A Paley-Wiener theorem for real reductive groups

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Contents

Introduction

Let G be a reductive Lie group with maximal compact subgroup K. The Paley-Wiener problem is to characterize the image of $C_c^\infty(G)$ under Fourier transform. It turns out to be more natural to look at the K finite functions in $C_c^\infty(G)$. This space, which we denote

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by $C_c^\infty(G, K)$, is sometimes called the Hecke algebra in analogy with p-adic groups. The goal of this paper is to characterize the image of $C_c^\infty(G, K)$ under Fourier transform. The problem was solved for real rank one by Campoli in his thesis [1]. This paper represents the generalization of Campoli's results to arbitrary rank.

If $f \in C_c^\infty(G, K)$ and π is an irreducible representation of G on a Banach space U_π, let

$$\pi(f) = \int_G f(x)\,\pi(x)\,dx.$$

Then

$$\pi \to \pi(f)$$

is a function whose domain is the set of irreducible representations of G, and which for any (π, U_π) takes values in the space of operators on U_π. The problem is to characterize which functions $\pi \to F(\pi)$ are of this form. For our introduction we shall fix a minimal parabolic subgroup B with Langlands decomposition $N_0 A_0 M_0^1$. Then we will have the (nonunitary) principal series, a family of representations $I_B(\sigma, \Lambda)$ of G induced from B, indexed by quasi-characters Λ of A_0, and irreducible representations σ of M_0^1. By a well known theorem of Harish-Chandra, any π is equivalent to a subquotient of some $I_B(\sigma, \Lambda)$. This means that $\pi(f)$ will be completely determined by the map

$$\hat{f}: (\sigma, \Lambda) \to \hat{f}_B(\sigma, \Lambda) = I_B(\sigma, \Lambda, f).$$

We will call $\hat{f}$ the *Fourier transform* of f.

What should the image of the Fourier transform be? The function $\hat{f}_B(\sigma, \Lambda)$ will have to satisfy certain growth conditions. It should also be an entire function of Λ. However, there is another, more complicated condition. It is that any linear relation among the matrix coefficients of the representations $I_B(\sigma, \Lambda)$ will have also to hold for the matrix coefficients of the operators $\hat{f}_B(\sigma, \Lambda)$. An adequate understanding of these linear relations would include a complete knowledge of all the irreducible subquotients of the principal series. Since this is not available, the third condition on $\hat{f}_B(\sigma, \Lambda)$ is not very explicit. In any case, following [1], we will define $PW(G, K)$ to be the space of all functions

$$F: (\sigma, \Lambda) \to F_B(\sigma, \Lambda)$$

which satisfy the three conditions above. Then our main result (Theorem III.4.1) is that the map $f \to \hat{f}$ is a topological isomorphism from $C_c^\infty(G, K)$ onto $PW(G, K)$.

An interesting consequence of our main result is the construction of an algebra of multipliers for $C_c^\infty(G, K)$. By a multiplier we mean a linear operator C on $C_c^\infty(G, K)$ such that

$$C(f * g) = C(f) * g = f * C(g)$$

for all f and g in $C_c^\infty(G, K)$. If C is such an operator, and π is an irreducible representation of G, there will be a scalar C_π such that

$$\pi(C(f)) = C_\pi \pi(f),$$

for every function $f \in C_c^\infty(G, K)$. In Theorem III.4.2 we will obtain an algebra of multipliers $\{C\}$ by constructing the corresponding algebra of scalar-valued functions $\{\pi \to C_\pi\}$. It is an analogue for real groups of a result (unpublished) of I. N. Bernstein, in which all the multipliers for the Hecke algebra of a p-adic group were constructed. We envisage using the theorem in the following way. Suppose that we happened to know that a given map

$$\pi \to F(\pi)$$

was represented by a function in $C_c^\infty(G, K)$. Then we could construct many other maps, each also represented by a function in $C_c^\infty(G, K)$, by taking

$$\pi \to C_\pi F(\pi).$$

If one studies the contribution of Eisenstein series to the trace formula, one is confronted with this very circumstance. In fact, the trace formula was our original motivation for working on the Paley-Wiener problem. In another paper, we will use Theorem III.4.2 to overcome a nasty convergence problem connected with Eisenstein series.

This paper is divided into three chapters. The first chapter is a collection of various results which are required for the proof of our main theorem. Much of the chapter contains familiar material, and discussion proceeds rather briskly. Chapters II and III contain the main body of the proof. It is a question of studying successive residues of certain meromorphic functions of Λ, in the spirit of Chapter 7 of Langlands' treatise [11 b] on Eisenstein series. The reader might find it easiest to start this paper at the beginning of Chapter II, referring to the sections in Chapter I only as they are needed.

We shall conclude the introduction by attemting to sketch the salient features of the proof of our main theorem. The theorem will actually be proved for $C_c^\infty(G, \tau)$, the space of smooth, compactly supported functions which are spherical with respect to a two-sided representation, τ, of K. Associated to τ we have the Eisenstein integral

$$E_B(x, \Phi, \Lambda), \quad x \in G,\ \Phi \in \mathcal{A}_0,\ \Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}.$$

$\mathcal{A}_0$ is the (finite dimensional) space of π_{M_0} spherical functions on M_0^1, while $\mathfrak{a}^*_{0,\mathbf{C}}$ is the space of quasi-characters on A_0, a complex vector space of dimension n, say. In this setting, $PW(G, \tau)$ will be defined as a certain space of entire functions from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathcal{A}_0$. The most difficult part of the theorem is to prove surjectivity. Given $F \in PW(G, \tau)$ we have to produce a function in $C_c^\infty(G, \tau)$. If f is such a function, we know from Harish-Chandra's Plancherel theorem that there will be a decomposition

$$f = \sum_{\mathcal{P}} f_{\mathcal{P}}$$

of f into components indexed by classes of associated standard parabolic subgroups of G. The component from the minimal parabolic subgroup will be

$$F^\vee_{\{B\}}(x) = |W_0|^{-1} \int_{i\mathfrak{a}_0^*} E_B(x, \mu_B(\Lambda) F(\Lambda), \Lambda)\, d\Lambda, \quad x \in G,$$

where $\mu_B(\Lambda)$ is Harish-Chandra's μ function and W_0 is the Weyl group of (G, A_0). We must somehow construct the other components and the function f.

Let $A_0(B)$ be the chamber in A_0 associated to B. If $a \in A_0(B)$, $E_B(a, \Phi, \Lambda)$ can be written as a sum

$$\sum_{s \in W_0} E_{B|B,s}(a, \Phi, \Lambda)$$

of functions indexed by the Weyl group. For any s, $E_{B|B,s}(\cdot, \Phi, \Lambda)$ is the unique function on $A_0(B)$ whose asymptotic expansion has leading term

$$(c_{B|B}(s, \Lambda)\Phi)(1) \cdot a^{(s\Lambda)} \cdot \delta_B(a)^{-1/2},$$

where δ_B is the modular function of B and $c_{B|B}(s, \Lambda)$ is Harish-Chandra's c function. We will extend $E_{B|B,s}(\cdot, \Phi, \Lambda)$ to a τ spherical function on

$$G_- = KA_0(B)K,$$

an open dense subset of G. It will then turn out that for $F \in PW(G, \tau)$ and $x \in G_-$,

$$E_{B|B,s}(x, \mu_B(\Lambda) F(\Lambda), \Lambda) = E_{B|B,1}(x, \mu_B(s\Lambda) F(s\Lambda), s\Lambda).$$

As a function of Λ this expression will be meromorphic with poles along hyperplanes of the form $(\beta, \Lambda) = r$, for β a root of (G, A_0) and $r \in \mathbf{R}$ (Corollary I.6.3). We shall also show that only finitely many of these poles intersect the negative chamber, $-\mathfrak{a}_0^*(B)$, in $\mathfrak{a}_0^*$ (Lemma I.5.3). For this introduction, let us assume that none of the poles meet the imaginary space $i\mathfrak{a}_0^*$. Then for $x \in G_-$,

$$F_{\{B\}}^\vee(x) = \int_{i\mathfrak{a}_0^*} E_{B|B,1}(x, \mu_B(\Lambda) F(\Lambda), \Lambda)\, d\Lambda.$$

Let X be a point in general position in the chamber $-\mathfrak{a}_0^*(B)$ which is far from any of the walls. In Theorem II.1.1 we will show that the function

$$F^\vee(x) = \int_{X+i\mathfrak{a}_0^*} E_{B|B,1}(x, \mu_B(\Lambda) F(\Lambda), \Lambda)\, d\Lambda, \quad x \in G_-,$$

is supported on a subset of G_- whose closure in G is compact. It will be our candidate for $f(x)$.

The difference

$$F^\vee(x) - F_{\{B\}}^\vee(x)$$

will be a sum of residues. Each one will be an integral over an affine space $X_T + i\mathfrak{b}$, where $X_T \in \mathfrak{a}_0^*$ and $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}_0^*$ of dimension $n-1$. (T will belong to some indexing set.) We will group the residual integrals into sums corresponding to the W_0-orbits among the spaces $\mathfrak{b}$. Now for any W_0-orbit of spaces $\mathfrak{b}$ there is an associated class $\mathcal{P}$ of parabolic subgroups of G, of parabolic rank $n-1$. We might expect that the corresponding sum of residual integrals should equal $f_{\mathcal{P}}(x)$. However, to have any hope of this we will need to replace each X_T by a vector which is orthogonal to $\mathfrak{b}$. Let Λ_T be the vector in $X_T + \mathfrak{b}$ which is orthogonal to $\mathfrak{b}$. Let $F_{\mathcal{P}}^\vee(x)$ be the sum of all the residual integrals, taken now over the contour $\Lambda_T + i\mathfrak{b}$, for which $\mathfrak{b}$ belongs to the W_0-orbit associated to $\mathcal{P}$. (Again, we make the simplifying assumption that each residual integrand is regular on $\Lambda_T + i\mathfrak{b}$.) Then

$$F^\vee(x) - F_{\{B\}}^\vee(x) - \sum_{\{\mathcal{P}:\, \mathrm{prk}\, \mathcal{P} = n-1\}} F_{\mathcal{P}}^\vee(x)$$

will be a new sum of residual integrals, each over an affine space of dimension $n-2$. We can repeat the process. In the end we will arrive at a formula

$$F^\vee(x) = \sum_{\mathcal{P}} F^\vee_{\mathcal{P}}(x), \quad x \in G_-,$$

where now $\mathcal{P}$ ranges over all classes of associated parabolic subgroups.

The difficulty is that we do not yet know that $F^\vee(x)$ extends from G_- to a smooth function on G. Suppose that for every $\mathcal{P} \neq \{G\}$ the functions $F^\vee_{\mathcal{P}}(x)$ could be extended smoothly to G. We could then use an argument of Campoli to extend $F^\vee_{\{G\}}$ to a smooth function on G. From this, we would be able to conclude that the function

$$F^\vee(x) = \sum_{\mathcal{P}} F^\vee_{\mathcal{P}}(x)$$

belonged to $C_c^\infty(G, \tau)$ and that its Fourier transform was the original function $F(\Lambda)$. Clearly, then, an induction hypothesis is in order. If P is any proper, standard parabolic subgroup of G, with Levi component M, we will assume that our main theorem holds for $C_c^\infty(M, \tau)$. In Lemma III.2.3 we will show that there is a natural injection

$$F \to F_P$$

of $PW(G, \tau)$ into $PW(M, \tau)$. By induction we will obtain a function $F_P^\vee$ in $C_c^\infty(M, \tau)$. Let A_P be the split component of P and let $\mathfrak{a}_P$ be the Lie algebra of A_P. If $\lambda \in i\mathfrak{a}_P^*$ let $F_{P,\mathrm{cusp}}(\lambda)$ be the function which maps any point $m \in M_-$ to

$$\int_{A_P} F^\vee_{P,\{M\}}(ma)\, e^{\lambda(H_P(ma))}\, da.$$

It will extend from M_- to a cuspidal, τ_M spherical function on M. We would be able to establish our main theorem for G if we could prove the formula

$$F^\vee_{\mathcal{P}}(x) = |\mathcal{P}|^{-1} \sum_{P \in \mathcal{P}} |W(\mathfrak{a}_P)|^{-1} \int_{i\mathfrak{a}_P^*} E_P(x, \mu_P(\lambda) F_{P,\mathrm{cusp}}(\lambda), \lambda)\, d\lambda, \tag{1}$$

for any class $\mathcal{P} \neq \{G\}$ and $x \in G$. Indeed, the function on the right is a wave packet of Eisenstein integrals and certainly extends to a smooth function on G. We would complete the argument as outlined above.

However this last formula turns out to be quite difficult. It hinges on a recent theorem of Casselman. Let $\mathfrak{Z}_M$ be the algebra of left and right invariant differential operators on M. In Chapter I, § 7 we will define a space $\mathscr{A}(M_-, \tau)$ of $\mathfrak{Z}_M$-finite, τ_M-spherical functions on M_-. (It seems likely that it is the space of all such functions.) $\mathscr{A}(M_-, \tau)$ will contain $\mathscr{A}_{\text{cusp}}(M, \tau)$, the space of functions which extend to cuspidal, A_P invariant functions on M. The essence of Casselman's theorem is that the Eisenstein integral $E_P(x, \varphi, \lambda)$, defined by Harish Chandra for $\varphi \in \mathscr{A}_{\text{cusp}}(M, \tau)$, can actually be extended to the space $\mathscr{A}(M_-, \tau)$. Then

$$E_P(x, \varphi, \lambda), \quad x \in G_-, \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}, \varphi \in \mathscr{A}(M_-, \tau),$$

becomes a meromorphic function of λ with values in the space of linear maps from $\mathscr{A}(M_-, \tau)$ to $\mathscr{A}(G_-, \tau)$. We will state Casselman's theorem formally in Chapter II, § 4. Then, in § 5 of the same chapter, we derive some consequences of the theorem. It turns out that all of the J, c and μ functions, defined by Harish-Chandra in [7 e] as linear maps on $\mathscr{A}_{\text{cusp}}(M, \tau)$, can also be extended to the space $\mathscr{A}(M_-, \tau)$. In particular the map

$$\varphi \to E_P(x, \mu_P(\lambda)\varphi, \lambda)$$

can be defined for $\varphi \in \mathscr{A}(M_-, \tau)$. This will allow us, in Theorem II.7.1, to prove a version of the formula (1). We will then combine it with the induction hypothesis in Chapter III to establish the main theorem.

A number of authors have proved Paley-Wiener theorems for particular classes of groups. The case of $SL_2(\mathbf{R})$ was solved by Ehrenpreis and Mautner [5 a], [5 b]. For the K bi-invariant functions on a general group the main problem was solved by Helgason [8 a], [8 b] and Gangolli [6]. They developed techniques which allowed for changes of contours of integration. To them is due the analysis on which sections I.5 and II.1 of this paper are based. The case of K bi-invariant functions is simplified by the fact that no residues are encountered during the necessary contour changes. Further results in this direction were later obtained by Helgason [8 c], [8 d]. We have already mentioned Campoli's contribution [1] for groups of real rank one. A Paley-Wiener theorem for complex semisimple groups was announced by Zelobenko in [14]. More recently, Delorme [4] established a Paley-Wiener theorem for any groups with one conjugacy class of Cartan subgroups. His techniques are algebraic in nature, and are completely different from ours. Finally, Kawazoe [10 a], [10 b] made significant progress in handling the residues on groups of rank greater than one. In particular, he established the main theorem for the group $SU(2, 2)$.

I am indebted to W. Casselman for many enlightening discussions. His theorem,

which is crucial for this paper, will be described in the forthcoming paper [2 b]. I would also like to thank N. Wallach for conversations on asymptotic expansions and J. Millson for telling me of some of Campoli's results.

Notational conventions: If H is any Lie group, we will denote the real Lie algebra of H by Lie (H), and universal enveloping algebra of Lie $(H)\otimes\mathbf{C}$ by $\mathscr{U}(H)$.

Our method for cross reference is as follows. Theorem III.4.2 means Theorem 2 of Section 4 of Chapter III. However, we will omit the numbers of chapters when referring to lemmas, theorems or formulas of a current chapter.

Chapter I

§ 1. The group G

Suppose that G is a reductive Lie group, with a fixed maximal compact subgroup K. We shall assume that G and K satisfy the general axioms of Harish-Chandra [7 e, § 3]. Then both G and its Lie algebra can be equipped with an involution θ as in [7 e, § 3]. Any parabolic subgroup P of G has a decomposition $P=N_P M_P$, where N_P is the unipotent radical of P and M_P is a reductive subgroup of G which is stable under θ ([7 e, § 4]). We shall call M_P the *Levi component* of P; we shall say that any group is a *Levi subgroup* (of G) if it is the Levi component of a parabolic subgroup of G.

Suppose that $M_* \subset M$ are two Levi subgroups of G. We shall denote the set of Levi subgroups of M which contain M_* by $\mathscr{L}^M(M_*)$. Let us also write $\mathscr{F}^M(M_*)$ for the set of parabolic subgroups of M which contain M_*, and $\mathscr{P}^M(M_*)$ for the set of groups in $\mathscr{F}^M(M_*)$ for which M_* is the Levi component. Each of these three sets is finite. If $M=G$ we shall usually denote the sets by $\mathscr{L}(M_*)$, $\mathscr{F}(M_*)$ and $\mathscr{P}(M_*)$. (In general, if a superscript M is used to denote the dependence of some object in this paper on a Levi subgroup, we shall often omit the superscript when $M=G$.) If $R\in\mathscr{P}^M(M_*)$ and $Q\in\mathscr{P}(M)$, we will let $Q(R)$ denote the unique group in $\mathscr{P}(M_*)$ which is contained in Q.

For the rest of this paper, M_0 will denote a fixed *minimal* Levi subgroup of G. If $M\in\mathscr{L}(M_0)$, let $M_K=M\cap K$. For induction arguments it will often be necessary to apply the notation and results of this paper to the group M. This poses no problem, for the triplet (M, K_M, M_0) satisfies the same hypotheses as (G, K, M_0).

Suppose that M is a group in $\mathscr{L}(M_0)$. Let A_M be the split component of M ([7 e, § 3]), and set $\mathfrak{a}_M=\mathrm{Lie}(A_M)$, the Lie algebra of A_M. Then $\mathfrak{a}_M$ is canonically isomorphic with

$$\mathrm{Hom}(X(M), \mathbf{R}),$$

where $X(M)$ is the group of all continuous homomorphisms from M to $\mathbf{R}^*$. As usual, we define a surjective homomorphism

$$H_M: M \to \mathfrak{a}_M$$

by setting

$$e^{\langle H_M(m), \chi \rangle} = |\chi(m)|, \quad \chi \in X(M),\ m \in M.$$

(In case $M=M_0$, we shall always write $\mathfrak{a}_0=\mathfrak{a}_{M_0}$, $A_0=A_{M_0}$ and $H_0=H_{M_0}$.) In general, M is the direct product of the kernel of H_M, which we denote by M^1, and A_M. Suppose that λ is an element in $\mathfrak{a}^*_{M,\mathbf{C}}$, the complexification of the dual space of $\mathfrak{a}_M$. Then λ defines quasi-characters

$$H \to \lambda(H), \quad H \in \mathfrak{a}_M,$$

and

$$a \to a^\lambda, \quad a \in A_M,$$

on each of the abelian groups $\mathfrak{a}_M$ and A_M. They are related by

$$a^\lambda = e^{\lambda(H_M(a))}, \quad a \in A_M.$$

Suppose that $P \in \mathcal{P}(M)$. We shall sometimes write $A_P=A_M$ and $\mathfrak{a}_P=\mathfrak{a}_M$. Associated to P are various real quasicharacters on these two groups. One arises from the modular function δ_P of P. Its restriction to A_P equals

$$\delta_P(a) = a^{2\varrho_P} = e^{2\varrho_P(H_M(a))}, \quad a \in A_P,$$

for a unique vector ϱ_P in $\mathfrak{a}^*_M$. There is also the set Σ_P of roots of (P, A_P), and the subset Δ_P of simple roots. We shall write $\mathbf{Z}(\Delta_P)$ for the abelian subgroup of $\mathfrak{a}^*_P$ generated by Δ_P, and $\mathbf{Z}^+(\Delta_P)$ for the subset of $\mathbf{Z}(\Delta_P)$ consisting of nonnegative integral combinations of elements in Δ_P.

If P is any group in $\mathcal{F}(M_0)$, we know that

$$G = PK = N_P M_P K.$$

For a given point x in G, let $N_P(x)$, $M_P(x)$, and $K_P(x)$ be the components of x in N_P, M_P and K relative to this decomposition. We shall write

$$H_P(x) = H_{M_P}(M_P(x)).$$

It is convenient, although not really necessary, to fix a G-invariant, symmetric bilinear form $(\,,)$ on $\mathrm{Lie}(G)$ such that, as in [7e], the quadratic form

$$-(X, \theta X), \quad X \in \mathrm{Lie}(G),$$

is positive definite. We will also write $(\,,)$ for the $\mathbf{C}$-linear extension of the bilinear form to $\mathrm{Lie}(G) \otimes \mathbf{C}$. Suppose that $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathrm{Lie}(G) \otimes \mathbf{C}$ such that $\mathfrak{h}_{\mathbf{C}} \cap \mathrm{Lie}(G)$ is a θ stable Cartan subalgebra of $\mathrm{Lie}(G)$. Then $(\,,)$ is nondegenerate on $\mathfrak{h}_{\mathbf{C}}$, and

$$H \to -(H, \theta H), \quad H \in \mathfrak{h}_{\mathbf{C}} \cap \mathrm{Lie}(G),$$

extends to a Hermitian norm $\|\cdot\|$ on $\mathfrak{h}_{\mathbf{C}}$. From the nondegenerate bilinear form on $\mathfrak{h}_{\mathbf{C}}$ we can define a bilinear form, which we also denote by $(\,,)$, on the dual space $\mathfrak{h}_{\mathbf{C}}^*$. We also obtain a Hermitian norm $\|\cdot\|$ on $\mathfrak{h}_{\mathbf{C}}^*$.

From now on we will take

$$\mathfrak{h}_{\mathbf{C}} = \mathfrak{h}_{K,\mathbf{C}} \otimes \mathfrak{a}_{0,\mathbf{C}},$$

where $\mathfrak{h}_K$ is a fixed Cartan subalgebra of $\mathrm{Lie}(K) \cap \mathrm{Lie}(M_0)$. Then by restriction we obtain a bilinear form $(\,,)$ and a Hermitian norm $\|\cdot\|$ on both $\mathfrak{a}_{0,\mathbf{C}}$ and $\mathfrak{a}_{0,\mathbf{C}}^*$. Suppose that $M \in \mathscr{L}(M_0)$. Then there are embeddings $\mathfrak{a}_{M,\mathbf{C}} \subset \mathfrak{a}_{0,\mathbf{C}}$ and $\mathfrak{a}_{M,\mathbf{C}}^* \subset \mathfrak{a}_{0,\mathbf{C}}^*$, so we can also restrict $(\,,)$ and $\|\cdot\|$ to these smaller spaces. As is customary, a singular hyperplane in $\mathfrak{a}_M^*$ will mean a subspace of the form

$$\{\lambda \in \mathfrak{a}_M^* : (\beta, \lambda) = 0\}$$

for some root β of (G, A_M). If $P \in \mathscr{P}(M)$, we shall write

$$\mathfrak{a}_M^*(P) = \{\lambda \in \mathfrak{a}_M^* : (\beta, \lambda) > 0,\ \beta \in \Sigma_P\},$$

$$\mathfrak{a}_M(P) = \{H \in \mathfrak{a}_M : \beta(H) > 0,\ \beta \in \Sigma_P\},$$

and

$$A_M(P) = \{a \in A_M : a^\beta > 1,\ \beta \in \Sigma_P\}.$$

Finally, we should say a word about Haar measures. From time to time we will want to integrate over various unimodular groups. Unless specified otherwise, the integrals will always be with respect to a fixed, but unnormalized, Haar measure. There will be two exceptions. On the compact group K_M, $M \in \mathscr{L}(M_0)$, we will always take the Haar measure for which the total volume is one. The second exception concerns groups connected with the spaces $\mathfrak{a}_M$. On $\mathfrak{a}_M$ we will take the Euclidean measure with respect to the fixed norm $\|\cdot\|$. The exponential map will transform this measure to a fixed Haar measure on the group A_M. Finally, on the real vector space $i\mathfrak{a}_M^*$ we will take the measure which is dual to the measure we fixed on $\mathfrak{a}_M$. Observe that if $\{\nu_1, \dots, \nu_r\}$ is any orthonormal basis of $\mathfrak{a}_M^*$, and h is a function in $C_c^\infty(i\mathfrak{a}_M^*)$,

$$\int_{i\mathfrak{a}_M^*} h(\lambda)\, d\lambda = \left(\frac{1}{2\pi i}\right)^r \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} h(u_1\nu_1 + \dots + u_r\nu_r)\, du_1 \dots du_r.$$

§ 2. Eisenstein integrals and associated functions

Throughout this section, and indeed for most of the paper, τ will be a fixed two sided representation of K on a finite dimensional vector space V_τ. This is the setting for Harish-Chandra's Eisenstein integral as well as his J, c and μ functions, which play such a central role in the harmonic analysis on G. We shall list some of the basic properties of these objects.

Fix a Levi subgroup M in $L(M_0)$. Let $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$ be the space of τ_M spherical functions on M/A_M which are cuspidal. This is the same as the space of *square integrable*, $\mathfrak{Z}_M$-finite functions

$$\varphi: M \to V_\tau$$

such that

(i) $\varphi(k_1 m k_2) = \tau(k_1)\varphi(m)\tau(k_2), \quad k_1, k_2 \in K_M,\ m \in M,$

and

(ii) $\varphi(ma) = \varphi(m), \quad m \in M,\ a \in A_M.$

Here $\mathfrak{Z}_M$ is the algebra of left and right invariant differential operators on M. The space $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$ is finite dimensional, and in fact equals $\{0\}$ unless M/A_M has a discrete series. Indeed, if ω is an equivalence class of square integrable representations of M/A_M, let $\mathcal{A}_\omega(M, \tau)$ be the space of functions φ in $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$ such that for any $\xi^* \in V_\tau^*$, the function

$$m \to \xi^*(\varphi(m)), \quad m \in M/A_M,$$

is a sum of matrix coefficients of ω. Then

$$\mathcal{A}_{\mathrm{cusp}}(M, \tau) = \bigoplus_\omega \mathcal{A}_\omega(M, \tau).$$

Let $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)$, $P \in \mathscr{P}(M)$, $x \in G$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. The Eisenstein integral is defined to be

$$E_P(x, \varphi, \lambda) = \int_{K_M \backslash K} \tau(k^{-1})\, \varphi_P(kx)\, e^{(\lambda+\rho_P)(H_P(kx))}\, dk,$$

where φ_P is the function on G such that

$$\varphi_P(nmk) = \varphi(m)\,\tau(k), \quad n \in N_P,\ m \in M,\ k \in K.$$

Then the function

$$E_P(\varphi, \lambda) : x \to E_P(x, \varphi, \lambda)$$

depends analytically on λ, and is a $\mathscr{Z} = \mathscr{Z}_G$ finite, τ spherical function on G.

Let W_0 be the Weyl group of (G, A_0). It is a finite group which acts on the vector spaces $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. Suppose that M_1 is another Levi subgroup in $\mathscr{L}(M_0)$. As is customary, we will write $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ for the set of distinct isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M_1}$ obtained by restricting elements in W_0 to $\mathfrak{a}_M$. (Recall that any two groups $P \in \mathscr{P}(M)$ and $P_1 \in \mathscr{P}(M_1)$ are said to be *associated* if this set is not empty.) If $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ we shall always let w_t denote some representative of t in K. Now, suppose that Σ is any subset of M such that $K_M \Sigma K_M = \Sigma$, and that φ is a τ_M spherical function from Σ to V_τ. If $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, define a τ_{M_1} spherical function on $\Sigma_1 = w_t \Sigma w_t^{-1}$ by

$$(t\varphi)(m_1) = \tau(w_t)\,\varphi(w_t^{-1} m_1 w_t)\,\tau(w_t), \quad m_1 \in \Sigma_1.$$

If $P \in \mathscr{P}(M)$ let tP be the group $w_t P w_t^{-1}$ in $\mathscr{P}(M_1)$. Then if $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)$, it is easily shown that

$$E_P(\varphi, \lambda) = E_{tP}(t\varphi, t\lambda). \tag{2.1}$$

More generally, if L is any Levi subgroup which contains both M and M_1, and $R \in \mathscr{P}^L(M)$, there is an identity

$$tE_R(\varphi, \lambda) = E_{tR}(t\varphi, t\lambda) \tag{2.1'}$$

for Eisenstein integrals on L and $tL = w_t L w_t^{-1}$.

Suppose that $P \in \mathscr{P}(M)$. If T is a Cartan subgroup of M, let $\Sigma_P(G, T)$ be the set of roots of (G, T) whose restrictions to A_M belong to Σ_P. Suppose that P' also belongs to $\mathcal{P}(M)$. Then the number

$$\beta_{P'|P} = \prod_{\alpha \in \Sigma_{\bar{P}}(G,T) \cap \Sigma_{P'}(G,T)} \left(\frac{(\alpha, \alpha)}{2}\right)^{1/2}$$

is independent of T. (As usual, $\bar{P}$ stands for the group in $\mathcal{P}(M)$ opposite to P.) Let dX be the Euclidean measure on $\mathfrak{n}_{P'} = \mathrm{Lie}(N_{P'})$ associated to the norm

$$\|X\|^2 = -(X, \theta X), \quad X \in \mathfrak{n}_{P'}.$$

We can normalize a Haar measure dn' on $N_{P'}$ by

$$\int_{N_{P'}} \varphi(n')\, dn' = \int_{\mathfrak{n}_{P'}} \varphi(\exp X)\, dX, \quad \varphi \in C_c^\infty(N_{P'}).$$

The same prescription gives us a Haar measure on the subgroup $N_{P'} \cap N_P$. From these two measures we then obtain an invariant quotient measure on the coset space $N_{P'} \cap N_P \backslash N_{P'}$. Now if $\varphi \in \mathcal{A}_{\mathrm{cusp}}(M, \tau)$, $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ and $m \in M$, define

$$(J^l_{P'|P}(\lambda)\, \varphi)(m) = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \tau(K_P(n))\, \varphi(M_P(n)\, m)\, e^{(\lambda + \varrho_P)(H_P(n))}\, dn,$$

and

$$(J^r_{P'|P}(\lambda)\, \varphi)(m) = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \varphi(m M_P(n))\, \tau(K_P(n))\, e^{(\lambda + \varrho_P)(H_P(n))}\, dn.$$

The integral converges if

$$(\mathrm{Re}\,\lambda + \varrho_P, \alpha) > 0$$

for each root α in $\Sigma_P \cap \Sigma_{\bar{P}'}$. Because the factor $\beta_{P'|P}$ is built into the definition, the integrals are independent of the measure on $N_{P'}$ and of the form $(\,,)$.

