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Derivative estimates for the Navier–Stokes equations in a three dimensional region

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1. Introduction

The mathematical theory of the Navier–Stokes equations has centered upon basic questions of the existence, uniqueness, and regularity of solutions of the initial value problem for fluid motions in all of space or in a subdomain of finite or infinite extent. Such solutions, when they can be constructed or shown to exist, represent flows of a viscous incompressible fluid. In two space dimensions the theorem of existence, uniqueness and regularity was essentially completed thirty years ago by the work of Leray [21], Lions [22] and Ladyzhenskaya [18] who showed that a smooth solution of the inital value problem exists for arbitrary square-integrable initial data.

For viscous, incompressible fluid motions in three space dimensions, to be considered in this paper, the theorem of existence uniqueness and regularity has been proved only for sufficiently small initial data or in special cases such as cylindrical symmetry that essentially reduce the problem to two space dimensions in some sense. In his 1934 paper [21] Leray considered the possibility of singular solutions in which momentum locally and temporarily overpowers the smoothing effects of viscosity. Subsequently the problem of the existence of singular solutions has been widely studied [3, 6, 7, 10, 11, 12, 15, 16, 18, 27, 28, 30, 32, 34, 35, 36, 37, 39, 40] and many results obtained, but no conclusive resolution of the question has yet been achieved.

The evidence favouring the existence of singular solutions, apart from the possible implications of repeated failure to disprove their existence, is substantial and has continued to mount. On the purely physical and observational side, the phenomena of atmospheric dust devils, tornadoes and other vortices tend to support the conjecture that singular solutions do occur and to lend a certain significance to the bettter understanding of them in both their pure and applied mathematical contexts [34]. Ladyzhenskaya [20] has given an example which falls short of being a full singular solution only in a specific limitation of its spatial domain at the singular instant. Scheffer [27, 28] and Caffarelli, Kohn and Nirenberg [3] have studied the singular point set in space time and have shown that it has one-dimensional Hausdorff measure equal to zero. Scheffer [29] has recently demonstrated a 'Navier–Stokes inequality' which indicates that the magnitudes and solenoidal vectorial properties of a possible singularity are compatible with the equations as well. Foias, Guillopé and Temam [10] have shown for flows on a three-dimensional torus T^3 that certain new estimates must hold for the space derivatives of solutions of the Navier–Stokes equations. These estimates involve fractional power integrability over the time variable of a space norm, and are a seemingly natural extension to the higher derivative level of the well known energy estimates for the Navier–Stokes solutions.

The main result of this paper is to establish such fractional estimates in the case of a domain with boundary upon which the solution components vanish—the appropriate fixed-boundary condition for viscous flows. It will be seen that the key to this extension is the consideration of time derivatives as well, so that a combined and extended set of estimates is obtained. The presence of a boundary brings in complications associated with the pressure variable, and in this paper the related scalar potential of the viscosity term is analyzed and estimated. Our estimates make possible a limited characterization of the pointwise behaviour of singular solutions, and also of the initial behaviour of solutions related to water hammer effects.

Because our estimates apply to Navier–Stokes solutions generally, with minimal regularity assumptions such as continuity off the singular set, it will follow that the behaviour of solutions generally will be restricted within a certain range of integrability and algebraic singular behaviour. Hopefully the more specific problems of characterization thus indicated will be found capable of still more precise resolution.

2. The Navier–Stokes equations

Let x_i $(i=1,2,3)$ denote Cartesian coordinates in $\mathbf{R}^3$ and let t be time. Let $u_i(x,t)$ $(i=1,2,3)$ be the vector field of velocity components of a fluid flow, and let $p(x,t)$ denote the pressure variable. The constant viscosity coefficient is denoted by ν. Then the equations of Navier–Stokes take the form [21]

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{\partial p}{\partial x_i}+\nu\Delta u_i \tag{2.1}$$

where $i=1,2,3$ and k is summed over $k=1,2,3$ by the Einstein convention for repeated indices.

The differential dx denotes the volume element $dx_1\,dx_2\,dx_3$ and the Laplacian operator in $\mathbf{R}^3$ is denoted by $\Delta\equiv\Sigma_{i=1}^3\partial^2/\partial x_i^2$. Together with the three momentum equations there holds the incompressibility equation

$$\operatorname{div} u=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=\frac{\partial u_i}{\partial x_i}=0. \tag{2.2}$$

The four equations together form a semi-linear elliptic-parabolic system.

