Harmonic volumes

by

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1. Introduction

Let X be a compact Riemann surface and dh_i, $i=1,2,3$, three real harmonic 1-forms on X satisfying

$$\int_X dh_i \wedge dh_j = 0, \quad i,j = 1,2,3, \tag{1.1}$$

$$\int_\gamma dh_i \in \mathbf{Z}, \quad \text{for any 1-cycle } \gamma \text{ on } x. \tag{1.2}$$

To this triple of harmonic 1-forms, we associate a point in $\mathbf{R}/\mathbf{Z}$, denoted $I(dh_1, dh_2, dh_3)$, which can be defined in two equivalent ways: first, as an iterated integral

$$\int_\gamma (h_1\, dh_2 - \eta_{12}) \operatorname{mod} \mathbf{Z}, \tag{1.3}$$

where γ is a path on X Poincaré-dual to the cohomology class of dh_3, h_1 is a function on γ obtained by integrating dh_1, and η_{12} is a 1-form on X satisfying $d\eta_{12} = dh_1 \wedge dh_2$ (and orthogonal to all closed 1-forms); second, as a volume $\operatorname{mod} \mathbf{Z}$: namely by (1.2), we can integrate the dh_i on X to obtain h_i: $X \to \mathbf{R}/\mathbf{Z}$ which are harmonic. Then $h = (h_1, h_2, h_3)$: $X \to \mathbf{R}^3/\mathbf{Z}^3 = T^3$, and it follows easily from (1.1) that $h(X)$, regarded as a singular 2-cycle, bounds a singular 3-chain c_3 (unique mod integral 3-cycles); we can then take the volume of c_3 ($\operatorname{mod} \mathbf{Z}$) to define $I(dh_1, dh_2, dh_3)$: we call this a "harmonic volume".

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Let $\mathcal{H}_{\mathbf{Z}}$ denote the free abelian group of rank $2g$ spanned by all harmonic 1-forms satisfying (1.2), form its third exterior power $\Lambda^3(\mathcal{H}_{\mathbf{Z}})$ and map it to $\mathcal{H}_{\mathbf{Z}}$ by

$$dh_1 \wedge dh_2 \wedge dh_3 \to \left[\int_X (dh_1 \wedge dh_2)\right] dh_3 + \left[\int_X (dh_2 \wedge dh_3)\right] dh_1 + \left[\int_X (dh_3 \wedge dh_1)\right] dh_2 \tag{1.4}$$

and let $\Lambda^3(\mathcal{H}_{\mathbf{Z}})'$ denote the kernel of (1.4). Let $\nu=2I$: then we will show that ν is a homomorphism,

$$\nu\colon \Lambda^3(\mathcal{H}_{\mathbf{Z}})' \to \mathbf{R}/\mathbf{Z}, \tag{1.5}$$

Let J be the Jacobian variety of X, and let X be imbedded in J with a base point of X going to the identity element of J. Our constructions will be independent of the choice of base point. If we identify $\mathcal{H}_{\mathbf{Z}}$, $H^1(X;\mathbf{Z})$ and $H^1(J;\mathbf{Z})$ then $\Lambda^3(\mathcal{H}_{\mathbf{Z}})$ can be identified with $H^3(J;\mathbf{Z})$ and the subgroup $\Lambda^3(\mathcal{H}_{\mathbf{Z}})'$ is just the subgroup P of *primitive* cohomology classes. The homomorphism ν of (1.5) can then be interpreted as a point on a complex torus, namely the complex dual of $P\otimes\mathbf{R}$ with the correct complex structure, modulo the lattice dual to P: thus ν is a point on an intermediate Jacobian of the manifold $J(X)$.

We next study the variation of I under change of conformal structure on X, i.e. we consider X as a point on Torelli space $\mathcal{T}$ (Riemann surfaces homeomorphic to X together with a homology basis). Fixing the integral cohomology classes of dh_i, $i=1,2,3$, makes $I(dh_1, dh_2, dh_3)$ an $\mathbf{R}/\mathbf{Z}$ valued function on $\mathcal{T}$ whose differential $\delta I(dh_1, dh_2, dh_3)$ is an $\mathbf{R}$-valued linear function on the tangent space to $\mathcal{T}$ at X, i.e. the real part of an element of the complex linear cotangent space at X. Since this cotangent space is just the space of holomorphic quadratic differentials on X, we may regard the differential δI of I as a function $(\Lambda^3\mathcal{H}_{\mathbf{Z}})' \to H^0(X;K^2)$ (the last group denoting the vector space of holomorphic quadratic differentials). We show that, in fact, this function extends to the $\mathbf{C}$-linear homomorphism of $\Lambda^3_{\mathbf{R}}(\mathcal{H})'$ given as follows: letting dh_i, $i=1,2,3$, satisfy (1.1) alone, then $dh_i \wedge dh_j = d\eta_{i,j}$, where $\eta_{i,j}$ is as described following (1.3) (so $\eta_{i,j}$ is a real non-harmonic 1-form). Now consider the non-holomorphic 1-form of type $(1,0)$

$$\eta_{i,j} + i(*\eta_{i,j}) = f_{i,j}(z)\,dz$$

as well as the holomorphic 1-form

$$dh_k + i(*dh_k) = \varphi_k(z)\,dz$$

so that their (symmetric) product $f_{i,j}(z)\,\varphi_k(z)\,(dz)^2$ is a non-holomorphic quadratic differential form on X. Then it turns out that the sum of these non-holomorphic quadratic differentials over the 3 cyclic permutations (i,j,k) of $(1,2,3)$

$$\Sigma(\eta_{ij}+i{*}\eta_{ij})(dh_k+i{*}dh_k) \tag{1.6}$$

is a holomorphic quadratic differential whose imaginary part (up to a factor of 2π) is just $\delta I(dh_1, dh_2, dh_3)$. To sum up, we have constructed a complex-analytic family of complex tori over Torelli space such that $2I=\nu$ is a cross-section and the explicit formula (1.6) for the differential δI shows that this cross-section is holomorphic.

In the last section of the paper we give explicit formulas for calculating the variational map (1.6) for *hyperelliptic* Riemann surfaces: it turns out that while harmonic volume takes on only values 0 or $\frac{1}{2}$ mod $\mathbf{Z}$, i.e. is constant on the hyperelliptic locus in Torelli space, its differential (1.6) is injective on the subspace of tangent vectors to Torelli space normal to this locus (at least with an additional condition on the surface).

My initial development of these results was based entirely on iterated integrals and the intermediate Jacobian, but Langlands remarked that the intermediate Jacobian suggested use of the cycle X in J and this stimulated me to give the harmonic volume definition of I. Later, Ron Donagi remarked that the results on δI at hyperelliptic curves together with recent results of Ceresa [1] suggested a relation of I with the algebraic cycle $X-i(X)$ which is homologous to zero in J (where i is the map on the group J taking each element to its inverse). It was then immediate to prove that the Abel-Jacobi image ν^* of $X-i(X)$ in the intermediate Jacobian of J, and I defined as harmonic volume, are related by

$$\nu^* = 2I. \tag{1.7}$$

Ceresa shows that for a *generic* algebraic curve X the cycle $X-i(X)$ is *not* algebraically equivalent to zero in J, thus answering a question of Weil ([8], p. 331): the main point is to show that ν is not identically zero as "normal function" on moduli space, which he does by evaluating it on a singular curve at the boundary of moduli space. The same result follows from my proof of the non-vanishing of $\delta\nu$. However, the detailed study of I and hence ν done above by iterated integrals allows one to go beyond the case of a "generic curve" and to study the same algebraic equivalence problem for a given curve, say defined over $\mathbf{Q}$: the case of the Fermat curve $x^4+y^4=1$ will be done in another paper [9].

The map (1.6) giving δI is generalized to automorphic forms of higher weight in [4].

Iterated integrals of differential forms on general differentiable or Riemannian manifolds have been studied by K. T. Chen and by D. Sullivan (see [7] and references given there) but without conditions such as (1.1–1.3): their work leads to real *nilpotent* Lie groups. Iterated integrals of holomorphic 1-forms on a Riemann surface were studied by Gunning [3] and Hwang-Ma (Ph.D. thesis at Brown University and [5]) the latter in the context of complex nilpotent Lie groups.

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2. Iterated integrals

The aim of this section is to show that $\nu=2I$, with I defined by (1.3), as an $\mathbf{R}/\mathbf{Z}$ valued function of certain triples dh_1, dh_2, dh_3 of real harmonic 1-forms (i.e. those satisfying 1.1, 1.2), can be regarded as a homomorphism of a certain subgroup P of $\Lambda^3H^1(X;\mathbf{Z})$ into $\mathbf{R}/\mathbf{Z}$. We identify real harmonic 1-forms on X with their cohomology classes in $H^1(X)$ (coefficients $\mathbf{Z}$ or $\mathbf{R}$), and will denote by $x\cdot y$ the pairing (with values in $\mathbf{Z}$ or $\mathbf{R}$) given by evaluating the cup product of the 1-dimensional cohomology classes x, y on the fundamental 2-dimensional homology class $\mathcal{O}_X$ of X. Throughout we assume genus $g\geqslant 3$.

Furthermore under inclusion $X\subset J$=Jacobian variety of X, we may identify $H^1(X;\mathbf{Z})$ with $H^1(J;\mathbf{Z})$ and $\Lambda^qH^1(X;\mathbf{Z})$ with $H^q(J;\mathbf{Z})$ for $0\leqslant q\leqslant 2g$. The alternating bilinear form $x\cdot y$ on H^1 extends to a bilinear form (non-degenerate, unimodular) on the whole of $H^*(J;\mathbf{Z})$ and corresponds to an element ω of $H^2(J;\mathbf{Z})$ (imaginary part of the Kähler metric). For $q\leqslant g$, the subspace of primitive elements in $H^q(J;\mathbf{Z})$ is defined as the kernel of the homomorphism $L^{g-q+1}\colon H^q(J;\mathbf{Z})\to H^{2g-q+2}(J;\mathbf{Z})$ where L is cup product with ω.

We now assert:

THEOREM. *Let the homomorphism* $\bar{p}\colon\Lambda^3H^1(X,\mathbf{Z})\to H^1(X;\mathbf{Z})$ *be defined by* $\bar{p}(x\wedge y\wedge z)=(x\cdot y)z+(y\cdot z)x+(z\cdot x)y$, *and let P denote the kernel of* $\bar{p}$. *Then*

(1) *P is generated by elements $x\wedge y\wedge z$ (belonging to P) such that $x\cdot y=y\cdot z=z\cdot x=0$. There is a unique homomorphism $\nu\colon P\to\mathbf{R}/\mathbf{Z}$ such that $\nu(x\wedge y\wedge z)$ is twice the iterated integral* (1.3) *for* $x, y, z=dh_1, dh_2, dh_3$.

(2) *If we identify $\Lambda^3H^1(X;\mathbf{Z})$ with $H^3(J;\mathbf{Z})$ then P is the subgroup of all primitive elements.*

The rest of this section will give the proof of this theorem. We will first give a proof of (2) of the theorem, then prove (1).

The proof of part 2 is just a review of Poincaré duality in X, J.

Let $A_i, B_i, i=1,\dots,g$, be a symplectic basis of $H^1(J;\mathbf{Z})=V$ and denote cup product by $\wedge$ in $H^*(J;\mathbf{Z})=\Lambda^*(V)$. Then $\omega=\Sigma_1^g B_i\wedge A_i$ and

$$\frac{\omega^k}{k!}=\sum_{i_1<\dots<i_k} B_{i_1}\wedge A_{i_1}\wedge\dots\wedge B_{i_k}\wedge A_{i_k}.$$

First consider Poincaré duality in X, given by cap product with $\mathcal{O}_X$: let $a_i=A_i\cap\mathcal{O}_X, b_i=B_i\cap\mathcal{O}_X$ be the Poincare duals in X of A_i, B_i, so that for all $\gamma\in H_1$ with Poincare dual $\Gamma\in H^1$,

$$\int_\gamma A_i=a_i\circ\gamma=\int_X A_i\wedge\Gamma=A_i\cdot\Gamma$$

and similarly for $\int B_i, b_i$. In particular for $\gamma=b_i$,

$$\int_{b_i} A_i=a_i\circ b_i=A_i\cdot B_i=1$$

and similarly, $\int_{a_i}B_i=b_i\circ a_i=B_i\cdot A_i=-1$.

