n$ fold iteration of $T_{-1}$ ($n$ arbitrarily large). Hence, by a limiting process, we conclude the existence of a point of type $\omega$ and $\alpha$ on this curve.

[1] That is, a chain of $\alpha$ (or $\omega$) points, each point arbitrarily near its successor, extending from $r = 0$ to $r = d$, can be found.

Thus we arrive at a set of points $A\Omega$ common to $A$ and $\Omega$, and connected with $r=0$, which reaches out at least to $r=\delta$. This set is clearly carried into itself by $T$ and forms the inner boundary of an open continuum. Thus the set $A\Omega$ forms a closed invariant curve within the scope of the definition.

However, in the regular case for a sufficiently restricted region $D$ such an invariant curve does not exist. Thus we have reached an absurdity, so that the italicized statement under consideration must be true.

An obvious consequence of this property is that the content of the set of $A(\Omega)$ points connected with $r=0$ is $0$.

We easily infer the following fact to be true:

*In the irregular case there exists a connected set of points $A\Omega$ reaching to $r=0$ from $r=\delta>0$ if $\delta$ is small enough. The set $A(\Omega)$ tend uniformly toward the set $A\Omega$ under indefinite iteration of $T_{-1}(T)$.*

Although the introduction of $\alpha$ and $\omega$ points was not necessary in the study of the unstable hyperbolic points, it is instructive to note that the above definitions hold there (and even in the stable elliptic case). In particular the $\omega$ points are the points on the analytic invariant curves tending toward $(0, 0)$ or at least not away from $(0, 0)$ under iteration of $T$. Similarly the $\alpha$ points lie on the invariant analytic curves which tend toward $(0, 0)$ on iteration of $T_{-1}$ or at least do not tend away from it. Thus the sets $A$ and $\Omega$ are analytic curves.

In general these two sets have only a finite set of points in common and the points $A(\Omega)$ tend uniformly toward $(0, 0)$ under iteration of $T_{-1}(T)$. This is the *regular* case. The *irregular* case arises when invariant point curves are present. These constitute the points $A\Omega$.

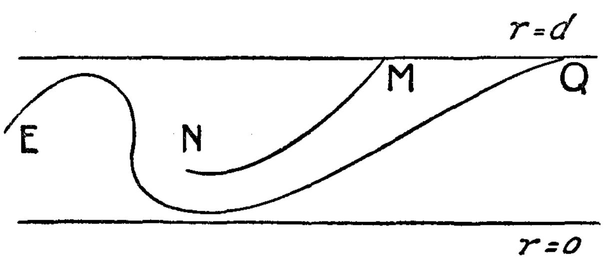
Thus our methods in the hyperbolic case have revealed the precise nature of the $\alpha$ and $\omega$ points — these fall along certain analytic branches ending at the invariant point. In the following paragraphs we shall extend the idea of branches to the unstable elliptic case.

## § 52. Further study of the regular case II′, $l$ finite.

Let us confine attention to the $r\varphi$-plane of polar coördinates and let us fix attention upon any point $Q$ which belongs to the set $A(\Omega)$ of points $\alpha(\omega)$ connected with $r=0$. At least one such point $Q$ lies upon $r=d$. Let $\Sigma$ denote the set of $\alpha(\omega)$ points connected with $Q$ for $r\geqq\delta>0$, where $\delta$ is an arbitrarily small quantity depending on the point of $\Sigma$ to be obtained.

If a continuum abutting on $r=d$ has the property that all of its points are accessible from $r=d$ along regular curves without double points and with tangent argument greater (less) than $-\frac{\pi}{2}$, its boundary will be said to be *left-handedly* (*right-handedly*) *accessible*. Thus the curve of the figure below ending at $Q$ is right-handedly accessible with $MN$ an auxiliary curve with tangent argument less than $-\frac{\pi}{2}$. With this definition we have the following:

*In the case II', l finite, the continuum of points accessible from $r=d$ along a regular curve to the left (right) of the set $\Sigma$ of $\alpha(\omega)$ points is right-handedly (left-handedly) accessible, and its boundary extends below $r=d$ indefinitely far to the left (right) (see figure).*




Take first $l=1$. The proof of the first part of this statement follows the line of argument already employed in § 44. If it is not true, there will exist above $\Sigma$ and on its left regions inaccessible from $r=d$ along regular curves without double points and with tangent argument less than $-\frac{\pi}{2}$. Since $T_{-1}$ rotates vertical lines positively such regions will be carried into similar regions toward $r=0$ by iteration of $T_{-1}$. This fact stands in contradiction with the existence of an invariant area integral.

In particular the point $Q$ is the point furthest to the right of the continuum so defined.

If now the second part of the statement fails to hold, the set $\Sigma$ does not extend indefinitely far to the left. Thus we have a connected set $\Sigma$ extending from $r=0$ to $r=d$ for which $\varphi$ is limited. But it has been shown that under iteration of $T_{-1}$ the range of variation of $\varphi$ along such a set increases indefinitely (§ 43). Hence an image exists which will cross $\Sigma$ from left to right. Here we allow the use of congruent images. But this shows that $\Sigma$ must be extended further to the left, since it contains all connected $\alpha$ points. This is absurd.

The properties stated in § 43 extends at once to the general case II', $l$

finite, so that our statement is true in this case also. It is probably true whenever a regular neighborhood surrounds the invariant point.

*Definition.* An unstable invariant point is *branched of $\alpha(\omega)$ type* if a continuos curve $C$ from $r=d$ to $r=0$ can be drawn in the $uv$-plane with no $\alpha(\omega)$ points on $C$ which are connected to the invariant point by $\alpha(\omega)$ points. In the contrary case it is *unbranched.*

*For the elliptic unstable case $II'$, $l$ finite, of $\alpha(\omega)$ branched type, the sets $A(\Omega)$ fall into a set of closed connected branches extending indefinitely far to the left (right) with $\lim r=0$ for $\lim \varphi=-\infty\ (+\infty)$, but only a finite distance to the right (left).*

Consider first the set $\Sigma$ and its congruent sets in the branched case. These divide the region $r\leqq d$ into component continua and their limit points. The continuous curve from $r=d$ to $r=0$ in the $uv$-plane which exists in the branched case becomes a continuous curve lying in one of these continua and approaching the line $r=0$. Since each of these continua lies to the right of an initial point such as $Q$ this auxiliary curve extends only a finite distance to the right and infinitely far to the left, approaching $r=0$. The *analysis situs* of the figure now renders it clear that each lower boundary curve of one of these continua tends uniformly toward $r=0$ as $\varphi$ becomes negatively infinite.

But the set $\Sigma$ cannot cross the auxiliary curve and its congruent images. Hence $\Sigma$ forms a closed connected set of $\alpha$ points having the properties specified for the branches.