Three initial conditions are appropriate:

$$u_i(x,0)=u_{i0}(x). \tag{2.3}$$

We consider initial values of integrable square on the spatial domain:

$$\|u_0\|_2^2=\sum_{i=1}^{3}\int_\Omega |u_{i0}(x,t)|^2dx<\infty. \tag{2.4}$$

Here and below integration will be taken over a suitable domain $\Omega \subseteq \mathbf{R}^3$. The boundary $\partial\Omega$ will when necessary be restricted as to smoothness, and three boundary conditions are also appropriate for viscous flow:

$$u_i(x,t) = 0, \quad x \in \partial\Omega. \tag{2.5}$$

The pressure $p=p(x,t)$ satisfies a Poisson type equation deduced from (2.1) by taking the divergence and applying (2.2):

$$\begin{aligned} \Delta p &= -\frac{\partial u_k}{\partial x_i}\frac{\partial u_i}{\partial x_k} \\ &= -u_{i,k}u_{k,i} \\ &= -(u_i u_k)_{,ik} \end{aligned} \tag{2.6}$$

where subscript commas denote derivatives with respect to x_i $(i=1,2,3)$. In view of (2.5) a boundary condition for p can be deduced from (2.1) by simply taking limits on approach to the boundary:

$$\frac{\partial p}{\partial x_i} = \nu \Delta u_i, \quad x \in \partial\Omega. \tag{2.7}$$

The normal component

$$\frac{\partial p}{\partial n} = p_{,i} n_i = \nu \Delta u_i n_i \tag{2.8}$$

alone provides a Neumann type boundary condition sufficient together with (2.6) to determine p up to a constant when $u_i(x,t)$ are known at any instant of time.

The Lebesgue space $L^p(\Omega)$ is the set of vector valued functions on Ω with finite norm $\|u\|_p$, where

$$\|u\|_p^p = \int_\Omega \sum_{i=1}^3 |u_i(x,t)|^p dx. \tag{2.9}$$

We shall also use the corresponding Lebesgue space of scalar functions. Throughout, these norms are all functions of time t. The inner product of two vector functions u_i, v_i is

$$(u,v) = \int_\Omega \sum_{i=1}^3 u_i v_i \, dx. \tag{2.10}$$

By Hölder's inequality, where p and q are dual indices: $1/p+1/q=1$, $p \geq 1$, $q \geq 1$, we have

$$|(u, v)| \leqslant \|u\|_p \|v\|_q. \tag{2.11}$$

Frequent use will be made of Young's inequality [14, Theorem 37]:

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \tag{2.12}$$

where $a>0$, $b>0$, $1/p+1/q=1$, and of a version with a replaced by $a\varepsilon$, b by b/ε.

If $a_j>0$, $j=1, \ldots, n$ and a_j has weight or degree w_j, then by a homogeneous extension of Young's inequality,

$$a_1 \ldots a_n \leqslant \sum_{j=1}^{n} \frac{1}{p_j} a_j^{p_j} \tag{2.13}$$

where $p_j = W/w_j$, $W = \Sigma_{j=1}^n w_j$. Every term on the right has weight $p_j w_j = W$.

We also use the inequality

$$\sum_{j=1}^{n} a_j^p \leqslant n\left(\sum_{j=1}^{n} a_j^{p/q}\right)^q \tag{2.14}$$

where $a_j>0, j=1, \ldots, n$ and $p>0$, $q>1$. It is easily seen that $n(\max_j a_j)^p$ lies between the left and right sides.

Inequalities of the Sobolev type in three space dimensions will be used, the most frequently employed being the first derivative inequality for a vector function of compact support or vanishing on $\partial\Omega$ [1, 19]:

$$\|u\|_q \leqslant C\|\nabla u\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{3} \geqslant 0. \tag{2.15}$$

Here ∇ denotes the gradient so that ∇u is a 9 component dyadic, with the norm including all such component derivatives. The constant C in (2.15) is independent of Ω [1, 19, 25]. For $q=\infty$ we use an inequality of Nirenberg [1, 9, 25]:

$$\text{ess.max}_{x\in\Omega} |u| = \|u\|_\infty \leqslant C(\|u\|_6^{1/2} \|\nabla u\|_6^{1/2} + \|u\|_6). \tag{2.16}$$

For a domain extending to infinity the second term can be omitted.

In the Hilbert space $L^2(\Omega)$ a vector field $w_i(x)$ can be expressed as a sum of gradient and solenoidal components. Suppose v_i is the component of w_i orthogonal to all gradient vectors $\nabla\phi$ in the scalar product $(u, v) = \int u_i v_i \, dx$:

$$(v_i, \nabla_i \phi) = 0$$

where

$$v_i = w_i - \nabla_i \phi_0$$

and ϕ is an arbitrary smooth scalar function. We have

$$(v_i, \nabla_i \phi) = \int_\Omega v_i \nabla_i \phi \, dx$$
$$= \int_{\partial\Omega} \phi v_i n_i \, ds - \int_\Omega \phi(\nabla_i v_i) \, dx = 0.$$

Since ϕ is arbitrary within Ω and on $\partial\Omega$ we conclude [19]

$$\operatorname{div} v_i = \nabla_i v_i = 0,$$

and

$$v_n = v_i n_i = 0$$

where $\{n_i\}$ is the unit normal vector to the boundary surface $\partial\Omega$. Therefore

$$w_i = v_i + \nabla_i \phi_0 \tag{2.17}$$

expresses w_i as a solenoidal vector field with vanishing normal component on $\partial\Omega$, plus a gradient field [19, p. 27].