Next denote by h the inclusion of X in J. Cap products in X, J are related by the formula

$$C\cap h_*(x)=h_*(h^*C\cap x)$$

for $C\in H^p(J), x\in H_r(X), p\leq r$. Furthermore, since J is a topological group, its homology is a ring with product denoted again by $\wedge$: in fact this is just the exterior algebra on $H_1(J;\mathbf{Z})$. For $C\in H^1(J)$, cap product with C will be denoted $i(C)$ and is a derivation of the homology ring:

$$C\cap(x_p\wedge y_q)=i(C)(x_p\wedge y_q)=[i(C)(x_p)]\wedge y_q+(-1)^p x_p\wedge i(C)(y_q)$$

where $x_p\in H_p, y_q\in H_q$. In particular we can now check that as element of $H_2(J)$,

$$h_*(\mathcal{O}_X)=\sum_{j=1}^g b_j\wedge a_j$$

since on identifying A_i, B_i with h^*A_i, h^*B_i and a_i, b_i with h_*a_i, h_*b_i we have

$$A_i \cap \mathcal{O}_X = a_i = i(A_i)(\Sigma\, b_j \wedge a_j),$$
$$B_i \cap \mathcal{O}_X = b_i = i(B_i)(\Sigma\, b_j \wedge a_j).$$

Finally if *we denote* $h_*(\mathcal{O}_X)$ by η, $\eta^g/g! = \pm\mathcal{O}_J$.

The following proposition will now imply (2) of the theorem:

PROPOSITION. *Let* $\eta = \Sigma b_i \wedge a_i = h_*(\mathcal{O}_X) \in H_2(J;\mathbf{Z})$ *and let* $\bar{p}\colon H^3(J;\mathbf{Z}) \to H^1(J;\mathbf{Z})$ *be defined as in the theorem. Then*

(1) *For* $\alpha \in H^3(J;\mathbf{Z})$, $(L^{g-2}(\alpha)/(g-2)!) \cap (\eta^g/g!) = (-1)^g \bar{p}(\alpha) \cap \eta$.

(2) *Since* $\cap\eta$ *is an isomorphism of* $H^1(J)$ *with* $H_1(J)$ *(corresponding to Poincaré duality in* X*), and since* $\cap(\eta^g/g!)$ *is an isomorphism (Poincaré duality in* J*) the kernel of* $\bar{p}$ *is the kernel of* L^{g-2} *on* $H^3(J)$.

Proof. Let $\alpha \in H^3$, so we may assume α has the form $C_1 \wedge C_2 \wedge C_3$ where $C_i \in H^1$, since it suffices to prove (1) for a basis of H^3.

$$\frac{L^{g-2}(\alpha)}{(g-2)!} \cap \frac{\eta^g}{g!} = \frac{\alpha \wedge \omega^{g-2}}{(g-2)!} \cap \frac{\eta^g}{g!} = \alpha \cap \left(\frac{\omega^{g-2}}{(g-2)!} \cap \frac{\eta^g}{g!}\right).$$

Since

$$\frac{\omega^{g-2}}{(g-2)!} = \sum_{i<j} B_1 \wedge A_1 \wedge \ldots \wedge \widehat{B_i \wedge A_i} \wedge \ldots \wedge \widehat{B_j \wedge A_j} \wedge \ldots \wedge B_g \wedge A_g$$

(where $\widehat{\ }$ denotes omission of the symbol $B \wedge A$ under it) and $\eta^g/g! = b_1 \wedge a_1 \wedge \ldots \wedge b_g \wedge a_g$ we find that

$$\frac{\omega^{g-2}}{(g-2)!} \cap \frac{\eta^g}{g!} = \pm\frac{\eta^2}{2!} = \pm\sum_{i<j} b_i \wedge a_i \wedge b_j \wedge a_j.$$

Finally,

$$\frac{L^{g-2}(\alpha)}{(g-2)!} \cap \frac{\eta^g}{g!} = \pm\alpha \cap \frac{\eta^2}{2!}.$$

Now replacing α by $C_1 \wedge C_2 \wedge C_3$ we get

$$\alpha \cap \frac{\eta^2}{2!} = i(C_1)\, i(C_2)\, i(C_3) \left(\frac{\eta^2}{2!}\right)$$

by the associativity of the cap product. By the derivation law,

$$i(C_2)\, i(C_3) \left(\frac{\eta^2}{2!}\right) = i(C_2)\,[i(C_3)(\eta)\,\eta] = [i(C_2)\, i(C_3)(\eta)]\,\eta - i(C_3)(\eta) \wedge i(C_2)(\eta),$$

$$i(C_1)\, i(C_2)\, i(C_3) \left(\frac{\eta^2}{2!}\right) = [i(C_2)\, i(C_3)(\eta)](i(C_1)(\eta)) - i(C_1)\, i(C_3)(\eta) \wedge i(C_2)(\eta)$$
$$+ i(C_3)(\eta) \wedge i(C_1)\, i(C_2)(\eta).$$

Since $i(C_j)\, i(C_k) = -i(C_k)\, i(C_j)$, we get

$$i(C_1)\, i(C_2)\, i(C_3) \left(\frac{\eta^2}{2!}\right) = \sum_{(1,2,3)} [i(C_1)\, i(C_2)(\eta)]\, i(C_3)\eta \text{ (sum over 3 cyclic permutations).}$$

However $i(C_1)\, i(C_2)(\eta) = (C_1 \wedge C_2) \cap \eta = C_1 \cdot C_2$ (this can also be checked directly by taking each of C_1, C_2 to be an A_i or B_i). Thus the sum over 3 cyclic permutations is just $\bar{p}(C_1 \wedge C_2 \wedge C_3) \cap \eta$, proving the proposition.

We now proceed to (1) of the theorem, and begin by obtaining a basis for P consisting of elements $x \wedge y \wedge z$ where $x, y, z \in H^1(X; \mathbf{Z})$ and their Poincare duals in $H_1(X; \mathbf{Z})$ are represented by three disjoint simple closed curves. Thus consider the following 3 sets of elements of H^3:

(a) $C_i \wedge C_j \wedge C_k$ where $1 \leq i < j < k \leq g$ and each $C_l = A_l$ or B_l. It is clear that such elements are as described above, are linearly independent, belong to P, and there are $2^3\binom{g}{3}$ of them.

(b) $(A_i + A_1) \wedge (B_i - B_1) \wedge C_k$ where i, k are distinct and >1. The curves corresponding to $A_i + A_1$, $B_i - B_1$ can be taken as, respectively, surrounding both the 1st and ith hole, or going around a "handle" separating these two holes (imagined next to one another), while C_k is again A_k or B_k and is attached to a third hole. There are $2(g-1)(g-2)$ such elements.

(c) $(A_i + A_2) \wedge (B_i - B_2) \wedge C_1$. Here $C_1 = A_1$ or B_1, and $i > 2$. These are of the same type as (b), and there are $2(g-2)$ of them.

It is clear that the type (a) elements are linearly independent among themselves, and span a direct summand of $H^3(J; \mathbf{Z})$. Modulo type (a) elements we may replace a type (b), $(A_i + A_1) \wedge (B_i - B_1) \wedge C_k$ by $(A_i \wedge B_i - A_1 \wedge B_1) \wedge C_k$, $i \neq k$, $i, k > 1$, and similarly for type (c) we take $(A_i \wedge B_i - A_2 \wedge B_2) \wedge C_1$ $(i>1)$. These new elements just defined are still in P and belong to the subspace of H^3 spanned by "monomials" $C_i \wedge C_j \wedge C_k$ where each C is A_l or B_l and two of the indices l are equal: thus they lie in a subspace linearly independent of that of the type (a) elements, in which each monomial involves three distinct indices. This shows that the type (a) elements are linearly independent of those of types (b) and (c). Now looking at types (b), (c) and replacing them mod type (a) by new elements as above, we see easily that the new elements are linearly independent among themselves and span a direct summand of P. Finally we have the desired basis of P, since we have a total of $2^3\binom{g}{3}+2(g-1)(g-2)+2(g-2)=\binom{2g}{3}-2g$ elements in P which are a basis for a direct summand of P; furthermore the homomorphism $\bar{p}$: $H^3 \to H^1$ is surjective (for instance, $\bar{p}(A_2 \wedge B_2 \wedge A_1)=A_1$) and H^1 is free over $\mathbf{Z}$ of rank $2g$, while H^3 is free of rank $\binom{2g}{3}$. Incidentally, this basis is easily checked to lie in both the kernel of $\bar{p}$ and the kernel of L^{g-2} which gives an independent proof of the fact that these kernels are equal, since they are easily seen to have the same rank.

The rest of the proof will go as follows. We will first prove that iterated integral can be considered as a homomorphism I: $(H^1 \otimes H^1)' \otimes H^1 \to \mathbf{R}/\mathbf{Z}$ where H^1 is $H^1(X;\mathbf{Z})$, and $(H^1 \otimes H^1)'$ is the kernel of the intersection number homomorphism $H^1 \otimes H^1 \to \mathbf{Z}$; however I will depend on a base point in X. This dependence on the base point is eliminated by restricting I to a subgroup of $(H^1 \otimes H^1)' \otimes H^1$, namely the kernel of the homomorphism $\bar{p}$: $H^1 \otimes$ ... $H^1 \otimes H^1 \otimes H^1 \to H^1 \oplus H^1 \oplus H^1$ given by $\bar{p}(x \otimes y \otimes z)=(x \cdot y)z \oplus (y \cdot z)x \oplus (z \cdot x)y$. Let the kernel of $\bar{p}$ be denoted $(H^1 \otimes H^1 \otimes H^1)'=K$. Within K we have elements $x \otimes y \otimes z$ corresonding to the $x \wedge y \wedge z$ which are a basis for P. Further we have a commutative diagram

$$\begin{array}{ccccc}
(H^1 \otimes H^1 \otimes H^1)' & \longrightarrow & H^1 \otimes H^1 \otimes H^1 & \xrightarrow{\bar{p}} & H^1 \oplus H^1 \oplus H^1 \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} \\
P & \longrightarrow & \Lambda^3 H^1 & \xrightarrow{\bar{p}} & H^1
\end{array}$$

where j_2 is the natural homomorphism of tensor product onto exterior product, $j_3(a \oplus b \oplus c)=a+b+c$ and since $j_3\bar{p}=\bar{p}j_2$, j_2 maps the kernel of $\bar{p}$ into the kernel of $\bar{p}$ thus defining j_1. Finally j_1 is surjective since we have exhibited a basis $x \wedge y \wedge z$ of P and corresponding pre-images $x \otimes y \otimes z$ in $(H^1 \otimes H^1 \otimes H^1)'$.

Next we will obtain a set of generators for $(H^1 \otimes H^1 \otimes H^1)'$, study the values of I on these and finally show that $\nu=2I$ on this group factors through the homomorphism j_1 onto P thus defining ν on P.

We begin with a discussion of iterated integrals. Let dh_i, $i=1,2$ be harmonic, as in the introduction and satisfy (1.1), (1.2), and let $\eta_{1,2}$ be the 1-form uniquely specified by the two conditions: $d\eta_{1,2}=dh_1\wedge dh_2$ and

$$\int_X \eta_{1,2}\wedge(*\alpha)=0 \quad \text{for all closed 1-forms } \alpha. \tag{2.1}$$

(We are considering *real* forms and functions here.) We define iterated integrals as in [7]: consider a closed path γ on X given as a function $\gamma(t)$, $0\leqslant t\leqslant 1$ with values in X. The iterated integral $\int_\gamma h_1\,dh_2=\int_0^1[\int_0^t dh_1(\gamma(s))]\,dh_2(\gamma(t))$ is *not* an invariant of the homotopy class of γ (with fixed base point) since $h_1\,dh_2$ is not a closed 1-form, but $h_1\,dh_2-\eta_{1,2}$ is closed and so $\int_\gamma(h_1\,dh_2-\eta_{1,2})$ gives a well-defined function on $\pi_1(X,x_0)$, to be denoted $I(dh_1,dh_2;\gamma)$. If $\gamma=\gamma'\gamma''$ then it is easily verified that

$$I(dh_1,dh_2;\gamma'\gamma'')=I(dh_1,dh_2;\gamma')+I(dh_1,dh_2;\gamma'')+\left(\int_{\gamma'}dh_1\right)\left(\int_{\gamma''}dh_2\right). \tag{2.2}$$

(This is valid even if the paths γ',γ'' are not closed.) Imposing condition (1.2) gives us then that for fixed dh_1,dh_2 satisfying (1.1) and (1.2), $I(dh_1,dh_2;\gamma')$ *modulo* $\mathbf{Z}$ is a homomorphism of π_1 into $\mathbf{R}/\mathbf{Z}$, and consequently it can be regarded as a homomorphism of the abelianized $\pi_1(X,x_0)$ into $\mathbf{R}/\mathbf{Z}$. Under change of base point from x_0 to x_1, this homomorphism is altered by addition of another homomorphism: let l denote any path from x_1 to x_0, γ a closed path based at x_0, $l\gamma l^{-1}$ one based at x_1, then

$$I(dh_1,dh_2,l\gamma l^{-1})=I(dh_1,dh_2,\gamma)+\int_l\left[\left(\int_\gamma dh_2\right)dh_1-\left(\int_\gamma dh_1\right)dh_2\right]$$

This is proved as follows: first, use the formula (2.2) with $\gamma'=l$, $\gamma''=l^{-1}$ and note that $I(dh_1,dh_2;l\cdot l^{-1})=0$ since $l\cdot l^{-1}$ is nullhomotopic, obtaining $I(dh_1,dh_2;l^{-1})=-I(dh_1,dh_2;l)+\int_l dh_1\int_l dh_2$. Next, use (2.2) again to expand the integral over the product $l\cdot\gamma\cdot l^{-1}$, and use the last formula for l^{-1}.