Two $\alpha$ points $A$ and $B$ connected with $r=0$ through $\alpha$ points will be said to belong to the same $\alpha$ branch if no auxiliary curve $C$ can be drawn between these points to $r=0$. Otherwise two such points belong to *different* branches. If $B$ lies to the right of $C$ and $A$ to the left then we will say that the $B$ branch lies to the right of the $A$ branch.

$A$ branch clearly includes all $\alpha$ points connected with one of its points for $r\geqq\delta>0$.

It is apparent that if the $A$ branch lies to the left of the $B$ branch and the $B$ branch to the left of another branch, then the $A$ branch lies to the left of this branch also. Thus there is a cyclic ordering of the branches.

The existence of a single auxiliary curve $C$ ensures the existence of an infinitude of distinct branches, occurring in congruent sets. Such branches have clearly the form specified, inasmuch as these lie between congruent curves $C$.

By the transformation $T$ any $\alpha$ branch is carried into a part within $r\leqq d$ and certain other portions outside. However, there is clearly an $\alpha$ branch of the image wholly within $r\leqq d$. If there were more than one, these branches together with the parts outside of $r \leqq d$ would enclose an area, and this area would tend toward $r = 0$ upon iteration of $T$. There must then be only one image branch under $T$.

By the transformation $T_{-1}$ an $\alpha$ branch is evidently carried into a part of such a branch or all of it.

Similar remarks hold for the $\omega$ branches.

*In the branched elliptic unstable type $II'$, $l$ finite, the transformation $T$ $(T_{-1})$ carries an $\alpha(\omega)$ branch into such a branch (as specified). The cyclic order of the branches is preserved.*

The last part of the statement is obvious and the first part has just been proved.

It is clear that we have associated with a branched point an $\alpha$ and $\omega$ rotation number, say $\theta_\alpha$ and $\theta_\omega$, indicating the rotation of the branches.

The identity of the branches in no way depends on $d$. Upon iteration by $T$ each branch is carried into $r \leqq \delta$ where $\delta$ is arbitrarily small.

Thus $\theta_\alpha$ and $\theta_\omega$ in no wise depend upon $d$.

*For a branched invariant point in the unstable elliptic case $II'$, $l$ finite, the rotation number of the $\alpha$ $(\omega)$ branches is at least (at most) $\theta$.*

First, we shall prove that if $\theta$ is positive the rotation number $\theta_\alpha$ is positive or zero. In fact if $\theta > 0$ the branch $\Sigma$ with terminal point $Q$ on $r = d$ goes into a branch $\Sigma_{-1}$ entirely to the left of $Q$. Now $\Sigma_{-1}$ cannot lie below $\Sigma$ for then the region made up of points below $\Sigma$ and to the left of $Q$ is carried by $T_{-1}$ into a region lying under $\Sigma$ and thus forming only part of the region below $\Sigma$ and to the left of $Q$. In the $uv$-plane we have a corresponding area which is carried into part of itself, which is not possible. Thus $\Sigma$ is taken into a branch to its left and above it, which shows that $\theta_\alpha$ is positive or zero in this case.

Consider now the general case and suppose if possible $\theta_\alpha < \theta$. Find an integer $m$ such that for an integer $k$

$$m\theta_\alpha < 2k\pi < m\theta.$$

Consider the transformation $T'$ obtained by following $T_m$ by a shift of the plane $2k\pi$ to the left. For $T'$ the rotation number of the invariant point is $\theta' = m\theta - 2k\pi$ and is positive, while $\theta'_\alpha = m\theta_\alpha - 2k\pi$ is the rotation number of the branches and is negative. But $T'$ satisfies all the conditions imposed on $T$ for $d$ sufficiently small. Hence we are brought back to the case proved impossible in the first place.

In the branchless case greater complexity exists. A discussion of this case is much to be desired.

### § 53. Interrelation of $\alpha$ and $\omega$ points.

Consider the $\alpha$ and $\omega$ points of $D$ which are connected with $r = 0$. The first set $A$ forms a closed connected set reaching from $r = d$ to $r = 0$. The second $\Omega$ has the same properties. In the $r\varphi$-plane it appears at once that the two sets must intersect infinitely often.

Let us develop briefly the proof of this fundamental fact. The basic reason which permits this conclusion is that if we have $\alpha$ and $\omega$ curves of the type $\Sigma$ specified in § 52, one to the left of a point $Q$ of $r = d$ and the other to the right of a point $P$ of $r = d$, and if $P$ is taken to the left of $Q$, there lies between $P$ and $Q$ a continuum with boundary points all of type $\alpha$ or $\omega$. Thus there are points of this closed boundary of both types, i. e. belonging to the boundaries of both regions.

*In the branched elliptic unstable case $II'$, $l$ finite, every $\alpha$ branch intersects every $\omega$ branch infinitely often.*

*In the unbranched case also the $A$ and $\Omega$ sets have infinitely many points in common.*

In the branched case then we have what may be described as a network of $\alpha$ and $\omega$ branches. In the general case it is clear that the $A$ and $\Omega$ sets together separate $r = 0$ from $r = d' > 0$ for $d'$ sufficiently small.

The lack of definiteness in our general conclusion for the elliptic case is in startling contrast with that found in the hyperbolic case. I believe, however, that this corresponds to the extremely general character of the situation. A fundamental distinction between the two cases is this: the natural domain in the hyperbolic case is the complex variable; in the elliptic case, the real.

## Chapter V. Recurrent point groups.

### § 54. Point groups.

Consider an analytic closed surface $S$ of any genus and for the present let $T$ denote any one-to-one, direct, analytic transformation of this surface into itself. The problem which we attack is that of determining the behavior of various classes of points of $S$ under indefinite iteration of $T$ and $T_{-1}$. Hitherto we have only considered points in the vicinity of an invariant point.

Let $P$ be any point of $S$ and consider the infinite sequence of points

$$\ldots, T_{-2}(P), T_{-1}(P), P, T(P), T_2(P), \ldots,$$

which will be termed the *point group* of $P$. If two members of this sequence are the same, say if $T_\alpha = T_\beta$, $\alpha < \beta$, then we have $T_{\beta-\alpha}(P) = P$. Here the point $P$ will periodically iterate through a set of $\beta - \alpha$ distinct points under $T$. Thus by considering $T_{\beta-\alpha}$ instead of $T$ we may apply our earlier results to the study of the points near this set of points under iteration of $T$.

The existence of infinitely many point sets of this particular type and of special properties may be considered as established by general theorems concerning the invariant points of such surfaces.[1] A set of points of this type forms a *periodic point group*.

Every limit point of the set $P, T(P), T_2(P), \ldots,$ will be termed an *$\omega$ limit point* of $P$, and every limit point of the set $P, T_{-1}(P), T_{-2}(P), \ldots,$ will be termed an *$\alpha$ limit point*. A point is counted as often as it appears. In the periodic case the finite set of points are all $\alpha$ and $\omega$ limit points, and there are no others.