The Stokes operator $\tilde{\Delta}$ is defined by the solenoidal part of the Laplacian, as applied to a general vector field $w_i(x, t)$, or to a Navier–Stokes flow $u_i(x, t)$:

$$\Delta u_i = \tilde{\Delta} u_i + \nabla_i f. \tag{2.18}$$

We reserve the symbol f for the scalar potential of the viscosity term, to be studied below. Since $\tilde{\Delta} u_i$ lies in the subspace $\mathscr{L}$ of $L^2(\Omega)$ solenoidal vector fields with vanishing normal component on $\partial\Omega$, we have $\operatorname{div} \Delta u_i \equiv (\tilde{\Delta} u_i)_{,i} = 0$, at least in a weak or generalized sense, and similarly $\tilde{\Delta} u_n = 0$ on $\partial\Omega$. If now $u_i \in C^2(\Omega)$, $u_i = 0$ on $\partial\Omega$ and $u_{i,i} = 0$ in Ω, then

$$\begin{aligned} (\tilde{\Delta} u_i, u_i) &= (\tilde{\Delta} u_i + \nabla_i f, u_i) = (\Delta u_i, u_i) \\ &= -(\nabla_j u_i, \nabla_j u_i) \\ &= -\|\nabla u\|_2^2. \end{aligned} \tag{2.19}$$

Hence if $\tilde{\Delta} u = 0$ it follows that $\|\nabla u\|_2^2 = 0$ so that $u = \text{constant} = 0$ since $u = 0$ on $\partial\Omega$. Consequently $\|\tilde{\Delta} u\|_2$ acts as a norm on the set of solenoidal vector fields u_i that vanish on $\partial\Omega$. Hence we can define [35] the completion H^2 of C^∞ solenoidal vector fields vanishing on $\partial\Omega$, in the $\|\tilde{\Delta}u\|_2$ norm. A Sobolev inequality of the special form

$$\|\nabla u\|_6 \leqslant C\|\tilde{\Delta}u\|_2 \tag{2.20}$$

holds in the space H^2; for a proof see [35, p. 194]. Then (2.16) becomes

$$\text{ess.max}_{x\in\Omega}|u| = \|u\|_\infty \leqslant C(\|\nabla u\|_2^{1/2}\|\tilde{\Delta}u\|_2^{1/2} + \|\nabla u\|_2), \quad u\in H^2. \tag{2.21}$$

Throughout our calculations the flow field u will satisfy an energy inequality $\|u\|_2 \leqslant K_1$ (see (3.1a) below) and a rate of dissipation inequality $\|\nabla u\|_2 \geqslant K_2^2$ (see (3.12) below). Under these conditions the second term in (2.21) can be dropped, for by (2.19) and (3.1a),

$$\|\nabla u\|_2^2 \leqslant \|u\|_2\|\tilde{\Delta}u\|_2 \leqslant K_1\|\tilde{\Delta}u\|_2.$$

Thus

$$\begin{aligned}\text{ess.max}\,|u| = \|u\|_\infty &\leqslant C\|\nabla u\|_2^{1/2}(\|\tilde{\Delta}u\|_2^{1/2} + \|\nabla u\|_2^{1/2})\\ &\leqslant C(\|\nabla u\|_2^{1/2}\|\tilde{\Delta}u\|_2^{1/2} + \frac{1}{K_2}\|\nabla u\|_2^{3/2})\\ &\leqslant C\left(1+\frac{K_1^{1/2}}{K_2}\right)\|\nabla u\|_2^{1/2}\|\tilde{\Delta}u\|_2^{1/2}\\ &\leqslant C'\|\nabla u\|_2^{1/2}\|\tilde{\Delta}u\|_2^{1/2}.\end{aligned} \tag{2.22}$$

A similar result holds if Ω satisfies a Poincaré inequality $\|u\|_2 \leqslant C\|\nabla u\|_2$ whence

$$\begin{aligned}\|\nabla u\|_2^2 &\leqslant \|u\|_2\|\tilde{\Delta}u\|_2\\ &\leqslant C\|\nabla u\|_2\|\tilde{\Delta}u\|_2\end{aligned}$$

and the result also follows from (2.21).

The mixed Lebesgue spaces $L^{p_1,p_2}(\Omega) = L^{p_2}(0,T;L^{p_1}(\Omega))$ are defined with norms

$$\|u\|_{p_1,p_2} = \left(\int_0^T \|u(\ ,t)\|_{p_1}^{p_2}\,dt\right)^{1/p_2}. \tag{2.23}$$

In effect our result will imply that u and its derivatives lie in certain mixed spaces of this type, usually with fractional values for p_2. When $p_2<1$ (2.23) is not a metric, as the triangle inequality does not hold for the mixed norm. We therefore study the space norms in $L^2(\Omega)$ of $u(x, t)$ and its various space and time derivatives as functions of time t.