Both $J^l_{P'|P}(\lambda)$ and $J^r_{P'|P}(\lambda)$ can be analytically continued as meromorphic functions from $\mathfrak{a}^*_{M,\mathbf{C}}$ to the finite dimensional space of endomorphisms of $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$. They satisfy all the usual properties of intertwining operators. In particular, let $d(P', P)$ be the number of singular hyperplanes which lie between the chambers $\mathfrak{a}_M(P)$ and $\mathfrak{a}_M(P')$. If P'' is a third group in $\mathcal{P}(M)$ such that

$$d(P'', P) = d(P'', P') + d(P', P),$$

one has

$$J^{\iota}_{P''|P}(\lambda) = J^{\iota}_{P''|P'}(\lambda)\, J^{\iota}_{P'|P}(\lambda), \quad \iota = l, r. \tag{2.2}$$

Suppose that M_* is a Levi subgroup which is contained in M, and that $R, R' \in \mathcal{P}^M(M_*)$. The functions

$$J^{\iota}_{R'|R}(\Lambda), \quad \Lambda \in \mathfrak{a}^*_{M_*,\mathbf{C}}, \; \iota = l, r,$$

associated to M and M_* (instead of G and M) can certainly be defined. They depend only on the projection of Λ onto the orthogonal complement of $\mathfrak{a}^*_{M,\mathbf{C}}$ in $\mathfrak{a}^*_{M_*,\mathbf{C}}$. We have the formula

$$J^{\iota}_{P(R')|P(R)}(\Lambda) = J^{\iota}_{R'|R}(\Lambda), \quad \iota = l, r. \tag{2.3}$$

If $\lambda_1 \in \mathfrak{a}^*_{M,\mathbf{C}}$ and $P_1, P_1' \in \mathcal{P}(M)$, the operators $J^{l}_{P_1'|P_1}(\lambda_1)$ and $J^{r}_{P'|P}(\lambda)$ commute. They also both commute with $\mathfrak{Z}_M$. Finally, we should recall that

$$\det(J^{\iota}_{P'|P}(\lambda)), \quad \iota = l, r,$$

does not vanish identically in λ; the inverse $J^{\iota}_{P'|P}(\lambda)^{-1}$ therefore exists as a meromorphic function of λ.

Actually, the J functions defined by Harish-Chandra in [7e] are intertwining operators between induced representations, rather than operators on $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$. It is in this context that the results we are discussing were proved, ([7e]). The difference, however, is minor and purely notational. For the convenience of the reader, we will spend § 3 reviewing the relations with induced representations.

Suppose that $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$. Suppose that $P \in \mathcal{P}(M)$ and $P_1 \in \mathcal{P}(M_1)$. The groups $s^{-1}P_1$ and $s^{-1}\bar{P}_1$ both belong to $\mathcal{P}(M)$. Define

$$c_{P_1|P}(s, \lambda) = s J^{l}_{s^{-1}P_1|P}(\lambda)\, J^{r}_{s^{-1}\bar{P}_1|P}(\lambda). \tag{2.4}$$

It is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ with values in the space of linear maps from $\mathcal{A}_{\mathrm{cusp}}(M, \tau)$ to $\mathcal{A}_{\mathrm{cusp}}(M_1, \tau)$. By [7d, Lemma 18.1] and the corollary to [7e, Lemma 18.1], it is just Harish-Chandra's c function. Harish-Chandra has defined other c functions

$$c^0_{P_1|P}(s, \lambda) = c_{P_1|P}(s, \lambda)\, c_{P|P}(1, \lambda)^{-1} \tag{2.5}$$

and

$$ {}^{0}c_{P_1|P}(s,\lambda) = c_{P_1|P_1}(1, s\lambda)^{-1}\, c_{P_1|P}(s,\lambda). \tag{2.6} $$

These also are meromorphic functions on $\mathfrak{a}^*_{M,\mathbf{C}}$ with values in the space of linear maps from $\mathscr{A}_{\mathrm{cusp}}(M,\tau)$ to $\mathscr{A}_{\mathrm{cusp}}(M_1,\tau)$. The following functional equations are satisfied:

$$ c^0_{P_2|P}(s_1 s,\lambda) = c^0_{P_2|P_1}(s_1, s\lambda)\, c^0_{P_1|P}(s,\lambda), \tag{2.7} $$

$$ {}^0c_{P_2|P}(s_1 s,\lambda) = {}^0c_{P_2|P_1}(s_1, s\lambda)\, {}^0c_{P_1|P}(s,\lambda), \tag{2.8} $$

$$ \begin{aligned} c_{P_2|P}(s_1 s,\lambda) &= c^0_{P_2|P_1}(s_1, s\lambda)\, c_{P_1|P}(s,\lambda) \\ &= c_{P_2|P_1}(s_1, s\lambda)\, {}^0c_{P_1|P}(s,\lambda), \end{aligned} \tag{2.9} $$

$$ E_P(x,\varphi,\lambda) = E_{P_1}(x, {}^0c_{P_1|P}(s,\lambda)\,\varphi, s\lambda), \tag{2.10} $$

for $s_1 \in W(\mathfrak{a}_{M_1}, \mathfrak{a}_{M_2})$ and $P_2 \in \mathscr{P}(M_2)$. Suppose that $t \in W_0$. If $M' \in \mathscr{L}(M_0)$,

$$ tM' = w_t M' w_t^{-1} $$

is another Levi subgroup; if $P \in \mathscr{P}(M')$, then $tP \in \mathscr{P}(tM')$. The restriction of t to $\mathfrak{a}_{M'}$ defines an element in $W(\mathfrak{a}_{M'}, \mathfrak{a}_{tM'})$, which we will denote also by t. It is an easy matter to show that for $P', P \in \mathscr{P}(M)$,

$$ tJ^{\iota}_{P'|P}(\lambda)\, t^{-1} = J^{\iota}_{tP'|tP}(t\lambda), \quad \iota = l, r. \tag{2.11} $$

One also has

$$ tc_{P_1|P}(s,\lambda) = c_{tP_1|P}(ts,\lambda), \tag{2.12} $$

$$ c_{P_1|P}(s,\lambda)\, t^{-1} = c_{P_1|tP}(st^{-1}, t\lambda), \tag{2.13} $$

and similar formulas for $c^0_{P_1|P}$ and ${}^0c_{P_1|P}$. From (2.11) one can also deduce alternate formulas

$$ c^0_{P_1|P}(s,\lambda) = sJ^l_{\bar{P}|s^{-1}P_1}(\lambda)^{-1} J^r_{s^{-1}\bar{P}_1|P}(\lambda) \tag{2.14} $$

and

$$ {}^0c_{P_1|P}(s,\lambda) = sJ^l_{s^{-1}P_1|P}(\lambda)\, J^r_{\bar{P}|s^{-1}P_1}(\lambda)^{-1} \tag{2.15} $$

for the supplementary c functions.

Finally, let us recall the definition of Harish-Chandra's μ functions. It is easy to see that for any P and P',

$$J^l_{P|P'}(\lambda)\, J^l_{P'|P}(\lambda) = J^r_{P'|P}(\lambda)\, J^r_{P|P'}(\lambda).$$

Let $\mu_{P'|P}(\lambda)$ be the inverse of this operator. It is a meromorphic function of λ with values in the space of endomorphisms of $\mathscr{A}_{\mathrm{cusp}}(M, \tau)$. For any λ, $\mu_{P'|P}(\lambda)$ commutes with any of the operators

$$J^{\iota}_{P_1'|P_1}(\lambda_1), \quad \lambda_1 \in \mathfrak{a}^*_{M,\mathbf{C}},\ P_1,\ P_1' \in \mathscr{P}(M).$$

Therefore $\mu_{P'|P}(\lambda)$ also commutes with $\mu_{P_1'|P_1}(\lambda_1)$. Analogues of (2.2) and (2.3) follow easily; one has

$$\mu_{P''|P}(\lambda) = \mu_{P''|P'}(\lambda)\, \mu_{P'|P}(\lambda) \tag{2.16}$$

if

$$d(P'', P) = d(P'', P') + d(P', P),$$

and

$$\mu_{P(R')|P(R)}(\Lambda) = \mu_{R'|R}(\Lambda), \tag{2.17}$$

if Λ, P, R and R' are as in (2.3). Now, for any $P \in \mathscr{P}(M)$, define

$$\mu_P(\lambda) = \mu_{\bar{P}|P}(\lambda).$$

It follows from the properties above that $\mu_P(\lambda)$ depends only on M and G, and not the group P. Moreover, for any $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ and $P_1 \in \mathscr{P}(M_1)$,

$$\mu_P(\lambda) = \mu_{P_1}(t\lambda).$$

LEMMA 2.1. *Suppose that* $M_* \in \mathscr{L}^M(M_0)$, $R \in \mathscr{P}^M(M_*)$ *and* $P \in \mathscr{P}(M)$. *Then if* $\Lambda \in (\mathfrak{a}_{M_*})^*_{\mathbf{C}}$ *and* $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$,

$$\mu_{\bar{P}(R)|P(R)}(\Lambda+\lambda) = \mu_{P(R)}(\Lambda+\lambda)\, \mu_R(\Lambda)^{-1}.$$

Proof. By (2.16) and (2.17) we have

$$\begin{aligned} \mu_{P(R)}(\Lambda+\lambda) &= \mu_{\bar{P}(\bar{R})|P(R)}(\Lambda+\lambda) \\ &= \mu_{\bar{P}(\bar{R})|\bar{P}(R)}(\Lambda+\lambda)\, \mu_{\bar{P}(R)|P(R)}(\Lambda+\lambda) \\ &= \mu_{\bar{R}|R}(\Lambda+\lambda)\, \mu_{\bar{P}(R)|P(R)}(\Lambda+\lambda). \end{aligned}$$

But

$$\mu_{\bar{R}|R}(\Lambda+\lambda)=\mu_{\bar{R}|R}(\Lambda)=\mu_R(\Lambda),$$

and this operator commutes with $\mu_{\bar{P}(R)|P(R)}(\Lambda+\lambda)$. The lemma follows. Q.E.D.

§ 3. Relation with induced representations

We shall remind ourselves of the connection of Eisenstein integrals with the theory of induced representations. The reader who is experienced in such things or who does not wish to track down the facts of § 2 in Harish-Chandra's paper [7e], could easily skip this section

Again, we fix a Levi subgroup $M\in\mathcal{L}(M_0)$. Let ω be an equivalence class of irreducible square integrable representations of M/A_M, and let (σ, U_σ) be a representation in the class of ω. (U_σ is the Hilbert space on which σ acts.) Suppose that $P\in\mathcal{P}(M)$. Define $\mathcal{H}(\sigma)$ to be the Hilbert space of measurable functions

$$\psi: K\rightarrow U_\sigma$$

such that

(i) $\psi(mk)=\sigma(m)\psi(k),\quad m\in K_M,\ k\in K,$

and

(ii) $\|\psi\|_2^2=\int_{K_M\backslash K}\|\psi(k)\|^2\,dk<\infty.$

If $\lambda\in\mathfrak{a}^*_{M,\mathbf{C}}$, there is the usual induced representation

$$(I_P(\sigma,\lambda,x)\psi)(k)=e^{(\lambda+\varrho_P)(H_P(kx))}\sigma(M_P(kx))\,\psi(K_P(kx)),$$

$\psi\in\mathcal{H}(\sigma)$, $x\in G$, which acts on $\mathcal{H}(\sigma)$.

Suppose that (τ, V) is an irreducible representation of K. Let $\mathcal{H}(\sigma)_\tau$ be the finite dimensional subspace of vectors in $\mathcal{H}(\sigma)$ under which the restriction of $I_P(\sigma,\lambda)$ to K is equivalent to τ. Suppose that $S\in\mathrm{Hom}_{K_M}(V, U_\sigma)$; that is, S is a map from V to U_σ such that

$$S(\tau(m)\xi)=\sigma(m)S(\xi),\quad \xi\in V,\ m\in K_M.$$

If $\xi\in V$, the function

$$\Psi_S(\xi): k\rightarrow S(\tau(k)\xi),\quad k\in K,$$

belong to $\mathcal{H}(\sigma)$. By Frobenius reciprocity, the map

$$S \to \Psi_S$$

is an isomorphism from $\mathrm{Hom}_{K_M}(V, U_\sigma)$ onto $\mathrm{Hom}_K(V, \mathcal{H}(\sigma))$. Notice that $\mathcal{H}(\sigma)_\tau$ is the space spanned by

$$\{\Psi_S(\xi): S \in \mathrm{Hom}_{K_M}(V, U_\sigma),\ \xi \in V\}.$$

It is isomorphic to $\mathrm{Hom}_{K_M}(V, U_\sigma) \otimes V$. Now, if $S \in \mathrm{Hom}_{K_M}(V, U_\sigma)$, and P' is another group in $\mathcal{P}(M)$, set

$$J_{P'|P}(\sigma, \lambda)\, S = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \sigma(M_P(n))\, S\tau(K_P(n))\, e^{(\lambda + \rho_P)(H_P(n))}\, dn.$$

Defined a priori only for those $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ for which the integral converges, $J_{P'|P}(\sigma, \lambda)$ can be continued as a meromorphic function from $\mathfrak{a}^*_{M,\mathbf{C}}$ to the space of endomorphisms of $\mathrm{Hom}_{K_M}(V, U_\sigma)$. The map

$$\Psi_S(\xi) \to \Psi_{J_{P'|P}(\sigma,\lambda)S}(\xi), \quad S \in \mathrm{Hom}_{K_M}(V, U_\sigma),\ \xi \in V,$$

which we can also denote by $J_{P'|P}(\sigma, \lambda)$, is just the restriction to $\mathcal{H}(\sigma)_\tau$ of the usual intertwining operator from $I_P(\sigma, \lambda)$ to $I_{P'}(\sigma, \lambda)$.

The contragredient representation of τ makes the dual space V^* of V into a K-module. Similarly, the dual Hilbert space U^*_σ is an M-module under the contragredient σ^* of σ. As above, we have an isomorphism

$$\mathrm{Hom}_{K_M}(V^*, U^*_\sigma) \xrightarrow{\sim} \mathrm{Hom}_K(V^*, \mathcal{H}(\sigma^*)).$$

On the one hand, the transpose gives a canonical isomorphism between $\mathrm{Hom}_{K_M}(V^*, U^*_\sigma)$ and $\mathrm{Hom}_{K_M}(U_\sigma, V)$. On the other hand, the K finite vectors in $\mathcal{H}(\sigma^*)$ and $\mathcal{H}(\sigma)$ are in duality under the pairing

$$\int_{K_M \backslash K} \langle \psi(k), \psi^*(k) \rangle\, dk, \quad \psi \in \mathcal{H}(\sigma),\ \psi^* \in \mathcal{H}(\sigma^*),$$

so that $\mathrm{Hom}_K(V^*, \mathcal{H}(\sigma^*))$ is canonically isomorphic to $\mathrm{Hom}_K(\mathcal{H}(\sigma), V)$. It follows that there is a canonical isomorphism

$$\Psi^*: \mathrm{Hom}_{K_M}(U_\sigma, V) \xrightarrow{\sim} \mathrm{Hom}_K(\mathcal{H}(\sigma), V).$$

It is given by

$$\Psi^*_{S^*}(\psi) = \int_{K_M \backslash K} \tau(k)^{-1} S^*(\psi(k))\, dk,$$

for $S^* \in \mathrm{Hom}_{K_M}(U_\sigma, V)$ and $\psi \in \mathscr{H}(\sigma)$. Notice that $\mathscr{H}(\sigma)^*_\tau$ is the space spanned by compositions

$$\{\xi^* \Psi^*_{S^*}: S^* \in \mathrm{Hom}_{K_M}(U_\sigma, V),\ \xi^* \in V^*\}.$$

It is isomorphic to $\mathrm{Hom}_{K_M}(U_\sigma, V) \otimes V^*$.

Suppose that

$$(\tau_i, V_i), \quad i = 1, 2,$$

is a pair of irreducible representations of K. We shall now take τ to be the *double* representation of K on

$$V_\tau = \mathrm{Hom}_{\mathbf{C}}(V_1, V_2),$$

defined by

$$\tau(k_2) X \tau(k_1) = \tau_2(k_2) \circ X \circ \tau_1(k_1),$$

for $k_1, k_2 \in K$, and $X \in \mathrm{Hom}_{\mathbf{C}}(V_1, V_2)$. Any double representation of K will be a direct sum of representations of this form. If $S_1 \in \mathrm{Hom}_{K_M}(V_1, U_\sigma)$ and $S_2^* \in \mathrm{Hom}_{K_M}(U_\sigma, V_2)$ then

$$\varphi(m) = S_2^* \sigma(m) S_1, \quad m \in M,$$

is a function in $\mathscr{A}_\omega(M, \tau)$. The J functions we have just defined are related to those of § 2 by the formulas

$$(J^l_{P'|P}(\lambda)\varphi)(m) = (J_{P|P'}(\sigma, \lambda)^* S_2^*)\sigma(m) S_1 \tag{3.1}$$

and

$$(J^r_{P'|P}(\lambda)\varphi)(m) = S_2^* \sigma(m)(J_{P'|P}(\sigma, \lambda) S_1). \tag{3.2}$$

Next, we shall show that for any $x \in G$,

$$\Psi^*_{S_2^*} I_P(\sigma, \lambda, x) \Psi_{S_1} = E_P(x, \varphi, \lambda). \tag{3.3}$$

Indeed, the left hand side is a composition of three operators, and is an element in $\mathrm{Hom}(V_1, V_2)=V_\tau$. Its value at any vector $\xi_1 \in V_1$ is

$$\int_{K_M\backslash K} \tau_2(k^{-1})\, S_2^*((I_P(\sigma,\lambda,x)\,\Psi_{S_1}(\xi_1))(k))\,dk$$

$$= \int_{K_M\backslash K} \tau(k^{-1})\,(S_2^*\sigma(M_P(kx))\,S_1)\,(\tau(K_P(kx))\,\xi_1)\,e^{(\lambda+\varrho_P)(H_P(kx))}\,dk$$

$$= \int_{K_M\backslash K} \tau(k^{-1})\,\varphi_P(kx)\,(\xi_1)\,e^{(\lambda+\varrho_P)(H_P(kx))}\,dk.$$

This is just the value of $E_P(x,\varphi,\lambda)$ at ξ_1.

Formula (3.3) provides a relation between Eisenstein integrals and matrix coefficients of $I_P(\sigma,\lambda)$. There is a slightly different way to express it. We have seen that the space of operators

$$\mathrm{Hom}\,(\mathscr{H}(\sigma)_{\tau_2}, \mathscr{H}(\sigma)_{\tau_1}) = \mathscr{H}(\sigma)_{\tau_1} \otimes \mathscr{H}(\sigma)^*_{\tau_2}$$

is canonically isomorphic with

$$\mathrm{Hom}_{K_M}(V_1, U_\sigma) \otimes V_1 \otimes \mathrm{Hom}_{K_M}(U_\sigma, V_2) \otimes V_2^*.$$

Now the correspondence

$$(S_1, S_2^*) \to \varphi$$

defines an isomorphism between

$$\mathrm{Hom}_{K_M}(V_1, U_\sigma) \otimes \mathrm{Hom}_{K_M}(U_\sigma, V_2)$$

and $\mathscr{A}_\omega(M,\tau)$, while $V_1 \otimes V_2^*$ is isomorphic to V_τ^*. Let End_τ be the double representation of K on the space

$$\mathrm{End}\,(V_\tau) = \mathrm{Hom}_{\mathbf{C}}\,(V_\tau, V_\tau)$$

given by

$$(\mathrm{End}_\tau(k_2)\cdot F\cdot \mathrm{End}_\tau(k_1))(X) = \tau(k_2)\,F(X)\,\tau(k_1),$$

$k_1, k_2 \in K$, $F \in \mathrm{End}\,(V_\tau)$ and $X \in V_\tau$. Then $\mathscr{A}_\omega(M,\tau)\otimes V_\tau^*$ equals $\mathscr{A}_\omega(M,\mathrm{End}_\tau)$. We therefore have an isomorphism

$$T \to \psi_T$$

from $\mathrm{Hom}(\mathscr{H}(\sigma)_{\tau_2}, \mathscr{H}(\sigma)_{\tau_1})$ onto $\mathscr{A}_\omega(M, \mathrm{End}_\tau)$. This mapping is essentially the one defined by Harish-Chandra in [7e, § 7]. Formula (3.3) leads to

$$\mathrm{tr}(I_P(\sigma, \lambda, x)\,T) = \mathrm{tr}(E_P(x, \psi_T, \lambda)), \quad T \in \mathrm{Hom}(\mathscr{H}(\sigma)_{\tau_2}, \mathscr{H}(\sigma)_{\tau_1}),$$

where the first trace is on the space $\mathscr{H}(\sigma)$ while the second is on V_τ. The relations between the J functions can be written

$$J^l_{P'|P}(\lambda)\,\psi_T = \psi_{TJ_{P|P'}(\sigma,\lambda)}$$

and

$$J^r_{P'|P}(\lambda)\,\psi_T = \psi_{J_{P'|P}(\sigma,\lambda)T}.$$

It is actually through these relations that we can extract formula (2.4) from the papers of Harish-Chandra.

§ 4. Asymptotic expansions

From now on τ will be as in § 2, a fixed two-sided representation of K on a finite dimensional vector space V_τ. For the results on asymptotic expansions that we shall quote the reader can refer to [7a], [7b] or [13].

If $\varepsilon \geqslant 0$ and $B \in \mathscr{P}(M_0)$, set

$$A_0^\varepsilon(B) = \{a \in A_0 \colon \alpha(H_0(a)) > \varepsilon,\ \alpha \in \Delta_B\}.$$

By a neighborhood of infinity in G we shall mean a set of the form

$$G_\varepsilon = K \cdot A_0^\varepsilon(B) \cdot K.$$

It is an open subset of G which is independent of B. If $\varepsilon = 0$, we shall write G_- for G_ε. Suppose that φ is a function defined on a neighborhood of infinity with values in V_τ which is $\mathscr{Z}$-finite and τ-spherical. It has a unique asymptotic expansion in any chamber $A_0(B)$. There is an ε such that

$$\varphi(a) = \sum_{i=1}^{n} \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} \varepsilon_{B,\zeta}(\Lambda_i, a)\, a^{\Lambda_i - \zeta},$$

for all $a \in A_0^\varepsilon(B)$. Here $\{\Lambda_i\}$ is a finite set of linear functions on $\mathfrak{a}_{0,\mathbf{C}}$ such that for any $i \neq j$, the function $\Lambda_i - \Lambda_j$ does not belong to $\mathbf{Z}^+(\Delta_B)$. For each i

$$a \to \varepsilon_{B,\zeta}(\Lambda_i, a), \quad \zeta \in \mathbf{Z}^+(\Delta_B),\ a \in A_0,$$

is a family of functions from A_0 to V_τ which are polynomials of bounded degree in $H_0(a)$ and such that

$$\tau(m)\,\varepsilon_{B,\zeta}(\Lambda_i, a)\,\tau(m)^{-1} = \varepsilon_{B,\zeta}(\Lambda_i, a)$$

for all $m \in M_0 \cap K$. The functions $\{\Lambda_i - \zeta\}$ are called the *exponents* of φ (with respect to B) while $\{\Lambda_i\}$ are called the *principal exponents*. Suppose that

$$B' = tB, \quad t \in W_0,$$

is another group in $\mathscr{P}(M_0)$. We leave the reader to check that there is a bijection

$$\Lambda_i \leftrightarrow \Lambda_i'$$

between the two sets of principal exponents such that

(i) $\Lambda_i' = t\Lambda_i$

and

(ii) $\varepsilon_{B', t\zeta}(\Lambda_i', w_t a w_t^{-1}) = \tau(w_t)\,\varepsilon_{B,\zeta}(\Lambda_i, a)\,\tau(w_t)^{-1}$ for all $\zeta \in \mathbf{Z}^+(\Delta_B)$.

Suppose that M is a Levi subgroup in $\mathscr{L}(M_0)$ and that $P \in \mathscr{P}(M)$. If $R \in \mathscr{P}^M(M_0)$ the group $B = P(R)$ belongs to $\mathscr{P}(M_0)$ and the set $\mathbf{Z}^+(\Delta_R)$ is contained in $\mathbf{Z}^+(\Delta_B)$. For φ as above, and a point a in $A_0^\varepsilon(R)$, define

$$E^P(a, \varphi) = \sum_{i=1}^{n} \sum_{\zeta \in \mathbf{Z}^+(\Delta_R)} \varepsilon_{B,\zeta}(\Lambda_i, a)\, a^{(\Lambda_i - \zeta + \varrho_P)}.$$

If t is an element in W_0, tB equals $(tP)(tR)$, and we have

$$\begin{aligned} E^{tP}(w_t a w_t^{-1}, \varphi) &= \sum_{i=1}^{n} \sum_{\zeta \in \mathbf{Z}^+(\Delta_R)} \varepsilon_{tB, t\zeta}(t\Lambda_i, w_t a w_t^{-1})\,(w_t a w_t^{-1})^{t(\Lambda_i - \zeta + \varrho_P)} \\ &= \tau(w_t)\, E^P(a, \varphi) \tau(w_t)^{-1}. \end{aligned}$$

Taking t to be an element in W_0^M, the Weyl group of (M, A_0), we see that $E^P(a, \varphi)$ extends to a τ_M-spherical function, $E^P(\varphi)$, on M_ε. If t is a general element in W_0, the formula above is just

$$E^{tP}(\varphi) = tE^P(\varphi). \tag{4.1}$$

Recall that there is a natural injective map

$$\gamma_M : \mathscr{Z} \to \mathscr{Z}_M.$$

Indeed, suppose that $P \in \mathscr{P}(M)$ and that $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are the Lie algebras of the unipotent radicals of P and $\bar{P}$ respectively. For $z \in \mathscr{Z}$, let $\gamma'_P(z)$ be the unique element in $\mathscr{U}(M)$ such that $z-\gamma'_P(z)$ belongs to $\mathfrak{n}\mathscr{U}(M)\bar{\mathfrak{n}}$. Then as a left invariant differential operator on M,

$$\gamma_M(z) = \delta_P(m)^{-1} \circ \gamma'_P(z) \circ \delta_P(m), \quad m \in M.$$

Now if φ is a function as above, we have differential equations

$$\gamma_M(z) E^P(\varphi) = E^P(z\varphi), \quad z \in \mathscr{Z}. \tag{4.2}$$

For the case that $M=M_0$ this formula is in the proof of [13, Lemma 9.1.4.5]. For general $M \subset G$ we refer the reader to [7 a] or the discussion in [11 a, pp. 91–97]. In any case, since $\mathscr{Z}_M$ is a finite module over $\gamma_M(\mathscr{Z})$, the function $E^P(\varphi)$ is $\mathscr{Z}_M$-finite.

