Acta Math., 188 (2002), 87–128

Smoothness of Lipschitz-continuous graphs with nonvanishing Levi curvature

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1. Introduction

In this paper we prove the C^∞-smoothness of Lipschitz-continuous graphs of $\mathbf{C}^2$ with smooth and nonvanishing Levi curvature.

Let Ω be an open subset of $\mathbf{R}^3$. Given a C^2-smooth function $u:\Omega\to\mathbf{R}$ the Levi curvature of its graph at the point $(\xi, u(\xi))$, $\xi\in\Omega$, is the real number

$$k(\xi,u):=\frac{\mathcal{L}u}{(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2}},\tag{1}$$

where

$$\mathcal{L}u:=u_{xx}+u_{yy}+2au_{xt}+2bu_{yt}+(a^2+b^2)u_{tt},\tag{2}$$

and $a=a(\nabla u), b=b(\nabla u)$ depend on the gradient of u as

$$a,b:\mathbf{R}^3\to\mathbf{R},\quad a(p)=\frac{p_2-p_1p_3}{1+p_3^2},\quad b(p)=\frac{-p_1-p_2p_3}{1+p_3^2}.\tag{3}$$

In (1), (2), $\xi=(x,y,t)$ denotes the point of $\mathbf{R}^3$, u_t is the first derivative of u with respect to t, and analogous notations are used for the other first- and second-order derivatives of u.

The notion of Levi curvature for a real manifold was introduced by E. E. Levi in 1909 in order to characterize the holomorphy domains of $\mathbf{C}^2$. Since then, it has played a crucial role in the geometric theory of several complex variables.

In looking for the polynomial hull of a graph, Slodkowski and Tomassini implicitly introduced in 1991 the following definition of Levi curvature for Lipschitz-continuous graphs [16].

Investigation supported by University of Bologna Funds for selected research topics.

Definition 1.1. Let Ω be an open subset of $\mathbf{R}^3$ and k a given function defined on $\Omega\times\mathbf{R}$. The graph of a Lipschitz-continuous function $u:\Omega\to\mathbf{R}$ will have Levi curvature $k(\xi,u(\xi))$ at any point $\xi\in\Omega$ if there exist a sequence (u_n) in $C^2(\Omega)$ and a sequence of positive numbers $\varepsilon_n\to 0$ satisfying the conditions:

(i) There exists $M>0$ such that $\|u_n\|_{L^\infty(\Omega)}+\|\nabla u_n\|_{L^\infty(\Omega)}\leqslant M$ for any $n\in\mathbf{N}$, and (u_n) uniformly converges to u.

(ii) $\mathcal{L}_{\varepsilon_n}u_n=k(\xi,u_n)H(\xi,\nabla u_n)$ in Ω for any $n\in\mathbf{N}$.

Here $\mathcal{L}_\varepsilon$ and H denote the operators

$$\mathcal{L}_\varepsilon u:=\mathcal{L}u+\varepsilon^2\frac{u_{tt}}{1+u_t^2} \tag{4}$$

and

$$H(\xi,\nabla u)=(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2}. \tag{5}$$

In (ii), u_n and ∇u_n are computed at the point ξ; a and b in (5) stand for $a(\nabla u)$ and $b(\nabla u)$, respectively. If the graph of u has Levi curvature $k(\xi,u(\xi))$ at every point $\xi\in\Omega$, we will also say that u is a strong viscosity solution of

$$\mathcal{L}u=k(\xi,u)H(\xi,\nabla u)\quad\text{in }\Omega. \tag{6}$$

If the function k, together with its first derivatives, satisfies some general growth conditions, the class of Lipschitz-continuous graphs with Levi curvature k is very large. Indeed, the existence of such graphs has been established by Slodkowski and Tomassini with viscosity techniques, starting from the key remark that the quasilinear operator $\mathcal{L}$ in (2) is degenerate elliptic as its characteristic form

$$\begin{aligned}A(p,\zeta)&=\zeta_1^2+\zeta_2^2+2a(p)\zeta_1\zeta_3+2b(p)\zeta_2\zeta_3+(a^2(p)+b^2(p))\zeta_3^2\\&=(\zeta_1+a(p)\zeta_3)^2+(\zeta_2+b(p)\zeta_3)^2,\end{aligned} \tag{7}$$

is nonnegative defined. Their result is the following: *Let $U\subset\subset\Omega$ be a strictly pseudo-convex domain with $\partial U\in C^{2,\alpha}$, $0<\alpha<1$. Let $k\in C^1(\bar\Omega\times\mathbf{R})$ satisfy the conditions of Proposition* 2 *and Theorem* 3 *in* [16]. *Then, for every $\phi\in C^{2,\alpha}(\bar\Omega)$ there exists $u\in\operatorname{Lip}(\bar U)$ whose graph has (generalized) Levi curvature $k(\xi,u(\xi))$ at any point $\xi\in U$. Moreover, $u(\xi)=\phi(\xi)$ for any $\xi\in\partial U$* (see [16, Theorem 4]).

The function u solves the equation

$$\mathcal{L}u=k(\xi,u)H(\xi,\nabla u)$$

in the weak viscosity sense of Crandall–Ishii–Lions (see [11]). Since the minimum eigenvalue of $A(p,\cdot)$ is equal to zero for every $p\in\mathbf{R}^3$, the operator $\mathcal{L}$ is not elliptic at any point, and the regularity results for viscosity solutions of nonlinear elliptic [3] and parabolic equations [18], [19] cannot be applied to our case. We have to introduce a completely different procedure, based on the particular structure of the Levi equation. This is well highlighted by some identities first explicitly written in [5], involving the two nonlinear vector fields, which appear in the characteristic form of $\mathcal{L}$, defined in (7):

$$X(p) := \partial_x + a(p)\partial_t, \quad Y(p) := \partial_y + b(p)\partial_t, \tag{8}$$

where a and b are defined in (3).

For a given function $u:\Omega\to\mathbf{R}$ we will write X instead of $X(\nabla u)$. Analogous abbreviations will be used for Y. Then the operator $\mathcal{L}$ can be written as

$$\mathcal{L} = (X^2u + Y^2u)(1+u_t^2),$$

and by relation (1) we call the following the prescribed Levi-curvature equation:

$$X^2u + Y^2u = k(\xi, u)\frac{(1+a^2+b^2)^{3/2}}{(1+u_t^2)^{1/2}}. \tag{9}$$

The Lie bracket of the first-order differential operators X and Y is

$$[X,Y] = -\frac{\mathcal{L}u}{1+u_t^2}\,\partial_t. \tag{10}$$

This structure has been very recently used by two of the authors in [8] to prove a first regularity result for viscosity solutions:

THEOREM. *Let us suppose that $k\in C^1(\Omega\times\mathbf{R})$. Let $u:\Omega\to\mathbf{R}$ be a Lipschitz-continuous function whose graph has Levi curvature k. Then $Xu, Yu\in H^1_{\mathrm{loc}}(\Omega)$ and u satisfies* (6) *pointwise almost everywhere.*

Here $H^1_{\mathrm{loc}}(\Omega)$ denotes the classical Sobolev space of order 1.

Without any extra condition on the curvature k it seems that the previous result cannot be improved. On the other hand, the following theorem was known ([5], see also [9]):

THEOREM. *If $k\in C^\infty(\Omega\times\mathbf{R})$ and never vanishes in $\Omega\times\mathbf{R}$, then every $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$-classical solution to* (6), *with $\alpha>\frac{1}{2}$, is of class C^∞ in Ω.*

In this paper we fill the gap between these results and prove a regularity theorem which has been announced in [6].

THEOREM 1.1. *Let $k\in C^{\infty}(\Omega\times\mathbf{R})$ be such that $k(\xi,s)\neq 0$ for every $(\xi,s)\in\Omega\times\mathbf{R}$. Then every Lipschitz-continuous graph having Levi curvature k is of class $C^{2,\alpha}$.*

Together with Theorem 4 in [16] and Theorem 1.1 in [5] Theorem 1.1 above immediately gives the following C^{∞}-solvability result for the Dirichlet problem related to the Levi operator.

COROLLARY 1.1. *Let Ω and k satisfy the hypotheses of Theorem 4 in [16]. Let us also assume that $k\in C^{\infty}(\Omega\times\mathbf{R})$ and $k(\xi,s)\neq 0$ for any $(\xi,s)\in\Omega\times\mathbf{R}$. Then, for every $\phi\in C^{2,\alpha}(\partial\Omega)$ the Dirichlet problem*

$$\begin{cases} \mathcal{L}u=k(\xi,u)(1+a^2+b^2)^{3/2}(1+u_t^2)^{1/2} & \text{in } \Omega,\\ u=\phi & \text{on } \partial\Omega \end{cases} \tag{11}$$

has a solution $u\in C^{\infty}(\Omega)\cap\mathrm{Lip}(\bar\Omega)$.

When k vanishes identically and Ω satisfies more restrictive hypotheses, a first existence result for (11) was proved by Bedford and Gaveau [1]. *If $k\equiv 0$, Ω is a regular pseudoconvex open set, $\phi\in C^{m+5}(\bar\Omega)$, $m\in\mathbf{N}$, and $\partial\Omega$ and ϕ satisfy some additional geometric conditions, then problem (11) has a solution $u\in C^{m+\alpha}(\Omega)\cap\mathrm{Lip}(\bar\Omega)$, $0<\alpha<1$. Besides, the graph is foliated in analytic complex curves.*

We would like to stress that the geometric arguments used in [1] do not work when $k\neq 0$. We emphasize some important differences between our Corollary 1.1 and the result of Bedford and Gaveau. The interior regularity result and the foliation phenomena of the solutions of the Dirichlet problem given in [1] for $k=0$ strictly depend on the regularity of boundary datum. The $C^{m+\alpha}$-regularity result cannot be improved, since every C^2-function u depending only on the variable t solves equation (6) with $k=0$. The foliation result has been extended in many directions (see [2], [4], [15]), but in all these papers it follows from the topology of the boundary of Ω. On the contrary, in Theorem 1.1 the local regularity property only follows from the structure of the operators $\mathcal{L}$ and H, since if k is of class C^{∞} and everywhere different from zero, any Lipschitz-continuous solution is of class C^{∞} independently of the regularity of boundary datum. Very recently, using a PDE technique similar to that introduced here, two of the present authors proved that also the foliation result for $k=0$ only depends on the structure of the operator, and in [7], [10] gave the following local version of it: *Every Lipschitz-continuous graph with Levi curvature $k\equiv 0$ is foliated in analytic curves.*

1.1. *Sketch of the proof*. The paper is organized as follows. In §2 we fix a solution u of the equation

$$\mathcal{L}_{\varepsilon}u=k(\xi,u)H(\xi,\nabla u)$$

in an open set Ω, and we denote by L_ε a linear operator formally defined as $\mathcal{L}_\varepsilon$:

$$L_\varepsilon = X^2+Y^2+T_\varepsilon^2,$$

where $T_\varepsilon=\varepsilon(1+u_t^2)^{-1/2}\partial_t$, and the coefficients of the vector fields X and Y depend on u. Then we prove that the coefficients a and b of the vector fields and the two functions

$$\omega=\partial_t u \quad \text{and} \quad v=\arctan(u_t)$$

are solutions of

$$L_\varepsilon z = f, \tag{12}$$

with different functions f.

The proof of Theorem 1.1 is based on the regularity of the solutions of this linear equation in some Sobolev spaces $W_\varepsilon^{m,p}$ naturally defined in terms of the vector fields X, Y and T_ε, but not explicitly on ∂_t. The classical elliptic regularization procedure is based on Sobolev inequalities and on a priori estimates of Caccioppoli type. In the present situation neither the Caccioppoli inequality holds, since the vector fields are not self-adjoint, nor the Sobolev inequality, since the coefficients of the vector fields are only bounded.

To overcome these difficulties we first prove an interpolation inequality, which will play a role similar to the Sobolev one.

PROPOSITION 1.1. *Let M be such that*

$$\|a\|_\infty+\|b\|_\infty+\|v\|_\infty \leqslant M.$$

For every function $z\in C^\infty$, $\phi\in C_0^\infty$, we have

$$\int |Xz|^3\phi^6 \leqslant c\int |\nabla_\varepsilon(Xz)|^2\phi^6+c\int(|\nabla_\varepsilon\phi|^6+\phi^6)(1+z^6), \tag{13}$$

where $c>0$ only depends on M and k. An analogous inequality is also satisfied if we replace Xz with Yz or $T_\varepsilon z$.

Only if the coefficients are much more regular we can establish a Sobolev-type inequality with optimal exponent (this is done is §3). In §4 we establish some a priori estimate in the intrinsic directions X and Y, weaker than the classical Caccioppoli one. Using these inequalities together with the interpolation ones, we prove a priori estimates in $W_\varepsilon^{m,p}$, for solutions z of (12) which holds under very general assumptions on the commutators of the vector fields, but requires some strong a priori estimates on the derivative $\partial_t z$, and this, up to now, has not been studied yet.

In §5 we conclude the proof of Theorem 1.1, starting with the estimates of the derivative ∂_t, which, by equality (10), can be expressed in terms of the commutator of the vector fields. We also use in an essential way the nonlinearity of the equation: the interpolation and Caccioppoli inequalities for the derivative ∂_t provide a gain of regularity only if applied to the function $\partial_t v$. In this way we obtain an L^2-estimate for Xv_t and Yv_t. Since $v_t=u_{tt}/(1+u_t^2)$, then, due to Definition 2.2 below, v_t has to be considered a derivative of weight 4 of u, while Xv_t and Yv_t are derivatives of weight 5 of the same function. Once the summability of these derivatives with respect to t is proved, it is possible to use the results in §4, and obtain analogous estimates for any derivation of weight 5 and 4. In particular, the coefficients a and b of the vector fields are now regular, and we can apply the Sobolev-type inequality proved in §3. It then follows that the derivatives of weight 4 belong to L^4, the derivatives of weight 3 belong to L^p for every p, and the derivatives of weight 2 belong to suitable classes C^α for every $\alpha\in]0,1[$. Now, using the results in [5], we deduce that $u\in C^{2,\alpha}$.

2. Properties of the coefficients a and b

Let us assume that u is a solution of class C^∞ of the regularized equation

$$\mathcal{L}_\varepsilon u = k(1+u_t^2)(1+a^2+b^2)^{3/2}, \tag{14}$$

on an open set Ω, where $\mathcal{L}_\varepsilon$ is the operator defined in (4). By simplicity let us denote by $a=a(\nabla u)$ and $b=b(\nabla u)$ the coefficients introduced in (3), and write X and Y instead of $X(\nabla u)$ and $Y(\nabla u)$, the vector fields defined in (8). Let us also write

$$T_\varepsilon = \varepsilon \frac{\partial_t}{\sqrt{1+u_t^2}}.$$

In this section we define some Sobolev spaces in terms of these vector fields, and a linear operator, formally defined as $\mathcal{L}_\varepsilon$:

$$L_\varepsilon = X^2+Y^2+T_\varepsilon^2.$$

Then we prove some properties of the coefficients a and b of the vector fields. In particular, we will prove that they are solutions of a linear equation of the type

$$L_\varepsilon z = f, \tag{15}$$

with different functions f. We will also introduce a new function $v=\arctan(u_t)$, which has properties similar to u_t, and satisfies the same equation, but with a simpler right-hand side.

2.1. *Natural Sobolev spaces.* It is natural to give the following definition:

Definition 2.1. If f is a $L^1_{\mathrm{loc}}(\Omega)$-function, we say that it is weakly differentiable with respect to X if there exists a function $g\in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int f X^*\phi = \int g\phi \quad \text{for all } \phi\in C_0^\infty(\Omega),$$

where X^* is the formal adjoint of X. The weak derivative with respect to any other vector field is defined in an analogous way.

