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Pluricomplex energy

by

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1. Introduction

This paper is a study of the complex Monge–Ampère operator $(dd^c)^n$. Let Ω be an open and bounded subset of $\mathbf{C}^n$. If $u_j \in C^2(\Omega)$, $1 \leqslant j \leqslant n$, then the Monge–Ampère operator operates on $(u_1, ..., u_n)$ and equals $dd^c u_1 \wedge ... \wedge dd^c u_n$, where $d=\partial+\bar\partial$ and $d^c=i(\bar\partial-\partial)$. If also each u_j is plurisubharmonic, then $dd^c u_1 \wedge ... \wedge dd^c u_n$ is a positive measure. This operator is of great importance in pluripotential theory, where it plays a role similar to that of the Laplace operator in classical potential theory. The Laplace operator is a linear, second-order differential operator and thus is defined on all distributions on Ω, while the complex Monge–Ampère operator is non-linear and cannot be defined on all plurisubharmonic functions on Ω, cf. [14], [20] and [8]. Moreover, the operator is discontinuous in the weak*-topology, cf. [9].

On the other hand, it was shown by Bedford and Taylor [2] that $(dd^c)^n$ is well-defined on all locally bounded plurisubharmonic functions. The problem of extending the domain of definition beyond $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}$ and describing the corresponding range has been studied by several authors: [2], [3], [8], [13], [15], [16] and [17]. See [1] for a survey on pluripotential theory. In particular, §4 of that paper contains a discussion of the domain of definition for $(dd^c)^n$. In this paper, we define certain classes $\mathcal{E}_p$ and $\mathcal{F}_p$ of plurisubharmonic functions, and study the complex Monge–Ampère operator $(dd^c)^n$ on them.

We prove:

(1) $\mathcal{E}_p$ and $\mathcal{F}_p$ are convex cones (Theorem 3.3).

(2) $(dd^c)^n$ is well-defined on $\mathcal{E}_p$ (Theorem 3.5).

(3) The comparison principle is valid in $\mathcal{F}_p$ (Theorem 4.5).

Our main result is to be found in §5, where we study the Dirichlet problem and give a complete description of $(dd^c\mathcal{F}_p)^n$, $p \geqslant 1$ (Theorem 5.1).

The remaining sections are based on the results from §5.

In §6, we consider the Dirichlet problem for $\mathcal{E}_p$ and also prove a decomposition theorem for positive and compactly supported measures. The last two sections are devoted to the Dirichlet problem with continuous boundary data.

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2. The classes $\mathcal{E}_p$ and $\mathcal{F}_p$

Let Ω be an open, bounded, connected and hyperconvex set in $\mathbf{C}^n$, $n\geqslant 2$, i.e., there is a continuous plurisubharmonic function h on Ω with $\{z\in\Omega : h(z)<c\}$ relatively compact in Ω for all $c<0$. We denote by $\mathcal{E}_0$ the class of negative and bounded plurisubharmonic functions φ on Ω such that $\lim_{z\to\xi}\varphi(z)=0$, $\forall\xi\in\partial\Omega$, and $\int(dd^c\varphi)^n<+\infty$.

Then $\mathcal{E}_0$ is a convex cone, for if $\varphi,\psi\in\mathcal{E}_0$ then $\int_{\varphi=\alpha\psi}(dd^c(\varphi+\psi))^n=0$ for some α, $1<\alpha<2$,

$$\begin{aligned}\int_\Omega (dd^c(\varphi+\psi))^n &= \int_{\varphi-\alpha\psi<0}(dd^c(\varphi+\psi))^n+\int_{\alpha\psi-\varphi<0}(dd^c(\varphi+\psi))^n\\ &=\int_{\frac{1+\alpha}{\alpha}\varphi<\varphi+\psi}(dd^c(\varphi+\psi))^n+\int_{(1+\alpha)\psi<\varphi+\psi}(dd^c(\varphi+\psi))^n\\ &\leqslant 3^n\int_\Omega (dd^c\varphi)^n+(dd^c\psi)^n,\end{aligned}$$

by the comparison principle. Cf. [3], [7].

Remark. Integration by parts in the class $\mathcal{E}_0$ is justified by the finite-mass assumption, cf. [12].

Definition 2.1. Given a Borel subset E of Ω, we define the relative extremal plurisubharmonic function for E (relative to Ω) as the smallest upper semicontinuous majorant $h_E^*(z)$ of

$$h_E(z):=\sup\{\varphi(z)\in \mathrm{PSH}(\Omega): -1\leqslant\varphi\leqslant 0,\ \varphi\leqslant -1 \text{ on } E\}.$$

Remark. The set $\{h_E<h_E^*\}$ is pluripolar, cf. [3].

Definition 2.2. For every $p\geqslant 1$, we define $\mathcal{E}_p$ $(=\mathcal{E}_p(\Omega))$ to be the class of plurisubharmonic functions φ on Ω such that there exists a sequence $\varphi_j\in\mathcal{E}_0$ with $\varphi_j\searrow\varphi$, $j\to+\infty$, and $\sup_j\int(-\varphi_j)^p(dd^c\varphi_j)^n<+\infty$. If also φ_j can be chosen so that $\sup_j\int(dd^c\varphi_j)^n<+\infty$, we say that $\varphi\in\mathcal{F}_p$.

Note that $\mathcal{E}_0\subset\mathcal{F}_p\subset\mathcal{E}_p$, $\forall p\geqslant 1$, and that $\mathcal{F}_q\subset\mathcal{F}_p$ if $q>p$ by Hölder's inequality.

In the unit ball, the classical energy of a function $\varphi\in\mathcal{E}_1$ is

$$\int -\varphi\Delta\varphi = 4^n \int -\varphi(dd^c\varphi)\wedge(dd^c(|z|^2-1))^{n-1}.$$

By Theorem 3.2 below, this can be estimated by a power of $\int -\varphi(dd^c\varphi)^n$, so all functions in $\mathcal{E}_1$ are of finite classical energy. We may say that the functions in $\mathcal{E}_1$ are the plurisubharmonic functions of finite pluricomplex energy.

Example 2.3. Consider $\Omega=B\left(0,\frac{1}{2}\right)$, the ball of radius $\frac{1}{2}$, and $v_\alpha=-(-\log|z|)^\alpha+(\log 2)^\alpha$, $0<\alpha<1$. Then $0\geqslant v_\alpha\in\mathrm{PSH}(\Omega)$ and

$$(dd^cv_\alpha)^n = n\alpha^n(1-\alpha)(-\log|z|)^{n(\alpha-1)-1}\,d\log|z|\wedge d^c\log|z|\wedge(dd^c\log|z|)^{n-1},$$

where $d\log|z|\wedge d^c\log|z|\wedge(dd^c\log|z|)^{n-1}=c\,dV/|z|^{2n}$, c a positive constant.

Thus $v_\alpha\in\mathcal{E}_p$ if and only if

$$\int_0^{1/2}\frac{(-\log r)^{\alpha p}r^{2n-1}}{(-\log r)^{n(1-\alpha)+1}r^{2n}}\,dr<+\infty,$$

which is true exactly when

$$n(1-\alpha)+1-\alpha p>1.$$

Thus $v_\alpha\in\mathcal{E}_p \Leftrightarrow n/p+n>\alpha$.

3. The operator $(dd^c)^n$ is well-defined on $\mathcal{E}_p$

In this section, we extend the domain of definition of $(dd^c)^n$ to $\mathcal{E}_p$.

LEMMA 3.1. *If $v\in\mathcal{E}_0$ then*

$$\int(-\varphi)^{n+1}(dd^cv)^n\leqslant(n+1)!\,[\sup(-v)]^n\int(-\varphi)(dd^c\varphi)^n,\quad \forall\varphi\in\mathcal{E}_0.$$

Proof. Cf. [4]. □

THEOREM 3.2. *Suppose $u,v\in\mathcal{E}_0$. If $p\geqslant1$ then*

$$\int(-u)^p(dd^cu)^j\wedge(dd^cv)^{n-j} \leqslant D_{j,p}\left(\int(-u)^p(dd^cu)^n\right)^{\frac{p+j}{p+n}}\left(\int(-v)^p(dd^cv)^n\right)^{\frac{n-j}{p+n}},\quad 0\leqslant j\leqslant n,$$

where $D_{j,p}$ equals $p(p+j)(n-j)/(p-1)$ for $p>1$, and 1 for $p=1$.

Proof. Cf. [12], [18]. □

THEOREM 3.3. *The classes $\mathcal{E}_p$ and $\mathcal{F}_p$ are convex cones.*

Proof. If $\alpha\geqslant 0$ and $u\in\mathcal{E}_p$, then obviously $\alpha u\in\mathcal{E}_p$.

If $u,v\in\mathcal{E}_p$ we have to prove that $u+v\in\mathcal{E}_p$. Suppose that $u_j\searrow u$, $v_j\searrow v$ as in the definition of $\mathcal{E}_p$. We have to estimate

$$\int(-u_j-v_j)^p(dd^c(u_j+v_j))^n.$$

Using Hölder's inequality, it is enough to estimate terms of the form

$$\int(-u_j)^p(dd^cu_j)^s\wedge(dd^cv_j)^{n-s},\quad 0\leqslant s\leqslant n,$$

and

$$\int(-v_j)^p(dd^cu_j)^s\wedge(dd^cv_j)^{n-s},\quad 0\leqslant s\leqslant n.$$

These terms can be estimated by

$$\int(-u_j)^p(dd^cu_j)^n\quad\text{and}\quad\int(-v_j)^p(dd^cv_j)^n$$

using Theorem 3.2.

But these two sequences are uniformly bounded by assumption. The statement about $\mathcal{F}_p$ follows now from the calculation in §2.

The proof of Theorem 3.3 is complete. □

LEMMA 3.4. *Suppose that $u\in\mathcal{E}_p$ (or $\mathcal{F}_p$), $0\geqslant v\in\mathrm{PSH}(\Omega)$. Then $w=\max(u,v)\in\mathcal{E}_p$ (or $\mathcal{F}_p$).*

Proof. Suppose that $u_j\searrow u$ as in the definition of $\mathcal{E}_p$. Put $w_j=\max(u_j,v)$. Then

$$\begin{aligned}\int(-w_j)^p(dd^cw_j)^n&\leqslant\int(-u_j)^p(dd^cw_j)^n\\&\leqslant D_{0,p}^p\left(\int(-u_j)^p(dd^cu_j)^n\right)^{\frac{p}{p+n}}\left(\int(-w_j)^p(dd^cw_j)^n\right)^{\frac{n}{p+n}}\end{aligned}$$

by Theorem 3.2. Therefore

$$\int(-w_j)^p(dd^cw_j)^n\leqslant D_{0,p}^{(p+n)/p}\int(-u_j)^p(dd^cu_j)^n.$$

Since $u\in\mathcal{E}_p$, the right-hand side is uniformly bounded, which proves the lemma. □

THEOREM 3.5. *Suppose $\mathcal{E}_0 \ni u_j \searrow u$, $j \to +\infty$, and*

$$\sup_j \int (-u_j)^p (dd^c u_j)^n < +\infty.$$

Then $(dd^c u_j)^n$ is weakly convergent and the limit is independent of the particular sequence.