In all cases the limit points of either class form a closed point set.

*The set of $\alpha$ ($\omega$) limit points of $P$ form a set of complete point groups. The distance of $T_k(P)$ from this limit set approaches* 0 *for lim* $k = -\infty$ $(+\infty)$.

Let $Q$ be an $\alpha$ limit point which $T_k(P)$ approaches for $k = k_1, k_2, \ldots$. Evidently $T_{k+1}(P)$ will approach $T(Q)$ at the same time. That is to say $T(Q)$, and likewise $T_{-1}(Q)$, are $\alpha$ limit points if $Q$ is. By repetition of this argument we infer that all points of the point group of $Q$ are $\alpha$ limit points.

To establish the second part of the theorem we employ an indirect argument. If $T_k(P)$ did not approach the set of $\alpha$ limit points uniformly for $\lim k = -\infty$ it would be possible to select an infinite set of negative values of $k$ such that $T_k(P)$ would be distant from any limit point of $P$ by at least a definite positive quantity $d$. There would then be at least one limit point $L$ of this set, and this point would be at least $d$ distant from any $\alpha$ limit point. By definition however $L$ is an $\alpha$ limit point, so that a contradiction results.

## § 55. Recurrent Point Groups.

Consider now an arbitrary closed set $\Sigma$ of complete point groups. It was observed above that the $\alpha$ or $\omega$ limit points form such a set of point groups.

[1] See my paper first cited.

More generally, if we take any set of complete point groups and adjoin to it the limit points, we obtain an enlarged set $\Sigma$.

If a set $\Sigma$ contains no proper closed subset $\Sigma'$ of the same type we shall say that $\Sigma$ is a *minimal set*. In this case if $P$ is any point of $\Sigma$ its $\alpha$ (or $\omega$) limit points form a closed set which must therefore coincide with $\Sigma$.

Any complete point group in a minimal set forms a *recurrent* point group.

The simplest type of recurrent point group is the periodic type referred to above.

In all cases but this simplest one, in which $\Sigma$ has only a finite number of points, a minimal set $\Sigma$ consists of a perfect point set. For suppose a closed minimal set to have an isolated point. This point is its own limit point under $T$ or $T_{-1}$. Hence this point is a member of a periodic point group, which must constitute the minimal set.

*In order that a point group generated by $P$ be recurrent it is necessary and sufficient that for any positive quantity $\varepsilon$, however small, there exists a positive integer $k$ so large that any $k$ successive points in the point group of $P$,*

$$T_m(P),\ T_{m+1}(P),\ \ldots,\ T_{m+k-1}(P),$$

*have representatives within distance $\varepsilon$ of every limit point of $P$.*

This condition is necessary.

If not there is a recurrent point group generated by $P$, and a positive $\varepsilon$ such that sequences of $k$ points ($k$ arbitrarily large) can be found no point of which comes with distance $\varepsilon$ of some limit point $Q$ of $P$. As $k$ increases the point $Q$ has at least one limit point $Q'$ and thus it is clear that for a properly taken set of sequences no point lies within distance $\frac{\varepsilon}{2}$ of $Q'$. Take $k$ odd and consider the middle point $L$ of such a sequence. It and its $\frac{k}{2}-1$ iterates under $T$ and $T_{-1}$ lie at a distance at least $\frac{\varepsilon}{2}$ from $Q'$. Consequently for a limiting position $L'$ of $L$ we infer that every point of the complete point group of $L$ is at distance at least $\frac{\varepsilon}{2}$ from $Q'$. Hence $L'$ defines a closed set of point groups lying within the closed minimal set defining the given recurrent motion, but forming only part of it, and in particular not containing $Q'$. This is absurd.

To prove the condition sufficient we note first that the set of $\alpha$ and $\omega$ limit points of a point group satisfying this condition must coincide. We need only to take $m > 0$ in the arbitrary set to see the truth of this fact. Call the set of these common $\alpha$ and $\omega$ limit points $\Sigma$.

If the set $\Sigma$ is not minimal it would contain a proper subset $\Sigma'$ of the same sort to which some point $Q$ of $\Sigma$ would not belong. Now, when one of the set of points $P$, $T(P)$, $T_2(P)$, ..., approaches sufficiently near to a point of $\Sigma'$ it will remain very near to this closed set for an arbitrarily large number of iterations under $T$, and so will not approach $Q$; it is to be kept in mind that $\Sigma'$ is a closed set of complete point groups. Thus the assumed condition would not be satisfied by the point group generated by $P$.

Hence $\Sigma$ is minimal and the point group of $P$ is recurrent.

### § 56. The general point group and recurrent point groups.

The importance of the complete point groups of recurrent type for the consideration of the general point group is evidenced by the following result:

*There exists at least one recurrent point group in the $\alpha$ ($\omega$) limit point group of any given point P.*

Let $\Sigma$ denote the closed set of $\alpha$ limit points. We need to prove that the set $\Sigma$ contains a minimal subset.

Divide the surface of $S$ into a large number of small regions of maximum span not greater than $d$, an assigned positive constant. Among the points of $\Sigma$ there will be one which enters a least set $S'$ of regions of $S$ under indefinite iteration of $T$ and $T_{-1}$. Let $\Sigma'$ be the corresponding closed set of complete limit point groups. This set is part of $\Sigma$ and lies wholly in the same regions $S'$. Divide $S'$ into regions of maximum span $\frac{d}{2}$. Among the points of $\Sigma'$ there will be one which enters a least set $S''$ of regions of $S'$ under indefinite iteration of $T$ and $T_{-1}$. Define $\Sigma''$ as the closed set of complete limit point groups, which is part of $\Sigma'$.

Proceeding in this way we determine an infinite sequence $\Sigma', \Sigma'', \dots$ of closed sets of complete point groups lying wholly upon $S', S'', \dots$ respectively. Now let $P^{(n)}$ be any point whatever of $\Sigma^{(n)}$ on $S^{(n)}$ and let $\overline{P}$ denote a limit point of the set $P', P'', \dots$ The point $\overline{P}$ belongs to $\Sigma$ of course since it is a limit point of points of $\Sigma$. Furthermore, since $P^{(n)}$ is contained in $\Sigma, \Sigma', \Sigma'', \dots$, the limit point $P$ lies on all of the regions $S, S', S'', \dots$. Likewise all of its images under $T$ or $T_{-1}$ lie on these regions. Thus the complete point group generated by $P$, and its $\alpha$ and $\omega$ limit points, do the same.

Moreover, every point lying in every region $S', S'', \dots$ is an $\alpha$ and $\omega$ limit point of $\overline{P}$. Otherwise for large positive (or large negative) $k$, $T_k(\overline{P})$ does not approach some point $Q$ in $S'$, $S''$, .... Hence it is apparent that the set of points $\overline{P}$, $T(\overline{P})$, ... will not enter into some one of the regions $S^{(k)}$, namely the particular one containing $Q$. But this set of points has a set of $\omega$ limit point groups, each with a representative in every one of the minimum set of subregions which make up $S^{(k)}$. Thus a contradiction results.