Definition 2.2. For every fixed ε we will denote

$$D_1 = X, \quad D_2 = Y, \quad D_3 = T_\varepsilon = \varepsilon \frac{1}{\sqrt{1+\omega^2}}\partial_t, \quad \nabla_\varepsilon = (X, Y, T_\varepsilon),$$

where $\omega = u_t$, and we will also define

$$D_4 = T = \frac{1}{\sqrt{1+\omega^2}}\partial_t.$$

We will define the weight of an index $i\in\{1, ..., 4\}$ as

$$|i| = 1 \quad \text{for every } i = 1, ..., 3,$$

and, due to identity (10),

$$|4| = 2.$$

In general, if $i=(i_1, ..., i_q)\in\{1,2,3,4\}^q$ we set $|i|=\sum_j |i_j|$ and

$$D_i = D_{i_1} ... D_{i_m}.$$

Then, for any open set $U\subset\Omega$ we call

$$W_\varepsilon^{m,p}(U) = \{f : D_i f\in L^p(U) \text{ for all } i \text{ such that } |i|\leqslant m\},$$

$$\|f\|_{W_\varepsilon^{m,p}(U)} = \sum_{|i|\leqslant m} \|D_i f\|_{L^p(U)}.$$

In particular,

$$\|f\|_{W_\varepsilon^{0,p}(U)} = \|f\|_{L^p(U)}.$$

We also say that $f\in W^{m,p}_{\varepsilon,\mathrm{loc}}(\Omega)$ if for every $\phi\in C_0^\infty(\Omega)$, $f\phi\in W_\varepsilon^{m,p}(\Omega)$.

Let us recall that the coefficients of the operator are the derivatives of the function u, in the direction of the vector fields:

$$Yu = a \quad \text{and} \quad Xu = -b. \tag{16}$$

From this equality it follows that

$$\mathcal{L}u = (X^2u + Y^2u)(1+u_t^2)$$

and

$$\mathcal{L}_\varepsilon u = (X^2u + Y^2u + T_\varepsilon^2 u)(1+u_t^2). \tag{17}$$

Moreover, if we introduce two new functions,

$$\omega = \partial_t u \quad \text{and} \quad v = \arctan(u_t), \tag{18}$$

the derivatives with respect to t of the coefficients a and b can be expressed as

$$\partial_t a = Yv - \omega Xv \quad \text{and} \quad \partial_t b = -Xv - \omega Yv. \tag{19}$$

As a consequence the formal adjoints of X, Y and T_ε become

$$X^* = -X - (Yv - \omega Xv)\cdot, \quad Y^* = -Y - (Xv + \omega Yv)\cdot, \quad T_\varepsilon^* = -T_\varepsilon + \omega Tv\cdot. \tag{20}$$

Also the commutators can be simply expressed in terms of v:

$$[X,Y] = T_\varepsilon v T_\varepsilon - k(1+a^2+b^2)^{3/2} T, \quad [X, T_\varepsilon] = -Yv T_\varepsilon, \quad [Y, T_\varepsilon] = Xv T_\varepsilon. \tag{21}$$

Finally we recall that for every $f \in C^\infty(\Omega)$, for every $\phi \in C_0^\infty(\Omega)$,

$$\int \partial_t X f \phi = -\int \partial_t f X\phi \quad \text{and} \quad \int \partial_t Y f \phi = -\int \partial_t f Y\phi. \tag{22}$$

All the assertions (16)–(22) are direct computations. We refer to [8] for a detailed proof of them.

2.2. *A linear equation.* We turn now to prove that a, b and v are solutions of the linear equation (15) for a suitable right-hand side f. We first note that, by (14) and (17), u is a solution of the equation

$$L_\varepsilon u = k \frac{(1+a^2+b^2)^{3/2}}{(1+\omega^2)^{1/2}}. \tag{23}$$

Now we prove that, if a function z is a solution of equation (15), then its intrinsic derivatives Xz, Yz and $T_\varepsilon z$ are solutions of the same equation, with different right-hand sides.

LEMMA 2.1. *If z is a solution of* (15) *then $s_1=Xz$ is a solution of the equation*

$$\begin{aligned} L_\varepsilon s_1 = {} & Xf + k(1+a^2+b^2)^{3/2} TYz + Y(k(1+a^2+b^2)^{3/2} Tz) \\ & + 2XvT_\varepsilon vT_\varepsilon z - 2Y(T_\varepsilon vT_\varepsilon z) + 2T_\varepsilon(YvT_\varepsilon z). \end{aligned} \tag{24}$$

Proof. It is a direct computation. Differentiating the equation with respect to X, we get

$$\begin{aligned} X^2 s_1 + Y^2 s_1 + T_\varepsilon^2 s_1 = {} & Xf - [X,Y]Yz - Y[X,Y]z - [X,T_\varepsilon]T_\varepsilon z - T_\varepsilon[X,T_\varepsilon]z \\ \overset{(21)}{=} {} & Xf - T_\varepsilon vT_\varepsilon Yz + k(1+a^2+b^2)^{3/2} TYz \\ & - Y(T_\varepsilon vT_\varepsilon z - k(1+a^2+b^2)^{3/2} Tz) + YvT_\varepsilon^2 z + T_\varepsilon(YvT_\varepsilon z). \end{aligned}$$

Note that

$$\begin{aligned} -T_\varepsilon vT_\varepsilon Yz + YvT_\varepsilon^2 z \overset{(21)}{=} {} & -T_\varepsilon vYT_\varepsilon z + XvT_\varepsilon vT_\varepsilon z + YvT_\varepsilon^2 z \\ = {} & -Y(T_\varepsilon vT_\varepsilon z) + YT_\varepsilon vT_\varepsilon z + XvT_\varepsilon vT_\varepsilon z + T_\varepsilon(YvT_\varepsilon z) - T_\varepsilon YvT_\varepsilon z \\ \overset{(21)}{=} {} & -Y(T_\varepsilon vT_\varepsilon z) + T_\varepsilon(YvT_\varepsilon z) + 2XvT_\varepsilon vT_\varepsilon z. \end{aligned}$$

Inserting this computation in the previous one we infer the thesis.

An analogous computation ensures

LEMMA 2.2. *If z is a solution of* (15) *then $s_2=Yz$ is a solution of the equation*

$$\begin{aligned} L_\varepsilon s_2 = {} & Yf - k(1+a^2+b^2)^{3/2} TXz - X(k(1+a^2+b^2)^{3/2} Tz) \\ & + 2YvT_\varepsilon vT_\varepsilon z + 2X(T_\varepsilon vT_\varepsilon z) - 2T_\varepsilon(XvT_\varepsilon z). \end{aligned} \tag{25}$$

LEMMA 2.3. *If z is a solution of* (15) *then $s_3=T_\varepsilon z$ is a solution of the equation*

$$\begin{aligned} L_\varepsilon s_3 = {} & T_\varepsilon f - 2YvXs_3 + 2XvYs_3 \\ & - ((Xv)^2 + (Yv)^2 + (T_\varepsilon v)^2)s_3 + k(1+a^2+b^2)^{3/2} Tvs_3. \end{aligned} \tag{26}$$

Proof. Differentiating the equation with respect to T_ε, we get

$$\begin{aligned} X^2 s_3 + Y^2 s_3 + T_\varepsilon^2 s_3 = {} & T_\varepsilon f - [T_\varepsilon,X]Xz - X[T_\varepsilon,X]z - [T_\varepsilon,Y]Yz - Y[T_\varepsilon,Y]z \\ \overset{(21)}{=} {} & T_\varepsilon f - YvT_\varepsilon Xz - X(YvT_\varepsilon z) + XvT_\varepsilon Yz + Y(XvT_\varepsilon z) \\ = {} & T_\varepsilon f - Yv[T_\varepsilon,X]z - YvXT_\varepsilon z - XYvT_\varepsilon z - YvXT_\varepsilon z \\ & + Xv[T_\varepsilon,Y]z + XvYT_\varepsilon z + YXvT_\varepsilon z + XvYT_\varepsilon z \\ = {} & [\text{using again (21) to sum the terms 4 and 8}] \\ = {} & T_\varepsilon f - 2YvXs_3 + 2XvYs_3 - ((Xv)^2 + (Yv)^2 + (T_\varepsilon v)^2)s_3 \\ & + k(1+a^2+b^2)^{3/2} Tvs_3. \end{aligned}$$

Let us finally turn to the principal properties of the functions a, b, v.

PROPOSITION 2.1. *The function a defined in* (3) *is a solution of the equation*

$$L_\varepsilon a = Y(k(1+a^2+b^2)^{3/2}(1+\omega^2)^{-1/2}) - k(1+a^2+b^2)^{3/2}TXu \\ - X(k(1+a^2+b^2)^{3/2}Tu). \tag{27}$$

The function b is a solution of the equation

$$L_\varepsilon b = -X(k(1+a^2+b^2)^{3/2}(1+\omega^2)^{-1/2}) - k(1+a^2+b^2)^{3/2}TYu \\ - Y(k(1+a^2+b^2)^{3/2}Tu). \tag{28}$$

The function v defined in (18) *is a solution of the equation*

$$L_\varepsilon v = T(k(1+a^2+b^2)^{3/2}). \tag{29}$$

Proof. First note that for every vector field D_i with $i=1,...,4$ we have

$$D_i(T_\varepsilon u) = \varepsilon D_i\left(\frac{\partial_t u}{(1+(u_t)^2)^{1/2}}\right) = \varepsilon\frac{D_i\partial_t u}{(1+(u_t)^2)^{3/2}} = \varepsilon\frac{D_i v}{(1+(u_t)^2)^{1/2}}. \tag{30}$$

Since u is a solution of equation (23), from Lemma 2.2 and (16) it follows that

$$L_\varepsilon a = L_\varepsilon Yu = Y(k(1+a^2+b^2)^{3/2}(1+\omega^2)^{-1/2}) - k(1+a^2+b^2)^{3/2}TXu \\ - X(k(1+a^2+b^2)^{3/2}Tu) + 2YvT_\varepsilon vT_\varepsilon u + 2X(T_\varepsilon vT_\varepsilon u) - 2T_\varepsilon(XvT_\varepsilon u).$$

On the other hand,

$$2YvT_\varepsilon vT_\varepsilon u + 2XT_\varepsilon vT_\varepsilon u + 2T_\varepsilon vXT_\varepsilon u - 2T_\varepsilon XvT_\varepsilon u - 2XvT_\varepsilon^2 u \\ \overset{(30)}{=} 2YvT_\varepsilon vT_\varepsilon u + 2[X,T_\varepsilon]vT_\varepsilon u + 2\varepsilon T_\varepsilon v\frac{Xv}{(1+(u_t)^2)^{1/2}} - 2\varepsilon Xv\frac{T_\varepsilon v}{(1+(u_t)^2)^{1/2}} = 0,$$

by (21). Hence assertion (27) follows. Assertion (28) can be proved in the same way, using the fact that $b=-Xu$.

Let us now prove (29). Differentiating (30) we get

$$D_i^2(T_\varepsilon u) = \varepsilon\frac{D_i^2 v}{(1+(u_t)^2)^{1/2}} - \varepsilon\frac{\omega(D_i v)^2}{(1+(u_t)^2)^{1/2}}.$$

From this relation and Lemma 2.3 we infer that

$$\begin{aligned} \varepsilon\frac{L_\varepsilon v}{(1+(u_t)^2)^{1/2}} &= ((Xv)^2+(Yv)^2+(T_\varepsilon v)^2)\frac{\varepsilon\partial_t u}{(1+(u_t)^2)^{1/2}} \\ &\quad + T_\varepsilon(k(1+a^2+b^2)^{3/2}(1+\omega^2)^{-1/2}) - 2YvXT_\varepsilon u + 2XvYT_\varepsilon u \\ &\quad - ((Xv)^2+(Yv)^2+(T_\varepsilon v)^2)T_\varepsilon u + k(1+a^2+b^2)^{3/2}TvT_\varepsilon u \\ &= [\text{the first and the fifth terms cancel, and, by relation (30),} \\ &\qquad \text{the terms 3 and 4 cancel}] \\ &= T_\varepsilon(k(1+a^2+b^2)^{3/2})(1+\omega^2)^{-1/2} + k(1+a^2+b^2)^{3/2}T_\varepsilon((1+\omega^2)^{-1/2}) \\ &\quad + k(1+a^2+b^2)^{3/2}TvT_\varepsilon u \\ &= T_\varepsilon(k(1+a^2+b^2)^{3/2})(1+\omega^2)^{-1/2}. \end{aligned}$$

This implies assertion (29).

3. Embedding theorems in the spaces $W_\varepsilon^{m,p}$

In this section we prove a Sobolev-type inequality in the spaces $W_\varepsilon^{m,p}(\Omega)$, under the assumption that

$$\|a\|_\infty + \|b\|_\infty + \|\omega\|_\infty \leqslant M_1 \tag{31}$$

and that

$$k(\xi, s) \neq 0 \quad \text{for all } (\xi, s) \in \Omega \times \mathbf{R}.$$

As we already noted in the introduction this assumption ensures that $X, Y, [X, Y] - T_\varepsilon v T_\varepsilon$ are linearly independent at every point, and that $\det(X, Y, [X, Y] - T_\varepsilon v T_\varepsilon)$ is uniformly bounded away from 0. It is known that a Sobolev inequality with optimal exponent holds if the coefficients of the operator are smooth. Here we will see that it is possible to prove the same assertion, under a weaker condition, which can be considered an "intrinsic" Lipschitz continuity. In particular, it is satisfied when the coefficients belong to suitable $W_\varepsilon^{m,p}$-spaces.

3.1. *Vector fields with Hölder-continuous coefficients.* If the coefficients a and b of the vector fields are Hölder continuous with respect to the Euclidean distance, and ω is bounded, we can associate to X, Y and T some frozen vector fields.

Definition 3.1. Let us fix three open sets $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, and assume that a, b are Hölder continuous in Ω. For every $\xi_0 \in \Omega$ we denote

$$X_{\xi_0} = \partial_x + (a(\xi_0) + 2(y - y_0))\partial_t, \quad Y_{\xi_0} = \partial_y + (b(\xi_0) - 2(x - x_0))\partial_t.$$

Since $[X_{\xi_0}, Y_{\xi_0}] = -4\partial_t$, ∂_t has the same direction as T.

The Lie algebra generated by X_{ξ_0} and Y_{ξ_0} is noncommutative, and free of step 2. Hence it is a Heisenberg algebra, and it is possible to introduce a canonical change of variable:

$$\phi_{\xi_0}(x, y, t) = (x, y, t - (a(\xi_0) - 2y_0)x - (b(\xi_0) + 2x_0)y),$$

which changes X_{ξ_0} and Y_{ξ_0} into two vector fields X_H and Y_H, independent of ξ_0. If we denote by d_H the control distance of these vector fields, then the control distance associated to X_{ξ_0} and Y_{ξ_0} is $d_{\xi_0} = d_H \circ \phi_{\xi_0}$ (see [14] for the definition of control distance). The distance d_{ξ_0} can be explicitly computed, and it is easy to see that d_{ξ_0} is locally equivalent to the distance

$$\tilde{d}_{\xi_0}(\xi, \zeta) = (((x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2)^2 + (t - t_0 - a(\xi_0)(x_\xi - x_\zeta) - b(\xi_0)(y_\xi - y_\zeta)^2)^{1/4},$$

in the sense that there exists a positive constant M_2, only dependent on Ω_2, such that

$$M_2^{-1} \tilde{d}_{\xi_0}(\xi, \zeta) \leqslant d_{\xi_0}(\xi, \zeta) \leqslant M_2 \tilde{d}_{\xi_0}(\xi, \zeta) \tag{32}$$

for every $\xi, \zeta \in \Omega_2$ (see [5] for a detailed proof).

It follows that, if a and b are Hölder continuous in classical sense, then there exists a constant M_3 such that

$$|a(\xi)-a(\xi_0)| \leqslant M_3 d_{\xi_0}^{\alpha}(\xi,\xi_0), \quad |b(\xi)-b(\xi_0)| \leqslant M_3 d_{\xi_0}^{\alpha}(\xi,\xi_0) \tag{33}$$

for every $\xi, \xi_0 \in \Omega_2$.