Proof. Let $\varepsilon > 0$ and $0 \leqslant \chi \in C_0^\infty(\Omega)$ be given. Define $\delta = \sup_{\operatorname{supp}\chi} u_1$ (which we assume to be <0). For each j, find $0 < r_j < r_{j-1}$ so that

$$r_j < \operatorname{dist}\big(\{u_j < \tfrac{1}{2}\delta\}, \complement\Omega\big)$$

and

$$\left| \int \chi (dd^c u_{r_j})^n - \int \chi (dd^c u_j)^n \right| < \varepsilon, \tag{1}$$

where $u_{r_j}(z) = \int u_j(z + r_j \xi)\, dV(\xi)$ (and where dV is the normalized Lebesgue measure on the unit ball).

Then $u_j \leqslant u_{r_j}$ and u_{r_j} is continuous and plurisubharmonic on $\{u_j < \frac{1}{2}\delta\}$. Define $\tilde{u}_j(z) = \max(u_{r_j} + \delta, 2u_j)$. Then $\{\tilde{u}_j\}$ is decreasing, $\tilde{u}_j \in \mathcal{E}_p$ by Lemma 3.4 and

$$\sup_j \int (-\tilde{u}_j)^p (dd^c \tilde{u}_j)^n < +\infty.$$

We now claim that $\lim_{j\to+\infty} \int \chi (dd^c \tilde{u}_j)^n$ exists. If we can prove this, the proof of the theorem is complete, since $\varepsilon > 0$ in (1) is arbitrary.

We first note that $\tilde{u} = \lim_{j\to+\infty} \tilde{u}_j \not\equiv -\infty$. For let h be an exhaustion function in $\mathcal{E}_0$ for Ω. Then

$$\begin{aligned} \int (-\tilde{u})^p (dd^c h)^n &= \lim_{j\to+\infty} \int (-\tilde{u}_j)^p (dd^c h)^n \\ &\leqslant D_{0,p} \sup_j \left(\int (-\tilde{u}_j)^p (dd^c \tilde{u}_j)^n \right)^{\frac{p}{n+p}} \left(\int (-h)^p (dd^c h)^n \right)^{\frac{n}{n+p}} < +\infty. \end{aligned} \tag{2}$$

Now, since $\tilde{u}_j$ is continuous near $\operatorname{supp}\chi$,

$$\begin{aligned} &\left| \int \chi (dd^c \tilde{u}_j)^n - \int \chi (dd^c \max(\tilde{u}_j, -k))^n \right| \\ &\quad = \left| \int_{\tilde{u} \leqslant -k} \chi (dd^c \tilde{u}_j)^n + \int_{\tilde{u} > -k} \chi (dd^c \tilde{u}_j)^n \right. \\ &\qquad \left. - \int_{\tilde{u} \leqslant -k} \chi (dd^c \max(\tilde{u}_j, -k))^n - \int_{\tilde{u} > -k} \chi (dd^c \max(\tilde{u}_j, -k))^n \right| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\tilde{u}\leqslant -k} \chi(dd^c\tilde{u}_j)^n + \int_{\tilde{u}\leqslant -k} \chi(dd^c \max(\tilde{u}_j,-k))^n \\
&\leqslant \frac{\sup\chi}{k^p}\int_{-\tilde{u}\geqslant k} k^p[(dd^c\tilde{u}_j)^n+(dd^c\max(\tilde{u}_j,-k))^n] \\
&\leqslant \frac{\sup\chi}{k^p}\int\{(-\tilde{u})^p(dd^c\tilde{u}_j)^n+(-\max(\tilde{u}_j,-k))^p(dd^c\max(\tilde{u}_j,-k))^n\} \\
&\leqslant \frac{\sup\chi}{k^p}\operatorname{const}\cdot\sup\int(-\tilde{u}_j)^p(dd^c\tilde{u}_j)^n
\end{aligned}
$$

by Theorem 3.2. This completes the proof of Theorem 3.5, since we have by [2] that $(dd^c\max(\tilde{u}_j,-k))^n$ converges weakly for each k.

Definition 3.6. For $u\in\mathcal{E}_p$, we define $(dd^cu)^n$ to be the non-negative measure found in Theorem 3.5.

THEOREM 3.7. *If $u_j\in\mathcal{E}_p$, $u_j\nearrow u$, $j\to+\infty$, then $u\in\mathcal{E}_p$ and*

$$(dd^cu_j)^n\to(dd^cu)^n,\quad j\to+\infty.$$

Proof. Since $u=\max(u,u_1)$, $u\in\mathcal{E}_p$ by Lemma 3.4. We can now use the ideas of Theorem 3.5, together with the monotone convergence theorem in [3], to prove Theorem 3.7. □

THEOREM 3.8. *If $u\in\mathcal{E}_1$, then $\int u(dd^cu)^n>-\infty$, and if $v_j\in\mathrm{PSH}(\Omega)$, $0\geqslant v_j\searrow u$, $j\to+\infty$, then $\int v_j(dd^cv_j)^n\searrow\int u(dd^cu)^n$, $j\to+\infty$.*

Proof. Since $u\in\mathcal{E}_1$, it follows from Lemma 3.4 that $v_j\in\mathcal{E}_1$, $\forall j\in\mathbf{N}$, and there is a decreasing sequence $u_j\in\mathcal{E}_0$ with

$$\lim_{j\to+\infty}u_j=u\quad\text{and}\quad\sup_j\int -u_j(dd^cu_j)^n=\alpha<+\infty.$$

Then

$$\int\max(u_j,v_k)(dd^c\max(u_j,v_k))^n\geqslant\int u_j(dd^cu_j)^n\geqslant-\alpha,\quad\forall j,k\in\mathbf{N},$$

so it is enough to prove that

$$\lim_{j\to+\infty}\int u_j(dd^cu_j)^n=\int u(dd^cu)^n.$$

We have for $k\geqslant j$,

$$
\begin{aligned}
\int -u_j(dd^cu_j)^n &\leqslant \int -u_j(dd^cu_k)^n \\
&= \int_{u_j\geqslant-\varepsilon} -u_j(dd^cu_k)^n+\int_{u_j<-\varepsilon} -u_j(dd^cu_k)^n
\end{aligned}
$$

for $\varepsilon>0$. Here

$$\begin{aligned}\int_{u_j\geqslant-\varepsilon}-u_j(dd^cu_k)^n&=\int_{u_j\geqslant-\varepsilon}-\sup(u_j,-\varepsilon)(dd^cu_k)^n\\&\leqslant\left(\int_\Omega-\sup(u_j,-\varepsilon)(dd^c\sup(u_j,-\varepsilon))^n\right)^{\frac{1}{1+n}}\left(\int_\Omega-u_k(dd^cu_k)^n\right)^{\frac{n}{n+1}}\\&\leqslant\left(\varepsilon\int(dd^cu_j)^n\right)^{\frac{1}{1+n}}\alpha^{n/(n+1)}\to0,\quad\varepsilon\to0.\end{aligned}$$

It follows from the proof of Theorem 3.5 that

$$\overline{\lim_{k\to+\infty}}\int_{u_j<-\varepsilon}-u_j(dd^cu_k)^n\leqslant\int_\Omega-u_j(dd^cu)^n.$$

On the other hand, since $-u_j$ is lower semicontinuous,

$$\underline{\lim}_{k\to+\infty}\int_\Omega-u_j(dd^cu_k)^n\geqslant\int_\Omega-u_j(dd^cu)^n.$$

Therefore, $\int u_j(dd^cu)^n=\lim_{k\to+\infty}\int u_j(dd^cu_k)^n$, $\forall j$.

Now

$$\begin{aligned}\lim_{j\to+\infty}\int u_j(dd^cu_j)^n&\geqslant\lim_{j\to+\infty}\lim_{k\to+\infty}\int u_j(dd^cu_k)^n\\&=\int u(dd^cu)^n\geqslant\overline{\lim_{k\to+\infty}}\int u(dd^cu_k)^n\\&=\overline{\lim_{k\to+\infty}}\lim_{j\to+\infty}\int u_j(dd^cu_k)^n\geqslant\lim_{j\to+\infty}\int u_j(dd^cu_j)^n.\end{aligned}$$

Hence $\lim_{j\to+\infty}\int u_j(dd^cu_j)^n=\int u(dd^cu)^n$, which completes the proof. □

Remark. The analogue of Theorem 3.8 for $p>1$ will be given in Theorem 5.6. The main difference is that for $p=1$, when $-u\leqslant-v$, integration by parts gives

$$\int(-u)(dd^cu)^n\leqslant\int(-v)(dd^cv)^n,$$

i.e., the constant $D_{0,1}$ in Theorem 3.2 equals 1, but for $1<p<\infty$, we only know that $D_{0,p}\geqslant1$.

We conclude this section with a few additional properties of $\mathcal{E}_p$.

LEMMA 3.9. *Suppose $h_j \in \mathcal{E}_0$,*

$$\int h_j (dd^c h_j)^n \to 0, \quad j \to \infty.$$

Then there is a subsequence $\{h_{k_j}\}$ such that

$$\sum h_{k_j} \in \mathcal{E}_1.$$

Proof. Suppose that h_{k_j}, $1 \leqslant j \leqslant N$, are chosen such that

$$\int \sum_{j=1}^{N} h_{k_j} \Big(dd^c \sum h_{k_j} \Big)^n > -1.$$

Choose $h_{k_{N+1}}$ such that

$$\int \sum_{j=1}^{N+1} h_{k_j} \Big(dd^c \sum_{j=1}^{N+1} h_{k_j} \Big)^n > -1,$$

that is,

$$\int \sum_{j=1}^{N} h_{k_j} \Big(dd^c \sum_{j=1}^{N+1} h_{k_j} \Big)^n + \int h_{k_{N+1}} \Big(dd^c \sum_{j=1}^{N+1} h_{k_j} \Big)^n > -1.$$

Note that the first term is the sum of $\int \sum_{j=1}^N h_{k_j} (dd^c \sum_{j=1}^N h_{k_j})^n$ and terms of the form $\int \sum_{j=1}^N h_{k_j} (dd^c \sum_{j=1}^N h_{k_j})^{n-p} \wedge (dd^c h_{k_{N+1}})^p$, $p \geqslant 1$. The first term is strictly greater than -1 by assumption and all the others together with the second term can be choosen as close to zero as we wish by Theorem 3.2.

In particular, we can choose h_{N+1} so that

$$\int \sum_{j=1}^{N+1} h_{k_j} \Big(dd^c \sum_{j=1}^{N+1} h_{k_j} \Big)^n > -1.$$

It follows that $h = \sum_{j=1}^{\infty} h_{k_j} \in \mathcal{E}_1$. □

PROPOSITION 3.10. *Suppose that E is a pluripolar subset of Ω. Then there is a $\psi \in \mathcal{E}_1$ such that $E \subset \{\psi = -\infty\}$.*

Proof. Choose a sequence of relatively compact open subsets θ_j such that every point of E is in all but finitely many θ_j, and such that

$$\int (dd^c h_{\theta_j})^n < \frac{1}{j}, \quad j \in \mathbf{N},$$

where h_{θ_j} is the relative extremal plurisubharmonic function for θ_j. We extract a subsequence of $(h_{\theta_j})_{j=1}^{\infty}$ such that $h=\sum h_{\theta_{k_j}}\in\mathcal{E}_1$, which is possible by Lemma 3.9. It follows that $E\subset\{h=-\infty\}$. □

Example 3.11. We construct a function $\gamma\in\mathcal{E}_p\setminus\mathcal{F}_p$, $\forall p\geqslant 1$. Let $\Omega=B$ be the unit ball and define $\gamma_j=\max(\log|z|,-1/2^j)$. Then $\int-\gamma_j(dd^c\gamma_j)^n=c/2^j$, where $0<c=\int(dd^c\gamma_j)^n$ and $\gamma=\sum_{k=1}^{\infty}\max(\log|z|,-1/2^{j_k})\in\mathcal{E}_1$ for some subsequence $(j_k)_{k=1}^{\infty}$ by Lemma 3.9.