The same argument shows that any point $P$ lying in every region $S'$, $S''$, ... has this complete set as its set of $\alpha$ or $\omega$ limit points. In other words the set of points common to $S'$, $S''$, ... form the desired minimal set.

The following further result shows that either a point $P$ generates a recurrent point group under $T$ or else that it successively approaches and recedes from such recurrent point groups:

*For any $\varepsilon > 0$ there exists a $k$ so large that any sequence of $k$ points $P, T(P), \ldots, T_k(P)$ contains at least one point within distance $\varepsilon$ of a recurrent point group.*

The proof is immediate.

If the theorem is not true it is possible to obtain points of this type not coming within distance $\varepsilon$ of any recurrent point group for $k$ arbitrarily large. Let then $Q$ denote the middle point of such a set ($k$ being taken odd). If $\overline{Q}$ is a limit point of points $Q$ for $\lim k = \infty$ evidently the complete point group generated by $\overline{Q}$ has none of its points within distance $\varepsilon$ of any recurrent point group. But the set of $\alpha$ and $\omega$ limit points of $Q$ contains a minimal set. Thus a contradiction appears, since every point group in a minimal set is by definition recurrent.

## § 57. Continuous recurrent point groups.

Recurrent point groups $\Sigma$ may be classified as follows: if a point $P$ of $\Sigma$ exists such that all sufficiently near points of $\Sigma$ are connected to $P$ through $\Sigma$ then $P$ is of *continuous type;* in the contrary case $\Sigma$ is of *discontinuous type.*

From every standpoint the first type is the simpler.

There are two extreme types of continuous recurrent point groups, namely the zero-dimensional or periodic type in which $\Sigma$ consists of a finite set of isolated points, and the two-dimensional type in which $\Sigma$ fills an area. But this area has no boundary since these boundaries would form a closed subset of point groups of the minimal set $\Sigma$. Hence this area comprises all of $S$. Consequently $S$ has no invariant points under $T$, and so has the connectivity of a torus, at least *if* $T$ can be generated by a deformation.[1]

[1] See my paper first cited.

If $\varphi, \psi$ are angular coördinates on a torus and if $\alpha, \beta$ are incommensurable with $2\pi$ and with each other, a transformation $T$ of this type is defined by

$$\varphi_1 = \varphi + \alpha, \quad \psi_1 = \psi + \beta.$$

Thus the two-dimensional type exists. The precise structure of this type is not here determined further.

The remaining one-dimensional continuous type arises when some but not all of the points of $S$ near $P$ belong to $\Sigma$, and are connected with it through nearby points of $\Sigma$.

Thus $\Sigma$ falls into a set of connected subsets, which undergo some sort of permutation under $T$ or $T_{-1}$. Since $P$ is carried into its own immediate neighborhood on sufficient iteration of $T$, the connected set containing $P$ is carried into itself after a finite set of iterations.

It appears then that $\Sigma$ consists of a set $\Sigma'_0$ containing $P$, and of its distinct successive images $\Sigma'_1, \Sigma'_2, \ldots, \Sigma'_{k-1}$, while $\Sigma'_k$ coincides with $\Sigma'_0$. Let us consider then $T_k$, which carries $\Sigma'_0$ into itself, and for which $\Sigma'_0$ is also a recurrent point group.

Now $\Sigma'_0$ is either a simply or multiply connected point set. By using a known theorem due to BROUWER[1] we will prove that it must be multiply connected. For, if not, $\Sigma'_0$ forms a simply connected set on a part of $S$ which can be represented in the plane, and is invariant under $T$. Moreover this set has no inner point, for the boundaries would then constitute a smaller closed set of complete point groups. Consequently by the theorem referred to there exists an invariant point of $\Sigma'_0$, which is absurd.

Hence the set $\Sigma'_0$ is multiply connected.

If $S$ has the connectivity of the sphere then $\Sigma'_0$ divides the surface of $S$ into two or more parts. But in one of these there is a point invariant under $T$ by another theorem also due to BROUWER.[1] Consequently its boundary is invariant under $T$ and must constitute all of $\Sigma'_0$. Here then $\Sigma$ consists of a finite set of closed two-sided curves, all outside of one another.

More generally, consider the neighborhood of a point near $\Sigma$ but not on it and follow along near $\Sigma$ until a complete circuit is made. The boundary so outlined is carried into itself or into one of a finite number of similar boundaries

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[1] *Continuous one-one transformations of surfaces in themselves, Proceedings of the Section of Sciences, Koninklijke Academie van Wetenschappen te Amsterdam,* vols. 11—15 (1908—1912). In the last part of this paper BROUWER develops the notion of *class* of a transformation, given later by myself in the paper first cited without knowledge of his work.

under $T_k$. Hence $T_{kl}$ carries this boundary and similarly its images under $T_{2k}$, $T_{3k}, \ldots, T_{(l-1)k}$ into themselves. Each boundary is thus recurrent under $T_{kl}$, and if two boundaries have any points in common all of their points are in common. Since all of the boundaries form a set which hangs together the images can consist only in the boundary of a single closed two-sided curve.

Thus continuous recurrent point groups lie in minimal sets which are either made up (1) of a finite set of points, (2) of a finite set of closed two-sided curves on $S$, or (3) of all the points of $S$.

In the one-dimensional case a single angular variable and a definite rotation number arise. A fundamental question is whether a similar representation in the two-dimensional case, by means of two angular variables and two characteristic rotation numbers, is possible.

## § 58. Discontinuous recurrent point groups.

An immediate division of the types of discontinuous recurrent point groups is possible. In the first case no point $P$ of the minimal set $\Sigma$ is connected with any other point through $\Sigma$; this is the *totally discontinuous type*. In the second this is not the case; here we have the *partially discontinuous type*.

For the second case $\Sigma$ falls into connected sets which are permuted among themselves by $T$ just as the points are in the first case. The existence of this second category of recurrent point groups is doubtful when $T$ has the properties which we have assumed. On the other hand the totally discontinuous type of recurrent point groups exists in important cases.

Inasmuch as analytic weapons are lacking we content ourselves merely with some examples and with making an attempt at classification in the totally discontinuous type.

Let $f(t)$ be a continuous increasing function of such that $f(t) - t$ is periodic of period $2\pi$. Then $t_1 = f(t)$ defines a one-to-one continuous direct transformation of a circle (on which $t$ is an angular coördinate) into itself. This is associated with a definite rotation number $\theta$ and defines at least one recurrent point group on the circle, which need not coincide with the whole circle. Its minimal set is represented by a perfect nowhere dense point set on the circle.[1] We limit attention to the corresponding values of $t$.

[1] See G. D. Birkhoff, *Quelques théorèmes sur le mouvement des systèmes dynamiques, Bulletin de la Société Mathématique de France,* vol. 40, 1912.