That the $\|u\|_\infty$ norm in (2.16), (2.21) and (2.22) is actually a maximum for almost all t in $(0, T)$ can be shown for unbounded Ω with the aid of the following auxiliary

PROPOSITION. *Let $\Omega \subseteq \mathbf{R}^3$ and let $v \in C(\bar{\Omega})$ be defined on Ω with $v=0$ on $\partial\Omega$. Also let $\|v\|_2$, $\|\nabla v\|_2$ and $\|\Delta v\|_2$ be finite. Then $v \to 0$ as $x \to \infty$ in Ω.*

Proof. Let $\{\phi_i\}$ be a periodic, C^∞, non-negative, locally finite partition of unity in $\mathbf{R}^3$: $\Sigma_i \phi_i \equiv 1$. Such a partition of unity can be constructed by periodic extension of a similar partition of unity on a three-dimensional torus or periodic parallelepiped T^3, every function ϕ_i having support within some translate of the parallelepiped. Let $v_i = v\phi_i$ so that $v = \Sigma_i \phi_i v = \Sigma_i v_i$ where at any point the sum is finite. Then

$$\begin{aligned} \sup_{x \in \Omega} |v_i| \equiv \|v_i\|_\infty &\leq C \|\nabla v_i\|_2^{1/2} (\|\Delta v_i\|_2^{1/2} + \|\nabla v_i\|_2^{1/2}). \\ &\leq C(\|v_i\|_2^{1/4} \|\Delta v_i\|_2^{3/4} + \|v_i\|_2^{1/2} \|\Delta v_i\|_2^{1/2}) \end{aligned} \tag{2.24}$$

since

$$\begin{aligned} \|\nabla v_i\|_2^2 &= \int_\Omega (\nabla v_i)^2 dx \\ &= -\int_\Omega v_i \Delta v_i \, dx \\ &\leq \|v_i\|_2 \, \|\Delta v_i\|_2. \end{aligned}$$

But

$$\begin{aligned} \|v\|_2^2 = \int_\Omega v^2 dx &= \sum_{i,j} \int_\Omega v_i v_j \, dx \\ &= \sum_i \int_\Omega v_i^2 \, dx + \sum_{j \neq i} \int v_i v_j \, dx. \end{aligned}$$

Since $\int_\Omega v_i v_j \, dx = \int_\Omega \phi_i \phi_j v^2 dx \geq 0$, we have $\Sigma_i \|v_i\|_2^2 < \|v\|_2^2 < \infty$ and consequently $\|v_i\|_2 \to 0$ as $i \to \infty$. But

$$\|\Delta v_i\|_2 \leq \max|\Delta\phi_i| \, \|v\|_2 + 2 \max|\nabla\phi_i| \, \|\nabla v\|_2 + \max|\phi_i| \, \|\Delta v\|_2 < K$$

where $\max|\Delta\phi_i|$, $\max|\nabla\phi_i|$ and $\max|\phi_i| \leq 1$, and therefore also K, are all bounded independently of i. It now follows from (2.24) that $\sup|v_i| \to 0$ as $i \to \infty$. Thus the uniformly finite sum $v = \Sigma_i v_i \to 0$ as $x \to \infty$ which proves the proposition. Hence finally by Sobolev's theorem $|v|$ is a continuous function that takes its maximum value at a finite point x.

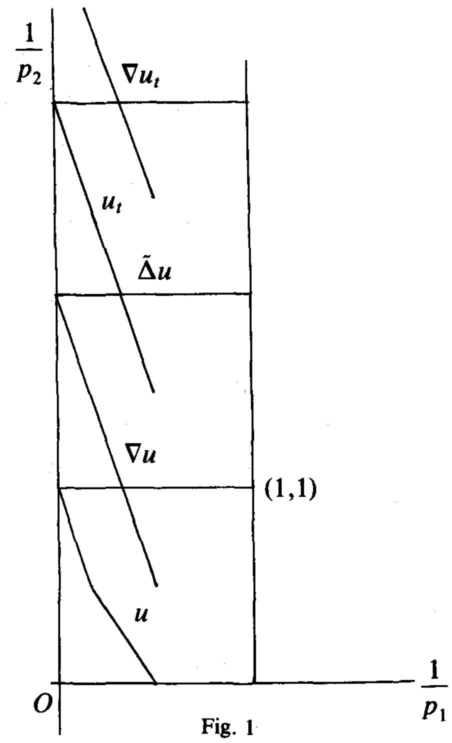

Fig. 1

3. Energy estimates

Multiplying the Navier–Stokes equations by u_i, contracting over $i=1,2,3$, and integrating over Ω, we obtain an identity which after use of the divergence and orthogonality relations reduces to

