A FINITENESS THEOREM FOR CUSPS

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Much is known about the topology and geometry of the quotient of hyperbolic 3-space $\mathbf{H}^3$ by the action of a group Γ of isometries which has a fundamental polyhedron with finite volume or (more generally) with a finite number of faces.(¹) On the other hand, if the group Γ is merely assumed to be finitely generated, the 3-dimensional fundamental polyhedron may be of *infinite geometrical complexity* (Greenberg, 1966) and few general results about $\mathbf{H}^3/\Gamma$ are known.

Salient among the known results is the Finiteness theorem of Ahlfors (1964) which describes how the 3-dimensional fundamental polyhedron of a finitely generated Kleinian group intersects the domain of discontinuity at infinity in a *finite* polygon. The intersection of the fundamental polyhedron with the limit set at infinity was shown to have spherical measure zero for the case of a finitely generated Kleinian group in Ahlfors (1965) and for the case of a general finitely generated discrete group in Sullivan (1978).

Conjecturally much more is true about this intersection: *the fundamental polyhedron of a finitely generated group should intersect the limit set in only finitely many inequivalent points*. For this conjecture to be true it is necessary that $\mathbf{H}^3/\Gamma$ have finitely many cusps. The *finiteness of the number of cusps* will be proved in this paper.

By definition a cusp of Γ or of $\mathbf{H}^3/\Gamma$ is a conjugacy class of non-trivial maximal parabolic subgroups. There are two types, rank one cusps and rank two cusps. Each (torsion free) rank one cusp determines a part of the manifold $\mathbf{H}^3/\Gamma$ homeomorphic to a cylinder $\times$ ray. Each (torsion free) rank two cusp determines an end of the manifold $\mathbf{H}^3/\Gamma$ homeomorphic to a torus $\times$ ray. This is due to Margulis and is discussed in Thurston's notes [7], section 5.10.

Our analysis begins with the simple fact that the inverse image in $\mathbf{H}^3$ (thought of as the unit ball B in Euclidean 3-spaces) of the various cusps in $\mathbf{H}^3/\Gamma$ consists of a disjoint collection of horoballs (smaller balls in B tangent to the boundary).

(¹) For a good survey see Marden (1977) and Thurston (1978).

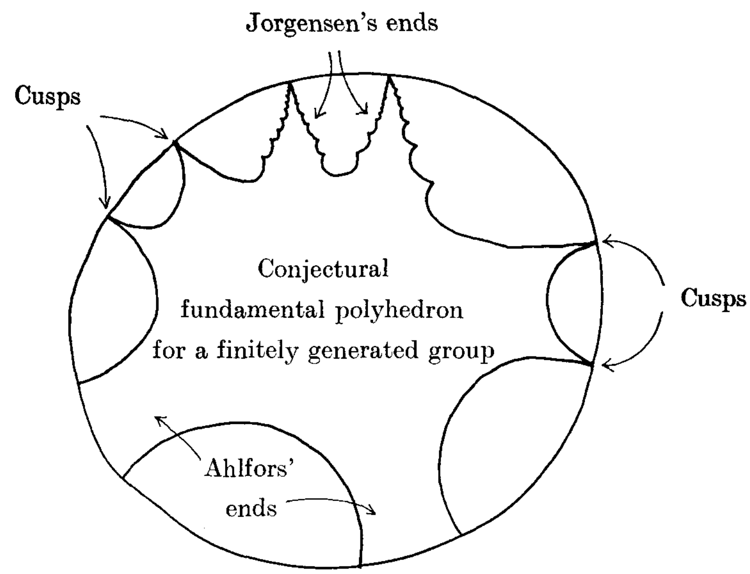


Figure 1

We follow Ahlfors [2] and construct Borel series $\sum a_n(z-b_n)^{-1}$ where b_n is the base of the nth horoball (after stereographic projection of the ∂B to $\mathbf{C}$) and a_n is a complex number whose size in the Kleinian case (Part I) is the volume of the horoball. These Borel series are used to determine crossed homomorphisms of Γ into fourth degree polynomials and Γ invariant automorphic forms of degree -2.