Poincaré duality assigns to each $\gamma\in H_1(X;\mathbf{Z})$ a harmonic 1-form $dh\in\mathscr{H}_{\mathbf{Z}}$ satisfying

$$\int_{\gamma'}dh=\int\int_X dh\wedge dh'=\gamma\circ\gamma'$$

where γ' corresponds to dh' and $\gamma\circ\gamma'$ is the intersection number. Thus, our formula on change of base point reads as follows if we write $\gamma=\gamma_3$ and let γ_i, $i=1,2,3$ correspond to dh_i (so $\gamma_1\circ\gamma_2=0$):

$$I(dh_1, dh_2, l\gamma_3 l^{-1}) = I(dh_1, dh_2, \gamma_3) + \int_l [(\gamma_1 \circ \gamma_2)\, dh_3 + (\gamma_3 \circ \gamma_1)\, dh_2 + (\gamma_2 \circ \gamma_3)\, dh_1].$$

It is immediate that we can also define iterated integrals not just for pairs dh_1, dh_2 in $\mathcal{H}_{\mathbf{Z}}$ satisfying (1.1) but more generally for bilinear combinations $\Sigma_i dh_{1_i} \otimes dh_{2_i}$ in $\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}}$ satisfying the analogue of (1.3), namely $\int_X \Sigma\, dh_{i1} \wedge dh_{2i} = 0$. We define η_{12} to satisfy $d\eta_{12} = \Sigma\, dh_{1_i} \wedge dh_{2_i}$, and for a path γ let $I(\Sigma\, dh_{1_i} \otimes dh_{2_i}; \gamma) = \int_\gamma [(\Sigma h_{1_i} dh_{2_i}) - \eta_{12}]$. Then all our formulas remain valid if we merely replace the indices $1, 2$ by $1i, 2i$ and sum over i. Let $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$ be the subgroup of $\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}}$ which is the kernel of the homomorphism to $\mathbf{Z}$ given by $dh \otimes dh' \to \int_X dh \wedge dh'$. Then we have shown that for fixed $\gamma \in \pi_1(X, x_0)$, $\Sigma\, dh_{1_i} \otimes dh_{2_i} \to I(\Sigma\, dh_{1_i} \otimes dh_{2_i}; \gamma)$ defines a homomorphism of $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})' \to \mathbf{R}/\mathbf{Z}$. Further, if we keep the same base point x_0 but vary $\gamma \in \pi_1(X, x_0)$, then for fixed $(\Sigma\, dh_{1_i} \otimes dh_{2_i} \in (\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$, $I(\Sigma\, dh_{1_i} \otimes dh_{2_i}; \gamma)$ gives a homomorphism of $\pi_1(X, x_0)$ into $\mathbf{R}/\mathbf{Z}$ which of course factors through the abelianized group $\pi_1(X, x_0)^{\mathrm{ab}}$ (formula (2.2)).

To summarize, $I(dh_1, dh_2; \gamma)$ is bilinear in dh_1 and dh_2 if (1.1), (1.2) are satisfied, i.e. if $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$ is the kernel of the intersection number homomorphism $\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \to \mathbf{Z}$ then I is a homomorphism

$$(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})' \otimes \pi_1(X, x_0)^{\mathrm{ab}} \to \mathbf{R}/\mathbf{Z}$$

and we may replace $\pi_1(X, x_0)^{\mathrm{ab}}$ by $\mathcal{H}_{\mathbf{Z}}$ by Poincaré duality getting

$$I: (\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})' \otimes \mathcal{H}_{\mathbf{Z}} \to \mathbf{R}/\mathbf{Z} \tag{2.4}$$

which *still depends* on the choice of base point. To eliminate the base point, we restrict the domain of I in (2.4) as follows: first note that the domain of (2.4) is the kernel of the homomorphism $i_{12}: \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \to \mathcal{H}_{\mathbf{Z}}$ given by $i_{12}(dh_1 \otimes dh_2 \otimes dh_3) = (\int_X dh_1 \wedge dh_2)\, dh_3$. Now let $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$ be the kernel of

$$\tilde{p}: \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \to \mathcal{H}_{\mathbf{Z}} \oplus \mathcal{H}_{\mathbf{Z}} \oplus \mathcal{H}_{\mathbf{Z}} \tag{2.5}$$

$$dh_1 \wedge dh_2 \wedge dh_3 \mapsto \left(\int_X dh_1 \wedge dh_2\right) dh_3 \oplus \left(\int_X dh_2 \wedge dh_3\right) dh_1 \oplus \left(\int_X dh_3 \wedge dh_1\right) dh_2$$

so that this kernel is contained in the kernel of i_{12} and I is defined on $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$. Then I is independent of base point.

(2.6) *Special condition on a triple* dh_1, dh_2, dh_3 *of elements of* $\mathcal{H}_Z$:

$$\iint_X dh_1 \wedge dh_j = 0 \text{ if } j=2,3$$

and dh_2, dh_3 are Poincaré duals to simple closed curves γ_j such that γ_2, γ_3 are either disjoint or meet transversely at just one point.

LEMMA 2.7. *Let* dh_1, dh_2, dh_3 *satisfy* (2.6). *Then* $I(dh_1 \otimes dh_2 \otimes dh_3) = I(dh_3 \otimes dh_1 \otimes dh_2)$. *If the* dh_i, $i=1,2,3$ *are Poincaré duals to three disjoint simple closed curves, then* $I(dh_1 \otimes dh_2 \otimes dh_3)$ *is invariant under cyclic permutations, changes sign under transposition, and vanishes if two* dh_i *are equal.*

Proof. Let dh_i be dual to the simple closed curve γ_i. If X is cut along γ_i giving a surface X_i with two boundary components γ_i', γ_i'', there is a harmonic function h_i on the bounded surface which takes values differing by 1 at corresponding boundary points: $\partial X_i = \gamma_i' \cup (-\gamma_i'')$, $h_i(p'') - h_i(p') = 1$.

We prove first that, if $d\eta_{1,2} = dh_1 \wedge dh_2$ then

$$\int_{\gamma_3} \eta_{1,2} = -\iint_{X_3} h_3\, dh_1 \wedge dh_2 \tag{2.8}$$

note first that $*\eta_{12}$ is exact and dh_3, $*dh_3$ are closed, so $\iint_X (*\eta_{12}) \wedge (*dh_3) = 0 = \iint_X \eta_{12} \wedge dh_3$ since $\alpha \wedge \beta = *\alpha \wedge *\beta$ for 1-forms. Then

$$\iint_{X_3} h_3\, dh_1 \wedge dh_2 = \iint_{X_3} h_3\, d\eta_{1,2} = \iint_{X_3} dh_3 \wedge \eta_{1,2} + h_3\, d\eta_{1,2} = \iint_{X_3} d(h_3 \eta_{1,2})$$
$$= \iint_{\gamma_3' \cup (-\gamma_3'')} h_3 \eta_{12} = -\int_{\gamma_3} \eta_{12},$$

the desired formula (2.8).

Now let $X_{2,3}$ denote X cut along both γ_2 and γ_3. Then $h_3\, dh_1 - \eta_{3,1}$ is closed on $X_{2,3}$ and

$$\iint_{X_{2,3}} dh_2 \wedge (h_3\, dh_1) = \iint_{X_{2,3}} dh_2 \wedge (h_3\, dh_1 - \eta_{3,1}) = \iint_{X_{2,3}} d(h_2 h_3\, dh_1 - h_2 \eta_{3,1})$$
$$= \int_{(\gamma_2') \cup (-\gamma_2'') \cup \gamma_3' \cup (-\gamma_3'')} (h_2 h_3\, dh_1 - h_2 \eta_{3,1})$$
$$= -\int_{\gamma_2} h_3\, dh_1 + \int_{\gamma_2} \eta_{3,1} - \int_{\gamma_3} h_2\, dh_1.$$

Rearranging terms, we have

$$\int\int_{X_{2,3}} h_3\, dh_1 \wedge dh_2 - \int_{\gamma_3} h_2\, dh_1 = \int_{\gamma_2} (h_3\, dh_1 - \eta_{3,1}).$$

Using (2.8), we get

$$-\int_{\gamma_3} (h_2\, dh_1 + \eta_{12}) = \int_{\gamma_2} h_3\, dh_1 - \eta_{3,1}.$$

But $\eta_{1,2} = -\eta_{2,1}$ so we have

$$-I(dh_2, dh_1, dh_3) = I(dh_3, dh_1, dh_2).$$

Since

$$\int_{\gamma_3} (h_1\, dh_2 - \eta_{1,2}) + (h_2\, dh_1 - \eta_{2,1}) = \int_{\gamma_3} d(h_1 h_2) = \int_{\gamma_3} dh_1 \int_{\gamma_3} dh_2 = 0,$$

$$I(dh_1, dh_2, dh_3) = -I(dh_2, dh_1, dh_3).$$

Thus, $I(dh_1, dh_2, dh_3) = I(dh_3, dh_1, dh_2)$, cyclic invariance, as well as skew-symmetry in the first two arguments, proving Lemma 2.7.

We return now to the commutative diagram involving $j_1 : (H^1 \otimes H^1 \otimes H^1)' \to P$. Write V for H^1 and $x \cdot y$ for the alternating product. Write T for $V^{\otimes 3}$ and T' for $(V^{\otimes 3})' =$ kernel of $\bar{p} : T \to V^{\oplus 3}$.

LEMMA 2.8. (1). *The kernel T' of $\bar{p}$ is generated by two types of elements:*

(a) *$x \otimes y \otimes z$ where x, y, z are Poincaré duals of three disjoint simple closed curves.*

(b) *Elements on which both j_1 and $2I$ vanish.*

(2) *$2I = v \circ j_1$ for a unique homomorphism $v : P \to \mathbf{R}/\mathbf{Z}$.*

Proof. $\bar{p} : T \to V^{\oplus 3}$ is equivariant relative to Σ_3 acting on T by signed permutations, $t \to (\operatorname{sgn} \sigma) \sigma(t)$, and acting on $V^{\oplus 3}$ by (unsigned) permutations. We define a cross-section $S : V^{\oplus 3} \to T$ (so $\bar{p} \circ S =$ identity), which commutes with cyclic permutations: let A_i, B_i, $i = 1, \ldots, g$ be a standard symplectic basis for V, let $C_i = A_i$ or B_i, and let

$$S(C_i \oplus 0 \oplus 0) = \begin{cases} A_1 \otimes B_1 \otimes C_i, & \text{if } i > 1 \\ A_2 \otimes B_2 \otimes C_1, & \text{if } i = 1 \end{cases}$$

(with S on $0\oplus C_i\oplus 0$ or $0\oplus 0\oplus C_i$ by cyclic permutation). For any basis $\{\gamma\}$ of T, a set of generators for $\ker \bar{p}=T'$ is given by all $\gamma-S\bar{p}(\gamma)$. We take the basis of T as $\gamma=x_i\otimes y_j\otimes z_k$ where x_r or y_r or z_r is either A_r or B_r (indices from 1 to g).

Now we distinguish several cases:

(1) $\gamma=x_i\otimes y_j\otimes z_k$ and either (i, j, k) are distinct in which instance we are in case (a) of the lemma, or else whenever two of the indices i, j, k are equal then the corresponding letters x_i or y_j or z_k are also equal: e.g., $A_i\otimes A_i\otimes B_k$ $(i\neq k)$, or $A_i\otimes A_i\otimes A_i$. For these latter elements it is clear that they are annihilated by j_1; furthermore, they satisfy (2.6) and Lemma 2.7 and are annihilated by I: e.g.,

$$I(A_i\otimes A_i\otimes B_k)=\frac{1}{2}(A_i\cdot B_k)(A_i\cdot B_k)=0.$$

(2) Suppose exactly two indices are equal but the corresponding elements are distinct: typically, $A_i\otimes B_i\otimes Z_k$, or $A_i\otimes Z_k\otimes B_i$, with i, k, 1 distinct.

$$\begin{aligned} &A_i\otimes B_i\otimes Z_k-Sp(A_i\otimes B_i\otimes Z_k)=(A_i\otimes B_i-A_1\otimes B_1)\otimes Z_k\\ &=(A_i+A_1)\otimes(B_i-B_1)\otimes Z_k+A_i\otimes B_1\otimes Z_k-A_1\otimes B_i\otimes Z_k. \end{aligned}$$

Each of the tree terms in the last sum is of type (a) of the lemma.

$$\begin{aligned} A_i\otimes Z_k\otimes B_i-S\bar{p}(A_i\otimes Z_k\otimes B_i)&=A_i\otimes Z_k\otimes B_i+(B_1\otimes Z_k\otimes A_1)\\ &=A_i\otimes Z_k\otimes B_i-A_1\otimes Z_k\otimes B_1+A_1\otimes Z_k\otimes B_1+B_1\otimes Z_k\otimes A_1\\ &=(A_i+A_1)\otimes Z_k\otimes(B_i-B_1)+A_i\otimes Z_k\otimes B_1-A_1\otimes Z_k\otimes B_i\\ &\qquad+A_1\otimes Z_k\otimes B_1+B_1\otimes Z_k\otimes A_1 \end{aligned}$$

The first three terms are each of type (a).

$$A_1\otimes Z_k\otimes B_1+B_1\otimes Z_k\otimes A_1=(A_1+B_1)\otimes Z_k\otimes(A_1+B_1)-A_1\otimes Z_k\otimes A_1-B_1\otimes Z_k\otimes B_1.$$

The last three terms are again of type (a) (or $I(A_1\otimes Z_k\otimes B_1)=-I(Z_k\otimes A_1\otimes B_1)=$(by Lemma 2.7)$-I(B_1\otimes Z_k\otimes A_1)$).

$$Z_k\otimes A_i\otimes B_i-S\bar{p}(Z_k\otimes A_i\otimes B_i)=Z_k\otimes A_i\otimes B_i-Z_k\otimes A_1\otimes B_1$$

both terms are covered by (2.6) and Lemma 2.7, and we can apply cyclic permutation to I to reduce to $B_i\otimes Z_k\otimes A_i-Sp(B_i\otimes Z_k\otimes A_i)$.

