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A lattice version of the KP equation

by

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1. Introduction

Let N and M be relatively prime integers. Let $V_{N,M}$ be the set of all real valued functions ψ on $\mathbf{Z}\times\mathbf{Z}$ satisfying $\psi(n+N,m)=\psi(n,m+M)=\psi(n,m)$. $V_{N,M}$ is a vector space of dimension NM over $\mathbf{R}$. Let A and B be functions from an interval $I=(a,b)$ to $V_{N,M}$. $A(n,m,t)$ will denote the value of $A(t)$ at the point $(n,m)\in\mathbf{Z}\times\mathbf{Z}$. In § 3, we will define two explicit real polynomial maps $f_{N,M}$ and $g_{N,M}$ on $V_{N,M}\times V_{N,M}\times\mathbf{R}^3$ to $V_{N,M}$. We will investigate solutions $A(t)$ and $B(t)$ to the following differential-difference equation:

$$\frac{dA(t)}{dt}=f_{N,M}(A(t),B(t),\alpha,\beta,\gamma) \tag{1.1}$$

$$\frac{dB(t)}{dt}=g_{N,M}(A(t),B(t),\alpha,\beta,\gamma) \tag{1.2}$$

for fixed α,β and γ. More intrisically, one may think of $f_{N,M}$ and $g_{N,M}$ as defining a vector field on $V_{N,M}\times V_{N,M}$ depending on parameters α,β and γ. Thus for any given t, $f_{N,M}(A(t),B(t),\alpha,\beta,\gamma)$ is a function on $\mathbf{Z}\times\mathbf{Z}$, and this function evaluated at (n,m) is a polynomial in α,β and γ and the numbers $A(i,j,t)$ and $B(i,j,t)$ which will turn out to be of degree 4, and $g_{N,M}(A(t),B(t),\alpha,\beta,\gamma)$ will turn out to be of degree 5. Actually these polynomials enjoy certain homogeneity properties explained in § 3.

These equations are derived from a certain algebro-geometric construction, which is in some sense a variant of a construction of Mumford and van Moerbeke (as will be explained in § 3). This construction starts with certain algebraic curves X with a distinguished point P (with certain additional structure). Using X and this structure, we

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will define a map Φ from an open subset U of the Jacobian of X, Jac(X), to $V_{N,M}\times V_{N,M}$. Let Ψ be the canonical map from X to Jac(X), this map being canonical up to translation. Let T_1, T_2 and T_3 be a basis of the osculating three space of the curve $\Psi(X)$ at the point $\Psi(P)$. We choose T_1 to be a tangent vector to the curve at $\Psi(P)$ and T_2 to be in the osculating plane of the curve at $\Psi(P)$. These vectors can be translated over the whole of Jac(X) to obtain vector fields again denoted by T_i. We will define $f_{N,M}$ and $g_{N,M}$ in such a way that the following holds: Suppose we start with a line bundle $\mathscr{L}\in U$ and allow it to flow along the vector field T_i for time t to a line bundle $\mathscr{L}_t$. Then there are α,β and γ (depending on i) so that if we let $Y=\Phi(\mathscr{L})$ flow along the vector field on $V_{N,M}\times V_{N,M}$ defined above by (1.1) and (1.2) to Y_t, then $\Phi(\mathscr{L}_t)=Y_t$. Thus the complicated flow defined by the non-linear equations (1.1) and (1.2) can be 'linearized' to a straight line flow on a Jacobian. Conjecturally, the generic A and B come from such a curve and line bundle. We also write down explicit conserved quantities of these flows.

Having derived these equations, we can ask about the behavior of solutions of these equations, especially with N and M large. One way of analyzing the behavior of such solutions is to try to construct a continuous model for these equations. Another way is to look graphically at numerical solutions of our equations. For an interesting account of these two ways, see [Z]. We have not attempted such an analysis in this paper. Instead, we will exhibit some solutions to our equations which do have interesting continuous models. Thus our results indicate that such an analysis would be interesting. One way of precisely defining these rather vague comments is the following rather *ad hoc* definition:

Definition 1.1. $\mathscr{C}$ is the class of all functions f on $\mathbf{R}^3$ satisfying the following properties:

(i) $f(x+1,y,t)=f(x,y+1,t)=f(x,y,t)$ for all $(x,y,t)\in\mathbf{R}^3$.

(ii) Given $\varepsilon>0$, there are $(\alpha,\beta,\gamma)\in\mathbf{R}^3$, an integer N, a constant C and functions $A(t)$ and $B(t)$ from $\mathbf{R}$ to V_{N,N^2+1} so that

$$\left|f\left(\frac{n}{N},\frac{m}{N^2+1},t\right)-NA(n,m,t)-C\right|<\varepsilon$$

and so that A and B satisfy the equations (1.1) and (1.2).

The main result of this paper (Theorem 2.5) is that $\mathscr{C}$ contains many of the solutions of the KP equation arising from algebraic geometry [D]. The definition of the class $\mathscr{C}$ is quite restrictive in that we require the discrete $NA(n,m,t)$ to be close to $f(x,y,t)$ for all t. It would be interesting to know what further conditions on the class $\mathscr{C}$

would imply that an $f \in \mathscr{C}$ satisfying these further conditions satisfies the KP equation. Another question unanswered here is whether a solution to the KP hierarchy (in three variables) belongs to some variant of the class $\mathscr{C}$. The definition of the class $\mathscr{C}$ is (to be frank) based on what we can prove.

§ 2 reviews the theory of curves and their Jacobians defined over $\mathbf{R}$. § 2 concludes with a precise statement of our main theorem. § 3 derives the expressions for $f_{N,M}$ and $g_{N,M}$. We conclude § 3 with a few observations on the relation of this work to the work of Mumford and van Moerbeke [MM] on spectral curves. In § 4, we give a proof of the following theorem:

THEOREM 1.1. *If C is non-hyperelliptic and V is a generic three dimensional subspace of $H^0(C,\Omega)$, then the map from $V\otimes H^0(C,\Omega)\rightarrow H^0(C,\Omega^{\otimes 2})$ is surjective.*

The proof of Theorem 1.1 was supplied by Lazarsfeld based on the ideas of [GL]. Green also supplied a proof, and Eisenbud provided a simpler proof by direct computation for trigonal curves, and so for a generic curve. § 5 develops some Kodaira–Spencer type deformation theory. In § 6 we show that a certain class of 'good' curves exists using a monodromy argument as well as our Kodaira–Spencer theory and Theorem 1.1. § 7 gives the proof of our main theorem.

The work in this paper was motivated by a hope of Trubowitz that understanding the spectral theory of lattice models of the KP equation might yield some insight into the transcendental spectral theory of the KP equation.

2. Curves defined over R

Let C be a non-singular curve defined over $\mathbf{C}$, i.e. a compact Riemann surface. Thus we can find a holomorphic embedding of C into $\mathbf{P}^n$ so that C is the locus of zeros of homogeneous polynomials with complex coefficients. We say that C is defined over $\mathbf{R}$ if we can choose the embedding so that the polynomials all have real coefficients. Note that $\mathbf{P}^n$ has a natural antiholomorphic involution

$$\iota(z_0,\ldots,z_n)=(\bar{z}_0,\ldots,\bar{z}_n)$$

and that ι leaves C invariant when C is defined over $\mathbf{R}$. We denote the restriction of ι to C by ι again. A function f on an ι invariant open set of C is said to be defined over $\mathbf{R}$ if $f(z^{\iota})=\overline{f(z)}$. A point or divisor is defined over $\mathbf{R}$ if it is invariant under ι. A holomorphic one form ω is defined over $\mathbf{R}$ if locally $\omega=df$, where f is defined over $\mathbf{R}$. If ω is defined over $\mathbf{R}$ and γ is a path on the surface, then

$$\int_{\gamma} \omega = \overline{\int_{\gamma^{\iota}} \omega},$$

as we can readily see by dividing the path γ into subpaths on which ω is exact.

Note that ι acts on the cohomology $H^1(C, \mathbf{Z})$. Let $\Lambda^+(C)$ be the elements of $H^1(C, \mathbf{Z})$ fixed by ι and let Λ^- be the elements γ of $H^1(C, \mathbf{Z})$ with $\gamma^{\iota} = -\gamma$. Note there is a natural integration map:

$$\int : H^1(C, \mathbf{Z}) \to H^0(C, \Omega)^*,$$

where $H^0(C, \Omega)^*$ is the dual pace of the holomorphic one forms of C. Let $H^0(C, \Omega)^*(\mathbf{R})$ be the set of real points of $H^0(C, \Omega)^*$. Then Λ^+ maps to $H^0(C, \Omega)^*(\mathbf{R})$ and Λ^- maps to $iH^0(C, \Omega)^*(\mathbf{R})$. $H^0(C, \Omega)^*(\mathbf{R})$ is a real vector space of dimension g, and the complex span of the vectors in $H^0(C, \Omega)^*(\mathbf{R})$ is just $H^0(C, \Omega)^*$. $H^1(C, \mathbf{Z})$ maps to a lattice in $H^0(C, \Omega)^*$. Thus we see that Λ^+ maps to a lattice in $H^0(C, \Omega)^*(\mathbf{R})$. The Jacobian of C has a natural real structure and the quotient of $H^0(C, \Omega)^*(\mathbf{R})$ by the image of Λ^+ is the component of the real points of the Jacobian of C which contains the identity of the Jacobian.