The reader will observe the complete analogy between the recurrent motions of that paper and recurrent point groups.

It may now be possible to represent the given recurrent point group in the form

$$u=\varphi(t), \quad v=\psi(t); \quad u_1=\varphi(t_1), \quad v_1=\psi(t_1),$$

where $\varphi$, $\psi$ are continuous functions, where $u, v$ are ordinary surface coördinates for $S$, and where $t$ ranges over the values specified. We shall say that the recurrent point group is of *rank* 1 in this case.

Or it may be possible to write

$$u=\varphi(t, w), \quad v=\psi(t, w); \quad u_1=\varphi(t_1, w_1), \quad v_1=\psi(t_1, w_1),$$

where $w$ has properties analogous to $t$. We then say that the recurrent point group is discontinuous of *rank* 2.

This definition obviously extends to any rank and is applicable to partially as well as totally discontinuous point groups.

It would be interesting to know whether or not the rank is finite in al cases which actually arise in applications.

## § 59. Unstable recurrent point groups.

Let us term a recurrent point group and its minimal set $\Sigma$ *unstable* if, for $\varepsilon > 0$ sufficiently small, it is impossible to find $\delta$ such that points $P$ within distance $\delta$ of $\Sigma$ remain within distance $\varepsilon$ of under indefinite iteration of $T$ and $T_{-1}$. In the contrary case let us call the point group and the set $\Sigma$ *stable*. This agrees with our earlier definition of stability in the case of an invariant point.

Let $\Sigma$ be an unstable minimal set and $P$ a point group such that the sequence of points $P, T(P), T_2(P), \ldots$ has $\Sigma$ as the only minimal set in the set of $\omega$ limit points. Then the point group of $P$ will be said to be *positively asymptotic* to $\Sigma$. Similarly if the sequence $P, T_{-1}(P), T_{-2}(P), \ldots$ has a single minimal set $\Sigma$ in its set of $\alpha$ limit points then the point group of $P$ will be said to be *negatively asymptotic to* $\Sigma$.

It is apparent that we cannot have the phenomenon of asymptotic point groups save when $\Sigma$ is unstable. For, if $P$ is any point at distance more than $\varepsilon$ from a stable set $\Sigma$, its iterates cannot approach to within some distance $\delta$ of $\Sigma$ by the definition of stability. Moreover our earlier work shows that for hyperbolic periodic point groups[1] such asymptotic point groups lie along hyper-continuous branches, while for regular elliptic periodic point groups other types of asymptotic point groups are present. In both of these cases the point $P$ tends toward $\Sigma$ asymptotically, under $T$ or $T_{-1}$, although such a state of affairs is not required by our definition.

*In the regular case an unstable periodic point group possesses positively and negatively asymptotic point groups forming connected sets of the kinds earlier specified.*

Moreover even in the irregular case the work of § 51 shows that we will have connected $\alpha$ and $\omega$ sets. These furnish asymptotic point groups unless there are other recurrent point groups in these sets. This follows by the last result of § 56.

*In the irregular case an unstable periodic point group possesses such asymptotic sets unless there are infinitely many recurrent point groups in its infinitesimal vicinity.*

It is this possibility which arises for a hyperbolic invariant point through which passes an invariant point curve. The nearby invariant points are the recurrent point groups in the vicinity.

Our initial conclusion for recurrent non-periodic point groups is the following:

*An unstable minimal set (not periodic) possesses positively and negatively asymptotic point groups forming a connected set, at least unless there are other recurrent point groups in its infinitesimal vicinity.*

In fact, if possible choose $\varepsilon$ so small that there are no other recurrent point groups within distance $\varepsilon$ of $\Sigma$. Now choose $\delta$ extremely small and consider the iterates of points within distance $\delta$ of $\Sigma$ under $T$. Because of the instability of $\Sigma$ these iterates reach out in a connected set to distance $\varepsilon$ in $N$ iterations ($N$ large). By a limiting process like that employed in § 51 we infer the existence of a closed set of points connected with $\Sigma$, reaching out to the boundary of this $\varepsilon$ vicinity, and remaining within this neighborhood under indefinite iteration of $T_{-1}$. But each point of this set has only the minimal set $\Sigma$ in its $\alpha$ point group. Hence these points approach $\Sigma$ uniformly often under iteration, by the last result of § 56, and are negatively asymptotic to $\Sigma$. The existence of a positively asymptotic set may be similarly established.

To advance further we introduce the notion of isomorphic recurrent point groups: Two recurrent point groups with minimal sets $\Sigma, \Sigma'$ are *isomorphic* if it

[1] A periodic point group of $q$ points $P, T(P), \ldots, T_{q-1}(P)$ is called hyperbolic if $P$ is hyperbolic under $T_q$. A similar terminology is employed in general.

is possible to establish a correspondence of closed point sets of $\Sigma$ to closed point sets of $\Sigma'$ which is maintained under $T$. It is assumed that there is more than a single set unless $\Sigma$ or $\Sigma'$ consists of a single point. Thus two periodic point groups of $k$ and $l$ points are isomorphic only if $k$ and $l$ have a common prime factor. Similarly two one-dimensional continuous recurrent point groups are isomorphic only if their rotation numbers are the same or if they fall into $k$ and $l$ curves, where $k$ and $l$ have a common prime factor.

*If there are not an infinitude of recurrent point groups in the neighborhood of $\Sigma$ and isomorphic with it, there will exist such connected asymptotic sets.*

The existence of infinitely many near by recurrent point groups is an evident necessary condition for the non-existence of asymptotic sets of this description. To show that infinitely many of these are isomorphic with $\Sigma$, we note that the earlier argument for existence of such positively asymptotic point groups only fails if the connected $\omega$ set obtained contains other minimal sets besides $\Sigma$. Let $\Sigma'$ be such a set. By operating with $T$ indefinitely often upon the set connecting $\Sigma$ and $\Sigma'$ we infer that there exist point sets connecting $\Sigma$ and $\Sigma'$, and remaining in the $\varepsilon$ neighborhood of $\Sigma, \Sigma'$ under indefinite iteration of $T$ and of $T_{-1}$. Let us establish a correspondence between the sets of points of $\Sigma$ and $\Sigma'$ so connected.

Now if all the points of $\Sigma$ and $\Sigma'$ are so connected we have a connected invariant set under $T$, and included by it an invariant point of course. If invariant points exist in every vicinity of $\Sigma$, there exists an invariant point on $\Sigma$, which must coincide with $\Sigma$. Hence in all cases the sets $\Sigma$ and $\Sigma'$ are isomorphic. If there are a finite number of connected sets we are led to isomorphic periodic point groups near $\Sigma$.

By letting $\varepsilon$ approach o we arrive at infinitely many periodic or other recurrent point groups having minimal sets isomorphic with $\Sigma$ and lying in its immediate vicinity. This is under the hypothesis that there are no asymptotic sets of the type described.