Note also that $\int(dd^c(\sum_{k=1}^{m}\max(\log|z|,-1/2^{j_k}))^n=mc$ by Stokes' theorem, so $\gamma\notin\mathcal{F}_1$. Since $0\geqslant\gamma\geqslant-1$, it follows that $\gamma\in\mathcal{E}_p\setminus\mathcal{F}_p$, $\forall p\geqslant 1$.

LEMMA 3.12. *Suppose that $u\in\mathcal{E}_1$, where Ω is a strictly pseudoconvex domain. Then*

$$\overline{\lim_{z\to\xi}}\, u(z)=0, \quad \forall\xi\in\partial\Omega.$$

Proof. Define $\overline{\lim}_{z\to\xi}\, u(z)$, $\xi\in\partial\Omega$.

Then γ is upper semicontinuous and less than or equal to zero. If there is a point ξ_0 where $\gamma(\xi_0)<0$, then we can find a continuous function h on $\partial\Omega$ such that $\gamma\leqslant h\leqslant 0$ and $h(\xi_0)<0$. Then there is a unique plurisubharmonic function v continuous up to the boundary, with vanishing Monge–Ampère mass and equal to h on $\partial\Omega$. Let $u_j\in\mathcal{E}_0$, $u_j\to u$ as in the definition of u.

Then $\lim_{j\to+\infty}\max(u_j,v)\in\mathcal{E}_1$, so $\lim_{j\to+\infty}\max(u_j,v)=\max(u,v)=v\in\mathcal{E}_1$. By Theorem 3.8,

$$\lim_{j\to+\infty}\int-\max(u_j,v)(dd^c\max(u_j,v))^n=0,$$

but

$$0\leqslant\int-\max(u_j,v)(dd^c\max(u_j,v))^n$$

is increasing in j, so $\max(u_j,v)\equiv 0$, $\forall j$, which is a contradiction. □

4. The comparison principle is valid

Here, we prove that the comparison principle is valid in $\mathcal{F}_p$. In particular, this means that we have uniqueness in $\mathcal{F}_p$ for the Dirichlet problem we are going to study in §5.

LEMMA 4.1. *Let U be an open subset of Ω and assume that $u,v\in\mathcal{E}_p$, $u=v$ near ∂U. Then*

$$\int_U(dd^cu)^n=\int_U(dd^cv)^n.$$

Proof. Choose $U'\subset\subset U$ so that $u=v$ near $\partial U'$, and consider the usual regularizations u_ε and v_ε. If $\varepsilon>0$ is small enough, $v_\varepsilon=u_\varepsilon$ near ∂U, and if $\chi\in C_0^\infty(U')$ with $\chi=1$ near

$\{u_\varepsilon \neq v_\varepsilon\}$ then $\int \chi (dd^c u_\varepsilon)^n = \int u_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{n-1} = \int v_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{n-1} = \int \chi (dd^c v_\varepsilon)^n$ since $dd^c\chi = 0$ where $v_\varepsilon \neq u_\varepsilon$. Hence

$$\int \chi (dd^c u)^n = \int \chi (dd^c v)^n,$$

so

$$\int_U (dd^c u)^n = \int_U (dd^c v)^n. \qquad \square$$

LEMMA 4.2. *If $u, v \in \mathcal{F}_p$ and if $u \leqslant v$ on Ω, then*

$$\int_\Omega (dd^c u)^n \geqslant \int_\Omega (dd^c v)^n.$$

Proof. Let u_j and v_j be as in the definition of $\mathcal{F}_p$ and let $h \in \mathcal{E}_0 \cap C(\Omega)$. Then

$$\begin{aligned}\int_\Omega -h(dd^c v_j)^n &\leqslant \int_\Omega -h(dd^c u_j)^n \\ &\leqslant \int_\Omega -h(dd^c u)^n + \varlimsup_{j\to+\infty} \int_{h>-\varepsilon} -h(dd^c u_j)^n \\ &\leqslant \int_\Omega -h(dd^c u)^n + \varepsilon \varlimsup_{j\to+\infty} \int_\Omega (dd^c u_j)^n.\end{aligned}$$

If we let ε tend to zero, we get that

$$\int_\Omega -h(dd^c v)^n \leqslant \int_\Omega -h(dd^c u)^n.$$

To complete the proof, we let h decrease to -1. $\square$

LEMMA 4.3. *Suppose that we have $\omega \in \mathcal{E}_p$, $\omega_j \searrow \omega$, $j \to +\infty$, as in the definition of $\mathcal{E}_p$. If $0 \geqslant u, v \in \mathrm{PSH}(\Omega)$ then*

$$\int_{\{u<v\}} (dd^c \omega)^n \leqslant \varliminf_{j\to\infty} \int_{\{u<v\}} (dd^c \omega_j)^n, \tag{3}$$

and if $u \geqslant v$ near $\partial\Omega$ then

$$\int_{\{u+\varepsilon\leqslant v\}} (dd^c \omega)^n \geqslant \varlimsup_{j\to+\infty} \int_{\{u+\varepsilon\leqslant v\}} (dd^c \omega_j)^n, \quad \forall \varepsilon > 0. \tag{4}$$

Proof. Let $\delta > 0$ be given. Since u and v are quasicontinuous ([3], [7, p. 37]), and since

$$\sup_j \int_\Omega (-\omega_j)^p (dd^c \omega_j)^n < +\infty,$$

it follows from Proposition 3.10 that there is an open set $\mathcal{O}_\delta$ with $\sup_j \int_{\mathcal{O}_\delta}(dd^c\omega_j)^n<\delta$, and there are two continuous functions $\tilde{u}$ and $\tilde{v}$ such that $\{u\neq\tilde{u}\}\cup\{v\neq\tilde{v}\}\subset\mathcal{O}_\delta$. Therefore

$$\{u<v\}\subset\{\tilde{u}<\tilde{v}\}\cup\mathcal{O}_\delta\subset\{u<v\}\cup\mathcal{O}_\delta$$

and

$$\{u+\varepsilon\leqslant v\}\subset\{\tilde{u}+\varepsilon\leqslant\tilde{v}\}\cup\mathcal{O}_\delta\subset\{u+\varepsilon\leqslant v\}\cup\mathcal{O}_\delta,$$

and so

$$\begin{aligned}\int_{\{u<v\}}(dd^c\omega)^n &\leqslant \int_{\{\tilde{u}<\tilde{v}\}\cup\mathcal{O}_\delta}(dd^c\omega)^n \leqslant \varliminf_{j\to+\infty}\int_{\{\tilde{u}<\tilde{v}\}\cup\mathcal{O}_\delta}(dd^c\omega_j)^n \\ &\leqslant \varliminf_{j\to+\infty}\int_{\{u<v\}\cup\mathcal{O}_\delta}(dd^c\omega_j)^n \leqslant \varliminf_{j\to+\infty}\int_{\{u<v\}}(dd^c\omega_j)^n+\delta.\end{aligned}$$

Also, if $u\geqslant v$ near the boundary of Ω then

$$\{u+\varepsilon\leqslant v\}\subset\subset\Omega$$

and

$$\begin{aligned}\varlimsup_{j\to+\infty}\int_{\{u+\varepsilon\leqslant v\}}(dd^c\omega_j)^n &\leqslant \varlimsup_{j\to+\infty}\int_{\{\tilde{u}+\varepsilon\leqslant\tilde{v}\}}(dd^c\omega_j)^n+\delta \\ &\leqslant \int_{\{\tilde{u}+\varepsilon\leqslant\tilde{v}\}}(dd^c\omega)^n+\delta \leqslant \int_{\{u+\varepsilon\leqslant v\}}(dd^c\omega)^n+2\delta.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\{u+\varepsilon<v\}}(dd^c\omega)^n &\leqslant \varliminf_{j\to+\infty}\int_{\{u+\varepsilon<v\}}(dd^c\omega_j)^n+\delta \\ &\leqslant \varlimsup_{j\to+\infty}\int_{\{u+\varepsilon\leqslant v\}}(dd^c\omega_j)^n+\delta \leqslant \int_{\{u+\varepsilon\leqslant v\}}(dd^c\omega)^n+3\delta.\end{aligned}$$ □

LEMMA 4.4. *Let $p\geqslant 1$ and suppose $u,v\in\mathcal{F}_p$. Then*

$$\int_{\{u<v\}}(dd^cv)^n\leqslant\int_{\{u<v\}}(dd^cu)^n.$$

Proof. Let $v_j\searrow v$, $u_j\searrow u$ as in the definition of $\mathcal{F}_p$, and choose an open set $\mathcal{O}_\delta$ as in the previous proof with $\sup_j\int_{\mathcal{O}_\delta}[(dd^cu_j)^n+(dd^cv_j)^n]<\delta$. Using (3) of Lemma 4.3, we get

$$\begin{aligned}\int_{\{u<v\}}(dd^cv)^n &\leqslant \varliminf_{j\to+\infty}\int_{\{u<v\}}(dd^cv_j)^n \\ &\leqslant \varliminf_{j\to+\infty}\lim_{k\to+\infty}\int_{\{u_k<v\}}(dd^cv_j)^n \leqslant \varliminf_{j\to+\infty}\varlimsup_{k\to+\infty}\int_{\{u_k<v_j\}}(dd^cv_j)^n.\end{aligned}$$

By Lemma V:3, p. 42, in [7], this can be estimated by

$$\begin{aligned}\varliminf_{j\to+\infty}\varlimsup_{k\to+\infty}\int_{\{u_k<v_j\}}(dd^cu_k)^n &\leqslant \varlimsup_{j\to+\infty}\varlimsup_{k\to+\infty}\int_{\{u\leqslant v_j\}}(dd^cu_k)^n\\ &\leqslant \varlimsup_{j\to+\infty}\lim_{k\to+\infty}\int_{\{u\leqslant v_j\}\cap CO_\delta}(dd^cu_k)^n+\delta\\ &\leqslant \int_{\{u\leqslant v\}}(dd^cu)^n+2\delta+\lim_{k\to+\infty}\int_\Omega g(dd^cu_k)^n\\ &\leqslant \int_{\{u\leqslant v\}}(dd^cu)^n+2\delta\\ &\qquad+\lim_{k\to+\infty}\int_\Omega (g-1)(dd^cu_k)^n+\int_\Omega(dd^cu)^n\\ &\leqslant \int_{\{u\leqslant v\}}(dd^cu)^n+2\delta+\int_\Omega g(dd^cu)^n,\end{aligned}$$

where g is any non-negative and continuous function which is bounded by 1 and equal to 1 close to the boundary of Ω. In the second last step, we have used the estimate

$$\int_\Omega (dd^cu_j)^n\leqslant\int_\Omega(dd^cu)^n,$$

which follows from Lemma 4.2. To complete the proof, we let g tend to zero. □

THEOREM 4.5 (the comparison principle). *Let $p\geqslant 1$ and suppose that $u,v\in\mathcal{F}_p$ with $(dd^cu)^n\leqslant(dd^cv)^n$. Then $v\leqslant u$ on Ω.*

Proof. Since Ω admits a continuous exhaustion function in $\mathcal{E}_0$, there is to every point $z_0\in\Omega$ a continuous exhaustion function P so that $(dd^cP)^n\geqslant dV$ near z_0, where dV denotes the Lebesgue measure. If there is a $z_0\in\Omega$ with $u(z_0)<v(z_0)$, take $\eta>0$ so small that $u(z_0)<v(z_0)+\eta P(z_0)$. Then the Lebesgue measure of $T=\{z\in\Omega: u<v+\eta P\}$ is strictly positive, and so is $\int_T(dd^cP)^n$.