For any $M \in \mathscr{L}(M_0)$, the abelian algebra

$$\mathfrak{h}_{\mathbf{C}} = \mathfrak{h}_{K,\mathbf{C}} \otimes \mathfrak{a}_{0,\mathbf{C}},$$

introduced in § 1, is a Cartan subalgebra of $\mathrm{Lie}(M)\otimes\mathbf{C}$. Let W^M be the Weyl group of $(\mathrm{Lie}(M)\otimes\mathbf{C}, \mathfrak{h}_{\mathbf{C}})$ and let

$$z \to p_z^M, \quad z \in \mathscr{Z}_M,$$

be the canonical isomorphism from $\mathscr{Z}_M$ onto the W^M-invariant polynomial functions on $\mathfrak{h}_{\mathbf{C}}^*$. Then

$$p^M_{\gamma_M(z)} = p^G_z$$

for all $z \in \mathscr{Z} = \mathscr{Z}_G$. Recall that any homomorphism from $\mathscr{Z}$ to $\mathbf{C}$ is of the form

$$z \to p_z^G(\nu) = p_z(\nu), \quad z \in \mathscr{Z},$$

and is uniquely determined by the orbit under $W=W^G$ of the linear function $\nu \in \mathfrak{h}_{\mathbf{C}}^*$. Suppose that φ is a $\mathscr{Z}_M$-finite, τ_M-spherical function defined on a neighborhood of infinity in M. Since $\gamma_M(\mathscr{Z})$ is a subalgebra of $\mathscr{Z}_M$, the vector space generated by $\{z\varphi : z \in \gamma_M(\mathscr{Z})\}$ is finite dimensional. Let $\{\varphi_i\}$ be a basis of this space. We can assume that each φ_i is a generalized eigenfunction of $\gamma_M(\mathscr{Z})$; that is,

$$(z-p_z^M(\nu_i))^d \varphi_i = 0, \quad z \in \gamma_M(\mathscr{Z}),$$

for a positive integer d and a function $\nu_i \in \mathfrak{h}_{\mathbf{C}}^*$. Let us write $\mathfrak{o}_G(\varphi)$ for the union over i of the orbits under W of the points ν_i. It is a finite set of linear functions on $\mathfrak{h}_{\mathbf{C}}$. Suppose that $M_* \in \mathcal{L}^M(M_0)$ and that $R \in \mathcal{P}^M(M_*)$. It follows from (4.2) that

$$\mathfrak{o}_G(\varphi) = \mathfrak{o}_G(E^R(\varphi)). \tag{4.3}$$

For convenience, we will denote the finite dimensional space $\mathcal{A}_{\mathrm{cusp}}(M_0, \tau)$ by $\mathcal{A}_0$. If $\Phi \in \mathcal{A}_0$ and $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}$, we have the function

$$E_{M_0}(\Phi, \Lambda) \colon m \to E_{M_0}(m, \Phi, \Lambda) = \Phi(m)\, e^{\Lambda(H_0(m))}, \quad m \in M_0.$$

Let $\mathfrak{o}_G(\tau, \Lambda)$ denote the union, over all $\Phi \in \mathcal{A}_0$, of the orbits $\mathfrak{o}_G(E_{M_0}(\Phi, \Lambda))$. It is precisely the set of W orbits of points $\eta + \Lambda$, where η is one of the finitely many linear functions on $\mathfrak{h}_{K,\mathbf{C}}$ such that

$$z\Phi = p_z^{M_0}(\eta)\, \Phi, \quad z \in \mathcal{Z}_{M_0},$$

for some $\Phi \in \mathcal{A}_0$. From formula (4.3) we have

LEMMA 4.1. *Suppose that φ is a $\mathcal{Z}_M$-finite, τ_M-spherical function defined on a neighborhood of infinity in M. Let $\Lambda_1, \ldots, \Lambda_n$ be the principal exponents of φ along a chamber $A_0(R)$, $R \in \mathcal{P}^M(M_0)$. Then for each i, $\mathfrak{o}_G(\varphi) \cap \mathfrak{o}_G(\tau, \Lambda_i + \varrho_R)$ is not empty. Moreover,*

$$\mathfrak{o}_G(\varphi) = \bigcup_{i=1}^{n} (\mathfrak{o}_G(\varphi) \cap \mathfrak{o}_G(\tau, \Lambda_i + \varrho_R)).$$

The $\mathcal{Z}$-finite functions on G of most interest in this paper are the Eisenstein integrals associated to minimal parabolics. For $B, B' \in \mathcal{P}(M_0)$ consider the expansions of the functions

$$E_B(x, \Phi, \Lambda), \quad \Phi \in \mathcal{A}_0,\ \Lambda \in \mathfrak{a}^*_{0,\mathbf{C}},$$

along the chamber $A_0(B')$. If ε is a small postive number and $a \in A_0^{\varepsilon}(B')$, $E_B(a, \Phi, \Lambda)$ can be written

$$\sum_{s \in W_0} \sum_{\zeta \in Z^+(\Delta_{B'})} (c_{B'|B,\zeta}(s, \Lambda)\Phi)(1)\, a^{(s\Lambda - \zeta - \varrho_{B'})},$$

where $c_{B'|B,\zeta}(s, \Lambda)$ is a meromorphic function of Λ with values in $\mathrm{End}(\mathcal{A}_0)$, the space of endomorphisms of $\mathscr{A}_0$. The functional equations for the Eisenstein integral give rise to the formulae

$$tc_{B'|B,\zeta}(s,\Lambda) = c_{tB'|B,t\zeta}(ts,\Lambda), \tag{4.4}$$

$$c_{B'|B,\zeta}(s,\Lambda)t^{-1} = c_{B'|tB,\zeta}(st^{-1},t\Lambda), \tag{4.5}$$

$$c_{B'|B,\zeta}(s_1 s,\Lambda) = c_{B'|B_1,\zeta}(s_1,s\Lambda)^0 c_{B_1|B}(s,\Lambda), \tag{4.6}$$

for elements $t, s_1 \in W_0$ and $B_1 \in \mathscr{P}(M_0)$. Suppose that x is an element in G_ε. Then

$$x = k_1 a k_2, \quad k_1, k_2 \in K,\ a \in A_0^\varepsilon(B').$$

Define

$$E_{B'|B,s}(x,\Phi,\Lambda) = \tau(k_1) \sum_{\zeta \in \mathbf{Z}^+(\Delta_{B'})} (c_{B'|B,\zeta}(s,\Lambda)\Phi)(1)\, a^{(s\Lambda-\zeta-\varrho_{B'})}\tau(k_2).$$

Then $E_{B'|B,s}(\Phi,\Lambda)$ is a τ-spherical function on G_ε. It is meromorphic in Λ, and

$$E_B(x,\Phi,\Lambda) = \sum_{s \in W_0} E_{B'|B,s}(x,\Phi,\Lambda).$$

From the three functional equations above, one obtains

$$E_{B'|B,s}(x,\Phi,\Lambda) = E_{tB'|B,ts}(x,\Phi,\Lambda), \tag{4.4'}$$

$$E_{B'|B,s}(x,\Phi,\lambda) = E_{B'|tB,st^{-1}}(x,t\Phi,t\Lambda), \tag{4.5'}$$

$$E_{B'|B,s_1 s}(x,\Phi,\Lambda) = E_{B'|B_1,s_1}(x,{}^0c_{B_1|B}(s,\Lambda)\Phi,s\Lambda). \tag{4.6'}$$

These functions are all $\mathscr{Z}$-finite. Indeed, if $z \in \mathscr{Z}$ let $\gamma_{M_0}(z,\Lambda)$ be the differential operator on M_0^1 obtained by evaluating $\gamma_{M_0}(z)$ at Λ. Then the equation

$$zE_{B'|B,s}(x,\Phi,\Lambda) = E_{B'|B,s}(x,\gamma_{M_0}(z,\Lambda)\Phi,\Lambda) \tag{4.7}$$

follows from the analogous formula for Eisenstein integrals. Since Φ is $\mathscr{Z}_M$-finite, $E_{B'|B,s}(\Phi,\Lambda)$ must be $\mathscr{Z}$-finite. Another consequence of (4.7) (or alternatively, of Lemma 4.1) is that the orbit $\mathfrak{o}_G(E_{B'|B,s}(\Phi,\Lambda))$ is contained in $\mathfrak{o}_G(\tau,\Lambda)$.

We shall prove a lemma for use in Chapter II. It is, I am sure, known to experts. Fix $M \in \mathscr{L}(M_0)$. We can, of course, define the functions

$$E_{R'|R,r}(\Phi,\Lambda), \quad r \in W_0^M,\ R, R' \in \mathscr{P}^M(M_0),$$

on a neighborhood of infinity in M. (W_0^M is the Weyl group of (M, A_0).) Fix $R \in \mathscr{P}^M(M_0)$. Also take groups $P, Q \in \mathscr{P}(M)$. We shall show that for a point Λ in general position in $\mathfrak{a}^*_{0,\mathbf{C}}$, the function

$$E_{R|R,1}(J^l_{Q(R)|P(R)}(\Lambda)\, J^r_{\bar{Q}(R)|P(R)}(\Lambda)\, \Phi, \Lambda) \tag{4.8}$$

can be expressed in terms of a $\mathscr{Z}$-finite, τ-spherical function on G_ε. Let $B = P(R)$. There is a unique coset s in W_0/W_0^M such that the group $P_1 = sQ$ contains B. If $M_1 = w_s M w_s^{-1}$ then P_1 belongs to $\mathscr{P}(M_1)$. Let s_B be the unique representative of s in W_0 such that $s_B(\alpha)$ is a root of (B, A_0) for every root α of (R, A_0).

LEMMA 4.2. *If Λ is a point in $\mathfrak{a}^*_{0,\mathbf{C}}$ in general position, the function* (4.8) *equals*

$$s^{-1}E^{P_1}(E_{B|B, s_B}(\Phi, \Lambda)).$$

Proof. Let $R_1 = B \cap M_1$. Notice that

$$s_B^{-1}B = s_B^{-1}P_1(R_1) = Q(R).$$

Also, $s_B^{-1}(\bar{B})$ equals $\bar{Q}(\bar{R})$. Now the leading term in the asymptotic expansion of

$$E^{P_1}(a_1, E_{B|B, s_B}(\Phi, \Lambda))$$

along $A_0(R_1)$ is

$$(c_{B|B}(s_B, \Lambda)\, \Phi)(1)\, a_1^{s_B(\Lambda - \varrho_R)}.$$

Therefore, the leading term in the expansion of the function $s^{-1}E^{P_1}(E_{B|B, s_B}(\Phi, \Lambda))$ at $a \in A_0^\varepsilon(R)$ is

$$\begin{aligned}
&(s_B^{-1} c_{B|B}(s_B, \Lambda)\, \Phi)(1) \cdot a^{\Lambda - \varrho_R} \\
&\quad = (J^l_{s_B^{-1}B|B}(\Lambda)\, J^r_{s_B^{-1}\bar{B}|B}(\Lambda)\, \Phi)(1) \cdot a^{\Lambda - \varrho_R} \\
&\quad = (J^l_{Q(R)|P(R)}(\Lambda)\, J^r_{\bar{Q}(\bar{R})|P(R)}(\Lambda)\, \Phi)(1) \cdot a^{\Lambda - \varrho_R},
\end{aligned}$$

by (2.4). On the other hand, the leading term in the expansion of the function (4.8) at $a \in A_0^\varepsilon(R)$ is

$$(J^r_{\bar{R}|R}(\Lambda)\, J^l_{Q(R)|P(R)}(\Lambda)\, J^r_{\bar{Q}(R)|P(R)}(\Lambda)\, \Phi)(1) \cdot a^{\Lambda - \varrho_R},$$

which by (2.2) and (2.3) also equals

$$(J^{l}_{Q(R)|P(R)}(\Lambda)\, J^{r}_{\bar{Q}(R)|P(R)}(\Lambda)\, \Phi)\,(1)\cdot a^{\Lambda-\varrho_R}.$$

It follows that if φ is the function

$$s^{-1}E^{P_1}(E_{B|B,s_B}(\Phi,\Lambda)) - E_{R|R,1}(J^{l}_{Q(R)|P(R)}(\Lambda)\, J^{r}_{\bar{Q}(R)|P(R)}(\Lambda)\,\Phi,\Lambda)$$

the coefficient of $a^{\Lambda-\varrho_R}$ in the expansion of $\varphi(a)$ along $A_0(R)$ vanishes.

Our aim is to show that φ itself vanishes. Suppose that this is not so. From what we have just shown, the principal exponents of φ along $A_0(R)$ are all of the form $\Lambda-\zeta-\varrho_R$, for nonzero elements $\zeta\in \mathbf{Z}^+(\Delta_R)$. By Lemma 4.1 any point in $\mathfrak{o}_G(\varphi)$ is contained in a set $\mathfrak{o}_G(\tau,\Lambda-\zeta)$ for some $\zeta\neq 0$. On the other hand,

$$\begin{aligned}\mathfrak{o}_G(s^{-1}E^{P_1}(E_{B|B,s_B}(\Phi,\Lambda))) &= \mathfrak{o}_G(E^{P_1}(E_{B|B,s_B}(\Phi,\Lambda)))\\ &= \mathfrak{o}_G(E_{B|B,s_B}(\Phi,\Lambda))\end{aligned}$$

by (4.3). As we observed above this set is contained in $\mathfrak{o}_G(\tau,\Lambda)$. Applied to the group M, the same observation tells us that

$$\mathfrak{o}_G(E_{R|R,1}(J^{l}_{Q(R)|P(R)}(\Lambda)\, J^{r}_{\bar{Q}(R)|P(R)}(\Lambda)\,\Phi,\Lambda))$$

is contained in $\mathfrak{o}_G(\tau,\Lambda)$. Thus, $\mathfrak{o}_G(\varphi)$ is contained in $\mathfrak{o}_G(\tau,\Lambda)$. It follows that there are elements $\eta_1,\eta_2\in\mathfrak{h}^*_{\mathfrak{k}}$, $t\in W$, and $\zeta\in\mathbf{Z}^+(\Delta_R)$, $\zeta\neq 0$, such that

$$\Lambda-\zeta+i\eta_1 = t(\Lambda+i\eta_2).$$

Since Λ is in general position, t must leave $\mathfrak{a}^*_{\mathfrak{d},\mathbf{C}}$ pointwise fixed. This contradicts the fact that ζ is a nonzero vector in $\mathfrak{a}^*_{\mathfrak{d},\mathbf{C}}$. It follows that φ vanishes. Q.E.D.

§ 5. Estimates

We will eventually want to study contour integrals of the functions $E_{B'|B,s}(a,\Phi,\Lambda)$. In this section we collect the needed estimates. We will assume from now on that the representation (τ, V_τ) is unitary. In particular, V_τ is a finite dimensional Hilbert space. The spaces $\mathscr{A}_{\mathrm{cusp}}(M,\tau)$, and particularly $\mathscr{A}_0$, will also be Hilbert spaces with the inner product

$$(\varphi_1,\varphi_2)=\int_{M/A_M}(\varphi_1(m),\varphi_2(m))\,dm,\quad \varphi_1,\varphi_2\in\mathscr{A}_{\mathrm{cusp}}(M,\tau).$$

The leading term in the expansion of

$$E_{B'|B,s}(a, \Phi, \Lambda), \quad a \in A_0^\varepsilon(B'),$$

is the function

$$(c_{B'|B,0}(s, \Lambda)\,\Phi)(1) \cdot a^{(s\Lambda - \varrho_{B'})}.$$

The operators $c_{B'|B,0}(s, \Lambda)$ on $\mathscr{A}_0$ are just the c functions, $c_{B'|B}(s, \Lambda)$ discussed in § 2. What about the other terms? For any $\zeta \in \mathbf{Z}^+(\Delta_{B'})$ it turns out that

$$c_{B'|B,\zeta}(s, \Lambda) = \Gamma_{B',\zeta}(s\Lambda - \varrho_{B'})\, c_{B'|B}(s, \Lambda),$$

where $\Gamma_{B',\zeta}$ is a rational function on $\mathfrak{a}^*_{0,\mathbf{C}}$ with values in $\mathrm{End}(\mathscr{A}_0)$. (See [13, § 9.1.4].)

LEMMA 5.1. *Fix $\Lambda_1 \in \mathfrak{a}_0^*$ and $H_0 \in \mathfrak{a}_0(B)$. Then we can find a polynomial $l(\Lambda)$ and constants c and n such that*

$$\|l(\Lambda)\,\Gamma_{B,\zeta}(\Lambda)\| \leq c(1+\|\Lambda\|)^n e^{\zeta(H_0)},$$

for all $\zeta \in \mathbf{Z}^+(\Delta_B)$ and $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}$ such that $\mathrm{Re}(\Lambda)$ *belongs to* $\Lambda_1 - \mathfrak{a}_0^*(B)$.

Proof. An estimate of this sort, without the dependence on Λ, was first proved by Helgason [8a]. (See also [13, Lemma 9.1.4.4].) It was derived from the recursion relations obtained from the radial component of the Casimir operator $\tilde{\omega}$ on G. This radial component, denoted $\delta'(\tilde{\omega})$, is a second order differential operator on A_0, with values in $\mathrm{End}(\mathscr{A}_0)$, such that if

$$\Phi_\Lambda(\exp H) = \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} \Gamma_{B,\zeta}(\Lambda - \varrho_B)\, e^{(\Lambda - \zeta - \varrho_B)(H)},$$

then

$$(\delta'(\tilde{\omega})\,\Phi_\Lambda)(\exp H) = \Phi_\Lambda(\exp H)\,\gamma_{M_0}(\tilde{\omega}, \Lambda).$$

It has been computed explicitly in Corollary 9.1.2.12 of [13], and is of the form

$$\delta'(\tilde{\omega}) = E + \delta_1' + \delta_0',$$

in which E is the Laplacian on A_0 and δ_0' and δ_1' are differential operators on A_0 of respective orders 0 and 1. A closer inspection of the formula in [13] (at the top of p. 279) reveals that the function δ_0' has an expansion

$$\sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} d'_\zeta e^{\zeta(H)}, \quad H \in \mathfrak{a}_0(B),$$

where the coefficients are endomorphisms of $\mathscr{A}_0$ whose norms have at most polynomial growth in ζ. The differential operator δ_1', on the other hand, is scalar valued and is actually independent of $\mathscr{A}_0$ (i.e. of the representation τ).

With the presence of the first order term δ_1' it is difficult to obtain uniform estimates.(¹) It is necessary to use the technique of Gangolli, as elaborated on p. 38–39 of [8b]. Let $\Delta^{1/2}(\exp H)$ be the product over all roots α of (B, A_0), repeated with multiplicities, of the functions

$$(e^{\alpha(H)} - e^{-\alpha(H)})^{1/2}, \quad H \in \mathfrak{a}_0(B).$$

The differential operators E and δ_1' are independent of τ, and we can eliminate the first order term exactly as in the K bi-invariant case [8b, p. 38]; we obtain

$$\Delta^{1/2}\delta'(\bar{\omega}) \circ \Delta^{-1/2} = E + \delta_0,$$

where δ_0 is a function with values in $\mathrm{End}(\mathscr{A}_0)$. As in [8b] there are expansions

$$\Delta^{-1/2}(\exp H) = e^{-\varrho_B(H)} \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} b_\zeta e^{-\zeta(H)}$$

$$\Delta^{1/2}(\exp H) = e^{\varrho_B(H)} \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} c_\zeta e^{-\zeta(H)},$$

and

$$\delta_0 = \delta_0(\exp H) = \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} d_\zeta e^{-\zeta(H)},$$

where $\{b_\zeta\}$ and $\{c_\zeta\}$ are complex numbers, while $\{d_\zeta\}$ are endomorphisms of $\mathscr{A}_0$. The norms of all the coefficients have at most polynomial growth in ζ. The function

$$\Psi_\Lambda(\exp H) = \Delta^{1/2}(\exp H)\, \Phi_\Lambda(\exp H) = \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} a_\zeta(\Lambda)\, e^{(\Lambda-\zeta)(H)}$$

(¹) I thank J. Carmona for pointing this out to me. My original proof was wrong.

where

$$a_\zeta(\Lambda) = \sum_{\{\mu \in \mathbf{Z}^+(\Delta_B):\, \zeta-\mu \in \mathbf{Z}^+(\Delta_B)\}} c_\mu \Gamma_{B,\zeta-\mu}(\Lambda - \varrho_B),$$

satisfies the differential equation

$$E\Psi_\Lambda(\exp H) - \Psi_\Lambda(\exp H)\gamma_{M_0}(\tilde{\omega}, \Lambda) + \delta_0(\exp H)\Psi_\Lambda(\exp H) = 0.$$

Regarded as an endomorphism in $\mathrm{End}(\mathscr{A}_0)$, $\gamma_{M_0}(\tilde{\omega}, \Lambda)$ equals the sum of (Λ, Λ) and an endomorphism which is independent of Λ. Since E is the Laplacian on $\mathscr{A}_0$, it follows readily that the coefficients $\{a_\zeta(\Lambda)\}$ satisfy recursion relations

$$(2(\Lambda, \zeta) - (\zeta, \zeta) + L)(a_\zeta(\Lambda)) = \sum_{\{\mu \neq 0,\, \zeta-\mu \in \mathbf{Z}^+(\Delta_B)\}} d_\mu a_{\zeta-\mu}(\Lambda),$$

where L is a linear transformation on the vector space $\mathrm{End}(\mathscr{A}_0)$ which is independent of Λ and ζ.

If

$$\zeta = \sum_{\alpha \in \Delta_B} n_\alpha \alpha,$$

set

$$m_\zeta = \sum n_\alpha$$

as in [8a] and [13]. Then the number

$$|2(\Lambda, \zeta) - (\zeta, \zeta)|$$

is bounded below by m_ζ, for all Λ whose real part belongs to $\Lambda_1 - \mathfrak{a}_0^*(B)$ and all but finitely many $\zeta \in \mathbf{Z}^+(\Delta_B)$. Thus, for all such Λ and all but finitely many ζ, the linear transformation

$$2(\Lambda, \zeta) - (\zeta, \zeta) + L$$

of $\mathrm{End}(\mathscr{A}_0)$ has an inverse, whose norm is bounded by m_ζ^{-1}. This means that

$$\|a_\zeta(\Lambda)\| \leq m_\zeta^{-1} \sum_{\{\mu \neq 0,\, \zeta-\mu \in \mathbf{Z}^+(\Delta_B)\}} |d_\mu| \cdot \|a_{\zeta-\mu}(\Lambda)\|$$

for all such Λ and all ζ with m_ζ greater than some constant N. Now each $a_\zeta(\Lambda)$ is a rational function of Λ. We can therefore choose a polynomial $l(\Lambda)$ and constants c and n such that

$$\|l(\Lambda)\, a_\zeta(\Lambda)\| \leq c(1+\|\Lambda\|)^n e^{\zeta(H_0)}$$

for all Λ with $\mathrm{Re}(\Lambda) \in \Lambda_1 - \mathfrak{a}_0^*(B)$ and for the finitely many ζ with

$$m_\zeta \leq \max\left\{N, \left(\sum_{\mu \in \mathbf{Z}^+(\Delta_B)} |d_\mu|\, e^{-\mu(H_0)}\right)\right\}.$$

This inequality then holds for all $\zeta \in \mathbf{Z}^+(\Delta_B)$ and all such Λ by induction on m_ζ. However,

$$\Gamma_{B,\zeta}(\Lambda - \varrho_B) = \sum_{\{\mu:\, \zeta-\mu \in \mathbf{Z}^+(\Delta_B)\}} b_\mu\, a_{\zeta-\mu}(\Lambda).$$

We therefore obtain the required estimate for the functions $\{\Gamma_{B,\zeta}\}$. Q.E.D.

The singularities of the rational functions $\Gamma_{B,\zeta}(\Lambda)$ lie along hyperplanes of the form $(\xi, \Lambda) = r$, for $\xi \in \mathbf{Z}^+(\Delta_B)$ and $r \in \mathbf{R}$. That is, for every $\Lambda_0 \in \mathfrak{a}_{0,\mathbf{C}}^*$, there is a polynomial $p(\Lambda)$ which is a product of linear forms $(\xi, \Lambda) - r$ such that $p(\Lambda)\Gamma_{B,\zeta}(\Lambda)$ is regular at Λ_0. Moreover, only finitely many of these linear forms vanish in any region $\Lambda_1 - \mathfrak{a}_0^*(B)$. (See [13, p. 287].) We see from the proof of Lemma 5.1 that $l(\Lambda)$ can be taken to be a product of such linear forms.

Define

$$r_B(x, \Phi, \Lambda) = E_{B|B,1}(x, \mu_B(\Lambda)\Phi, \Lambda)$$

for $\Phi \in \mathscr{A}_0$, $\Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*$ and x in a neighborhood of infinity in G. This function will be of particular concern to us. The leading term of its expansion along $A_0(B)$ is

$$(c_{B|B}(1, \Lambda)\mu_B(\Lambda)\Phi)(1)\, a^{(\Lambda - \varrho_B)}, \quad a \in A_0.$$

LEMMA 5.2. *The singularities of the functions* $c_{B|B}(1, \Lambda)$ *and* $\mu_B(\Lambda)$ *all lie along hyperplanes of the form* $(\beta, \Lambda) = r$, *where* β *is a root of* (G, A_0) *and* r *is a real number. Moreover, for every* $\Lambda_1 \in \mathfrak{a}_0^*$ *we can find a polynomial* $l(\Lambda)$ *and constants* c *and* n *such that*

$$\|l(\Lambda)\, c_{B|B}(1, \Lambda)\mu_B(\Lambda)\| \leq c(1+\|\Lambda\|)^n$$

whenever $\mathrm{Re}(\Lambda)$ *belongs to* $\Lambda_1 - \mathfrak{a}_0^*(B)$.

Proof. It follows from (2.2), (2.3) and (2.4) that $c_{B|B}(1,\Lambda)$ can be expressed as a product of operators $c_{B_\beta|B_\beta}(1,\Lambda)$, where β is a root in Σ_B and $B_\beta = B\cap M_\beta$ for a group M_β in $\mathscr{L}(M_0)$ which modulo the center has real rank one. In particular, $c_{B_\beta|B_\beta}(1,\Lambda)$ depends only on (β,Λ). As observed in [12], any matrix coefficient of an operator $c_{B_\beta|B_\beta}(1,\Lambda)$ can be expressed as a linear combination of functions of the form

$$\frac{\Gamma(r^{-1}(\beta,\Lambda)+m_1)}{\Gamma(r^{-1}(\beta,\Lambda)+n_1)}\cdot\frac{\Gamma(r^{-1}(\beta,\Lambda)+m_2)}{\Gamma(r^{-1}(\beta,\Lambda)+n_2)}$$

where m_1, m_2, n_1, n_2 and r are real numbers, and r is positive. The poles of the functions

$$\frac{\Gamma(r^{-1}z+m_i)}{\Gamma(r^{-1}z+n_i)},\quad z\in\mathbf{C},$$

all lie on the real axis. Similarly, $\mu_B(\Lambda)$ is a product of μ functions $\mu_{B_\beta}(\Lambda)$ associated to M_β. Each function $\mu_{B_\beta}(\Lambda)$ depends only on (β,Λ), and we know from its explicit formula ([7e]) that it has poles only when (β,Λ) is real. Therefore, the singularities of both $c_{B|B}(1,\Lambda)$ and $\mu_B(\Lambda)$ are of the required form.

Now

$$c_{B|B}(1,\Lambda)\mu_B(\Lambda)$$

is a product of operators

$$c_{B_\beta|B_\beta}(1,\Lambda)\mu_{B_\beta}(\Lambda),$$

where β is a root of (B, A_0). It is enough to prove the estimate if G is replaced by M_β. An estimate of the sort we need appeared in [1], but the proof was omitted. Write

$$\begin{aligned} c_{B_\beta|B_\beta}(1,\Lambda)\mu_{B_\beta}(\Lambda) &= \mu_{B_\beta}(\Lambda)J_{\bar{B}_\beta|B_\beta}(\Lambda)\\ &= J_{B_\beta|\bar{B}_\beta}(\Lambda)^{-1}\\ &= c_{\bar{B}_\beta|\bar{B}_\beta}(1,\Lambda)^{-1}. \end{aligned}$$

We need to estimate the norm of this operator when $(\beta,\mathrm{Re}(\Lambda))<(\beta,\Lambda_1)$. This is the same as estimating the norm of $c_{B_\beta|B_\beta}(1,\Lambda)^{-1}$ in any region

$$(\beta,\mathrm{Re}\,\Lambda)>b,\quad b\in\mathbf{R}.$$

It follows from the results of [3] that the inverse of the determinant of $c_{B_\beta|B_\beta}(1,\Lambda)$ is a constant multiple of a product of functions

$$\frac{\Gamma(r^{-1}(\beta,\Lambda)+\mu)}{\Gamma(r^{-1}(\beta,\Lambda)+\nu)},$$

with $\mu,\nu\in\mathbf{C}$ and r a positive real number. Therefore by the result we quoted above, any matrix coefficient of $c_{B_\beta|B_\beta}(1,\Lambda)^{-1}$ is a linear combination of such products. It is known that

$$\lim_{|z|\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\log z}=1,\quad |\arg z|\leqslant\pi-\delta$$

for any $a\in\mathbf{C}$ and $\delta>0$. It follows easily that for any b we can find constants c and n such that

$$\left|\frac{\Gamma(r^{-1}z+\mu)}{\Gamma(r^{-1}z+\nu)}\right|\leqslant c(1+|z|)^n,$$

whenever $|z|$ is sufficiently large and $\operatorname{Re}(z)>b$. Since $\Gamma(r^{-1}z+\mu)/\Gamma(r^{-1}z+\nu)$ has only finitely many poles in the region $\operatorname{Re}(z)>b$, we can choose a polynomial $l(z)$ and constants c and n such that

$$\left|l(z)\frac{\Gamma(r^{-1}z+\mu)}{\Gamma(r^{-1}z+\nu)}\right|\leqslant c(1+|z|)^n,$$

whenever $\operatorname{Re}(z)>b$. The lemma follows. Q.E.D.

LEMMA 5.3. *Fix $\varepsilon>0$ and $\Lambda_1\in\mathfrak{a}_0^*$. Then we can find a polynomial $l(\Lambda)$, which is a product of linear forms*

$$(\zeta,\Lambda)-r,\quad \zeta\in\mathbf{Z}^+(\Delta_B),\ r\in\mathbf{R},$$

and constants c and n, such that

$$\|l(\Lambda)\,r_B(a,\Phi,\Lambda)\|\leqslant c(1+\|\Lambda\|)^n\,\|\Phi\|\,a^{(\operatorname{Re}\Lambda-\varrho_B)}$$

for all $\Phi\in\mathcal{A}_0$, $a\in A_0^\varepsilon(B)$ and $\Lambda\in\mathfrak{a}_{0,\mathbf{C}}^*$ such that $\operatorname{Re}\Lambda\in\Lambda_1-\mathfrak{a}_0^*(B)$.

Proof. We can choose H_0 such that for any $\alpha\in\Delta_B$, $\alpha(H_0)$ equals $\varepsilon/2$. Then for $\zeta\in\mathbf{Z}^+(\Delta_B)$ and m_ζ as in the proof of Lemma 5.1, $e^{\zeta(H_0)}$ equals $e^{\frac{1}{2}\varepsilon m_\zeta}$. If a belongs to $A_0^\varepsilon(B)$,

$$a^{-\zeta}e^{\zeta(H_0)}\leqslant e^{-\varepsilon m_\zeta}e^{\frac{1}{2}\varepsilon m_\zeta}=e^{-\frac{1}{2}\varepsilon m_\zeta}.$$

It is clear that

$$\sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} e^{-\frac{1}{2}\varepsilon m_\zeta}$$

is finite. Now, if a is any point in $A_0(B)$, we have

$$\|r_B(a, \Phi, \Lambda)\| \leq \sum_{\zeta \in \mathbf{Z}^+(\Delta_B)} \|(c_{B|B,\zeta}(1, \Lambda)\, \mu_B(\Lambda)\, \Phi)(1)\|\, a^{(\mathrm{Re}(\Lambda)-\zeta-\varrho_B)}$$

$$\leq \|c_{B|B}(1, \Lambda)\, \mu_B(\Lambda)\| \cdot \|\Phi\| \cdot a^{(\mathrm{Re}(\Lambda)-\varrho_B)} \sum_\zeta \|\Gamma_{B,\zeta}(\Lambda-\varrho_B)\|\, a^{-\zeta}.$$

Our result follows from Lemmas 5.1 and 5.2. Q.E.D.

Suppose that $B, B' \in \mathcal{P}(M_0)$ and that $s \in W_0$. The first part of the proof of Lemma 5.2 can be applied to $c_{B'|B}(s, \Lambda)$ to show that the singularities of this function also lie along hyperplanes $(\beta, \Lambda) = r$, for β a root of (G, A_0) and $r \in \mathbf{R}$.

LEMMA 5.4. *Suppose that C is a compact subset of $A_0(B')$, that C^* is a compact subset of $\mathfrak{a}^*_{0,\mathbf{C}}$ and that $s \in W_0$. Then there is a polynomial $l(\Lambda)$, which is a product of linear factors*

$$(\zeta, \Lambda) - r, \quad \zeta \in \mathbf{Z}(\Delta_B),\ r \in \mathbf{R},$$

such that the series

$$l(\Lambda)\, E_{B'|B,s}(a, \Phi, \Lambda) = \sum_{\zeta \in \mathbf{Z}^+(\Delta_{B'})} l(\Lambda)\, (c_{B'|B,\zeta}(s, \Lambda)\, \Phi)(1) \cdot a^{(s\Lambda-\zeta-\varrho_{B'})}$$

converges absolutely uniformly for $a \in C$ and $\Lambda \in C^*$. In particular, the function

$$l(\Lambda)\, E_{B'|B,s}(a, \Phi, \Lambda)$$

is defined and bounded for $a \in C$ and $\Lambda \in C^*$.