These objects are treated using the finite generation of Γ, the Ahlfors finiteness theorem [1], the tangent ergodic theorem in Sullivan [6], and an elementary packing estimate for disjoint spheres resting on a plane, Proposition 2.

We arrive at the following restriction on the number of cusps. Let Γ be a discrete group of isometries of $\mathbf{H}^3$ with N generators.

THEOREM (finiteness of cusps). *The number N_c of cusps of Γ is finite and satisfies $N_c \leqslant 5N-4$.*

Acknowledgement. The paper was directly inspired by a conversation with Fernando C. Rocha about the proof that $\mathbf{H}^3/\Gamma$ has only finitely many topological ends (Γ finitely generated, see Addendum).

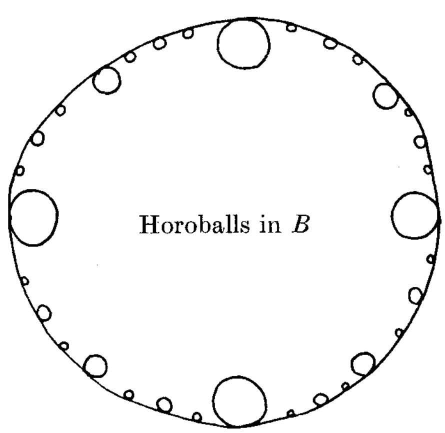
Also hidden behind the construction of $\psi_\alpha(\xi)$ (Part II) is an interpretation of Ahlfors calculation [2] in terms of holomorphic quadratic vector fields which arose from a conversation with Bill Thurston.

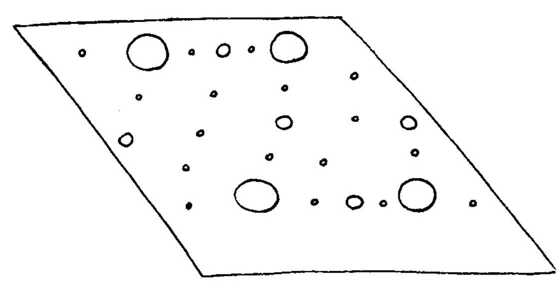
Finally, the reader of Ahlfors [2] will realize the great debt owed to the argument of page 11 there. This argument has intrigued me for some time.

Added (February 1981). After writing this paper I received an announcement of Bill Abikoff in which he describes a sharper inequality (the 5 becomes a 3) valid under a topological regularity condition on the three manifold. Abikoff's proof is based on topological work of Scott and the geometrical work of Thurston.

Demonstration of the theorem

We represent $\mathbf{H}^3$ as the unit ball B in Euclidean 3-space and write $\|\gamma'\xi\|$ for the linear distortion of the Euclidean metric by $\gamma\in\Gamma$ at $\xi\in S^2=\partial B$. By the Margulis decomposition of an arbitrary hyperbolic manifold M each cusp ($\equiv$conjugacy class of maximal parabolic subgroups) determines a certain region in M which lifts to a family of disjoint horoballs in hyperbolic space. Furthermore these families are disjoint for the different cusps. (See Thurston [7], section 5.10.) We may suppose that the center of the ball B lies outside all the horoballs ([7], section 5.10).





Horoballs on the plane

Figure 2

If z_α in S^2 represents one cusp stabilized by the parabolic group P_α, let Γ_α denote the coset space Γ/P_α. Note that $\|\gamma'(z_\alpha)\|$ only depends on the coset of γ in Γ_α.

PROPOSITION 1. $\sum_{\Gamma_\alpha}\|\gamma'(z_\alpha)\|^3<\infty$.

Proof. The Euclidean volumes of the horoballs at z_α and γz_α are in the ratio $\|\gamma' z_\alpha\|^3$ (up to a universal factor). Since all these horoballs are *disjoint*, the sum of their volumes is finite. These horoballs are labeled by Γ_α so the series converges. Q.E.D.