(3) $i=j=k$. If $x_i=y_i=z_i$ we are in case (1). Otherwise typical terms are $A_i\otimes B_i\otimes A_i$, $A_i\otimes A_i\otimes B_i$, $B_i\otimes A_i\otimes A_i$ $(i>1)$.

$$A_i \otimes B_i \otimes A_i - S\bar{p}(A_i \otimes B_i \otimes A_i) = A_i \otimes B_i \otimes A_i - A_1 \otimes B_1 \otimes A_i + A_i \otimes A_1 \otimes B_1$$
$$= (A_i + A_1) \otimes (B_i - B_1) \otimes A_i + A_i \otimes B_1 \otimes A_i - A_1 \otimes B_i \otimes A_i + A_i \otimes A_1 \otimes B_1.$$

On applying I, each term falls under Lemma 2.7 so can be cyclically rotated to bring the last letter to the first position: this gives

$$I[A_i \otimes B_i \otimes A_i - S\bar{p}(A_i \otimes B_i \otimes A_i)] = I[A_i \otimes A_i \otimes B_i - A_i \otimes A_1 \otimes B_1 + B_1 \otimes A_i \otimes A_1].$$

Again by Lemma 2.7, I of the last two terms gives 0 and $I(A_i \otimes A_i \otimes B_i) = \frac{1}{2}$. Thus this last term is in the kernel of $v = 2I$. Similar calculations apply to the other cases. This proves part (1) of Lemma 2.8. Part (2) follows from this and from Lemma 2.7: the invariance of I under the Σ_3 action on type (a) elements shows that I on the subgroup of T' generated by type (a) elements factors through the quotient by this Σ_3 action, which is not yet $P = (\Lambda^3)'$ but an extension of $(\Lambda^3)'$ by elements of order 2, and then $2I$ annihilates these elements of order 2 as well as the type (b) elements which give the rest of the kernel of j_1.

3. Volumes

Let dh_1, dh_2, dh_3 satisfy (1.1) and (1.2), and let the homology dual of dh_3 be a simple closed curve γ_3 (which does not divide X). Let X_3 be X cut along γ_3, with boundary $\gamma_3' \cup (-\gamma_3'')$, as in the proof of Lemma 2.7, with $h_3(p'') - h_3(p') = 1$ where $p' \in \gamma_3', p'' \in \gamma_3''$ correspond to $p \in \gamma_3$. We may regard $(h_1, h_2, h_3) = \mathbf{h}$ as a map $X \to T^3$, and similarly, as a map $H: X_3 \to T^2 \times \mathbf{R}$ which gives $\mathbf{h}$ after mapping X_3 to X and $T^2 \times \mathbf{R}$ to $T^2 \times (\mathbf{R}/\mathbf{Z}) = T^3$. First we will "close up" $H(X_3)$ to a 2-cycle in $T^2 \times \mathbf{R}$ which bounds a 3-chain, then show that this 3-chain projects to a 3-chain with the same volume in T^3 whose boundary is $\mathbf{h}(X)$. Finally we compute the volume of the 3-chain in $T^2 \times \mathbf{R}$ and show it is just $I(dh_1, dh_2, dh_3)$.

The boundary of $H(X_3)$ is $H(\gamma_3') \cup H(-\gamma_3'')$. To show that the singular 1-cycle $H(\gamma_3')$ bounds in $T^2 \times \mathbf{R}$, it suffices to see that $\int_{H(\gamma_3')} dx_i = 0$ for dx_i, $i = 1, 2$, the invariant 1-forms on T^2 with period 1 over the two factors T^1. But $H^*(dx_i) = dh_i$ and

$$\int_{H(\gamma_3')} dx_i = \int_{\gamma_3'} dh_i = \int_{\gamma_3} dh_i = \int\int_X dh_i \wedge dh_3 = 0.$$

Let now $H(\gamma_3') = \partial d_2'$, d_2' a 2-chain in $T^2 \times \mathbf{R}$. Since $H(\gamma_3'') = H(\gamma_3') + (0, 0, 1)$ in $T^2 \times \mathbf{R}$, $H(\gamma_3'') = \partial d_2''$ where $d_2'' = d_2' + (0, 0, 1)$. Now, using addition of singular chains (not addition in $T^2\times\mathbf{R}$), $H(X_3)-d_2'+d_2''=Y$ is a 2-cycle in $T^2\times\mathbf{R}$ which, under the covering map $T^2\times\mathbf{R}\to T^2\times\mathbf{R}/\mathbf{Z}$ goes into $h(X_3)$ (since d_2'', d_2' differ only by translation by $(0,0,1)$). We will show Y bounds a 3-chain c_3 in $T^2\times\mathbf{R}$, so the image c_3 of c_3' in the 3-chains on T^3 has boundary $\mathbf{h}(X)$, the image chain of $H(X_3)$, and the volume of c_3 is by definition

$$\int_{c_3} dh_1\wedge dh_2\wedge dh_3=\int_{c_3} dh_1\wedge dh_2\wedge dh_3.$$

Y bounds in $T^2\times\mathbf{R}$ since

$$\int_Y dx_1\wedge dx_2=\int_{X_3} dh_1\wedge dh_2-\int_{d_2'} dh_1\wedge dh_2+\int_{d_2''} dh_1\wedge dh_2.$$

The last two terms give 0 since $dh_1\wedge dh_2$ is translation invariant and d_2'' differs from d_2' only by translation. The first term is just $\int_X dh_1\wedge dh_2=0$.

Now $Y=\partial c_3'$, and the volume of c_3' is

$$\int_{c_3} dh_1\wedge dh_2\wedge dh_3=\int_{\partial c_3} h_3\,dh_1\wedge dh_2=\int_{X_3} h_3\,dh_1\wedge dh_2-\int_{d_2'} h_3\,dh_1\wedge dh_2+\int_{d_2''} h_3\,dh_1\wedge dh_2.$$

The first term is $-\int_{\gamma_3}\eta_{1,2}$ by (2.8). Since $h_3(p'')=h_3(p')+1$, and $dh_1\wedge dh_2$ is the same on d_2' and d_2'', the last two terms give just $\int_{d_2''} dh_1\wedge dh_2$, and we want to show this equals $\int_{\gamma_3} h_1\,dh_2$. The 2-form $dh_1\wedge dh_2$ on $T^2\times\mathbf{R}$ arises by pulling back $dh_1\wedge dh_2$ on T^2 by the projection p, thus $\int_{d_2''} dh_1\wedge dh_2=\int_{p_\#(d_2'')} dh_1\wedge dh_2$ where $p_\#(d_2'')=d_2^\#$ is the 2-chain on T^2 obtained by projecting d_2'' to T^2. Furthermore, the boundary $\partial(d_2^\#)=p_\#(\gamma_3'')=$image of γ_3 under the map $(h_1,h_2):\gamma_3\to T^2$. This map (h_1,h_2) on γ_3 lifts to a map $\tilde{H}_{1,2}:\gamma_3\to\mathbf{R}^2$ since $\int_{\gamma_3} dh_i=0$, $i=1,2$. Thus, if $\tilde{d}_2$ is any 2-chain on $\mathbf{R}^2$ with boundary $\tilde{H}_{1,2}(\gamma_3)$ then

$$\int_{d_2^\#} dh_1\wedge dh_2\equiv\int_{\tilde{d}_2} dh_1\wedge dh_2 \mod \mathbf{Z}$$

(since projection of $\tilde{d}_2$ on T^2 and $d_2^\#$ are 2-chains on T^2 with the same boundary $(h_1,h_2)(\gamma_3)$). However, on $\mathbf{R}^2$ $dh_1\wedge dh_2=d(h_1\,dh_2)$ so

$$\int_{\tilde{d}_2} dh_1\wedge dh_2\equiv\int_{\tilde{H}_{1,2}(\gamma_3)} h_1\,dh_2 \mod \mathbf{Z}$$
$$=\int_{\gamma_3} h_1\,dh_2.$$

This concludes the proof that $I(dh_1, dh_2, dh_3)$ defined as iterated integral equals the volume bounded by $(h_1, h_2, h_3)\,(X)$ in T^3.

Consider now the cycle $X-i(X)$ in J discussed in the introduction, which is clearly homologous to zero since i acts as -1 on $H_1(J)$. The inclusion α of X in J is given by $(h_1, \ldots, h_{2g})$ for a basis h_i of harmonic integrals with periods in $\mathbf{Z}$, with (h_1, h_2, h_3) as above. Let $\pi: J \to T^3$ be the projection on the first three coordinates, and let $\bar{i}$ denote inverse in T^3: then $\pi \circ i = \bar{i} \circ \pi$ and $\pi \circ \alpha = \mathbf{h}$. Let $X-i(X)=\partial D_3$, D_3 a 3-chain in J. Then

$$\begin{aligned}\partial(\pi D_3) &= \pi(\partial D_3) = \pi(X) - \pi i(X)\\ &= \pi(X) - \bar{i}\pi(X) \quad \text{(where } X \text{ means } \alpha(X))\\ &= \mathbf{h}(X) - \bar{i}\mathbf{h}(X)\\ &= \partial c_3 - \bar{i}\partial c_3 = \partial(c_3 - \bar{i}c_3).\end{aligned}$$

Thus $\pi D_3 = (c_3 - \bar{i}c_3) \bmod$ 3-cycles.

Now ν^* is the linear function on P defined (mod periods) by integration over the 3-chain D_3 in J, and

$$\begin{aligned}\nu^*(dh_1 \wedge dh_2 \wedge dh_3) &= \text{volume of } \pi D_3 \quad (\operatorname{mod} \mathbf{Z})\\ &= \text{volume of } (c_3 - \bar{i}c_3)\\ &= 2 \cdot (\text{volume of } c_3)\end{aligned}$$

since τ acts as -1 on the volume form of T^3. This shows that $\nu^* = 2I$.

4. Complex tori

In this section we discuss complex structures on the vector spaces $H^3(J;\mathbf{R})$ and $P_{\mathbf{R}} = P \otimes \mathbf{R}$ and show that the homomorphism $\nu: P \to \mathbf{R}/\mathbf{Z}$ can be regarded as a point on the complex dual space of $P_{\mathbf{R}}$ modulo a lattice (dual lattice to P), so we obtain an interpretation of ν as a point on a complex torus: an intermediate Jacobian of the Kähler manifold J. One choice of complex structure make this complex torus into an Abelian variety, namely the primitive part of Weil's intermediate Jacobian which does not vary holomorphically with moduli of the Riemann surface X. Another complex structure is not an Abelian variety but gives the complex tori studied by Griffiths [2] and shown by him to vary holomorphically. In later sections we will use the second complex structure. We finally define an $\mathbf{R}$-linear map

$$D\colon H^3(J;\mathbf{R}) \to H^{1,0}(J;\mathbf{C}) \underset{\mathbf{C}}{\otimes} H^{1,1}(J;\mathbf{C})$$

which is in fact C-linear for *both* of the complex structures on $H^3(J;\mathbf{R})$. The restriction of D to P will be related in the next section to the variational formula for I.