By Lemma 4.4 we have that

$$\int_T(dd^c(v+\eta P))^n\leqslant\int_T(dd^cu)^n,$$

but the right-hand side is assumed to be smaller than or equal to $\int_T(dd^cv)^n$. Hence $\int_T(dd^cv)^n+\eta^n\int_T(dd^cP)^n\leqslant\int_T(dd^cv)^n$, so $\int_T(dd^cP)^n=0$, which is a contradiction. □

Remark. Except for the above result, Lemma 4.4 is sometimes also called "the comparison principle". There is also a comparison principle for bounded plurisubharmonic functions: Suppose that u and v are bounded plurisubharmonic functions which are continuous and equal at the boundary of the domain Ω. If $(dd^cu)^n\leqslant(dd^cv)^n$ on Ω then $u\geqslant v$ on Ω. Cf. [6].

5. The Dirichlet problem

We now prove the main theorem of this paper.

THEOREM 5.1. *Let Ω be a bounded and hyperconvex set in $\mathbf{C}^n$, $n\geqslant 2$, $p\geqslant 1$ and μ a positive measure with finite total mass on Ω. Then there is a (uniquely determined) function $u\in\mathcal{F}_p$ with $(dd^c u)^n=\mu$ if and only if there is a constant A such that*

$$\int(-\varphi)^p\,d\mu\leqslant A\left(\int(-\varphi)^p(dd^c\varphi)^n\right)^{\frac{p}{n+p}},\quad \forall\varphi\in\mathcal{E}_0. \tag{5}$$

Remark. Note that if μ is a measure satisfying (5) for some $p\geqslant 1$, then μ puts no mass on pluripolar sets.

LEMMA 5.2. *Suppose that μ is a positive and compactly supported measure satisfying (5) with $p>n/(n-1)$. If $u_j\in\mathcal{E}_0\cap C(\bar\Omega)$, $u_j\to u\in\mathrm{PSH}(\Omega)$, $j\to+\infty$, a.e. dV, and if $\sup_j\int(dd^c u_j)^n<+\infty$, then $\lim_{j\to+\infty}\int u_j\,d\mu=\int u\,d\mu$.*

Proof. Note that $\overline{\lim}_{j\to+\infty}\int u_j\,d\mu\leqslant\int u\,d\mu$, so it is enough to prove that $\int u\,d\mu\leqslant\underline{\lim}_{j\to+\infty}\int u_j\,d\mu$. For each $N\in\mathbf{N}$, write $A_N^j=\{z\in\operatorname{supp}\mu: u_j<-N\}$. Then

$$\int_{A_N^j}d\mu\leqslant\int(-h_{A_N^j})^p\,d\mu\leqslant A\left(\int_\Omega(dd^c h_{A_N^j})^n\right)^{\frac{p}{n+p}},$$

where $h_E(z)$ is the relative extremal plurisubharmonic function for E. By Lemma 4.4,

$$\begin{aligned}\int_\Omega(dd^c h_{A_N^j})^n&=\int_{\bar A_N^j}(dd^c h_{A_N^j})^n\leqslant\int_{2u_j/N<h_{A_N^j}}(dd^c h_{A_N^j})^n\\&\leqslant\frac{2^n}{N^n}\int(dd^c u_j)^n\leqslant\frac{2^n}{N^n}\alpha,\end{aligned}\tag{6}$$

where $\alpha=\sup\int(dd^c u_j)^n$. Hence

$$\int_{A_N^j}d\mu\leqslant A(2^n\alpha)^{p/(n+p)}\frac{1}{N^{np/(n+p)}}.$$

Since $p>n/(n-1)$, $\gamma=np/(n+p)>1$. Therefore

$$\int_{A_{2^N}^j}-u_j\,d\mu=\sum_{k=N}^{\infty}\int_{-2^{k+1}<u_j\leqslant-2^k}-u_j\,d\mu\leqslant A(2^n\alpha)^{p/(n+p)}\sum_{k=N}^{\infty}\frac{2^{k+1}}{2^{k\gamma}}\to 0,\quad N\to+\infty.$$

Thus

$$\begin{aligned}\int_\Omega-u_j\,d\mu&=\int_{u_j\geqslant-2^N}-u_j\,d\mu+\int_{u_j<-2^N}-u_j\,d\mu\\&\leqslant\int 2^N\,d\mu+A(2^n\alpha)^{p/(n+p)}\sum_{k=N}^{\infty}\frac{2^{k+1}}{2^{k\gamma}},\quad N\in\mathbf{N}.\end{aligned}\tag{7}$$

In particular, $\sup_j \int_\Omega -u_j\,d\mu<+\infty$, and we see that it is enough to prove that $\int -\max(u_j,-N)\,d\mu \to \int -\max(u,-N)\,d\mu$, $j\to+\infty$. In other words, we can assume that $\{u_j\}$ is uniformly bounded.

In this case, since $\sup_j \int u_j^2\,d\mu<+\infty$, there is a $v\in L^2(d\mu)$ and a subsequence u_{j_t} so that $(1/M)\sum_{t=1}^M u_{j_t}\to v$ in $L^2(d\mu)$. Then there is a subsequence M_q such that $f_q=(1/M_q)\sum_{t=1}^{M_q} u_{j_t}\to v$ a.e. $d\mu$, $q\to+\infty$. But $f_q\to u$ in $L^2(dV)$ so $(\sup_{r\geqslant q} f_r)^*\searrow u$ everywhere, and

$$\int \big(\sup_{r\geqslant q} f_r\big)^* \,d\mu = \int \sup_{r\geqslant q} f_r\,d\mu \to \int v\,d\mu, \quad q\to\infty,$$

from the remark above and the fact that $f_r\to v$ a.e. $d\mu$. Thus we have $\int u\,d\mu=\int v\,d\mu=\lim \int u_{j_t}\,d\mu$. □

Lemma 5.3. *If we have that $u_s\in\mathcal{E}_0\cap C(\Omega)$, $u\in\mathrm{PSH}$, $u_s\to u$, $s\to+\infty$, a.e. dV, $\sup\int -u_s(dd^cu_s)^n<+\infty$ and if $\int |u-u_s|(dd^cu_s)^n\to 0$, then $(dd^cu_s)^n\to(dd^cu)^n$.*

Proof. We can assume $\int |u-u_s|(dd^cu_s)^n<1/s^2$. Then, for $0\leqslant\chi\in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\left|\int \chi(dd^cu)^n-\int \chi(dd^cu_s)^n\right|\\
&\quad=\left|\int \chi[(dd^cu)^n-(dd^c(\max(u_s+1/s,u)-1/s))^n\right.\\
&\qquad\left.+(dd^c(\max(u_s+1/s,u)-1/s))^n-(dd^cu_s)^n]\right|\\
&\quad\leqslant\left|\int \chi[(dd^cu)^n-(dd^c\max((u_s+1/s,u)-1/s))^n]\right|\\
&\qquad+\left|\int_{u_s+1/s\leqslant u} \chi[(dd^c(\max(u_s+1/s,u)-1/s))^n-(dd^cu_s)^n]\right|\\
&\quad\leqslant\left|\int \chi[(dd^cu)^n-(dd^c\max((u_s+1/s,u)-1/s))^n]\right|\\
&\qquad+2\sup\chi\left|\int_{u_s+1/s\leqslant u}(dd^cu_s)^n\right|.
\end{aligned}$$

Since

$$\int_{u_s+1/s\leqslant u}(dd^cu_s)^n\leqslant s\int |u-u_s|(dd^cu_s)^n\to 0, \quad s\to+\infty,$$

it is enough to prove that

$$(dd^c(\max(u_s+1/s,u)-1/s))^n\to(dd^cu)^n, \quad s\to+\infty.$$

Define $g_s = \max(u_s + 1/s, u) - 1/s$. Then

$$\int_{g_s<-N} (dd^c g_s)^n \leqslant \frac{1}{N} \int (-u_s)(dd^c u_s)^n.$$

Hence $\int_{g_s<-N} (dd^c g_s)^n \to 0$ uniformly in s when $N \to +\infty$, so it is enough to prove that

$$(dd^c \max(g_s, -N))^n \to (dd^c \max(u, -N))^n, \quad s \to +\infty, \ \forall N \in \mathbf{N}.$$

It follows from the construction of g_s that

$$\max(g_s, -N) \to \max(u, -N)$$

in C_n-capacity (cf. [21]). We can therefore apply Theorem 1 in [21] to conclude that

$$(dd^c \max(g_s, -N))^n \to (dd^c \max(u, -N))^n, \quad s \to +\infty,$$

which completes the proof of Lemma 5.3. □

LEMMA 5.4. *Suppose that $u \in \mathcal{E}_p$ and that ψ is a negative, continuous and plurisubharmonic function on Ω. Then*

$$\chi_A (dd^c u)^n = \chi_A (dd^c \max(u, \psi))^n,$$

where $A = \{z \in \Omega : u > \psi\}$.