Proof. We know that

$$c_{B'|B,\zeta}(s, \Lambda) = \Gamma_{B',\zeta}(s\Lambda - \varrho_{B'})\, c_{B'|B}(s, \Lambda).$$

We can certainly estimate $\Gamma_{B',\zeta}(s\Lambda-\varrho_{B'})$ on C^* by Lemma 5.1. The lemma is then proved the same way as Lemma 5.3. Q.E.D.

This last lemma tells us that the functions $E_{B'|B,s}(x,\Phi,\Lambda)$ are defined for all x in $G_-=K\cdot A_0(B)\cdot K$.

§ 6. Further properties of the functions $E_{B'|B,s}$

Let $\mathcal{M}(\mathcal{A}_0)$ be the space of meromorphic functions from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathcal{A}_0$ whose singularities lie along hyperplanes of the form $(\beta,\Lambda)=r$, where β is a root of (G,A_0) and $r\in\mathbf{R}$. Suppose that $B,B'\in\mathcal{P}(M_0)$ and $s\in W_0$. We are going to study the class of $\mathcal{Z}$-finite, τ-spherical functions on G_- obtained by differentiating functions

$$E_{B'|B,s}(x,\Phi(\Lambda),\Lambda),\quad \Phi(\Lambda)\in\mathcal{M}(\mathcal{A}_0),$$

with respect to Λ. For most later applications it will suffice to take $B'=B$ and $s=1$.

LEMMA 6.1. *Suppose that* $\Phi(\Lambda)\in\mathcal{M}(\mathcal{A}_0)$. *The meromorphic function* $E_{B'|B,s}(x,\Phi(\Lambda),\Lambda)$ *is regular at* $\Lambda=\Lambda_0$ *if and only if each of the functions*

$$c_{B'|B,\zeta}(s,\Lambda)\,\Phi(\Lambda),\quad \zeta\in\mathbf{Z}^+(\Delta_{B'}),$$

is regular at $\Lambda=\Lambda_0$. *Suppose that this is the case and that* $D=D_\Lambda$ *is a differential operator on* $\mathfrak{a}^*_{0,\mathbf{C}}$. *Then for* $a\in A_0(B')$,

$$\lim_{\Lambda\to\Lambda_0} D_\Lambda E_{B'|B,s}(a,\Phi(\Lambda),\Lambda)$$

equals

$$\sum_{\zeta\in\mathbf{Z}^+(\Delta_{B'})}\lim_{\Lambda\to\Lambda_0} D_\Lambda(c_{B'|B,\zeta}(s,\Lambda)\,\Phi(\Lambda))(1)\,a^{(\Lambda-\zeta-\varrho_{B'})}.$$

Proof. The poles of the functions $c_{B'|B,\zeta}(s,\Lambda)\,\Phi(\Lambda)$ all lie along hyperplanes $(\xi,\Lambda)=r$, for $\xi\in\mathbf{Z}(\Delta_{B'})$ and $r\in\mathbf{R}$. By Lemma 5.4, the same is true of the function $E_{B'|B,s}(x,\Phi(\Lambda),\Lambda)$. Suppose that Λ is a point in general position on the hyperplane $(\xi,\Lambda)=r$. Set

$$\Lambda_u=u(\xi,\xi)^{-1/2}\xi+\Lambda,\quad u\in\mathbf{C}.$$

Let Γ be a small, positively oriented circle about the origin in the complex plane. The function $E_{B'|B,s}(x, \Phi(\Lambda), \Lambda)$ will be regular along the hyperplane in question if and only if

$$\int_\Gamma u^n E_{B'|B,s}(a, \Phi(\Lambda_u), \Lambda_u)\, du$$

vanishes for all $a \in A_0(B')$ and all nonnegative integers n. By Lemma 5.4 this integral equals

$$\sum_{\zeta \in \mathbf{Z}^+(\Delta_{B'})} \int_\Gamma u^n (c_{B'|B,\zeta}(s, \Lambda_u)\, \Phi(\Lambda_u))(1) \cdot a^{(s\Lambda_u - \zeta - \varrho_{B'})}\, du.$$

By the uniqueness of the asymptotic expansion, the first integral will vanish if and only if each term in the second series vanishes. This happens for all $n \geq 0$ if and only if each function $c_{B'|B,\zeta}(s, \Lambda)\, \Phi(\Lambda)$ is regular along the hyperplane in question. The first statement of the lemma follows. The second statement follows without difficulty from Lemma 5.4 and Cauchy's integral formula. Q.E.D.

Functions of the form

$$\varphi(x) = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{B'|B,s}(x, \Phi(\Lambda), \Lambda), \quad x \in G_-,$$

with $\Phi(\Lambda)$ and D_Λ as in the lemma, will arise later. It is an easy consequence of formula (4.7) that the orbit $\mathfrak{o}_G(\varphi)$ is contained in $\mathfrak{o}_G(\tau, \Lambda_0)$.

N. Wallach has shown that the only singular hyperplanes of the functions $c_{B'|B,\zeta}(s, \Lambda)$ are actually of the form $(\beta, \Lambda) = r$, where β is a root of (G, A_0). His proof (not yet published) uses Verma modules. We will need this result, so we shall include a different proof which is based on the differential equations (4.2).

LEMMA 6.2. *Suppose that $\Phi(\Lambda)$ is a function in $\mathcal{M}(\mathcal{A}_0)$. Then the singularities of the function*

$$E_{B'|B,s}(x, \Phi(\Lambda), \Lambda)$$

lie along hyperplanes of the form $(\beta, \Lambda) = r$, where β is a root of (G, A_0) and $r \in \mathbf{R}$.

Proof. Fix a singular hyperplane of the function. It is of the form $(\xi, \Lambda) = r$, for $\xi \in \mathbf{Z}(\Delta_{B'})$ and $r \in \mathbf{R}$. We must show that ξ is a multiple of a root of (G, A_0). Suppose that this is not so. Let Λ be a point in general position on the hyperplane; define Λ_u,

$u \in \mathbf{C}$, and Γ as in the proof of the last lemma. Then there is a nonnegative integer n such that the function

$$\varphi(x) = \int_\Gamma u^n E_{B'|B,s}(x, \Phi(\Lambda_u), \Lambda_u)\, du, \quad x \in G_-,$$

does not vanish. Suppose that $a \in A_0(B')$. In view of Lemma 5.4 we can write $\varphi(a)$ as

$$\sum_{\zeta \in \mathbf{Z}^+(\Delta_{B'})} \int_\Gamma u^n (c_{B'|B,\zeta}(s, \Lambda_u)\, \Phi(\Lambda_u))(1) \cdot a^{(s\Lambda_u - \zeta - \varrho_{B'})}\, du.$$

Remember we are assuming that ξ is not a multiple of a root of (G, A_0). Neither of the functions $c_{B'|B}(s, \cdot)$ or $\Phi(\cdot)$ has a singularity along the hyperplane in question, so that

$$(c_{B'|B}(s, \Lambda_u)\, \Phi(\Lambda_u))(1)$$

is regular at $u=0$. It follows that the first term in the expansion for $\varphi(a)$ vanishes. Therefore the principal exponents of φ are all of the form $s\Lambda - \zeta - \varrho_{B'}$ for nonzero elements $\zeta \in \mathbf{Z}(\Delta_{B'})$. Then by Lemma 4.1 any point in $\mathfrak{o}_G(\varphi)$ is contained in a set $\mathfrak{o}_G(\tau, s\Lambda - \zeta)$, for some $\zeta \neq 0$. This latter set can also be written as $\mathfrak{o}_G(\tau, \Lambda + \zeta_1)$ for another nonzero element ζ_1 in $\mathfrak{a}_0^*$.

On the other hand, we can write

$$\varphi(x) = \lim_{u \to 0} \frac{2\pi i}{(k+1)!} \left(\frac{d}{du}\right)^k (u^{n+k+1} E_{B'|B,s}(x, \Phi(\Lambda_u), \Lambda_u)),$$

for any large integer k. By the remark following the last lemma, $\mathfrak{o}_G(\varphi)$ is contained in $\mathfrak{o}_G(\tau, \Lambda)$. It follows from this that there are elements $\eta_1, \eta_2 \in \mathfrak{h}_K^*$, $t \in W$ and $\zeta_1 \in \mathfrak{a}_0^*$, $\zeta_1 \neq 0$, such that

$$\Lambda + \zeta_1 + i\eta_1 = t(\Lambda + i\eta_2).$$

Now we can write

$$\Lambda = X + \lambda$$

where λ is a point in general position in the subspace of $\mathfrak{a}_{0,\mathbf{C}}^*$ orthogonal to ξ, and X is a vector parallel to ξ. The fact that λ is in general position means that t must leave the subspace pointwise fixed. Since the subspace is of co-dimension 1 in $\mathfrak{a}_{0,\mathbf{C}}^*$ and ξ is not a multiple of a root, t must leave $\mathfrak{a}_{0,\mathbf{C}}^*$ pointwise fixed. In particular, $\Lambda = t\Lambda$. Therefore

$$\zeta_1 + i\eta_1 = it\eta_2.$$

Since ζ_1 is a nonzero vector in $\mathfrak{a}_0^*$, while $it\eta_2 - i\eta_1$ belongs to $i\mathfrak{h}_K^*$ we have a contradiction. The vector ξ must then be a multiple of a root. Q.E.D.

COROLLARY 6.3. *Suppose that $F_B(\Lambda)$ is an entire function from $\mathfrak{a}_{0,\mathbf{C}}^*$ to $\mathscr{A}_0$. Then the singularities of the function*

$$r_B(x, F_B(\Lambda), \Lambda)$$

lie along hyperplanes of the form $(\beta, \Lambda) = r$, where β is a root of (G, A_0) and $r \in \mathbf{R}$.

Proof. This follows immediately from Lemmas 5.2 and 6.2. Q.E.D.

In fact, it follows from Cauchy's integral formula that the polynomials $l(\Lambda)$ in Lemmas 5.1, 5.2, 5.3 and 5.4 can all be taken to be products of linear factors

$$(\beta, \Lambda) - r,$$

where once again β is a root of (G, A_0) and $r \in \mathbf{R}$.

§ 7. The space $\mathscr{A}(G_-, \tau)$

Suppose that for a function $\Phi(\Lambda) \in \mathscr{M}(\mathscr{A}_0)$ and a group $B \in \mathscr{P}(M_0)$, the function

$$\Lambda \to E_{B|B,1}(x, \Phi(\Lambda), \Lambda), \quad \Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*,$$

is regular at Λ_0. Then if $D = D_\Lambda$ is a differential operator (with respect to Λ) on $\mathfrak{a}_{0,\mathbf{C}}^*$, the function

$$\varphi(x) = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{B|B,1}(x, \Phi(\Lambda), \Lambda)$$

is a $\mathscr{Z}$-finite, τ-spherical function from G_- to V_τ. Let $\mathscr{A}(G_-, \tau)$ be the space spanned by functions of this form. It is infinite dimensional. However, suppose that $\mathfrak{o}$ is any finite union of W-orbits in $\mathfrak{h}_{\mathbf{C}}^*$ and that d is an integer. Let $\mathscr{A}_{\mathfrak{o},d}(G_-, \tau)$ be the space of functions of the form

$$\varphi_1 + \ldots + \varphi_n$$

where for each i, φ_i is a function in $\mathscr{A}(G_-, \tau)$ with the property that

$$(z - p_z(\nu_i))^d \varphi_i = 0, \quad z \in \mathscr{Z},$$

for some point ν_i in $\mathfrak{o}$. It is a finite dimensional space. If φ is as above, and $d=\deg D$, then φ belongs to $\mathscr{A}_{\mathfrak{o}_G(\tau,\Lambda_0),d}(G_-,\tau)$.

The space $\mathscr{A}(G_-,\tau)$ is independent of B. Suppose for each i, $1\leq i\leq n$, that $\Phi_i(\Lambda)\in\mathscr{M}(\mathscr{A}_0)$, that D_i is an analytic differential operator on $\mathfrak{a}^*_{0,\mathbf{C}}$, that B_i, $B_i'\in\mathscr{P}(M_0)$ and that s_i, $s_i'\in W_0$. Suppose they are such that the function

$$\sum_i (D_i)_\Lambda E_{B_i'|B_i,s_i}(x,\Phi_i(\Lambda),s_i'\Lambda),\quad \Lambda\in\mathfrak{a}^*_{0,\mathbf{C}},$$

is regular at Λ_0. We shall show that its value at Λ_0 belongs to $\mathscr{A}(G_-,\tau)$. Fix $B\in\mathscr{P}(M_0)$. Then in view of (4.4′) and (4.5′), we may assume that $B_i'=B$ and that $s_i'=1$. But

$$\sum_i D_i E_{B|B_i,s_i}(x,\Phi_i(\Lambda),\Lambda)$$

equals

$$\sum_i D_i E_{B|B,1}(x,{}^0c_{B|B_i}(s_i,\Lambda)\,\Phi_i(\Lambda),\Lambda).$$

Let $l_N(\Lambda)$ be the product, over all roots β of (B,A_0), of the factors $(\Lambda-\Lambda_0,\beta)^N$. For every positive integer N there is a differential operator D_N such that

$$\lim_{\Lambda\to\Lambda_0} D_N(l_N(\Lambda)f(\Lambda))=f(\Lambda_0)$$

for every function f which is regular at Λ_0. (For example, if $(u_1,\dots,u_k)$ is a system of co-ordinates on $\mathfrak{a}^*_{0,\mathbf{C}}$ around Λ_0, and $cu_1^{n_1}\dots u_k^{n_k}$ is the lowest term of $l_N(\Lambda)$ relative to the lexicographic order on the monomials in $(u_1,\dots,u_k)$, we could take

$$D_N=c^{-1}((n_1)!\dots(n_k)!)^{-1}\left(\frac{\partial}{\partial u_1}\right)^{n_1}\dots\left(\frac{\partial}{\partial u_k}\right)^{n_k}.)$$

If D is an analytic differential operator of degree no greater than d,

$$\tilde{D}=l_N(\Lambda)\,D\circ l_{N-d}(\Lambda)^{-1}$$

is an analytic differential operator. Choose $d=\max\{\deg D_i\}$. Then

$$\lim_{\Lambda\to\Lambda_0}\sum_i (D_i)_\Lambda E_{B|B_i,s_i}(x,\Phi_i(\Lambda),\Lambda) \tag{7.1}$$

equals

$$\lim_{\Lambda\to\Lambda_0}\sum_i (D_N\tilde{D}_i)_\Lambda E_{B|B,1}(x, l_{N-d}(\Lambda)\,{}^0c_{B|B_i}(s_i,\Lambda)\,\Phi_i(\Lambda), s_i\Lambda).$$

The function whose value at $s_i\Lambda$ is

$$l_{N-d}(\Lambda)\,{}^0c_{B|B_i}(s_i,\Lambda)\,\Phi_i(\Lambda)$$

certainly belongs to $\mathcal{M}(\mathcal{A}_0)$. If N is sufficiently large

$$E_{B|B,1}(x, l_{N-d}(\Lambda)\,{}^0c_{B|B_i}(s_i,\Lambda)\,\Phi_i(\Lambda), s_i\Lambda)$$

is regular at $\Lambda=\Lambda_0$. This means that the function (7.1) belongs to $\mathcal{A}(G_-,\tau)$, or more precisely, to $\mathcal{A}_{o(\tau,\Lambda_0),d}(G_-,\tau)$.

As a particular case, we have

LEMMA 7.1. *Suppose that $\Phi(\Lambda)$ is an analytic function from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathcal{A}_0$. Then if $D=D_\Lambda$ is a differential operator (with respect to Λ) on $\mathfrak{a}^*_{0,\mathbf{C}}$, and $B\in\mathcal{P}(M_0)$, the function*

$$\varphi(x)=\lim_{\Lambda\to\Lambda_0} D_\Lambda E_B(x,\Phi(\Lambda),\Lambda),\quad x\in G_-, \tag{7.2}$$

belongs to $\mathcal{A}(G_-,\tau)$.

Proof. The function $\varphi(x)$ equals

$$\lim_{\Lambda\to\Lambda_0} D_\Lambda\left(\sum_{s\in W_0} E_{B|B,s}(x,\Phi(\Lambda),\Lambda)\right).$$

The lemma follows from the discussion above. Q.E.D.

The function $\varphi(x)$ defined by (7.2) is the restriction to G_- of a $\mathcal{Z}$-finite, τ-spherical function on G. Let $\mathcal{A}(G,\tau)$ be the space spanned by functions on G of this form. Lemma 7.1 tells us that by restricting functions in $\mathcal{A}(G,\tau)$ to G_- we obtain an embedding of $\mathcal{A}(G,\tau)$ into $\mathcal{A}(G_-,\tau)$. We can define

$$\mathcal{A}_{o,d}(G,\tau)=\mathcal{A}(G,\tau)\cap\mathcal{A}_{o,d}(G_-,\tau).$$

One would expect $\mathcal{A}(G_-,\tau)$ to be the space of all $\mathcal{Z}$-finite, τ-spherical functions defined in a neighborhood of infinity in G, while $\mathcal{A}(G,\tau)$ ought to be the subspace of such functions whose domains extend to all of G. However, we will not investigate this question here.

There is the third space, $\mathcal{A}_{\mathrm{cusp}}(G,\tau)$, which we introduced in § 2.

LEMMA 7.2. *Any function* $\varphi \in \mathcal{A}_{\mathrm{cusp}}(G,\tau)$ *is a sum of functions of the form*

$$E_B(x,\Phi,\Lambda), \quad B\in\mathcal{P}(M_0),\ \Phi\in\mathcal{A}_0,\ \Lambda\in\mathfrak{a}^*_{0,\mathbf{C}}.$$

Proof. The lemma is essentially Harish-Chandra's subquotient theorem. We can assume, as in § 3, that $V_\tau=\mathrm{Hom}(V_1,V_2)$ for irreducible representations (τ_1,V_1) and (τ_2,V_2) of K. We can also assume that

$$\varphi(x)=s_2^*\pi(x)s_1,$$

for an irreducible square integrable representation (π,U_π) of G/A_G, and maps $s_1\in\mathrm{Hom}_K(V_1,U_\pi)$ and $s_2^*\in\mathrm{Hom}_K(U_\pi,V_2)$. Then π is equivalent to a subquotient of $I_B(\sigma,\Lambda)$, for some B, σ and Λ. More precisely, if $\mathfrak{g}=\mathrm{Lie}(G)$, there is a $(\mathfrak{g},K)$ isomorphism A from the $(\mathfrak{g},K)$ module associated to π and a subquotient of the $(\mathfrak{g},K)$ module associated to $I_B(\sigma,\Lambda)$. It follows that if $k\in K$ and $X\in\mathfrak{g}$,

$$\begin{aligned} s_2^*\pi(k)\pi(X)s_1 &= s_2^*A^{-1}I_B(\sigma,\Lambda,k)I_B(\sigma,\Lambda,X)As_1\\ &=\Sigma_2^*I_B(\sigma,\Lambda,k)I_B(\sigma,\Lambda,X)\Sigma_1,\end{aligned}$$

for elements $\Sigma_1\in\mathrm{Hom}_K(V_1,\mathcal{H}_B(\sigma))$ and $\Sigma_2^*\in\mathrm{Hom}_K(\mathcal{H}_B(\sigma),V_2)$. By taking exponentials we obtain

$$\begin{aligned}\varphi(x)&=s_2^*\pi(x)s_1\\ &=\Sigma_2^*I_B(\sigma,\Lambda,x)\Sigma_1.\end{aligned}$$

In the notation of § 3 we have $\Sigma_1=\Psi_{S_1}$ and $\Sigma_2^*=\Psi^*_{S_2^*}$, for unique elements $S_1\in\mathrm{Hom}_{M_0}(V_1,U_\sigma)$ and $S_2^*\in\mathrm{Hom}_{M_0}(U_\sigma,V_2)$. It follows from (3.3) that

$$\varphi(x)=E_B(x,\Phi,\Lambda),$$

if $\Phi\in\mathcal{A}_0$ is defined by

$$\Phi(m)=S_2^*\sigma(m)S_1,\quad m\in M_0. \qquad \text{Q.E.D.}$$

Thus, $\mathcal{A}_{\mathrm{cusp}}(G,\tau)$ is a subspace of $\mathcal{A}(G,\tau)$. We have inclusions

$$\mathcal{A}_{\mathrm{cusp}}(G,\tau)\subset\mathcal{A}(G,\tau)\subset\mathcal{A}(G_-,\tau).$$

If $M \in \mathscr{L}(M_0)$ we can of course define the spaces $\mathscr{A}(M, \tau)$ and $\mathscr{A}(M_-, \tau)$. They each consist of $\mathscr{Z}_M$-finite, τ_M-spherical functions on M. We can define the Eisenstein integral of any $\varphi \in \mathscr{A}(M, \tau)$ by the familiar formula

$$E_P(x, \varphi, \lambda) = \int_{K_M \backslash K} \tau(k)^{-1} \varphi_P(kx)\, e^{(\lambda+\varrho_P)(H_P(kx))}\, dk,$$

for $P \in \mathscr{P}(M)$, $x \in G$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. Suppose that M_* is a Levi subgroup which is contained in M, and that $R \in \mathscr{P}^M(M_*)$. Then $P(R) \in \mathscr{P}(M_*)$. The Eisenstein integral has the transitivity property

$$E_P(E_R(\varphi, \lambda_*), \lambda) = E_{P(R)}(x, \varphi, \lambda_* + \lambda),$$

for $\varphi \in \mathscr{A}(M_*, \tau)$, $\lambda_* \in \mathfrak{a}^*_{M_*,\mathbf{C}}$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. If we let $M_* = M_0$, and look back at the last lemma (with G replaced by M) we see that the definition of Eisenstein integral is consistent with the definition for the subspace $\mathscr{A}_{\mathrm{cusp}}(M, \tau)$ of $\mathscr{A}(M, \tau)$. The integral formula, however, will not extend to functions $\varphi \in \mathscr{A}(M_-, \tau)$. Nevertheless, a theorem of Casselman, which we will discuss later, gives another method of extending the definition to functions $\mathscr{A}(M_-, \tau)$. It will be crucial to us.

LEMMA 7.3. *Suppose that φ is a function in $\mathscr{A}_{\mathrm{cusp}}(M, \tau)$ which equals $E_R(\Phi, \Lambda)$, for some $R \in \mathscr{P}^M(M_0)$, $\Phi \in \mathscr{A}_0$ and $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}$. Then if λ is a point in $\mathfrak{a}^*_{M,\mathbf{C}}$ in general position and P, $P' \in \mathscr{P}(M)$, we have*

$$J^\iota_{P'|P}(\lambda)\, \varphi = E_R(J^\iota_{P'(R)|P(R)}(\Lambda+\lambda)\, \Phi, \Lambda), \quad \iota = l, r.$$

Proof. In the notation of the proof of Lemma 7.2 (but with G replaced by M), we can assume that

$$\varphi(m) = \Psi^*_{S_2^*} I_R(\sigma, \Lambda, m)\, \Psi_{S_1}, \quad m \in M.$$

Now $I_R(\sigma, \Lambda)$ is of course a representation of M. By the transitivity of induction,

$$I_P(I_R(\sigma, \Lambda), \lambda) = I_{P(R)}(\sigma, \Lambda+\lambda)$$

for any $P \in \mathscr{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. Moreover, for the intertwining operators between induced representations, described in § 3, we have

$$J_{P'|P}(I_R(\sigma, \Lambda), \lambda) = J_{P'(R)|P(R)}(\sigma, \Lambda+\lambda).$$

This is an immediate consequence of the integral formula for the intertwining operators. The lemma follows from formulas (3.1) and (3.2). Q.E.D.

Chapter II

§ 1. A function of bounded support

We are now ready to begin the discussion which will eventually culminate in the Paley-Wiener theorem. The main problem will be to prove surjectivity of the Fourier transform. By means of a kind of inverse Fourier transform we will have to produce a smooth function of compact support. In this section we will verify the compactness of support. The proof of smoothness will have to wait until Chapter III. (The natural domain of the function obtained in this section will be G_-. The task of Chapter III will be to show that the function has a smooth extension to G. The support in G_- will actually only be bounded, in the sense that its closure in G is compact.)

Suppose that $B \in \mathscr{P}(M_0)$. We shall say that a point $X_B \in \mathfrak{a}_0^*$ is *sufficiently regular in* $-\mathfrak{a}_0^*(B)$ if for each $\alpha \in \Delta_B$ the number $-(\alpha, X_B)$ is sufficiently large. Recall that the function $r_B(x, \Phi, \Lambda)$, introduced in I, § 5, is meromorphic in Λ. By Lemma I.5.3 it has only a finite set of singular hyperplanes, independent of x and Φ, which meet any translate of $-\mathfrak{a}_0^*(B)$. It follows that $r_B(x, \Phi, \Lambda)$ is analytic for all Λ such that $\mathrm{Re}(\Lambda)$ is sufficiently regular in $-\mathfrak{a}_0^*(B)$. Suppose that

$$F_B \colon \mathfrak{a}_{0,\mathbf{C}}^* \to \mathscr{A}_0$$

is an entire function which is *rapidly decreasing on vertical cylinders*. By this we mean that for every pair of positive integers A and n,

$$\sup_{\{\Lambda \in \mathfrak{a}_{0,\mathbf{C}}^* : \|\mathrm{Re}(\Lambda)\| \leq A\}} (\|F_B(\Lambda)\| (1 + \|\Lambda\|)^n) < \infty .$$

Let X_B be a point in $\mathfrak{a}_0^*$ at which $r_B(x, \Phi, \Lambda)$ is analytic. Then the function is analytic on $X_B + i\mathfrak{a}_0^*$, and for any $x \in G_-$ the integral

$$\int_{X_B + i\mathfrak{a}_0^*} r_B(x, F_B(\Lambda), \Lambda)\, d\Lambda$$

converges, by Lemma I.5.3. It follows from this lemma and Cauchy's theorem that as long as X_B is sufficiently regular in $-\mathfrak{a}_0^*(B)$, the integral is independent of X_B.

Now suppose that

$$F = \{F_B(\Lambda) : B \in P(M_0)\}$$

is a collection of entire functions from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathscr{A}_0$ indexed by $\mathscr{P}(M_0)$. Assume that each function is rapidly decreasing on vertical cylinders. Define

$$F^\vee(x) = \sum_{B\in\mathscr{P}(M_0)} |\mathscr{P}(M_0)|^{-1} \int_{X_B+i\mathfrak{a}_0^*} r_B(x, F_B(\Lambda), \Lambda)\, d\Lambda, \quad x\in G_-,$$

where for each B, X_B is a point in $\mathfrak{a}_0^*$ which is sufficiently regular in $-\mathfrak{a}_0^*(B)$. (We let $|\mathscr{P}(M_0)|$ denote the number of elements in the set $\mathscr{P}(M_0)$.) Then $F^\vee$ is a smooth τ-spherical function from G_- to V_τ. It is independent of the points $\{X_B\}$.

If N is a positive number, let $G(N)$ denote the subset of G consisting of all points

$$k_1\cdot \exp H\cdot k_2, \quad k_1, k_2\in K,\ H\in\mathfrak{a}_0,$$

for which $\|H\|\leq N$.

THEOREM 1.1. *Suppose that* $F=\{F_B(\Lambda): B\in\mathscr{P}(M_0)\}$ *is a collection of entire functions from* $\mathfrak{a}^*_{0,\mathbf{C}}$ *to* $\mathscr{A}_0$. *Assume that there exists an* N *such that*

$$\|F\|_{N,n} = \sup_{\{\Lambda\in\mathfrak{a}^*_{0,\mathbf{C}},\, B\in\mathscr{P}(M_0)\}} (\|F_B(\Lambda)\|\, e^{-N\|\mathrm{Re}\,\Lambda\|}(1+\|\Lambda\|)^n)$$

is finite for every integer n. *Then the support of the function* $F^\vee$ *is contained in* $G(N)$.

Proof. Fix $B\in\mathscr{P}(M_0)$. Let H be any point in $\mathfrak{a}_0(B)$ such that $\|H\|>N$. Since $G_-=K\cdot A_0(B)\cdot K$, the theorem will be proved if we can show that the function

$$\int_{X_B+i\mathfrak{a}_0^*} r_B(\exp H, F_B(\Lambda), \Lambda)\, d\Lambda \tag{1.1}$$

vanishes. Here X_B can be any point in $\mathfrak{a}_0^*$ which is sufficiently regular in $-\mathfrak{a}_0^*(B)$. By Lemma I.5.3 there are constants c_0 and n_0 such that the norm in V_τ of the expression (1.1) is bounded by

$$c_0\, e^{(X_B-\varrho_B)(H)} \cdot \int_{X_B+i\mathfrak{a}_0^*} (1+\|\Lambda\|)^{n_0}\, \|F_B(\Lambda)\|\, d\Lambda.$$

This expression is in turn bounded by

$$c_0\, \|F\|_{N,n}\, e^{(X_B-\varrho_B)(H)}\, e^{N\|X_B\|} \cdot \int_{X_B+i\mathfrak{a}_0^*} (1+\|\Lambda\|)^{n_0-n}\, d\Lambda,$$

for any n. These inequalities are true uniformly for all points X_B sufficiently regular in $-\mathfrak{a}_0^*(B)$. We can choose n such that

$$\int_{X_B+i\mathfrak{a}_0^*} (1+\|\Lambda\|)^{n_0-n}\, d\Lambda$$

is bounded independently of X_B. Now, if $\mathfrak{a}_0$ were identified with its dual space $\mathfrak{a}_0^*$ by means of the inner product $(\,,)$ we could always choose X_B to be a large negative multiple of H. Then we would have

$$X_B(H) = -\|X_B\|\cdot\|H\|.$$

By taking this negative multiple to be large enough we could ensure both that X_B is sufficiently regular and that the norm is as large as we want. It follows that the norm of (1.1) is bounded by a constant multiple of

$$e^{\|X_B\|(N-\|H\|)}.$$

Since $\|H\|>N$, and $\|X_B\|$ can be made arbitrarily large, the expression (1.1) vanishes. Q.E.D.

§ 2. The residue scheme

The function $F^\vee(x)$ is a sum of integrals over contours $X_B+i\mathfrak{a}_0^*$. Our aim is to deform these contours to new contours, $\varepsilon_B+i\mathfrak{a}_0^*$, where ε_B is a point in $\mathfrak{a}_0^*$ which is very close to the origin. In this section, we shall set up a formal procedure for doing this. We will obtain residues of the functions $r_B(x, F(\Lambda), \Lambda)$, new functions which could reasonably be called Eisenstein systems, in analogy with Chapter 7 of [11 b]. The procedure we follow does bear some formal resemblance to that of Langlands. However, ours is much the easier, for here there are few of the serious analytic difficulties that arise in the theory of Eisenstein series. Moreover, we will eventually be able to appeal to Harish-Chandra's spectral decomposition of $L^2(G, \tau)$, whereas the main purpose of Chapter 7 of Langlands' treatise is to establish the spectral decomposition of the underlying Hilbert space.