We next discuss theta functions following [M]. We regard $H^1(C, \mathbf{Z})$ as a subgroup of $H^0(C, \Omega)^*$. The involution ι extends to an antiholomorphic involution on $H^0(C, \Omega)^*$ again denoted by ι. Let $\mathbf{C}_1^*$ be the set of all complex numbers of absolute value one. Choose a map

$$\alpha: H^1(C, \mathbf{Z}) \to \mathbf{C}_1^*$$

so that

$$\alpha(u^{\iota}) = \overline{\alpha(u)}, \tag{2.1}$$

and

$$\frac{\alpha(u_1 + u_2)}{\alpha(u_1)\,\alpha(u_2)} = e^{i\pi\langle u_1, u_2\rangle}. \tag{2.2}$$

There is a unique Hermitian form H on $H^0(C, \Omega)^*$ so that

$$\operatorname{Im} H(x, y) = \langle x, y \rangle.$$

We see that since

$$\langle x^{\iota}, y^{\iota} \rangle = -\langle x, y \rangle,$$

we have that

$$\overline{H(x^\iota, y^\iota)} = H(x, y).$$

Let ϑ defined on $H^0(C, \Omega)^*$ be the function satisfying the functional equation

$$\vartheta(z+u) = \alpha(u)\, e^{\pi H(z,u)+\pi H(u,u)/2}\vartheta(z)$$

for $z \in H^0(C, \Omega)^*$ and $u \in H^1(C, \mathbf{Z})$. The function ϑ' defined by

$$\vartheta'(z) = \overline{\vartheta(z^\iota)}$$

satisfies the same functional equation as ϑ. Since ϑ is defined up to a constant multiple by its functional equation, we see that we can choose ϑ to be real on the fixed set of ι, which is $H^0(C, \Omega)(\mathbf{R})^*$. Consider $K_1 \in H^0(C, \Omega)^*$ and suppose that $K_1^\iota \in K_1 + H^1(C, \mathbf{Z})$. Choose a point $P \in C$ and a parameter z around P. We can define linear functionals in $H^0(C, \Omega)(\mathbf{R})^*$ by the formulas:

$$v_i(\omega) = \left(\frac{d^{i-1}}{dz^{i-1}} \frac{\omega}{dz}\right)_{z=0}.$$

The v_i form a Frenet frame for the natural map ϕ of a neighborhood of P in C to $H^0(C, \Omega)^*$ defined by the formula

$$\phi(Q)(\omega) = \int_P^Q \omega.$$

We call the span of v_1, v_2 and v_3 the osculating three space at P.

Define a meromorphic function f on $\mathbf{R}^3$ by the formula

$$f(x, y, t) = \frac{\partial^2}{\partial x^2} \log \vartheta(xv_1 + yv_2 + tv_3 + K_1). \tag{2.3}$$

Then

$$\begin{aligned}
\overline{f(x, y, t)} &= \frac{\partial^2}{\partial x^2} \log \vartheta(xv_1 + yv_2 + tv_3 + K_1^\iota) \\
&= \frac{\partial^2}{\partial x^2} \log \vartheta(xv_1 + yv_2 + tv_3 + K_1 + u), \\
&= \frac{\partial^2}{\partial x^2} \log \vartheta(xv_1 + yv_2 + tv_3 + K_1) \\
&= f(x, y, t)
\end{aligned}$$

for some $u \in H^1(C, \mathbf{Z})$. We have used that the second logarithmic derivative of the ϑ function is periodic, as follows from differentiating the functional equation.

Let $\mathcal{M}_{g,1}$ be the set of all (C, P), where C is a curve of genus g and P is a point on C. $\mathcal{M}_{g,1}$ has the natural structure of an analytic space. Let $\mathcal{M}_{g,1}(\mathbf{R})$ be the subset of $\mathcal{M}_{g,1}$ consisting of all curves C defined over $\mathbf{R}$ and P a point of C defined over $\mathbf{R}$. Let $(C, P) \in \mathcal{M}_{g,1}$.

Definition 2.1. $v_i \in H^0(C, \Omega((i+1)P))$ for i from 1 to 3 are called adapted if the v_i map to linearly independent elements of $H^0(C, \Omega)^*$.

Define

$$\phi: H^0(C, \Omega(-P)) \oplus H^0(C, \Omega(-2P)) \oplus H^0(C, \Omega(-3P)) \to H^0(C, \Omega^{\otimes 2}(P))$$

by

$$\phi(\omega_1, \omega_2, \omega_3) = v_1 \omega_1 + v_2 \omega_2 + v_3 \omega_3,$$

where $H^0(C, \Omega^{\otimes 2}(P))$ is the set of quadratic differentials which have a pole at P.

Definition 2.2. The v_i are acceptable if the map ϕ is injective.

Let V_j be the annihilator of $H^0(C, \Omega(-jP))$ in $H^0(C, \Omega)^*$, where $H^0(C, \Omega(-jP))$ is the set of all one forms which vanish j times at P. We assume that V_3 has dimension 3, which is the same as assuming that there is not a non-trivial function having a pole of order 3 or less at P. Let $\Lambda^+(\mathbf{R})$ be the real span of the vectors in Λ^+. Note that both $\Lambda^+(\mathbf{R})$ and $H^0(C, \Omega)(\mathbf{R})$ are in $H^1(C, \mathbf{C})$ and that cup product on $H^1(C, \mathbf{C})$ induces a perfect pairing between these two real vector spaces and hence that we have a natural isomorphism from $\Lambda^+(\mathbf{R})$ to $H^0(C, \Omega)^*(\mathbf{R})$. Choose v_1, v_2 and v_3 in $\Lambda^+(\mathbf{R})$ so that $v_i \in V_i$. $H^0(C, \Omega((j+1)P))$ is also included in $H^1(C, \mathbf{C})$, since all the differentials in $H^0(C, \Omega((j+1)P))$ are of the second kind. Further $H^0(C, \Omega((j+1)P))$ is the annihilator of $H^0(C, \Omega(-jP))$ under cup product. It follows that $v_j \in H^0(C, \Omega((j+1)P))$.

Definition 2.3. $(C, P) \in \mathcal{M}_{g,1}(\mathbf{R})$ is good if:

(i) $\Lambda^+ \cap H^0(C, \Omega(2P))$ has rank one in Λ^+.

(ii) $\Lambda^+ \cap H^0(C, \Omega(3P))$ has rank two in Λ^+.

(iii) There are adapted $v_i \in \Lambda^+$ which are acceptable.

(iv) The dimension of V_3 is three.

PROPOSITION 2.4. *The good points of $\mathcal{M}_{g,1}(\mathbf{R})$ are dense (in the classical topology) if $g > 2$.*

We next state our main theorem.

THEOREM 2.5. *Suppose that K_1-K_1' is in the image of $H^1(C,\mathbf{Z})$ and that the pair (C,P) is good. Let the v_i be an adapted acceptable set in Λ^+ and let $\bar{v}_i$ be the image of v_i in $H^0(C,\Omega)^*$. Suppose that the function*

$$\vartheta(x\bar{v}_1+y\bar{v}_2+t\bar{v}_3+K_1)$$

does not vanish on $\mathbf{R}^3$. Let

$$f(x,y,t)=\frac{\partial^2}{\partial x^2}\log\vartheta(x\bar{v}_1+y\bar{v}_2+t\bar{v}_3+K_1).$$

Then f is in the class $\mathscr{C}$.

Under the hypotheses of Theorem 2.5, it is well known that there is a constant K so that $f+K$ satisfies the KP equation.

3. Equations of motion

Before deriving the formulas for $f_{N,M}$ and $g_{N,M}$, we define the homogeneity properties of these polynomials mentioned in the Introduction. Define an $\mathbf{R}^*$ action on the space of all polynomials on $V_{N,M}\times V_{N,M}\times\mathbf{R}^3$ by

$$P^{\lambda(s)}(A,B,\alpha,\beta,\gamma)=P(sA,s^2B,s\alpha,s^2\beta,s^3\gamma).$$

We say P has weight r if

$$P^{\lambda(s)}=s^rP.$$

Thus the polynomial $P_{i,j}$ defined by $P_{i,j}(A,B,\alpha,\beta,\gamma)=A(i,j)$ has weight 1. By abuse of notation, we will denote $P_{i,j}$ by $A(i,j)$. Similarly, $B(i,j)$ will denote the analogous polynomial of weight 2, while α,β and γ will denote the analogous polynomial of degree $1,2$, and 3 respectively. We will show that our expressions for $f_{N,M}$ and $g_{N,M}$ will have weights four and five respectively.

Let X be a smooth curve defined over $\mathbf{R}$ of genus g, and let P and Q be real points of X. Suppose that $N(P-Q)$ is linearly equivalent to 0. Thus there is a function α on X having a pole of order N at P and a zero of order N at Q, and having no other poles or zeros. Let R_i and S_i be points of X for $i=1,\ldots,M$ so that R_i+S_i is defined over $\mathbf{R}$.

Suppose there is a function β having divisor

$$M(P+Q)-\sum_{i=1}^{M}(R_i+S_i).$$

For general i, we define R_i and S_i by periodicity, $R_i=R_{[i]}$ and $S_i=S_{[i]}$, where $[i]$ is the positive residue of $i \bmod m$. Let $\mathscr{L}$ be a line bundle of degree g on X. Let

$$\mathscr{L}_{n,m}=\mathscr{L}((n+m)P+(m-n)Q+D_m).$$

Here $D_0=0$ and $D_{m+1}=D_m-R_{m+1}-S_{m+1}$.

Definition 3.1. $\mathscr{L}$ is non-degenerate if $H^0(X,\mathscr{L}_{n,m}(-P))=0$.

The Riemann–Roch theorem then implies that $h^0(X,\mathscr{L}_{n,m})=1$ if $\mathscr{L}$ is non-degenerate. Assume $\mathscr{L}$ is nondegenerate. Let z be a parameter defined at P and choose a section $s_{0,0}$ of $\mathscr{L}$. There is a nonzero section $s_{n,m}$ of $H^0(X,\mathscr{L}_{n,m})$, which is defined up to a constant, and $s_{n,m}$ considered as a meromorphic section of $\mathscr{L}$ has a pole of order exactly $n+m$ at P. Let

$$f_{n,m}=\frac{s_{n,m}}{s_{0,0}}.$$

We normalize $s_{n,m}$ so that

$$f_{n,m}z^{n+m}(P)=1,$$

i.e. the leading term in the Laurent expansion of f in terms of z is one. We can also normalize α and β so that $(\alpha z^N)(P)=1$ and $(\beta z^M)(P)=1$

Given a non-degenerate line bundle $\mathscr{L}$ and the parameter z, we can form several functions on $\mathbf{Z}\times\mathbf{Z}$, namely

$$d_i(n,m)=\left(\frac{d^if_{n,m}z^{n+m}}{i!\,dz^i}\right).$$

In this paper, we will be mostly considering d_1, d_2 and d_3. They are the coefficients of the Laurent expansion of $f_{n,m}$.