$$\frac{d}{dt}\|u\|_2^2+2\nu\|\nabla u\|_2^2=0. \tag{3.1}$$

The decrease of kinetic energy is equal to the rate of dissipation of energy by viscosity. If u is a weak solution, the relation may be an inequality with dominant right hand side [32]. Integrating over $(0, T)$, we find

$$\|u\|_2^2+2\nu\int_0^T \|\nabla u\|_2^2\,dt \leqslant \|u_0\|_2^2, \quad 0<T\leqslant\infty \tag{3.1 a}$$

and so deduce $u\in L^{2,\infty}(0,T)$, $\nabla u\in L^{2,2}(0,T)$ and by Sobolev's inequality, $u\in L^{6,2}(0,T)$. The inclusion of u and ∇u in these spaces can be schematized by means of an index diagram, Figure 1. The reciprocals $(1/p_1, 1/p_2)$ are plotted as Cartesian coordinates so the points $(1/2, 0)$ and $(1/6, 1/2)$ are marked as u-points and $(1/2, 1/2)$ as a ∇u-point in the diagram. By the standard interpolation theorems the set of points for which u belongs to the indicated space is convex, so u belongs to the spaces indicated by the line segment $3/p_1+2/p_2=3/2$, $p_2 \geqslant 2$. Another property of the index diagram is that the space to which a product uv belongs, say L^{r_1,r_2}, is given by vector addition on the diagram: $1/r_i=1/p_i+1/q_i$ where $u \in L^{p_1,p_2}$, $v \in L^{q_1,q_2}$. These properties of convexity and vector addition remain valid if $0<p_2<1$, for which the diagram extends to a semi-infinite vertical strip $0 \leqslant 1/p_1 \leqslant 1$, $0 \leqslant 1/p_2 < \infty$. Leray, Serrin and others have shown that if $u \in L^{p_1,p_2}$ where $3/p_1+2/p_2<1$ then u is a regular solution without singularities, at least for $0<t<T$ [33].

A second integral estimate or inequality, as is well known, [4, 15, 21, 33] can be derived by multiplying the Navier–Stokes equations by $\tilde{\Delta}u$; and integrating over Ω. As the calculation is typical of others to follow below, we present it in detail. Since $u_{i,t}$ is also solenoidal and vanishes on $\partial\Omega$, we find

$$\begin{aligned}\int_\Omega \tilde{\Delta}u_i u_{i,t}\,dV &= \int_\Omega \Delta u_i u_{i,t}\,dV \\ &= -\int_\Omega \nabla u_i \nabla u_{i,t}\,dV \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla u)^2 dV \\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2.\end{aligned} \tag{3.2}$$

Also $\int_\Omega \tilde{\Delta}u_i p_{,i}\,dV=0$ by the orthogonality of the solenoidal and gradient subspaces, while

$$\int_\Omega \tilde{\Delta}u_i \Delta u_i\,dV = \int_\Omega \tilde{\Delta}u_i \tilde{\Delta}u_i\,dV = \|\tilde{\Delta}u\|_2^2.$$

The nonlinear convective terms will be estimated by means of the inequalities of Hölder, Sobolev, and Young:

$$\begin{aligned}\int_\Omega \tilde{\Delta}u_i u_k u_{i,k}\,dV &\leqslant \|\tilde{\Delta}u\|_2 \|u\|_6 \|\nabla u\|_3 \\ &\leqslant C\|\tilde{\Delta}u\|_2 \|\nabla u\|_2 \|\nabla u\|_6^{1/2} \|\nabla u\|_2^{1/2} \\ &\leqslant C\|\tilde{\Delta}u\|_2^{3/2} \|\nabla u\|_2^{3/2} \\ &\leqslant \frac{\nu}{2}\|\tilde{\Delta}u\|_2^2 + C\nu^{-3}\|\nabla u\|_2^6.\end{aligned} \tag{3.3}$$

After a cancellation of terms in $\|\tilde{\Delta} u\|_2^2$, the resulting inequality becomes

$$\frac{d}{dt}\|\nabla u\|_2^2+\nu\|\tilde{\Delta} u\|_2^2 \leqslant K\|\nabla u\|_2^6. \tag{3.4}$$

This inequality has yielded much of the known behaviour of singular solutions and asymptotic behaviour for long times.