It is to be hoped that a more complete analysis of the notion of isomorphism will be made.

Let us say that a point is *positively* (*negatively*) *asymptotic* to a set of isomorphic recurrent point groups if these and these alone form the recurrent point groups among its $\omega$ ($\alpha$) limit points.

The above argument then enables us to state the following:

*For a given recurrent point group in any continuum $D$ there exist connected sets positively and negatively asymptotic to a set of isomorphic recurrent point groups containing the given point group unless there is such an isomorphic point group with a point on the boundary of $D$.*

## § 60. Stable recurrent point groups.

The simplest type of continuous recurrent point groups is the periodic type. If this is stable each of the $k$ points of the group is clearly surrounded by infinitely many neighboring curves which are permuted by $T$. These curves are invariant under $T$ and their form has been partially determined (§§ 44—47).

The two-dimensional continuous type is stable by definition since its points fill $S$.

Suppose finally that we have a stable continuous one-dimensional recurrent point group with minimal set $\Sigma$. On either side of the curve $\Sigma$ it is readily inferred (see § 42) that we have an infinite succession of nearby invariant curves. If the rate of rotation of nearby points exceeds that along the curve (as in the case of a regular neighborhood of an invariant point) the nature of these curves can be discussed more fully, but we will not attempt such a discussion.

Thus a stable one-dimensional continuous recurrent point group is surrounded by infinitely many neighboring invariant curves on either side.

In the case of a discontinuous recurrent point group with minimal set $\Sigma$ we are led similarly to a set of nearby invariant sets of continua containing the set $\Sigma$ as inner points and lying within distance $\varepsilon$ of $\Sigma$. Clearly $\iint Q\,du\,dv$ taken over any of these continua is the same, so their number is finite, and they are carried into themselves by $T_k$. Thus there is an invariant point under $T_k$ within each of them. Such a point $P$ lying near a point of $\Sigma$ and in the same continuum clearly remains nearby under iteration of $T$ or $T_{-1}$. By letting $\varepsilon$ decrease the number of these continua increases indefinitely. At each stage $k$ is unaltered or changes to a multiple of itself.

*A stable periodic point group of $k$ points has in its neighborhood infinitely many invariant sets of $k$ enclosing curves as specified.*

*A stable one-dimensional recurrent point group has in its neighborhood infinitely many invariant rings within which it lies.*

*A stable discontinuous recurrent point group has in its neighborhood infinitely many periodic point groups and invariant sets of enclosing curves. A point of a periodic point group approximates uniformly to any nearby point $P$ of the given group under all iterations of $T$ and $T_{-1}$.*

# Chapter VI. The general point group.

## § 61. Classification of transformations $T$.

Before entering upon further discussion of the behavior of points under $T$, we shall effect a classification which is fundamental.

A transformation $T$ will be called *transitive* if, for any pair of points $P$ and $Q$ on $S$ nearby points $P'$ and $Q'$ respectively can be found such that $Q' = T_n(P')$.

A transformation $T$ is *intransitive* in the contrary case.

It seems highly probable that the transitive case is to be regarded as the general case.

## § 62. The transitive case.

We commence with the transitive case.

*In the transitive case all of the recurrent point groups are unstable.*[1]

In fact it has been observed earlier that a stable recurrent point group leads to continua forming part of $S$, which are invariant as a set under $T$ and lie near the point group. Hence if we take a point $P$ outside of these continua and a point $Q$ within one of them, the condition given in the definition of transitivity cannot be fulfilled.

We note that invariant sets of continua cannot exist in the transitive case for the same reason.

*In the transitive case the asymptotic $\alpha$ or $\omega$ point groups connected with any recurrent point group and its isomorphic recurrent point groups, together with these recurrent groups, are everywhere dense throughout the surface $S$.*

For suppose that there is no such asymptotic point in some small region $\sigma$ for a recurrent point group with minimal set $\Sigma$.

Take then a small vicinity of $\Sigma$ and consider the regions into which it goes by $T$. Evidently this set of region must ultimately overlap part of $\sigma$ or we shall be led to invariant continua, such as can not exist in the transitive case.

Applying then precisely the same considerations that we have used earlier, i. e. considering smaller and smaller neighborhoods of $\Sigma$, we derive the existence

[1] The exceptional case in which there is a single recurrent point group whose minimal set fills $S$ is left out of consideration.

of a connected $\alpha$ set reaching from $\Sigma$ to the boundary of $\sigma$ at $P$. Either $P$ belongs to a point group isomorphic with $\Sigma$, or its point group is positively asymptotic to $\Sigma$, or to a set of isomorphic recurrent point groups, by the preceding paragraph.

*In the transitive case any positively asymptotic connected set of points has infinitely many points in common with any negatively asymptotic set, at least if there exists a single elliptic periodic point group $II^l$ with $l$ finite.*

This follows at once from the immediately preceding propositions and from the structure of the network of asymptotic sets $A$ and $\Omega$ about such an invariant point (§ 53).

For, consider the transformation $T_q$ which leaves such a point $P$ of an elliptic periodic point group unchanged.

The given connected asymptotic $\alpha$ set reaches into this network indefinitely near to the invariant point $P$ without meeting the $A$ set. The negatively connected asymptotic $\omega$ set reaches into this network without meeting the $\Omega$ set. Consequently the two sets have infinitely many points in common.

Thus there exist point groups positively and negatively asymptotic to assigned periodic point groups.

Suppose now that we designate any point whose $\alpha$ or $\omega$ limit points do not form all of $S$ as a *special* point. All of the points belonging to recurrent point groups or points asymptotic to such point groups are of this type.

Points which are not special evidently pass into the neighborhood of all points of $S'$ under iteration of $T$ or $T_{-1}$. Such points we term *general.*

*In the transitive case the general points are everywhere dense in $S$.*

To see this we divide $S$ into a large number of regions $S'$ of small diameter $d$, and consider the set of points $P$ whose iterates do not enter within all of the regions $S'$. Such points $P$ evidently form a closed set of points, $M$ say.

This set $M$ is nowhere dense in $S$. In the contrary case suppose $M$ to fill a small region $\sigma'$. Now there are only a finite set of regions $S'$ and thus only a finite number of combinations of less than all of them. Divide the points of $\sigma'$ into the finite number of closed sets according to the regions $S'$ which the points enter. Thus $\sigma'$ is divided into a finite number of closed sets, at least one of which therefore fills some neighborhood $\sigma''$ of $\sigma'$ densely. We recall that a finite or denumerably infinite set of nowhere dense closed sets cannot fill a complete neighborhood. But the existence of such a region $\sigma''$ contradicts the condition that $T$ is transitive. Thus $M$ is nowhere dense.

Again choose a set of subregions $S''$ of the regions $S'$ of diameter less than

$\frac{d}{2}$ leading to a set $M'$ which includes $M$ by a similar process. The set $M'$ is nowhere dense.