First, let's notice that we can write down a linear relation between $s_{n+1,m}, s_{n,m+1}$, $s_{n,m}$ and $s_{n-1,m}$ in terms of the functions d_1 and d_2. Such a relation must exist, since $s_{n+1,m}, s_{n,m+1}, s_{n,m}$ and $s_{n-1,m}$ are all in $H^0(\mathscr{L}_{n,m}(P+Q))$ and $h^0(\mathscr{L}_{n,m}(P+Q))=3$ by Riemann–Roch and our assumptions on non-degeneracy. Fixing n and m for the moment, we can write

$$af_{n+1,m}+bf_{n,m+1}=cf_{n,m}+df_{n-1,m}.$$

Both $f_{n+1,m}$ and $f_{n,m+1}$ have a pole of order $n+m+1$ at P and $f_{n+1,m}/f_{n,m+1}$ has value 1 there. Both $f_{n,m}$ and $f_{n-1,m}$ have poles of order less than $n+m+1$ at P. Thus we have $a=b\neq 0$. So we may choose $a=-b=1$. Thus we may write

$$s_{n+1,m}-s_{n,m+1}=A(\mathscr{L},n,m)\,s_{n,m}+B(\mathscr{L},n,m)\,s_{n-1,m}.$$

A and B are uniquely determined. We denote $A(\mathscr{L},n,m)$ by $A(n,m)$ when $\mathscr{L}$ is understood. By comparing the Laurent expansions of the above equations around $z=0$, we see that we have the following recursion relations:

$$d_1(n+1,m)-d_1(n,m+1)=A(n,m) \tag{3.1}$$

$$d_2(n+1,m)-d_2(n,m+1)=A(n,m)\,d_1(n,m)+B(n,m) \tag{3.2}$$

$$d_3(n+1,m)-d_3(n,m+1)=A(n,m)\,d_2(n,m)+B(n,m)\,d_1(n,m). \tag{3.3}$$

The d_i also have periodicity properties with respect to translation by $(N,0)$ and $(0,M)$. Specifically, let

$$\alpha=z^{-N}+a_1z^{-N+1}+\dots$$

and

$$\beta=z^{-M}+b_1z^{-M+1}+\dots$$

be the Laurent expansions of α and β. Note that $\alpha s_{n,m}\in H^0(\mathscr{L}_{n+N,m})$. So $\alpha s_{n,m}$ is a constant multiple of $s_{n+N,m}$. By our normalization, this constant must be 1 so

$$\alpha s_{n,m}=s_{n+N,m}. \tag{3.4}$$

Similarly,

$$\beta s_{n,m}=s_{n,m+M}. \tag{3.5}$$

Consider the Laurent expansions of the equation (3.4). Comparing coefficients we see that

$$d_1(n+N,m)=a_1+d_1(n,m)$$

$$d_2(n+N,m)=a_2+a_1d_1(n,m)+d_2(n,m)$$

and

$$d_3(n+N, m) = a_3+a_2d_1(n, m)+a_1d_2(n, m)+d_3(n, m).$$

We have similar formulas for $d_i(n, m+M)$. Note that $A(n+N, m)=A(n, m)=A(n, m+M)$ and that $B(n, m+M)=B(n, m)=B(n+N, m)$.

The key observation here is that given the a_i for i from 1 to 3, we can compute the d_i and the b_i for i from 1 to 3 in terms of A and B by universal polynomials which depend only on N and M. Since $f_{0,0}=1$, we have $d_i(0,0)=0$. Hence we can use the recurrence relation (3.1) to solve for $d_1(l, -l)$ directly. Since N and M are relatively prime, by the Chinese remainder theorem for any n and m, we can find a and b so that

$$n = l+aN$$

and

$$m = -l+bM.$$

So

$$d_1(n, m) = d_1(l, -l)+a_1a+b_1b.$$

We have

$$d_1(NM, -NM) = Ma_1-Nb_1.$$

But $d_1(NM, -NM)$ is expressed directly in terms of the A's. So we can determine b_1 in terms of the A's and a_1. We see b_1 and d_1 have weight 1. Having determined d_1, we can now determine $d_2(n, m)$ from the recurrence relation (3.2) and the Chinese remainder theorem. We can similarly find an expression for b_2 in terms of the A's, the B's, and a_1 and a_2. Finally, d_3 and b_3 are determined in the same way. Note that these formulas only involve the a_i, and A and B, and not X or $\mathscr{L}$. Further, the d_i and b_i have weight i.

Actually, one can continue this process and find that all the b_i can be expressed in terms of the A's, the B's and the a_i. If we allow the $\mathscr{L}$ to evolve while fixing the curve, the points P, Q, and the R_i and S_i as well as the parameter z, the b_i will of course remain constant. This means that the b_i are conserved quantities of such an evolution.

Let D_0 be a fixed divisor of degree g on X, and let J_g be the Jacobian of all line bundles of degree g on X. If z is a point of X close to P, we can define a linear functional $\phi(z)$ on $H^0(X, \Omega)$ by the formula:

$$\phi(z)(\omega) = \int_P^z \omega.$$

There is a natural analytic homomorphism

$$\Phi\colon H^0(X,\Omega)^* \to J_g$$

so that $\Phi(0)=\mathcal{O}(D_0)$, and so that $\Phi(\phi(z)+a)=\Phi(a)(z-P)$. The kernel of Φ is just $H^1(X,\mathbf{Z})$. On the other hand, we have the previously introduced linear functionals defined on $H^0(X,\Omega)$:

$$v_i(\omega)=\left(\frac{d^{i-1}}{dz^{i-1}}\frac{\omega}{dz}\right)_{z=0}.$$

Let

$$A'_{n,m}(f)=A(\Phi(f),n,m)$$

for $f\in H^0(X,\Omega)^*$. Our aim is to compute

$$\nabla A'_{n,m}\cdot v_3,$$

which is the directional derivative of A' in the direction v_3, in terms of polynomials in the A's and B's. For t small, let $D(t)$ be the divisor $D_1+D_2+D_3$ on X, where the $z(D_i)$ are the three cube roots of t. We first show that we have the formula:

$$\nabla A'_{n,m}(f)\cdot v_3=2\frac{d}{dz}A(\Phi(f)(D(z)-3P),n,m)_{z=0}.$$

Let $w(x)$ be the point of X so that $z(w(x))=x$ for x small. Let $\psi(x)=f+\Phi^{-1}(D(x)-3P)$ so that $\Phi(\psi(x))=\Phi(f)(D(x)-3P)$. Then

$$\psi(x^3)=\phi(x)+\phi(\zeta x)+\phi(\zeta^2 x)+f, \tag{3.6}$$

where ζ is a primitive cube root of 1. We have

$$\frac{d}{dz}A(\Phi(f)(D(z)-3P),n,m)_{z=0}=\nabla A'(f)_{n,m}\cdot\psi'(0)$$

and

$$v_1=\phi'(0),$$

$$v_2=\phi''(0),$$

and

$$v_3=\phi'''(0).$$

So our claim will follow from

$$\phi'''(0) = 2\psi'(0).$$

However, this follows by differentiating the identity (3.6) three times and setting $x=0$.

Note that $s_{n,m}$, $s_{n+1,m}$, $s_{n+2,m}$ and $s_{n+3,m}$ are a basis of $H^0(\mathscr{L}_{n,m}(3P))$. For t small, let $\mathscr{L}_{n,m,t}$ be the line bundle $\mathscr{L}_{n,m}(3P-D_1-D_2-D_3)$, where the $z(D_i)$ are the three cube roots of t. Note that $h^0(\mathscr{L}_{n,m,t}(-P))=0$, for t sufficiently small, as $\mathscr{L}_{n,m,t}(-P)\to\mathscr{L}_{n,m}(-P)$ as $t\to 0$. Let $s_{n,m,t}$ be a non-zero section of $\mathscr{L}_{n,m,t}$ varying holomorphically with t. We can write

$$s_{n,m,t} = \sum_{i=0}^{3} a_i(n,m,t)\, s_{n+i,m}, \tag{3.7}$$

where the a_i are all holomorphic in t and one is non-zero at $t=0$. In fact, since $s_{n,m,t}\to s_{n,m}$ as $t\to 0$ modulo multiplication by constants, we see that a_0 is non-zero.