Since (2.19) implies

$$\|\nabla u\|_2^2 \leqslant \|u\|_2\|\tilde{\Delta} u\|_2 \tag{3.5}$$

we can deduce from (3.4) that

$$\frac{d}{dt}\|\nabla u\|_2^2+\nu\frac{\|\nabla u\|_2^4}{\|u\|_2^2} \leqslant K\|\nabla u\|_2^6. \tag{3.6}$$

This self-contained inequality for $\|\nabla u\|_2$ in terms of $\|u\|_2$ can be linearized by an exchange of independent and dependent variables [5, 6]. With $\|u\|_2=z$ and use of (3.1) we find

$$\nu^2\frac{d^2t}{dz^2}+Kz^2 \geqslant 0. \tag{3.7}$$

From this inequality two successive integrations with respect to z show that

$$t+cz^4=C(z), \quad c=\frac{K}{12\nu^2} \tag{3.8}$$

is a convex function of z. Also the expression

$$\frac{dt}{dz}+4cz^3=C'(z) \tag{3.9}$$

is an increasing function of z. From (3.1) we have

$$\frac{dt}{dz}=\frac{-z}{\nu\|\nabla u\|_2^2}<0$$

with equality at singular values z_T where $\|\nabla u\|_2 \to \infty$ so that $C'(z_T)=4cz_T^3>0$. For $z>z_T$ (or $t<T$) we have [6]

$$\|\nabla u\|_2^2 \geqslant \frac{z}{4c\nu(z^3-z_T^3)}, \quad z>z_T. \tag{3.10}$$

This is a modified form of an inequality first derived by Leray [21].

From (3.6) it follows that if

$$K\|\nabla u\|_2^2\,\|u\|_2^2 < \nu \tag{3.11}$$

then $\|\nabla u\|_2$ will be monotonic decreasing in t. Since $\|u\|_2$ is also monotonic decreasing in t, by (3.1), it is apparent that (3.11) will continue to hold thereafter, so that no singular instants can occur subsequently. Consequently, during the time interval in which singular instants may occur, before (3.11) takes effect say at T_1, we have the lower bound

$$\|\nabla u\|_2^2 \geqslant \frac{\nu}{K\|u\|_2^2} > \frac{\nu}{K\|u_0\|_2^2}, \quad 0<t<T_1. \tag{3.12}$$

Observe that

$$\|u_0\|_2^2 \geqslant 2\nu \int_0^{T_1} \|\nabla u\|_2^2\,dt > \frac{2\nu^2 T_1}{K\|u_0\|_2^2}$$

so that [21]

$$T_1 < \frac{K}{2\nu^2}\|u_0\|_2^4. \tag{3.13}$$

Behaviour as time tends to infinity has been studied in [13].

4. The main theorem

While three dimensional viscous incompressible flows apparently do not enjoy a complete boundedness and regularity property, we will show in the following sections that the space and time derivatives of a solution do satisfy estimates similar to the energy estimates discussed above for $\|u\|_2$ and $\|\nabla u\|_2$. In consequence it can be shown that these derivatives do lie in mixed Lebesgue spaces in which the space index p_1 is 2 and the time index p_2 is in general a positive fraction. The Sobolev inequalities and interpolation then imply similar results for $p_1>2$. Let $D_t=\partial/\partial t$, $D_i=\partial/\partial x_i$, $D_x^s=D_1^{s_1}D_2^{s_2}D_3^{s_3}$, and let $\|D_x^s u\|_p^p$ denote the pth power sum over all derivative norms of order s.

THEOREM. *Let $u_i(x,t)$ be a Navier–Stokes flow with finite energy in a suitable three-dimensional domain $\Omega\subset\mathbf{R}^3$. Let $u_i(x,t)$ be smooth except on a singular set. Over any time interval $(0,T)$ where $0\leqslant T<\infty$ we have, for $r, s_j=0,1,2,\ldots$; $s=s_1+s_2+s_3$,*

$$\|D_t^r D_x^s u\|_2 \in L^{2(4r+2s-1)^{-1}}(0,T) \tag{4.1}$$

for r or an $s_j>0$, and

$$\max_{x\in\Omega} |D_t^r D_s^x u| \in L^{(2r+s+1)^{-1}}(0,T). \tag{4.2}$$

The corresponding integrals are bounded by constants depending only on r, s_j ($j=1,2,3$), ν, T, Ω and $\|u_0\|_2$.

Note that when $r=0$ and the multi-index $s=(s_1,s_2,s_3)=0$ the pattern of (4.1) does not hold precisely. When $r=s=0$ in (4.2) we obtain the known result $\max|u| \in L^1(0,T)$. This was obtained with the aid of (2.12), (2.15) and (2.21) from the estimate $\|\tilde{\Delta} u\|_2 \in L^{2/3}(0,T)$ derived in [4]. For $r=0$ and arbitrary $s=(s_1,s_2,s_3)$ the result was established by Foias, Guillopé and Temam [10] for the three-dimensional torus T^3.

Since T the time interval is arbitrary, these estimates will hold through any singular instant, since by (3.13) all singular instants lie in a bounded time interval.

As $t\to 0+$ the various norms considered are in general unbounded. Thus $\lim_{t\to 0}\|\nabla u\|_2$ can not be finite if u_0 is nondifferentiable on an open subset of Ω. Our estimates nonetheless apply near the lower limit $t=0$. The physical counterpart is known as "water hammer": when the tap is turned on the entire plumbing system vibrates to pressure shocks of unlimited speed. Compatibility conditions are also involved and have been studied by Heywood [15] and Rautmann [26].