The Lebesgue measure of a sphere $B_{\xi_0}(\xi, R)$ in the metric d_{ξ_0} is $R^4|B_0(0,1)|$. In what follows we set $N=4$, and we call this number the homogeneous dimension of $\mathbf{R}^3$ with respect to L_{ξ_0}. This implies in particular that for every ball such that $B_{\xi_0}(\xi, R) \subset\subset \Omega_2$ and for every function $f \in C([0,R])$,

$$\int_{B_{\xi_0}(\xi,R)} f(d_{\xi_0}(\xi,\zeta))\, d\zeta = C\int_0^R f(\varrho)\varrho^{N-1}\, d\varrho, \quad N=4. \tag{34}$$

Let us also recall that the operator $X_H^2+Y_H^2$ is the Kohn Laplacian on the Heisenberg group, and it has a fundamental solution Γ_H, explicitly computed by Folland [12]. The fundamental solution of the operator $X_{\xi_0}^2+Y_{\xi_0}^2$ with pole at ξ is then the function $\Gamma_{\xi_0}(\xi,\zeta)=\Gamma_H(\phi_{\xi_0}(\xi),\phi_{\xi_0}(\zeta))$. As a consequence, the fundamental solution satisfies the relation

$$\Gamma_{\xi_0}(\xi,\zeta)=\Gamma_H(\phi_{\xi_0}(\xi),\phi_{\xi_0}(\zeta)) \leqslant C d_H^{-N+2}(\phi_{\xi_0}(\xi),\phi_{\xi_0}(\zeta)) \leqslant C d_{\xi_0}^{-N+2}(\xi,\zeta) \quad \text{for all } \xi,\zeta,$$

for a constant C only dependent on Γ_H.

Remark 3.1. From the definition of fundamental solution we can deduce the following assertion: for every $z \in C_0^\infty(\Omega_2)$,

$$z(\xi)=\int X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)X_{\xi_0}z(\zeta)\, d\zeta+\int Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)Y_{\xi_0}z(\zeta)\, d\zeta.$$

Analogously, adapting to the Kohn Laplacian a standard argument known for the classical Laplacian, it is possible to prove the following Morrey-type estimate for the vector fields X, Y and T. Let us denote by $z_{\xi_0,B(\xi,R)}$ the mean value of the function z on $B_{\xi_0}(\xi,R)$, and let $\xi_0 \in \Omega_1$. If $R>0$ satisfies $R<\frac{1}{4}d_{\xi_0}(\xi_0,\partial\Omega_2)$, and $\xi \in B_{\xi_0}(\xi_0,R)$, then we have

$$\begin{aligned} |z(\xi)-z_{\xi_0,B(\xi,R)}| &\leqslant c\int_{B_{\xi_0}(\xi,2R)} d_{\xi_0}^{-N+1}(\xi,\zeta)(|Xz(\zeta)|+|Yz(\zeta)|)\, d\zeta \\ &+c\int_{B_{\xi_0}(\xi,2R)} d_{\xi_0}^{-N+1}(\xi,\zeta)(|a(\zeta)-a(\xi_0)|+|b(\zeta)-b(\xi_0)|)(1+\omega^2)^{1/2}|Tz(\zeta)|\, d\zeta. \end{aligned} \tag{35}$$

Then the following theorem holds:

THEOREM 3.1. *Assume that a, b and ω satisfy conditions* (31) *and* (33), *and that p,q are real numbers such that $p,q>1$ and $N<\min(p,(1+\alpha)q)$. Then there exists a constant C only dependent on M_1, M_3, Ω_1 and Ω_2 such that for every $z\in C^\infty(\Omega)$*

$$\|z\|_{L^\infty(\Omega_1)} \leqslant C(\|z\|_{L^1(\Omega_2)}+\|Xz\|_{L^p(\Omega_2)}+\|Yz\|_{L^p(\Omega_2)}+\|Tz\|_{L^q(\Omega_2)}).$$

Besides, for every $\xi,\xi_0\in\Omega_1$,

$$|z(\xi)-z(\xi_0)| \leqslant Cd^r_{\xi_0}(\xi,\xi_0)(\|Xz\|_{L^p(\Omega_2)}+\|Yz\|_{L^p(\Omega_2)}+\|Tz\|_{L^q(\Omega_2)}),$$

where $r=\min(1-N/p,\alpha+1-N/q)$. In particular, if $p=\infty$ and $N<\alpha q$, then

$$|z(\xi)-z(\xi_0)| \leqslant Cd_{\xi_0}(\xi,\xi_0)(\|Xz\|_{L^p(\Omega_2)}+\|Yz\|_{L^p(\Omega_2)}+\|Tz\|_{L^q(\Omega_2)})$$

for every $\xi,\xi_0\in\Omega_1$, where we have denoted by p' the exponent conjugate of p in the sense that $1/p+1/p'=1$.

Proof. It is quite standard to deduce these assertions from formula (35). Hence we will prove only the first one. With the same notations as in (35), for every $\xi\in B_{\xi_0}(\xi_0,2R)$ we have

$$\begin{aligned}
|z(\xi)| &\leqslant \frac{C}{R^4}\int_{B_{\xi_0}(\xi,2R)}|z(\zeta)|\,d\zeta \\
&\quad + C\biggl(\int_0^{2R}\varrho^{(-N+1)p'+N-1}\,d\varrho\biggr)^{1/p'}(\|Xz\|_{L^p(B_{\xi_0}(\xi,2R))}+\|Yz\|_{L^p(B_{\xi_0}(\xi,2R))}) \\
&\quad + C\biggl(\int_0^{2R}\varrho^{(-N+1+\alpha)q'+N-1}\,d\varrho\biggr)^{1/q'}\|Tz\|_{L^q(B_{\xi_0}(\xi,2R))} \\
&\leqslant [\text{since } B_{\xi_0}(\xi,2R)\subset\Omega_2] \\
&\leqslant C(\|z\|_{L^1(\Omega_2)}+R^{1-N/p}(\|Xz\|_{L^p(\Omega_2)}+\|Yz\|_{L^p(\Omega_2)})+R^{1+\alpha-N/q}\|Tz\|_{L^q(\Omega_2)}).
\end{aligned}$$

3.2. *Intrinsic Lipschitz-continuous coefficients.*

PROPOSITION 3.1. *If condition* (33) *holds with $\alpha=1$, then the function*

$$d(\xi,\xi_0)=d_\xi(\xi,\xi_0)+d_{\xi_0}(\xi,\xi_0) \tag{36}$$

is a pseudodistance, and the functions a and b are Lipschitz continuous with respect to it.(1)

(1) We recall that a pseudodistance is a function $d\colon \mathbf{R}^3\times\mathbf{R}^3\to\mathbf{R}$ satisfying the same conditions as a distance, but with the triangular inequality replaced by the requirement that there exists a constant $C>0$ such that for every x,y,z

$$d(x,y)\leqslant C(d(x,z)+d(z,y)).$$

Proof. It is a consequence of the estimates (32). Indeed,

$$\begin{aligned}d_{\xi_0}(\xi,\xi_0) &\leqslant C\tilde d_{\xi_0}(\xi,\xi_0)\\&\leqslant C(\tilde d_\xi(\xi,\xi_0)+|a(\xi)-a(\xi_0)|^{1/2}|x-x_0|^{1/2}+|b(\xi)-b(\xi_0)|^{1/2}|y-y_0|^{1/2})\\&\leqslant C\tilde d_\xi(\xi,\xi_0)\leqslant Cd_\xi(\xi,\xi_0).\end{aligned}$$

Then

$$\begin{aligned}d_{\xi_0}(\xi,\xi_0) &\leqslant C\tilde d_{\xi_0}(\xi,\xi_0)\\&\leqslant C(|x-x_1|+|x_1-x_0|+|y-y_1|+|y_1-y_0|)\\&\quad +C(|t-t_1-a(\xi)(x-x_1)-b(\xi)(y-y_1)|^{1/2}\\&\qquad +|t_1-t_0-a(\xi_0)(x_1-x_0)-b(\xi_0)(y_1-y_0)|^{1/2})\\&\quad +C(|a(\xi_0)-a(\xi_1)|^{1/2}|x-x_1|^{1/2}+|b(\xi_0)-b(\xi_1)|^{1/2}|y-y_1|^{1/2})\\&\leqslant C(\tilde d_\xi(\xi,\xi_1)+\tilde d_{\xi_0}(\xi_0,\xi_1))\leqslant C(d(\xi,\xi_1)+d(\xi_1,\xi_0)).\end{aligned}$$

Definition 3.2. If condition (33) holds with $\alpha=1$, then we will say that a and b are Lipschitz continuous with respect to the intrinsic distance, and we will denote by $C^\alpha_d(\Omega)$ the class of functions Hölder continuous with respect to d.

THEOREM 3.2. *Assume that a and b are Lipschitz continuous with respect to the intrinsic distance, and that there exists a constant $M_4>0$ such that*

$$\|Xa\|_{L^\infty(\Omega_2)}+\|Ya\|_{L^\infty(\Omega_2)}+\|Xb\|_{L^\infty(\Omega_2)}+\|Yb\|_{L^\infty(\Omega_2)}+\|\nabla_\varepsilon v\|_{L^4(\Omega_2)}\leqslant M_4.$$

Let p be a real number such that $N/(N-1)<p<N$. Then there exists a constant C only dependent on the Lipschitz coefficients M_3 and M_4 such that for every $z\in C^\infty_0(\Omega_1)$

$$\|z\|_{L^r}\leqslant C\|\nabla_\varepsilon z\|_{L^p}\quad \textit{and}\quad r=\frac{Np}{N-p}.$$

Proof. Using Remark 3.1 we get

$$\begin{aligned}z(\xi)=\int X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)Xz(\zeta)\,d\zeta+\int Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)Yz(\zeta)\,d\zeta&\\+\int X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-a(\xi_0))\partial_t z(\zeta)\,d\zeta&\\+\int Y_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(b(\zeta)-b(\xi_0))\partial_t z(\zeta)\,d\zeta.&\end{aligned}$$

Evaluating the function z at the point ξ_0, and using identity (21), we get

$$\begin{aligned}
z(\xi_0)=&\int X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)Xz(\zeta)\,d\zeta+\int Y_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)Yz(\zeta)\,d\zeta\\
&+\int X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-a(\xi_0))\frac{(1+\omega^2)^{1/2}}{(1+a^2+b^2)^{3/2}}[X,Y]z(\zeta)\,d\zeta\\
&-\int X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-a(\xi_0))\frac{(1+\omega^2)^{1/2}}{(1+a^2+b^2)^{3/2}}T_\varepsilon vT_\varepsilon z(\zeta)\,d\zeta\\
&+\int Y_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-b(\xi_0))\frac{(1+\omega^2)^{1/2}}{(1+a^2+b^2)^{3/2}}[X,Y]z(\zeta)\,d\zeta\\
&-\int Y_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(b(\zeta)-b(\xi_0))\frac{(1+\omega^2)^{1/2}}{(1+a^2+b^2)^{3/2}}T_\varepsilon vT_\varepsilon z(\zeta)\,d\zeta.
\end{aligned}$$

These terms have similar behavior, so that we will study only one of them. Let us choose, for example,

$$I_3(\xi_0)=\int X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-a(\xi_0))\frac{(1+\omega^2)^{1/2}}{(1+a^2+b^2)^{3/2}}XYz(\zeta)\,d\zeta.$$

If we denote by X^ζ the derivative with respect to the variable ζ, and use identity (20), then we get

$$\begin{aligned}
I_3(\xi_0)=&-\int X^\zeta\Big(X_{\xi_0}\Gamma_{\xi_0}(\xi,\zeta)(a(\zeta)-a(\xi_0))\frac{1}{(1+a^2+b^2)^{3/2}}\Big)(1+\omega^2)^{1/2}Yz(\zeta)\,d\zeta\\
&-\int X_{\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)(a(\zeta)-a(\xi_0))\frac{1}{(1+a^2+b^2)^{3/2}}\\
&\qquad\times(X(1+\omega^2)^{1/2}+(Yv-\omega Xv)(1+\omega^2)^{1/2})Yz(\zeta)\,d\zeta,
\end{aligned}$$

and so

$$|I_3(\xi_0)|\leqslant C\int d^{-N+1}(\xi_0,\zeta)|Yz(\zeta)|\,d\zeta+C\int d^{-N+2}(\xi_0,\zeta)|Xv(\zeta)|\,|Yz(\zeta)|\,d\zeta.$$

Inserting this estimate in the previous expression we obtain

$$|z(\xi_0)|\leqslant C\Big(\int d^{-N+1}(\xi_0,\zeta)|\nabla_\varepsilon z(\zeta)|\,d\zeta+C\int d^{-N+2}(\xi_0,\zeta)|\nabla_\varepsilon z(\zeta)|\,|\nabla_\varepsilon v(\zeta)|\,d\zeta\Big).$$

Since the pseudodistance d is doubling, then from this relation the asserted inequality holds, see [17, pp. 13, 354].

THEOREM 3.3. *Assume that a and b are Lipschitz continuous with respect to the intrinsic distance. Let z be a function such that*

$$Xz, Yz \in C_d^{\beta}(\Omega_2), \quad Tz \in L^{N/(1-\beta)}(\Omega_2) \quad \text{with } 0<\beta<1.$$

Then there exists a constant M_4 only dependent on M_3 and M such that for every $\xi, \xi_0 \in \Omega_2$

$$\begin{aligned}|z(\xi)-z(\xi_0)-Xz(\xi_0)(x-x_0)-Yz(\xi_0)(y-y_0)|\\ \leqslant M_4 d^{1+\beta}(\xi,\xi_0)(\|Xz\|_{C^\beta(\Omega_1)}+\|Yz\|_{C^\beta(\Omega_1)}+\|Tz\|_{L^q(\Omega_1)}).\end{aligned}$$

Proof. Applying inequality (35) to the function

$$z_1(\xi)=z(\xi)-Xz(\xi_0)(x-x_0)-Yz(\xi_0)(y-y_0),$$

we get

$$\begin{aligned}
&|z(\xi)-z(\xi_0)-Xz(\xi_0)(x-x_0)-Yz(\xi_0)(y-y_0)|=|z_1(\xi)-z_1(\xi_0)|\\
&\quad\leqslant \int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))} d_{\xi_0}^{-N+1}(\xi,\zeta)(|Xz(\zeta)-Xz(\xi_0)|+|Yz(\zeta)-Yz(\xi_0)|)\,d\zeta\\
&\qquad+\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))} d_{\xi_0}^{-N+1}(\xi,\zeta)|a(\zeta)-a(\xi_0)|\,|Tz(\zeta)|\,d\zeta\\
&\qquad+\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))} d_{\xi_0}^{-N+1}(\xi,\zeta)|b(\zeta)-b(\xi_0)|\,|Tz(\zeta)|\,d\zeta \qquad (37)\\
&\qquad+\int_{B_{\xi_0}(\xi,2d_{\xi_0}(\xi,\xi_0))} d_{\xi_0}^{-N+2}(\xi,\zeta)|Tz(\zeta)|\,d\zeta\\
&\quad\leqslant \text{[since } Xz, Yz\in C_d^\beta\text{, and by the assumptions on } a \text{ and } b\text{, setting } r=d_{\xi_0}(\xi,\xi_0)]\\
&\quad\leqslant c\int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+1}(\xi,\zeta)d_{\xi_0}^{\beta}(\xi_0,\zeta)\,d\zeta+\int_{B_{\xi_0}(\xi,2r)} d_{\xi_0}^{-N+2}(\xi,\zeta)|Tz(\zeta)|\,d\zeta\\
&\quad\leqslant \int_0^r(\varrho^\beta+r^\beta)\varrho\,d\varrho+\left(\int_0^r \varrho^{(-N+1)q'+N-1}\,d\varrho\right)^{1/q'}\|Tz\|_q.
\end{aligned}$$

COROLLARY 3.1. *Assume that (31) is satisfied and that there exists a constant M_5 such that*

$$\begin{aligned}&\|a\|_{W_\varepsilon^{2,6}(\Omega_2)}+\|Ta\|_{W_\varepsilon^{1,3}(\Omega_2)}+\|T^2a\|_{L^2(\Omega_2)}\\ &+\|b\|_{W_\varepsilon^{2,6}(\Omega_2)}+\|Tb\|_{W_\varepsilon^{1,3}(\Omega_2)}+\|T^2b\|_{L^2(\Omega_2)}+\|T\omega\|_{L^3(\Omega_2)}\leqslant M_5.\end{aligned} \qquad (38)$$

Then the function d defined in (36) is a distance, a and b are Lipschitz continuous with respect to it, and the following inequalities hold:

(i) *If $N/(N-1)<p<N$, $r=Np/(N-p)$ then*

$$\|z\|_{L^r} \leqslant c\|\nabla_\varepsilon z\|_{L^p}$$

for all $z\in C_0^\infty(\Omega_1)$.