Let

$$a_i(n,m,t) = \sum_j a_{ij}(n,m)\, t^j \tag{3.8}$$

be the Taylor expansion of $a_i(n,m,t)$, where we may assume that $a_{0,0}(n,m)=1$. We have the identity

$$0 = s_{n,m,t^3}(t), \tag{3.9}$$

since s_{n,m,t^3} vanishes at all the cube roots of t^3, including t. On the other hand, we have

$$s_{n+i,m}(z) = \sum_{j=0}^{\infty} d_j(n+i,m)\, z^{-n-i-m+j} s_{0,0}. \tag{3.10}$$

Now substitute (3.10) and (3.8) into (3.7) and (3.7) into (3.9) and compute the first few nonzero coefficients of t. The coefficient of t^{-n-m} in $s_{n,m,t^3}(t)$ is just $a_{3,1}(n,m)+a_{0,0}(n,m)$, since

$$a_{3,0}(n,m) = a_{2,0}(n,m) = a_{1,0}(n,m) = 0.$$

Thus $a_{3,1}(n,m)=-1$, since $a_{0,0}=1$. We can replace $s_{n,m,t}$ by $s_{n,m,t}\,t/a_3(n,m,t)$ and assume that $a_3(n,m,t)=-t$ for all n and m. With our new choice, we have

$$\frac{s_{n,m,t}\, z^{n+m}}{s_{0,0,t}}(P) = 1.$$

So we can write

$$s_{n+1,m,t}-s_{n,m+1,t}=A(\mathscr{L}_t, n, m)\, s_{n,m,t}+B(\mathscr{L}_t, n, m)\, s_{n-1,m,t}. \tag{3.11}$$

The coefficient of t^{-n-m+1} in $s_{n,m,t^3}(t)$ is just $a_{2,1}(n,m)-d_1(n+3,m)+d_1(n,m)$, so

$$a_{2,1}(n,m)=d_1(n+3,m)-d_1(n,m). \tag{3.12}$$

So $a_{2,1}$ has weight 1. The coefficient of t^{-n-m+2} in $s_{n,m,t^3}(t)$ is just

$$a_{1,1}(n,m)+a_{2,1}(n,m)\, d_1(n+2,m)-d_2(n+3,m)+d_2(n,m),$$

so

$$a_{1,1}(n,m)=-d_1(n+2,m)\, a_{2,1}(n,m)+d_2(n+3,m)-d_2(n,m). \tag{3.13}$$

So $a_{1,1}(n,m)$ has weight 2. The coefficient of t^{-n-m+3} in $f_{n,m,t^3}(t)$ is just

$$a_{0,1}(n,m)+a_{1,1}(n,m)\, d_1(n+1,m)+a_{2,1}(n,m)\, d_2(n+2,m)-d_3(n+3,m)+d_3(n,m),$$

so

$$a_{0,1}(n,m)=-a_{1,1}(n,m)\, d_1(n+1,m)-a_{2,1}(n,m)\, d_2(n+2,m)+d_3(n+3,m)-d_3(n,m). \tag{3.14}$$

So $a_{0,1}(n,m)$ has weight 3.

Let's look at the expression

$$\Psi=s_{n+1,m,t}-s_{n,m+1,t}-A(\mathscr{L}_t, n, m)\, s_{n,m,t}-B(\mathscr{L}_t, n, m)\, s_{n-1,m,t}.$$

Equation (3.7) allows us to express the $s_{a,b,t}$ in terms of the $a_i(a,b,t)$ and in terms of the $s_{c,d}$. Further,

$$s_{n+i,m+1}=s_{n+i+1,m}-A(n+i,m)\, s_{n+i,m}-B(n+i,m)\, s_{n+i-1,m}.$$

We can therefore express Ψ as a linear combination of the $s_{n+i,m}$. But the vectors $s_{n+i,m}$ for fixed m are all linearly independent. Since $\Psi=0$, all the coefficients of this expresson for Ψ must be zero. In particular, since the coefficients are power series in t, the individual terms in this power series are zero. If $Q(t)=\Sigma_i Q_i(t)\, s_{n+i,m}$, let $p(Q)$ be the coefficient of t in the power series expansion of the coefficient of Q_0 and let $q(Q)$ be the coefficient of t in the power series expansion of Q_{-1}. Note that $p(s_{n+1,m,t})=0$, since $s_{n+1,m,t}$ does not involve $s_{n,m}$. Next, let's compute $p(s_{n,m+1,t})$. We have

$$s_{n,m+1,t}=-ts_{n+3,m+1}+a_2(n,m+1,t)\, s_{n+2,m+1}+a_1(n,m+1,t)\, s_{n+1,m+1}+a_0(n,m+1,t)\, s_{n,m+1}.$$

The coefficient of $s_{n,m}$ in $s_{n,m+1}$ is $-A(\mathscr{L}, n, m)$ and that the coefficient of $s_{n,m}$ in $s_{n+1,m+1}$ is $-B(\mathscr{L}, n+1, m)$ so

$$-p(s_{n,m+1,t}) = a_{1,1}(n, m+1)\, B(\mathscr{L}, n+1, m) + a_{0,1}(n, m+1)\, A(\mathscr{L}, n, m).$$

Let $\dot{A}(\mathscr{L}, n, m)$ denote

$$\frac{d}{dt} A(\mathscr{L}_t, n, m)_{t=0}.$$

Next we compute

$$p(A(\mathscr{L}_t, n, m)\, s_{n,m,t}) = \dot{A}(\mathscr{L}, n, m) + a_{0,1}(n, m)\, A(\mathscr{L}, n, m).$$

We further compute

$$p(B(\mathscr{L}_t, n, m)\, s_{n-1,m,t}) = B(\mathscr{L}, n, m)\, a_{1,1}(n-1, m).$$

So we obtain from $p(\Psi)=0$,

$$\begin{aligned}\dot{A}(\mathscr{L}, n, m) = a_{1,1}(n, m+1)\, B(\mathscr{L}, n+1, m) + a_{0,1}(n, m+1)\, A(\mathscr{L}, n, m)& \\ -a_{0,1}(n, m)\, A(\mathscr{L}, n, m)& \\ -B(\mathscr{L}, n, m)\, a_{1,1}(n-1, m)&\end{aligned} \tag{3.15}$$

thus $\dot{A}(n, m)$ has weight four. Similarly,

$$q(s_{n,m+1,t}) = -B(\mathscr{L}, n, m)\, a_{0,1}(n, m+1)$$

and

$$q(B(\mathscr{L}_t, n, m)\, s_{n-1,m,t}) = \dot{B}(n, m) + a_{0,1}(n-1, m)\, B(\mathscr{L}, n, m)$$

so

$$\dot{B}(n, m) = -a_{0,1}(n-1, m)\, B(\mathscr{L}, n, m) + B(\mathscr{L}, n, m)\, a_{0,1}(n, m+1).$$

Taking into account the formulas (3.12), (3.13) and (3.14) for $a_{i,j}$, we have formulas for $\dot{A}(\mathscr{L}, n, m)$ and $\dot{B}(\mathscr{L}, n, m)$. So we have formulas for

$$\nabla A'_{n,m}(f) \cdot v_3 = -2\dot{A}(n, m)$$

and

$$\nabla B'_{n,m}(f) \cdot v_3 = -2\dot{B}(n, m).$$

These formulas are the $f_{N,M}$ and $g_{N,M}$ referred to in the Introduction. Note that $\dot{B}$ has weight five.

We can express the functions A and B in terms of ϑ functions. For simplicity, let us assume that all the R_i are the same point R and all the S_i are the same point S, and Q, S and R are all close to P. The formula

$$\phi(z)(\omega) = \int_P^z \omega$$

always gives well defined element of J_g. If $\{\vartheta=0\}+K$ does not contain the image of ϕ, then we get a well defined divisor $\mathcal{D}_K$ on X by pulling back this divisor by ϕ locally. This is well defined, since a choice of a different path from P to z would yield the same divisor. There is a constant $K_{\mathcal{L}} \in H^0(X, \Omega)^*$ so that

$$\mathcal{D}_{K_{\mathcal{L}}}$$

is the divisor of a non-zero section of $\mathcal{L}$. Fix a divisor D_0 of degree $g-1$. For x near P let

$$C_x = K_{\mathcal{O}(x+D_0)}.$$

Notice that

$$(-n-m)\, C_P + (n-m)\, C_Q + m(C_R + C_S) + K_{\mathcal{L}} = K_{\mathcal{L}_{n,m}}.$$

Consider the following meromorphic function on $H^0(X, \Omega)^*$:

$$g_{n,m}(Z) = \vartheta^{-n-m}(Z+C_P)\, \vartheta^{n-m}(Z+C_Q)\, \vartheta^m(Z+C_R)\, \vartheta^m(Z+C_S)\, \vartheta(Z+K_{\mathcal{L}_{n,m}})\, (\vartheta(Z+K_{\mathcal{L}}))^{-1}$$

This function is periodic on $H^0(X, \Omega)^*$ and so $h_{n,m} = g_{n,m} \circ \phi$ is a well defined rational function on X. The divisor of $h_{n,m}$ is just the same as the divisor of $f_{n,m}$. So to compute $d_1(n, m)$ all we have to do is to compute the logarithmic derivative of $h_{n,m}$ at P with respect to z. We have that the derivative of $f \circ \phi$ is just $\nabla f \cdot v_1$. So we obtain

$$d_1(n, m) = C' + C_1\, n + C_2\, m + v_1\, \nabla \log \vartheta(K_{\mathcal{L}_{n,m}})$$

for suitable constants C_1 and C_2 independent of $\mathcal{L}$ and a constant C' dependent on $\mathcal{L}$. Thus we have

$$A(n, m) = C + v_1\, \nabla \log \vartheta(K_{\mathcal{L}_{n+1,m}}) - v_1\, \nabla \log \vartheta(K_{\mathcal{L}_{n,m+1}})$$

for a suitable C independent of $\mathcal{L}$. There is a similar formula for B.

Given A and B in $V_{N,M}$, here is a conjectural construction of a curve X, points

$$P, Q, R_1, \dots, R_M, S_1, \dots, S_M \in X$$

and a line bundle on X which will give back A and B when we apply the construction of this section, at least for generic A and B. We can consider the following difference operator

$$L(\psi) = \psi(n+1, m) - \psi(n, m+1) - A(n, m)\,\psi(n, m) - B(n, m)\,\psi(n-1, m)$$

on the space of all complex functions on $\mathbf{Z}^2$. If $\alpha, \beta \in \mathbf{C}^*$, let $\mathcal{M}_{\alpha,\beta}$ be the set of all ψ so that $L(\psi)=0$ and $\psi(n+N, m)=\alpha\psi(n, m)$ and $\psi(n, m+M)=\beta\psi(n, m)$. Let $\mathcal{B}$ be the set of (α, β) so that the dimension of $\mathcal{M}_{\alpha,\beta}$ is positive. If there is a curve X as in this section having associated A and B, let X' be $X - \{P, Q, R_1, \dots, R_M, S_1, \dots, S_M\}$. Then there is a natural map $\pi\colon X' \to \mathcal{B}$ by sending $x \in X'$ to $(\alpha(x), \beta(x))$. To see that the image of π is in $\mathcal{B}$, we choose an isomorphism of the fiber $\mathcal{L}_x$ with $\mathbf{C}$ and let $\psi(n, m)=s_{n,m}(x) \in \mathcal{L}_x$. Further, there is even a line bundle on the subset $\mathcal{B}_1$ of $\mathcal{B}$ on which the dimension of $\mathcal{M}_{\alpha,\beta}$ is exactly one. This suggests that for generic A and B, that $\mathcal{B}_1 = \mathcal{B}$ and that $\mathcal{B}$ can be compactified to a curve X by adding points $\{P, Q, R_1, \dots, R_M, S_1, \dots, S_M\}$. Further extending the line bundle to a line bundle of X and applying the construction of this section will give back the original generic A and B. But we have not worked out here this conjectural correspondence.