The conditions to be imposed on our finitely connected domain Ω shall be as follows. The boundary $\partial\Omega$ shall be piecewise C^∞ with a finite number of edges and corners in any bounded subregion, and shall satisfy a weak cone condition [1]. If $\partial\Omega$ extends to infinity the order condition

$$\int_{\partial\Omega \cap S(R)} dS \leqslant KR^2 \tag{4.3}$$

is also imposed with a fixed constant K, where $S(R)$ denotes the ball $|x|\leqslant R$. This condition is required for Lemma 2 and prevents the boundary surface from being too tightly coiled at large distances. It is not satisfied, for example, by the region enclosed by rotating the curve $x=e^z(2+\sin e^{2z})$ about the z-axis.

5. The scalar viscosity potential

As will be seen below, it is possible to obtain estimates for the Stokes term $\tilde{\Delta} u_i$ on a region Ω with boundary. However to find estimates of second order space derivatives, and higher order derivatives, knowledge of the Laplacian term Δu_i is required. From (2.18) we see that estimates will be necessary for the gradient ∇f of the scalar viscosity potential function f. In this section we begin the investigation of properties of f and ∇f.

Since $\Delta u_i = \tilde{\Delta} u_i + \nabla_i f$ and $(\Delta u_i)_{,i} = 0$, $(\tilde{\Delta} u_i)_{,i} = 0$ we find by taking divergences that

$$\Delta f = \nabla_i(\nabla_i f) = 0. \tag{5.1}$$

Hence f is a harmonic function that depends on time t as a parameter. Since the normal component of $\tilde{\Delta} u_i$ on $\partial\Omega$ is zero, by the basic property of the solenoidal Stokes term as orthogonal to all gradients, it follows that on $\partial\Omega$,

$$\frac{\partial f}{\partial n} = \frac{\partial f}{\partial x_i} = \Delta u_i n_i. \tag{5.2}$$

By the divergence theorem

$$\int_{\partial\Omega} \Delta u_i n_i \, dS = \int_{\Omega} (\Delta u_i)_{,i} \, dV = 0 \tag{5.3}$$

so the boundary datum $\Delta u_i n_i$ satisfies the necessary integral condition for the Neumann problem of classical potential theory [17]. Thus $f = f(x_i, t)$ may be regarded as defined by (5.1) and (5.2) up to an additive constant. For simplicity we choose this constant so that

$$\int_{\Omega} f(x_i, t) \, dx = 0, \quad 0 \leq t < \infty. \tag{5.4}$$

LEMMA 1. *For ∇f there holds the estimate*

$$\|\nabla f\|_2^2 \leq C \|\nabla u\|_2 (\|\tilde{\Delta} u\|_2 + C_1 \|\nabla u\|_2) \tag{5.5}$$

where C and C_1 are independent of u and f. Also $\|\nabla f\|_2 \in L^1(0, T)$.

Proof. Let $N = N(x, y)$ be the harmonic Neumann function for Ω: thus $\partial N / \partial n_x = 0$ on $\partial\Omega$ while

$$-\Delta_x N = \delta(x, y) - \frac{1}{V} \tag{5.6}$$

with $V = \int_\Omega 1 dx$ [2, 8]. Likewise let $G(x, y)$ be the harmonic Green's function for Ω so that $G(x, y) = 0$ for $y \in \partial\Omega$, $-\Delta_x G(x, y) = \delta(x, y)$ and $G(x, y)$ like $N(x, y)$ is symmetric in x and y. The Bergman kernel for Ω is defined as [2]

$$K(x, y) = N(x, y) - G(x, y) \tag{5.7}$$

so that

$$\Delta_x K(x, y) = \frac{1}{V}. \tag{5.8}$$

The solution of a Neumann problem for $\Delta u=0$ is given by a surface integral representation with $N(x, y)$ as kernel. Thus

$$f(x, t) = \int_{\partial\Omega} N(x, y)\, \Delta u_i(y, t)\, n_i(y)\, dS_y. \tag{5.9}$$

Since $G(x, y)$ vanishes for $y \in \partial\Omega$, we may write

$$\begin{aligned} f(x, t) &= \int_{\partial\Omega} K(x, y)\, \Delta u_i(y, t)\, n_i(y)\, dS_y \\ &= \int_{\Omega} (K(x, y)\, \Delta u_i(y, t))_{,i}\, dy \\ &= \int_{\Omega} K_{,i}(x, y)\, \Delta u_i(y, t)\, dy \end{aligned} \tag{5.10}$$

by the divergence theorem and (2.2). Here the subscript comma and index i refer to the y variable. Applying Green's second theorem to (5.10) we find two terms vanish and

$$f(x, t) = \int_{\partial\Omega} K_{,i} \frac{\partial u_i}{\partial n}\, dS. \tag{5.11}$$

From (2.2) and (2.5) we see that if coordinates x_α $(\alpha=1, 2)$ tangential to $\partial\Omega$ are chosen, and the normal coordinate is denoted by n, then on $\partial\Omega$

$$\frac{\partial u_n}{\partial n} = -\frac{\partial u_\alpha}{\partial x_\alpha} = 0.$$

Hence

$$f(x, t) = \int_{\partial\Omega} K(x, y)_{,\alpha} \frac{\partial u_\alpha}{\partial n}\, dS. \tag{5.12}$$

Here α runs through tangential indices 1, 2 only, with respect to the y variable.