In particular,

$$\chi_A (dd^c u)^n \leqslant (dd^c \max(u, \psi))^n.$$

Proof. The lemma is trivially true when u is continuous. Let K be a given compact subset of Ω, and $\mathcal{O}$ a relatively compact open subset of Ω containing K. Following the proof of Theorem 3.5, given $\delta < 0$, choose $v_j \in \mathcal{E}_p$, v_j decreasing to $\max(u+\delta, 2u)$ on Ω, and v_j decreasing to $u+\delta$ on $\mathcal{O}$, and v_j continuous on $\mathcal{O}$. Given $\varepsilon > 0$, choose $\mathcal{O}_1$ open in $\mathcal{O}$, containing K, and K_1 compact in $A \cap K$, such that

$$\int (dd^c h_{\mathcal{O}_1 \setminus K_1})^n < \varepsilon.$$

Then, with

$$A^j = \{z \in \mathcal{O} : v_j > \psi + \delta\},$$

we have

$$\chi_{A^j} (dd^c \max(v_j, \psi+\delta))^n = \chi_{A^j} (dd^c v_j)^n.$$

So

$$\chi_{K\cap A}(dd^c v_j)^n = \chi_{K\cap A}(dd^c \max(v_j, \psi+\delta))^n,$$

and therefore,

$$\chi_{A\cap K}(dd^c \max(v_j, \psi+\delta))^n = \chi_{\mathcal{O}_1}(dd^c v_j)^n + (\chi_{A\cap K} - \chi_{\mathcal{O}_1})(dd^c v_j)^n.$$

Here,

$$\begin{aligned}\int (\chi_{\mathcal{O}_1} - \chi_{A\cap K})(dd^c v_j)^n &\leqslant \int -h_{\mathcal{O}_1\setminus K_1}(dd^c v_j)^n \\ &\leqslant D_{0,p}\left(\int (dd^c h_{\mathcal{O}_1\setminus K_1})^n\right)^{\frac{p}{n+p}} \left(\int (-v_j)^p (dd^c v_j)^n\right)^{\frac{n}{n+p}} \\ &\leqslant \mathrm{const}\cdot \varepsilon^{p/(n+p)}.\end{aligned}$$

Since $\chi_{\mathcal{O}_1}$ is lower semicontinuous, we can now use

$$\begin{aligned}&\chi_{\mathcal{O}_1}(dd^c v_j)^n + (\chi_{A\cap K} - \chi_{\mathcal{O}_1})(dd^c v_j)^n \\ &\quad = \chi_{A\cap K}(dd^c \max(v_j, \psi+\delta)^n \\ &\quad = \chi_{K_1}(dd^c \max(v_j, \psi+\delta))^n + (\chi_{A\cap K} - \chi_{K_1})(dd^c \max(v_j, \psi+\delta))^n\end{aligned}$$

to conclude that

$$\chi_{\mathcal{O}_1}(dd^c u)^n \leqslant \chi_{K_1}(dd^c \max(u+\delta, \psi+\delta))^n + d\mu_\varepsilon,$$

where

$$\int d|\mu_\varepsilon| \leqslant \varepsilon.$$

Therefore,

$$\chi_{A\cap K}(dd^c u)^n \leqslant \chi_{A\cap K}(dd^c \max(u, \psi))^n$$

and the reverse inequality can be obtained in a similar way using

$$\begin{aligned}&\chi_{K_1}(dd^c v_j)^n + (\chi_{A\cap K} - \chi_{K_1})(dd^c v_j)^n \\ &\quad = \chi_{A\cap K}(dd^c v_j)^n = \chi_{A\cap K}(dd^c \max(v_j, \psi+\delta))^n \\ &\quad = \chi_{\mathcal{O}_1}(dd^c \max(v_j, \psi+\delta))^n + (\chi_{A\cap K} - \chi_{\mathcal{O}_1})(dd^c \max(v_j, \psi+\delta))^n. \qquad \square\end{aligned}$$

Proof of Theorem 5.1. Suppose first that $p>n/(n-1)$ and that μ has compact support in Ω. For each s large enough, we consider a subdivision I^s of $\operatorname{supp}\mu$ consisting of

$c\cdot 3^{2ns}$ isomorphic, semi-open cubes I_j^s with side $\left(\frac{1}{3}\right)^s$, $1\leqslant j\leqslant c3^{2ns}$. By [6] or [11] we can find $u_s\in\mathrm{PSH}(\Omega)\cap C(\bar{\Omega})$ with

$$\lim_{z\to\xi} u_s(z)=0, \quad \forall\xi\in\partial\Omega,$$

and

$$(dd^c u_s)^n=\sum_j\left(\int_{I_j^s} d\mu\right)\chi_{I_j^s}\frac{1}{d_s^{2n}}\,dV,$$

where $d_s=\left(\frac{1}{3}\right)^s=$ length of side of I_j^s, and where dV is the Lebesgue measure. Using the super mean-value property for superharmonic functions, we have

$$\int -u_s(dd^c u_s)^n\leqslant \mathrm{const}\cdot\int -u_s\,d\mu,$$

which is uniformly bounded since $\int(dd^c u_s)^n=\mu(1)<+\infty$, as already noted in (7) in the proof of Lemma 5.2. It follows from (2) that

$$\sup_s\int_{\Omega'} -u_s\,dV<+\infty, \quad \forall\Omega'\subset\subset\Omega,$$

so we can pick a subsequence $(u_{s_j})_{j=1}^{\infty}$, again denoted by (u_s), $u_s\to u\in\mathrm{PSH}(\Omega)$, $s\to+\infty$, a.e. dV. Since $u=\lim_{j\to\infty}(\sup_{s>j}u_s)^*$ we have that $u\in\mathcal{F}_1$. Define

$$V_s(x)=\frac{1}{B(nd_s)}\int_{|\xi|\leqslant nd_s}|u(x+\xi)-u_s(x+\xi)|\,dV,$$

where $B(r)$ is the volume of the ball with radius r. Then

$$\begin{aligned}\int|u-u_s|(dd^c u_s)^n&=\sum_j\left(\int_{I_j^s}d\mu\right)d_s^{-2n}\int_{I_j^s}|u-u_s|\,dV\\&\leqslant\sum_j\frac{B(nd_s)}{d_s^{2n}}\int_{I_j^s}V_s(x)\,d\mu(x)\leqslant\mathrm{const}\cdot\int V_s(x)\,d\mu(x).\end{aligned}$$

Now,

$$\begin{aligned}V_s(x)&=\frac{1}{B(nd_s)}\int_{|\xi|<nd_s}|u(x+\xi)-u_s(x+\xi)|\,dV\\&=\frac{1}{B(nd_s)}\int_{[\xi|\leqslant nd_s}\Big|u(x+\xi)-\sup_{j\geqslant s}u_j(x+\xi)+\sup_{j\geqslant s}u_j(x+\xi)-u_s(x+\xi)\Big|\,dV\\&\leqslant\frac{1}{B(nd_s)}\int_{|\xi|\leqslant nd_s}\big(\sup_{j\geqslant s}u_j(x+\xi)-u(x+\xi)\big)\,dV\\&\quad+\frac{1}{B(nd_s)}\int_{|\xi|\leqslant nd_s}\sup_{j\geqslant s}u_j(x+\xi)\,dV-\frac{1}{B(nd_s)}\int_{|\xi|\leqslant nd_s}u_s(x+\xi)\,dV\end{aligned}$$

$$\leqslant \frac{1}{B(nd_s)} \int_{|\xi|\leqslant nd_s} \left[\left(\sup_{j\geqslant s} u_j(x+\xi)\right)^* - u(x+\xi) \right] dV$$
$$+ \frac{1}{B(nd_s)} \int_{|\xi|\leqslant nd_s} \left(\sup_{j\geqslant s} u_j(x+\xi)\right)^* dV - u_s(x).$$

It follows now from monotone convergence and Lemma 5.2 that

$$\int V_s(x)\, d\mu(x) \to 0, \quad s \to +\infty,$$

and then from Lemma 5.3 that

$$(dd^c u_j)^n \to (dd^c u)^n, \quad j \to +\infty.$$

But $(dd^c u_s)^n \to \mu$, $s \to +\infty$, by construction, so $\mu = (dd^c u)^n$.

It remains to prove that $u \in \mathcal{E}_p$. Define χ_N as the characteristic function for

$$\{z \in \Omega : u \geqslant -N\}.$$

By what we have just proved, we can find $\varphi_N \in \mathcal{F}_1$ with

$$(dd^c \varphi_N)^n = \chi_N (dd^c u)^n.$$

By Lemma 5.4, $\varphi_N \geqslant \max(u, -N)$, so it follows from Lemma 4.2 and from Theorem 3.3 in [10] that $\varphi_N \in \mathcal{E}_0$. Thus

$$\int (-\varphi_N)^p (dd^c \varphi_N)^n = \int (-\varphi_N)^p \chi_N (dd^c u)^n$$
$$\leqslant \int (-\varphi_N)^p (dd^c u)^n \leqslant A \left(\int (-\varphi_N)^p (dd^c \varphi_N)^n \right)^{\frac{p}{n+p}}.$$

Therefore

$$\int (-\varphi_N)^p (dd^c \varphi_N)^n \leqslant A^{(n+p)/n},$$

so $\lim_{N\to+\infty} \varphi_N \in \mathcal{F}_p$ and $u = \lim_{N\to+\infty} \varphi_N$ by Theorem 4.5.

Suppose now that $p \geqslant 1$. Fix $q > n/(n-1)$ and choose

$$E > D_{q,0} \left(\int (dd^c h_K)^n \right)^{\frac{n}{n+q}},$$

where K is the support of μ. Define

$$M = \left\{ \nu \geqslant 0 : \operatorname{supp} \nu \subset K,\ \int (-\varphi)^q \, d\nu \leqslant E \left(\int (-\varphi)^q (dd^c \varphi)^n \right)^{\frac{q}{n+q}},\ \forall \varphi \in \mathcal{E}_0 \right\}.$$

If $L\subset K$ then, by Theorem 3.2, $(dd^c h_L)^n\in M$, so if G is any Borel subset of K with $\nu(G)=0$ for all $\nu\in M$, then G is pluripolar.

Fix $0\neq\nu_0\in M$ and define

$$R=\Big\{\nu\geqslant 0:\nu(1)=1,\operatorname{supp}\nu\subset K,$$
$$\int(-\varphi)^q\,d\nu\leqslant\left(\frac{E}{T}+\frac{E}{\nu_0(1)}\right)\left(\int(-\varphi)^q(dd^c\varphi)^n\right)^{\frac{q}{n+q}},\forall\varphi\in\mathcal{E}_0\Big\},$$

where $T=\sup\{\nu(1):\nu\in M\}$. Then

$$\frac{(T-\nu(1))\nu_0+\nu_0(1)\nu}{T\nu_0(1)}\in R$$

for all $\nu\in M$. Obviously, R is a weak*-compact convex set of probability measures. By a generalization of the Radon–Nikodym theorem in [19], there is a $\nu\in R$, $f\in L^1(d\nu)$ and a positive measure ν_s which is orthogonal to R, such that

$$\mu=f\,d\nu+\nu_s.$$

Note that if $\nu(G)=0$ for all $\nu\in R$, then G is pluripolar.

From the remark after Theorem 5.1, μ has no mass on pluripolar sets, so $\nu_s=0$ and $\mu=f\,d\nu$. We have already proved that there is, to every N, a unique

$$u_N\in\mathcal{F}_q$$

with $(dd^cu_N)^n=f_N\,d\nu$ where $f_N=\inf(f,N)$. Then $u_N\geqslant u_{N+1}$ by Theorem 4.5, and repeating the corresponding argument above, we conclude that $\lim u_N\in\mathcal{F}_p$, which completes the proof if μ has compact support.

Finally, if only $\mu(1)<+\infty$, consider $\chi_{k_N}\mu$, where

$$k_N=\{z\in\Omega:\operatorname{dist}(z,\complement\Omega)\geqslant 1/N\},$$

and repeat the argument above.

The "only if" part of Theorem 5.1 follows from Theorem 3.2, which completes the proof of Theorem 5.1. □

COROLLARY 5.5. *Suppose that μ is a positive and compactly supported measure such that*

$$\mu(K)\leqslant A\left(\int(dd^ch_K)^n\right)^{p/n},\quad\forall K\subset\subset\Omega,$$

for some $p>1$ and some A. Then there is a $u\in\mathcal{F}_1$ with $(dd^cu)^n=\mu$.

Proof. With notations as in Lemma 5.2, it follows from the inequality (6) and the assumption that

$$\int_{A_N^j}d\mu\leqslant A(2^n\alpha)^{p/(n+p)}\frac{1}{N^{np/(n+p)}}.$$

Therefore, the crucial inequality (7) holds true and the proof of the corollary can be completed using the first part of the proof of Theorem 5.1. □

We can now prove a generalization of Theorem 3.8.