Now we want an estimate about how close the disjoint horoballs can pack around the base of a given one, B_0. Let $V(n)$ denote the volume of those horoballs which are based at points whose distance from the base of B_0 lies between $1/(n+1)$ and $1/n$.

PROPOSITION 2. *There is a constant c and an integer N_0 depending on the radius of B_0 so that* $V(n) \leqslant c/n^5$ for $n > N_0$.

Proof.

Figure 3

Any such ball for n sufficiently large has radius $\leqslant c_0/n^2$. The total available volume under B_0 is that of a circular solid ring like region whose width is no more than $(1/n - 1/(n+1)) + 2c_0/n^2$ and whose height is no more than $(1/n + c_0/n^2)^2$. Since each of these has order $1/n^2$, and the circumference of the ring has order $1/n$, this volume is less than c/n^5 and the proposition is proved.

We transform the group by stereographic projection to the complex plane. In the Kleinian case ($\equiv$the domain of discontinuity in S^2, $D_\Gamma \neq \varnothing$) we assume ∞ corresponds to a point of D_Γ.

Part I (The Kleinian case, $D_\Gamma \neq \varnothing$). We now follow the calculation of Ahlfors [2] for Kleinian groups, and then show how it may be extended to the case of discrete non-Kleinian groups. Let $\gamma' z$ denote the complex derivative of $z \mapsto \gamma z$. The first step of the proof is to form the Borel series

$$\varphi_\alpha(\xi) = \sum_{\Gamma_\alpha} \frac{(\gamma'(z_\alpha))^3}{\gamma z_\alpha - \xi}$$

where Γ_α as before is the coset space of the parabolic group fixing z_α.

PROPOSITION 3. *If $\sum_n |a_n| < \infty$, then the series*

$$\sum_n \frac{a_n}{b_n - z}$$

converges a.e. *in* $\mathbf{C}$. *On every compact disk the series converges in L^1. If the limit function equals zero* a.e. *and the b_n are distinct then all the a_n are zero.*

Proof. The first part follows from the proof of the second. Since the integral of $|a/(z-b)|$ over a fixed disk is less than a constant times $|a|$, the L^1 convergence of $\sum |a_n/(b_n-z)|$ is clear. Then by monotone convergence we have absolute convergence a.e. If $\delta(x)$ denotes the unit dirac mass at x, in the sense of distributions,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{a}{z-b}\right)=a\delta(b).$$

Applying dominated convergence yields

$$\frac{\partial}{\partial \bar{z}}\left(\sum_{n=1}^{\infty}\frac{a_n}{z-b_n}\right)=\sum_{n=1}^{\infty}a_n\,\delta(b_n)$$

in the sense of distributions. This proves the a_n are all zero if $\sum_n a_n/(z-b_n)$ equals zero a.e. and the b_n are distinct.(1) Q.E.D.

COROLLARY. *The series for $\varphi_\alpha(\xi)$ converges* a.e., *and in L^1 of every compact disk* $\{|z|<R\}$.

Proof. Stereographic projection has bounded distortion in the compact limit set so the absolute convergence of $\sum_{\Gamma_\alpha}(\gamma' z_\alpha)^3$ follows from Proposition 1. Then the corollary follows from Proposition 3. Q.E.D.

The *second step* of Ahlfors argument [2] is to consider the difference (or coboundary)

$$\begin{aligned}\Delta_\eta^\alpha(\xi)=\varphi_\alpha(\eta\xi)\frac{1}{(\eta'\xi)^2}-\varphi_\alpha(\xi)&=\frac{1}{(\eta'\xi)^2}\sum_{\Gamma_\alpha}\frac{(\gamma' z_\alpha)^3}{(\gamma z_\alpha-\eta\xi)}-\sum_{\Gamma_\alpha}\frac{(\gamma' z_\alpha)^3}{\gamma z_\alpha-\xi}\\&=\sum_{\Gamma_\alpha}\left(\frac{1}{(\eta'\xi)^2}\frac{(\gamma' z_\alpha)^3}{\gamma z_\alpha-\eta\xi}-\frac{((\eta^{-1}\gamma)'(z_\alpha))^3}{(\eta^{-1}\gamma z_\alpha-\xi)}\right).\end{aligned}$$

PROPOSITION 4. *For each $\eta\in\Gamma$, $\Delta_\eta^\alpha(\xi)$ agrees* a.e. *with a polynomial in ξ of degree* $\leqslant 4$.