(ii) *If $p>N$, $q>\frac{1}{2}N$ and $\beta=\min(1-N/p, 2-N/q)$ then*

$$|z(\xi)-z(\xi_0)| \leqslant cd^\beta(\xi,\xi_0)(\|\nabla_\varepsilon z\|_{L^p(\Omega_2)}+\|Tz\|_{L^q(\Omega_2)})$$

for all $z\in C^\infty(\Omega_2)$ and for every $\xi,\xi_0\in\Omega_1$.

Proof. Let us first note that, by the standard Sobolev embedding theorem, there exists a constant only dependent on M_5 such that

$$\|Ta\|_{L^6(\Omega_2)}+\|Tb\|_{L^6(\Omega_2)} \leqslant C.$$

By identity (19) we also have

$$\|\nabla_\varepsilon \omega\|_{L^6(\Omega_2)} \leqslant \|Ta\|_{L^6(\Omega_2)}+\|Tb\|_{L^6(\Omega_2)} \leqslant M_5.$$

Besides, all the other second-order Euclidean derivatives are bounded:

$$\begin{aligned}\|\partial_{tt}^2 a\|_{L^2(\Omega_2)} &\leqslant C\left\|T\left(\frac{\partial_t a}{(1+\omega^2)^{1/2}}\right)+\frac{\omega\partial_t aT\omega}{(1+\omega^2)^{3/2}}\right\|_{L^2(\Omega_2)}\\ &\leqslant \|T^2a\|_{L^2(\Omega_2)}+\|Ta\|_{L^6(\Omega_2)}\|T\omega\|_{L^3(\Omega_2)} \leqslant C.\end{aligned}$$

Since

$$\begin{aligned}\|X\partial_t a\|_{L^2(\Omega_2)} &= \|XTa\|_{L^2(\Omega_2)}+\|X\omega Ta\|_{L^2(\Omega_2)}\\ &\leqslant \|XTa\|_{L^2(\Omega_2)}+\|X\omega\|_{L^3(\Omega_2)}\|Ta\|_{L^6(\Omega_2)} \leqslant C,\end{aligned}$$

then

$$\|\partial_{xt}a\|_{L^2(\Omega_2)} \leqslant \|X\partial_t a\|_{L^2(\Omega_2)}+\|\partial_{tt}^2 a\|_{L^2(\Omega_2)} \leqslant C$$

and

$$\begin{aligned}\|\partial_{xx}a\|_{L^2(\Omega_2)} &\leqslant \|(X-a\partial_t)^2 a\|_{L^2(\Omega_2)}\\ &= \|X^2a-Xa\partial_t a-aX\partial_t a-a(\partial_t a)^2-aX\partial_t a+a^2\partial_{tt}a\|_{L^2(\Omega_2)} \leqslant C.\end{aligned}$$

Hence a, b belong to the classical Sobolev space $H^1(\Omega_2)$, and there exists a constant M_3 only dependent on M_5 in (38) such that (33) holds with $\alpha=\frac{1}{2}$. Now we choose $\Omega_3, ..., \Omega_5$ such that $\Omega_1 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega_2$. Hence

$$\begin{aligned}
&\|Xa\|_{L^1(\Omega_2)}+\|\nabla_\varepsilon Xa\|_{L^6(\Omega_2)}+\|TXa\|_{L^3(\Omega_2)} \\
&\qquad \leqslant \|a\|_{W^{2,6}(\Omega_2)}+\|\nabla_\varepsilon vTa\|_{L^3(\Omega_2)}+\|XTa\|_{L^3(\Omega_2)} \\
&\qquad \overset{(21)}{\leqslant} \|a\|_{W^{2,6}(\Omega_2)}+\|Ta\|_{L^6(\Omega_2)}+\|Ta\|_{W^{1,3}(\Omega_2)} \leqslant C.
\end{aligned}$$

Hence the first assertion of Proposition 3.1 with $p=6$, $q=3$ and $\alpha=\frac{1}{2}$ ensures that there exists a constant C only dependent on M_i and Ω_i, $i=1, ..., 5$, such that $\|Xa\|_{L^\infty(\Omega_4)} \leqslant C$. By the second assertion in Proposition 3.1, using the fact that

$$\|Xa\|_{L^\infty(\Omega_4)}+\|Ta\|_{L^6(\Omega_4)} \leqslant C,$$

we deduce that

$$|a(\xi)-a(\xi_0)| \leqslant C d_{\xi_0}^{\alpha}(\xi, \xi_0) \quad \text{for all } \xi, \xi_0 \in \Omega_3,$$

where $\alpha=\frac{5}{6}$, and again C only depends on M_i and Ω_i. Applying the third assertion of the same proposition we now get

$$|a(\xi)-a(\xi_0)| \leqslant C d_{\xi_0}(\xi, \xi_0) \quad \text{for all } \xi, \xi_0 \in \Omega_1.$$

The thesis now follows from Theorem 3.3.

4. L^p-estimates for the linear equation

In this section we prove the following a priori estimates, in the Sobolev spaces $W^{m,p}_{\varepsilon,\mathrm{loc}}(\Omega)$ for solutions of equation (15), under the assumption that there exists a constant M such that

$$\|u\|_{L^\infty(\Omega)}+\|\nabla_\varepsilon u\|_{L^\infty(\Omega)}+\|\partial_t u\|_{L^\infty(\Omega)}+\|\nabla_\varepsilon a\|_{L^2(\Omega)}+\|\nabla_\varepsilon b\|_{L^2(\Omega)}+\|\nabla_\varepsilon v\|_{L^2(\Omega)} \leqslant M. \tag{39}$$

THEOREM 4.1. *Let $p \geqslant 3$ and m be a fixed positive integer. Assume that $f \in C^\infty(\Omega)$, and let z be a solution of equation* (15) *in Ω. If $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ then there exist constants C and $\widetilde{C}$ which depend on p, Ω_i and on M in* (39), *but are independent of ε or z, such* *that the solution satisfies the estimate*

$$\begin{aligned}
&\|z\|^p_{W^{m+1,p}_\varepsilon(\Omega_1)}+\sum_{|i|=m+1}\||D_iz|^{(p-1)/2}\|^2_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\qquad\leqslant C\big(\|f\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|v\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|z\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\big)\\
&\qquad\quad+\widetilde{C}\big(\|Tz\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|Tz\|^{p}_{W^{m-1,p}_\varepsilon(\Omega_2)}\\
&\qquad\qquad+\|a\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|b\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|(1+a^2+b^2)^{3/2}\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\big).
\end{aligned}$$

If $k=0$, *then we can choose* $\widetilde{C}=0$.

In view of further applications, we have stated here a result more general than strictly necessary in this context. In particular, we do not make any assumption on the curvature k.

The proof of this result is a modification of the classical Moser argument, which uses a Sobolev-type theorem and a Caccioppoli inequality. In our context the Caccioppoli inequality still holds, but the coefficients of the vector fields are not regular, and no embedding theorems hold in these spaces. In particular, we cannot apply the results just proved in the first steps of the regularization procedure. On the contrary we prove an interpolation inequality which will take the place of the embedding theorems. This is done in §4.1. In §4.2 we prove the Caccioppoli inequality. In §4.3 we perform an iterative procedure, and we end the section with the proof of Theorem 4.1.

4.1. *Interpolation inequalities.* Let us start with a simple remark:

PROPOSITION 4.1. *For every function* $\phi\in C^\infty_0(\Omega)$ *we have*

$$\int|\nabla_\varepsilon v|^2\phi^2\leqslant C\int(k^2+|Tk|)\phi^2+C\int|\nabla_\varepsilon\phi|^2,$$

for a suitable constant C *depending only on the constant* M *in* (39).

Proof. Let us first note that

$$\partial_t aYv-\partial_t bXv\overset{(19)}{=}(Yv-\omega Xv)Yv+(Xv+\omega Yv)Xv=(Xv)^2+(Yv)^2. \tag{40}$$

Then we have

$$\begin{aligned}
\int ((Xv)^2+(Yv)^2)\phi^2+\int (T_\varepsilon v)^2\phi^2 &\overset{(40)}{=} \int (\partial_t aYv-\partial_t bXv)\phi^2+\int (T_\varepsilon v)^2\phi^2\\
&\overset{(16)}{=} \int \partial_t YuYv\phi^2+\int \partial_t XuXv\phi^2+\int (T_\varepsilon v)^2\phi^2\\
&\overset{(22)}{=} -\int \partial_t uY^2v\phi^2-2\int \partial_t uYv\phi Y\phi-\int \partial_t uX^2v\phi^2\\
&\quad -2\int \partial_t uXv\phi X\phi+\int (T_\varepsilon v)^2\phi^2\\
&\overset{(29)}{=} \int \partial_t uT_\varepsilon^2 v\phi^2-\int \omega T(k(1+a^2+b^2)^{3/2})\phi^2\\
&\quad +\int (T_\varepsilon^2 v)^2\phi^2-2\int \omega Yv\phi Y\phi-2\int \omega Xv\phi X\phi\\
&= [\text{using (20) in the first term,}\\
&\quad \text{and the definition of } T \text{ and (19) in the second}]\\
&= \int \omega^2(T_\varepsilon v)^2\phi^2-\int T_\varepsilon\omega T_\varepsilon v\phi^2\\
&\quad -2\int \omega T_\varepsilon v\phi T_\varepsilon\phi+\int (|Tk|+|k|\,|Xv|+|k|\,|Yv|)\phi^2\\
&\quad +\int (T_\varepsilon v)^2\phi^2-2\int \omega Yv\phi Y\phi-2\int \omega Xv\phi X\phi\\
&= [\text{since } v=\arctan(\omega) \text{ the terms 1, 2 and 5 cancel}]\\
&\leqslant \int |\nabla_\varepsilon v|\,|\phi|\,|\nabla_\varepsilon\phi|+\int (|k|\,|Xv|+|k|\,|Yv|+|Tk|)\phi^2.
\end{aligned}$$

The thesis now follows with a Hölder inequality.

Now we can prove our main interpolation inequality:

PROPOSITION 4.2. *For every $p\geqslant 3$, there exists a constant C_p, dependent on p and the constant M in* (39), *such that for every function $z\in C^\infty(\Omega)$ and for every $\phi\in C_0^\infty(\Omega)$,*

$$\begin{aligned}
&\int |Xz|^p\phi^{2p}\\
&\quad\leqslant C\Big(\int |z|^{2p}\phi^{2p}+\int |\nabla_\varepsilon(|Xz|^{(p-1)/2})|^2|\phi|^{2p}+\int |Xz|^{p-1}(|X\phi|^2+\phi^2)|\phi|^{2p-2}\Big),
\end{aligned}$$

where the function v is defined in (18). *Analogous relations hold if we replace X with Y or T_ε.*

Proof. We have

$$
\begin{aligned}
\int |Xz|^p\phi^{2p} &= \int Xz|Xz|^{p-1}\operatorname{segn}(Xz)\phi^{2p}\\
&= [\text{integrating by parts and using the fact that } X^* = -X-Yv+\omega Xv]\\
&= -\int (Yv-\omega Xv)z|Xz|^{p-1}\operatorname{segn}(Xz)\phi^{2p}\\
&\quad -2\int zX(|Xz|^{(p-1)/2})|Xz|^{(p-1)/2}\phi^{2p}\\
&\quad -2p\int z|Xz|^{p-1}\operatorname{segn}(Xz)\phi^{2p-1}X\phi\\
&\leqslant [\text{by a Hölder inequality}] \qquad\qquad (41)\\
&\leqslant C\biggl(\int |\nabla_\varepsilon v|^2|Xz|^{p-1}\phi^{2p}+\int |z|^{2p}\phi^{2p}+\delta\int |Xz|^p\phi^{2p}\\
&\quad +\int |X(|Xz|^{(p-1)/2})|^2\phi^{2p}+\int |Xz|^{p-1}\phi^{2p-2}|X\phi|^2\biggr)\\
&\leqslant [\text{by Proposition 4.1}]\\
&\leqslant C\biggl(\int |Xz|^{p-1}\phi^{2p}+\int |z|^{2p}\phi^{2p}+\delta\int |Xz|^p\phi^{2p}\\
&\quad +\int |\nabla_\varepsilon(|Xz|^{(p-1)/2})|^2\phi^{2p}+\int |Xz|^{p-1}\phi^{2p-2}|X\phi|^2\biggr),
\end{aligned}
$$

and choosing δ sufficiently small we get the assertion.

4.2. *Caccioppoli-type inequalities.* Let us start with a Caccioppoli-type inequality for the derivative with respect to T.

THEOREM 4.2. *Assume that $f\in C^\infty(\Omega)$, and that z is a solution of* (15). *Then there exists a constant $C>0$ dependent on M such that for every $\phi\in C_0^\infty(\Omega)$*

$$
\begin{aligned}
\int (|\nabla_\varepsilon Tz|^2+|T\nabla_\varepsilon z|^2)\phi^6+\int |\nabla_\varepsilon v|^2(Tz)^2\phi^6&\\
\leqslant C\int (\phi^2(|k|+|Tk|)+|\nabla_\varepsilon\phi|^2)|Tz|^2\phi^4-\int TfTz\phi^6.&
\end{aligned}
\tag{42}
$$

We will make use of the following simple property:

Remark 4.1. From identity (22) and the definition of T it immediately follows that for every function $f,\phi\in C_0^\infty(\Omega)$ we have

$$\int TXf\phi=\int\frac{\partial_t Xf}{(1+\omega^2)^{1/2}}\phi=\int\omega TfXv\phi-\int TfX\phi.$$

Analogously

$$\int TYf\phi=\int\omega TfYv\phi-\int TfY\phi$$

and

$$\int TT_\varepsilon f\phi=\int\omega TfT_\varepsilon v\phi-\int TfT_\varepsilon\phi.$$

Proof. We differentiate equation (15) with respect to T, then we multiply by $Tz\phi^6$ and integrate:

$$\begin{aligned}\int TfTz\phi^2&=\int T(X^2z+Y^2z+T_\varepsilon^2)Tz\phi^6\\&=\text{[by Remark 4.1]}\\&=-\int TXzXTz\phi^6+\int TXz\omega XvTz\phi^6-6\int TXzTz\phi^5X\phi\\&\quad-\int TYzYTz\phi^6+\int TYz\omega YvTz\phi^6-6\int TYzTz\phi^5Y\phi\\&\quad-\int TT_\varepsilon zT_\varepsilon Tz\phi^6+\int TT_\varepsilon z\omega T_\varepsilon vTz\phi^6-6\int TT_\varepsilon zTz\phi^5T_\varepsilon\phi\\&=I_1+\ldots+I_9.\end{aligned}$$

Let us consider a few terms separately:

$$\begin{aligned}I_1+I_4&=-\frac12\int([T,X]z+XTz)XTz\phi^6-\frac12\int TXz([X,T]z+TXz)\phi^6\\&\quad-\frac12\int([T,Y]z+YTz)YTz\phi^6-\frac12\int TYz([Y,T]z+TYz)\phi^6\\&\overset{(21)}{=}-\frac12\int((TXz)^2+(XTz)^2+(TYz)^2+(YTz)^2)\phi^6\\&\quad-\frac12\int YvTz(XTz-TXz)\phi^6+\frac12\int XvTz(YTz-TYz)\phi^6\\&=-\frac12\int((TXz)^2+(XTz)^2+(TYz)^2+(YTz)^2)\phi^6\\&\quad+\frac12\int((Xv)^2+(Yv)^2)(Tz)^2\phi^6.\end{aligned}$$

On the other hand, using identity (21) in I_2 and I_5, we get

$$\begin{aligned} I_2+I_5+I_8 = &\int YvTz\omega XvTz\phi^6 - \int XvTz\omega YvTz\phi^6 \\ &+\frac{1}{2}\int X((Tz)^2)Xv\omega\phi^6 + \frac{1}{2}\int Y((Tz)^2)Yv\omega\phi^6 \\ &+\frac{1}{2}\int T_\varepsilon((Tz)^2)T_\varepsilon v\omega\phi^6. \end{aligned}$$