Let us call a subspace $\mathfrak{b}$ of $\mathfrak{a}_0^*$ a *root subspace* if it is of the form $\mathfrak{a}_M^*$ for some Levi subgroup $M\in\mathscr{L}(M_0)$. Then the root subspaces of $\mathfrak{a}_0^*$ are precisely those subspaces which are intersections of hyperplanes of the form $\{\Lambda\in\mathfrak{a}_0^*: (\alpha, \Lambda)=0\}$, for a root α of (G, A_0). If $\mathfrak{b}$ is a root subspace, we shall write $\mathfrak{b}^\perp$ for the orthogonal complement, with respect to $(\,,)$, of $\mathfrak{b}$ in $\mathfrak{a}_0^*$. As always, we shall write $\mathfrak{b}_\mathbb{C}$ and $\mathfrak{b}_\mathbb{C}^\perp$ for the complexifications of $\mathfrak{b}$ and $\mathfrak{b}^{\perp}$. We shall also write $\mathrm{cham}(\mathfrak{b})$ for the set of chambers in $\mathfrak{b}$; these are the connected components of the complement in $\mathfrak{b}$ of the hyperplanes $\{\Lambda \in \mathfrak{b}: (\alpha, \Lambda)=0\}$, where now α is a root of (G, A_0) which is not orthogonal to $\mathfrak{b}$. (Let us denote the set of such roots by $\Sigma_{\mathfrak{b}}(G, A_0)$.) For each pair $(\mathfrak{b}, c)$, $c \in \mathrm{cham}(\mathfrak{b})$, there is a unique parabolic subgroup $P \in \mathscr{F}(M_0)$ such that $\mathfrak{b}=\mathfrak{a}^*_{M_P}$ and $c=\mathfrak{a}^*_{M_P}(P)$.

We will be defining meromorphic functions on the spaces $\mathfrak{b}_{\mathbf{C}}$ with possible poles along hyperplanes $(\alpha, \Lambda)=0$, α being a root which lies in $\Sigma_{\mathfrak{b}}(G, A_0)$. This will prevent us from integrating over the imaginary spaces $i\mathfrak{b}$. We are therefore forced to fix, for each root subspace $\mathfrak{b}$, a finite nonempty set $\mathscr{E}(\mathfrak{b})$ of points in $\mathfrak{b}$. We assume that each point in $\mathscr{E}(\mathfrak{b})$ is very close to the origin in $\mathfrak{b}$, but does not lie on any of the hyperplanes $(\alpha, \Lambda)=0$. Assume also that each chamber in $\mathfrak{b}$ contains an equal number of points in $\mathscr{E}(\mathfrak{b})$. Finally, suppose that the sets are such that if $\mathfrak{b}'=s\mathfrak{b}$ for some $s \in W_0$, then $\mathscr{E}(\mathfrak{b}')=s\mathscr{E}(\mathfrak{b})$. For a typical example, take $\mathscr{E}(\mathfrak{a}_0^*)$ to be the orbit under W_0 of a regular point in $\mathfrak{a}_0^*$ of very small norm. We could then define $\mathscr{E}(\mathfrak{b})$ in the following way. Given a chamber c in $\mathfrak{b}$, let c_0 be a chamber in $\mathfrak{a}_0^*$ whose closure contains c. Let ε_c be the projection onto $\mathfrak{b}$ of the unique point in $\mathscr{E}(\mathfrak{a}_0^*) \cap c_0$. It belongs to c and does not depend on c_0. Take $\mathscr{E}(\mathfrak{b})$ to be the set of points $\{\varepsilon_c: c \in \mathrm{cham}(\mathfrak{b})\}$. This example would almost suffice. We have taken a more general definition of the sets $\{\mathscr{E}(\mathfrak{b})\}$ only to accommodate a later induction argument.

In addition to the sets $\{\mathscr{E}(\mathfrak{b})\}$, our procedure will depend on a group B in $\mathscr{P}(M_0)$ and a point $X=X_B$ in $\mathfrak{a}_0^*$. We do not yet need to take X to be a sufficiently regular point in $-\mathfrak{a}_0^*(B)$, as in § 1. We will insist, however, that it be in sufficiently general position in $\mathfrak{b}$, in a sense to be made precise presently. For every root subspace $\mathfrak{b}$ we are going to define a finite collection $\mathscr{T}_B(\mathfrak{b}, X)$ of triplets $T=(\mathfrak{A}_T, X_T, r_T)$. We shall first describe the triplets and then give their definition. The first component of T will be an affine subspace

$$\mathfrak{A}_T = \Lambda_T + \mathfrak{b},$$

of $\mathfrak{a}_0^*$, the translate of $\mathfrak{b}$ by a point Λ_T in $\mathfrak{b}^{\perp}$. The second component will be a point X_T in $\mathfrak{A}_T$. The third component will be a function

$$r_T = r_T(x, \Phi, \Lambda)$$

with values in V_τ. The variable x belongs to G_-, Λ belongs to $\mathfrak{A}_{T,\mathbf{C}}=\Lambda_T+\mathfrak{b}_{\mathbf{C}}$, and Φ belongs to $\mathrm{Hom}(S(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$, the space of linear maps from the symmetric algebra on $\mathfrak{b}_{\mathbf{C}}^{\perp}$ to $\mathscr{A}_0$. This vector space is infinite dimensional. However, let $S_d(\mathfrak{b}_{\mathbf{C}}^{\perp})$ be the space of symmetric tensors of degree at most d. We will be able to choose d, independent of x and Λ, so that $r_T(x, \Phi, \Lambda)$ depends only on the projection of Φ onto the finite dimensional space $\mathrm{Hom}(S_d(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$. The function will be linear in Φ and meromorphic in Λ. Its singularities will lie along hyperplanes of the form

$$\{\Lambda \in \mathfrak{A}_T: (\alpha, \Lambda) = r\}, \quad \alpha \in \Sigma_{\mathfrak{b}}(G, A_0),\ r \in \mathbf{R}.$$

As a function of x, $r_T(x, \Phi, \Lambda)$ will belong to $\mathscr{A}(G_-, \tau)$.

Our definition will be one of decreasing induction on $\dim \mathfrak{b}$. We take $\mathscr{T}_B(\mathfrak{a}_0^*, X)$ to consist of the one triplet,

$$(\mathfrak{a}_0^*, X, r_B(x, \Phi, \Lambda)), \quad x \in G_-,\ \Phi \in \mathscr{A}_0, \Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*.$$

In general, let us write $\mathscr{T}_B(k, X)$ for the union over all spaces $\mathfrak{b}$ of dimension k of the sets $\mathscr{T}_B(\mathfrak{b}, X)$. Assume inductively that $\mathscr{T}_B(k+1, X)$ has been defined, and that each function

$$r_{T_1}(x, \Phi_1, \Lambda_1), \quad T_1 \in \mathscr{T}_B(k+1, X),$$

has the properties described above. Then $\mathscr{T}_B(k, X)$ is defined to be the disjoint union over all root spaces $\mathfrak{b}_1$ of dimension $k+1$, over all triplets $T_1 \in \mathscr{T}_B(\mathfrak{b}_1, X)$ and over all points ε in $\mathscr{E}(\mathfrak{b}_1)$ of certain sets. The set indexed by $\mathfrak{b}_1$, T_1 and ε will be the collection of all triplets $T = (\mathfrak{A}_T, X_T, r_T)$, in which $\mathfrak{A}_T$ ranges over the singular hyperplanes of

$$r_{T_1}(x, \Phi_1, \Lambda_1), \quad \Lambda_1 \in \mathfrak{b}_{1,\mathbf{C}},$$

which meet the line segment joining X_{T_1} and $\Lambda_{T_1} + \varepsilon$, X_T is the intersection of $\mathfrak{A}_T$ with this line segment, and r_T is the residue of $-|\mathscr{E}(\mathfrak{b}_1)|^{-1} r_{T_1}$ along $\mathfrak{A}_T$ at X_T. More precisely, suppose the singular hyperplane $\mathfrak{A}_T$ equals $\Lambda_T + \mathfrak{b}$, for a root subspace $\mathfrak{b}$ of dimension k. Let ν be the real unit vector in $\mathfrak{b}_1$, orthogonal to $\mathfrak{b}$, whose inner product with the vector $\Lambda_{T_1} + \varepsilon - X_{T_1}$ is positive. It defines a basis of the one dimensional complex vector space $\mathfrak{b}_{\mathbf{C}}^{\perp} / \mathfrak{b}_{1,\mathbf{C}}^{\perp}$, and allows us to identify any vector Φ in $\mathrm{Hom}(S(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$ with a formal power series in one variable,

$$\sum_{n=0}^{\infty} \Phi_n z^n,$$

with coefficients Φ_n in $\mathrm{Hom}(S(\mathfrak{b}_{1,\mathbf{C}}^{\perp}), \mathscr{A}_0)$. If Λ is a point in general position in $\mathfrak{A}_{T,\mathbf{C}}$, $r_T(x, \Phi, \Lambda)$ is defined to equal

$$-|\mathscr{E}(\mathfrak{b}_1)|^{-1}(2\pi i)^{-1}\sum_{n=0}^{\infty}\int_{\Gamma} z^n r_{T_1}(x, \Phi_n, \Lambda+\nu z)\,dz, \tag{2.1}$$

where Γ is a small positively oriented circle about the origin in the complex plane. It is clear that the series is actually finite. It is also clear that the new functions $r_T(x, \Phi, \Lambda)$ possess the properties described above.

The points in each set $\mathscr{E}(\mathfrak{b})$ were assumed to be regular, and very close to the origin. In view of the nature of their singularities, the functions

$$r_T(x, \Phi, \Lambda), \quad T \in \mathscr{T}_B(\mathfrak{b}, X),$$

are all regular at each point in $\mathscr{E}(\mathfrak{b})$. Notice that a small perturbation of X will induce corresponding small perturbations in each of the points X_T. The precise property that we will require of the general position of X is that each of the functions $r_T(x, \Phi, \Lambda)$ be regular at $\Lambda = X_T$, and that all the singular hyperplanes which meet the line segments joining X_T and $\Lambda_T + \varepsilon$, $\varepsilon \in \mathscr{E}(\mathfrak{b})$, do so at distinct points.

For any $T \in \mathscr{T}_B(\mathfrak{b}, X)$ there is an integer d such that the function

$$r_T(\Phi, \Lambda): x \to r_T(x, \Phi, \Lambda), \quad x \in G_-,$$

belongs to $\mathscr{A}_{o(\tau,\Lambda),d}(G_-, \tau)$. Notice also that

$$r_T(xa, \Phi, \Lambda) = r_T(x, \Phi, \Lambda)\, a^{\Lambda}, \tag{2.2}$$

for any a in A_G, the split component of the center of G. We can give estimates for these functions. With the aid of the positive definite form $(\,,\,)$, we can define the norm, $\|\Phi\|_d$, of the projection of any vector $\Phi \in \mathrm{Hom}(S(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$ onto $\mathrm{Hom}(S_d(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$. Then we have

LEMMA 2.1. *Suppose that C is a compact subset of G_-, and C^* is a compact subset of $\mathfrak{A}_T$. Then we can find a polynomial $l(\Lambda)$, which is a product of linear factors*

$$(\alpha, \Lambda) - r, \quad \alpha \in \Sigma_{\mathfrak{b}}(G, A_0),\ r \in \mathbf{R},$$

and constants c and n, such that

$$\|l(\Lambda)\, r_T(x, \Phi, \Lambda)\| \leq c(1 + \|\Lambda\|)^n \cdot \|\Phi\|_d,$$

for all $\Phi \in \mathrm{Hom}(S(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathscr{A}_0)$, $x \in C$ and $\Lambda \in C^* + i\mathfrak{b}$.

Proof. The lemma is a consequence of Lemma I.5.3, the remark following Corollary I.6.3, and the inductive definition of $r_T(x, \Phi, \Lambda)$. Q.E.D.

§ 3. The functions $F_{\mathcal{P}}^{\vee}$

The notion of residue we have adopted was copied from [11 b]. At first glance it might seem odd, but it is designed to accommodate the residues of functions $r_B(x, F_B(\Lambda), \Lambda)$ where, as in § 1,

$$F_B: \mathfrak{a}_{0,\mathbf{C}}^* \to \mathcal{A}_0$$

is an entire function which is rapidly decreasing on vertical cylinders. Given $T \in \mathcal{T}_B(\mathfrak{b}, X)$ and $\Lambda \in \mathfrak{A}_{T,\mathbf{C}}$, we can expand the analytic function

$$\eta \to F_B(\eta+\Lambda), \quad \eta \in \mathfrak{b}_{\mathbf{C}}^{\perp},$$

on $\mathfrak{b}_{\mathbf{C}}^{\perp}$ as a Taylor series about $\eta=0$. This results in a vector in $\mathrm{Hom}(S(\mathfrak{b}_{\mathbf{C}}^{\perp}), \mathcal{A}_0)$, which we denote by $(d_T F_B)(\Lambda)$, or simply by $dF_B(\Lambda)$. Suppose that as in the inductive definition of § 2, T is obtained from a singular hyperplane of the function $r_{T_1}(x, \Phi_1, \Lambda_1)$ which meets the line segment joining X_{T_1} and $\Lambda_{T_1}+\varepsilon$. Then if $\Phi=(d_T F_B)(\Lambda)$, the residue (2.1) is just

$$-|\mathcal{E}(\mathfrak{b}_1)|^{-1}(2\pi i)^{-1}\int_{\Gamma} r_{T_1}(x, (d_{T_1} F_B)(\Lambda+\nu z), \Lambda+\nu z)\, dz.$$

Any derivative of $F_B(\Lambda)$ can be estimated by the Cauchy integral formula in terms of $F_B(\Lambda)$ itself. It follows from Lemma 2.1 that for a given $T \in \mathcal{T}_B(\mathfrak{b}, X)$ and a positive integer N, there exists a constant c_N such that

$$\|l(\Lambda)\, r_T(x, dF_B(\Lambda), \Lambda)\| \leq c_N(1+\|\Lambda\|)^{-N}, \tag{3.1}$$

for any $x \in C$ and $\Lambda \in C^*+i\mathfrak{b}$. Here C, C^* and $l(\Lambda)$ are as in Lemma 2.1. We can assume that $l(\Lambda)$ vanishes only on the singular hyperplanes of r_T which meet C^*. In particular, if T_1 is as above, and $\varepsilon \in \mathcal{E}(\mathfrak{b}_1)$, the integrals

$$\int_{X_{T_1}+i\mathfrak{b}_1} r_{T_1}(x, (d_{T_1} F_B)(\Lambda_1), \Lambda_1)\, d\Lambda_1$$

and

$$\int_{\Lambda_{T_1}+\varepsilon+i\mathfrak{b}_1} r_{T_1}(x,(d_{T_1}F_B)(\Lambda_1),\Lambda_1)\,d\Lambda_1$$

both converge. Their difference, when divided by $|\mathscr{E}(\mathfrak{b}_1)|$, is a sum of integrals

$$\int_{X_T+i\mathfrak{b}} r_T(x,(d_TF_B)(\Lambda),\Lambda)\,d\Lambda, \tag{3.2}$$

the sum ranging over the $T\in\mathscr{T}_B(k,X)$ indexed as in § 2 by $\mathfrak{b}_1$, T_1 and ε.

Now suppose that

$$F=\{F_B(\Lambda): B\in\mathscr{P}(M_0)\}$$

is a collection of entire functions, each rapidly decreasing in vertical cylinders. For each B, choose a point X_B to be both suitably regular in $-\mathfrak{a}_0^*(B)$ as in § 1 and in general position as in § 2. We have defined the function

$$F^\vee(x)=|\mathscr{P}(M_0)|^{-1}\sum_{B\in\mathscr{P}(M_0)}\int_{X_B+i\mathfrak{a}_0^*} r_B(x,F_B(\Lambda),\Lambda)\,d\Lambda,\quad x\in G_-.$$

In this expression, and in the subsequent residues as well, we shall move the contours of integration. We will be left with a profusion of integrals over contours $\Lambda_T+\varepsilon+i\mathfrak{b}$. Suppose that $\mathscr{P}$ is a class of associated parabolic subgroups in $\mathscr{F}(M_0)$. Write $\operatorname{prk}(\mathscr{P})$ for the dimension of A_P, P being any group in $\mathscr{P}$. Let $\operatorname{rt}(\mathscr{P})$ be the set of root spaces $\mathfrak{b}$ such that $\mathfrak{b}=\mathfrak{a}_{M_P}^*$ for some group $P\in\mathscr{P}$. For any $x\in G_-$, define $F_{\mathscr{P}}^\vee(x)$ to equal

$$|\mathscr{P}(M_0)|^{-1}\sum_{B\in\mathscr{P}(M_0)}\ \sum_{\mathfrak{b}\in\operatorname{rt}(\mathscr{P})}\ \sum_{T\in\mathscr{T}_B(\mathfrak{b},X_B)}|\mathscr{E}(\mathfrak{b})|^{-1}\sum_{\varepsilon\in\mathscr{E}(\mathfrak{b})}\int_{\Lambda_T+\varepsilon+i\mathfrak{b}} r_T(x,dF_B(\Lambda),\Lambda)\,d\Lambda.$$

Then $F_{\mathscr{P}}^\vee$ is a τ-spherical function from G_- to V_τ.

LEMMA 3.1. *For any integer n, $F^\vee(x)$ equals the sum of*

$$\sum_{\{\mathscr{P}:\,\operatorname{prk}\mathscr{P}>n\}} F_{\mathscr{P}}^\vee(x) \tag{3.3}$$

and

$$|\mathscr{P}(M_0)|^{-1} \sum_B \sum_{\{\mathfrak{b}:\,\dim \mathfrak{b}=n\}} \sum_{T\in \mathscr{T}_B(\mathfrak{b}, X_B)} \int_{X_T+i\mathfrak{b}} r_T(x, dF_B(\Lambda), \Lambda)\, d\Lambda. \tag{3.4}$$

(In the summations, $\mathscr{P}$ stands for classes of associated parabolic subgroups, and $\mathfrak{b}$ *stand for root spaces.)*

Proof. The lemma is established by decreasing induction on n. If $n=\dim \mathfrak{a}_0^*$, the first of the two given terms vanishes, while the second one, (3.4), is by definition equal to $F^\vee(x)$. Suppose then that $n<\dim \mathfrak{a}_0^*$ and that the lemma holds with n replaced by $(n+1)$. In expression (3.4) (with n replaced by $(n+1)$), decompose the integral into $|\mathscr{E}(\mathfrak{b})|$ equal parts, one for each point ε in $\mathscr{E}(\mathfrak{b})$. In each of these, change the contour of integration from $X_T+i\mathfrak{b}$ to $\Lambda_T+\varepsilon+i\mathfrak{b}$. The contribution to (3.3) from integrals taken over the new contours equals

$$\sum_{\{\mathscr{P}:\,\mathrm{prk}(\mathscr{P})=n+1\}} F_{\mathscr{P}}^\vee(x).$$

The residues, terms of the form (3.2), just add up to the expression (3.4) (with the original n). The lemma is proved. Q.E.D.

If we take $n<\dim \mathfrak{a}_G^*$, the expression (3.4) vanishes. We obtain

COROLLARY 3.2. $$F^\vee(x) = \sum_{\mathscr{P}} F_{\mathscr{P}}^\vee(x).$$

It follows from (2.2), and the classical Paley-Wiener theorem applied to A_G, that for any $\mathscr{P}$ and $x\in G_-$, the function

$$a \to F_{\mathscr{P}}^\vee(xa), \quad a\in A_G,$$

is of compact support. Define

$$F_{\mathrm{cusp}}(\lambda, x) = \int_{A_G} F_{\{G\}}^\vee(xa)\, e^{-\lambda(H_G(xa))}\, da, \quad \lambda\in \mathfrak{a}_{G,\mathbf{C}}^*.$$

In the formula

$$|\mathscr{P}(M_0)|^{-1} \sum_B \sum_{T\in \mathscr{T}_B(\mathfrak{a}_G^*, X_B)} |\mathscr{E}(\mathfrak{a}_G^*)|^{-1} \sum_{\varepsilon\in \mathscr{E}(\mathfrak{a}_G^*)} \int_{\Lambda_T+\varepsilon+i\mathfrak{a}_G^*} r_T(xa, dF_B(\Lambda), \Lambda)\, d\Lambda$$

for $F^\vee_{\{G\}}(xa)$, we can replace each contour $\Lambda_T+\varepsilon+i\mathfrak{a}_G^*$ by $\Lambda_T+i\mathfrak{a}_G^*$. We obtain

$$F^\vee_{\{G\}}(xa) = |\mathscr{P}(M_0)|^{-1} \sum_B \sum_T \int_{\Lambda_T+i\mathfrak{a}_G^*} r_T(xa, dF_B(\Lambda), \Lambda)\, d\Lambda.$$

It follows from the Fourier inversion formula on A_G that

$$F_{\mathrm{cusp}}(\lambda, x) = |\mathscr{P}(M_0)|^{-1} \sum_B \sum_{T\in \mathscr{T}_B(\mathfrak{a}_G^*, X_B)} r_T(x, dF_B(\Lambda_T+\lambda), \Lambda_T).$$

In particular, the function

$$F_{\mathrm{cusp}}(\lambda)\colon x \to F_{\mathrm{cusp}}(\lambda, x), \quad x\in G_-,$$

belongs to $\mathscr{A}(G_-,\tau)$. It is also invariant under A_G.

The notation, however, is only a promise of things to come; at the moment, $F_{\mathrm{cusp}}(\lambda)$ does not extend to a smooth function on G_-, so it does not belong to $\mathscr{A}_{\mathrm{cusp}}(G,\tau)$.

All the discussion so far in Chapter II can be applied to any Levi subgroup M in $\mathscr{L}(M_0)$. A root space, $\mathfrak{b}$, for M is just a root space for G which contains $\mathfrak{a}_M^*$. We have already fixed the sets $\mathscr{C}(\mathfrak{b})$ of points in $\mathfrak{b}$, so for $R\in\mathscr{P}^M(M_0)$ we can define the collection $\mathscr{T}_R(\mathfrak{b}, X)$ of triplets $T=(\mathfrak{A}_T, X_T, r_T(m,\Phi,\Lambda))$ exactly as in § 2, but associated to M. Suppose that $P\in\mathscr{F}(M_0)$. Given the collection $F=\{F_B(\Lambda)\colon B\in\mathscr{P}(M_0)\}$, let

$$F_P = \{F_R(\Lambda) = F_{P(R)}(\Lambda)\colon R\in\mathscr{P}^{M_P}(M_0)\},$$

the subset of F indexed only by those $B\in\mathscr{P}(M_0)$ such that $B\subset P$. We can then define the function $F_P^\vee(m)$ on M_- as in § 1. Similarly, we can take the subset $\{X_R=X_{P(R)}\colon R\in\mathscr{P}^{M_P}(M_0)\}$ of $\{X_B\colon B\in\mathscr{P}(M_0)\}$. For each associated class $\mathscr{R}$ of parabolic subgroups of M_P we have the functions $F^\vee_{P,\mathscr{R}}$. We also have the functions

$$F_{P,\mathrm{cusp}}(\lambda), \quad \lambda\in i\mathfrak{a}_{M_P}^*,$$

on $\mathscr{A}(M_-,\tau)$.

§ 4. The theorem of Casselman

Suppose that M is a Levi subgroup in $\mathscr{L}(M_0)$. The Eisenstein integral provides a natural lifting of functions in $\mathscr{A}(M,\tau)$ to functions in $\mathscr{A}(G,\tau)$. A recent theorem of Casselman generalizes this lifting to one between the spaces $\mathscr{A}(M_-,\tau)$ and $\mathscr{A}(G_-,\tau)$.

Let s be a coset in W_0/W_0^M and let P be a group in $\mathscr{P}(M)$. Choose any group $B\in\mathscr{P}(M_0)$ which is contained in P and let $R=B\cap M$. Then $B=P(R)$. Let s_B be the unique representative of s in W_0 such that $s_B(\alpha)$ is a root of (B,A_0) for every root α of (R,A_0).

THEOREM 4.1 (Casselman). (i) *Suppose that $\Phi(\Lambda)\in\mathscr{M}(\mathscr{A}_0)$ is such that the function*

$$\Lambda\to E_{R|R,1}(m,\Phi(\Lambda),\Lambda),\quad \Lambda\in\mathfrak{a}_{0,\mathbf{C}}^*,\ m\in M_-,$$

is regular at $\Lambda=\Lambda_0$. Then if λ is a point in general position in $\mathfrak{a}_{M,\mathbf{C}}^*$ the function

$$\Lambda\to E_{B|B,s_B}(x,\Phi(\Lambda),\Lambda+\lambda),\quad \Lambda\in\mathfrak{a}_{0,\mathbf{C}}^*,\ x\in G_-,$$

is also regular at $\Lambda=\Lambda_0$.

(ii) *If $D=D_\Lambda$ is any differential operator on $\mathfrak{a}_{0,\mathbf{C}}^*$,*

$$E_{P,B,s}(x,\varphi,\lambda)=\lim_{\Lambda\to\Lambda_0}D_\Lambda E_{B|B,s_B}(x,\Phi(\Lambda),\Lambda+\lambda)$$

depends, as the notation suggests, only on the function

$$\varphi(m)=\lim_{\Lambda\to\Lambda_0}D_\Lambda E_{R|R,1}(m,\Phi(\Lambda),\Lambda),\tag{4.1}$$

and not on its realization in terms of $\Phi(\Lambda)$.

(iii) *The map*

$$\varphi\to E_{P,B,s}(x,\varphi,\lambda)$$

extends to an injective linear map from $\mathscr{A}(M_-,\tau)$ to the space of meromorphic functions of λ with values in $\mathscr{A}(G_-,\tau)$.

The usual treatment of asymptotic expansions of Eisenstein integrals via recursion formulas ([13]) does not seem to lead to the theorem. In fact, from this point of view the theorem is quite surprising. Casselman actually finds a new formula for the asymptotic expansion of the Eisenstein integrals

$$E_B(a,\Phi,\Lambda+\lambda),\quad a\in A_0(B),\ \Phi\in\mathscr{A}_0,$$

in terms of the asymptotic expansion of $E_R(a,\Phi,\Lambda)$. The leading terms of his formula are just the analytic continuations of the Knapp-Stein intertwining integrals. The other terms are obtained from more general integrals. If Λ is allowed to approach Λ_0, the formula for $E_B(a, \Phi(\Lambda), \Lambda+\lambda)$ makes it possible to express the function $E_{P,B,s}(x, \varphi, \lambda)$ in terms only of the function $\varphi(m)$, and not its realization (4.1). (See [2a], [2b].)

Suppose that $\mathfrak{o}$ is a finite union of W^M-orbits in $\mathfrak{h}_{\mathbf{C}}^*$. If $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$, let $\mathfrak{o}_G(\mathfrak{o}, \lambda)$ be the set of orbits under W of the points $\nu+\lambda$, where ν is a point in $\mathfrak{o}$. It is a finite union of W-orbits in $\mathfrak{h}_{\mathbf{C}}^*$. If φ belongs to $\mathscr{A}_{\mathfrak{o},d}(M_-, \tau)$ then the function

$$E_{P,B,s}(\varphi, \lambda): x \to E_{P,B,s}(x, \varphi, \lambda), \quad x \in G_-,$$

belongs to $\mathscr{A}_{\mathfrak{o}_G(\mathfrak{o},\lambda),d}(G_-, \tau)$.

Suppose that t is an element in W_0. Retaining the notation of the theorem, we set $M_1 = tM$. Then tst^{-1} is a coset in $W_0/W_0^{M_1}$. We shall show that

$$E_{tP,tB,tst^{-1}}(t\varphi, t\lambda) = E_{P,B,s}(\varphi, \lambda). \tag{4.2}$$

For if φ is given by (4.1), $t\varphi$ equals

$$\lim_{\Lambda \to \Lambda_0} D_\Lambda(tE_{R|R,1}(\Phi(\Lambda), \Lambda)) = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{tR|tR,1}(t\Phi(\Lambda), t\Lambda),$$

in view of (I.4.4′) and (I.4.5′). (See also (I.2.1′).) Now,

$$(tst^{-1})_{tB} = ts_B t^{-1},$$

so that

$$\begin{aligned} E_{tP,tB,tst^{-1}}(t\varphi, t\lambda) &= \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{tB|tB,ts_Bt^{-1}}(t\Phi(\Lambda), t\Lambda + t\lambda) \\ &= \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{B|B,s_B}(\Phi(\Lambda), \Lambda+\lambda) = E_{P,B,s}(\varphi, \lambda), \end{aligned}$$

again by (I.4.4′) and (I.4.5′). This establishes (4.2). Consider the special case that t is an element of W_0^M. Then $t\varphi = \varphi$, $t\lambda = \lambda$, and the coset tst^{-1} equals ts. We have

$$E_{P,B,s}(\varphi, \lambda) = E_{P,tB,ts}(\varphi, \lambda). \tag{4.3}$$

Thus, the map $E_{P,B,s}$ does actually depend on the minimal parabolic subgroup B, and not just on P.

We have agreed that $\mathscr{A}(M, \tau)$ is a subspace of $\mathscr{A}(M_-, \tau)$, so $E_{P,B,s}(\varphi, \lambda)$ is defined for $\varphi \in \mathscr{A}(M, \tau)$. In this case, the Eisenstein integral $E_P(\varphi, \lambda)$ is also defined. They are related by the next lemma.

LEMMA 4.2. *If* $\varphi \in \mathcal{A}(M, \tau)$ *we have*

$$E_P(\varphi, \lambda) = \sum_{s \in W_0/W_0^M} E_{P,B,s}(\varphi, \lambda),$$

for any group $B \in \mathcal{P}(M_0)$ *with* $B \subset P$.