Proof. Since $a(b-\eta\xi)^{-1}$ has a pole at $\xi=\eta^{-1}b$ with residue $-a/\eta'(\eta^{-1}b)$ we see the term in the parenthesis has a pole at $\eta^{-1}\gamma(z_\alpha)$ with zero residue, namely,

$$[(\gamma' z_\alpha)^3/(\eta'(\eta^{-1}\gamma(z_\alpha))^2\eta'(\eta^{-1}\eta'(\eta^{-1}\gamma z_\alpha)]-[(\eta^{-1}\gamma)'(z_\alpha)]^3.$$

Thus the $\partial/\partial\bar{z}$ distributional derivative of $\Delta_\eta^\alpha(\xi)$ is zero. Since $1/(\eta'(\xi))^2=(c\xi+d)^4$ and $\varphi_\alpha(\xi)=O(1/|\xi|)$ near ∞, $\Delta_\eta^\alpha(\xi)$ is $O(\xi^4)$ near infinity. Since $\Delta_\eta^\alpha(\xi)$ is holomorphic it must be a polynomial in ξ of degree $\leqslant 4$. Q.E.D.

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(1) Ahlfors [2] could appeal to Denjoy's theorem for this uniqueness. This requires $\sum |a_n||\log a_n|<\infty$ which causes a difficulty for us in case the limit set is all of S^2.

Ahlfors *third step* is to interpret $\eta \mapsto \Delta_\eta^\alpha(\xi)$ as a crossed homomorphism from Γ into the vector space P_4 of 4th degree polynomials. Namely Γ acts on P_4 by $p(\xi) \mapsto (p(\xi))^\gamma = p(\gamma\xi)/(\gamma'\xi)^2$. The expression for Δ_η^α is formally the coboundary of φ_α and so satisfies the cocycle or crossed homomorphism condition $\Delta_{\eta\nu}^\alpha = \Delta_\eta^\alpha + (\Delta_\nu^\alpha)^\eta$.

Since crossed homomorphisms are determined by the values on generators, the dimension of the space of all such is $\leqslant 5N$ if Γ is generated by N elements. Thus we have

PROPOSITION 5. *If a Kleinian group has N generators and more than $5N$ cusps $z_1, z_2, \ldots, z_\alpha, \ldots$, a non-trivial linear combination $\varphi = \sum_{i=1}^k c_i \varphi_\alpha$ satisfies, for all $\gamma \in \Gamma$, $\varphi(\gamma\xi)/(\gamma'\xi)^2 = \varphi(\xi)$* a.e.

Now we show the set of φ satisfying $\varphi(\gamma\xi)/(\gamma'\xi)^2 = \varphi(\xi)$, $\gamma \in \Gamma$, is limited.

PROPOSITION 6. *Let Γ be any finitely generated discrete group of hyperbolic isometries, and let φ be any complex valued measurable function defined* a.e. *on the limit set which satisfies $\varphi(\gamma\xi)/(\gamma'\xi)^d = \varphi(\xi)$, $\gamma \in \Gamma$ (for any integer $d \neq 0$). Then φ vanishes* a.e. *on the limit set.*

Proof. Actually we will prove a stronger assertion. By first taking absolute values and then taking arguments we obtain two systems of equations. Each of these systems is contradictory.

The absolute value equations are impossible for any positive real d. For if we have a solution $|\varphi|$ let $\psi(\xi) = |\varphi(\xi)|^{-2/d}$. Then $\psi(\gamma\xi)|\gamma'\xi|^2 = \psi(\xi)$ or in other words, the measure $\psi(\xi) \times$ (Lebesgue measure) is invariant by Γ. This contradicts section VI of [4] which asserts there is no σ-finite measure on the limit set which is Γ invariant and absolutely continuous with respect to Lebesgue measure.