Canceling the first two terms and integrating by parts the last three terms by means of the identities (20), we get

$$\begin{aligned} I_2+I_5+I_8 = &-\frac{1}{2}\int (Tz)^2 XvX\omega\phi^6 - 3\int (Tz)^2 Xv\omega\phi^5 X\phi \\ &-\frac{1}{2}\int (Tz)^2 Xv\omega(Yv-\omega Xv)\phi^6 - \frac{1}{2}\int (Tz)^2 YvY\omega\phi^6 \\ &-3\int (Tz)^2 Yv\omega\phi^5 Y\phi + \frac{1}{2}\int (Tz)^2 Yv\omega(Xv+\omega Yv)\phi^6 \\ &-\frac{1}{2}\int (Tz)^2 T_\varepsilon v T_\varepsilon\omega\phi^6 - 3\int (Tz)^2 T_\varepsilon v\omega\phi^5 T_\varepsilon\phi \\ &-\frac{1}{2}\int (Tz)^2\omega^2(T_\varepsilon v)^2\phi^6 - \frac{1}{2}\int (Tz)^2\omega L_\varepsilon v\phi^6. \end{aligned}$$

Using the fact that $v=\arctan u_t$ in the terms 1, 4 and 7, and using Proposition 2.1 in the last term, we arrive at

$$\begin{aligned} I_2+I_5+I_8 = &-\frac{1}{2}\int (Tz)^2\omega T(k(1+a^2+b^2)^{3/2})\phi^6 \\ &-3\int (Tz)^2\omega(XvX\phi+YvY\phi+T_\varepsilon vT_\varepsilon\phi)\phi^5 \\ &-\frac{1}{2}\int (Tz)^2((Xv)^2+(Yv)^2+(T_\varepsilon v)^2)\phi^6 \\ = &-\frac{1}{2}\int (Tz)^2\omega Tk(1+a^2+b^2)^{3/2}\phi^6 \\ &-\frac{3}{2}\int (Tz)^2\omega k(1+a^2+b^2)^{1/2}(aTa+bTb)\phi^6 \\ &-3\int (Tz)^2\omega(XvX\phi+YvY\phi+T_\varepsilon vT_\varepsilon\phi)\phi^5 \\ &-\frac{1}{2}\int (Tz)^2((Xv)^2+(Yv)^2+(T_\varepsilon v)^2)\phi^6. \end{aligned}$$

Summing up all terms we get

$$\begin{aligned}
&\frac{1}{2}\int((TXz)^2+(XTz)^2+(TYz)^2+(YTz)^2)\phi^6+\int(T_\varepsilon Tz)^2\phi^6+\frac{1}{2}\int(Tz)^2(T_\varepsilon v)^2\phi^6\\
&\qquad\leqslant-\int TfTz\phi^6+\frac{1}{2}C\int(Tz)^2|\omega|\,|Tk|\phi^6\\
&\qquad\quad+C\delta\int(Tz)^2((Ta)^2+(Tb)^2)\phi^6+\frac{C}{\delta}\int(Tz)^2k^2\phi^6\\
&\qquad\quad+\delta\int(Tz)^2((Xv)^2+(Yv)^2+(T_\varepsilon v)^2)\phi^6\\
&\qquad\quad+\frac{1}{\delta}\int(Tz)^2((X\phi)^2+(Y\phi)^2+(T_\varepsilon\phi)^2)\phi^4.
\end{aligned}$$

By condition (19) and the boundedness of ω it follows that $(Ta)^2+(Tb)^2\leqslant(Xv)^2+(Yv)^2$, and by condition (21) we deduce that

$$|Xv|\,|Tz|+|Yv|\,|Tz|\leqslant|[X,T]z|+|[Y,T]z|.$$

Hence we get inequality (42), choosing δ sufficiently small.

Let us now prove another Caccioppoli-type inequality, more general than the preceding one, in the directions X,Y,T_ε, for the solutions of the linear equation (15). By Lemmas 2.1–2.3, if z is a solution of that equation, then its derivative is a solution of an equation of the form

$$L_\varepsilon z=f_0+f_1Xz+f_2Yz+f_3T_\varepsilon z. \tag{43}$$

Hence, in view of the iteration, we will study solutions of this equation.

LEMMA 4.1. *Assume that $f_0,...,f_3\in L^r_{\mathrm{loc}}(\Omega)$ and $f_4,...,f_6\in W^{1,r}_{\varepsilon,\mathrm{loc}}(\Omega)$ with $r>2$, and that $z\in W^{2,2}_{\varepsilon,\mathrm{loc}}(\Omega)\cap W^{1,3}_{1,\mathrm{loc}}(\Omega)$ is a solution of the equation*

$$L_\varepsilon z=\tilde f_0+\tilde f_1Xz+\tilde f_2Yz+\tilde f_3T_\varepsilon z+X\tilde f_4+Y\tilde f_5+T_\varepsilon\tilde f_6+z\tilde f_7. \tag{44}$$

For every $p\geqslant3$ there exist constants C_1,C_2,C_3,C_4 depending only on p and the constant M in (39), and independent of ε and z, such that for every $\phi\in C_0^\infty(\Omega)$, $\phi>0$, we have

$$\begin{aligned}
\int|\nabla_\varepsilon(|z|^{(p-1)/2})|^2\phi^2\leqslant C_1&\int|z|^{p-1}(\phi^2+|\nabla_\varepsilon\phi|^2)-\int\tilde f_0|z|^{p-3}z\phi^2-\int\tilde f_7|z|^{p-1}\phi^2\\
&+C_2\int|z|^{p-1}|\nabla_\varepsilon v|^2\phi^2+C_3\int|z|^{p-1}(|\tilde f_1|^2+|\tilde f_2|^2+|\tilde f_3|^2)\phi^2\\
&+C_4\int|z|^{p-3}(|\tilde f_4|^2+|\tilde f_5|^2+|\tilde f_6|^2)\phi^2\\
&+C_4\int|z|^{p-2}(|\tilde f_4|+|\tilde f_5|+|\tilde f_6|)|\nabla_\varepsilon v|\phi^2.
\end{aligned}$$

If the curvature $k=0$, we can choose $C_2=0$. If $k=0$, and $f_1=-2\partial_t a$, $f_2=-2\partial_t b$, $f_3=2\omega T_\varepsilon v T_\varepsilon z$, we can choose $C_2=C_3=0$.

Proof. Let us multiply both members of equation (44) by $|z|^{p-3}z\phi^2$, and integrate. Then we get

$$\begin{aligned}
&\int (\tilde f_0+X\tilde f_4+Y\tilde f_5+T_\varepsilon \tilde f_6+\tilde f_1 Xz+\tilde f_2 Yz+\tilde f_3 T_\varepsilon z+z\tilde f_7)|z|^{p-3}z\phi^2\\
&\quad=\int (X^2z+Y^2z+T_\varepsilon^2 z)|z|^{p-3}z\phi^2\\
&\quad=[\text{since } X^*=-X-\partial_t a,\ Y^*=-Y-\partial_t b \text{ and } T_\varepsilon^*=-T_\varepsilon+\omega T_\varepsilon v]\\
&\quad=-\int \partial_t a Xz|z|^{p-3}z\phi^2-(p-2)\int (Xz)^2|z|^{p-3}\phi^2-2\int |z|^{p-3}zXz\phi X\phi\\
&\qquad-\int \partial_t b Yz|z|^{p-3}z\phi^2-(p-2)\int (Yz)^2|z|^{p-3}\phi^2-2\int |z|^{p-3}zYz\phi Y\phi\\
&\qquad+\int \omega T_\varepsilon v T_\varepsilon z|z|^{p-3}z\phi^2-(p-2)\int (T_\varepsilon z)^2|z|^{p-3}\phi^2-2\int T_\varepsilon z|z|^{p-3}z\phi T_\varepsilon\phi\\
&\quad\leqslant \int \langle(-\partial_t a,-\partial_t b,\omega T_\varepsilon v),\nabla_\varepsilon z\rangle |z|^{p-3}z\phi^2-\frac{4(p-2)}{(p-1)^2}\int (\nabla_\varepsilon(|z|^{(p-1)/2}))^2\phi^2\\
&\qquad-\frac{4}{p-1}\int \langle \nabla_\varepsilon(|z|^{(p-1)/2})\phi,\nabla_\varepsilon\phi|z|^{(p-1)/2}\rangle,
\end{aligned}\tag{45}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbf{R}^3$. This obviously implies that there exists a constant $C>0$ such that

$$\begin{aligned}
\frac{4(p-2)}{(p-1)^2}\int |\nabla_\varepsilon(|z|^{(p-1)/2})|^2\phi^2 \leqslant C\int |z|^{p-1}|\nabla_\varepsilon\phi|^2&\\
-\int \langle(\partial_t a,\partial_t b,-\omega T_\varepsilon v),\nabla_\varepsilon z\rangle|z|^{p-3}z\phi^2&\\
-\int \langle(\tilde f_1,\tilde f_2,\tilde f_3),\nabla_\varepsilon z\rangle|z|^{p-3}z\phi^2&\\
-\int \tilde f_7|z|^{p-1}\phi^2-\int \tilde f_0|z|^{p-3}z\phi^2&\\
-\int (X\tilde f_4+Y\tilde f_5+T_\varepsilon\tilde f_6)|z|^{p-3}z\phi^2.&
\end{aligned}\tag{46}$$

Let us denote by $I_0,\dots,I_5$ the terms on the right-hand side. We have to study only I_1, I_2 and I_5. Integrating by parts the last term we have

$$
\begin{aligned}
I_5 = & \int \partial_t a \tilde{f}_4 |z|^{p-3} z \phi^2 + (p-2) \int \tilde{f}_4 X(|z|) |z|^{p-3} \phi^2 + 2 \int \tilde{f}_4 |z|^{p-3} z \phi X \phi \\
& + \int \partial_t b \tilde{f}_5 |z|^{p-3} z \phi^2 + (p-2) \int \tilde{f}_5 Y(|z|) |z|^{p-3} \phi^2 + 2 \int \tilde{f}_5 |z|^{p-3} z \phi Y \phi \\
& - \int \omega T_\varepsilon v \tilde{f}_6 |z|^{p-3} z \phi^2 + (p-2) \int \tilde{f}_6 T_\varepsilon(|z|) |z|^{p-3} \phi^2 + 2 \int \tilde{f}_6 |z|^{p-3} z \phi T_\varepsilon \phi.
\end{aligned}
$$

Using relation (19) in the integrals 3, 6 and 9 we get

$$
\begin{aligned}
I_5 \leqslant & \int |\nabla_\varepsilon v| \, |\tilde{f}_4| \, |z|^{p-2} \phi^2 + \frac{2(p-2)}{p-1} \int |\tilde{f}_4| X(|z|^{(p-1)/2}) |z|^{(p-3)/2} \phi^2 \\
& + 2 \int |\tilde{f}_4| \, |z|^{p-2} \phi |X\phi| \\
& + \int |\nabla_\varepsilon v| \, |\tilde{f}_5| \, |z|^{p-2} \phi^2 + \frac{2(p-2)}{p-1} \int |\tilde{f}_5| Y(|z|^{(p-1)/2}) |z|^{(p-3)/2} \phi^2 \\
& + 2 \int |\tilde{f}_5| \, |z|^{p-2} \phi |Y\phi| \\
& + \int |\nabla_\varepsilon v| \, |\tilde{f}_6| \, |z|^{p-2} \phi^2 + \frac{2(p-2)}{p-1} \int |\tilde{f}_6| T_\varepsilon(|z|^{(p-1)/2}) |z|^{(p-3)/2} \phi^2 \\
& + 2 \int |\tilde{f}_6| \, |z|^{p-2} \phi |T_\varepsilon \phi|,
\end{aligned}
$$

and with a Hölder inequality we arrive at

$$
\begin{aligned}
I_5 \leqslant & \, C_4 \int |z|^{p-2} (|\tilde{f}_4| + |\tilde{f}_5| + |\tilde{f}_6|) |\nabla_\varepsilon v| \phi^2 + C_4 \int |z|^{p-3} (|\tilde{f}_4|^2 + |\tilde{f}_5|^2 + |\tilde{f}_6|^2) \phi^2 \\
& + \delta \int |\nabla_\varepsilon (z^{(p-1)/2})|^2 \phi^2 + C \int |z|^{p-1} |\nabla_\varepsilon \phi|^2,
\end{aligned}
$$

where δ will be chosen sufficiently small.

Finally we have to consider I_1 and I_2 in (46). If we do not have any hypotheses on k, we get

$$
\begin{aligned}
I_1 + I_2 = & -\frac{2}{p-1} \int (\partial_t a X(|z|^{(p-1)/2}) + \partial_t b Y(|z|^{(p-1)/2}) - \omega T_\varepsilon v T_\varepsilon(|z|^{(p-1)/2})) |z|^{(p-1)/2} \phi^2 \\
& - \frac{2}{p-1} \int (\tilde{f}_1 X(|z|^{(p-1)/2}) + \tilde{f}_2 Y(|z|^{(p-1)/2}) + \tilde{f}_3 T_\varepsilon(|z|^{(p-1)/2})) |z|^{(p-1)/2} \phi^2 \\
& \leqslant \text{[using equation (19) and a Hölder inequality]} \\
& \leqslant C_1 \int |z|^{p-1} |\nabla_\varepsilon v|^2 \phi^2 + C_2 \int |z|^{p-1} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |\tilde{f}_3|^2) \phi^2 \\
& + \delta \int |\nabla_\varepsilon (|z|^{(p-1)/2})|^2 \phi^2.
\end{aligned}
$$

Now the thesis follows, inserting all terms in (46).

Note that, when $k=0$, then by (21), $T_\varepsilon v T_\varepsilon z=[X,Y]z$. Hence, using that $\partial_t a=\partial_t Yu$ and $\partial_t b=-\partial_t Xu$ from (16), and then (22),

$$\begin{aligned}
&\int \partial_t a Xz|z|^{p-3}z\phi^2+\int \partial_t b Yz|z|^{p-3}z\phi^2-\int \omega T_\varepsilon v T_\varepsilon z|z|^{p-3}z\phi^2\\
&\quad=-\int \omega YXz|z|^{p-3}\phi^2-(p-2)\int \omega XzYz|z|^{p-3}\phi^2-2\int \omega X|z|\,|z|^{p-2}\phi Y\phi\\
&\qquad+\int \omega XYz|z|^{p-3}z\phi^2-(p-2)\int \omega YzXz|z|^{p-3}\phi^2-2\int \omega Y|z|\,|z|^{p-2}\phi X\phi\\
&\qquad-\int \omega[X,Y]z|z|^{p-3}z\phi^2\\
&\quad\leqslant \delta\int |\nabla_\varepsilon(|z|^{(p-1)/2})|^2\phi^2+\frac{C}{\delta}\int |z|^{p-1}|\nabla_\varepsilon\phi|^2,
\end{aligned}$$

where in the last step we used that the integrals 1, 4 and 7 cancel, as do the integrals 2 and 5. To the other we applied a Hölder inequality.

Again, inserting all terms in (46), we get the stated assertion, for $k=0$.

4.3. *Iterative procedure.* We can now conclude the proof of Theorem 4.1 using iteratively the interpolation and the Caccioppoli inequalities. We first deduce from the preceding lemmas some a priori estimates for the derivatives of a function z, solution of equation (43).

THEOREM 4.3. *Let $p\geqslant 3$ be fixed, let $f_0,\dots f_3\in C^\infty(\Omega)$, and let z be a solution of equation* (43). *Then there exist two constants C and $\widetilde{C}$ which depend on p and the constant M in* (39), *but are independent of ε and z, such that for every $\phi\in C_0^\infty(\Omega)$, $\phi>0$,*

$$\begin{aligned}
&\int |\nabla_\varepsilon z|^p\phi^{2p}+\int |\nabla_\varepsilon(|\nabla_\varepsilon z|^{(p-1)/2})|^2\phi^{2p}\\
&\qquad\leqslant C\int |z|^{2p}\phi^{2p}+C\int(\phi^2+|\nabla_\varepsilon\phi|^2)^p+C\int |\nabla_\varepsilon v|^{2p}\phi^{2p}\\
&\qquad\quad+C\int(|f_0|^{2p/3}+|f_1|^{2p}+|f_2|^{2p}+|f_3|^{2p})\phi^{2p}\\
&\qquad\quad+\widetilde{C}\biggl(\int |Tz|^{2p/3}\phi^{2p}+\int(|\nabla_\varepsilon a|+|\nabla_\varepsilon b|)^{p/2}|Tz|^{p/2}\phi^{2p}\biggr).
\end{aligned}$$

If $k=0$, we can choose $\widetilde{C}=0$.