THEOREM 5.6. *Suppose that* $u\in\mathcal{F}_1$ *and* $p>1$. *If* $\int(-u)^p(dd^cu)^n<+\infty$ *then* $u\in\mathcal{F}_p$, *and conversely, if* $u\in\mathcal{F}_p$ *then there exists a decreasing sequence* $u_j\in\mathcal{E}_0$ *with* $\lim u_j=u$ *and*

$$\lim_{j\to+\infty}\int(-u_j)^p(dd^cu_j)^n=\int(-u)^p(dd^cu)^n<+\infty.$$

Furthermore, if $\{v_j\}$ *is any sequence of functions in* $\mathcal{E}_0$, *decreasing to* $u\in\mathcal{E}_p$, *then*

$$\sup_j\int(-v_j)^p(dd^cv_j)^n<+\infty.$$

Proof. The last statement follows from the proof of Lemma 3.4. Suppose $u\in\mathcal{F}_p$. Since $(-u)^p$ is lower semicontinuous, $\int(-u)^p(dd^cu)^n<+\infty$.

Suppose that $u\in\mathcal{F}_1$ and $\int(-u)^p(dd^cu)^n<+\infty$. With notations as in Lemma 5.4, we use Theorem 5.1 to find $u_N\in\mathcal{E}_0$, $(dd^cu_N)^n=\chi_{A_N}(dd^cu)^n$. Since $u\in\mathcal{F}_1$, u_N decreases to u by Theorem 4.5. Now,

$$\int(-u_N)^p(dd^cu_N)^n=\int(-u_N)^p\chi_{A_N}(dd^cu)^n\to\int(-u)^p(dd^cu)^n,\quad N\to+\infty,$$

by monotone convergence. Therefore, $u\in\mathcal{F}_p$ and the theorem is proved. □

THEOREM 5.7. *Suppose that* μ *is a positive and compactly supported measure on* $\Omega\subset\mathbf{C}^n$, $n\geqslant2$. *If there is a constant* A *so that for some* $p\geqslant1$,

$$\int(-\varphi)^p\,d\mu\leqslant A\left(\int(-\varphi)(dd^c\varphi)^n\right)^{\frac{p}{n+1}},\quad\forall\varphi\in\mathcal{E}_0,$$

then there is a $u\in\mathcal{F}_p$ *with*

$$(dd^cu)^n=\mu.$$

Furthermore, if $0\leqslant f\in L^{p/(p-1)}(d\mu)$ *then there is a* $v\in\mathcal{F}_1$ *with*

$$(dd^cv)^n=f\,d\mu.$$

Proof. It follows from Theorem 5.1 that there is a $u\in\mathcal{F}_1$ with

$$(dd^cu)^n=\mu.$$

Now,

$$\int(-u)^p(dd^cu)^n\leqslant A\Big(\int(-u)(dd^cu)^n\Big)^{\frac{p}{n+1}}\leqslant A\Big(\int(-u)^p(dd^cu)^n\Big)^{\frac{1}{n+1}}\Big(\int d\mu\Big)^{\frac{p-1}{n+1}}$$

by Theorem 3.8 and Hölder's inequality. It follows that

$$\int(-u)^p(dd^cu)^n\leqslant\Big(\int d\mu\Big)^{\frac{p-1}{n}}A^{(n+1)/n},$$

so $u\in\mathcal{F}_p$ by Theorem 5.6. Now, if $0\leqslant f\in L^{p/(p-1)}(d\mu)$,

$$\begin{aligned}\int(-\varphi)f\,d\mu&\leqslant\Big(\int(-\varphi)^p\,d\mu\Big)^{\frac{1}{p}}\Big(\int f^{p/(p-1)}\,d\mu\Big)^{\frac{p-1}{p}}\\&\leqslant A^{1/p}\Big(\int(-\varphi)(dd^c\varphi)^n\Big)^{\frac{1}{n+1}}\Big(\int f^{p/(p-1)}\,d\mu\Big)^{\frac{p-1}{p}}.\end{aligned}$$

So another application of Theorem 5.6 completes the proof. □

6. Some applications

PROPOSITION 6.1. *Let Ω be a hyperconvex domain. Suppose that μ is a positive measure with finite mass on Ω such that $\mu\leqslant(dd^c\psi)^n$, where ψ is a bounded plurisubharmonic function on Ω. Then there is a uniquely determined bounded plurisubharmonic function $\varphi\in\mathcal{F}_1$ with $(dd^c\varphi)^n=\mu$.*

Remark. This is Theorem A in [15] in the case of boundary data zero. See also Theorem 8.1.

Proof. It is no restriction to assume $-1\leqslant\psi\leqslant0$. Consider $h_N=\max(\psi,Nh)$ where $h\in\mathcal{E}_0$ is an exhaustion function for Ω. It follows from Theorems 3.2, 3.4, 4.5 and 5.1 that there is a uniquely determined $\psi_N\in\mathcal{E}_0$ with $(dd^c\psi_N)^n=\chi_{A_N}d\mu$, where

$$A_N=\{z\in\Omega:Nh<-1\}.$$

Then

$$0\geqslant\psi_N\geqslant h_N\geqslant\psi,$$

so $\lim_{N\to+\infty}\psi_N\in\mathcal{F}_1\cap L^\infty$ since we have assumed that μ has bounded total mass. □

Next, we extend Theorem 5.1 to $\mathcal{E}_p$, $p\geqslant1$.

THEOREM 6.2. *Let Ω be a hyperconvex domain and suppose that μ is a positive measure on Ω such that* (5) *holds for some $p\geqslant 1$. Then there is a uniquely determined $u\in\mathcal{E}_p$ with $(dd^cu)^n=\mu$.*

Proof. Let $(K_j)_{j=1}^{\infty}$ be an increasing sequence of compact subsets of Ω with $\bigcup_{j=1}^{\infty} K_j=\Omega$. It follows from (5) that there is a uniquely determined $u_j\in\mathcal{F}_p$ with $(dd^cu_j)^n=\chi_{K_j}\,d\mu$. Then u_j is a decreasing sequence of functions in $\mathcal{F}_p$ and it follows from Theorem 5.6 that

$$\int(-u_j)^p(dd^cu_j)^n=\int(-u_j)^p\chi_{K_j}\,d\mu\leqslant\int(-u_j)^p\,d\mu\leqslant A\biggl(\int(-u_j)^p(dd^cu_j)^n\biggr)^{\frac{p}{n+p}}.$$

Therefore,

$$\lim_{j\to+\infty}u_j=u\in\mathcal{E}_p$$

and $(dd^cu)^n=\mu$.

Let now h be a continuous exhaustion function for Ω in $\mathcal{E}_0$ and define

$$A_m=\{z\in\Omega:v>-m(-h)^{1/p}\},$$

where $v\in\mathcal{E}_p$ and $(dd^cv)^n=\mu$. We have then by Lemma 5.4,

$$\chi_{K_j}\chi_{A_m}\,d\mu\leqslant(dd^c\max(v,-m(-h)^{1/p}))^n.$$

Thus

$$U(\chi_{K_j}\chi_{A_m},0)\geqslant\max(v,-m(-h)^{1/p}),$$

where $U(\chi_{K_j}\chi_{A_m}\,d\mu,0)$ denotes the unique function in $\mathcal{E}_0$ with

$$(dd^cU(\chi_{K_j}\chi_{A_m}\,d\mu,0))^n=\chi_{K_j}\chi_{A_m}\,d\mu.$$

(See §7 for this notation.)

Therefore, $U(\chi_{K_j}\chi_{A_m}\,d\mu,0)\geqslant u_j\geqslant v$ for all m, so $u\geqslant v$. In other words, if $v\in\mathcal{E}_p$, $(dd^cv)^n=d\mu$, then $u\geqslant v$. It remains to prove the reverse inequality.

We know from Lemma 5.4 that

$$(dd^c\max(v,-m(-h)^{1/p}))^n=\chi_{A_m}(dd^cv)^n+\chi_{\{v\leqslant-m(-h)^{1/p}\}}(dd^c\max(v,-m(-h)^{1/p}))^n.$$

Write

$$\mu_m=\chi_{\{v\leqslant-m(-h)^{1/p}\}}(dd^c\max(v,-m(-h)^{1/p}))^n$$

and $g_m=U(\mu_m,0)$. Then by the comparison principle,

$$\max(v,-m(-h)^{1/p})\geqslant u+g_m\quad\text{for all } m\geqslant 1,$$

and it is enough to prove that

$$\varlimsup_{m\to\infty} g_m = 0 \quad \text{a.e. } dV.$$

Define $U_m = (\sup\{g_j : j \geqslant m\})^*$. Using Theorem 3.4, we have for any $j \geqslant m$,

$$\begin{aligned}\int (-U_m)^p (dd^c U_m)^n &\leqslant m^p \int (-h)(dd^c U_m)^n \leqslant m^p \int (-h)\, d\mu_j \\ &\leqslant \left(\frac{m}{j}\right)^p \int (-v)^p (dd^c \max(v, -j(-h)^{1/p}))^n \leqslant \text{const}\cdot\left(\frac{m}{j}\right)^p.\end{aligned}$$

Therefore, $(dd^c U_m)^n = 0$, so since U_m is a bounded plurisubharmonic function with boundary values equal to zero, $U_m = 0$, which completes the proof of the theorem.

We conclude this section with a decomposition theorem for positive and compactly supported measures.

THEOREM 6.3. *Suppose that μ is a positive and compactly supported measure in a hyperconvex domain Ω. Then there exist $\psi \in \mathcal{E}_0$, $0 \leqslant f \in L^1((dd^c\psi)^n)$ and a positive measure ν_s carried by a pluripolar set, such that $\mu = f(dd^c\psi)^n + \nu_s$. In particular, if μ vanishes on all pluripolar sets, then there is an increasing sequence of measures $(dd^c u_j)^n$ tending to μ as $j \to +\infty$, where $u_j \in \mathcal{E}_0$.*

Proof. It follows from the last part of the proof of Theorem 5.1 that there exist $\varphi \in \mathcal{F}_p$, $0 \leqslant f \in L^1((dd^c\varphi)^n)$ and ν_s, carried by a pluripolar set, with $\mu = f(dd^c\varphi)^n + \nu_s$. Since μ has compact support, it is no restriction to assume that $(dd^c\varphi)^n$ has compact support. Consider

$$g = (-\varphi)^{-1} \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega).$$

Then a calculation shows that $(-\varphi)^{-2n}(dd^c\varphi)^n \leqslant \text{const}\cdot(dd^c g)^n$, and since $(dd^c\varphi)^n$ has compact support, we can modify g outside the support of $(dd^c\varphi)^n$ so that $g \in \mathcal{E}_0$. By Proposition 6.1, there is a $\psi \in \mathcal{E}_0$ with $(-\varphi)^{-2n}(dd^c\varphi)^n = (dd^c\psi)^n$, which gives $\mu = f(-\varphi)^{2n}(dd^c\psi)^n + \nu_s$.

Finally, if μ vanishes on all pluripolar sets, $\nu_s = 0$. Use Proposition 6.1 to solve

$$u_j \in \mathcal{E}_0, \quad (dd^c u_j)^n = \inf(f(-\varphi)^{2n}, j)(dd^c\psi)^n. \qquad \square$$

7. The Dirichlet problem with smooth boundary data

In this section, we use the results from the previous sections to study the Dirichlet problem with smooth boundary data.