The construction of this section is very close to that of Mumford and van Moerbeke [MM], although we have not worked out the exact relation here. Start with a complex function ϕ on $\mathbf{Z}$. Define ψ on $\mathbf{Z}^2$ by the following inductive procedure on m:

$$\psi(n, 0) = \phi(n)$$

and

$$\psi(n, m+1) = -A(n, m)\,\psi(n, m) - B(n, m)\,\psi(n-1, m) + \psi(n+1, m).$$

Define $L_1(\phi)(n)=\psi(n, M)$. Then $\mathcal{B}$ is the set of (α, β) so that there is a nonzero ϕ with $\phi(n+N)=\alpha\phi(n)$ and $L_1(\phi)=\beta\phi$. Thus $\mathcal{B}$ is the spectral curve associated by Mumford and van Moerbeke to the operator L_1. Note that in Mumford and van Moerbeke's theory, the operator L_1 can be reconstructed from the curve, the line bundle, and the points added to compactify, while the construction here depends on a choice of a decomposition of the zeros of β into M divisors of degree 2. Note that the case $M=1$ is the classical Toda lattice case.

4. Proof of Theorem 1.1

We will prove Theorem 1.1 following Lazarsfeld. We will show that there is a rank two vector bundle E on C with $\det E=\Omega$, $h^0(E)=3$, $h^0(E^*)=0$, and E is generated by global sections. Suppose that we are given such an E. Then setting $H=H^0(E)$, there is a canonical exact sequence

$$0 \to \Omega^{-1} \to H\otimes\mathcal{O} \to E \to 0$$

Next, set $V=H^*$, and dualize this to get:

$$0 \to E^* \to V\otimes\mathcal{O} \to \Omega \to 0. \tag{4.1}$$

Twisting by Ω and take cohomology.

$$H^0(\Omega)\otimes V \to H^0(\Omega^{\otimes 2}) \to H^1(E^*\otimes\Omega) \to V\otimes H^1(\Omega) \to 0.$$

Since $h^0(E)=h^1(E^*\otimes\Omega)=3$, we see that the map $H^1(E^*\otimes\Omega)\to V\otimes H^1(\Omega)$ is an isomorphism. On the other hand, (4.1) lets us view V as a subspace of $H^0(\Omega)$. So the theorem will follow from the existence of such an E.

To construct such an E, we fix a line bundle A on C so that the degree of A is g, $h^0(A)=2$, and A is generated by global sections. Indeed, let $A=\Omega(-P_1-\ldots-P_{g-2})$, where the P_i are chosen generically. Since the map of C to projective space via the canonical map is an embedding, A has the required properties. Note that there is a unique section of $\Omega\otimes A^*$ which vanishes at the P_i. Consider the kernel K of the natural map

$$\alpha\colon \mathrm{Ext}^1(A,\Omega\otimes A^*) \to \mathrm{Hom}(H^0(A), H^1(\Omega\otimes A^*)),$$

which takes an extension

$$0 \to \Omega\otimes A^* \to E \to A \to 0,$$

to the connecting homomorphism it determines. $\mathrm{Ext}^1(A,\Omega\otimes A^*)$ is dual to $H^0(A^{\otimes 2})$ and

$$\mathrm{Hom}(H^0(A), H^1(\Omega\otimes A^*))$$

is dual to $H^0(A)^*\otimes H^0(A)^*$, and α is dual to the multiplication map

$$H^0(A)\otimes H^0(A) \to H^0(A^{\otimes 2}).$$

The base point free pencil trick shows that the cokernel of multiplication has dimension

$h^0(A^{\otimes 2})-3=g-2$. So K is a vector space of dimension $g-2$. On the other hand, any non-trivial extension in K gives a vector bundle E satisfying the desired properties, except that E might not be generated by its global sections.

We will show that if we choose a generic element of K, then the resulting E will be generated by global sections. Suppose E is not generated by global sections. The three sections of E do generate a subsheaf E' of E, which sits in a diagram of extension as follows:

$$\begin{tikzcd} 0 \arrow[r] & \Omega\otimes A^*(-D) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega\otimes A^* \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Since E is generated by global sections away from $P_1 \dots P_{g-2}$, we have that $D \subset \bigcup P_i$. So if E comes from an element $e \in K$ that fails to be generated by global sections, then there is a point P among the P_i and an extension

$$0 \to \Omega\otimes A^*(-P) \to E'' \to A \to 0,$$

so that e is induced from this extension. Note that then such extensions are necessarily surjective on global sections. But such extensions are classified by elements in

$$\ker(H^0(A^{\otimes 2}(P)))^* \to H^0(A)^* \otimes H^0(A(P))^*.$$

Noting that $h^0(A(P))=3$ (since $P \in P_1+\dots+P_{g-2}$), the base point free pencil trick shows that the cokernel of

$$H^0(A)\otimes H^0(A(P)) \to H^0(A^{\otimes 2}(P))$$

has dimension $g-3$. Hence the extensions in K which fail to be generated by global sections have codimension at least 1 and so an extension with all the required properties exists.

5. Kodaira–Spencer theory

Suppose that U is a simply connected neighborhood of 0 in $\mathbf{C}^n$. We will denote the coordinates on $\mathbf{C}^n$ by $z_1, \dots, z_n$. Let $\pi: \mathscr{X} \to U$ be a proper smooth map from an $n+1$ dimensional $\mathscr{X}$ so the fibers $\mathscr{X}_s$ of π are smooth curves of genus g. Let Q be a section of π. Let $\gamma_1, \dots, \gamma_g$ be elements of $H^1(\mathscr{X}, \mathbf{Z})$. Let $\omega_1, \dots, \omega_g$ be global sections of $\Omega_{\mathscr{X}/U}$, the relative one forms on $\mathscr{X} \to U$. Let $\omega_{i,s}$ denote the restriction of ω_i to $H^0(\mathscr{X}_s, \Omega)$. We assume that the $\omega_{i,s}$ form a basis of $H^0(\mathscr{X}_s, \Omega)$ which is dual to the restrictions of the γ_j to $H^1(\mathscr{X}_s, \mathbf{C})$.

Let W_s be the complex span of the γ_i in $H^1(\mathscr{X}_s, \mathbf{C})$. Let us further assume that there are functions A_j on U for $j=2, \ldots, g$, B_j on U for $j=3, \ldots, g$ and C_j on U for $j=4, \ldots, g$ so that if we set

$$\delta_1(s) = \gamma_1 + \sum_{j=2}^{g} A_j(s)\gamma_j$$

$$\delta_2(s) = \gamma_2 + \sum_{j=3}^{g} B_j(s)\gamma_j$$

$$\delta_3(s) = \gamma_3 + \sum_{j=4}^{g} C_j(s)\gamma_j,$$

then for each $s \in U$, $\delta_1(s)$ is a basis for the annihilator of $H^0(\mathscr{X}_s, \Omega(-Q(s)))$ in W_s, so that $\delta_1(s)$ and $\delta_2(s)$ are a basis for the annihilator of $H^0(\mathscr{X}_s, \Omega(-2Q(s)))$ in W_s and so that $\delta_1(s)$ and $\delta_2(s)$ and $\delta_3(s)$ are a basis for the annihilator of $H^0(\mathscr{X}_s, \Omega(-3Q(s)))$ in W_s. Our aim is to compute the partials of the A_j, B_j and C_j in terms of Kodaira–Spencer theory. In particular, we wish to know when the map Φ from U to $\mathbf{C}^{3g-6}$ defined by sending s to the vector

$$(A_2(s), \ldots, A_g(s), B_3(s), \ldots, B_g(s), C_4(s), \ldots, C_g(s))$$

has maximal rank.

To compute these partials, we introduce the following functions a_i, b_i and c_i so that

$$\omega'_{2,s} = \omega_{2,s} - a_2(s)\,\omega_{1,s} \in H^0(\mathscr{X}_s, \Omega(-Q(s)))$$

$$\omega'_{3,s} = \omega_{3,s} - a_3(s)\,\omega_{1,s} - b_3(s)\,\omega_{2,s} \in H^0(\mathscr{X}_s, \Omega(-2Q(s)))$$

and

$$\omega'_{j,s} = \omega_{j,s} - a_j(s)\,\omega_{1,s} - b_j(s)\,\omega_{2,s} - c_j(s)\,\omega_{3,s} \in H^0(\mathscr{X}_s, \Omega(-3Q(s)))$$

for $j > 3$. By evaluating the identities

$$\langle \delta_1(s), \omega'_{j,s} \rangle = 0,$$

$$\langle \delta_2(s), \omega'_{j,s} \rangle = 0$$

for $j>2$ and

$$\langle \delta_3(s), \omega'_{j,s} \rangle = 0$$

for $j>3$, we see that the a_i, b_i and c_i can be expressed in term of the A_i, B_i and C_i. For instance, $c_i=C_i$. On the other hand, the functions $a'_i=\langle \delta_1(0), \omega'_j \rangle$, $b'_i=\langle \delta_2(0), \omega'_j \rangle$, $c'_i=\langle \delta_3(0), \omega'_j \rangle$ can all be expressed in terms of the functions a_i, b_i and c_i by expanding their definitions. So it suffices to determine when the map ϕ from U to $\mathbf{C}^{3g-6}$ defined by sending u to the vector

$$(a'_2(s), \ldots, a'_g(s), b'_3(s), \ldots, b'_g(s), c'_4(s), \ldots, c'_g(s))$$

has maximal rank.