Proof. Since z is a solution of equation (43) then by Lemma 2.1, $s_1=Xz$ satisfies equation (44), with coefficients

$$\tilde f_0 = k(1+a^2+b^2)^{3/2}TYz+2XvT_\varepsilon vT_\varepsilon z, \quad \tilde f_4 = f_0+f_1Xz+f_2Yz+f_3T_\varepsilon z,$$
$$\tilde f_1=\tilde f_2=\tilde f_3=\tilde f_7=0, \quad \tilde f_5=-2T_\varepsilon vT_\varepsilon z+k(1+a^2+b^2)^{3/2}Tz, \quad \tilde f_6=2YvT_\varepsilon z.$$

Using Lemma 4.1 we deduce

$$\begin{aligned}
\int & |\nabla_\varepsilon(|s_1|^{(p-1)/2})|^2\phi^{2p} \\
&\leqslant C_1\int |s_1|^{p-1}(|\nabla_\varepsilon\phi|^2+\phi^2)\phi^{2p-2}-\int |s_1|^{p-3}s_1k(1+a^2+b^2)^{3/2}TYz\phi^{2p} \\
&\quad -2\int |s_1|^{p-3}s_1XvT_\varepsilon vT_\varepsilon z\phi^{2p}+C_2\int |s_1|^{p-1}|\nabla_\varepsilon v|^2\phi^{2p} \\
&\quad +C_4\int |s_1|^{p-3}(f_0^2+|f_1|^2|Xz|^2+|f_2|^2|Yz|^2+|f_3|^2|T_\varepsilon z|^2)\phi^{2p} \\
&\quad +C_4\int |s_1|^{p-3}|\nabla_\varepsilon v|^2|T_\varepsilon z|^2\phi^{2p}+C_4\int k^2(1+a^2+b^2)^3(Tz)^2|s_1|^{p-3}\phi^{2p} \\
&\leqslant \delta\int |s_1|^p\phi^{2p}+C\int(\phi^2+|\nabla_\varepsilon\phi|^2)^p-\int |s_1|^{p-3}s_1k(1+a^2+b^2)^{3/2}TYz\phi^{2p} \\
&\quad +\frac{C}{\delta}\int |\nabla_\varepsilon v|^{2p}\phi^{2p}+\delta\int |\nabla_\varepsilon z|^p\phi^{2p}+C\int |f_0|^{2p/3}\phi^{2p} \\
&\quad +C\int(|f_1|^{2p}+|f_2|^{2p}+|f_3|^{2p})\phi^{2p}+C_4\int k^2|Tz|^{2p/3}\phi^{2p}.
\end{aligned} \tag{47}$$

Integrating by parts the second term on the right-hand side by means of Remark 4.1, we get

$$\begin{aligned}
-\int & |s_1|^{p-3}s_1k(1+a^2+b^2)^{3/2}TYz\phi^{2p} \\
&=\int |s_1|^{p-3}s_1k(1+a^2+b^2)^{3/2}\omega T_\varepsilon vTz\phi^{2p} \\
&\quad +\frac{2(p-2)}{p-1}\int Y(|s_1|^{(p-1)/2})|s_1|^{(p-3)/2}k(1+a^2+b^2)^{3/2}Tz\phi^{2p} \\
&\quad +\int |s_1|^{p-3}s_1Y(k(1+a^2+b^2)^{3/2})Tz\phi^{2p} \\
&\quad +2p\int |s_1|^{p-3}s_1k(1+a^2+b^2)^{3/2}Tz\phi^{2p-1}Y\phi \\
&\leqslant \delta\int |s_1|^p+C\int |T_\varepsilon v|^{2p}\phi^{2p}+\frac{C}{\delta}\int |Tz|^{2p/3}\phi^{2p}+\delta\int |Y(|s_1|^{(p-1)/2})|^2\phi^{2p} \\
&\quad +\int k^2(|\nabla_\varepsilon a|+|\nabla_\varepsilon b|)^{p/2}|Tz|^{p/2}\phi^{2p}+C\int |\nabla_\varepsilon\phi|^{2p}.
\end{aligned} \tag{48}$$

By Proposition 4.2, we have

$$
\begin{aligned}
&\int |Xz|^p\phi^{2p}+\int |\nabla_\varepsilon(|Xz|^{(p-1)/2})|^2\phi^{2p}\\
&\quad\leqslant C\int |z|^{2p}\phi^{2p}+C\int |X(|Xz|^{(p-1)/2})|^2\phi^{2p}+C\int |Xz|^{p-1}(|\nabla_\varepsilon\phi|^2+\phi^2)\phi^{2p-2}\\
&\quad\leqslant \text{[using (47), (48) and the fact that } s_1=Xz]\\
&\quad\leqslant C\int |z|^{2p}\phi^{2p}+C\int (\phi^2+|\nabla_\varepsilon\phi|^2)^p+\delta\int |\nabla_\varepsilon z|^p\phi^{2p}+C\int |\nabla_\varepsilon v|^{2p}\phi^{2p}\\
&\qquad+C\int |f_0|^{2p/3}\phi^{2p}+C\int (|f_1|^{2p}+|f_2|^{2p}+|f_3|^{2p})\phi^{2p}\\
&\qquad+C\int k^2|Tz|^{2p/3}\phi^{2p}+C\int k^2(|\nabla_\varepsilon a|+|\nabla_\varepsilon b|)^{p/2}|Tz|^{p/2}\phi^{2p}.
\end{aligned}\tag{49}
$$

Analogous relations hold for Yz and $T_\varepsilon z$, and hence

$$
\begin{aligned}
&\int |\nabla_\varepsilon z|^p\phi^{2p}+\int |\nabla_\varepsilon(|\nabla_\varepsilon z|^{(p-1)/2})|^2\phi^{2p}\\
&\quad\leqslant C\int |z|^{2p}\phi^{2p}+C\int (\phi^2+|\nabla_\varepsilon\phi|^2)^p\\
&\quad\leqslant C\int |\nabla_\varepsilon v|^{2p}\phi^{2p}+\delta\int |\nabla_\varepsilon z|^p\phi^{2p}\\
&\qquad+C\int (|f_0|^{2p/3}+|f_1|^{2p}+|f_2|^{2p}+|f_3|^{2p})\phi^{2p}\\
&\qquad+\widetilde{C}\left(\int |Tz|^{2p/3}\phi^{2p}+\int (|\nabla_\varepsilon a|+|\nabla_\varepsilon b|)^{p/2}|Tz|^{p/2}\phi^{2p}\right).
\end{aligned}
$$

Choosing δ sufficiently small, we get the stated assertion.

THEOREM 4.4. *Let $p\geqslant 3$ be fixed, let $f_0,\dots f_3\in C^\infty(\Omega)$, let z be a solution of equation* (43), *and let $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$. Then there exist constants C and $\widetilde{C}$ which depend on p, on Ω_i and on the constant M in* (39), *but are independent of ε or z, such that*

$$
\begin{aligned}
&\|z\|^p_{W^{2,p}_\varepsilon(\Omega_1)}+\sum_{|i|=2}\|\,|D_iz|^{(p-1)/2}\|^2_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\quad\leqslant C\Big(\|f_0\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)}+\sum_{i=1}^{3}(\|f_i\|^p_{W^{1,p}_\varepsilon(\Omega_2)}+\|f_i\|^{2p}_{L^{2p}(\Omega_2)})+\|v\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)}+\|z\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)}\Big)\\
&\qquad+\widetilde{C}\big(\|Tz\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)}+\|Tz\|^p_{L^p(\Omega_2)}+\|a\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)}+\|b\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)}\big).
\end{aligned}
$$

If $k=0$, we can choose $\widetilde{C}=0$.

Proof. If z is a solution of (43), then by Lemma 2.3 the function $s_3 = T_\varepsilon z$ is a solution of the same equation with coefficients

$$\begin{aligned} \tilde{f}_0 &= T_\varepsilon f_0 + T_\varepsilon f_1 Xz + T_\varepsilon f_2 Yz + T_\varepsilon f_3 T_\varepsilon z + k(1+a^2+b^2)^{3/2} T_e v Tz \\ &\quad -((Xv)^2+(Yv)^2+(T_\varepsilon v)^2) T_\varepsilon z + f_1 Yv T_\varepsilon z - f_2 Xv T_\varepsilon z \\ \tilde{f}_1 &= f_1 - 2Yv, \quad \tilde{f}_2 = f_2 + 2Xv, \quad \tilde{f}_3 = f_3. \end{aligned} \tag{50}$$

Let us choose Ω_3 such that $\Omega_1 \subset\subset \Omega_3 \subset\subset \Omega_2$. By Theorem 4.3 there exist constants C and $\widetilde{C}$ independent of ε such that

$$\begin{aligned} &\|s_3\|^p_{W^{1,p}_\varepsilon(\Omega_3)} + \sum_{i=1}^3 \| |D_i s_3|^{(p-1)/2} \|^2_{W^{1,2}_\varepsilon(\Omega_3)} \\ &\quad \leqslant C\Big(\|\tilde{f}_0\|^{2p/3}_{L^{2p/3}(\Omega_2)} + \sum_{i=1}^3 \|\tilde{f}_i\|^{2p}_{L^{2p}(\Omega_2)} + \|v\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|s_3\|^{2p}_{L^{2p}(\Omega_2)} \Big) \\ &\quad\quad + \widetilde{C}\big(\|Ts_3\|^{2p/3}_{L^{2p/3}(\Omega_2)} + \|a\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|b\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} \big) \\ &\quad \leqslant C\Big(\|f_0\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)} + \sum_{i=1}^3 \big(\|f_i\|^p_{W^{1,p}_\varepsilon(\Omega_2)} + \|f_i\|^{2p}_{L^{2p}(\Omega_2)} \big) + \|v\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|z\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} \Big) \\ &\quad\quad + \widetilde{C}\big(\|Tz\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)} + \|Tz\|^p_{L^p(\Omega_2)} + \|a\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|b\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} \big). \end{aligned} \tag{51}$$

Analogously, by Lemma 2.1, the function $s_1 = Xz$ is a solution of

$$L_\varepsilon s_1 = \tilde{f}_0 + f_1 X s_1 + f_2 Y s_1 + f_3 T_\varepsilon s_1 + Y(k(1+a^2+b^2)^{3/2} Tz), \tag{52}$$

where

$$\begin{aligned} \tilde{f}_0 &= k(1+a^2+b^2)^{3/2} TYz + Xf_0 + Xf_1 Xz + Xf_2 Yz + Xf_3 \\ &\quad + f_2 T_\varepsilon v T_\varepsilon z + f_2 k(1+a^2+b^2)^{3/2} Tz + f_3 Yv T_\varepsilon z + 2Yv T_\varepsilon^2 z - 2T_\varepsilon v Y T_\varepsilon z \end{aligned}$$

and, by Theorem 4.3,

$$\begin{aligned} &\|s_1\|^p_{W^{1,p}_\varepsilon(\Omega_1)} + \| |D_i s_1|^{(p-1)/2} \|^2_{W^{1,2}_\varepsilon(\Omega_1)} \\ &\quad \leqslant C\Big(\|f_0\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_3)} + \sum_{i=1}^3 \big(\|f_i\|^p_{W^{1,p}_\varepsilon(\Omega_3)} + \|f_i\|^{2p}_{L^{2p}(\Omega_3)} \big) + \|v\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_3)} + \|z\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_3)} \Big) \\ &\quad\quad + C\|s_3\|^p_{W^{1,p}_\varepsilon(\Omega_3)} + \widetilde{C}\big(\|Ts_1\|^{2p/3}_{L^{2p/3}(\Omega_3)} + \|a\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_3)} + \|b\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_3)} \big) \\ &\quad \overset{(51)}{\leqslant} C\Big(\|f_0\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)} + \sum_{i=1}^3 \big(\|f_i\|^p_{W^{1,p}_\varepsilon(\Omega_2)} + \|f_i\|^{2p}_{L^{2p}(\Omega_2)} \big) + \|v\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|z\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} \Big) \\ &\quad\quad + \widetilde{C}\big(\|Tz\|^{2p/3}_{W^{1,2p/3}_\varepsilon(\Omega_2)} + \|Tz\|^p_{L^p(\Omega_2)} + \|a\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} + \|b\|^{2p}_{W^{1,2p}_\varepsilon(\Omega_2)} \big). \end{aligned}$$

Finally, arguing exactly in the same way with Yz, we deduce the thesis.

Proof of Theorem 4.1. We will prove by induction that, if z is a solution of equation (43), then

$$\begin{aligned}
&\|z\|^p_{W^{m+1,p}_\varepsilon(\Omega_1)}+\sum_{|i|=m+1}\||D_iz|^{(p-1)/2}\|^2_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\quad\leqslant C\Big(\|f_0\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\sum_{i=1}^3\big(\|f_i\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|f_i\|^p_{W^{m-1,p}(\Omega_2)}\big)\\
&\quad\quad+\|v\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|z\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\Big)\\
&\quad\quad+\widetilde{C}\big(\|Tz\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|Tz\|^p_{W^{m-1,p}_\varepsilon(\Omega_2)}+\|a\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\\
&\quad\quad+\|b\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|(1+a^2+b^2)^{3/2}\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\big),
\end{aligned}$$

for suitable constants C and $\widetilde{C}$ depending only on Ω_i and M and such that $\widetilde{C}=0$ if $k=0$. By Theorem 4.4 the assertion is true for $m=1$. Let us assume that it is true for $m-1$. Since z is a solution of (15) then $T_\varepsilon z$ is a solution of (44), with coefficients described in (50), and there exists a constant independent of ε such that

$$\begin{aligned}
&\|T_\varepsilon z\|^p_{W^{m,p}_\varepsilon(\Omega_1)}+\sum_{|i|=m}\||D^iT_\varepsilon z|^{(p-1)/2}\|_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\leqslant C\Big(\|\nabla_\varepsilon f_0\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\sum_{i=1}^3\|\nabla_\varepsilon f_i\nabla_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}\\
&\quad+\|k(1+a^2+b^2)^{3/2}T_\varepsilon vT_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\|(\nabla_\varepsilon v)^2\nabla_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}\\
&\quad+\sum_{i=1}^3\big(\|f_i\nabla_\varepsilon v\nabla_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\|f_i\|^p_{W^{m-1,p}_\varepsilon(\Omega_2)}+\|f_i\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\big)\\
&\quad+\|v\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|z\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|v\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}+\|z\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\Big)\\
&\quad+\widetilde{C}\big(\|TT_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\|TT_\varepsilon z\|^p_{W^{m-2,p}_\varepsilon(\Omega_2)}+\|a\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\\
&\quad+\|b\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}+\|(1+a^2+b^2)^{3/2}\|^{2p}_{W^{m-2,2p}_\varepsilon(\Omega_2)}\big)\\
&\leqslant C\Big(\|f_0\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\sum_{i=1}^3\big(\|f_i\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|f_i\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\big)\\
&\quad+\|v\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|z\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\Big)\\
&\quad+\widetilde{C}\big(\|Tz\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|Tz\|^p_{W^{m-1,p}_\varepsilon(\Omega_2)}+\|a\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\\
&\quad+\|b\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}+\|(1+a^2+b^2)^{3/2}\|^{2p}_{W^{m-2,2p}_\varepsilon(\Omega_2)}\big).
\end{aligned}\tag{53}$$

Analogous relations hold for X and Y, and the thesis follows.