The real vector space $H^1(X;\mathbf{R})$ is given a complex structure j_1 by identifying it with $\mathscr{H}_{\mathbf{R}}$, the space of real harmonic 1-forms, and taking $j_1(dh)=-{*dh}$. Further $\mathscr{H}_{\mathbf{R}}$ is put in $1-1$ correspondence with the complex vector space $H^{1,0}(X)$ of holomorphic 1-forms by letting dh correspond to $-{*dh}+idh=j_1(dh)+idh$ (i.e. dh is the imaginary part of the holomorphic 1-form). Under this correspondence, multiplication by $j_1=-*$ on $\mathscr{H}_{\mathbf{R}}$ correspond to multiplication by i on $H^{1,0}$, i.e. this is a complex-linear isomorphism. We may also then identify $H^1(X;\mathbf{R}), H^{1,0}(X), H^1(J;\mathbf{R})$ and $H^{1,0}(J)$. Of course $H^1(X;\mathbf{R})\otimes_{\mathbf{R}}\mathbf{C}$ is a direct sum of two subspaces $H^{1,0}$ and $H^{0,1}$ on which $j_1\otimes 1$ acts as multiplication by $1\otimes i$, $1\otimes(-i)$ respectively. Next we consider $\Lambda^3 H^1(J;\mathbf{R})=H^3(J;\mathbf{R})$, and consider two complex structures j_3' and j_3:

$$j_3'=j_1\wedge j_1\wedge j_1$$

$$j_3=\tfrac{1}{2}[j_3'+j_3'']=\tfrac{1}{2}[j_1\wedge j_1\wedge j_1+(j_1\wedge 1\wedge 1+1\wedge j_1\wedge 1+1\wedge 1\wedge j_1)]$$

(thus $j_3''(A\wedge B\wedge C)=j_1(A)\wedge B\wedge C+A\wedge j_1(B)\wedge C+A\wedge B\wedge j_1(C)$). j_3', j_3 are complex structures (i.e. satisfy $j_3^2=-\mathrm{Id}$) but j_3'' is not. On the complexification $H^3(J;\mathbf{R})\otimes\mathbf{C}=H^{3,0}\oplus H^{2,1}\oplus H^{1,2}\oplus H^{0,3}$, $j_3'\otimes 1$, which we will abbreviate as j_3', acts as multiplication by i^{a-b} on $H^{a,b}$, so as i on $H^{2,1}\oplus H^{0,3}$ (and $-i$ on $H^{3,0}\oplus H^{1,2}$). These subspaces $H^{2,1}\oplus H^{0,3}$ yield Weil's Jacobians. Similarly j_3 acts as i on $H^{3,0}\oplus H^{2,1}$ (and $-i$ on $H^{1,2}\oplus H^{0,3}$): this is the decomposition used by Griffiths and will be the one we need.

Each of the operators j_3', j_3 could thus have been defined as multiplication by i or $-i$ on each $H^{a,b}$. Since the primitive subspace P was defined as the kernel of multiplication by ω^{g-2} on $H^3(J;\mathbf{R})$, where $\omega\in H^2(J;\mathbf{Z})$, there are primitive subspaces $P_{\mathbf{R}}=P\otimes\mathbf{R}$ and $P_{\mathbf{C}}=P\otimes\mathbf{C}$ defined in the same way, and $P_{\mathbf{C}}=\oplus_{a,b}P_{\mathbf{C}}\cap H^{a,b}$ since ω has type (1,1) and multiplication by ω commutes with decomposition into bigraded summands $H^{a,b}$. In particular if we set

$$H_+^{2q+1}=\bigoplus_{\substack{r+s=2q+1\\ r>s}} H^{r,s}$$

then multiplication by ω sends these subpsaces into one another so that, for $2q+1=3$, $P\otimes\mathbf{C}=P_+\oplus P_-$ where P_+ is the projection of $P\otimes\mathbf{C}$ on H_+^3 and P_- is the projection on the complementary subspace. Further $P_{\mathbf{R}}\to P_+$ is an isomorphism of real vector spaces, and of complex vector spaces as well if we use the complex structure j_3 on $P_{\mathbf{R}}$.

Griffiths has shown that the periods of the 3-forms in P_+ vary holomorphically with the complex structure on J and hence with that on X: we recall now the precise meaning of this statement. Consider the group $\mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{Z}) = P^*$ (it can be considered a subgroup of $H_3(J; \mathbf{Z})$), and fix a basis c_ν, d_ν for P^*, $\nu = 1, \ldots, n$, such that the c_ν are a $\mathbf{C}$-basis for the $\mathbf{C}$-dual of $P_{\mathbf{R}}$ ($P_{\mathbf{R}}$ is isomorphic to P_+ since we are using j_3 on $P_{\mathbf{R}}$). If $\omega_\alpha, \alpha = 1, \ldots, n$, is a set of harmonic 3-forms on J which are a basis for $P_+^3 = P^{3,0} \oplus P^{2,1}$ then the period matrix is $F = [\omega_\alpha(c_\nu); \omega_\alpha(d_\nu)]$ a matrix with n rows and $2n$ columns in which the left hand half is invertible since the c_ν are a $\mathbf{C}$-basis for the dual of P_+. We call F *normalized* if the submatrix $[\omega_\alpha(c_\nu)]$ of F is the identity matrix. The Plücker coordinates of F are the *ratios* of n rowed minors of F, and so are given by the entries of F and polynomials in them if F is normalized. Now consider a holomorphic family X_s of compact Riemann surfaces where the parameter s varies in a polydisk (say contained in Torelli space, as in the next section), and $s=0$ corresponds to a fixed reference surface $X_0 = X$, namely the one we were considering above. Then the integral cohomology of X_s or $J(X_s) = J_s$ may be considered as fixed but the complex structure on J_s and in particular the differential forms $\omega_\alpha(s)$ are smooth functions of s. The theorem of Griffiths [2, II; Theorem 1.27] in particular asserts that the mapping Φ from the complex s-polydisk to the Grassmannian of complex n-dimensional subspaces of the fixed $2n$ complex dimensional space $H^3(J_0; \mathbf{C})$ which assigns to s the subspace $P_+^3(J_s)$ is holomorphic: in terms of coordinates this subspace is the subspace of $\mathbf{C}^{2n}$ spanned by the rows of the period matrix $F(s)$. If the left hand half of the matrix $F(0)$ is non-singular, the same will still be true for $F(s)$ for s near 0. We may then replace the basis $\omega_\alpha(s)$ by a new basis which makes the period matrix of ω_α over c_ν the identity matrix for all s near 0, thus normalizing $F(s)$ to have the form $[I\, Z(s)]$ where $Z(s) = [\omega_\alpha(d_\nu)]$ is *holomorphic in s*, since the definition of Φ to be holomorphic is that the Plücker coordinates are holomorphic.

Now the equations $\omega_\alpha(c_\nu) = \delta_{\alpha,\nu}$, $\omega_\alpha(d_\nu) = Z_{\alpha,\nu}(s)$ imply that in the dual space of P_+ we have

$$d_\nu = \sum_\alpha c_\alpha Z_{\alpha,\nu}(s) \tag{4.1}$$

and

$$Z_{\alpha,\nu}(s)\, c_\alpha = X_{\alpha,\nu}(s)\, c_\alpha + Y_{\alpha,\nu}(s)\, (j_s c_\alpha); \quad X_{\alpha,\nu}, Y_{\alpha,\nu} \in \mathbf{R}.$$

Let θ_s be the complex linear isomorphism of the dual space $(P_+)^*$ of P_+ with $\mathbf{C}^n$ such that $\theta_s(c_\nu)$ is the standard basis vector $\mathbf{e}_\nu$ of $\mathbf{C}^n, \nu=1,\ldots,n$, where $P_+=P^3_+(J_s)$. Then

$$\theta_s(d_\nu)=\sum_{\alpha}\mathbf{e}_\alpha Z_{\alpha,\nu}(s) \tag{4.2}$$

so that θ_s takes the lattice in $(P_+)^*$ spanned by the c_ν, d_ν to the lattice in $\mathbf{C}^n$ spanned by the $2n$ column vectors of $[I_n, Z(s)]$. This then gives us a specific picture of the family of complex tori $(P_+)^*(s)$ modulo lattice $\{c_\nu, d_\nu\}$ in terms of the family $\mathbf{C}^n/\{$lattice of column vectors $I, Z(s)\}$, where $Z(s)$ is holomorphic in s. Furthermore a map φ from the parameter space $\{s\}$ to the family $\mathbf{C}^n/$(lattice $I, Z(s)$) such that $\varphi(s)\in\mathbf{C}^n/$ (lattice $I, Z(s)$) for each s will be holomorphic if its lift to a map into $\mathbf{C}^n$ (which exists locally) has the form

$$\varphi(s)=\sum\varphi_\nu(s)\,\mathbf{e}_\nu \quad \text{with } \varphi_\nu(s) \text{ holomorphic in } s.$$

Suppose now for each parameter value s we are given a homomorphism $\tilde{f}_s$ of the abelian group P into $\mathbf{R}/\mathbf{Z}$: we will write $\tilde{f}_s(\lambda)=\tilde{f}(\lambda;s)\in\mathbf{R}/\mathbf{Z}$ for $\lambda\in P$. We want to identify $\tilde{f}_s$ with a point on the complex torus $(P_+)^*/\{c_\nu, d_\nu\}$ and then to give a simple criterion for this point to vary holomorphically with s. Since $\operatorname{Hom}(P,\mathbf{R}/\mathbf{Z})=\operatorname{Hom}(P,\mathbf{R})/\operatorname{Hom}(P,\mathbf{Z})$ where $\operatorname{Hom}(P,\mathbf{Z})$ is the lattice P^*, we just have to identify $\operatorname{Hom}(P,\mathbf{R})$ with $\operatorname{Hom}_\mathbf{R}(P_\mathbf{R};\mathbf{R})$ where $P_\mathbf{R}=P\otimes\mathbf{R}$ has complex structure j_s depending on s, and then identify $\operatorname{Hom}_\mathbf{R}(P_\mathbf{R},\mathbf{R})$ with $\operatorname{Hom}_\mathbf{C}(P_\mathbf{R};\mathbf{C})$. The latter identification will make $l\colon P_\mathbf{R}\to\mathbf{R}$ correspond to $\tilde{l}\colon P_\mathbf{R}\to\mathbf{C}$ via $\tilde{l}(p)=l(j_s p)+il(p)$.

So now $\tilde{f}_s\in\operatorname{Hom}_\mathbf{R}(P_\mathbf{R},\mathbf{R})/\operatorname{Hom}(P,\mathbf{Z})$. Choose a coset representative $f_s\in\operatorname{Hom}_\mathbf{R}(P_\mathbf{R},\mathbf{R})$. Using $\theta_s\colon\operatorname{Hom}_\mathbf{R}(P_\mathbf{R},\mathbf{R})\to\mathbf{C}^n$, $\theta_s(c_\nu)=\mathbf{e}_\nu$ as before, let

$$\theta_s(f_s)=\sum_\nu f_\nu(s)\,\mathbf{e}_\nu.$$

For $\lambda\in P$, write $f_s(\lambda)=f(\lambda;s)\in\mathbf{R}$.

We state the following criterion for the $\mathbf{C}$-valued functions $f_\nu(s)$ to be holomorphic in s:

PROPOSITION 4.3. *Let s vary in a one-complex-dimensional disk with center $s=0$. Suppose that for $s=0$ we are given a $\mathbf{C}$-linear homomorphism $Q_1\colon P\otimes\mathbf{R}\to\mathbf{C}$ relative to* *the complex structure j_0 on $P\otimes\mathbf{R}$ such that $(\partial/\partial\bar{s})f(\lambda;s)|_{s=0}=\overline{Q_1(\lambda)}$ for all $\lambda\in P\otimes\mathbf{R}$. Then $(\partial/\partial\bar{s})f_\nu(s)|_{s=0}=0$.*

Proof. Let θ_s^{tr} be the transpose of θ_s, i.e. a $\mathbf{C}$-linear map $(\mathbf{C}^n)^*\to P\otimes\mathbf{R}$. Let C_μ, D_μ be the $\mathbf{Z}$-basis of P dual (over $\mathbf{Z}$) to c_ν, d_ν and let $\mathbf{e}_\nu^*$ be the basis of $(\mathbf{C}^n)^*$ dual to the basis $\mathbf{e}_\mu$ of $\mathbf{C}^n$. For $\mathbf{v}\in\mathbf{C}^n$, let $\mathbf{e}_\nu^*(\mathbf{v})=\varepsilon_\nu(i\mathbf{v})+i\varepsilon_\nu(\mathbf{v})$, so $\varepsilon_\nu=\operatorname{Im}\mathbf{e}_\nu^*$. Then

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu)(c_\mu)=\varepsilon_\nu(\theta_s c_\mu)=\varepsilon_\nu(\mathbf{e}_\mu)=0$$

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu)(d_\mu)=\varepsilon_\nu\left(\sum_\varrho Z_{\mu,\varrho}(s)\,\mathbf{e}_p\right)=Y_{\mu,\nu}(s)$$

(by (4.2), and the definition of ε_ν as the imaginary part of $\mathbf{e}_\nu^*$).