The argument equations are also impossible. Define a d-ray to be d-symmetrical rays at a point. Extend the argument of φ at each point to a field of d-rays. The argument equation

$$\arg \varphi(\gamma\xi) + d \arg (\gamma' z) = \arg \varphi(\xi) \pmod{2\pi}$$

implies the field of d-rays is Γ invariant. Such a field is impossible for any integer d by the second corollary, section VI of [6].

Remarks. (1) The argument of [6] only require that Γ be any discrete group so that the action on the limit set has no fundamental set of positive measure. Thus Proposition 6 is valid in this more general case.

(2) The appeal to [6] could be avoided here in the Kleinian case by using an argument like Ahlfors [2] to bound the dimension of the space of solutions.

To study φ on the domain of discontinuity D_Γ we have

PROPOSITION 7. *The Borel series* $\varphi_\alpha(\xi)$ *are holomorphic in the domain of discontinuity* D_Γ.

Proof. The poles are in the limit set and the convergence is uniform on compact subsets of the domain of discontinuity.

PROPOSITION 8. *If* D_Γ/Γ *is compact any holomorphic function* φ *on* D_Γ *satisfying* $\varphi(\gamma\xi)/(\gamma'\xi)^2=\varphi(\xi)$, $\gamma\in\Gamma$, *must be identically zero on* D_Γ.

Proof (well-known). Such a φ determines a holomorphic cross section of the square of the tangent line bundle. (If there are elliptic elements we pass to a finite branched cover.) Since the Euler class of this bundle is negative the cross section must be zero, for otherwise the intersection with the zero section would be $\geqslant 0$. Q.E.D.

Now we consider the case when D_Γ/Γ has punctures. Let c denote a parabolic cusp associated to one of these punctures. Choose a disk D tangent at c lying in D_Γ so that the parabolic group fixing c stabilizes D and so that the further images by elements of Γ are disjoint.

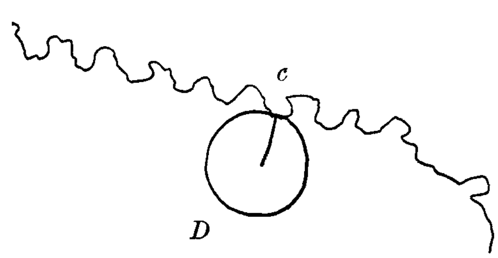

Figure 4

Such a disk is constructed by lifting a neighborhood of the cusp in D_Γ/Γ.

We consider the growth of $\varphi_\alpha(\xi)$ along the radius r from the center of D to c.

PROPOSITION 9. *Along the radius* r *of* D *approaching* c (*thought of as the origin of* ξ-*plane*) $|\varphi_\alpha(\xi)|\leqslant C/|\xi|$.

Proof. At a point x of distance d from c all the poles of $\varphi_\alpha(\xi)$ are at least distance d from x because they lie outside D. Thus the absolute value of $\varphi_\alpha(\xi)$ is no more than $1/d\times$ (total mass of the poles) $\leqslant$constant$/d$. Q.E.D.

So consider linear combinations of the $\varphi_\alpha(\xi)$ satisfying $\varphi(\gamma\xi)/\gamma'(\xi)^2=\varphi(\xi)$. Let $\overline{D_\Gamma}/\Gamma$ denote the Riemann surface D_Γ/Γ compactified by adding in the punctures.

PROPOSITION 10. *Such a φ determines a holomorphic section of the square of the tangent bundle over $\overline{D/\Gamma}$.*

Proof. If t is the variable around a puncture and ξ is the variable near the corresponding cusp c, we can write $\xi = 1/\log t^{-1}$. Then

$$\varphi(\xi)(\partial/\partial\xi)^2 = \varphi(1/\log t^{-1})(\partial/\partial(1/\log t^{-1}))^2 = \varphi(1/\log t^{-1})(\log t^{-1})^4 t^2(\partial/\partial t)^2.$$

So the absolute value is $\leqslant |\log t^{-1}|^5 |t|^2$. A holomorphic function in the punctured disk satisfying such an inequality has a zero at the origin $t=0$ and is holomorphic there. Q.E.D.