Remark 4.2. Let $p\geqslant 3$ and let m be a fixed positive integer. Assume that $f\in C^\infty(\Omega)$, let z be a solution of equation (15) in Ω, and let $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$. Note that

$$\begin{aligned}
&\|(\nabla_\varepsilon v)^2\nabla_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}\\
&\quad\leqslant \|\nabla_\varepsilon(\nabla_\varepsilon v)\nabla_\varepsilon v\nabla_\varepsilon z\|^{2p/3}_{W^{m-2,2p/3}_\varepsilon(\Omega_2)}+\|(\nabla_\varepsilon v)^2(\nabla_\varepsilon)^2 z\|^{2p/3}_{W^{m-2,2p/3}_\varepsilon(\Omega_2)}\\
&\quad\leqslant \|v\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|z\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|v\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}+\|z\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}.
\end{aligned}$$

Then, by Theorem 4.1 and (53),

$$\begin{aligned}
&\|T_\varepsilon z\|^p_{W^{m,p}_\varepsilon(\Omega_1)}+\sum_{|i|=m}\||D^iT_\varepsilon z|^{(p-1)/2}\|_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\quad\leqslant C\big(\|f\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\|v\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|z\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|v\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}+\|z\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}\big)\\
&\qquad+\widetilde{C}\big(\|TT_\varepsilon z\|^{2p/3}_{W^{m-1,2p/3}_\varepsilon(\Omega_2)}+\|TT_\varepsilon z\|^p_{W^{m-2,p}_\varepsilon(\Omega_2)}+\|a\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}+\|b\|^{2p}_{W^{m-1,2p}_\varepsilon(\Omega_2)}\big).
\end{aligned}$$

Analogous relations hold for X and Y, and we get

$$\begin{aligned}
&\|z\|^p_{W^{m+1,p}_\varepsilon(\Omega_1)}+\sum_{|i|=m+1}\||D_iz|^{(p-1)/2}\|^2_{W^{1,2}_\varepsilon(\Omega_1)}\\
&\quad\leqslant C\big(\|f\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|v\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}+\|v\|^p_{W^{m,p}_\varepsilon(\Omega_2)}+\|z\|^{4p}_{W^{m-1,4p}_\varepsilon(\Omega_2)}+\|z\|^p_{W^{m,p}_\varepsilon(\Omega_2)}\big)\\
&\qquad+\widetilde{C}\big(\|Tz\|^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)}+\|Tz\|^p_{W^{m-1,p}_\varepsilon(\Omega_2)}+\|a\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}+\|b\|^{2p}_{W^{m,2p}_\varepsilon(\Omega_2)}\big).
\end{aligned}$$

5. Regularity of solutions of the nonlinear equation

In this section we conclude the proof of Theorem 1.1. In order to do so, we first prove an a priori estimate for the solutions of the nonlinear regularized equation (14) in the space $W^{m,p}_\varepsilon$, independent of ε. By the Sobolev Embedding Theorem 3.2 this leads to an estimate in the space $C^\alpha_{d,\varepsilon}$. Letting ε go to 0 we deduce that the function u has all the weak Euclidean derivatives of order 2 in C^α_d. Then, by the results in [5], we conclude the proof of Theorem 1.1.

5.1. *$W^{m,p}_\varepsilon$-regularity of solutions of the regularized equation.* Let u be a solution of equation (14) satisfying conditions (39). In order to prove an a priori estimate in the spaces $W^{m,p}_\varepsilon$, for the function u, we will make use of the a priori estimates established in §3, together with a new interpolation inequality, based on the hypothesis on k:

PROPOSITION 5.1. *Let $k\neq 0$ in $\Omega\times\mathbf{R}$. If $Tz\in W^{1,2}_{\varepsilon,\mathrm{loc}}(\Omega)$ then for all $\phi\in C_0^\infty$,*

$$\int |Tz|^3\phi^6 \leqslant C\int |\nabla_\varepsilon Tz|^2|\phi|^6+\int((\nabla_\varepsilon v)^2+|\nabla_\varepsilon z|^2)(Tz)^2\phi^6+\int|\nabla_\varepsilon\phi|^6. \tag{54}$$

Proof. Let $s_4=Tz$. Then

$$\begin{aligned}
&\int |s_4|^3\phi^6\\
&\leqslant \sup\frac{1}{|k|}\int |k|(1+a^2+b^2)^{3/2}|s_4|^3\phi^6\\
&=\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\int k(1+a^2+b^2)^{3/2}Tz\operatorname{sign}(s_4)|s_4|^2\phi^6\\
&\overset{(21)}{=}\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\int(-[X,Y]z+T_\varepsilon vT_\varepsilon z)\operatorname{sign}(s_4)|s_4|^2\phi^6\\
&=\text{[integrating by parts by using (20) we get]}\\
&=\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\left(\int YzX(\operatorname{sign}(s_4)|s_4|^2\phi^6)-\int XzY(\operatorname{sign}(s_4)|s_4|^2\phi^6)\right)\\
&\quad+\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\int(Yz(Yv-\omega Xv)+Xz(Xv+\omega Yv)+T_\varepsilon zT_\varepsilon v)\operatorname{sign}(s_4)|s_4|^2\phi^6\\
&=\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\left(2\int YzXs_4|s_4|\phi^6+6\int Yzs_4^2\phi^5X\phi\right.\\
&\quad\left.-2\int XzYs_4|s_4|\phi^6-6\int Xzs_4^2\phi^5Y\phi\right)\\
&\quad+\operatorname{sign}(k)\sup\left(\frac{1}{|k|}\right)\int(Yz(Yv-\omega Xv)+Xz(Xv+\omega Yv)+T_\varepsilon zT_\varepsilon v)\operatorname{sign}(s_4)|s_4|^2\phi^6,
\end{aligned}$$

and the thesis follows.

Remark 5.1. Differentiating equation (19) we deduce that

$$\begin{aligned}
T^2a=T\left(\frac{Yv-\omega Xv}{(1+\omega^2)^{1/2}}\right)&=\frac{TYv-\omega TXv-T\omega Xv-(Yv-\omega Xv)\omega Tv}{(1+\omega^2)^{1/2}}\\
&=\frac{TYv-\omega TXv-TvXv-\omega YvTv}{(1+\omega^2)^{1/2}}
\end{aligned}\tag{55}$$

and

$$T^2b=-\frac{TXv+\omega TYv+TvYv-\omega XvTv}{(1+\omega^2)^{1/2}}. \tag{56}$$

Applying the previous result, we verify that a, b and v satisfy all the assumption necessary to apply our Sobolev embedding. For technical reasons we start with the derivatives of the function v:

LEMMA 5.1. *If $\Omega_1 \subset\subset \Omega$ and u is a solution of equation* (14) *in Ω, with $k \neq 0$, there exists a positive constant C depending only on the constant M introduced in* (39) *and Ω_1 such that*

$$\|Tv\|_{L^3(\Omega_1)} + \|\nabla_\varepsilon Tv\|_{L^2(\Omega_1)} + \|T\nabla_\varepsilon v\|_{L^2(\Omega_1)} + \|\nabla_\varepsilon v Tv\|_{L^2(\Omega_1)} \leqslant C, \tag{57}$$

where $v = \arctan u_t$ is the function defined in (18).

Proof. By Proposition 2.1 the function v is a solution of equation (29). Then by Theorem 4.2 we have that for every $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}
&\int (|\nabla_\varepsilon Tv|^2 + |T\nabla_\varepsilon v|^2)\phi^6 + \int |\nabla_\varepsilon v|^2 (Tv)^2 \phi^6 \\
&\leqslant C \int (\phi^2(|k| + |Tk|) + |\nabla_\varepsilon \phi|^2)|Tv|^2 \phi^4 + C \int T^2(k(1+a^2+b^2))Tv\phi^6 \\
&\leqslant C \int (\phi^2 + |\nabla_\varepsilon \phi|^2)|Tv|^2\phi^4 + C \int (1 + |T^2 a| + |T^2 b| + |Ta|^2 + |Tb|^2)|Tv|\phi^6 \\
&\leqslant \text{[by Remark 5.1 and (19)]} \\
&\leqslant C \int (\phi^2 + |\nabla_\varepsilon \phi|^2)|Tv|^2\phi^4 + C \int (1 + |T\nabla_\varepsilon v| + |\nabla_\varepsilon v|\,|Tv| + |\nabla_\varepsilon v|^2)|Tv|\phi^6 \\
&\leqslant C \int (\phi^2 + |\nabla_\varepsilon \phi|^2)|Tv|^2\phi^4 + C \int |Tv|\phi^6 + \frac{C}{\delta} \int (|Tv|^2 + |\nabla_\varepsilon v|^2)\phi^6 \\
&\qquad + \delta \int (|T\nabla_\varepsilon v|^2 + |\nabla_\varepsilon v|^2 |Tv|^2)\phi^6,
\end{aligned}$$

for $\delta \in]0, 1[$ to be fixed later. Choosing δ sufficiently small we have

$$\begin{aligned}
&\int (|\nabla_\varepsilon Tv|^2 + |T\nabla_\varepsilon v|^2)\phi^6 + C \int |\nabla_\varepsilon v|^2 (Tv)^2 \phi^6 \\
&\leqslant C \int (\phi^2 + |\nabla_\varepsilon \phi|^2)|Tv|^2\phi^4 + C \int |Tv|\phi^6 + C \int |\nabla_\varepsilon v|^2 \phi^6 \\
&\leqslant \text{[for a value of } \delta \text{ which can be different from the preceding one]} \\
&\leqslant \frac{C}{\delta} \int (\phi^6 + |\nabla_\varepsilon \phi|^6) + \delta \int |Tv|^3 \phi^6 + C \int |\nabla_\varepsilon v|^2 \phi^2 \\
&\leqslant \text{[by Proposition 5.1]} \\
&\leqslant \frac{C}{\delta} \int (\phi^2 + |\nabla_\varepsilon \phi|^2)^3 + \delta \int (|\nabla_\varepsilon Tz|^2 + (\nabla_\varepsilon v)^2 (Tz)^2)\phi^6 + C \int |\nabla_\varepsilon v|^2 \phi^2.
\end{aligned}$$

Choosing δ sufficiently small, and $\phi \equiv 1$ in Ω_1, we get the thesis.

Remark 5.2. We explicitly note that, if $\Omega_1 \subset\subset \Omega$, then from the previous lemma, and (55) and (56), it follows that

$$\|T^2 a\|_{L^2(\Omega_1)} + \|T^2 b\|_{L^2(\Omega_1)} \leqslant C, \tag{58}$$

for a constant C only dependent on M and Ω_1.

Let us estimate the derivatives of the functions a and b:

LEMMA 5.2. *For every $\Omega_1 \subset\subset \Omega$, there exists a positive constant C depending only on the constant M in* (39) *and Ω_1 such that*

$$\begin{aligned} &\|\nabla_\varepsilon Ta\|_{L^2(\Omega_1)} + \|T\nabla_\varepsilon a\|_{L^2(\Omega_1)} + \|\nabla_\varepsilon v Ta\|_{L^2(\Omega_1)} \\ &\quad + \|\nabla_\varepsilon Tb\|_{L^2(\Omega_1)} + \|T\nabla_\varepsilon b\|_{L^2(\Omega_1)} + \|\nabla_\varepsilon v Tb\|_{L^2(\Omega_1)} \leqslant C. \end{aligned} \tag{59}$$

Proof. By Proposition 2.1 the function a is a solution of equation (27). If we denote the right-hand side f_a, we have

$$|f_a| \leqslant |\nabla_\varepsilon a| + |\nabla_\varepsilon b| + |\nabla_\varepsilon v|. \tag{60}$$

Choosing $\phi \in C_0^\infty(\Omega_1)$, $\phi \equiv 1$ in Ω, by Theorem 4.2, we get

$$\begin{aligned} &\int (|\nabla_\varepsilon Ta|^2 + |T\nabla_\varepsilon a|^2)\phi^6 + \int |\nabla_\varepsilon v|^2 (Ta)^2 \phi^6 \\ &\leqslant C\int (\phi^2(|k| + |Tk|) + |\nabla_\varepsilon \phi|^2)|Ta|^2\phi^4 + C\int Tf_a Ta\phi^6 \\ &\leqslant [\text{since } Ta \text{ is bounded in } L^2(\Omega_1) \text{ by the constant } M, \text{ and } \phi \text{ is fixed}] \\ &\leqslant C\left(1 + \int Tf_a Ta \phi^6\right) \\ &= [\text{integrating by parts with respect to } T, \text{ and using (20) of the adjoint}] \\ &= C\left(1 - \int f_a T^2 a\phi^6 - 6\int f_a Ta\phi^5 T\phi + \int f_a Ta\omega Tv\phi^6\right) \\ &\leqslant [\text{since } |Ta| \leqslant |\nabla_\varepsilon a|] \\ &\leqslant C\left(1 + \int (|f_a|^2 + |T^2 a|^2 + |Ta|^2 + |Ta|^2|Tv|^2)\phi^4\right) \leqslant C, \end{aligned}$$

by (60), Lemma 5.1 and Remark 5.2. This inequality provides an estimate for the derivatives of a, and arguing in the same way with the function b, we conclude the proof of the lemma.

Remark 5.3. We explicitly note that, if $\Omega_1 \subset\subset \Omega$, then

$$\|\nabla_\varepsilon v\|_{L^6(\Omega_1)} + \|Ta\|_{L^6(\Omega_1)} + \|Tb\|_{L^6(\Omega_1)} \leqslant C, \tag{61}$$

for a constant C only dependent on M and Ω_1.

Indeed, by (19) we have that

$$Xv = -\frac{Tb+\omega Ta}{(1+\omega^2)^{1/2}}, \quad Yv = \frac{Ta-\omega Tb}{(1+\omega^2)^{1/2}}, \quad T_\varepsilon v = \varepsilon Tv,$$

and hence,

$$\begin{aligned}\|X\nabla_\varepsilon v\|_{L^2(\Omega_1)} + \|Y\nabla_\varepsilon v\|_{L^2(\Omega_1)} \leqslant{}& \|XTa\|_{L^2(\Omega_1)} + \|XTb\|_{L^2(\Omega_1)} + \|XvTa\|_{L^2(\Omega_1)} \\ &+ \|XvTb\|_{L^2(\Omega_1)} + \|XTv\|_{L^2(\Omega_1)} + \|YTa\|_{L^2(\Omega_1)} \\ &+ \|YTb\|_{L^2(\Omega_1)} + \|YvTa\|_{L^2(\Omega_1)} \\ &+ \|YvTb\|_{L^2(\Omega_1)} + \|YTv\|_{L^2(\Omega_1)} \leqslant C,\end{aligned}$$

for a constant C only dependent on M and Ω_1, by Lemma 5.1 and Lemma 5.2. In particular,

$$\|v\|_{W_\varepsilon^{2,2}(\Omega_1)} \leqslant C. \tag{62}$$

On the other hand, always by Lemma 5.1 we have

$$\|T\nabla_\varepsilon v\|_{L^2(\Omega_1)} \leqslant C.$$

Hence by the classical Sobolev embedding theorem there exists a constant C only dependent on M and Ω_1 such that

$$\|\nabla_\varepsilon v\|_{L^6(\Omega_1)} \leqslant C.$$

By (19) we also have

$$\|Ta\|_{L^6(\Omega_1)} + \|Tb\|_{L^6(\Omega_1)} \leqslant C.$$

LEMMA 5.3. *For every $\Omega_1 \subset\subset \Omega$ there exists a positive constant C depending only on M and the choice of Ω_1 such that*

$$\|a\|_{W_\varepsilon^{3,2}(\Omega_1)} + \|b\|_{W_\varepsilon^{3,2}(\Omega_1)} + \|v\|_{W_\varepsilon^{2,3}(\Omega_1)} + \|v\|_{W_\varepsilon^{3,2}(\Omega_1)} \leqslant C.$$

Proof. Applying Theorem 4.3 to the function $z=a$, we get

$$\begin{aligned}\int |\nabla_\varepsilon a|^3\phi^6 + \int |\nabla_\varepsilon(\nabla_\varepsilon a)|^2\phi^6 \leqslant C\Big(1 + \int a^6\phi^6 + \int |\nabla_\varepsilon v|^6\phi^6 + \int |f_a|^2\phi^6 \\ + \int |Ta|^2\phi^6 + \int |\nabla_\varepsilon a|^2\phi^6 + \int |\nabla_\varepsilon b|^2\phi^6\Big).\end{aligned}$$

Choosing $\phi\equiv 1$ in Ω_1, and using the estimate (60) for f_a, we deduce that

$$\|a\|_{W^{2,2}_\varepsilon(\Omega_1)} \leqslant C.$$

Also, using Lemma 5.2 we have

$$\|T\nabla_\varepsilon a\|_{L^2(\Omega_1)} + \|X\nabla_\varepsilon a\|_{L^2(\Omega_1)} + \|Y\nabla_\varepsilon a\|_{L^2(\Omega_1)} \leqslant C.$$

Using the classical Sobolev embedding theorem, we get

$$\|a\|_{W^{1,6}_\varepsilon(\Omega_1)} \leqslant C,$$

and in the same way

$$\|b\|_{W^{1,6}_\varepsilon(\Omega_1)} \leqslant C. \tag{63}$$

Applying Theorem 4.1 we deduce

$$\begin{aligned}\|a\|_{W^{2,3}_\varepsilon(\Omega_1)} + \|a\|_{W^{3,2}_\varepsilon(\Omega_1)} &\leqslant \|f_a\|_{W^{1,2}_\varepsilon(\Omega_1)} + \|v\|_{W^{1,6}_\varepsilon(\Omega_1)} + \|Ta\|_{W^{1,2}_\varepsilon(\Omega_1)} \\ &\quad + \|Ta\|_{L^3(\Omega_1)} + \|a\|_{W^{1,6}_\varepsilon(\Omega_1)} + \|b\|_{W^{1,6}_\varepsilon(\Omega_1)} \leqslant C,\end{aligned}$$

by Remark 5.3 and Lemma 5.1. Arguing in the same way with the functions b and v, the assertion is proved.