Let Ω be a bounded pseudoconvex domain and assume that f is a continuous real-valued function on $\partial\Omega$. We are going to define classes $\mathcal{F}_p(f)$ of plurisubharmonic functions and study the problem when there is a $v\in\mathcal{F}_p(f)$ with

$$\begin{cases} \overline{\lim}_{z\to\xi} v(z) = f(\xi), & \forall \xi\in\partial\Omega, \\ (dd^c v)^n = \mu & \text{on } \Omega. \end{cases} \tag{8}$$

In particular, we will prove that if Ω is strictly pseudoconvex, then there exists a uniquely determined $v\in\mathcal{F}_p(f)$ satisfying (8) if and only if μ satisfies (5).

Suppose first that μ is a positive measure on Ω such that the class of plurisubharmonic functions

$$B(\mu,f) = \{v\in \mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(\Omega) : (dd^c v)^n \geqslant \mu,\ \overline{\lim_{z\to\xi}}\, v(z)\leqslant f(\xi), \forall\xi\in\partial\Omega\}$$

is non-empty. Then

$$U(\mu,f) = \sup\{v : v\in B(\mu,f)\}\in B(\mu,f),$$

cf. [10]. Sometimes we also write $U(\mu,0)$ for the solution obtained in Theorem 5.1. Also, if Ω is strictly pseudoconvex and if $\mu = g\,dV$, where $0\leqslant g\in L^2(\Omega)$, then $U(g\,dV,f)$ solves (8) and is continuous on $\bar\Omega$, cf. [11]. If Ω is smoothly bounded and strictly pseudoconvex, $f\in C^\infty(\bar\Omega)$, and if $0<\varepsilon\leqslant g\in C^\infty(\bar\Omega)$ for some $\varepsilon>0$, then $U(g\,dV,f)\in C^\infty(\bar\Omega)$, cf. [5]. Then, by Lemma 4.2,

$$\int_\Omega (dd^c(U(0,f)+U(0,-f)))^n \leqslant \int_\Omega (dd^c(U(dV,f)+U(dV,-f)))^n < +\infty,$$

so $U(0,f)+U(0,-f)\in\mathcal{E}_0$, and if $\varphi\in\mathcal{E}_0$, $\mu\leqslant(dd^c\varphi)^n$, then

$$\int_\Omega (dd^c U(\mu,f))^n \leqslant \int_\Omega (dd^c(\varphi+U(0,f))^n \leqslant \int_\Omega (dd^c(\varphi+U(0,f)+U(0,-f)))^n < +\infty$$

since $\mathcal{E}_0$ is a convex cone. Thus, if $\varphi\in\mathcal{E}_0$ and $(dd^c\varphi)^n\geqslant\mu$, we have $U(0,f)\geqslant U(\mu,f)\geqslant \varphi+U(0,f)$ and $U(\mu,f)+U(0,-f)\in\mathcal{E}_0$. This leads us to the following definition.

Definition 7.1. Suppose that Ω is a hyperconvex domain. We consider functions $f\in C(\partial\Omega)$ such that $\lim_{z\to\xi} U(0,f)(z) = f(\xi)$ for all $\xi\in\partial\Omega$. For such functions we then denote by $\mathcal{E}_0(f)$ (or $\mathcal{F}_p(f)$), $p\geqslant 1$, the class of plurisubharmonic functions u such that there exists $\varphi\in\mathcal{E}_0$ (or $\mathcal{F}_p$) with

$$U(0,f)\geqslant u\geqslant \varphi+U(0,f). \tag{9}$$

This can be thought of as a type of analogy with the Riesz decomposition theorem of a subharmonic function as a sum of a potential and a harmonic function.

Remark. Note that since $\mathcal{E}_0$ (or $\mathcal{F}_p$) is convex, so is $\mathcal{E}_0(f)$ (or $\mathcal{F}_p(f)$).

THEOREM 7.2. *The Monge–Ampère operator* $(dd^c)^n$ *is well-defined on* $\mathcal{F}_p(f)$ *for all* $p\geqslant 1$.

Proof. It is no restriction to assume that $f\leqslant 0$. Let $u\in\mathcal{F}_p(f)$ be given. Then there exists $\varphi\in\mathcal{F}_p$ such that

$$U(0,f)\geqslant u\geqslant \varphi+U(0,f).$$

The sequence of functions $\max(u,\varphi_j+U(0,f))$ in $\mathcal{E}_0(f)$ decreases to u, where φ_j decreases to φ as in the definition of $\mathcal{F}_p$. Let now $\{u_j\}\subset\mathcal{E}_0(f)$ be any given sequence decreasing to u as $j\to+\infty$. Let K be any given compact subset of Ω and choose c so large that $U(0,f)>c\varphi$ on K. Then $u\geqslant(c+1)\varphi$ near K, so $v_j=\max(u_j,(c+1)\varphi_j)\in\mathcal{E}_0$ and v_j decreases to $\max(u,(c+1)\varphi)\in\mathcal{F}_p$, $j\to+\infty$. It follows now from Theorem 3.5 that $(dd^cv_j)^n$ converges weakly, $j\to+\infty$, and since K is an arbitrarily chosen compact set, $(dd^cu_j)^n$ converges weakly, $j\to+\infty$, which proves the theorem. □

To make sure that $\lim_{z\to\xi}U(0,f)(z)=f(\xi)$, $U(\mu,f)+U(0,-f)\in\mathcal{E}_0$ and to avoid regularity problems, we assume in the rest of this section that Ω is a smoothly bounded strictly pseudoconvex set and that $f\in C^\infty(\bar{\Omega})$.

LEMMA 7.3. *Let* $p\geqslant 1$ *and assume that* $u,v\in\mathcal{F}_p(f)$ *satisfy* $u=v$ *near* $\partial\Omega$. *Then*

$$\int_\Omega (dd^cu)^n=\int_\Omega (dd^cv)^n.$$

Proof. The proof of Lemma 4.1 applies. □

LEMMA 7.4. *Let* $p\geqslant 1$ *and assume that* $u,v\in\mathcal{F}_p(f)$ *satisfy* $u\leqslant v$ *on* Ω. *Then*

$$\int_\Omega (dd^cu)^n\geqslant\int_\Omega (dd^cv)^n.$$

Proof. Suppose that $u_j\leqslant v_j$, $u_j\searrow u$, $v_j\searrow v$, $j\to+\infty$, as in the definition of $\mathcal{F}_p(f)$, and assume that $h\in\mathcal{E}_0$.

If $1\leqslant p\leqslant n$, then

$$\int h(dd^cu_j)^p\wedge(dd^cv_j)^{n-p}\leqslant\int h(dd^cu_j)^{p-1}\wedge(dd^cv_j)^{n-p+1},$$

so in particular, $\int h(dd^cu_j)^n\leqslant\int h(dd^cv_j)^n$.

For, by Stokes' theorem,

$$\begin{aligned}0&=\int_{\partial\Omega} h\,d^cu_j\wedge(dd^cu_j)^{p-1}\wedge(dd^cv_j)^{n-p}\\&=\int dh\wedge d^cu_j\wedge(dd^cu_j)^{p-1}\wedge(dd^cv_j)^{n-p}+\int h(dd^cu_j)^p\wedge(dd^cv_j)^{n-p}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_\Omega h(dd^c u_j)^p\wedge(dd^c v_j)^{n-p} &= -\int_\Omega du_j\wedge d^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &= \int_\Omega u_j\, dd^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &\quad -\int_{\partial\Omega} u_j\, d^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &= \int_\Omega u_j\, dd^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &\quad -\int_{\partial\Omega} v_j\, d^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &= \int_\Omega u_j\, dd^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &\quad -\int_\Omega dv_j\wedge d^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &\quad -\int_\Omega v_j\, dd^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &\leqslant -\int dv_j\wedge d^c h\wedge(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p}\\ &= \int h(dd^c u_j)^{p-1}\wedge(dd^c v_j)^{n-p+1},\end{aligned}$$

where we have used that $u_j=v_j=f$ on $\partial\Omega$. Hence,

$$\begin{aligned}\int -h(dd^c v)^n \leqslant \varliminf \int -h(dd^c v_j)^n &\leqslant \varlimsup \int -h(dd^c u_j)^n\\ &\leqslant \int -h(dd^c u)^n+\varepsilon\varlimsup\int_{-\varepsilon<h}(dd^c u_j)^n\\ &\leqslant \int_\Omega -h(dd^c u)^n+\varepsilon\sup_j\int(dd^c u_j)^n.\end{aligned}$$

But since we are assuming that $\sup_j\int(dd^c u_j)^n<+\infty$, it follows that $\int -h(dd^c v)^n\leqslant \int -h(dd^c u)^n$, so letting $h\searrow -1$, we get the desired conclusion. □

LEMMA 7.5. *Let $p\geqslant 1$. If $u,v\in\mathcal{F}_p(f)$, then*

$$\int_{\{u<v\}}(dd^c v)^n \leqslant \int_{\{u<v\}}(dd^c u)^n.$$

Proof. The proofs of Lemmas 4.3 and 4.4 go through without changes. We only need to observe that $u+U(0,-f), v+U(0,-f)\in\mathcal{F}_p$, so Theorem 3.2 gives

$$\int(-\varphi)[(dd^cu)^n+(dd^cv)^n]\leqslant \text{const}\cdot\left(\int -\varphi(dd^c\varphi)^n\right)^{\frac{1}{n+1}},\quad \forall\varphi\in\mathcal{E}_0. \qquad \square$$

THEOREM 7.6. *Let $p\geqslant 1$ and suppose that $u,v\in\mathcal{F}_p(f)$ satisfy $(dd^cu)^n\leqslant(dd^cv)$. Then $v\leqslant u$ on Ω.*

Proof. The proof of Theorem 4.5 goes through if we use Lemma 7.5 instead of Lemma 4.4. $\square$

THEOREM 7.7. *Let Ω be a smoothly bounded, strictly pseudoconvex domain in $\mathbf{C}^n$, $n\geqslant 2$, $p\geqslant 1$, μ a positive measure on Ω with finite mass and $f\in C^\infty(\partial\Omega)$. Then there is a uniquely determined $u\in\mathcal{F}_p(f)$ with $(dd^cu)^n=\mu$ if and only if there is a constant A such that*

$$\int(-\varphi)^p\,d\mu\leqslant A\left(\int(-\varphi)^p(dd^c\varphi)^n\right)^{\frac{p}{n+p}},\quad \forall\varphi\in\mathcal{E}_0. \tag{5}$$

Proof. Suppose $u\in\mathcal{F}_p(f)$, $(dd^cu)^n=\mu$. Then $U(0,f)\geqslant u\geqslant\varphi+U(0,f)$ for some $\varphi\in\mathcal{F}_p$, so $0\geqslant u+U(0,-f)\geqslant\varphi+U(0,f)+U(0,-f)$. By Lemma 3.4, $u+U(0,-f)\in\mathcal{F}_p$, since we have $\varphi+U(0,f)+U(0,-f)\in\mathcal{F}_p$. Therefore, $(dd^c(u+U(0,-f)))^n$ satisfies (5), and since $\mu=(dd^cu)^n\leqslant(dd^c(u+U(0,-f)))^n$ so does μ. Thus (5) is a necessary condition for the Dirichlet problem (8) to have a solution. To be able to complete the proof of Theorem 7.7 we need two lemmas. $\square$

LEMMA 7.8. *Suppose that μ is a positive measure with compact support in Ω such that μ satisfies (5) for some $p>n/(n-1)$. Assume that $u_j\in\mathcal{E}_0(f)\cap C(\bar{\Omega})$, $u_j\to u\in\mathrm{PSH}(\Omega)$ a.e. dV, $j\to+\infty$, and that $\sup_j\int_\Omega(dd^cu_j)^n<+\infty$.*

Then $\lim_{j\to+\infty}\int u_j\,d\mu=\int u\,d\mu$.