Let us assume for the moment that $n=\dim U$ is one and that $z=z_1$ is the coordinate. By shrinking U, we can find a cover $\{U_\alpha\}$ of $\mathscr{X}_0$ and holomorphic embeddings h_α: $U_\alpha \times U \to \mathscr{X}$ so that $\pi \circ h_\alpha$ is just the projection of $U_\alpha \times U$ to U and so that if $Q(0) \in U_\alpha$, then $Q(s)=h_\alpha(Q(0), s)$. Let Θ be the sheaf of holomorphic derivations on $\mathscr{X}_0$, i.e. the dual of the sheaf of holomorphic one forms. If f is a function on an open subset V of U_α, define $T_\alpha(f)$ to be the function on $h_\alpha(V \times U)$ defined by $T_\alpha(f)(h_\alpha(v, s))=f(v)$. There is a cocycle $D_{\alpha,\beta} \in H^0(U_\alpha \cap U_\beta, \Theta)$ so that if f is a function on a non-empty open V of $\mathscr{X}_0$ then

$$\lim_{z \to 0} \left(\frac{T_\beta(f) - T_\alpha(f)}{z} \right) = D_{\alpha,\beta}(f),$$

where both sides are defined. Thus we get a Kodaira–Spencer class

$$KS \in H^1(\mathscr{X}_0, \Theta(-Q(0))),$$

which is easily seen to be independent of the choices of covers we have made.

Let ω be a meromorphic form on $\mathscr{X}_0$ which is of the second kind, i.e. it is locally exact. By choosing the U_α simply connected, we can write $\omega=df_\alpha$, where f_α is meromorphic on U_α. Then $c_{\alpha,\beta}=\{f_\alpha - f_\beta\}$ defines a cocycle with values in $\mathbf{C}$ and gives a well defined element $L_\omega \in H^1(\mathscr{X}_0, \mathbf{C})=H^1(\mathscr{X}_s, \mathbf{C})$. Let ω' be a section of $\Omega_{\mathscr{X}/D}((-k+1)\,Q(U))$. So for each $s \in U$, ω'_s is a holomorphic one form on $\mathscr{X}_s$ vanishing $k-1$ times at $Q(s)$. Suppose that $\omega \in H^0(\mathscr{X}_0, \Omega(kQ(0)))$. We have a function defined on U by the following process: The product of L_ω and ω_s is a well defined element denoted $\langle \omega, \omega' \rangle_s$ of $H^1(\mathscr{X}_s, \Omega)=\mathbf{C}$. Note that $\langle w, w' \rangle_0=0$, since L_ω maps to zero in $H^1(\mathscr{O}(k-1)(Q(0)))$ (L_ω is the coboundary of the $\{f_\alpha\}$). We have the following formula:

$$\left(\frac{d}{dz} \langle \omega, \omega' \rangle_z \right)_{z=0} = KS(\omega)(\omega').$$

We have used the multiplication maps

$$H^1(\Theta(-Q(0)))\times H^0(\Omega(k(Q(0))))\to H^1(\mathcal{O}(k-1)(Q(0)))$$

to evaluate $KS(\omega)$ and

$$H^1(\mathcal{O}((k-1)(Q(0))))\times H^0(\Omega((1-k)(Q(0))))\to H^1(\Omega)=\mathbf{C}$$

to evaluate $KS(\omega)(\omega')$.

This formula is easily proved. For let $\lambda_{\alpha,\beta}=T_\alpha(f_\alpha)-T_\beta(f_\beta)-c_{\alpha,\beta}$ define a cohomology class Λ in $H^1(\mathcal{O}((k-1)\,Q(U)))$. Note that the $\lambda_{\alpha,\beta}$ all vanish when $z=0$. So

$$\lim_{z\to 0}\frac{\lambda_{\alpha,b}}{z}=D_{\alpha,\beta}(f_\alpha),$$

since

$$T_\beta(f_\alpha)=T_\beta(f_\beta)-c_{\alpha,\beta}.$$

Consequently,

$$\lim_{z\to 0}\left\langle\frac{\Lambda}{z},\omega'\right\rangle_z=KS(\omega)(\omega').$$

On the other hand, for $z\neq 0$, then Λ is the image of L_ω in $H^1(\mathscr{X}_s,\mathcal{O}((k-1)\,Q(s)))$. Thus $\langle\Lambda,\omega'\rangle_z=\langle\omega,\omega'\rangle_z$, and so our formula is established.

Let us return to the case of U of dimension n. We can apply the analysis of the preceding paragraph to the curve C_i defined by be setting all the $z_j=0$ for $j\neq i$. This will give an element $KS(\partial/\partial z_i)\in H^1(\mathscr{X}_0,\Theta(-Q(0)))$. We have

$$\left(\frac{\partial a_j'}{\partial z_i}\right)_{z=0}=KS\left(\frac{\partial}{\partial z_i}\right)(\delta_1(0))(\omega_j')$$

for $j>1$

$$\left(\frac{\partial b_j'}{\partial z_i}\right)_{z=0}=KS\left(\frac{\partial}{\partial z_i}\right)(\delta_2(0))(\omega_j')$$

for $j>2$

$$\left(\frac{\partial c_j'}{\partial z_i}\right)_{z=0}=KS\left(\frac{\partial}{\partial z_i}\right)(\delta_3(0))(\omega_j')$$

for $j>3$.

THEOREM 5.1. *Suppose that the $KS(\partial/\partial z_i)$ actually span $H^1(\mathscr{X}_0, \Theta(-Q(0)))$ and that the δ_i are acceptable. Then the map Φ defined above has maximal rank.*

Proof. By duality, the map from $H^1(\mathscr{X}_0, \Theta(-Q(0)))$ to

$$H^1(\mathscr{X}_0, \mathscr{O}(Q(0)))\oplus H^1(\mathscr{X}_0, \mathscr{O}(2Q(0)))\oplus H^1(\mathscr{X}_0, \mathscr{O}(3Q(0)))$$

defined by $\theta \to \Sigma\, \delta_i \theta$ is surjective.

6. A monodromy argument

There are smooth analytic manifolds U_1 and $\mathscr{X}$ and a proper smooth morphism $\pi: \mathscr{X} \to U_1$ and a section Q of π so that the dimension of U_1 is $3g-2$ and the dimension of $\mathscr{X}$ is $3g-1$ and so that the induced map $G: U_1 \to \mathscr{M}_{g,1}$ is surjective, where G is defined by $G(u)=(\pi^{-1}(u), Q(u))$. Further we may choose U_1 and π so that they are defined over $\mathbf{R}$ and so that for any point $u \in U_1$, there is a coordinate system z_i so that the Kodaira–Spencer classes $KS(\partial/\partial z_i)$ generate $H^1(\mathscr{X}_u, \Theta(-Q(u)))$. In such a situation there is a monodromy map $T: \pi_1(U_1, s) \to \mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{Z}))$, where $\mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{Z}))$ is the group of symplectic automorphisms of $H^1(\mathscr{X}_s, \mathbf{Z})$. We can assume that the image of T is a subgroup of finite index in $\mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{Z}))$. We will establish the following later in this section:

PROPOSITION 6.1. *There is a dense set of points $u \in U_1(\mathbf{R})$ so that if v_1, v_2 and v_3 are adapted and in $\Lambda^+(\mathscr{X}_u)(\mathbf{R})$, then the v_i are acceptable.*

Let U_2 be the set of all (u, Λ) so that $u \in U_1$ and Λ is a complex subspace of $H^1(\mathscr{X}_u, \mathbf{C})$ of dimension g. Note that U_2 inherits the natural structure of a complex manifold of dimension $3g-2$ so that the projection map P_1 from U_2 to U_1 is a covering map. Indeed, let W be simply connected neighborhood of $u \in U_1$. Then $H^1(\mathscr{X}_w, \mathbf{C})$ for $w \in W$ form a local system of vector spaces on W, which is trivial, since W is simply connected. Thus we get an identification ϕ_w of $H^1(\mathscr{X}_u, \mathbf{C})$ with $H^1(\mathscr{X}_w, \mathbf{C})$. The map $\psi: w \mapsto (w, \phi_w(\Lambda))$ is a section of P_1 and defines a chart for the holomorphic structure of U_2, by definition. Note that U_2 also inherits a real structure. Indeed if $(u, \Lambda) \in U_2$, then the antiholomorphic involution on $\mathscr{X}$ restricts to an antiholomorphic map from $\mathscr{X}_u$ to $\mathscr{X}_{u^\iota}$. Let Λ' be the image of Λ under this map. We define ι on U_2 by $(u, \Lambda)^\iota = (u^\iota, \Lambda')$. In particular, if $(u, \Lambda) \in U_2$ is a real point, then Λ is invariant under ι and so $\Lambda = \Lambda^+(\mathscr{X}_u)\cdot \mathbf{C}$.

If $(u, \Lambda) \in U_2$, there is a natural map from Λ to $H^0(\mathscr{X}_u, \Omega)^*$ induced by cup product. Let $U_3 \subset U_2$ be the set of all (u, Λ) so that this map is an isomorphism. Note that the real points of U_2 are all in U_3. Let U_4 be the set of all $(u, \Lambda) \in U_3$ so that there are adapted $v_i \in \Lambda$ which are acceptable. U_4 is an open subset of U_3 whose complement is defined by analytic equations.

We wish to show

PROPOSITION 6.2. $\bar{U}_4 = \bar{U}_3$.