By continuing in this fashion we get an infinite set of closed sets $M, M', \ldots,$ each containing its predecessor. Every point $P$ which has not all of $S$ for its set of limit points evidently belongs to some one of these sets.

But by the theorem quoted above the set of all points belonging to some $M^{(k)}$ nowhere fills a complete neighborhood. Hence the stated property holds.

It would appear to be a very important and difficult question to determine the relative measure of the special points and general points. The above argument renders it clear that both of these sets are measurable in the sense of Lesbesgue, but sheds no light on their relative measures. One naturally conjectures that the special points are of measure o.

## § 63. The intransitive case.

In the intransitive case there exists at least one pair of points $P, Q$ such that no point very near to $P$ goes into a point very near to $Q$ under iteration of $T$ or $T_{-1}$. Obviously this state of affairs implies the existence of invariant sets of two-sided curves forming the boundaries of open continua on opposite sides of which $P$ and $Q$ lie.

We term a transformation $T$ for which there exist only a finite number $k > 0$ of such curves *finitely intransitive;* otherwise, *infinitely intransitive.*

Within one of the invariant sets of continua bounded by these curves in such a finitely transitive case, the condition for transitivity is satisfied i. e. for any pair of points $P, Q$ within, nearby points $P', Q'$ respectively can be found such that $Q' = T_n(P')$ for some $n$.

*In the finitely intransitive case the theorems stated for the transitive case hold within each invariant set of continua.*

The infinitely intransitive case obviously includes the integrable case when the points move along analytic curves. More generally, it includes the case when there is at least one stable recurrent point group. Indeed it seems possible that the existence of such a stable recurrent point group is a necessary as well as a sufficient condition for infinite intransitivity. But we have not been able to establish this conjecture.

In order to satisfactorily describe the point groups and their interrelations in the intransitive case it is essential to know the possible types of invariant sets of curves. Lacking such information save for the neighborhood of a periodic point group of elliptic type $\mathrm{I}l'$, $l$ finite, we do not attempt to go further.

# Chapter VII. Dynamical applications.

## § 64. The equations of motion.

For definiteness we consider a dynamical system with equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial x'}-\frac{\partial L}{\partial x}=0,\quad \frac{d}{dt}\frac{\partial L}{\partial y'}-\frac{\partial L}{\partial y}=0 \tag{32}$$

when $L$ is a function of the two coördinates $x, y$ and their time derivatives $x', y'$. This differential system is of the fourth order. If then we regard $x, y, x', y'$ as the coördinates of a point in four-dimensional space the motions of the dynamical system are represented by a set of curves, one through each point of the space.

Now we have the well-known integral relation

$$x'\frac{\partial L}{\partial x'}+y'\frac{\partial L}{\partial y'}=\text{const.} \tag{33}$$

Hence these curves lie on $\infty^1$ three-dimensional manifolds. We fix attention on any one of these.

We assume this three-dimensional manifold to lie in the finite part of the four-dimensional space and to be without singularity.

## § 65. Periodic motions.

To periodic motions correspond closed curves of the three-dimensional spread above obtained.

Suppose we take a point $P$ of such a stream line and consider a small element of an analytic surface containing $P$ and cutting the closed curve at an angle not $0$. If we take any point $A$ on this element near to $P$, and follow along the unique curve through it in the sense of increasing time $t$, the element will be crossed again later at a point $Q$. The transformation of the element which takes $P$ into $Q$ is the transformation $T$ which we shall consider.

The conservative transformation $T$[1] thus defined is clearly essentially independent of the particular surface element employed, since any other transformation so obtained can be derived from $T$ by a proper change of variables.

We classify the periodic motions into types I′, I″, II′, II″, II‴, III′, III″ according as $T$ is of such a type (§ 2). We define a periodic motion to be *elliptic* or *hyperbolic* according as the transformation $T$ is elliptic or hyperbolic; and the integer $l$ is similarly defined. The periodic motion is *stable* if nearby motions remain nearby for all $t$. This means that $T$ is stable. In the contrary case the periodic motion and $T$ are *unstable.*

Finally we will term the dynamical problem *integrable* if $T$ is integrable.

## § 66. The integrable case.

In the integrable case $T$ leaves a family of curves $F^* = \text{const.}$ invariant. Thus in the three-dimensional representing space there is a one-parameter analytic family of surfaces in the vicinity of the closed curve representing the periodic motion, each surface being made up of curves of motion. If this motion is of elliptic type there is a family of closed annular surfaces of which the curve of motion forms a degenerate member. If this motion is of hyperbolic type these surfaces are open and the curve lies on one or more of them. This much is obvious.

*The necessary and sufficient condition for integrability of the dynamical problem is the existence of an integral relation $G(x, y, x', y') = 0$ where $G$ is analytic in its indicated arguments, and where $G = 0$ is not an identity in virtue of the known integral relation.*

The condition is evidently sufficient. This relation yields an invariant family of surfaces in the three-dimensional representing space and these cut the surface elements used to define $T$ in a family of invariant analytic curves in the vicinity of the invariant point under $T$. Consequently $T$ is integrable.

Conversely, if $T$ is integrable we obtain an analytic family of surfaces on which the curves of motion lie. These may be represented in the form $H(u, v, \varphi) = 0$, where $u, v$ are coördinates for the surface elements and $\varphi$ is an angular coördinate. Also $H$ is analytic in its three variables. On account of the fact that the four-dimensional manifold under consideration consists of a

[1] See my paper first cited.

one-parameter analytic family of the three-dimensional manifolds which we have under consideration, it is apparent that these variables $u$, $v$, $\varphi$ may be expressed as analytic functions of $x$, $y$, $x'$, $y'$. By this means a relation of the desired type is obtained.

*In the hyperbolic integrable subcase there exist $k > 0$ one-parameter analytic families of motions asymptotic to the given periodic motions for $\lim t = \pm \infty$ (or else periodic) whose analytic representation we will not specify.*[1] *All other nearby motions first approach and then recede from the periodic orbit.*

This conclusion is an immediate consequence of the form of $T$ near a hyperbolic invariant point.

*In the elliptic integrable subcase nearby motions have coördinates $x$, $y$ representable as analytic functions of variables $e^{\sqrt{-1}\alpha\tau}$, $e^{\sqrt{-1}\beta\tau}$, while $t = c\tau +$ another function of this type.*

The curve of motion lies on a torus and a point on such of curve increases its angular coördinates by a fixed amount as a single circuit of the torus is made (§ 50). Evidently an analytic distortion takes this torus into an ordinary right circular cylinder on which the curves of motion are the spirals making a fixed angle with the generators. Now $x$, $y$ can be expressed as periodic analytic functions of the angular coördinates $p$, $q$ on this torus. But this establishes the stated form of representation for $x$, $y$. Also $\frac{dt}{d\tau}$ is a similar function of $p$, $q$, whence the form of the expression for $t$ in terms of $\tau$.