Proof. Since $u_j\in\mathcal{E}_0(f)$, we have already found that $(dd^c(u_j+U(0,-f)))^n$ satisfies (5) and so does $(dd^cu_j)^n$. It follows then from Theorem 5.1 that $(dd^cU((dd^cu_j)^n,0))^n=(dd^cu_j)^n$. Again,

$$\begin{aligned}\int(dd^c(u_j+U(0,-f)))^n&\leqslant\int(dd^c(u_j+U(0,-f)+U(0,f)))^n\\&\leqslant 3^n\left[\int(dd^cu_j)^n+\int(dd^c(U(0,f)+U(0,-f)))^n\right],\end{aligned}$$

so

$$\sup_j\int(dd^c(u_j+U(0,-f)))^n=\alpha<+\infty.$$

It follows from Lemma 5.2 that

$$\lim_{j\to\infty}\int (u_j+U(0,-f))\,d\mu=\int (u+U(0,-f))\,d\mu,$$

which proves the lemma. □

Note also that it follows from (7) in Lemma 5.2 that

$$\int -(u_j+U(0,-f))\,d\mu\leqslant 2\int d\mu+2(2^n\alpha)^{n/(n+p)}\sum_{k=1}^{\infty}\frac{2^{k+1}}{2^{k\gamma}},$$

where $\gamma=np/(n+p)$.

LEMMA 7.9. *Suppose that* $u_s\in\mathcal{E}_0(f)\cap C(\bar\Omega)$, $u_s\to u\in\mathrm{PSH}(\Omega)$, *a.e.* dV, $s\to+\infty$, $\sup_s\int_\Omega -u_s(dd^cu_s)^n<+\infty$, *and that* $\int_\Omega |u-u_s|(dd^cu_s)^n\to 0$, $s\to+\infty$.
Then $(dd^cu_s)^n$ *tends weakly to* $(dd^cu)^n$, $s\to+\infty$.

Proof. The proof of Lemma 5.3 applies. □

End of the proof of Theorem 7.7. Assume that $p>n/(n-1)$ and that μ has compact support. We can then copy the proof of Theorem 5.1 to find $u_s\in\mathrm{PSH}(\Omega)$, $u_s\to u\in\mathrm{PSH}(\Omega)$, a.e. dV, $s\to+\infty$, $(dd^cu_s)^n$ converges weakly to μ,

$$\sup_s\int -u_s(dd^cu_s)^n\leqslant \text{const}$$

and

$$U(0,f)\geqslant u_s\geqslant U((dd^cu_s)^n,0)+U(0,f),$$

where $\overline{\lim}_{s\to+\infty}U((dd^cu_s)^n,0)=w\in\mathcal{F}_p$. Therefore, $u=\overline{\lim}_{s\to+\infty}u_s\in\mathcal{F}_p(f)$ and $U(0,f)\geqslant u\geqslant w+U(0,f)$. If we form

$$V_s(x)=\frac{1}{B(nd_s)}\int_{|\xi|<nd_s}|u(x+\xi)-u_s(x+\xi)|\,dV,$$

as in the proof of Theorem 5.1, then it follows from monotone convergence and Lemma 7.8 that $\int_\Omega V_s(x)\,d\mu(x)\to 0$, $s\to+\infty$, and then from Lemma 7.9 that $(dd^cu_s)^n$ tends weakly to $(dd^cu)^n$, $s\to+\infty$.

Assume now that $p\geqslant 1$. Let K_j be an increasing sequence of compact subsets of Ω with $\bigcup_{j=1}^\infty K_j=\Omega$. By Theorem 6.3 there exist $\psi_j\in\mathcal{E}_0$ such that $\chi_{K_j}\,d\mu=g_j(dd^c\psi_j)^n$ for some $0\leqslant g_j\in L^1((dd^c\psi_j)^n)$. We have already proved that there exist $u_j^s\in\mathcal{E}_0(f)$ with $(dd^cu_j^s)^n=\inf(g_j,s)\,(dd^c\psi_j)^n$. Then

$$U(0,f)\geqslant u_j^s\geqslant U(\mu,0)+U(0,f),$$

so $\lim_{s\to+\infty} u_j^s = u_j \in \mathcal{F}_p(f)$, and finally $u = \lim_{j\to+\infty} u_j \in \mathcal{F}_p(f)$ since we know from Theorem 5.1 that $U(\mu, 0) \in \mathcal{F}_p$. Since $(dd^c u_j)^n = \chi_{K_j}\, d\mu$, it follows that $(dd^c u)^n = d\mu$, which completes the proof of Theorem 7.7. □

Remark. It follows from Lemma 3.12 that $\overline{\lim}_{z\to\xi} U(\mu, 0)(z) = 0$, $\forall \xi \in \partial\Omega$, so we have solved the Dirichlet problem (8).

8. The Dirichlet problem with continuous boundary data

In this last section, we consider the Dirichlet problem (8) for continuous boundary data on hyperconvex sets.

First, we prove that Theorem A in [15] can be deduced from Theorem 7.7.

THEOREM 8.1. *Suppose that Ω is a bounded pseudoconvex domain, $f \in C(\partial\Omega)$, and that μ is a positive measure on Ω, such that $U(\mu, f) \in \mathrm{PSH} \cap L^\infty(\Omega)$ and such that $\lim_{z\to\xi} U(\mu, f)(z) = \lim_{z\to\xi} U(0, f)(z) = f(\xi)$, $\forall \xi \in \partial\Omega$.*

Then for every positive measure ν dominated by μ, $(dd^c U(\nu, f))^n = \nu$ and $U(\nu, f)$ satisfies the inequality $U(0, f) \geqslant U(\nu, f) \geqslant U(\mu, f)$.

Proof. Suppose $0 \leqslant \nu \leqslant \mu$. It is no restriction to assume that ν has compact support, since ν can be approximated by an increasing sequence of compactly supported measures. Assume first that Ω is smoothly bounded and strictly pseudoconvex, and that $f \in C^\infty(\partial\Omega)$. Then, by considering $U(\nu, f) + U(0, -f)$ we see that $B(\nu, 0) \neq \varnothing$, so $U(\nu, 0) \in \mathcal{E}_0$ and $\nu \leqslant (dd^c U(\nu, 0))^n$, and so ν satisfies (5) for any $1 \leqslant p < +\infty$, by Theorem 3.2. By Corollary 7.10, there is a uniquely determined v, namely $v = U(\nu, f)$, with $(dd^c v)^n = \nu$ and $\lim_{z\to\xi} v(z) = f(\xi)$, $\forall \xi \in \partial\Omega$.

Assume now that Ω is pseudoconvex and let $(\Omega_j)_{j=1}^\infty$ be an increasing sequence of smoothly bounded strictly pseudoconvex domains with $\bigcup_{j=1}^\infty \Omega_j = \Omega$, where $\operatorname{supp} \nu \subset\subset \Omega_1$. Since each $f_j = U(0, f)|_{\partial\Omega_j}$ is upper semicontinuous, there exist $f_{jk} \in C^\infty(\partial\Omega_j)$ with $f_{jk} \searrow f_j$, $k \to +\infty$. By the first part of the proof, there exist uniquely determined functions $u_{jk} \in \mathrm{PSH} \cap L^\infty(\Omega_j)$ with $(dd^c u_{jk})^n = \nu$ and $\lim_{z\to\xi} u_{jk}(z) = f_{jk}(\xi)$, $\forall \xi \in \partial\Omega_j$.

Also, $U(\nu, f)|_{\Omega_j} \leqslant u_{jk}$ since

$$\varlimsup_{z\to\xi} U(\nu, f)(z) \leqslant \varlimsup_{z\to\xi} U(0, f_{jk})(z) = f_{jk}(\xi), \quad \forall \xi \in \partial\Omega_j.$$

Since $u_{jk} \searrow u_j$, $k \to +\infty$, we have $(dd^c u_j)^n = \nu$ and $U(\mu, f) \leqslant U(\nu, f) \leqslant u_j \leqslant U(0, f_j)$ on Ω_j.

Finally, $u_{j+1}|_{\partial\Omega_j} \leqslant U(0, f_j)|_{\partial\Omega_j} = U(0, f)|_{\partial\Omega_j} = f_j$, so $(u_j)_{j=1}^\infty$ is a decreasing sequence; since $(dd^c u_j)^n = \nu$ and $U(0, f) \geqslant u_j \geqslant U(\mu, f)$, the proof of the theorem is complete. □

THEOREM 8.2. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$, $n\geqslant 2$, $p\geqslant 1$, μ a positive measure with finite mass on Ω. Then, to every $f\in C(\partial\Omega)$ such that $\lim_{z\to\xi} U(0,f)(z)=f(\xi)$ for all $\xi\in\partial\Omega$, there is a function $u\in\mathcal{F}_p(f)$ with $(dd^c u)^n=\mu$ if and only if there is a constant A such that*

$$\int(-\varphi)^p\,d\mu\leqslant A\left(\int(-\varphi)^p(dd^c\varphi)^n\right)^{\frac{p}{n+p}},\quad \forall\varphi\in\mathcal{E}_0. \tag{5}$$

Proof. By choosing $f=0$, it follows from Theorem 3.2 that (5) is a necessary condition. To prove that (5) is sufficient, we first note that it follows from Theorem 5.1 that there is a $\varphi\in\mathcal{F}_p$ with $(dd^c\varphi)^n=\mu$. Define, as in Lemma 5.4,

$$A_N=\{z\in\Omega:\varphi>-N\}$$

and

$$k_N=\{z\in A_N:\operatorname{dist}(z,\complement\Omega)\geqslant 1/N\}.$$

Then there is a uniquely determined $\varphi_N\in\mathcal{E}_0$ with

$$(dd^c\varphi_N)^n=\chi_{A_N}\,d\mu.$$

Thus

$$(dd^c(\varphi_N+U(0,f)))^n\geqslant\chi_{A_N}\,d\mu$$

and $\lim_{z\to\xi}(\varphi_N+U(0,f)(z)=f(\xi)$, for all $\xi\in\partial\Omega$. It follows from Theorem 8.1 that $(dd^cU(\chi_{A_N}\,d\mu,f))^n=\chi_{A_N}d\mu$. Thus

$$U(0,f)\geqslant U(\chi_{A_N}\,d\mu,f)\geqslant\varphi+U(0,f),$$

and since $U(\chi_{A_N}\,d\mu,f)$ is a decreasing sequence of functions in $\mathcal{E}_0$, it follows that

$$\lim_{N\to\infty}U(\chi_{A_N}\,d\mu,f)=u\in\mathcal{F}_p(f)$$

and $(dd^cu)^n=\mu$ by Theorem 7.2. $\square$

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