Proof. Both U_4 and U_3 are defined locally by the non-vanishing of analytic equations. If the proposition were false, there is a whole component U_5 of $\bar{U}_3$ contained in the complement of U_4. $\pi_1(U_1, s)$ acts on U_5, since the projection from U_5 to U_1 is a covering map. If $\gamma \in \pi_1(U_1, s)$, the image of (s, Λ) under γ, which is just $(s, T(\gamma)(\Lambda))$, would be in U_5. Thus if (s, Λ') is any point in U_5, then $(s, T(\gamma)(\Lambda')) \in U_5$. In particular, if s is a generic point of U_5, we would have that $A(\Lambda)$ would not contain an acceptable adapted set for all A in some subgroup H of finite index in $\mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{Z}))$. But H is Zariski dense in $\mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{C}))$. It follows that for all $A \in \mathrm{Sp}(H^1(\mathscr{X}_s, \mathbf{C}))$, we would have that $A(\Lambda)$ would not contain an acceptable adapted set.

Consider the transvection

$$T_w(v) = v + \langle v, w \rangle \, w,$$

where w is a holomorphic one form on $\mathscr{X}_s$. If v_i' for i from one to three form a basis of $H^0(\mathscr{X}_s, \Omega(4Q(s))) \cap \Lambda'$, the $T_w(v_i')$ form a basis of $H^0(\mathscr{X}_s, \Omega(4Q(s))) \cap T_w(\Lambda')$. We say that v_i' for i from one to three satisfy a nontrivial relation if there are nontrivial ω_i in $H^0(\mathscr{X}_s, \Omega(-iQ(s)))$ so that $v_1'\omega_1 + v_2'\omega_2 + v_3'\omega_3 = 0$. Let's suppose that if $w_i \in H^0(\mathscr{X}_s, \Omega)$ are chosen generically, then the w_i do not satisfy any relation and that $(s, \Lambda) \in U_5$. Suppose the v_i are an adapted set in Λ. We may choose the v_i and the w_i so that $\langle v_1, w_2 \rangle = 0$, $\langle v_1, w_3 \rangle = 0$, and $\langle v_2, w_3 \rangle = 0$, but that $\langle v_1, w_1 \rangle \neq 0$, $\langle v_2, w_2 \rangle \neq 0$, and $\langle v_3, w_3 \rangle \neq 0$. Let $S_t = t^2(T_{t^{-1}w_3} \circ T_{t^{-1}w_2} \circ T_{t^{-1}w_1})$. Then

$$S_t(v_1) = t^2 v_1 + \langle v_1, w_1 \rangle \, w_1$$

$$S_t(v_2) = t^2 v_2 + \langle v_2, w_1 \rangle \, w_1 + \langle v_2, w_2 \rangle \, w_2$$

$$S_t(v_3) = t^2 v_3 + \langle v_3, w_1 \rangle \, w_1 + \langle v_3, w_2 \rangle \, w_2 + \langle v_3, w_3 \rangle \, w_3.$$

The $S_t(v_i)$ are adapted and $(s, S_t(\Lambda)) \in U_5$, so $S_t(v_i)$ satisfy a nontrivial relation for all t. By taking the limit as $t \to 0$, we see that the w_i would satisfy a non-trivial relation. Thus if U_5 is nonempty, w_i chosen generically would satisfy a nontrivial relation.

We know that the map $\psi: H^0(\mathscr{X}_s, \Omega)^3 \to H^0(\mathscr{X}_s, \Omega^{\otimes 2})$ defined by $(\omega_1, \omega_2, \omega_3) \to$

$w_1\omega_1+w_2\omega_2+w_3\omega_3$ is surjective for generic s and w_i. We may assume that w_1 does not vanish at $Q(s)$, that w_2 vanishes exactly once at $Q(s)$, and that w_3 vanishes exactly twice at $Q(s)$. Since the vectors $(w_2, -w_1, 0)$, $(w_3, 0, -w_1)$ and $(0, w_3, -w_2)$ are in the kernel of ψ, these vectors generate the kernel of ψ. So if $w_1\omega_1+w_2\omega_2+w_3\omega_3=0$ is a nontrivial relation, we must have

$$\omega_1 = aw_2+bw_3$$

$$\omega_2 = -aw_1+cw_3$$

$$\omega_3 = -bw_1-cw_2.$$

But ω_3 vanishes two times at $Q(s)$, so $b=c=0$. So $a=0$. So U_5 is empty.

Proof of Proposition 6.1. Note that Proposition 6.2 implies that $U_4(\mathbf{R})$ is dense in $U_2(\mathbf{R})$. Indeed, U_4 is dense in U_2 and the complement of U_4 is defined locally by analytic equations. If $U_4(\mathbf{R})$ were not dense in $U_2(\mathbf{R})$, then the equations defining the complement of $U_4(\mathbf{R})$ would vanish of an open subset of $U_2(\mathbf{R})$ and hence on an open subset of U_2. Further, $U_2(\mathbf{R})$ actually maps onto $\mathcal{M}_{g,1}(\mathbf{R})$, and if $(u, \Lambda)\in U_2(\mathbf{R})$, then Λ is the lattice fixed by the antiholomorphic involution of $\mathcal{X}_u$. It follows that for a dense set of points (C, P) in $\mathcal{M}_{g,1}(\mathbf{R})$ so that if the $v_i\in\Lambda^+$ adapted, then the v_i are acceptable.

Proof of Proposition 2.4. Let $u\in U_4(\mathbf{R})$. We can choose a basis γ_i of $\Lambda^+(\mathcal{X}_u)$ so that for suitable choice of a_i, b_i and c_i we have that

$$V_1 = \gamma_1+\sum_{j=2}^{g} a_j\gamma_j$$

$$V_2 = \gamma_2+\sum_{j=3}^{g} b_j\gamma_j$$

$$V_3 = \gamma_3+\sum_{j=4}^{g} c_j\gamma_j$$

are an adapted and therefore acceptable set v_i. We can therefore find $A_j(s), B_j(s), C_j(s)$ locally as in section 3 so that $V_i=\delta_i(u)$, and so that the $\delta_i(s)$ satisfy the condition in the first paragraph of section 5. The map Φ restricts locally to a map of maximal rank from $U_4(\mathbf{R})$ to $\mathbf{R}^{3g-6}$. In particular, we can find points s near to u so $A_j(s)$ and $B_j(s)$ are rational. Thus we see that $(\mathcal{X}_s, Q(s))$ is good.

7. Proof of Theorem 2.5

We use the notation of §4. Assume that γ_1, γ_2 and γ_3 are adapted at 0, i.e. that $\bar{\gamma}_i \in H^0(\mathscr{X}_0, \Omega(i+1)\,Q(0))$ and that they form a basis of $H^0(\mathscr{X}_0, \Omega(4(Q(0))))$. It follows that $\omega_{2,0} \in H^0(\Omega(-Q(0)))$ and that $\omega_{3,0} \in H^0(\Omega(-2Q(0)))$. Assume the rest of the γ_i defined over $\mathbf{R}$ and that $(\mathscr{X}_0, Q(0))$ is good. Then we can assume that γ_1 and γ_2 are both in $H^1(\mathscr{X}_0, \mathbf{Z})$. By replacing U by a smaller neighborhood of 0, we can find a neighborhood of $Q(u)$ and a function z defined over $\mathbf{R}$ so that $z=0$ is the defining equation for $Q(U)$ and so that $dz=\omega_1$ as relative forms. If $s \in U$ and $R \in \mathscr{X}_s$ so that $z(R)=\zeta$ and ω_s is a holomorphic one form on $\mathscr{X}_s$, we denote

$$\int_{Q(s)}^{R} \omega_s$$

by

$$\int_0^{\zeta} \omega_s,$$

where the integral is to be taken on a path connecting $Q(s)$ and R lying close to $Q(s)$.

If $e \in \mathbf{C}$, there are $e^+(z)$ and $e^-(z)$ so that $e^+(z)+e^-(z)=z$ and $z^2-(e^+(z))^2-(e^-(z))^2=2z^2e$ for z small. Consider the following functions:

$$A_j'(s,z,e) = \frac{\int_0^z \omega_j}{z},$$

where $j=2,\ldots,g$,

$$B_j'(s,z,e) = \left(1+\frac{1}{z^2}\right)\left(\int_0^z \omega_j - \int_0^{e^+(z)} \omega_j - \int_0^{e^-(z)} \omega_j\right)$$

and

$$C_j' = \left(\frac{d^2}{dz^2}\left(\frac{\omega_j}{dz}\right)\frac{d}{dz}\left(\frac{\omega_2}{dz}\right) - \frac{d^2}{dz^2}\left(\frac{\omega_2}{dz}\right)\frac{d}{dz}\left(\frac{\omega_j}{dz}\right)\right)_{z=0}.$$

These functions are defined on $U \times (D-\{0\}) \times D$, where D is some neighborhood of $0 \in \mathbf{C}$ so that all the integrals above are defined. The significance of these functions is the following: Suppose we have a point $(s, 1/N, e)$ with $N \in \mathbf{Z}$ with $N >> 0$ and that $A_j'(s,1/N,e)=0$ for $j>1$, $B_j'(s,1/N,e)=0$ for $j>2$, $B_2'(s,1/N,e)=1$ and $C_j'(s,1/N,e)=0$ for $j>3$. Then there are points P, Q, S and R all in $\mathscr{X}_s$ so that $z(P)=0$, $z(Q)=1/N$,

$z(S)=e^{+}(1/N)$, and $z(R)=e^{-}(1/N)$. Further, the divisor $P-Q$ is a point of order N in the Jacobian of $\mathscr{X}_s$, and the divisor $P+Q-R-S$ is a point of order N^2+1, since $B_1'(s,z,e)=1$. Finally, the linear functional L on $H^0(\Omega)$:

$$\omega \to \left(\frac{d^2}{dz^2}\left(\frac{\omega}{dz}\right)\frac{d}{dz}\left(\frac{\omega_2}{dz}\right)-\frac{d^2}{dz^2}\left(\frac{\omega_2}{dz}\right)\frac{d}{dz}\left(\frac{\omega}{dz}\right)\right)_{z=0}$$

vanishes on the span of $\omega_1, \omega_2, \omega_4 \ldots$ and so L must to a multiple of

$$\omega \to \langle \omega, \gamma_3 \rangle.$$

Note that L is in the osculating three space of the curve $\mathscr{X}_s$ at $Q(s)$.