Finally let us verify that the last condition on the coefficients a and b required by the Sobolev embedding theorem is satisfied:

Remark 5.4. For every $\Omega_1 \subset\subset \Omega$ there exists a positive constant C depending only on M and the choice of Ω_1 such that

$$\|Ta\|_{W^{1,3}_\varepsilon(\Omega_1)} + \|Ta\|_{W^{2,2}_\varepsilon(\Omega_1)} + \|Tb\|_{W^{1,3}_\varepsilon(\Omega_1)} + \|Tb\|_{W^{2,2}_\varepsilon(\Omega_1)} \leqslant C$$

and

$$\|a\|_{W^{2,6}_{\varepsilon,\mathrm{loc}}(\Omega)} + \|b\|_{W^{2,6}_{\varepsilon,\mathrm{loc}}(\Omega)} \leqslant C.$$

Proof. By (19) and Lemma 5.3,

$$\begin{aligned}\|Ta\|_{W^{1,3}_\varepsilon(\Omega_1)} + \|Ta\|_{W^{2,2}_\varepsilon(\Omega_1)} + \|Tb\|_{W^{1,3}_\varepsilon(\Omega_1)} + \|Tb\|_{W^{2,2}_\varepsilon(\Omega_1)} \\ \leqslant \|\nabla_\varepsilon v\|_{W^{1,3}_\varepsilon(\Omega_1)} + \|\nabla_\varepsilon v\|_{W^{2,2}_\varepsilon(\Omega_1)} \leqslant C.\end{aligned}$$

In particular, we get

$$\begin{aligned}\|T(\nabla_\varepsilon)^2 a\|_{L^2(\Omega_1)} &= \|[T,\nabla_\varepsilon]\nabla_\varepsilon a\|_{L^2(\Omega_1)} + \|\nabla_\varepsilon [T,\nabla_\varepsilon]a\|_{L^2(\Omega_1)} + \|(\nabla_\varepsilon)^2 Ta\|_{L^2(\Omega_1)} \\ &\overset{(21)}{\leqslant} \|\nabla_\varepsilon v T\nabla_\varepsilon a\|_{L^2(\Omega_1)} + \|\nabla_\varepsilon(\nabla_\varepsilon v Ta)\|_{L^2(\Omega_1)} + \|(\nabla_\varepsilon)^2 Ta\|_{L^2(\Omega_1)}.\end{aligned}$$

Writing the first term as $(\nabla_\varepsilon v)^2 Ta + \nabla_\varepsilon v \nabla_\varepsilon Ta$ by means of (21) and using a Hölder inequality we arrive at

$$\|T(\nabla_\varepsilon)^2 a\|_{L^2(\Omega_1)} \leqslant \|\nabla_\varepsilon v\|_{W_\varepsilon^{1,6}(\Omega_1)} + \|Ta\|_{L^6(\Omega_1)} + \|Ta\|_{W_\varepsilon^{1,3}(\Omega_1)} + \|v\|_{W_\varepsilon^{2,3}(\Omega_1)} + \|Ta\|_{W_\varepsilon^{2,2}(\Omega_1)} \leqslant M.$$

This last inequality, together with Lemma 5.3 and the classical Sobolev embedding theorem, ensures that

$$\|a\|_{W_\varepsilon^{2,6}(\Omega_1)} + \|b\|_{W_\varepsilon^{2,6}(\Omega_1)} \leqslant C.$$

Note that, by Remarks 5.2, 5.3 and 5.4, we can apply the Sobolev embedding theorem stated in Corollary 3.1, and we deduce

THEOREM 5.1. *For every $\Omega_1 \subset\subset \Omega$ there exists a positive constant C depending only on M and the choice of Ω_1 such that*

$$\|u\|_{W_\varepsilon^{5,2}(\Omega_1)} \leqslant C.$$

Proof. Applying Theorem 4.1 to the function a we get

$$\begin{aligned}
&\|a\|_{W_\varepsilon^{3,3}(\Omega_1)} + \|a\|_{W_\varepsilon^{4,2}(\Omega_1)} \\
&\quad \leqslant \|f_a\|_{W_\varepsilon^{2,2}(\Omega_1)} + \|v\|_{W_\varepsilon^{2,3}(\Omega_1)} + \|v\|_{W_\varepsilon^{1,12}(\Omega_1)} + \|a\|_{W_\varepsilon^{2,3}(\Omega_1)} + \|a\|_{W_\varepsilon^{1,12}(\Omega_1)} \\
&\qquad + \|Ta\|_{W_\varepsilon^{2,2}(\Omega_1)} + \|Ta\|_{W_\varepsilon^{1,3}(\Omega_1)} + \|a\|_{W_\varepsilon^{2,6}(\Omega_1)} + \|b\|_{W_\varepsilon^{2,6}(\Omega_1)} \\
&\quad \leqslant [\text{by Corollary 3.1}] \\
&\quad \leqslant \|f_a\|_{W_\varepsilon^{2,2}(\Omega_1)} + \|v\|_{W_\varepsilon^{3,2}(\Omega_1)} + \|a\|_{W_\varepsilon^{3,2}(\Omega_1)} + \|Ta\|_{W_\varepsilon^{2,2}(\Omega_1)} \\
&\qquad + \|b\|_{W_\varepsilon^{3,2}(\Omega_1)} + \|Tb\|_{W_\varepsilon^{2,2}(\Omega_1)} + \|a\|_{W_\varepsilon^{2,6}(\Omega_1)} + \|b\|_{W_\varepsilon^{2,6}(\Omega_1)}.
\end{aligned}$$

Analogously, arguing in the same way with b and εv, we get

$$\|\nabla_\varepsilon u\|_{W_\varepsilon^{4,2}(\Omega_1)} \leqslant C,$$

which is equivalent to the thesis.

5.2. *$C^{2,\alpha}$-regularity of viscosity solutions.* Let u be a strong viscosity solution, and (u_j) its approximating sequence, as defined in Definition 1.1. For each function u_j we will denote $a_j = a_{u_j}$ and $b_j = b_{u_j}$, the coefficients introduced in (3); X_j and Y_j the corresponding vector fields, defined in (8); ∇_{ε_j} and $W_{\varepsilon_j}^{k,p}(\Omega)$ the related gradient and Sobolev spaces, introduced in Definition 2.2. Besides, a, b, X, Y, ∇_0 will be the coefficients and vector fields associated to the limit function u, while $W_0^{k,p}(\Omega)$ will be the associated Sobolev space. By definition of viscosity solutions we have

$$\|u_j\|_{L^\infty(\Omega)}+\|\nabla_{\varepsilon_j}u_j\|_{L^\infty(\Omega)}+\|\partial_t u_j\|_{L^\infty(\Omega)}\leqslant \widetilde{M}.$$

By this assumption and the results in [9], setting as in (18) $v_j=\arctan(\partial_t u_j)$, we have

$$\|\nabla_{\varepsilon_j}a_j\|_{L^2(\Omega_1)}+\|\nabla_{\varepsilon_j}b_j\|_{L^2(\Omega_1)}+\|\nabla_{\varepsilon_j}v_j\|_{L^2(\Omega_1)}\leqslant \widetilde{M},$$

for a constant $\widetilde{M}$ independent of j. Besides,

$$a_j\to a,\ b_j\to b \quad \text{as } j\to+\infty \text{ in } L^2_{\mathrm{loc}}(\Omega).$$

THEOREM 5.2. *If $u\in\mathrm{Lip}(\Omega)$ is a strong viscosity solution of* (6), *then for every $\alpha\in]0,1[$, a,b belong to the space $C^\alpha_{\mathrm{eucl,loc}}(\Omega)$ of Hölder-continuous functions in Euclidean sense. Besides, a and b admit Taylor developments of first order with respect to the intrinsic distance d: if $z=a$, or $z=b$, and $\alpha\in]0,1[$, then for every $\xi\in\Omega$,*

$$z(\xi)=z(\xi_0)+Xz(\xi_0)(x-x_0)+Yz(\xi_0)(y-y_0)+O(d^{1+\alpha}(\xi,\xi_0)).$$

Proof. Let $\Omega_1\subset\subset\Omega_2\subset\subset\Omega_3\subset\subset\Omega$, let $\alpha\in]0,1[$, and let $p<N/(1-\alpha)$. Since $a_j=Y_ju_j$ and $b_j=-X_ju_j$, by Theorem 5.1,

$$\|a_j\|_{W^{4,2}_{\varepsilon_j}(\Omega_3)}+\|b_j\|_{W^{4,2}_{\varepsilon_j}(\Omega_3)}+\|v_j\|_{W^{3,2}_{\varepsilon_j}(\Omega_3)}\leqslant C,$$

for C independent of j. By Sobolev Embedding Corollary 3.1 there exists a constant C only dependent on $\widetilde{M}$ and p such that

$$\begin{aligned}\|\partial_t a_j\|_{L^p(\Omega_1)}+\|X_ja_j\|_{L^p(\Omega_1)}+\|Y_ja_j\|_{L^p(\Omega_1)}&\leqslant C\|a_j\|_{W^{2,p}_{\varepsilon_j}(\Omega_1)}\leqslant C\|a_j\|_{W^{3,4}_{\varepsilon_j}(\Omega_1)}\\ &\leqslant C\|a_j\|_{W^{4,2}_{\varepsilon_j}(\Omega_1)}\leqslant C.\end{aligned}\tag{64}$$

Consequently, by the classical Sobolev embedding theorem, (a_j) is bounded in $C^\alpha_{\mathrm{eucl}}(\Omega_1)$, and the limit a belongs to this space. By Sobolev Embedding Corollary 3.1 it also follows that there exists a constant $C>0$ independent of j such that for every $\xi,\xi_0\in\Omega_1$

$$|X_ja_j(\xi)-X_ja_j(\xi_0)|\leqslant Cd_j^\alpha(\xi,\xi_0),$$

where d_j is defined as in (36), in terms of u_j.

By Theorem 3.2 the Taylor expansion follows.

THEOREM 5.3. *If* $u \in \mathrm{Lip}(\Omega)$ *is a strong viscosity solution of* (6), *then* $u \in H^2_{\mathrm{loc}}(\Omega)$, *and for every* $\alpha \in]0,1[$, *for every multiindex* I *of weight* 2, $D_I u \in C^{\alpha}_{d,\mathrm{loc}}$.

Proof. Applying Theorem 5.1 and Sobolev Embedding Corollary 3.1 to every element of the approximating sequence, and letting j go to ∞, it follows that for every multiindex I of weight 2, $D_I u \in C^{\alpha}_{d,\mathrm{loc}}$ for every $\alpha \in]0,1[$, while $u \in W^{5,2}_0$, from which it follows that $u \in H^2_{\mathrm{loc}}(\Omega)$.

5.3. *Proof of the main theorem.* We can now apply to the solution u of class $C^{2,\alpha}_d$ just found the regularity results stated in [5], and conclude the proof of the main result.

This approach is an iteration of the method used in §3. Since a and b have Taylor developments of order 1, it is possible to introduce the following vector fields, which approximate X and Y much better than the analogous vectors introduced in §3:

$$X_{\xi_0} = \partial_x + P^1_{\xi_0} a(\xi)\partial_t, \quad Y_{\xi_0} = \partial_y + P^1_{\xi_0} b(\xi)\partial_t,$$

where $P^1_{\xi_0} a(\xi) = a(\xi_0) + Xa(\xi_0)(x-x_0) + Ya(\xi_0)(y-y_0)$, and $P^1_{\xi_0} b(\xi)$ is defined in an analogous way. It then follows that

$$[X_{\xi_0}, Y_{\xi_0}] = -\frac{k(\xi_0)(1+u_t^2)^{1/2}}{(1+a^2+b^2)^{3/2}}(\xi_0)\partial_t. \tag{65}$$

In order to use the theorems stated in [5], we first recognize that the distance used here is equivalent to the control distance associated to the vector fields used in [5], and that a function with weak derivative of class C^{α}_d has also the Lie derivatives in C^{α}_d, which is the notion of derivative used in [5].

Remark 5.5. If condition (33) holds with $\alpha = 1$, then the pseudodistance d is equivalent to the pseudodistance

$$\tilde{d}(\xi,\xi_0) = \inf\{((\theta_1^2+\theta_2^2)^2+\theta_3^2)^{1/4} : \gamma_\theta \in E(\xi,\xi_0)\},$$

where

$$E(\xi,\xi_0) = \{\gamma_\theta : [0,1] \to \mathbf{R}^3 : \gamma_\theta(0) = \xi_0,\ \gamma_\theta(1) = \xi,\ \gamma'_\theta = \theta_1 X + \theta_2 Y + \theta_3 \partial_t,\ \theta \in \mathbf{R}^3\}.$$

Remark 5.6. Assume that $f \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in]0,1[$, and its weak derivatives $Xf, Yf \in C^{\alpha}_{\mathrm{loc}}(\Omega)$, $\partial_t f \in L^p_{\mathrm{loc}}(\Omega)$ with $p > 1/\alpha$. Let $\xi \in \Omega$, and let γ be an integral curve of X such that $\gamma(0) = \xi$. Then

$$Xf(\xi) = \frac{d}{dh}(f \circ \gamma)\Big|_{h=0}.$$

We refer to [10] for the proof of these two remarks.

Proof of Theorem 1.1. The function u is a solution of class $C_d^{2,\alpha}$ of the equation

$$X^2u+Y^2u-(Xa+Yb)\partial_t u=f(1+u_t^2)^{1/2},$$

with $f=k(1+a^2+b^2)^{3/2}\in C_d^{1,\alpha}$ for every $\alpha<1$. By (65),

$$X\partial_t u(\xi_0)=\frac{(1+u_t^2(\xi_0))^{1/2}}{k(\xi_0)(1+a^2+b^2)^{3/2}}(\xi_0)X(X_{\xi_0}Y_{\xi_0}-Y_{\xi_0}X_{\xi_0})u(\xi_0).$$

Then by Theorem 3.3 in [5], $\partial_t u\in C_d^{1,\alpha}$. Since $u\in H^2$, the derivative $\partial_t a$ belongs to L^2. By relation (19), and the regularity of $\partial_t u$, this derivative belongs to C_d^α. Then a, b and f are of class $C_d^{1,\alpha}$ and partially differentiable with respect to t, with derivatives of class C_d^α. Then by Theorem 3.2 in [5], $\partial_t u\in C_d^{2,\alpha}$. In particular, $u\in C_{\mathrm{eucl}}^{2,\alpha}$.

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Received April 26, 2000