Proof. Let $R = M \cap B$. We can assume that

$$\begin{aligned}\varphi(m) &= \lim_{\Lambda \to \Lambda_0} D_\Lambda E_R(m, \Phi(\Lambda), \Lambda)\\ &= \lim_{\Lambda \to \Lambda_0} D_\Lambda \left(\sum_{r \in W_0^M} E_{R|R,r}(m, \Phi(\Lambda), \Lambda) \right).\end{aligned}$$

In the notation used in the discussion prior to Lemma I.7.1 (but with G replaced by M), this equals

$$\sum_{r \in W_0^M} \lim_{\Lambda \to \Lambda_0} (D_N \tilde{D})_\Lambda E_{R|R,1}(m, l_{N-d}(\Lambda)\, {}^0c_{B|B}(r, \Lambda)\, \Phi(\Lambda), r\Lambda).$$

Then

$$\sum_{s \in W_0/W_0^M} E_{P,B,s}(x, \varphi, \lambda)$$

equals

$$\begin{aligned}&\sum_s \sum_r \lim_{\Lambda \to \Lambda_0} (D_N \tilde{D})_\Lambda E_{B|B,s_B}(x, l_{N-d}(\Lambda)\, {}^0c_{B|B}(r, \Lambda)\, \Phi(\Lambda), r\Lambda + \lambda)\\ &\qquad = \lim_{\Lambda \to \Lambda_0} (D_N \tilde{D})_\Lambda \left(\sum_{s,r} E_{B|B,s_Br}(x, l_{N-d}(\Lambda)\, \Phi(\Lambda), \Lambda + \lambda) \right),\end{aligned}$$

by (I.4.6′). This is just

$$\begin{aligned}&\lim_{\Lambda \to \Lambda_0} ((D_N)_\Lambda \circ l_N(\Lambda) \circ D_\Lambda) E_B(x, \Phi(\Lambda), \Lambda + \lambda)\\ &\quad = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_B(x, \Phi(\Lambda), \Lambda + \lambda)\\ &\quad = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_P(x, E_R(\Phi(\Lambda), \Lambda), \lambda)\\ &\quad = E_P(x, \varphi, \lambda).\end{aligned}$$

The lemma is proved. Q.E.D.

Motivated by the lemma, we define for each $\varphi \in \mathscr{A}(M_-, \tau)$,

$$E_P(x, \varphi, \lambda) = \sum_{s \in W_0/W_0^M} E_{P,B,s}(x, \varphi, \lambda),$$

for any group $B \in \mathscr{P}(M_0)$ with $B \subset P$. It is independent of B. For any other group $B_1 \in \mathscr{P}(M_0)$, with $B_1 \subset P$, will equal tB for some $t \in W_0^M$. By (4.3),

$$\sum_{s \in W_0/W_0^M} E_{P,B_1,s}(x, \varphi, \lambda) = \sum_{s \in W_0/W_0^M} E_{P,B,t^{-1}s}(x, \varphi, \lambda).$$

A change of variable in the sum over s shows that this is just $E_P(x, \varphi, \lambda)$. Thus, the Eisenstein integral can be extended to a map on $\mathscr{A}(M_-, \tau)$. However, if $\varphi \in \mathscr{A}(M_-, \tau)$, $E_P(x, \varphi, \lambda)$ will in general have poles in λ, unlike the Eisenstein integral. They lie along hyperplanes $(\alpha, \lambda) = c$, where α is a root of (G, A_M) and $c \in \mathbf{C}$.

§ 5. Some definitions

In this section we shall unravel some interesting consequences of the theorem of Casselman. It turns out that all the maps defined in I, § 2 on $\mathscr{A}_{\mathrm{cusp}}(M, \tau)$ can be extended to the space $\mathscr{A}(M_-, \tau)$. Fix groups P and P' in $\mathscr{P}(M)$. The first step is to show that the operators

$$J^{\iota}_{P'|P}(\lambda)\colon \mathscr{A}_{\mathrm{cusp}}(M, \tau) \to \mathscr{A}_{\mathrm{cusp}}(M, \tau), \quad \iota = l, r,$$

can be extended to the space $\mathscr{A}(M_-, \tau)$.

THEOREM 5.1. (i) *Suppose that $\Phi(\Lambda) \in \mathscr{M}(\mathscr{A}_0)$ and $R \in \mathscr{P}^M(M_0)$ are such that the function*

$$\Lambda \to E_{R|R,1}(m, \Phi(\Lambda), \Lambda), \quad \Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}, \ m \in M_-,$$

is regular at $\Lambda = \Lambda_0$. Then if λ is a point in general position in $\mathfrak{a}^*_{M,\mathbf{C}}$, and $\iota = l$ or r, the function

$$\Lambda \to E_{R|R,1}(m, J^{\iota}_{P'(R)|P(R)}(\Lambda + \lambda)\, \Phi(\Lambda), \Lambda), \quad \Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}, \ m \in M_-,$$

is also regular at $\Lambda = \Lambda_0$.

(ii) *If $D = D_\Lambda$ is any differential operator on $\mathfrak{a}^*_{0,\mathbf{C}}$,*

$$(J^{\iota}_{P'|P}(\lambda)\, \varphi)(m) = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{R|R,1}(m, J^{\iota}_{P'(R)|P(R)}(\Lambda + \lambda)\, \Phi(\Lambda), \Lambda) \tag{5.1}$$

depends, as the notation suggests, only on the function

$$\varphi(m) = \lim_{\Lambda \to \Lambda_0} D_\Lambda E_{R|R,1}(m, \Phi(\Lambda), \Lambda) \tag{5.2}$$

and not its realization in terms of $\Phi(\Lambda)$.

(iii) *The map*

$$\varphi \to (J^{\iota}_{P'|P}(\lambda)\,\varphi)(m)$$

extends to an injective linear map from $\mathscr{A}(M_-, \tau)$ *to the space of meromorphic functions of* λ *with values in* $\mathscr{A}(M_-, \tau)$.

Proof. We shall prove the theorem first in the special case that $P' = \bar{P}$. Define another group $Q \in \mathscr{P}(M)$ by

$$Q = \begin{cases} \bar{P} & \text{if} \quad \iota = l \\ P & \text{if} \quad \iota = r. \end{cases}$$

Then the function

$$E_{R|R,1}(m, J^{\iota}_{\bar{P}(R)|P(R)}(\Lambda+\lambda)\,\Phi(\Lambda), \Lambda)$$

is the product of $e^{-\lambda(H_M(m))}$ with the value at m of

$$E_{R|R,1}(J^{l}_{Q(R)|P(R)}(\Lambda+\lambda)\,J^{r}_{\bar{Q}(R)|P(R)}(\Lambda+\lambda)\,\Phi(\Lambda), \Lambda+\lambda). \tag{5.3}$$

Let $B = P(R)$. There is a unique coset s in W_0/W_0^M such that the group $P_1 = sQ$ contains B. Then if $\Lambda+\lambda$ is in general position, (5.3) equals

$$s^{-1}E^{P_1}(E_{B|B,s_B}(\Phi(\Lambda), \Lambda+\lambda)) \tag{5.4}$$

by Lemma I.4.2. Theorem II.4.1 tells us that for λ in general position, $E_{B|B,s_B}(\Phi(\Lambda), \Lambda+\lambda)$ is regular at $\Lambda = \Lambda_0$. By Lemma I.6.1 each of the terms in the asymptotic expansion of

$$E_{B|B,s_B}(a, \Phi(\Lambda), \Lambda+\lambda), \quad a \in A_0(B),$$

is regular at $\Lambda = \Lambda_0$. It follows from the definition of E^{P_1} that the function (5.4) is regular at $\Lambda = \Lambda_0$. This proves the first statement of the theorem in the special case under consideration.

Suppose that λ_0 is a linear function on $\mathfrak{a}_{M,\mathbf{C}}$. Let $\mathscr{A}(M_-, \tau, \lambda_0)$ be the subspace of

$\mathscr{A}(M_-,\tau)$ generated by functions φ of the form (5.2), for which the restriction of Λ_0 to $\mathfrak{a}_M$ is λ_0. Define a map $J^{\iota}_{\tilde{P}|P}(\lambda)$ from $\mathscr{A}(M_-,\tau,\lambda_0)$ to itself by setting

$$J^{\iota}_{\tilde{P}|P}(\lambda)\,\psi = s^{-1}E^{P_1}(E_{P,B,s}(\psi,\lambda))\cdot e^{-\lambda(H_M(\cdot))},$$

for $\psi\in\mathscr{A}(M_-,\tau,\lambda_0)$. From Theorem II.4.1 and the definition of E^{P_1} it follows that the map is well defined and injective. We would like to show that it is linear. This is not trivial, for although $E_{P,B,s}(\psi,\lambda)$ is linear in ψ, the map E^{P_1} is in general *not* linear. Suppose then that

$$\psi=\sum_{i=1}^{n}\psi_i,$$

where for each i,

$$\psi_i(m)=\lim_{\Lambda\to\Lambda_i}(D_i)_\Lambda\,E_{R|R,1}(m,\Phi_i(\Lambda),\Lambda)$$

is a function in $\mathscr{A}(M_-,\tau,\lambda_0)$. Let $M_1=w_sM\,w_s^{-1}$. Then $P_1\in\mathscr{P}(M_1)$. We shall show that for λ in general position, the restriction of any principal exponent of $E_{P,B,s}(\psi,\lambda)$ to $\mathfrak{a}_{M_1}$ equals $s(\lambda_0+\lambda)-\varrho_{P_1}$. The case that $n=1$ will of course be included, so the principal exponents of $E_{P,B,s}(\psi_i,\lambda)$ will also all restrict to $s(\lambda_0+\lambda)-\varrho_{P_1}$ on $\mathfrak{a}_{M_1}$. With these properties, the formula

$$E^{P_1}(E_{P,B,s}(\psi,\lambda))=\sum_{i=1}^{n}E^{P_1}(E_{P,B,s}(\psi_i,\lambda))$$

will be an immediate consequence of the definition of E^{P_1}.

Any principal exponent of $E_{P,B,s}(\varphi,\lambda)$ will be of the form

$$\mu=s_B(\Lambda_i+\lambda)-\zeta-\varrho_B,\quad 1\leq i\leq n,\ \zeta\in\mathbf{Z}^+(\Delta_B).$$

By Lemma I.4.1 the set $\mathfrak{o}_G(\tau,\mu+\varrho_B)$ intersects $\mathfrak{o}_G(E_{P,B,s}(\psi,\lambda))$. However, according to the remark following Lemma I.6.1, $\mathfrak{o}_G(E_{P,B,s}(\psi,\lambda))$ is contained in the union over j of the sets $\mathfrak{o}_G(\tau,\Lambda_j+\lambda)$. It follows that there are elements η_i, $\eta_j\in\mathfrak{h}_{\mathfrak{k}}^*$ and $t\in W$ such that

$$\mu+\varrho_B+i\eta_i=s_B(\Lambda_i+\lambda)-\zeta+i\eta_i=t(\Lambda_j+\lambda+i\eta_j).$$

Since λ is a point in general position in $\mathfrak{a}^*_{M,\mathbf{C}}$, $t^{-1}s_B$ must leave $\mathfrak{a}^*_M$ pointwise fixed. Now suppose that $H_1 \in \mathfrak{a}_{M_1}$. Then $H_1 = tH = sH$ for a unique point $H \in \mathfrak{a}_M$. We have

$$\begin{aligned}\mu(H_1) &= (t(\Lambda_j+\lambda+i\eta_j))(H_1)-(\varrho_B+i\eta_i)(H_1)\\ &= (\Lambda_j+\lambda+i\eta_j)(H)-\varrho_{P_1}(H_1)\\ &= (\lambda_0+\lambda)(H)-\varrho_{P_1}(H_1)\\ &= (s(\lambda_0+\lambda)-\varrho_{P_1})(H_1).\end{aligned}$$

Therefore the restriction of μ to $\mathfrak{a}_{M_1}$ does equal $s(\lambda_0+\lambda)-\varrho_{P_1}$. It follows that $J^{\iota}_{\bar{P}|P}(\lambda)$ is a linear operator on $\mathscr{A}(M_-, \tau, \lambda_0)$. It clearly extends to a linear operator on

$$\mathscr{A}(M_-, \tau) = \bigoplus_{\lambda_0} \mathscr{A}(M_-, \tau, \lambda_0).$$

Finally, suppose that φ is the function (5.2). We must verify the relation (5.1). By the discussion above, (5.1) is equivalent to the formula

$$E^{P_1}(E_{P,B,s}(\varphi, \lambda)) = \lim_{\Lambda\to\Lambda_0} D_\Lambda E^{P_1}(E_{B|B,s_B}(\Phi(\Lambda), \Lambda+\lambda)). \tag{5.5}$$

If $a \in A_0(B)$, $E_{P,B,s}(a, \varphi, \lambda)$ equals

$$\sum_{\zeta\in \mathbf{Z}^+(\Delta_B)} \lim_{\Lambda\to\Lambda_0} D_\Lambda(c_{B|B,\zeta}(s_B, \Lambda+\lambda)\,\Phi(\Lambda))(1)\cdot a^{(s_B(\Lambda+\lambda)-\zeta-\varrho_B)},$$

by Theorem II.4.1 and Lemma I.6.1. Then (5.5) follows from the fact, just proved, that for λ in general position the principal exponents of $E_{P,B,s}(\varphi, \lambda)$ have the same restrictions to $\mathfrak{a}_{M_1}$ as the function $s_B(\Lambda_0+\lambda)-\varrho_{P_1}$. This completes the proof of the theorem when $P' = \bar{P}$.

The proof of the theorem for general P' can be deduced from this special case. For suppose that P'' is a third group in $\mathscr{P}(M)$ such that

$$d(P'', P) = d(P'', P') + d(P', P).$$

Then

$$d(P''(R), P(R)) = d(P''(R), P'(R)) + d(P'(R), P(R)),$$

so

$$J^{\iota}_{P''(R)|P(R)}(\Lambda+\lambda) = J^{\iota}_{P''(R)|P'(R)}(\Lambda+\lambda)\, J^{\iota}_{P'(R)|P(R)}(\Lambda+\lambda).$$

We therefore need only consider the case that $d(P', P)=1$. As a bonus, we will obtain the usual functional equations

$$J^{\iota}_{P''|P}(\lambda) = J^{\iota}_{P''|P'}(\lambda)\, J^{\iota}_{P'|P}(\lambda).$$

However, if $d(P', P)=1$ there will be a maximal parabolic subgroup P_L in $\mathcal{P}^L(M)$, for a Levi subgroup $L \in \mathcal{L}(M)$, such that

$$J^{\iota}_{P'(R)|P(R)}(\Lambda+\lambda) = J^{\iota}_{\bar{P}_L(R)|P_L(R)}(\Lambda+\lambda)$$

(see (I.2.3)). Thus, we are reduced to the case that $P'=\bar{P}$ (but with G replaced by L), which was established above. Q.E.D.

COROLLARY 5.2. *If $P, P' \in \mathcal{P}(M)$ and $\iota=l$ or r, $J^{\iota}_{P'|P}(\lambda)$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ which for any $\mathfrak{o}$ and d takes values in the finite dimensional space of endomorphisms of $\mathcal{A}_{\mathfrak{o},d}(M_-, \tau)$. It is independent of the group $R \in \mathcal{P}^M(M_0)$ used in the definition. Moreover, if P'' is also in $\mathcal{P}(M)$,*

$$J^{\iota}_{P''|P}(\lambda) = J^{\iota}_{P''|P'}(\lambda)\, J^{\iota}_{P'|P}(\lambda).$$

Proof. If Q is any group in $\mathcal{P}(M)$, it follows from the proof of the theorem that for any φ,

$$(J^{l}_{Q|P}(\lambda)\, J^{r}_{\bar{Q}|P}(\lambda)\, \varphi)(m)$$

equals the product of $e^{-\lambda(H_M(m))}$ with the value at m of

$$s^{-1}E^{P_1}(E_{P,B,s}(\varphi, \lambda)). \tag{5.6}$$

Suppose that R was replaced by rR, for some $r \in W^M_0$. Then B, s, and P_1 would all have to be replaced by rB, rs and rP_1 respectively. The expression (5.6) would have to be replaced by

$$s^{-1}r^{-1}E^{rP_1}(E_{P,rB,rs}(\varphi, \lambda)).$$

By (I.4.1) and (II.4.3), this equals

$$s^{-1}E^{P_1}(E_{P,B,s}(\varphi, \lambda)),$$

which is just (5.6). Therefore $J^{l}_{Q|P}(\lambda)\, J^{r}_{\bar{Q}|P}(\lambda)$ is independent of R. The independence of $J^{\iota}_{P'|P}(\lambda)$ from R follows from the arguments used to prove the theorem. The functional equations and all the other statements of the corollary follow also from the proof of the theorem. Q.E.D.

For any $\mathfrak{o}$ and d we can take the determinant of the restriction of $J^l_{P'|P}(\lambda)$ to $\mathcal{A}_{\mathfrak{o},d}(M_-,\tau)$. It is a meromorphic complex valued function of λ. In view of the injectivity assertion in the theorem, the determinant will not vanish identically. Thus, the inverse, $J^l_{P'|P}(\lambda)^{-1}$ is defined; as with $J^l_{P'|P}(\lambda)$, we regard it as a meromorphic function which for any $\mathfrak{o}$ and d takes values in the finite dimensional space of endomorphisms of $\mathcal{A}_{\mathfrak{o},d}(M_-,\tau)$. We can proceed merrily to define all the funcitons of I, § 2:

$$c_{P_1|P}(s,\lambda) = sJ^l_{s^{-1}P_1|P}(\lambda)\, J^r_{s^{-1}\bar{P}_1|P}(\lambda),$$

$$c^0_{P_1|P}(s,\lambda) = c_{P_1|P}(s,\lambda)\, c_{P|P}(1,\lambda)^{-1} = sJ^l_{\bar{P}|s^{-1}P_1}(\lambda)^{-1} J^r_{s^{-1}\bar{P}_1|P}(\lambda),$$

$${}^0c_{P_1|P}(s,\lambda) = c_{P_1|P_1}(1,s\lambda)^{-1} c_{P_1|P}(s,\lambda) = sJ^l_{s^{-1}P_1|P}(\lambda)\, J^r_{P|s^{-1}P_1}(\lambda)^{-1},$$

and

$$\mu_P(\lambda) = J^l_{\bar{P}|P}(\lambda)^{-1} J^l_{P|\bar{P}}(\lambda)^{-1} = J^r_{P|\bar{P}}(\lambda)^{-1} J^r_{\bar{P}|P}(\lambda)^{-1}.$$

Here s is an element in $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$; $c_{P_1|P}(s,\lambda)$, $c^0_{P_1|P}(s,\lambda)$ and ${}^0c_{P_1|P}(s,\lambda)$ all map $\mathcal{A}(M_-,\tau)$ to $\mathcal{A}((M_1)_-,\tau)$, while $\mu_P(\lambda)$ maps $\mathcal{A}(M_-,\tau)$ to itself. In each case, if φ is of the form (5.2), the value of the operator at φ can be expressed by a formula akin to (5.1). The only such formula we will need is for the operator $\mu_P(\lambda)$.

LEMMA 5.3. *Suppose, as in Theorem* 5.1, *that* $E_{R|R,1}(m,\Phi(\Lambda),\Lambda)$ *is regular at* $\Lambda=\Lambda_0$ *and that*

$$\varphi(m) = \lim_{\Lambda\to\Lambda_0} D_\Lambda E_{R|R,1}(m,\Phi(\Lambda),\Lambda), \quad m\in M_-.$$

Then if λ *is a point in general position in* $\mathfrak{a}^*_{M,\mathbf{C}}$, *and* $P\in\mathcal{P}(M)$, *the function*

$$\Lambda \to E_{R|R,1}(m, \mu_{P(R)}(\Lambda+\lambda)\,\mu_R(\Lambda)^{-1}\Phi(\Lambda),\Lambda)$$

is also regular at $\Lambda=\Lambda_0$, *and*

$$(\mu_P(\lambda)\,\varphi)(m) = \lim_{\Lambda\to\Lambda_0} D_\Lambda E_{R|R,1}(m, \mu_{P(R)}(\Lambda+\lambda)\,\mu_R(\Lambda)^{-1}\Phi(\Lambda),\Lambda).$$

Proof. By Lemma I.2.1 we have

$$\mu_{P(R)}(\Lambda+\lambda)\,\mu_R(\Lambda)^{-1} = \mu_{\bar{P}(R)|P(R)}(\Lambda+\lambda)$$
$$= J_{P(R)|\bar{P}(R)}(\Lambda+\lambda)^{-1} J_{\bar{P}(R)|P(R)}(\Lambda+\lambda)^{-1}.$$

Our lemma then follows from Theorem 5.1 and the definition of $\mu_P(\lambda)$. Q.E.D.

We should point out that the restriction to $\mathscr{A}_{\mathrm{cusp}}(M,\tau)$ of any of the operators defined in this section equals the corresponding operator defined in I, § 2. This follows from Lemma I.7.3. All the functional equations of I, § 2 hold for these more general operators. As functions of λ, their poles all lie along hyperplanes $(\alpha,\lambda)=c$, where α is a root of (G, A_M) and $c\in\mathbf{C}$.

§ 6. Application to the residues

We want to use the theorem of Casselman to compare the functions $r_T(x,\Phi,\Lambda)$ with analogous functions on Levi subgroups. Fix a Levi subgroup M in $\mathscr{L}(M_0)$, a coset $s\in W_0/W_0^M$ and a group $P\in\mathscr{P}(M)$. Fix also a group $B\in\mathscr{P}(M_0)$, $B\subset P$. Then the group $R=B\cap M$ belongs to $\mathscr{P}^M(M_0)$ and $B=P(R)$.

Suppose that $\mathfrak{b}$ is a root subspace of $\mathfrak{a}_0^*$ such that $\mathfrak{b}_1=s_B^{-1}\mathfrak{b}$ contains $\mathfrak{a}_M^*$. Let X be a point in general position in $\mathfrak{a}_0^*$. Then $X_1=s_B^{-1}X$ will also be a point in general position in $\mathfrak{a}_0^*$. We are going to construct a bijection between $\mathscr{T}_B(\mathfrak{b},X)$ and $\mathscr{T}_R(\mathfrak{b}_1,X_1)$. Suppose that F_B is an analytic function from $\mathfrak{a}_{0,\mathbf{C}}^*$ to $\mathscr{A}_0$. To make the notation simpler, we will assume that

$$F_B(s_B\Lambda) = {}^0c_{B|B}(s_B,\Lambda)\,F_B(\Lambda),\quad \Lambda\in\mathfrak{a}_{0,\mathbf{C}}^*.$$

LEMMA 6.1. *There is a bijection*

$$T\to T_1,\quad T\in\mathscr{T}_B(\mathfrak{b},X),$$

from $\mathscr{T}_B(\mathfrak{b},X)$ *onto* $\mathscr{T}_R(\mathfrak{b}_1,X_1)$ such that

(i) $\mathfrak{A}_{T_1}=s_B^{-1}\mathfrak{A}_T$

(ii) $X_{T_1}=s_B^{-1}X_T$

(iii) $E_{P,B,s}(\mu_P(\lambda)\,r_{T_1}(\Phi_1,\Lambda_1),\lambda) = r_T(\Phi, s_B(\Lambda_1+\lambda))$,

where $\Lambda_1\in\mathfrak{A}_{T_1}$, $\lambda\in\mathfrak{a}_{M,\mathbf{C}}^*$, *and for any* $F_B(\Lambda)$ *as above,*

$$\Phi = (d_T F_B)(s_B(\Lambda_1+\lambda))$$

and

$$\Phi_1 = (d_{T_1} F_B)(\Lambda_1 + \lambda).$$

Remarks. (1) The third equation is of course an equality between meromorphic functions in Λ_1 and λ.

(2) Since $s_B(\alpha)$ is positive for each root α of (R, A_0), there is no singular hyperplane of either ${}^0c_{B|B}(s_B, \Lambda)$ or ${}^0c_{B|B}(s_B, \Lambda)^{-1}$ which contains $\mathfrak{a}^*_{M,\mathbf{C}}$. In other words, if we first fix Λ_1 and then fix a point $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ in general position, ${}^0c_{B|B}(s_B, \Lambda_1+\lambda)$ and ${}^0c_{B|B}(s_B, \Lambda_1+\lambda)^{-1}$ will both be defined. It follows that given a positive integer d and *any* vector $\Phi \in \mathrm{Hom}(S(\mathfrak{b}^\perp_{\mathbf{C}}), \mathscr{A}_0)$, we will be able to choose a function F_B as above such that Φ and $(d_T F_B)(s_B(\Lambda_1+\lambda))$ both have the same projections onto $\mathrm{Hom}(S_d(\mathfrak{b}^\perp_{\mathbf{C}}), \mathscr{A}_0)$. On the other hand, if it is $\Phi_1 \in \mathrm{Hom}(S(\mathfrak{b}^\perp_{1,\mathbf{C}}), \mathscr{A}_0)$ that we are given, we can choose F_B as above so that Φ_1 and $(d_{T_1} F_B)(\Lambda_1+\lambda)$ both have the same projections onto $\mathrm{Hom}(S_d(\mathfrak{b}^\perp_{1,\mathbf{C}}), \mathscr{A}_0)$. In particular, $\Phi \leftrightarrow \Phi_1$ defines an isomorphism between $\mathrm{Hom}(S_d(\mathfrak{b}^\perp_{\mathbf{C}}), \mathscr{A}_0)$ and $\mathrm{Hom}(S_d(\mathfrak{b}^\perp_{1,\mathbf{C}}), \mathscr{A}_0)$. There is a d such that the functions in (iii) above depend only on the projections of Φ and Φ_1 onto these respective spaces. It follows that the map $T \to T_1$ is uniquely defined by the conditions of the lemma.

Proof of Lemma 6.1. Suppose that $T_1 \in \mathscr{T}_R(\mathfrak{b}_1, X_1)$. Fix $\Lambda_1 \in \mathfrak{A}_{T_1}$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. As always, we assume that these points are in sufficiently general position. It follows from the inductive definition of $\mathscr{T}_R(\mathfrak{b}_1, X_1)$ that there is a nested sequence of root subspaces

$$\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \ldots \subset \mathfrak{b}_{r+1} = \mathfrak{a}^*_0,$$

and for each i a triple $T_i \in \mathscr{T}_R(\mathfrak{b}_i, X_1)$ such that $r_T(\Phi_1, \Lambda_1)$ is obtained from the function

$$r_R(F_B(\Lambda+\lambda), \Lambda) = E_{R|R,1}(\mu_R(\Lambda) F_B(\Lambda+\lambda), \Lambda)$$

by successive residues along the hyperplanes $\mathfrak{A}_{T_i}$. More precisely, for each i, $1 \leqslant i \leqslant r$, there is a unit vector $v_i \in \mathfrak{b}_{r-i+2}$, orthogonal to $\mathfrak{b}_{r-i+1}$, such that if

$$\Lambda_u = \Lambda_1 + u_1 v_1 + \ldots + u_r v_r, \quad u = (u_1, \ldots, u_r) \in \mathbf{C}^r,$$

then

$$r_{T_1}(m, \Phi_1, \Lambda_1), \quad m \in M_-,$$

equals the product of

$$|\mathscr{C}(\mathfrak{b}_2)|^{-1} \dots |\mathscr{C}(\mathfrak{b}_{r+1})|^{-1}(2\pi i)^{-r}$$

with

$$\varphi(m) = \int_{\Gamma_r} \dots \int_{\Gamma_1} r_R(m, F_B(\Lambda_u+\lambda), \Lambda_u)\, du_1 \dots du_r.$$

Here $\Gamma_1, \dots, \Gamma_r$ are small positively oriented circles about the origin in the complex plane such that for each i the radius of Γ_i is much smaller than that of Γ_{i+1}. We shall find a formula for $E_{P,B,s}(\mu_R(\lambda)\, r_{T_1}(\Phi_1, \Lambda_1), \lambda)$.

Let

$$l_N(\Lambda), \quad \Lambda \in \mathfrak{a}^*_{0,\mathbf{C}},$$

be the product, over the roots β of (R, A_0), of the factors $(\beta, \Lambda-\Lambda_1)^N$. For every positive integer N there is a differential operator D_N on $\mathfrak{a}^*_{0,\mathbf{C}}$ such that

$$\lim_{\Lambda \to \Lambda_1} D_N(l_N(\Lambda) f(\Lambda)) = \int_{\Gamma_r} \dots \int_{\Gamma_1} f(\Lambda_u)\, du_1 \dots du_r,$$

for every meromorphic function on $\mathfrak{a}^*_{0,\mathbf{C}}$ such that $l_N(\Lambda) f(\Lambda)$ is regular at $\Lambda=\Lambda_1$. (For example, if $cu_1^{(n_1+1)} \dots cu_r^{(n_r+1)}$ is the lowest term in $l_N(\Lambda_u)$ relative to the lexicographic order on the monomials in $(u_1, \dots, u_r)$, we could take D_N to be the differential operator which in the co-ordinates $(u_1, \dots, u_r)$ is

$$c^{-1}(2\pi i)^r((n_1)! \dots (n_r)!)^{-1} \left(\frac{\partial}{\partial u_1}\right)^{n_1} \dots \left(\frac{\partial}{\partial u_r}\right)^{n_r}.)$$

We will take N so large that

$$l_N(\Lambda)\, r_R(m, F_B(\Lambda+\lambda), \Lambda)$$

is regular at $\Lambda=\Lambda_1$. Then

$$\varphi(m) = \lim_{\Lambda \to \Lambda_1} (D_N)_\Lambda E_{R|R,1}(m, \Phi(\Lambda), \Lambda),$$

where

$$\Phi(\Lambda) = l_N(\Lambda)\, \mu_R(\Lambda)\, F_B(\Lambda+\lambda).$$

By Lemma 5.3, $(\mu_P(\lambda)\varphi)(m)$ equals

$$\lim_{\Lambda\to\Lambda_1}(D_N)_\Lambda E_{R|R,1}(m,\Phi'(\Lambda),\Lambda),$$

where

$$\begin{aligned}\Phi'(\Lambda)&=\mu_B(\Lambda+\lambda)\mu_R(\Lambda)^{-1}\Phi(\Lambda)\\&=l_N(\Lambda)\mu_B(\Lambda+\lambda)F_B(\Lambda+\lambda).\end{aligned}$$

Therefore

$$E_{P,B,s}(x,\mu_P(\lambda)\varphi,\lambda)$$

equals

$$\begin{aligned}&\lim_{\Lambda\to\Lambda_1}(D_N)_\Lambda E_{B|B,s_B}(x,\Phi'(\Lambda),\Lambda+\lambda)\\&\quad=\int_{\Gamma_r}\dots\int_{\Gamma_1}E_{B|B,s_B}(x,\mu_B(\Lambda_u+\lambda)F_B(\Lambda_u+\lambda),\Lambda_u+\lambda)\,du_1\dots du_r.\end{aligned}$$

By (I.4.6′), the integrand here equals

$$\begin{aligned}&E_{B|B,1}(x,{}^0c_{B|B}(s_B,\Lambda_u+\lambda)\mu_B(\Lambda_u+\lambda)F_B(\Lambda_u+\lambda),s_B(\Lambda_u+\lambda))\\&\quad=E_{B|B,1}(x,\mu_B(\Lambda_u+\lambda)F_B(s_B(\Lambda_u+\lambda)),s_B(\Lambda_u+\lambda))\\&\quad=r_B(x,F_B(s_B(\Lambda_u+\lambda)),s_B(\Lambda_u+\lambda)).\end{aligned}$$

Therefore

$$E_{P,B,s}(x,\mu_P(\lambda)r_{T_1}(\Phi_1,\Lambda_1),\lambda)$$

equals the product of

$$|\mathscr{C}(\mathfrak{b}_2)|^{-1}\dots|\mathscr{C}(\mathfrak{b}_{r+1})|^{-1}(2\pi i)^{-r}=|\mathscr{C}(s_B\mathfrak{b}_2)|^{-1}\dots|\mathscr{C}(s_B\mathfrak{b}_{r+1})|^{-1}(2\pi i)^{-r}$$

with

$$\int_{\Gamma_r}\dots\int_{\Gamma_1}r_B(x,F_B(s_B(\Lambda_u+\lambda)),s_B(\Lambda_u+\lambda))\,du_1\dots du_r.$$

This function is obtained from successive residues along the hyperplanes

$$s_B\mathfrak{A}_{T_i}=s_B\Lambda_{T_i}+s_B\mathfrak{b}_i.$$

From the injectivity of the maps $\mu_P(\lambda)$ and $E_{P,B,s}(\cdot,\lambda)$, we know that it does not vanish. It follows from the inductive definition of $\mathcal{T}_B(\mathfrak{b},X)$ that there is a triple $T \in \mathcal{T}_B(\mathfrak{b},X)$ such that

$$\mathfrak{A}_T = s_B \mathfrak{A}_{T_1},$$

$$X_T = s_B X_{T_1},$$

and

$$r_T(\Phi, s_B(\Lambda_1+\lambda)) = E_{P,B,s}(\mu_P(\lambda)\, r_{T_1}(\Phi_1, \Lambda_1), \lambda).$$

On the other hand, suppose that T is any triple in $\mathcal{T}_B(\mathfrak{b},X)$. We can follow the argument backwards to produce a $T_1 \in \mathcal{T}_R(\mathfrak{b}_1, X_1)$. This gives us the required bijection. Q.E.D.