Similarly writing $(\varepsilon_\nu\circ i)(\mathbf{v})=\varepsilon_\nu(i\mathbf{v})$,

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu\circ i)(d_\mu)=\varepsilon_\nu\left(i\sum_\varrho Z_{\mu,e}(s)\,\mathbf{e}_\varrho\right)=X_{\mu,\nu}(s)$$

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu\circ i)(c_\mu)=\varepsilon_\nu(i e_\mu)=\delta_{\mu,\nu}.$$

Expressing $\theta_s^{\mathrm{tr}}(\varepsilon_\nu)$, $\theta_s^{\mathrm{tr}}(\varepsilon_\nu\circ i)$ in the $\mathbf{R}$-basis C_μ, D_μ of $P\otimes\mathbf{R}$,

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu)=\sum_\mu D_\mu\,Y_{\mu,\nu}(s)$$

$$\theta_s^{\mathrm{tr}}(\varepsilon_\nu\circ i)=C_\nu+\sum_\mu D_\mu X_{\mu,\nu}(s).$$

Since θ_s, θ_s^{tr} are $\mathbf{C}$-linear, $\theta_s^{\mathrm{tr}}(\varepsilon_\nu\circ i)=j_s\,\theta_s^{\mathrm{tr}}(\varepsilon_\nu)$, so that

$$C_\nu=-\sum_\mu(X_{\mu,\nu}-Y_{\mu,\nu}\,j_s)D_\mu=-\sum_\mu\overline{Z_{\mu,\nu}(s)}\,D_\mu. \tag{4.4}$$

We had defined

$$\theta_s(f_s)=\sum_\nu f_\nu(s)\,\mathbf{e}_\nu.$$

Applying the $\mathbf{C}$-linear function $\mathbf{e}_\mu^*$ with real and imaginary parts $\varepsilon_\mu\circ i$ and ε_μ respectively we get

$$\begin{aligned}\mathrm{Re} f_\mu(s) &= (\varepsilon_\mu \circ i)(\theta_s f_s) = f_s(\theta_s^{\mathrm{tr}}(\varepsilon_\mu \circ i))\\ &= f_s\left(C_\mu + \sum_\nu X_{\nu,\mu}(s) D_\nu\right) = f_s(c_\mu) + \sum_\nu X_{\nu,\mu}(s) f_s(D_\nu)\\ \mathrm{Im} f_\mu(s) &= \varepsilon_\mu(\theta_s f_s) = f_s(\theta_s^{\mathrm{tr}} \varepsilon_\mu)\\ &= f_s\left(\sum_\nu Y_{\nu,\mu}(s) D_\nu\right) = \sum_\nu Y_{\nu,\mu}(s) f_s(D_\nu)\end{aligned}$$

$$f_\mu(s) = \mathrm{Re} f_\mu(s) + i\,\mathrm{Im} f_\mu(s) = f_s(C_\mu) + \sum_\nu Z_{\nu,\mu}(s) f_s(D_\nu).$$

Now take $\partial/\partial\bar{s}$ at $s=0$ and use the hypothesis

$$\frac{\partial}{\partial\bar{s}} f_s(\lambda)|_{s=0} = \overline{Q_1(\lambda)}, \quad \text{for } \lambda = C_\mu, D_\nu, \frac{\partial}{\partial\bar{s}} Z_{\nu,\mu}(s) = 0.$$

$$\frac{\partial}{\partial\bar{s}} f_\mu(s)|_{s=0} = \overline{Q_1(C_\mu)} + \sum_\nu Z_{\nu,\mu}(0)\,\overline{Q_1(D_\nu)}.$$

Since Q_1 is **C**-linear (relative to j_0 acting on $P \otimes \mathbf{R}$),

$$\begin{aligned}\frac{\partial}{\partial\bar{s}} f_\mu(s)|_{s=0} &= \overline{Q_1(C_\mu) + \sum_\nu \overline{Z_{\nu,\mu}(0)}\, Q_1(D_\nu)}\\ &= \overline{Q_1\left(C_\mu + \sum_\nu \overline{Z_{\nu,\mu}(0)}\, D_\nu\right)}\\ &= 0 \quad \text{by (4.4).}\end{aligned}$$

Thus we have proved that if the complex structure of the tori considered above varies holomorphically, i.e. the matrix $Z_{\mu,\nu}(s)$ is holomorphic in s, and if at all points a, $(\partial/\partial\bar{s})f_s(\lambda)$ is given by a **C**-linear function $Q_1(\lambda)$ as in the proposition then $\tilde{f}_s$ varies holomorphically with s, or in other words defines a holomorphic section of the holomorphic family of tori over the parameter manifold. In the next section such a function $Q_1(\lambda)$ will be obtained by combining integration over the Riemann surface X with the following linear map D: $H^3(J;\mathbf{R}) \to H^{1,0}(J) \otimes_{\mathbf{C}} H^{1,1}(J)$.

Identifying $H^3(J;\mathbf{R})$ with $\Lambda^3_{\mathbf{R}} H^1(J;\mathbf{R})$ and $H^1(J;\mathbf{R})$ with the space of real harmonic 1-forms denoted dh, and further making correspond to dh the $(1,0)$ form

$$\omega = (-\ast dh) + i\,dh = j_1(dh) + i\,dh$$

and the (0, 1) form

$$\bar{\omega}=j_1(dh)-i\,dh$$

we define $D(dh_1 \wedge dh_2 \wedge dh_3)$ as the expression

$$\sum_{(1,2,3)} \omega_1 \otimes_{\mathbf{C}} (\omega_2 \wedge \bar{\omega}_3 - \omega_3 \wedge \bar{\omega}_2) \quad \text{(sum over 3 cyclic permutations of indices 1, 2, 3)}$$

which we will also write as a 3×3 determinant

$$\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \bar{\omega}_1 & \bar{\omega}_2 & \bar{\omega}_3 \end{vmatrix}. \tag{4.5}$$

The above expression is clearly $\mathbf{R}$-trilinear and alternating in the dh_i or equivalently in the ω_i. Furthermore the operator j_3'' on $\Lambda^3_{\mathbf{R}} H^1(J;\mathbf{R})$ which was defined as

$$j_3''(dh_1 \wedge dh_2 \wedge dh_3) = j_1(dh_1) \wedge dh_2 \wedge dh_3 + dh_1 \wedge j_1(dh_2) \wedge dh_3 + dh_1 \wedge dh_2 \wedge j_1(dh_3)$$

(where $j_1 = -\ast$ and $j_1(\omega) = j_1[(-\ast dh + i dh] = i\omega,\ j_1(\bar{\omega}) = -i\bar{\omega})$ is related to D by the equation

$$D \circ j_3'' = iD.$$

To prove this we have to take the determinant (4.5), apply j_1 to each column in turn, and add the 3 determinants: this gives us

$$\begin{aligned} D(j_3''(dh_1 \wedge dh_2 \wedge dh_3)) = {} & i[\omega_1 \otimes (\omega_2 \wedge \bar{\omega}_3 - \omega_3 \wedge \bar{\omega}_2) + \omega_2 \otimes (-\omega_3 \wedge \bar{\omega}_1 - \omega_1 \wedge \bar{\omega}_3) \\ & + \omega_3 \otimes (\omega_1 \wedge \bar{\omega}_2 + \omega_2 \wedge \bar{\omega}_1) + \omega_1 \otimes (\omega_2 \wedge \bar{\omega}_3 + \omega_3 \wedge \bar{\omega}_2) \\ & + \omega_2 \otimes (\omega_3 \wedge \bar{\omega}_1 - \omega_1 \wedge \bar{\omega}_3) + \omega_3 \otimes (-\omega_1 \wedge \bar{\omega}_2 - \omega_2 \wedge \bar{\omega}_1) \\ & + \omega_1 \otimes (-\omega_2 \wedge \bar{\omega}_3 - \omega_3 \wedge \bar{\omega}_2) + \omega_2 \otimes (\omega_3 \wedge \bar{\omega}_1 + \omega_1 \wedge \bar{\omega}_3) \\ & + \omega_3 \otimes (\omega_1 \wedge \bar{\omega}_2 - \omega_2 \wedge \bar{\omega}_1)] \\ = {} & i[\omega_1 \otimes (\omega_2 \wedge \bar{\omega}_3 - \omega_3 \wedge \bar{\omega}_2) + \omega_2 \otimes (\omega_3 \wedge \bar{\omega}_1 - \omega_1 \wedge \bar{\omega}_3) \\ & + \omega_3 \otimes (\omega_1 \wedge \bar{\omega}_2 - \omega_2 \wedge \bar{\omega}_1)] \\ = {} & iD(dh_1 \wedge dh_2 \wedge dh_3). \end{aligned}$$

Also, if $j_3'(dh_1\wedge dh_2\wedge dh_3)=j_1(dh_1)\wedge j_1(dh_2)\wedge j_1(dh_3)$ then $D\,j_3'=iD$: this is clear since j_3' multiplies each ω in the determinant by i and each $\bar\omega$ by $-i$. The complex structure j_3 was defined as $\frac{1}{2}(j_3'+j_3'')$ so that we also have $D\circ j_3=iD$. D can be extended to $H^3(J;\mathbf{R})\otimes\mathbf{C}$ as $D\otimes 1$: if we continue to denote $D\otimes 1$ as D then we note that *D is zero on $H^{a,b}$ if $(a,b)\neq(2,1)$* (if $a+b=3$). The last statement follows from the fact that $H^{2,1}$ is the only one of the spaces $H^{a,b}$ on which both j_3 and j_3' act as i. D can also be defined, by using the multiplication m: $J\times J\to J$, as $D=(\text{projection on } H^{1,0}(J)\otimes H^{1,1}(J))\circ m^*$.

We can also use D to obtain a C-linear map

$$\bar D\colon \Lambda^3_{\mathbf{R}}H^1(X;\mathbf{R})\to H^{1,0}(X)\underset{\mathbf{C}}{\otimes} A^{1,1}(X) \tag{4.6}$$

where $A^{1,1}(X)$ denotes the space of differentiable 2-forms on X of type $(1,1)$, simply by composing D with the complex-analytic map h: $X\to J$. Since in $A(X)$ we have $*dh_j\wedge *dh_k=dh_j\wedge dh_k$ and consequently

$$\omega_j\wedge\bar\omega_k-\omega_k\wedge\bar\omega_j=4\,dh_j\wedge dh_k$$

we can write $\bar D$ as the following determinant

$$\bar D(dh_1\wedge dh_2\wedge dh_3)=2\begin{vmatrix}\omega_1 & \omega_2 & \omega_3\\ dh_1 & dh_2 & dh_3\\ dh_1 & dh_2 & dh_3\end{vmatrix}\in H^{1,0}\otimes A^{1,1} \tag{4.7}$$

(remembering that terms in the top row multiply others terms by $\otimes$ product). Of course we can restrict the domain of D to the primitive subspace $P\otimes\mathbf{R}$, and can continue to use the same formulas since $P\otimes\mathbf{R}$ has a basis of elements of the form $dh_1\wedge dh_2\wedge dh_3$.

5. Variational formula

We study now the variation of harmonic volumes I when we vary the conformal structure on the Riemann surface X: more precisely, X varies within Torelli space $\mathcal{T}$, which is defined as the set of pairs (X, canonical homology basis for $H_1(X;\mathbf{Z})$) modulo conformal homeomorphisms preserving the canonical homology basis. Teichmüller space is a covering space of Torelli space, but the latter is more convenient for us since we use only homology and not the fundamental group of X.

According to Bers (Bulletin Amer. Math. Soc., Vol. 67, 1961) the normalized holomorphic differentials of the first kind dw_k are holomorphic functions on Teichmüller space, as are their integrals w_k and their periods. The real harmonic differentials dh_i, $i=1, \dots, 2g$, Poincaré dual to the homology basis are obtained from the real and imaginary parts of the dw_k by a non-singular matrix determined by the period matrix of the dw_k, thus, the dh_i are real analytic functions on Teichmüller space, and on Torelli space as well. Finally, if we fix 3 integral 1-dimensional homology classes with mutual intersection numbers zero on the surface X, then the corresponding harmonic volume I varies real-analytically as X varies in Torelli space, and to compute its differential, it suffices to choose a suitable set of tangent vectors to Torelli space at X and compute these partial derivatives of I. We will do this using variational formulas given in [6], Chapter 8, Section 1 ("Schiffer variations"). Thus, we recall the notation of this book (mainly Chapters 3 and 8):

$K_1, \dots, K_{2g}$ denote a canonical basis of 1-cycles: $K_{2i-1} \circ K_{2i}=1$, $K_p \circ K_q=0$ if $(p,q) \neq (2i-1, 2i)$ or $(2i, 2i-1)$. α_i is the real harmonic 1-form dual to K_i, so $\int_{K_j} \alpha_i = K_i \circ K_j$.

$\Omega_{q_0, q_1}(p)$ is a multivalued analytic function of p whose real part is *single-valued*, and $d\Omega_{q_0, q_1}(p)$ is the elementary differential of the 3rd kind with simple poles at q_0, q_1 of respective residues -1, $+1$. $dw_j, j=1, \dots, g$ are the holomorphic differentials of the first kind, normalized by

$$\int_{K_{2k-1}} dw_j = \delta_{j,k}, \quad \int_{K_{2k}} dw_j = X_{j,k} + i\, Y_{j,k}$$

$X_{j,k}$, $Y_{j,k}$ are both real (real and imaginary parts of the period matrix).