PROPOSITION 11. *Such a φ satisfying $\varphi(\gamma\xi)/(\gamma'\xi)^2 = \varphi(\xi)$ must be identically zero on D_Γ.*

Proof. φ determines a holomorphic section of the square of the tangent bundle over $\overline{D_\Gamma/\Gamma}$. Thus it must be zero by Proposition 8. Q.E.D.

Now we can finish the proof of the theorem for the Kleinian case.

THEOREM 1. *Let Γ be a Kleinian group with N generators. Then the total number of cusps for Γ does not exceed $5(N-1)$.*

Proof. The Borel series $\varphi_\alpha(\xi)$ generate a linear space of dimension equal to the number of cusps using Proposition 3 and its corollary. This linear space maps into the space of crossed homomorphisms from Γ into P_4, the space of 4th order polynomials, using Propositions 4 and 5. The kernel consists of linear combinations of the $\varphi_\alpha(\xi)$, $\varphi(\xi)$ satisfying $\varphi(\gamma\xi)/(\gamma'\xi)^2 = \varphi(\xi)$.

Such functions vanish a.e. in the limit set by Proposition 6. If there are no punctures in D_Γ/Γ (namely $t=0$) such a φ is zero on D_Γ by Propositions 7 and 8. Otherwise, their restrictions to D_Γ vanish by Propositions 9, 10, 11. Thus if there are more than $5N$ cusps we violate the third part of Proposition 3; applied to the $\varphi_\alpha(\xi)$.

To improve $5N$ to $5(N-1)$ we add the space of 4th degree polynomials to the space of Borel series generated by the cusps. Their coboundaries generate a 5-dimensional space of cocycles. Q.E.D.

Part II (The limit set is all of S^2). Now we consider finitely generated groups discrete groups whose limit set is all of S^2 (or $\mathbf{C} \cup \infty$).

We assume stereographic projection has been chosen so that 0 and ∞ are cusps equivalent by Γ. Let $a_\gamma = (\gamma' z_\alpha/\gamma z_\alpha)^3$ and form the Borel series

$$\varphi_\alpha(\xi) = \sum_{\Gamma_\alpha} \frac{a_\gamma}{\gamma(z_\alpha) - \xi}.$$

Convergence a.e. is assured by the following.

PROPOSITION 12. *The series* $\sum_{\Gamma(z_\alpha)} a_\gamma$ *is absolutely convergent.*

Proof. Break the series into (a) the part near the origin, (b) the part near infinity and (c) the rest.

For part (c) the terms are comparable in size to the cubes of the spherical derivatives, $\|\gamma'(z_\alpha)\|^3$. So Proposition 1 yields absolute convergence.

For the terms between distance $1/(n+1)$ and $1/n$ of the origin we have the estimate of Proposition 2 which yields

$$\sum_{1/(n+1)\leqslant|\gamma z_\alpha|<1/n} |a_\gamma| \leqslant n^3 V(n) \leqslant n^3/n^5.$$

We have used the fact that spherical and planar derivatives are comparable near the origin. So for part (a) the terms are less than $\sum_n 1/n^2 < \infty$.

For the terms whose corresponding points on the sphere are at a distance between $1/(n+1)$ and $1/n$ from the north pole, we can again apply Proposition 2. The planar derivative has a size equal to n^2 times the size of the spherical derivative. The denominators of these a_γ have size $1/n^3$. Thus all in all these terms are no more than

$$(n^2)^3 \frac{1}{n^3} V(n) \leqslant \frac{n^6}{n^8} = \frac{1}{n^2}.$$

So we have absolute convergence for part (c), the terms near infinity. Q.E.D.

COROLLARY. *The Borel series* $\varphi_\alpha(\xi)$ *converges* a.e. *and in* L^1 *of any compact disk. The distributional derivative* $\partial(\varphi_\alpha(\xi))/\partial\bar{\xi}$ *is the measure* $\sum_{\Gamma_\alpha} a_\gamma \delta(\gamma z_\alpha)$.