§ 7. Application to the functions $F_{\mathcal{P}}^{\vee}$

Suppose that F is a collection of entire functions

$$F_B\colon \mathfrak{a}_{0,\mathbf{C}}^* \to \mathcal{A}_0, \quad B \in \mathcal{P}(M_0),$$

each rapidly decreasing on vertical cylinders. We will assume in addition, that the collection has the symmetry property

$$F_{B'}(s\Lambda) = {}^0c_{B'|B}(s,\Lambda)\, F_B(\Lambda)$$

for $s \in W_0$ and B, $B' \in \mathcal{P}(M_0)$. Then if $P=NM$ is a parabolic subgroup in $\mathcal{F}(M_0)$, the collection

$$F_P = \{F_R(\Lambda) = F_{P(R)}(\Lambda) \colon R \in \mathcal{P}^M(M_0)\}$$

has the same symmetry property for M. As in II, § 3, we choose for each $B \in \mathcal{P}(M_0)$ a point X_B which is both suitably regular in $-\mathfrak{a}_0^*(B)$ and in general position.

If M is any group in $\mathcal{L}(M_0)$, we shall write $W(\mathfrak{a}_M)$ for $W(\mathfrak{a}_M, \mathfrak{a}_M)$.

THEOREM 7.1. (i) *If $P=NM$ is a parabolic subgroup in $\mathcal{F}(M_0)$ the function $F_{P,\mathrm{cusp}}(\lambda)$ is independent of any of the points $\{X_B\}$.*

(ii) *For any class $\mathcal{P}$ of associated parabolic subgroups in $\mathcal{F}(M_0)$ the function $F_{\mathcal{P}}^{\vee}(x)$* equals

$$|P|^{-1} \sum_{P \in \mathcal{P}} |W(\mathfrak{a}_P)|^{-1} \int_{\varepsilon_P + i\mathfrak{a}_P^*} E_P(x, \mu_P(\lambda) F_{P,\mathrm{cusp}}(\lambda), \lambda)\, d\lambda,$$

where ε_P is any point in the chamber $\mathfrak{a}_P^*(P)$ of sufficiently small norm. In particular, $F_{\mathcal{P}}^{\vee}(x)$ is also independent of the points $\{X_B\}$.

Proof. Assume by induction on $\dim G$ that the theorem holds if G is replaced by any proper Levi subgroup. If $P=NM$ is a parabolic subgroup in $\mathcal{F}(M_0)$, $F_{P,\mathrm{cusp}}(\lambda)$ is defined in terms of the collection $\{F_P\}$ associated to M. Then if $P \neq G$, the function $F_{P,\mathrm{cusp}}(\lambda)$ is independent of the points $\{X_B\}$. Now $F_{P,\mathrm{cusp}}(\lambda)$ equals the sum over $\{B \in \mathcal{P}(M_0): B \subset P\}$ of the product of $|\mathcal{P}^M(M_0)|^{-1}$ with

$$\sum_{T \in \mathcal{T}_{B \cap M}(\mathfrak{a}_M^*, X_B)} r_T(dF_B(\Lambda_T + \lambda), \Lambda_T). \tag{7.1}$$

If we change *one* of the points $\{X_B\}$, it will change at most one of the terms in the sum over B. It follows that the expression (7.1) is independent of the point X_B.

Suppose that $\mathcal{P}$ is a class of associated proper parabolic subgroups in $\mathcal{F}(M_0)$. Then $F_{\mathcal{P}}^{\vee}(x)$ equals the sum over $B \in \mathcal{P}(M_0)$, $\mathfrak{b} \in \mathrm{rt}(\mathcal{P})$, $T \in \mathcal{T}_B(\mathfrak{b}, X_B)$ and $\varepsilon \in \mathcal{E}(\mathfrak{b})$ of

$$|\mathcal{P}(M_0)|^{-1} |\mathcal{E}(\mathfrak{b})|^{-1} \int_{\Lambda_T + i\mathfrak{b} + \varepsilon} r_T(x, dF_B(\Lambda), \Lambda)\, d\Lambda.$$

Consider a summand, indexed by B, $\mathfrak{b}$, T and ε. The point ε belongs to a chamber in $\mathfrak{b}$. There corresponds a certain group in $\mathcal{P}$, which is in turn conjugate to a unique group $P=NM$ in $\mathcal{P}$ which contains B. There is also a unique element $s \in W_0/W_0^M$ such that $s^{-1}\mathfrak{b} = \mathfrak{a}_M^*$ and such that the point $\varepsilon_1 = s^{-1}\varepsilon$ belongs to the chamber $\mathfrak{a}_M^*(P)$. We can therefore write $F_{\mathcal{P}}^{\vee}(x)$ as the sum over $B \in \mathcal{P}(M_0)$, $\{P=NM \in \mathcal{P}: P \supset B\}$, $s \in W_0/M_0^M$ and $\varepsilon_1 \in \mathcal{E}(\mathfrak{a}_M^*) \cap \mathfrak{a}_M^*(P)$ of

$$|\mathcal{P}(M_0)|^{-1} |\mathcal{E}(\mathfrak{a}_M^*)|^{-1} \sum_{T \in \mathcal{T}_B(s\mathfrak{a}_M^*, X_B)} \int_{\Lambda_T + s(\varepsilon_1 + i\mathfrak{a}_M^*)} r_T(x, dF_B(\Lambda), \Lambda)\, d\Lambda.$$

For each T let T_1 be the unique triplet in $\mathcal{T}_{B\cap M}(\mathfrak{a}_M^*, s_B^{-1}X_B)$ given by Lemma II.6.1. Since $\Lambda_T = s_B\Lambda_{T_1}$ we can make a change of variables in the integral over Λ. The integral becomes

$$\int_{\varepsilon_1+i\mathfrak{a}_M^*} r_T(x, (d_T F_B)(s_B(\Lambda_{T_1}+\lambda)), s_B(\Lambda_{T_1}+\lambda))\, d\lambda.$$

By Lemma II.6.1, this in turn equals

$$\int_{\varepsilon_1+i\mathfrak{a}_M^*} E_{P,B,s}(x, \mu_P(\lambda)\, r_{T_1}((d_{T_1} F_B)(\Lambda_{T_1}+\lambda), \Lambda_{T_1}), \lambda)\, d\lambda.$$

The correspondence $T \to T_1$ is a bijection, so we may replace the sum over T by a sum over $T_1 \in \mathcal{T}_{B\cap M}(\mathfrak{a}_M^*, s_B^{-1}X_B)$. Since the maps $E_{P,B,s}$ and $\mu_P(\lambda)$ are linear, we will be confronted with an expression

$$\sum_{T_1 \in \mathcal{T}_{B\cap M}(\mathfrak{a}_M^*, s_B^{-1}X_B)} r_{T_1}(dF_B(X_{T_1}+\lambda), \Lambda_{T_1}). \tag{7.2}$$

If α is a root of $(B\cap M, A_0)$,

$$(\alpha, s_B^{-1}X_B) = (s_B\alpha, X_B)$$

will be a large negative number. Thus, $s_B^{-1}X_B$ is a suitably regular point in $-\mathfrak{a}_0^*(B\cap M)$ in general position. It follows from our induction assumption that the expression (7.2) equals (7.1). In particular it is independent of s. The only thing which does depend on s is the map $E_{P,B,s}$. When we sum over $s \in W_0/W_0^M$ we will obtain the map E_P, which is in turn independent of B.

Up to this point, we have shown that $F_{\mathcal{P}}^{\vee}(x)$ equals the sum over $P=NM$ in $\mathcal{P}$ and ε_1 in $\mathcal{E}(\mathfrak{a}_M^*)\cap\mathfrak{a}_M^*(P)$ of the product of $|\mathcal{P}(M_0)|^{-1}|\mathcal{E}(\mathfrak{a}_M^*)|^{-1}$ with

$$\int_{\varepsilon_1+i\mathfrak{a}_M^*} E_P\left(x, \mu_P(\lambda)\left[\sum_{B\subset P}\ \sum_{T\in\mathcal{T}_{B\cap M}(\mathfrak{a}_M^*, X_B)} r_T(dF_B(\Lambda_T+\lambda), \Lambda_T)\right], \lambda\right) d\lambda.$$

The expression inside the square brackets is just equal to

$$|\mathcal{P}^M(M_0)|\, F_{P,\mathrm{cusp}}(\lambda).$$

The function $E_P(x, \mu_P(\lambda) F_{P,\mathrm{cusp}}(\lambda), \lambda)$ is regular for all λ in the tube over the elements in $\mathfrak{a}_M^*(P)$ of sufficiently small norm. Therefore, we can deform each contour $\varepsilon_1 + i\mathfrak{a}_M^*$ to the contour $\varepsilon_P + i\mathfrak{a}_M^*$. It follows that $F_{\mathscr{P}}^{\vee}(x)$ equals the sum over $P \in \mathscr{P}$ of the product of

$$\int_{\varepsilon_P + i\mathfrak{a}_P^*} E_P(x, \mu_P(\lambda) F_{P,\mathrm{cusp}}(\lambda), \lambda)\, d\lambda,$$

with

$$|\mathscr{P}(M_0)|^{-1}\, |\mathscr{P}^{M_P}(M_0)|, \tag{7.3}$$

and

$$|\mathscr{C}(\mathfrak{a}_P^*)|^{-1}\, |\mathscr{C}(\mathfrak{a}_P^*) \cap \mathfrak{a}_P^*(P)|. \tag{7.4}$$

Now the reciprocal of (7.3) equals the number of groups in $\mathscr{P}$ which are conjugate to P. The reciprocal of (7.4) is just equal to the number of chambers in $\mathfrak{a}_P^*$. When divided by the order of $W(\mathfrak{a}_P)$, this equals the number of conjugacy classes within the associated class $\mathscr{P}$. Therefore the product of (7.3) and (7.4) equals

$$|\mathscr{P}|^{-1}\, |W(\mathfrak{a}_P)|^{-1}.$$

We have obtained the required formula for $F_{\mathscr{P}}^{\vee}(x)$. In particular $F_{\mathscr{P}}^{\vee}(x)$ is independent of the points $\{X_B\}$.

We saw in II, § 1 that the function $F^{\vee}(x)$ is also independent of $\{X_B\}$. It follows from Corollary II.3.2 that

$$F_{\{G\}}^{\vee}(x) = F^{\vee}(x) - \sum_{\mathscr{P} \neq \{G\}} F_{\mathscr{P}}^{\vee}(x)$$

is itself independent of the points $\{X_B\}$. Therefore

$$F_{G,\mathrm{cusp}}(\lambda, x) = \int_{A_G} F_{\{G\}}^{\vee}(xa)\, e^{-\lambda(H_G(xa))}\, da, \quad \lambda \in \mathfrak{a}_{G,\mathbf{C}}^*,$$

is independent of $\{X_B\}$. This completes the proof of part (i) of the theorem. The only thing remaining in part (ii) is the formula for $F_{\{G\}}^{\vee}(x)$ in terms of $F_{G,\mathrm{cusp}}(\lambda, x)$. This follows from the formula just quoted by Fourier inversion on the group A_G. Q.E.D.

Chapter III

§ 1. A review of the Plancherel formula

The Plancherel formula for reductive groups is due, of course, to Harish-Chandra. The version we will use pertains to $\mathscr{C}(G,\tau)$, the space of τ-spherical *Schwartz* functions from G to V_τ (see [7c]). The results we recall here are well known, and can be extracted from ([7c], [7d], [7e]).

Suppose that $\mathscr{P}$ is an associated class of parabolic subgroups in $\mathscr{F}(M_0)$. Suppose that for each $P \in \mathscr{P}$, we are given a Schwartz function $F_{P,\text{cusp}}$ on $i\mathfrak{a}_{M_P}^*$ with values

$$F_{P,\text{cusp}}(\lambda): m \to F_{P,\text{cusp}}(\lambda, m), \quad \lambda \in i\mathfrak{a}_{M_P}^*, \ m \in M_P,$$

in the finite dimensional vector space $\mathscr{A}_{\text{cusp}}(M_P, \tau)$ of functions on M_P. Then for $x \in G$, the function

$$|\mathscr{P}|^{-1} \sum_{P \in \mathscr{P}} |W(\mathfrak{a}_P)|^{-1} \int_{i\mathfrak{a}_P^*} E_P(x, \mu_P(\lambda) F_{P,\text{cusp}}(\lambda), \lambda)\, d\lambda \tag{1.1}$$

belongs to $\mathscr{C}(G,\tau)$. The closed subspace of $\mathscr{C}(G,\tau)$ generated by such functions is denoted by $\mathscr{C}_{\mathscr{P}}(G,\tau)$. Then there is a decomposition

$$\mathscr{C}(G,\tau) = \bigoplus_{\mathscr{P}} \mathscr{C}_{\mathscr{P}}(G,\tau).$$

Let $\mathscr{C}(\hat{G},\tau)$ be the space of collections

$$F = \{F_{P,\text{cusp}}: P \in \mathscr{F}(M_0)\}$$

of Schwartz functions $F_{P,\text{cusp}}$ from $i\mathfrak{a}_{M_P}^*$ to $\mathscr{A}_{\text{cusp}}(M_P,\tau)$ with the following symmetry condition: if t is an element in $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, for groups $P, P' \in \mathscr{F}(M_0)$, then

$$F_{P',\text{cusp}}(t\lambda) = {}^0c_{P'|P}(t,\lambda) F_{P,\text{cusp}}(\lambda).$$

Given $F \in \mathscr{C}(\hat{G},\tau)$, let $F_{\mathscr{P}}^\vee(x)$ be the function in $\mathscr{C}_{\mathscr{P}}(G,\tau)$ defined by (1.1), and let

$$F^\vee(x) = \sum_{\mathscr{P}} F_{\mathscr{P}}^\vee(x).$$

The Plancherel formula can be taken to be the assertion that $F \to F^\vee$ is a topological isomorphism from $\mathscr{C}(\hat{G},\tau)$ onto $\mathscr{C}(G,\tau)$.

Suppose that $f \in \mathscr{C}(G, \tau)$. There is a unique $F \in \mathscr{C}(\hat{G}, \tau)$ such that $F^\vee(x) = f(x)$. For any class $\mathscr{P}$, set

$$f_{\mathscr{P}}(x) = F^\vee_{\mathscr{P}}(x).$$

It is the projection of f onto $\mathscr{C}_{\mathscr{P}}(G, \tau)$. If $P \in \mathscr{F}(M_0)$, set

$$\hat{f}_{P, \mathrm{cusp}}(\lambda) = F_{P, \mathrm{cusp}}(\lambda), \quad \lambda \in i\mathfrak{a}^*_{M_P}.$$

It can be recovered from f by the formula

$$(\hat{f}_{P, \mathrm{cusp}}(\lambda), \psi) = \int_G (f(x), E_P(x, \psi, \lambda))\, dx, \tag{1.2}$$

valid for any $\psi \in \mathscr{A}_{\mathrm{cusp}}(M_P, \tau)$. The collection

$$\{\hat{f}_{P, \mathrm{cusp}}(\lambda) = F_{P, \mathrm{cusp}}(\lambda) \colon P \in \mathscr{F}(M_0)\}$$

can be regarded as the Fourier transform of f. We shall usually write $\hat{f}_{\mathrm{cusp}}(\lambda)$ for $\hat{f}_{G, \mathrm{cusp}}(\lambda)$. Then we have

$$\hat{f}_{\mathrm{cusp}}(\lambda, x) = \int_{A_G} f_{\{G\}}(xa)\, e^{-\lambda(H_G(xa))}\, da.$$

Similarly, we will write $F_{\mathrm{cusp}}(\lambda)$ for the function

$$F_{G, \mathrm{cusp}}(\lambda) = \int_{A_G} F^\vee_{\{G\}}(xa)\, e^{-\lambda(H_G(xa))}\, da.$$

Suppose that $P \in \mathscr{F}(M_0)$. There is another interpretation of the function $\hat{f}_{P, \mathrm{cusp}}(\lambda)$. If $f \in \mathscr{C}(G, \tau)$, define

$$f_P(m) = \delta_P(m)^{1/2} \int_{N_P} f(mn)\, dn, \quad m \in M_P,$$

where δ_P is the modular function of P. Then $f \to f_P$ is a continuous map from $\mathscr{C}(G, \tau)$ to $\mathscr{C}(M_P, \tau)$ (see [7c]). On the other hand, if $F \in \mathscr{C}(\hat{G}, \tau)$, consider the collection

$$F_P = \{F_R(\lambda) = F_{P(R)}(\lambda) \colon R \in \mathscr{F}^{M_P}(M_0)\}.$$

It is clear that $F \to F_P$ is a continuous map from $\mathscr{C}(\hat{G}, \tau)$ to $\mathscr{C}(\hat{M}_P, \tau)$. It is not hard to show that if F is the Fourier transform of f, then F_P is the Fourier transform of f_P. That is,

$$(F^\vee)_P(m) = (F_P)^\vee(m). \tag{1.3}$$

We shall write $F_P^\vee(m)$ for this common value. It is clear that $\hat{f}_{P,\,\mathrm{cusp}}(\lambda)$, as defined above, is also equal to the function

$$(f_P)^\wedge_{M_P,\,\mathrm{cusp}}(\lambda) = (f_P)^\wedge_{\mathrm{cusp}}(\lambda).$$

Incidentally, there are no proper parabolic subgroups of M_0, so if $B \in \mathscr{P}(M_0)$ there is no need to include "cusp" in the notation. We shall write

$$\hat{f}_B(\Lambda) = \hat{f}_{B,\,\mathrm{cusp}}(\Lambda),$$

and

$$F_B(\Lambda) = F_{B,\,\mathrm{cusp}}(\Lambda)$$

for $\Lambda \in i\mathfrak{a}_0^*$.

We have been a little bit compulsive with the notation. Our aim has been to make it mesh with the notation for the collections

$$F = \{F_B(\Lambda) \colon B \in \mathscr{P}(M_0),\ \Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*\}$$

introduced in Chapter II.

§ 2. The space $PW(G, \tau)$

If N is a positive number, let $C_N^\infty(G, \tau)$ denote the space of smooth, τ-spherical functions from G to V_τ which are supported on the set

$$G(N) = \{k_1 \cdot \exp H \cdot k_2 \colon k_1, k_2 \in K, H \in \mathfrak{a}_0, \|H\| \leq N\}.$$

It is a complete topological vector space with the usual seminorms. Let $C_c^\infty(G, \tau)$ be the space of all smooth, τ-spherical functions from G to V_τ which are compactly supported. As a topological space it is the direct limit, as N approaches ∞, of the spaces $C_N^\infty(G, \tau)$.

Our main problem is to characterize the image of $C_c^\infty(G, \tau)$ under Fourier transform. For a compactly supported function the Fourier transform will be defined as a collection of functions indexed only by the minimal parabolic subgroups $B \in \mathscr{P}(M_0)$. Suppose that $f \in C_c^\infty(G, \tau)$. If $\Phi \in \mathscr{A}_0$, define

$$(\hat{f}_B(\Lambda), \Phi) = \int_G (f(x), E_B(x, \Phi, -\bar{\Lambda}))\, dx, \quad \Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*.$$

Then $\hat{f}_B(\Lambda)$ is an entire function from $\mathfrak{a}_{0,\mathbf{C}}^*$ with values in $\mathscr{A}_0$ which is rapidly decreasing on vertical cylinders. Unfortunately, the image cannot be described very explicitly. The reason is that any identity between Eisenstein integrals will show up in the collection

$$\{\hat{f}_B(\Lambda): B \in \mathscr{P}(M_0)\}.$$

Indeed, suppose that for all $x \in G$ and $v \in V_\tau$ a relation

$$\sum_{k=1}^n D_k(v, E_{B_k}(x, \Phi_k, -\bar{\Lambda}_k)) = 0 \tag{2.1}$$

holds, for groups $B_k \in \mathscr{P}(M_0)$, vectors $\Phi_k \in \mathscr{A}_0$, points $\Lambda_k \in \mathfrak{a}_{0,\mathbf{C}}^*$ and differential operators D_k of constant coefficients on $\mathfrak{a}_0^*$. Then it is obvious that the relation

$$\sum_{k=1}^n D_k(\hat{f}_{B_k}(\Lambda_k), \Phi_k) = 0$$

will also hold. (In each case, D_k acts through the variable Λ_k.)

Suppose that N is a positive number. Let $PW_N(G, \tau)$ be the space of collections

$$F = \{F_B(\Lambda): B \in \mathscr{P}(M_0)\}$$

of entire functions F_B from $\mathfrak{a}_{0,\mathbf{C}}^*$ to $\mathscr{A}_0$ which satisfy two conditions. First, whenever a relation of the form (2.1) holds, the relation

$$\sum_{k=1}^n D_k(F_{B_k}(\Lambda_k), \Phi_k) = 0$$

must also hold. Secondly, for every integer n, the semi-norm

$$\|F\|_{N,n} = \sup_{\{\Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*, B \in \mathscr{P}(M_0)\}} (\|F_B(\Lambda)\| e^{-N\|\mathrm{Re}\,\Lambda\|}(1+\|\Lambda\|)^n)$$

is finite. With these semi-norms, $PW_N(G, \tau)$ becomes a topological vector space. We define $PW(G, \tau)$ to be the direct limit, as N approaches ∞, of the spaces $PW_N(G, \tau)$.

Observe that as a special case of a relation of the form (2.1) we have the functional equation

$$E_B(x, \Phi, -\bar{\Lambda}) = E_{B'}(x, {}^0c_{B'|B}(s, -\bar{\Lambda})\,\Phi, -s\bar{\Lambda})$$

for any $s \in W_0$. It implies that

$$F_{B'}(s\Lambda) = {}^0c_{B'|B}(s, \Lambda)\, F_B(\Lambda) \tag{2.2}$$

if $F = \{F_B(\Lambda)\}$ belongs to $PW(G, \tau)$. In particular, all the results of Chapter II hold for collections F in $PW(G, \tau)$.

Suppose that for each j in a finite indexing set J, S_j is a finite dimensional subspace of $S(\mathfrak{a}^*_{0,\mathbf{C}})$. Suppose also that for each $j \in J$, we are given a group $B_j \in \mathscr{P}(M_0)$ and a point $\Lambda_j \in \mathfrak{a}^*_{0,\mathbf{C}}$. If F_{B_j} is an analytic function from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathscr{A}_0$ we shall write $d_{S_j} F_{B_j}(\Lambda_j)$ for the projection of the vector

$$dF_{B_j}(\Lambda_j) \in \operatorname{Hom}(S(\mathfrak{a}^*_{0,\mathbf{C}}), \mathscr{A}_0)$$

onto the finite dimensional vector space $\operatorname{Hom}(S_j, \mathscr{A}_0)$.

LEMMA 2.1. *Suppose that $F = \{F_B(\Lambda) : B \in \mathscr{P}(M_0)\}$ is a collection in $PW(G, \tau)$. Then the vector*

$$\bigoplus_{j \in J} d_{S_j} F_{B_j}(\Lambda_j)$$

belongs to the subspace

$$U_j = \left\{ \bigoplus_{j \in J} d_{S_j} \hat{f}_{B_j}(\Lambda_j) : f \in C^\infty_c(G, \tau) \right\}$$

of $\oplus_{j \in J} \operatorname{Hom}(S_j, \mathscr{A}_0)$.

Proof. If $v \in V_\tau$ and $x \in G$, define a vector $e_B(x, v, \Lambda)$ in $\mathscr{A}_0$ by

$$(e_B(x, v, \Lambda), \Phi) = (v, E_B(x, \Phi, -\bar{\Lambda})).$$

Then

$$e(x, v) = \bigoplus_{j \in J} d_{S_j} e_{B_j}(x, v, \Lambda_j)$$

is a smooth function from $G\times V_\tau$ to $\oplus_j \mathrm{Hom}(S_j, \mathscr{A}_0)$. Suppose that f_0 is any smooth, compactly supported function from G to V_τ. Then the function

$$f(x)=\int_K\int_K \tau(k_1)^{-1}f_0(k_1xk_2)\,\tau(k_2)^{-1}\,dk_1\,dk_2$$

belongs to $C_c^\infty(G,\tau)$. We have

$$\begin{aligned}\int_G (e_B(x,f_0(x),\Lambda),\Phi)\,dx &= \int_G (f_0(x),E_B(x,\Phi,-\bar{\Lambda}))\,dx\\ &= \int_G (f(x),E_B(x,\Phi,-\bar{\Lambda}))\,dx\\ &= (\hat{f}_B(\Lambda),\Phi),\end{aligned}$$

for any $\Phi\in\mathscr{A}_0$. Therefore,

$$\int_G e(x,f_0(x))\,dx=\bigoplus_{j\in J} d_{S_j}\hat{f}_{B_j}(\Lambda_j).$$

It follows that U_J is the subspace of $\oplus_j\mathrm{Hom}(S_j,\mathscr{A}_0)$ spanned by

$$\{e(x,v):x\in G,\,v\in V_\tau\}.$$

Now $\mathscr{A}_0$ is a (finite dimensional) Hilbert space, so there is a nondegenerate pairing between $\oplus_j\mathrm{Hom}(S_j,\mathscr{A}_0)$ and $\oplus_j(S_j\otimes\mathscr{A}_0)$. When $e(x,v)$ is paired with the vector

$$\bigoplus_j\left(\sum_i X_{ij}\otimes\Phi_{ij}\right),\quad X_{ij}\in S_j,\ \Phi_{ij}\in\mathscr{A}_0, \tag{2.3}$$

the result is

$$\sum_{i,j} D(X_{ij})\,(v,E_{B_j}(x,\Phi_{ij},-\bar{\Lambda}_j)), \tag{2.4}$$

where $D(X_{ij})$ is the differential operator on $\mathfrak{a}_0^*$ associated to X_{ij}. Now, suppose that (2.3) is an arbitrary vector in the annihilator of U_J in $\oplus_j(S_j\otimes\mathscr{A}_0)$. Then (2.4) will vanish. It follows from the definition of $PW(G,\tau)$ that

$$\sum_{i,j} D(X_{ij})\,(F_{B_j}(\Lambda_j),\Phi_{ij})=0.$$

In other words, the vector (2.3) annihilates

$$\bigoplus_{j\in J} d_{S_j} F_{B_j}(\Lambda_j).$$

It follows that this latter vector belongs to U_J. The lemma is proved. Q.E.D.

This lemma gives another interpretation of the first condition in the definition of $PW(G,\tau)$. The condition is equivalent to demanding that for each finite set $\{(S_j, B_j, \Lambda_j): j\in J\}$ there be an $f\in C_c^\infty(G,\tau)$such that

$$d_{S_j} F_{B_j}(\Lambda_j) = d_{S_j} \hat{f}_{B_j}(\Lambda_j), \quad j\in J.$$

In other words, $F=\{F_B(\Lambda)\}$ must locally be a Fourier transform of a function in $C_c^\infty(G,\tau)$.

The following corollary is the form of the lemma we will actually need to apply.

COROLLARY 2.2. *Suppose for each* $j\in J$ *that the point* Λ_j *belongs to* $(\mathfrak{a}_G^*)_{\mathbf{C}}^{\perp}$, *and that* S_j *equals* $S_d((\mathfrak{a}_G^*)_{\mathbf{C}}^{\perp})$, *the space of symmetric tensors on* $(\mathfrak{a}_G^*)_{\mathbf{C}}^{\perp}$ *of degree at most d. Then for any* $F\in PW(G,\tau)$ *there is a function* $h\in C_c^\infty(G,\tau)$ *such that*

$$\bigoplus_j d_{S_j} \hat{h}_{B_j}(\Lambda_j+\lambda) = \bigoplus_j d_{S_j} F_{B_j}(\Lambda_j+\lambda)$$

for every point $\lambda\in\mathfrak{a}_{G,\mathbf{C}}^*$.