$d\omega_{q_0, q_1}(p)$ is the elementary analytic differential of the 3rd kind with the same poles and residues at q_0, q_1 as $d\Omega_{q_0, q_1}(p)$ but normalized by $\int_{K_{2i-1}} d\omega_{q_0, q_1} = 0$, $i=1, \dots, g$. Then

$$d\Omega_{q, q_0}(p) = d\omega_{q, q_0}(p) + \sum_{j=1}^{g} 2\pi i \left(\int_{q_0}^{q} \alpha_{2j-1} \right) dw_j.$$

The change of conformal structure of X is described in [6], Chapter 8, § 1: fix a point $t \in X$ and fix a local coordinate z in a neighborhood of t, with $z(t)=0$. Let z^* be a local coordinate in a region of X which overlaps the domain of z in an annulus but which *excludes* the disk $|z| \leq \varrho$. X is obtained by identifying the domains of z and z^* on their overlap by the identity map $z^*=z$, while a new surface X^* is obtained by the new identification $z^*=z+s/z$ where $s=e^{2i\varphi}\varrho^2$ is any sufficiently small complex number (so X corresponds to $s=0$). A vector field in a neighborhood of the point t, i.e. of $z=0$, is given by $\partial/\partial s|_{s=0}=(1/z)\, d/dz$: this can be regarded as an element of $H^1(X, K^{-1})$, $K=$canonical (i.e. cotangent) bundle of X, which is the dual vector space to $H^0(X, K^2)$=space of holomorphic quadratic differentials $q(z)\,(dz)^2$, the dual pairing being

$$\left(q(z)\,(dz)^2, \frac{1}{z}\frac{d}{dz}\right) = \operatorname*{Res}_t q(z)\,(dz)^2\frac{1}{z}\frac{d}{dz} = \operatorname*{Res}_t q(z)\frac{dz}{z} = q(t),$$

i.e. just "evaluation at the point t" of the holomorphic quadratic differential written in the coordinate z.

Now for the two surfaces X and $X^* = X^*_s$ we have real harmonic 1-forms α_μ, α^*_μ dual to the canonical homology basis K_μ (supposed to be the same for X, X^*). Let q be any point on X distinct from t, and regard α_μ, α^*_μ as 1-forms in the variable q. Out first aim is the following formula:

$$\alpha^*_\mu(q) - \alpha_\mu(q) = \operatorname{Re}\left[2s\, d_q\, \Omega'_{q,q_0}(t)\, \alpha'_\mu(t)\right] + o(s) \tag{5.1}$$

where $\alpha'_\mu(t) = (\alpha_\mu/dz)(t)$, α_μ/dz is a function denoted α'_μ which is evaluated at t: if $\alpha_\mu + i{*}\alpha_\mu = dA_\mu$ is a holomorphic differential, then $\alpha_\mu/dz = \frac{1}{2}(dA_\mu/dz.)$ Similarly, if d_q denotes differential with respect to the variable q, then $d_q\, \Omega_{q,q_0}(z)$ is a differential in q and a function of z and $d_q\, \Omega'_{q,q_0}(t)$ denotes the value of its derivative with respect to z, evaluated at t, which is then a differential in q. The proof of (5.1) uses the following formulas, in which we denote quantities such as $w^*_\mu(q) - w_\mu(q)$ by $\delta w_\mu(q)$ (formulas 8.1.4, 8.1.5 loc cit.; 8.1.5 as given there is missing a minus sign):

$$\delta w_\mu(q) - \delta w_\mu(q_0) = s w'_\mu(t)\, \omega'_{q,q_0}(t) + o(s) \tag{5.2}$$

and if $\gamma_{\mu,\nu} = X_{\mu,\nu} + i\, Y_{\mu,\nu}$ is the period matrix

$$\delta\gamma_{\mu,\nu} = -s \cdot 2\pi i\, w'_\mu(t)\, w'_\nu(t) + o(s). \tag{5.3}$$

With these formulas, the proof of 5.1 is easy: let $dw_\mu = du_\mu + i dv_\mu$, $\mu = 1, \ldots, g$; we have

$$dw_\mu = -\alpha_{2\mu} + \sum_\nu \gamma_{\mu,\nu}\, \alpha_{2\nu-1}$$

$$du_\mu = -\alpha_{2\mu} + \sum_\nu X_{\mu,\nu}\, \alpha_{2\nu-1}$$

$$dv_\mu = \sum_\nu Y_{\mu,\nu}\, \alpha_{2\nu-1}$$

so that

$$\begin{aligned}
\alpha_{2\mu-1} &= \sum_{\lambda} Y^{-1}_{\mu,\lambda}\, dv_\lambda \\
\alpha'_{2\mu-1}(t) &= \sum_{\lambda} Y^{-1}_{\mu,\lambda} \frac{dv_\lambda}{dz}(t) = -\frac{i}{2}\sum_{\lambda} Y^{-1}_{\mu,\lambda}\, w'_\lambda(t) \qquad (5.4)\\
\alpha_{2\mu} &= -du_\mu + \sum_{\lambda} (XY^{-1})_{\mu,\lambda}\, dv_\lambda = -du_\mu + \sum_{j} X_{\mu,j}\,\alpha_{2j-1} \\
\alpha'_{2\mu}(t) &= -\frac{1}{2}\left[w'_\mu(t) + i\sum_{\lambda} (XY^{-1})_{\mu,\lambda}\, w'_\lambda(t)\right].
\end{aligned}$$

Since δ may be treated as a derivation (mod $o(s)$), we will temporarily drop the term $o(s)$ from our formulas:

$$\delta\alpha_{2\mu-1} = \sum_{\lambda} (Y^{-1})_{\mu,\lambda}\,\delta(dv_\lambda) + \sum_{\lambda} (-Y^{-1}\cdot\delta Y\cdot Y^{-1})_{\mu,\lambda}\, dv_\lambda$$

(using (5.2) and taking imaginary parts)

$$\begin{aligned}
&= \sum_{\lambda} Y^{-1}_{\mu,\lambda}\,\mathrm{Im}\left[s w'_\lambda(t)\, d_q\,\omega'_{q,q_0}(t)\right] - \sum_{\gamma,\lambda} (Y^{-1}\cdot\delta Y)_{\mu,\gamma}\, Y^{-1}_{\gamma,\lambda}\, dv_\lambda \\
&= \mathrm{Im}\left[\sum_{\lambda} Y^{-1}_{\mu,\lambda}\, w'_\lambda(t)\, s d_q\,\omega'_{q,q_0}(t) - (Y^{-1}\delta Y)_{\mu,\lambda}\,\alpha_{2\lambda-1}\right]
\end{aligned}$$

(using 5.3 now and taking imaginary parts)

$$\begin{aligned}
&= \mathrm{Im}\left[s\left(\sum_{\lambda} Y^{-1}_{\mu,\lambda}\, w'_\lambda(t)\right)\left(d_q\,\omega'_{q,q_0}(t) + \sum_{\gamma} 2\pi i w'_\gamma(t)\,\alpha_{2\gamma-1}\right)\right] \\
&= \mathrm{Im}\left[s\left(\sum_{\lambda} Y^{-1}_{\mu,\lambda}\, w'_\lambda(t)\right) d_q\,\Omega'_{q,q_0}(t)\right] \\
&= \mathrm{Im}\left[2is\alpha'_{2\mu-1}(t)\, d_q\,\Omega'_{q,q_0}(t)\right]
\end{aligned}$$

which proves (5.1) for odd indices $2\mu-1$. The proof for $\delta(\alpha_{2\mu})$ is the same, starting with (5.4), so we omit it. Finally, (5.1) is valid, not just for the basis α_μ *of* $\mathscr{H}_Z$, but for any linear combination of the α_μ as long as the coefficients in the linear combination are independent of the conformal structure.

We can now consider a general infinitesimal variation of the conformal structure of

X: consider not just one point t in X, but $3g-3$ *general* points t_i such that any holomorphic quadratic differential vanishing at all these points must be identically zero. Further let s_i be a suitably small complex number at t_i (so that the various disks do not overlap), and form X^* by using the gluing map $z_i^*=z_i+s_i/z_i$ near t_i for all i. This gives us a $3g-3$ dimensional family of deformations of X parametrized by $\mathbf{s}=(s_1,\ldots,s_{3g-3})$ (see Kodaira-Morrow, "Complex Manifolds" Chapter 2, Section 3). The tangent space to this parameter space at X consists of all tangent vectors $\Sigma_\nu c_\nu(\partial/\partial s_\nu)$, and the value of such a tangent vector on a holomorphic quadratic differential $q(z)(dz)^2$ on X is just $\Sigma_\nu c_\nu q(t_\nu), \nu=1,\ldots,3g-3$. Thus the tangent space to the parameter space S of this family is mapped isomorphically, by the Kodaira-Spencer map, onto $H^1(X,\Theta), \Theta=K^{-1}$. By the Kodaira-Spencer completeness theorem, this family contains all deformations of X sufficiently near X (loc. cit.), so can replace a neighborhood of X in Torelli (or Teichmüller) space.

We may now calculate the infinitesimal variation of harmonic volume $I(dh_1, dh_2, dh_3)$ where the homology classes of the dh_i are fixed but the conformal structure of X varies (and so the dh_i vary). Recall that $\mathbf{h}=(h_1,h_2,h_3)$ is a map of X into $\mathbf{R}^3/\mathbf{Z}^3=T^3$. The element of area dA on this surface in T^3 is $\frac{1}{2}d\mathbf{h}\times d\mathbf{h}$ (cross-product of 3-dimensional vectors together with wedge product of 1-forms giving a vector-valued 2-form). If now $\mathbf{h}^*$ denotes the corresponding vector for X^*, then for $\mathbf{h}^*$ close to $\mathbf{h}$, the volume between the surfaces X and X^* is a sum of volume elements $\frac{1}{2}(\mathbf{h}^*-\mathbf{h})\cdot(d\mathbf{h}\times d\mathbf{h})$ (triple product of vector analysis) representing the volume of a small parallelepiped with base $d\mathbf{h}\times d\mathbf{h}$ in $\mathbf{h}(X)$. Thus we get as change in volume

$$\int_X \tfrac{1}{2}(\mathbf{h}^*-\mathbf{h})\cdot(d\mathbf{h}\times d\mathbf{h}). \tag{5.5}$$

For $\mathbf{h}^*-\mathbf{h}$ we may use (5.1) *integrated* with respect to q, namely

$$h_i^*(q)-h_i(q)=\operatorname{Re}\left[2s\Omega'_{q,q_0}(t)\,h_i'(t)\right]+o(s) \tag{5.6}$$

(where $\Omega'_{q,q_0}(t)=(d\Omega_{q,q_0}/dz)(t)$. Inserting (5.6) in (5.5), we get the integral over X (with respect to the variable q)

$$\frac{1}{2}\int_{q\in X}\Omega_{q,q_0}(t)\,(d\mathbf{h}\times d\mathbf{h})(q).$$

Now $\frac{1}{2}(d\mathbf{h}\times d\mathbf{h})$ has components $dh_2\wedge dh_3$, $dh_3\wedge dh_1$, $dh_1\wedge dh_2$, and $dh_i\wedge dh_j=d\eta_{i,j}$. Let $g_{i,j}$ be the real function on X defined (up to addition of a constant) by $dg_{i,j}=*\eta_{i,j}$, so

$-d{*}dg_{i,j}=dh_i\wedge dh_j$. Thus $g_{i,j}$ is the function whose Laplacian (in any metric in the conformal class of X) is the function corresponding under $*$ to $dh_i\wedge dh_j$ (it is not actually necessary to introduce a metric, since $*$ on 1-forms suffices here). It is classical that $g_{i,j}$ is obtained from $dh_i\wedge dh_j$ by integrating against the kernel $(1/2\pi)\operatorname{Re}\Omega_{q,q_0}$:

$$g_{i,j}(p)-g_{i,j}(p_0)=\frac{1}{2\pi}\int_{q\in X}\operatorname{Re}\left[\Omega_{q,q_0}(p)-\Omega_{q,q_0}(p_0)\right]dh_i\wedge dh_j(q)$$

$(p_0, q_0$ fixed on $X)$. Taking differentials with respect to p,

$$dg_{ij}=\frac{1}{2\pi}\int_{q\in X}\operatorname{Re}(d\Omega_{q,q_0})\cdot dh_i\wedge dh_j(q)$$

$$dg_{i,j}+i{*}dg_{ij}=\frac{1}{2\pi}\int_{q\in X}d\Omega_{q,q_0}\cdot dh_i\wedge dh_j(q)={*}\eta_{i,j}-i\eta_{i,j}=-i(\eta_{i,j}+i{*}\eta_{i,j}).$$

Thus the change in volume (5.5) can be written as follows, using determinant notation and replacing $2h_i'(t)$ by $F_i'(t)=dF_i/dz(t)$, where

$$dF_i=dh_i+i{*}dh_i$$

$$\begin{aligned}&\operatorname{Re}\left[\frac{s}{2}\int_{q\in X}\Omega'_{q,q_0}(t)\begin{vmatrix}F_1'(t) & F_2'(t) & F_3'(t)\\ dh_1(q) & dh_2(q) & dh_3(q)\\ dh_1(q) & dh_2(q) & dh_3(q)\end{vmatrix}\right]\\ &=\operatorname{Im}\left[2\pi s\sum_{(1,2,3)}\left(\frac{dh_1+i{*}dh_1}{dz}\right)\left(\frac{\eta_{2,3}+i{*}\eta_{2,3}}{dz}\right)(t)\right].\end{aligned}\tag{5.7}$$