We claim that the A_j' and B_j' can be extended as holomorphic functions to $U\times D\times D$, and that $A_j'(s,0,e)$ and $B_j'(s,0,e)$ can be computed in terms of e and the a_j and b_j of § 3. We can write near $Q(s)$

$$\omega_{i,s}=\alpha_i(s)\,dz+\beta_i(s)\,z\,dz+\frac{1}{2}\varepsilon_i(s)\,z^2\,dz+D_i z^3\,dz,$$

where the α_i and β_i are functions on U and the D_i are functions on $\mathscr{X}$ defined locally. Note that

$$\alpha_1=1.$$

Similarly, we see that

$$\beta_1=\varepsilon_1=0,$$

and that

$$\alpha_2(0)=\alpha_3(0)=\beta_3(0)=0.$$

Note that β_2 is nonzero near $s=0$, since the form $\omega_{2,0}$ is in $H^0(\mathscr{X}_0, \Omega(-Q(0)))$, but not in $H^0(\mathscr{X}_0, \Omega(-2Q(0)))$. Similarly, $\varepsilon_3(0)\neq 0$. Since $\omega_{2,s}-a_2(s)\,\omega_{1,s}$ vanishes at $Q(s)$, we see that $\alpha_2(s)=a_2(s)$. Similarly, $\omega_{3,s}-a_3(s)\,\omega_{1,s}-b_3(s)\,\omega_{2,s}$ vanishes twice at $Q(s)$, so

$$\alpha_3(s)-a_3(s)-b_3(s)\,\alpha_2(s)=0$$

and

$$\beta_3(s)-b_3(s)\,\beta_2(s)=0.$$

For $i>3$ we have

$$\alpha_i(s) - a_i(s) - b_i(s)\,\alpha_2(s) - c_i(s)\,\alpha_3(s) = 0$$

and

$$\beta_i(s) - b_i(s)\,\beta_2(s) - c_i(s)\,\beta_3(s) = 0,$$

and

$$\varepsilon_i(s) - b_i(s)\,\varepsilon_2(s) - c_i(s)\,\varepsilon_3(s) = 0.$$

Further, we have

$$\lim_{z\to 0} A_j = \alpha_j$$

and

$$\lim_{z\to 0} B_j = e\beta_j.$$

These can be seen by the formulas:

$$\int_0^{\zeta} \omega_{i,s} = \zeta\alpha_{i,s} + \frac{\zeta^2}{2}\beta_{i,s} + \dots .$$

Note that

$$C_j' = \varepsilon_j\beta_2 - \varepsilon_2\beta_j.$$

By shrinking U, we may assume that β_2 and ε_3 never vanish on U. Examining these equations, we see that the subvariety of U defined by the vanishing of α_i for $i>1$, the β_i for $i>2$ and the $C_i'>3$ is contained in the subvariety of U defined by the vanishing of a_i for $i>1$, the b_i for $i>2$ and the c_i for $i>3$. Using the fact that the a_j for $j>1$, the b_j for $j>2$ and the c_j for $j>3$ all have independent gradients, we see that the α_j for $j>1$, the β_j for $j>2$ and the C_j' for $j>3$ all have independent gradients. Thus, A_j' for $j>1$, the B_j' for $j>2$ and the C_j' for $j>3$ have independent gradients when restricted to the set $z=0$ near $s=0$. So if

$$R = \left(0, 0, \frac{1}{\beta_2(0)}\right),$$

then the equations $A_j'=0$ for $j>1$, $B_j'=0$ for $j>2$, $B_2'=1$ and $C_j'=0$ for $j>0$ defines a smooth manifold $W \subset U \times D \times \mathbf{C}$ in a neighborhood of R and that the function z defines a smooth map from W to D near R. We have established:

PROPOSITION 7.1. *If (C, P) is a good pair and v_i for i from 1 to 3 is an adapted set with v_1 and v_2 in $\Lambda^+(C)$ and $v_3 \in \Lambda^+(C)(\mathbf{R})$, then there is a family $\pi: \mathcal{Y}_s \to D$, which is proper and smooth over D and sections $P_i: D \to \mathcal{Y}$ such that if $\mathcal{Y}_z$ denotes the fiber of $\mathcal{Y}$ over z, then $(C, P) = (\mathcal{Y}_0, P_i(0))$. Further, if we denote by v_i the element of $H^1(\mathcal{Y}_z, \mathbf{C})$ obtained by transport of v_i, then for $z \neq 0$, $\bar{v}_3$ is in the osculating three space of $\mathcal{Y}_z$, and for all $\omega \in H^0(\mathcal{Y}_z, \Omega)$, we have*

$$\int_{P_1(z)}^{P_2(z)} \omega = z\langle \omega, v_1 \rangle,$$

and

$$\int_{P_1(z)}^{P_2(z)} \omega - \int_{P_1(z)}^{P_3(z)} \omega - \int_{P_1(z)}^{P_4(z)} \omega = \frac{z^2\langle \omega, v_2 \rangle}{1+z^2}.$$

Further, if z is real, then $\mathcal{Y}_z$ is defined over $\mathbf{R}$ and the points $P_1(z)$, $P_2(z)$ and the divisor $P_3(z)+P_4(z)$ are defined over $\mathbf{R}$.

Choose $\gamma_1, \dots, \gamma_g$ so that $\gamma_i = v_i$ for i from 1 to 3 and $\gamma_i \in \Lambda^+(\mathbf{R})$ and choose

$$\alpha: H^1(\mathcal{Y}, \mathbf{Z}) \to \mathbf{C}_1^*$$

satisfying 2.1 and 2.2. Let ϑ_z be the theta function on $H^0(\mathcal{Y}_z, \Omega)^*$ attached to α. Let

$$f(z, x_1, \dots, x_g; K_1) = \vartheta_z\left(\sum x_i \bar{\gamma}_i + K_1\right),$$

where $\bar{\gamma}_i$ is the image of γ_i in $H^0(\Omega)^*$. We assume that we have chosen K_1 so that $K_1 - K_1^t \in H^1(C, \mathbf{Z})$ and that $f(0; x_1, \dots, x_g; K_1)$ never vanishes for $(x_1, \dots, x_g) \in \mathbf{R}^g$. Define

$$\begin{aligned} H(z; x_1, \dots, x_g; K_1) = &\frac{\partial \log f(z; x_1+z, x_2, \dots, x_g; K_1)}{z \partial x_1} \\ &- \frac{\partial \log f(z; x_1, x_2+z^2/(1+z^2), x_3, \dots, x_g; K_1)}{z \partial x_1} \\ &- \frac{\partial^2 \log f(0; x_1, x_2, \dots, x_g; K_1)}{\partial x_1^2}. \end{aligned}$$

For fixed K_1, note that $H(z; x_1, \dots, x_g; K_1)$ is periodic on $\mathbf{R}^g$ for each z with respect to a lattice in $\mathbf{R}^g$ independent of z. Further, $H(z;x_1,\dots,x_g;K_1)$ can be extended to an analytic function on $D\times\mathbf{R}^g$, which vanishes on $\{0\}\times\mathbf{R}^g$. Thus given ε, there is a δ so that if $|z|<\delta$, then $|H(z;x_1,\dots,x_g;K_1)|<\varepsilon$. Fix $N\in\mathbf{Z}$ so that $1/N<\delta$. Let $\mathscr{L}$ be the line bundle on $C=\mathscr{Y}_{1/N}$ associated to K_1. Let $P=P_1(1/N)$, $Q=P_2(1/N)$, $R=P_3(1/N)$ and $S=P_4(1/N)$. Choose a parameter z around P so that the Frenet frame associated to z is the image of the v_i. Let $A(n,m,\mathscr{L})$ be the functions introduced in section 3. The line bundle $\mathscr{L}_{n,m}$ is associated to

$$K_1+\frac{n}{N}\gamma_1+\frac{m}{N^2+1}\gamma_2=K_{\mathscr{L}_{n,m}},$$

and the equations say that $\mathcal{O}(Q-P)$ is associated to γ_1/N and $\mathcal{O}(P+Q-R-S)$ is associated to

$$\frac{\gamma_2}{N^2+1}.$$

Furthermore,

$$\frac{\partial\log f(1/N;x_1\dots,x_g;K_1)}{\partial x_1}=v_1\,\nabla\log\vartheta_{1/N}\left(\sum x_i\gamma_i+K_1\right).$$

Let

$$h_1(x,y;K_1)=\frac{\partial^2\log f(0;x,y,0,\dots,0;K_1)}{\partial x^2}.$$

Then we have the following formula:

$$H\left(\frac{1}{N};\frac{n}{N},\frac{m}{N^2+1},0,\dots;K_1\right)=NA(n,m,\mathscr{L})+C-h_1\left(\frac{n}{N},\frac{m}{N^2+1};K_1\right).$$

In particular,

$$\left|NA(n,m,\mathscr{L})+C-h_1\left(\frac{n}{N},\frac{m}{N^2+1};K_1\right)\right|<\varepsilon.$$

Fix K_1 and apply the preceding discussion to $K_1+t\gamma_3$, noting that

$$|H(z,x_1,\dots,x_g;K_1+t\gamma_3)|<\varepsilon$$

independent of t. Let

$$h(x,y,t)=h_1(x,y;K_1+t\gamma_3)=\frac{\partial^2\log(0;x,y,t,\dots,0;K_1)}{\partial x^2}.$$

Let $\mathscr{L}_t$ be the line bundle associated to $K_1+t\gamma_3$. Then we have

$$\left|NA(n, m, \mathscr{L}_t)+C-h\left(\frac{n}{N}, \frac{m}{N^2+1}, t\right)\right| < \varepsilon.$$

Thus Theorem 2.5 is established.

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