Proof. G is the direct product of A_G and

$$G^1 = \{x\in G: H_G(x)=0\}.$$

It follows that

$$C_c^\infty(G,\tau) = C_c^\infty(A_G)\otimes C_c^\infty(G^1,\tau)$$

and

$$PW(G,\tau) = PW(A_G)\otimes PW(G^1,\tau).$$

Moreover,

$$U_J = \left\{\bigoplus_{j\in J} d_{S_j}(f^1)^\wedge_{B_j}(\Lambda_j): f^1\in C_c^\infty(G^1,\tau)\right\}.$$

Applying the lemma to G^1, and recalling the classical Paley-Wiener theorem, we see that

$$\bigoplus_{j\in J} d_{S_j} F_{B_j}(\Lambda_j+\lambda), \quad \lambda \in \mathfrak{a}^*_{G,\mathbf{C}},$$

is the Fourier transform (on A_G) of a smooth, compactly supported function from A_G to the finite dimensional space U_J. The corollary follows. Q.E.D.

Suppose that $P \in \mathscr{F}(M_0)$. We can certainly define the space $PW(M_P, \tau)$. It consists of collections of functions from $\mathfrak{a}^*_{0,\mathbf{C}}$ to $\mathscr{A}_0$ indexed by the groups in $\mathscr{P}^{M_P}(M_0)$. The next lemma will prepare the way for a key inductive argument.

LEMMA 2.3. *Suppose that $F=\{F_B(\Lambda): B \in \mathscr{P}(M_0)\}$ belongs to $PW(G,\tau)$. Then the collection*

$$F_P = \{F_R(\Lambda) = F_{P(R)}(\Lambda): R \in \mathscr{P}^{M_P}(M_0)\}$$

belongs to $PW(M_P, \tau)$.

Proof. Suppose that we have any identity

$$\sum_{k=1}^{n} D_k(v, E_{R_k}(m, \Phi_k, -\bar{\Lambda}_k)) = 0, \quad m \in M_P,\ v \in V_\tau,$$

between Eisenstein integrals on M_P. If $E_{R_k}(\cdot, \Phi_k, -\bar{\Lambda}_k)$ is extended to a function on G in the usual way, then

$$E_{P(R_k)}(x, \Phi_k, -\bar{\Lambda}_k) = \int_{K\cap M_P\backslash K} \tau(u)^{-1} E_{R_k}(ux, \Phi_k, -\bar{\Lambda}_k)\, du.$$

It follows that

$$\sum_{k=1}^{n} D_k(v, E_{P(R_k)}(x, \Phi_k, -\bar{\Lambda}_k)) = 0, \quad x \in G,\ v \in V_\tau.$$

Since $F \in PW(G,\tau)$, we will have

$$\sum_{k=1}^{n} D_k(f_{P(R_k)}(\Lambda_k), \Phi_k) = 0 = \sum_{k=1}^{n} D_k(F_{R_k}(\Lambda_k), \Phi_k).$$

Therefore F_P belongs to $PW(M_P, \tau)$. Q.E.D.

§ 3. The main theorem

We are now ready for our main result. We shall show that the Fourier transform maps $C_c^\infty(G,\tau)$ isomorphically onto $PW(G,\tau)$.

LEMMA 3.1. *The map*

$$f \to \{\hat{f}_B : B \in \mathcal{P}(M_0)\}, \quad f \in C_N^\infty(G,\tau),$$

is a continuous, injective map from $C_N^\infty(G,\tau)$, *to* $PW_N(G,\tau)$.

Proof. If $f \in C_N^\infty(G,\tau)$ and $\Phi \in \mathcal{A}_0$, $(\hat{f}_B(\Lambda),\Phi)$ is the integral over x of the inner product of $f(x)$ with

$$\int_K \Phi_B(kx)\, e^{(-\bar{\Lambda}+\varrho_B)(H_B(kx))}\, dk.$$

The integral will vanish unless

$$x = k_1 \cdot \exp H \cdot k_2, \quad k_1, k_2 \in K, \ \|H\| \leq N.$$

It is known that the point

$$H_B(kx) = H_B(kk_1 \exp H)$$

lies within the convex hull of

$$\{sH : s \in W_0\}.$$

Consequently

$$\begin{aligned} \left|e^{-\bar{\Lambda}(H_B(kx))}\right| &\leq e^{\|\mathrm{Re}\,\Lambda\|\cdot\|H_B(kx)\|} \\ &\leq e^{\|\mathrm{Re}\,\Lambda\|\cdot\|H\|} \\ &\leq e^{N\|\mathrm{Re}\,\Lambda\|}. \end{aligned}$$

It follows that there is a continuous seminorm $\|\ \|_N$ on $C_N^\infty(G,\tau)$ such that

$$|(\hat{f}_B(\Lambda),\Phi)| \leq \|f\|_N \|\Phi\|\, e^{N\|\mathrm{Re}\,\Lambda\|},$$

for all f and Φ. Now, for any n it is possible to choose an element z in $\mathfrak{Z}_G$ such that

$$(1+\|\Lambda\|)^n \|\hat{f}_B(\Lambda)\| \leq \|(zf)^\wedge_B(\Lambda)\|$$

for all $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}$. But

$$f \to \|zf\|_N$$

is a continuous seminorm on $C^\infty_N(G, \tau)$, so $f \to \{\hat{f}_B\}$ is a continuous map from $C^\infty_N(G, \tau)$ to $PW_N(G, \tau)$.

If G is compact modulo A_G the injectivity of the map follows from the classical Fourier inversion formula on A_G. Suppose then that G/A_G is not compact. We will assume by induction that the lemma holds if G is replaced by M_P, for any proper parabolic subgroup P of G. Let f be a function in $C^\infty_c(G, \tau)$ such that $\hat{f}_B(\Lambda)=0$ for every B and Λ. If P is a proper parabolic subgroup of G, we have

$$(f_P)^\wedge_R(\Lambda) = \hat{f}_{P(R)}(\Lambda) = 0$$

for any group $R \in \mathscr{P}^{M_P}(M_0)$. It follows from our induction assumption that the function f_P in $C^\infty_c(M_P, \tau)$ vanishes. In particular,

$$\hat{f}_{P,\mathrm{cusp}}(\lambda) = 0, \quad \lambda \in i\mathfrak{a}^*_{M_P}.$$

From (1.1) and (1.3) we see that if $\mathscr{P}$ is an associated class of parabolic subgroups, $\mathscr{P} \neq \{G\}$, the function

$$f_{\mathscr{P}}(x) = |\mathscr{P}|^{-1} \sum_{P \in \mathscr{P}} |W(\mathfrak{a}_P)|^{-1} \int_{i\mathfrak{a}^*_P} E_P(x, \mu_P(\lambda) \hat{f}_{P,\mathrm{cusp}}(\lambda), \lambda)\, d\lambda$$

equals zero. Therefore

$$f(x) = \sum_{\mathscr{P}} f_{\mathscr{P}}(x) = f_{\{G\}}(x).$$

Applying a Fourier transform on A_G, we obtain

$$\hat{f}_{\mathrm{cusp}}(\lambda, x) = \int_{A_G} f(xa)\, e^{-\lambda(H_G(xa))}\, da.$$

As a function of x, the expression on the right has compact support modulo A_G. The expression on the left, however, is a $\mathscr{Z}$-finite, τ-spherical function on G. In particular, it is analytic. Since G/A_G is not compact, both functions must vanish identically in x and λ. By Fourier inversion on A_G, $f(x)$ vanishes. This establishes injectivity. Q.E.D.

The next theorem is the culmination of everything. It tells us that any collection $F \in PW(G,\tau)$ can be extended to a collection in $\mathscr{C}(\hat{G},\tau)$ so that the functions $F_{P,\mathrm{cusp}}(\lambda,x)$, $F^\vee(x)$ and $F^\vee_{\mathscr{P}}(x)$, which we defined differently for $PW(G,\tau)$ and $\mathscr{C}(\hat{G},\tau)$, actually coincide.

THEOREM 3.2. *For every $F=\{F_B\}$ in $PW(G,\tau)$ there is a unique function $f \in C_c^\infty(G,\tau)$ such that*

$$\hat{f}_B(\Lambda)=F_B(\Lambda),\quad B\in\mathscr{P}(M_0),\ \Lambda\in\mathfrak{a}^*_{0,\mathbf{C}}.$$

It has the additional properties

(i) $\hat{f}_{P,\mathrm{cusp}}(\lambda)=F_{P,\mathrm{cusp}}(\lambda),\quad P\in\mathscr{F}(M_0),\ \lambda\in i\mathfrak{a}^*_{M_P},$

(ii) $f(x)=F^\vee(x),$

(iii) $f_{\mathscr{P}}(x)=F^\vee_{\mathscr{P}}(x),$

for each associated class $\mathscr{P}$.

Proof. The uniqueness of f was established in the last lemma so we can concentrate on its existence. If G is compact modulo A_G, the theorem is an immediate consequence of the classical Paley-Wiener theorem.

Suppose then that G/A_G is not compact. We will assume by induction that the theorem holds if G is replaced by M_P, for any proper parabolic subgroup P in $\mathscr{F}(M_0)$. By Lemma 2.3 the collection

$$F_P=\{F_R(\Lambda)=F_{P(R)}(\Lambda): R\in\mathscr{P}^{M_P}(M_0)\}$$

belongs to $PW(M_P,\tau)$. Then $F^\vee_P$, the associated τ_{M_P}-spherical function on $(M_P)_-$ defined in II, § 1, extends to a function in $C_c^\infty(M_P,\tau)$ with the properties demanded by the theorem. By (i),

$$F_{P,\mathrm{cusp}}(\lambda)=(F^\vee_P)^\wedge_{\mathrm{cusp}}(\lambda).$$

In particular, $F_{P,\mathrm{cusp}}(\lambda)$ belongs to $\mathscr{A}_{\mathrm{cusp}}(M_P,\tau)$. Now suppose that $\mathscr{P}$ is associated class, $\mathscr{P}\neq\{G\}$. We shall use the formula for $F^\vee_{\mathscr{P}}(x)$ in Theorem II.7.1. Since $F_{P,\mathrm{cusp}}(\lambda)$ belongs to $\mathscr{A}_{\mathrm{cusp}}(M_P,\tau)$, for $P\in\mathscr{P}$, the function $E_P(x,F_{P,\mathrm{cusp}}(\lambda),\lambda)$ is entire in λ. Therefore

$$E_P(x,\mu_P(\lambda)F_{P,\mathrm{cusp}}(\lambda),\lambda)$$

is holomorphic in a neighborhood of $i\mathfrak{a}_P^*$. We can therefore change the contour in the integral in the formula for $F_{\mathscr{P}}^{\vee}(x)$ from $\varepsilon_P+i\mathfrak{a}_P^*$ to $i\mathfrak{a}_P^*$. We obtain

$$F_{\mathscr{P}}^{\vee}(x)=\frac{1}{|\mathscr{P}|}\sum_{P\in\mathscr{P}}\frac{1}{|W(\mathfrak{a}_P)|}\int_{i\mathfrak{a}_P^*}E_P(x,\mu_P(\lambda)F_{P,\,\mathrm{cusp}}(\lambda),\lambda)\,d\lambda. \tag{3.1}$$

Thus $F_{\mathscr{P}}^{\vee}$ belongs to $\mathscr{C}_{\mathscr{P}}(G,\tau)$. In particular, it extends across G_- to a smooth function of G.

Suppose, given what we have shown so far, that we were able to find a function $f\in C_c^\infty(G,\tau)$ such that $\hat{f}_B(\Lambda)=F_B(\Lambda)$ for each B. We will show how the other conditions of the lemma follow. We must first check a compatibility condition. If P is a proper parabolic subgroup of G we have two functions in $C_c^\infty(M_P,\tau)$; the function $F_P^{\vee}$ given by our induction assumption above, and the function

$$f_P(m)=\delta_P(m)^{1/2}\int_{N_P}f(mn)\,dn.$$

We must show they are the same. If $R\in\mathscr{P}^{M_P}(M_0)$,

$$(f_P)_R^{\wedge}(\Lambda)=\hat{f}_{P(R)}(\Lambda)=F_{P(R)}(\Lambda).$$

However, $F_P^{\vee}$ is by definition the unique function in $C_c^\infty(M_P,\tau)$ with this property, so the two functions are in fact the same. Now, if $P\neq G$,

$$\hat{f}_{P,\,\mathrm{cusp}}(\lambda)=(F_P^{\vee})_{\mathrm{cusp}}^{\wedge}(\lambda)=F_{P,\,\mathrm{cusp}}(\lambda).$$

This is property (i). It follows from (3.1) that if $\mathscr{P}\neq\{G\}$, $f_{\mathscr{P}}(x)$ equals $F_{\mathscr{P}}^{\vee}(x)$. Therefore if $x\in G_-$,

$$\begin{aligned}f(x)-F^{\vee}(x)&=\sum_{\mathscr{P}}f_{\mathscr{P}}(x)-\sum_{\mathscr{P}}F_{\mathscr{P}}^{\vee}(x)\\&=f_{\{G\}}(x)-F_{\{G\}}^{\vee}(x).\end{aligned}$$

Applying a Fourier transform on A_G, we obtain

$$\hat{f}_{\mathrm{cusp}}(\lambda,x)-F_{\mathrm{cusp}}(\lambda,x)=\int_{A_G}(f(xa)-F^{\vee}(xa))\,e^{-\lambda(H_G(xa))}\,da.$$

As a function of x, the expression on the right has compact support modulo A_G. The expression on the left, however, is $\mathscr{Z}$-finite and τ-spherical, and is an analytic function on G_-. Since G/A_G is not compact, both functions must vanish identically in x and λ. Therefore

$$\hat{f}_{\text{cusp}}(\lambda, x) = F_{\text{cusp}}(\lambda, x)$$

for all x and λ. In particular, $F_{\text{cusp}}(\lambda)$ belongs to $\mathscr{A}_{\text{cusp}}(G, \tau)$ for each λ. By Fourier inversion on A_G, we see also that $f_{\{G\}}(x)=F^{\vee}_{\{G\}}(x)$ and that $f(x)=F^{\vee}(x)$. This verifies all the conditions (i), (ii), and (iii) for f.

However, we have not yet proved the existence of f. To proceed, recall that there is an integer d, independent of λ and F, such that the function

$$F_{\text{cusp}}(\lambda, x) = |\mathscr{P}(M_0)|^{-1} \sum_{B \in \mathscr{P}(M_0)} \sum_{T \in \mathscr{T}_B(\mathfrak{a}_G^*, X_B)} r_T(x, dF_B(\Lambda_T+\lambda), \Lambda_T)$$

depends only on the projection of the vector

$$\bigoplus_B \bigoplus_T (d_T F_B)(\Lambda_T+\lambda), \quad \lambda \in \mathfrak{a}^*_{G,\mathbf{C}}, \tag{3.2}$$

onto the finite dimensional vector space

$$\bigoplus_B \bigoplus_T \operatorname{Hom}(S_d((\mathfrak{a}_G^*)^{\perp}_{\mathbf{C}}), \mathscr{A}_0). \tag{3.3}$$

It follows from Corollary 2.2 that there is an h in $C_c^\infty(G, \tau)$ such that for all $\lambda \in \mathfrak{a}^*_{G,\mathbf{C}}$, the vector

$$\bigoplus_B \bigoplus_T (d_T \hat{h}_B)(\Lambda_T+\lambda)$$

has the same projection onto the space (3.3) as does (3.2). Therefore, $F_{\text{cusp}}(\lambda, x)$ equals

$$|\mathscr{P}(M_0)|^{-1} \sum_{B \in \mathscr{P}(M_0)} \sum_T r_T(x, d\hat{h}_B(\Lambda_T+\lambda), \Lambda_T).$$

From the discussion above, (with f replaced by h), we obtain

$$\hat{h}_{\text{cusp}}(\lambda, x) = F_{\text{cusp}}(\lambda, x),$$

and then by Fourier inversion on A_G,

$$h_{\{G\}}(x) = F^{\vee}_{\{G\}}(x).$$

In particular, $F^{\vee}_{\{G\}}(x)$ extends to a function in $\mathscr{C}_{\{G\}}(G,\tau)$. Therefore the function

$$F^{\vee}(x)=\sum_{\mathscr{P}} F^{\vee}_{\mathscr{P}}(x),$$

defined, a priori only for $x\in G_-$, extends to a function in $\mathscr{C}(G,\tau)$. This will be the required function f. Since it has bounded support, f belongs to $C_c^\infty(G,\tau)$. For any $\mathscr{P}$, $F^{\vee}_{\mathscr{P}}$ is just the projection of f onto $\mathscr{C}_{\mathscr{P}}(G,\tau)$, so it equals $f_{\mathscr{P}}$.

Let $\mathscr{P}_0=\mathscr{P}(M_0)$, and take a group $B\in\mathscr{P}_0$. It follows easily from (1.2) that if $\mathscr{P}$ is an associated class distinct from $\mathscr{P}_0$, and Λ is a point in $i\mathfrak{a}_0^*$,

$$(f_{\mathscr{P}})^{\wedge}_{B}(\Lambda)=0.$$

Therefore,

$$\hat{f}_B(\Lambda)=(f_{\mathscr{P}_0})^{\wedge}_{B}(\Lambda).$$

Consider the collection

$$F_{\mathscr{P}_0}=\{F_{\mathscr{P}_0,P,\mathrm{cusp}}(\lambda):P\in\mathscr{F}(M_0)\}$$

in which $F_{\mathscr{P}_0,P,\mathrm{cusp}}$ equals F_P if P belongs to $\mathscr{P}_0$, and equals 0 otherwise. The collection belongs to $\mathscr{C}(\hat{G},\tau)$; in fact it is the Fourier transform (in the sense of § 1) of $f_{\mathscr{P}_0}$. That is,

$$F_{\mathscr{P}_0,P,\mathrm{cusp}}(\lambda)=(f_{\mathscr{P}_0})^{\wedge}_{P}(\lambda),$$

for each $P\in\mathscr{F}(M_0)$. It follows that

$$\hat{f}_B(\Lambda)=F_B(\Lambda)$$

for each $B\in\mathscr{P}(M_0)$, and all $\Lambda\in i\mathfrak{a}_0^*$. By analytic continuation the formula is true for all $\Lambda\in\mathfrak{a}^*_{0,\mathbf{C}}$. Thus, f is the function required by the lemma. Q.E.D.

If we combine Theorems 3.1 and 3.2 of this section with Theorem II.1.1 we obtain

THEOREM 3.3. *The map*

$$f\to\{\hat{f}_B: B\in\mathscr{P}(M_0)\},\quad f\in C_c^\infty(G,\tau),$$

is a topological isomorphism from $C_c^\infty(G,\tau)$ onto $PW(G,\tau)$. For any N, the image of the space $C_N^\infty(G,\tau)$ is *$PW_N(G,\tau)$.*

§ 4. The Hecke algebra and multipliers

It is a simple matter to reformulate our results for complex valued K-finite functions on G. Suppose that N is a positive number and that Γ is a finite set of classes of irreducible representations of K. Let $C_N^\infty(G)_\Gamma$ be the set of smooth complex valued functions on G, which are supported on $G(N)$, and whose left and right translates by K each span a space which under the action of K is a direct sum of representations in Γ. Define $C_c^\infty(G, K)$ to be the direct limit over N and Γ of the spaces $C_N^\infty(G)_\Gamma$. It is just the space of K finite functions in $C_c^\infty(G)$, but it is a complete topological vector space under the direct limit topology.

We shall let $\mathrm{Rep}(G)$ denote the set of irreducible admissible representations of G. Suppose that N and Γ are as above. If (σ, U_σ) belongs to $\mathrm{Rep}(M_0/A_0)$ and $B \in \mathscr{P}(M_0)$, let $\mathscr{H}_B(\sigma)_\Gamma$ be the sum, over all irreducible representations τ of K which belong to an equivalence class in Γ, of the spaces $\mathscr{H}_B(\sigma)_\tau$ defined in I, § 3. Let $PW_N(G)_\Gamma$ be the space of collections

$$F = \{F_B(\sigma) : B \in \mathscr{P}(M_0),\ \sigma \in \mathrm{Rep}(M_0/A_0)\}$$

of entire functions

$$F_B(\sigma) : \Lambda \to F_B(\sigma, \Lambda), \quad \Lambda \in \mathfrak{a}_{0,\mathbf{C}}^*,$$

from $\mathfrak{a}_{0,\mathbf{C}}^*$ to $\mathscr{H}_B(\sigma)_\Gamma$ which satisfy two conditions. First, suppose that for all $x \in G$ a relation

$$\sum_{k=1}^{n} D_k(I_{B_k}(\sigma_k, \Lambda_k, x)\Psi_k, \tilde{\Psi}_k) = 0 \tag{4.1}$$

holds, for differential operators D_k of constant coefficients on $\mathfrak{a}_{0,\mathbf{C}}^*$ and vectors Ψ_k, $\tilde{\Psi}_k$, in $\mathscr{H}_{B_k}(\sigma_k)_\Gamma$. Then the relation

$$\sum_{k=1}^{n} D_k(F_{B_k}(\sigma_k, \Lambda_k)\Psi_k, \tilde{\Psi}_k) = 0$$

must also hold. (As in III, § 2, it is understood that D_k acts through the variable Λ_k.) Secondly, for every integer n the semi-norm

$$\|F\|_{N,n} = \sup_{\{B,\sigma,\Lambda\}} (\|F_B(\sigma, \Lambda)\| e^{-N\|\mathrm{Re}\,\Lambda\|}(1 + \|\Lambda\|)^n)$$

is finite. With these semi-norms, $PW_N(G)_\Gamma$ becomes a topological vector space. Define $PW(G, K)$ to be the direct limit, over N and Γ, of the spaces $PW_N(G)_\Gamma$.

If $f \in C_c^\infty(G, K)$, set

$$\hat{f}_B(\sigma, \Lambda) = I_B(\sigma, \Lambda, f) = \int_G f(x) I_B(\sigma, \Lambda, x)\, dx.$$

Then the collection

$$\hat{f} = \{\hat{f}_B(\sigma, \Lambda): B \in \mathscr{P}(M_0),\ \sigma \in \mathrm{Rep}\,(M_0/A_0)\}$$

belongs to $PW(G, K)$. It will be called the *Fourier transform* of f.

In I, § 3 we gave a dictionary between Eisenstein integrals and matrix coefficients of induced representations. The translation of Theorem III.3.3 is

THEOREM 4.1. *The map*

$$f \to \hat{f}, \quad f \in C_c^\infty(G, K),$$

is a topological isomorphism from $C_c^\infty(G, K)$ onto $PW(G, K)$. For any N and Γ, the image of $C_N^\infty(G)_\Gamma$ is $PW_N(G)_\Gamma$.

The space $C_c^\infty(G, K)$ is an algebra under convolution. It is sometimes called the *Hecke algebra*, in analogy with the theory of p-adic groups. It is clear that $PW(G, K)$ is also an algebra, and that the isomorphism of the last theorem preserves the multiplication.

The space $C_c^\infty(G, K)$ also has the structure of a left and right module over the universal enveloping algebra $\mathscr{U}(G)$. Let $\mathrm{End}_{\mathscr{U}(G)}(C_c^\infty(G, K))$ be the algebra of left and right $\mathscr{U}(G)$ endomorphisms of $C_c^\infty(G, K)$. It is just the algebra of linear operators C on $C_c^\infty(G, K)$ such that

$$C(f * g) = C(f) * g = f * C(g)$$

for all f, $g \in C_c^\infty(G, K)$. We shall see how to explicitly exhibit a large number of such operators.

We have introduced the Cartan subalgebra $\mathfrak{h}_{\mathbf{C}} = \mathfrak{h}_{K,\mathbf{C}} \oplus \mathfrak{a}_{0,\mathbf{C}}$ of $\mathrm{Lie}\,(G) \otimes \mathbf{C}$. Set

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{a}_0.$$

It is a real abelian Lie algebra, which remains invariant under the Weyl group W of $(\mathrm{Lie}\,(G) \otimes \mathbf{C}, \mathfrak{h}_{\mathbf{C}})$. Let $\mathscr{E}(\mathfrak{h})^W$ be the space of compactly supported distributions on $\mathfrak{h}$

which are invariant under W. It is an algebra under convolution. Any $\gamma \in \mathscr{E}(\mathfrak{h})^W$ has a Fourier-Laplace transform

$$\hat{\gamma}(\nu), \quad \nu \in \mathfrak{h}_{\mathbb{C}}^*.$$

It is an entire, W invariant function on $\mathfrak{h}_{\mathbb{C}}^*$. There exist integers N_γ and n_γ such that the semi-norm

$$\sup_{\nu \in \mathfrak{h}_{\mathbb{C}}^*} (|\hat{\gamma}(\nu)| e^{-N_\gamma \|\mathrm{Re}\, \nu\|}(1+\|\nu\|)^{-n_\gamma}) \tag{4.2}$$

is finite. Here $\mathrm{Re}(\nu)$ stands for the real part of ν relative to the decomposition

$$\mathfrak{h}_{\mathbb{C}}^* = \mathfrak{h}^* + i\mathfrak{h}^*.$$

An example of a function

$$\hat{\gamma}(\nu), \quad \gamma \in \mathscr{E}(\mathfrak{h})^W, \ \nu \in \mathfrak{h}_{\mathbb{C}}^*,$$

is a W-invariant polynomial function on $\mathfrak{h}_{\mathbb{C}}^*$. Such a function is of the form p_z, for a unique differential operator z in $\mathscr{Z}$. There corresponds an operator

$$f \to zf, \quad f \in C_c^\infty(G, K),$$

in $\mathrm{End}_{\mathscr{U}(G)}(C_c^\infty(G, K))$. If π belongs to $\mathrm{Rep}(G)$, let $\{\nu_\pi\}$ be the W-orbit in $\mathfrak{h}_{\mathbb{C}}^*$ associated to the infinitesimal character of π. Then by definition

$$\pi(zf) = p_z(\nu_\pi)\pi(f), \quad f \in C_c^\infty(G, K).$$

The function zf is uniquely determined by this formula. The next theorem, which is the second major result of this paper, is a generalization of this example.

THEOREM 4.2. *For every distribution γ in $\mathscr{E}(\mathfrak{h})^W$ and every function $f \in C_c^\infty(G, K)$, there is a unique function f_γ in $C_c^\infty(G, K)$ such that*

$$\pi(f_\gamma) = \hat{\gamma}(\nu_\pi)\pi(f)$$

for any $\pi \in \mathrm{Rep}(G)$.

Proof. It is clear that f_γ is uniquely determined by this condition. We need only establish its existence. Fix $f \in C_N^\infty(G)_\Gamma$. Define a collection

$$F = \{F_B(\sigma, \Lambda) : B \in \mathscr{P}(M_0), \ \sigma \in \mathrm{Rep}(M_0/A_0)\}$$

by setting

$$F_B(\sigma, \Lambda) = \hat{\gamma}(\nu_\sigma + \Lambda) \hat{f}_B(\sigma, \Lambda), \quad \Lambda \in \mathfrak{a}^*_{\mathfrak{d},\mathbf{C}}.$$

We shall show that it belongs to $PW_{N+N_\gamma}(G)_\Gamma$.

The growth condition is easy. For any n, the semi-norm $\|F\|_{N+N_\gamma, n}$ is bounded by the product of (4.2) and $\|\hat{f}\|_{N, n+n_\gamma}$, and is in particular finite. Next, suppose that for all x, the relation (4.1) holds. We must show that

$$\sum_{k=1}^{n} D_k(F_{B_k}(\sigma_k, \Lambda_k) \Psi_k, \tilde{\Psi}_k) = 0.$$

Now $\hat{\gamma}$ is an entire function on $\mathfrak{h}^*_{\mathbf{C}}$; its Taylor series converges uniformly on compact subsets. Since $\hat{\gamma}$ is W-invariant, its Taylor series will be a sum of W-invariant polynomial functions on $\mathfrak{h}^*_{\mathbf{C}}$. It follows that there is a sequence $\{z_j\}_{j=1}^{\infty}$ of operators in $\mathscr{Z}$ such that

$$\hat{\gamma}(\nu) = \sum_{j=1}^{\infty} p_{z_j}(\nu), \quad \nu \in \mathfrak{h}^*_{\mathbf{C}},$$

with absolutely uniform convergence on compact subsets of $\mathfrak{h}^*_{\mathbf{C}}$. If π is the representation $I_B(\sigma, \Lambda)$, $\{\nu_\pi\}$ will be the W-orbit of the point $\nu_\sigma + \Lambda$. It follows that

$$\begin{aligned} F_B(\sigma, \Lambda) &= \hat{\gamma}(\nu_\sigma + \Lambda) \hat{f}_B(\sigma, \Lambda) \\ &= \sum_{j=1}^{\infty} p_{z_j}(\nu_\sigma + \Lambda) I_B(\sigma, \Lambda, f) \\ &= \sum_{j=1}^{\infty} I_B(\sigma, \Lambda, z_j f). \end{aligned}$$

Since any Taylor series can be differentiated term by term,

$$\sum_{k=1}^{n} D_k(F_{B_k}(\sigma_k, \Lambda_k) \Psi_k, \tilde{\Psi}_k)$$

equals

$$\sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{n} D_k I_{B_k}(\sigma_k, \Lambda_k, z_j f) \right\}.$$

The expression in the brackets will vanish, by virtue of the relation (4.1). We conclude that F does belong to $PW_{N+N_\gamma}(G)_\Gamma$.

Let f_γ be the unique function in $C^\infty_{N+N_\gamma}(G)_\Gamma$ whose Fourier transform is F. Any $\pi \in \mathrm{Rep}(G)$ will be equivalent to a subquotient of some representation $I_B(\sigma, \Lambda)$. Then $\pi(f_\gamma)$ will be equivalent to the action of the operator

$$I_B(\sigma, \Lambda, f_\gamma) = \hat{\gamma}(\nu_\sigma + \Lambda)\, I_B(\sigma, \Lambda, f)$$

on an invariant subquotient of $\mathscr{H}_B(\sigma)$. It follows that

$$\begin{aligned} \pi(f_\gamma) &= \hat{\gamma}(\nu_\sigma + \Lambda)\, \pi(f) \\ &= \hat{\gamma}(\nu_\pi)\, \pi(f). \end{aligned}$$

This proves the theorem. Q.E.D.

In the proof of the theorem we observed

COROLLARY 4.3. *If f belongs to $C^\infty_N(G)_\Gamma$, f_γ will belong to $C^\infty_{N+N_\gamma}(G)_\Gamma$.*

The following is also clear.

COROLLARY 4.4. *Define*

$$C_\gamma(f) = f_\gamma, \quad \gamma \in \mathscr{E}(\mathfrak{h})^W, f \in C^\infty_c(G, K).$$

Then the map

$$\gamma \to C_\gamma, \quad \gamma \in \mathscr{E}(\mathfrak{h})^W,$$

is a homomorphism from the algebra $\mathscr{E}(\mathfrak{h})^W$ to the algebra $\mathrm{End}_{\mathscr{U}(G)}(C^\infty_c(G, K))$.

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