Now define a quadratic differential depending on dh_1, dh_2, dh_3 by

$$Q(dh_1, dh_2, dh_3)=\sum_{(1,2,3)}(dh_1+i{*}dh_1)(\eta_{2,3}+i{*}\eta_{23})$$

this sum of 3 differentials each of the form $\varphi(z)(dz)^2$ is in fact a holomorphic quadratic differential, as is easily checked. Furthermore the formula for Q as an integral of a determinant shows that Q is obtained as the composite of the map $\bar{D}$ of (4.7), namely the determinant, integration with kernel $d\Omega_{q,q_0}$ which maps a subspace $A^{1,1}(X)'$ of $A^{1,1}(X)$ to $A^{1,0}(X)$, and finally the symmetric product of differentials $A^{1,0}(X)\otimes A^{1,0}(X)\to$ (quadratic differentials on X). The properties proved in Section 4 for the maps D and $\bar{D}$ now yield

THEOREM 5.8. (1) *Q: $P\otimes\mathbf{R}\to H^0(X;K^2)$ is a complex linear map if we use the complex structure j_3 on $P\otimes\mathbf{R}\subset H^3(J;\mathbf{R})$, and the induced map $Q\otimes 1$ on $P\otimes\mathbf{C}$ vanishes on $(P\otimes\mathbf{C})\cap H^{a,b}(J)$ if $(a,b)\neq(2,1)$.*

(2) *Let $\lambda=dh_1\wedge dh_2\wedge dh_3\in P$ and consider the harmonic volume $I(\lambda;s)=I(dh_1,dh_2,dh_3;s)$ for $s=(s_1,\ldots,s_{3g-3})$ near $s=0$. Then*

$$I(\lambda;s)-I(\lambda;0)=\textit{Imaginary part of}\left[2\pi\sum_{j=1}^{3g-3}s_j\frac{Q(\lambda)}{(dz_j)^2}(t_j)\right]+o(s)$$

(where z_j is the complex coordinate on $X=X_0$ near t_j).

(3) *Harmonic volume I varies holomorphically with s.*

Proof. (1) follows from the remarks just before the statement of the theorem.

(2) is a restatement of formula (5.7) and the definition of Q, in the general case of a variation at $3g-3$ points at each of which $Q(\lambda)$ is evaluated.

(3) is a restatement of Proposition 4.3, in which we considered a 1-complex variable variation, say s_i at t_i, and $Q_1(\lambda)$ then denotes $(Q(\lambda)/(dz_i)^2)(t_i)$.

6. Hyperelliptic Riemann surfaces

Let X be hyperelliptic, with involution σ. Then $\sigma=-1$ on $\mathcal{H}_{\mathbf{Z}}$. The images of X in T^3 under $\mathbf{h}=(h_1,h_2,h_3)$ and under $\mathbf{h}\circ\sigma=-\mathbf{h}$ are the same (as singular 2-chains) and so the volumes bounded are the same $(\mathrm{mod}\,\mathbf{Z})$. On the other hand, the volume form on T^3 reverses sign under $\mathbf{x}\mapsto-\mathbf{x}$, so we conclude that $I(dh_1,dh_2,dh_3)=-I(dh_1,dh_2,dh_3)\bmod\mathbf{Z}$ or: *I takes values 0 or $\frac{1}{2}\bmod\mathbf{Z}$ for hyperelliptic Riemann surfaces.*

Next we discuss the map δI for hyperelliptic surfaces. Let the surface be given by the equation $u^2=\Pi_1^n(z-e_i)$, $n=2g+2$, e_i distinct in $\mathbf{C}$. Put $z=re^{i\theta}$, $z-e_j=r_je^{i\theta_j}$ in the z-plane. We assume $\Pi_1^n r_j$ is an *even* function of θ (which holds if the e_j are symmetrically distributed about the real axis).

A basis for the holomorphic 1-forms is

$$\begin{aligned} dw_k&=\frac{z^k dz}{\sqrt{(z-e_1)\ldots(z-e_n)}}\quad\left(k=0,1,\ldots,\frac{n}{2}-2\right)\\ &=\frac{r^k}{(r_1\ldots r_n)^{1/2}}e^{ik\theta}e^{-i(\theta_1+\ldots+\theta_n)/2}e^{i\theta}(dr+ird\theta). \end{aligned}$$

Then

$$\begin{aligned}
\operatorname{Re} dw_k \wedge \operatorname{Re} dw_l &= \frac{(dw_k+\overline{dw_k})}{2} \wedge \frac{(dw_l+\overline{dw_l})}{2} \\
&= \frac{1}{4}(dw_k \wedge \overline{dw_l} - dw_l \wedge \overline{dw_k} \\
&= \frac{r^{k+l}}{\Pi_1^n r_i}\left(\frac{e^{i(k-l)\theta}-e^{i(l-k)\theta}}{4}\right)(-2i)(r\,dr\,d\theta) \\
&= \frac{r^{k+l}}{r_1 \dots r_n}(\sin(k-l)\,\theta)\, r\,dr\,d\theta,
\end{aligned}$$

which is a 2-form on the Riemann sphere $\mathbf{P}^1$ pulled up to X by the projection $p: X \to \mathbf{P}^1$. Thus since this projection has degree 2,

$$\begin{aligned}
\int_X \operatorname{Re}(dw_k) \wedge \operatorname{Re}(dw_l) &= 2 \int\int_{\mathbf{P}^1} \left(\frac{r^{k+l}}{r_1 \dots r_n} \sin(k-l)\,\theta\right) r\,dr\,d\theta \\
&= 2 \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{r^{k+l}}{r_1 \dots r_n} \sin(k-l)\,\theta\, r\,dr\,d\theta.
\end{aligned}$$

The r_i are functions of both r and θ and their product *will be assumed* to be an *even* function of θ, thus integrating first in θ gives zero:

$$\int_X \operatorname{Re}(dw_k) \wedge \operatorname{Re}(dw_l) = 0.$$

Now consider the elementary differential of the third kind

$$d\Omega_{q,q_0} = 2\pi(dU_{q,q_0} + idV_{q,q_0})$$

on X (beginning of Section 5) with poles at q, q_0 with residues -1, $+1$. Choose q_0 to be a fixed point of the involution σ (q_0 lies over some e_i). Then $d\Omega_{q,q_0} + d\Omega_{\sigma(q),q_0} = d\Omega_{q,q_0} + \sigma^* d\Omega_{q,q_0}$ has poles at $q, \sigma(q), q_0, \sigma(q_0) = q_0$ and is invariant under the involution, and also is determined by its poles and residues (since its periods are pure imaginary). Thus it equals the image under p^* of a differential on $\mathbf{P}^1$, namely

$$d\Omega_{q,q_0} + \sigma^* d\Omega_{q,q_0} = p^*\left[\frac{dz}{z-z(q_0)} - \frac{dz}{z-z(q)}\right]. \tag{6.1}$$

Let $d\eta_{k,l} = (\operatorname{Re} dw_k) \wedge (\operatorname{Re} dw_l)$, then from Section 5

$$\eta_{k,l} + i{*}\eta_{k,l} = \frac{i}{2\pi} \int_{q \in X} d\Omega_{q,q_0} (\operatorname{Re} dw_k \wedge \operatorname{Re} dw_l)(q). \tag{6.2}$$

However, since $\operatorname{Re} dw_k \wedge \operatorname{Re} dw_l$ is invariant under σ, so is $\eta_{k,l}$ and therefore so are both sides of (6.2). Thus in the integral on the right hand side of (6.2), we may replace $d\Omega_{q,q_0}$ by $\frac{1}{2}(d\Omega_{q,q_0}+\sigma^* d\Omega_{q,q_0})$ which is given by (6.1). Furthermore, since we integrate with respect to the variable q, the term $dz/(z-z(q_0))$ is constant as function of q and so contributes zero to the integral since $\int_X \operatorname{Re} dw_k \wedge \operatorname{Re} dw_l = 0$. Thus (6.2) becomes, *on putting* $z(q)=re^{i\theta}$ (which is the variable of integration and not to be confused with z)

$$\eta_{k,l}+i{*}\eta_{k,l}=\frac{-i}{2\pi}dz\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{1}{(z-re^{i\theta})}\,\frac{r^{k+l}\sin(k-l)\,\theta}{r_1\ldots r_n}\,r\,dr\,d\theta. \tag{6.3}$$

We defined the quadratic differential as

$$\delta I(dh_1, dh_2, dh_3)=2\pi\sum_{\text{cyclic}}(dh_j+i{*}dh_j)(\eta_{k,l}+i{*}\eta_{k,l})$$

which, in our case, gives, if $dw_j=z^j\,dz/\sqrt{(z-e_1)\ldots(z-e_n)}$

$$\begin{aligned}&\delta I(\operatorname{Re} dw_j\wedge \operatorname{Re} dw_k\wedge \operatorname{Re} dw_l)\\ &\quad=-i\frac{(dz)^2}{\sqrt{(z-e_1)\ldots(z-e_n)}}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{(\Sigma z^j r^{k+l}\sin(k-l)\,\theta)\,r\,dr\,d\theta}{(z-re^{i\theta})\,r_1\ldots r_n}.\end{aligned}$$

The cyclic sum in the numerator is a determinant

$$\begin{vmatrix} z^j & z^k & z^l\\ r^j\sin j\theta & r^k\sin k\theta & r^l\sin l\theta\\ r^j\cos j\theta & r^k\cos k\theta & r^l\cos l\theta\end{vmatrix}=-\frac{i}{2}\begin{vmatrix} z^j & z^k & z^l\\ \zeta^j & \zeta^k & \zeta^l\\ \bar{\zeta}^j & \bar{\zeta}^k & \bar{\zeta}^l\end{vmatrix}$$

where $\zeta=re^{i\theta}=z(q)$; in the special case $k=j+1$, $l=j+2$ this equals

$$-\frac{i}{2}(\zeta\bar{\zeta})^j z^j(\zeta-z)(\bar{\zeta}-z)(\bar{\zeta}-\zeta)=-z^j r^{2j}(\zeta-z)(\bar{\zeta}-z)\sin\theta.$$

Thus

$$\begin{aligned}&\delta I(\operatorname{Re} dw_j\wedge \operatorname{Re} dw_{j+1}\wedge \operatorname{Re} dw_{j+2})\\ &\quad=-i\frac{z^j(dz)^2}{\sqrt{(z-e_1)\ldots(z-e_n)}}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{(re^{-i\theta}-z)\,r^{2j+1}\sin\theta\,r\,dr\,d\theta}{r_1\ldots r_n}.\end{aligned}$$

Breaking up the integral into a sum of 3 terms corresponding to $re^{-i\theta}-z=r\cos\theta-ir\sin\theta-z$ and noting that symmetry in θ of $r_1\ldots r_n$ makes the first and third vanish, gives us

$$\delta I(\operatorname{Re} dw_j \wedge \operatorname{Re} dw_{j+1} \wedge \operatorname{Re} dw_{j+2}) = -\frac{z^j (dz)^2}{\sqrt{(z-e_1)\dots(z-e_n)}} \left(\int_0^\infty \int_0^{2\pi} \frac{r^{2j+2} \sin^2\theta \, r \, dr \, d\theta}{r_1 \dots r_n} \right) \tag{6.4}$$

where the integral is >0, $n=2g+2$ and $j \leqslant g-3$. We can state now:

THEOREM 6.5. *Let X be the hyperelliptic Riemann surface of the function $\sqrt{(z-e_1)\dots(z-e_n)}$, $n=2g+2$, $g \geqslant 3$. Assume that the (distinct) complex numbers $e_1, \dots, e_n$ satisfy the condition:*

$$\prod_{i=1}^{n} |z-e_i| \quad \textit{is a symmetric function of } \theta = \operatorname{Arg} z.$$

Then δI: $(\Lambda^3_{\mathbf{R}} \mathcal{H})' \to H^0(X; K^2)$ has as image the whole (-1)-eigenspace of the hyperelliptic involution.

Proof. This (-1)-eigenspace has as basis the quadratic differentials listed in (6.4) with $0 \leqslant j \leqslant g-3$. The map δI commutes with automorphisms of X and in particular with σ which acts as -1 on $\Lambda^3 \mathcal{H}$, thus its image lies in the (-1)-eigenspace of σ. This proves (6.5).

We may sum up our results on hyperelliptic X now: I takes only values 0 or $\frac{1}{2}$ in $\mathbf{R}/\mathbf{Z}$, and is thus constant as X varies in the $2g-1$ dimensional hyperelliptic locus. The space of tangent vectors to Torelli space at X which are normal to the hyperelliptic locus is $g-2$ dimensional and dual to the (-1)-eigenspace in the quadratic differentials. The differential of I_s regarded as a linear function on this (-1)-subspace of the tangent space to S is *injective* i.e. its transpose δI is surjective (at least in the special case where $e_1, \dots, e_{2g+2}$ satisfy a symmetry property).

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