Proof. This follows from Propositions 3 and 12. Q.E.D.

Now consider the function $\psi_\alpha(\xi) = \xi^3\varphi_\alpha(\xi)$, and form the coboundary as before

$$\Delta_\eta^\alpha(\xi) = \psi_\alpha(\eta\xi)\frac{1}{(\eta'\xi)^2} - \psi_\alpha(\xi).$$

PROPOSITION 13. *The function* $\Delta_\eta^\alpha(\xi)$ *agrees* a.e. *with a polynominal of degree at most* 4.

Proof. At $\eta^{-1}\gamma z_\alpha$ the second term on the right hand side has a pole with residue $(\eta^{-1}\gamma(z_\alpha))^3 a_{\eta^{-1}\gamma} = ((\eta^{-1}\gamma)'(z_\alpha))^3$. The first term on the right hand side has a pole at $\eta^{-1}\gamma z_\alpha$ with residue

$$[1/\eta'(\eta^{-1}\gamma z_\alpha)]^2[a_\gamma/\eta'(\eta^{-1}\gamma z_\alpha)](\gamma z_\alpha)^3 = [1/\eta'(\eta^{-1}\gamma z_\alpha)]^3[\gamma'(z_\alpha)]^3.$$

These cancel just as before in Proposition 4. We see $\Delta_\eta^\alpha(\xi)$ is a sum of polynomials of degree 4. Namely,

$$\sum_{\gamma\in\Gamma_\alpha}\left((\eta\xi)^3\left(\frac{a_\gamma}{\gamma_\alpha-\eta\xi}\right)\frac{1}{(\eta^1\xi)^2}-(\xi^3)\frac{a_{\eta^{-1}\gamma}}{\eta^{-1}\gamma(z_\alpha)-\xi}\right).$$

Since we have a sequence of 4th degree polynomials converging in L^1 (any disk) and pointwise a.e. to $\Delta_\eta^\alpha(\xi)$ an application of the Cauchy integral theorem shows $\Delta_\eta^\alpha(\xi)$ must be a 4th degree polynomial. Q.E.D.

THEOREM 2. *Let Γ be a discrete group of isometries of $\mathbf{H}^3$ which is generated by N elements and whose limit set is all of S^2. Then the number of cusps is $\leqslant 5N-4$.*

Proof. If there are more than $5N$ cusps (not counting the orbit containing 0 and ∞), some nontrivial linear combination ψ of the ψ^α will satisfy $\psi(\eta\xi)/(\eta'\xi)^2-\psi(\xi)=0$. Such ψ must be zero a.e. by Proposition 6. By the corollary to Proposition 12 the distributional derivative $\partial\psi/\partial\bar\xi=\partial(\xi^3\varphi)/\partial\bar\xi=\xi^3\partial\varphi/\partial\bar z$ would be a locally finite measure which is nontrivial because the atoms are disjoint (if there are enough cusps).

This is impossible if $\psi\equiv 0$ (a.e.). We conclude there are no more than $5N$ cusps besides the one whose orbit contains 0 and ∞.

As in Theorem 1 we subtract 5 from this upper bound by adding the space of 4th degree polynomials to the space generated by the $\psi_\alpha(\xi)$. Q.E.D.

Addendum. (1) The argument above shows more generally that the number of cusps of Γ is finite if

(i) the domain of discontinuity satisfies D_Γ/Γ is a countable union of finite type Riemann surface,

(ii) there is no wandering set of positive measure in the limit set,

(iii) the cohomology group H^1 (Γ, 4th degree polynomials) is finite dimensional.

(2) A homological argument shows a 3-manifold with a finitely generated fundamental group and a contractible covering space has only finitely many topological ends. (By Scott there is a compact submanifold with the same homotopy type. In particular $H_2(M,\mathbf{Z})$ is finitely generated. But if the number of ends is $k+1$ the rank of H_2 is at least k.)

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