## § 67. Formal series in the non-integrable case.

Evidently the results of the first part of the present paper may be interpreted as results for the formal series representing the motion in the dynamical problem. The asymptotic validity of such series can be readily established. We will only remark upon the following fact: Inasmuch as there exists a formally invariant series $F^*$ in the non-integrable case (§ 10), *there exists always a formally invariant integral relation of the type $G = 0$ considered above.* Thus the dynamical problem is 'formally integrable' in the vicinity of an elliptic or hyperbolic periodic motion. In the elliptic case this means that periodic power series in two periods may be employed to represent nearby motions.

If I am not mistaken it has never yet been demonstrated that integrability

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[1] In the simplest and general case I', $x, y$ may be expressed as convergent power series in $\rho^{\pm\tau}$ while we have $t = c\tau +$ another power series of the same sort.

in the above sense cannot always prevail, although such a possibility appears remote. POINCARÉ has merely shown that integrability does not exist uniformly throughout certain domains with variation of a parameter $\mu$.[1] The particular example of § 31 yields a non-integrable conservative transformation, but it is not yet established that such a transformation arises in a dynamical problem.

### § 68. Periodic motions in the non-integrable case.

The results of Chapter II when interpreted in the general hyperbolic case show at once:

*The results stated above for the integrable hyperbolic case hold also in the non-integrable case, the analytic families of asymptotic motions being representable by means of hypercontinuous functions.*

Interpreting the results of Chapter III for the stable elliptic case II', $l$ finite, we conclude:

*In the non-integrable stable elliptic case II', $l$ finite, there exist an infinite number of continuous closed one parameter families of nearby motions, representable by means of continuous biperiodic functions of limited variation, and which are invariant as a family upon a circuit of the periodic motion.*

*In the unstable elliptic case II', $l$ finite, there exist connected families of asymptotic motions for both $\lim t = -\infty$ and $\lim t = +\infty$, each containing the given periodic motion. The family of the one type has infinitely many doubly asymptotic motions in common with any family of the other type. The motions not in any such family are everywhere dense near the periodic motion.*

As before we omit details.

### § 69. Surfaces of section.

In very many if not in all cases an analytic *surface of section* $S$ in the three-dimensional spread representing the motions may be found with the property that it is cut by every curve of motion in one and the same sense and has boundaries formed by closed curves representing periodic motion.

By following along a curve of motion from a point $P$ of such a surface to the next point $Q$, in the sense of increasing time a transformation $T$ for which $Q = T(P)$ is defined. This transformation is one-to-one, analytic and conservative.

We consider the totality of motions by the aid of such a surface $S$.

[1] H. POINCARÉ, *Les méthodes nouvelles de la mécanique céleste*, vol. 1, Paris 1892, Chap. 5.

## § 70. Recurrent motions.

A *recurrent motion* may be defined as one which comes arbitrarily near all its phases during *any* sufficiently large interval of time (from $t = -\infty$ to $t = +\infty$). Evidently such a motion corresponds to a recurrent point group on $S$. Hence we find:

*Every motion has at least one recurrent limit motion for* $\lim t = +\infty$ *(and for* $\lim t = -\infty$*). It recurs uniformly often arbitrarily near some one of these limit recurrent motions (not necessarily the same one).*[1]

Recurrent motions may either be periodic, biperiodic (representable on a square or torus) or triperiodic (representable in a cube), or discontinuous.[2] We will not follow out the classification suggested by §§ 57, 58 further.

## § 71. Asymptotic motions.

A motion will be said to be *positively (negatively) asymptotic* to a recurrent motion if it has only this recurrent motion as a limiting recurrent motion for $\lim t = +\infty$ ($\lim t = -\infty$). Furthermore we will say that two recurrent motions are *isomorphic* if the corresponding point groups are isomorphic. The direct application of the results of § 59 gives then:

*Unless there are infinitely many nearby isomorphic recurrent motions, any recurrent motion has connected families of motions asymptotic to it for* $\lim t = +\infty$ *and for* $\lim t = -\infty$.

## § 72. Transitive and intransitive systems.

If a motion can be found passing from nearly one prescribed phase to any second prescribed phase the dynamical system is *transitive*. Here $T$ is transitive also, and conversely (§ 62). Otherwise the dynamical system is *intransitive*.

*In the transitive case the motions asymptotic in either sense to a given recurrent motion or set of isomorphic recurrent motions, together with these motions, are everywhere dense.*

*Infinitely many motions exist doubly asymptotic to any two prescribed recurrent motions (or isomorphic sets of such motions) for* $\lim t = \pm\infty$, *at least if there exists*

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[1] See my paper last cited.

[2] The existence of recurrent motions of discontinuous type has been established by H. C. M. Morse, *Certain types of geodesic motion on a surface of negative curvature*, Harvard Dissertation, 1917.

*a single periodic motion of the elliptic type II', l finite. There exist also a dense set of general motions which approach every possible phase arbitrarily closely for both lim* $t = +\infty$ *and for lim* $t = -\infty$.

The *intransitive* case includes the integrable case. The simplest possibility is the *finitely intransitive* case when the curves of motion fall into $k > 0$ types filling out regions in the three-dimensional manifold. This corresponds exactly to the finitely intransitive type of transformation $T$.

*In the finitely intransitive case each type of motions has the same properties stated above for the intransitive type.*

We do not consider the infinitely intransitive type of dynamical system except as covered in the general results stated above. Here we have infinitely many types of motion, and, in default of a knowledge of the types which may exist, the results to be obtained are necessarily vague.

## § 73. Conclusion.

The varying degree of definiteness of the results above obtained for dynamical systems is striking. The catalogue of types of motion according to their degree of simplicity appears to run as follows: ordinary periodic motions, biperiodic motions representable analytically by convergent trigonometric series in two arguments, triperiodic motions representable by three arguments; motions asymptotic to periodic motions of hyperbolic type, motions asymptotic to periodic motions of elliptic type and of the other types just referred to; recurrent motions of biperiodic or triperiodic type (not representable by convergent trigonometric series); recurrent motions of discontinuous type; motions asymptotic to recurrent motions of these new types (or to sets of isomorphic recurrent motions); *special* motions (i. e. not passing near all phases for both lim $t = +\infty$ and lim $t = -\infty$) not of above types; *general* motions.

The degree of definiteness attained has varied with the analytic instruments at hand, and will probably be found to correspond to the nature of the case, at least unless entirely new analytic instruments are discovered.

The remarkable diversity and complexity of structure possible in dynamical systems with two degrees of freedom is likely to stand permanently in the way of approach to any definitive form for the theory of such systems. As has appeared above, many of the most vital questions are still without an answer. Progress with these questions and progress with the theory of the conservative transformations $T$ which we have studied will go hand in hand.