We now consider values of x on the boundary $\partial\Omega$. While $K(x, y)$ unlike $N(x, y)$ or $G(x, y)$ is regular in the interior of Ω, this is no longer true if x and y both lie on the boundary.

LEMMA 2. *The kernels*

$$\frac{\partial K(x,y)}{\partial y_\alpha}, \quad \frac{\partial^2 K(x,y)}{\partial n_x \partial y_\alpha} \tag{5.13}$$

are Calderon–Zygmund kernels on the boundary manifold, and hence define bounded integral operators on $L^2(\partial\Omega)$.

Proof. For x sufficiently close to $\partial\Omega$, x falls within a boundary neighbourhood fibred by normals. Draw the tangent plane to $\partial\Omega$ touching at the foot of the normal through x; and let x' be the mirror image of x in this plane. Similarly, given $r(x,y)=|x-y|$, let $r'=r(x,y')=|x-y'|$.

Set

$$\begin{aligned} N(x,y) &= \frac{1}{4\pi r}+\frac{1}{4\pi r'}+n(x,y) \\ G(x,y) &= \frac{1}{4\pi r}-\frac{1}{4\pi r'}+g(x,y). \end{aligned} \tag{5.14}$$

Here

$$\Delta_x n(x,y)=\frac{1}{V}, \quad \Delta_x g(x,y)=0 \tag{5.15}$$

and

$$\frac{\partial N(x,y)}{\partial n_x}=0, \quad G(x,y)=0, \quad x\in\partial\Omega. \tag{5.16}$$

Applying the standard representation formulas to $n(x,y)$ and $g(x,y)$ we find that

$$\begin{aligned} n(x,y) &= \frac{1}{4\pi}\int_{\partial\Omega} N(x,z)\frac{\partial}{\partial n_z}\left(-\frac{1}{r}-\frac{1}{r'}\right) dS_z \\ g(x,y) &= \frac{1}{4\pi}\int_{\partial\Omega} \frac{\partial G(x,z)}{\partial n_z}\left(\frac{1}{r}-\frac{1}{r'}\right) dS_z \end{aligned} \tag{5.17}$$

where $r=r(z,y)$, $r'=r'(z,y)$ in the integrands.

We now consider y as lying on the boundary near x, so $r'=r$, and invoke a lemma of classical potential theory: as $z\to x$ on $\partial\Omega$,

$$\frac{\partial}{\partial n_z}\left(\frac{1}{r}\right)=\frac{B(x,z)}{r}, \quad r=r(x,z) \tag{5.18}$$

where $B(x,z)$ is smooth and uniformly bounded [17].

It follows that the functions

$$n(x,y),\quad g(x,y),\quad \frac{\partial n(x,y)}{\partial n_x},\quad \frac{\partial g(x,y)}{\partial n_x},$$

where $x, y\in\partial\Omega$, can all be represented by integrals of the form

$$\int_{\partial\Omega}\frac{F_1(x,z)}{r(x,z)}\frac{F_2(z,y)}{r(z,y)}\,dS_z \tag{5.19}$$

where $F_1(x,z)$ and $F_2(z,y)$ are smooth bounded functions on $\partial\Omega$. By dimensional considerations, and by direct estimation, it can be shown that such an integral has the form

$$\ln r(x,y)\,F_3(x,y)+F_4(x,y)$$

where again $F_3(x,y)$ and $F_4(x,y)$ are smooth bounded functions on $\partial\Omega$ [17].

Consequently it now follows that the derivatives (5.13) of $K(x,y)$ have the local behaviour as $x\to y$, $r=r(x,y)$,

$$\begin{aligned}\frac{\partial K(x,y)}{\partial y_\alpha}&=\frac{1}{2\pi}\frac{\partial}{\partial y_\alpha}\left(\frac{1}{r}\right)+O\left(\frac{1}{r}\right)\\ \frac{\partial^2 K(x,y)}{\partial n_x\,\partial y_\alpha}&=\frac{1}{2\pi}\frac{\partial}{\partial y_\alpha}\left(\frac{B(y,x)}{r}\right)+O\left(\frac{1}{r}\right)\end{aligned} \tag{5.20}$$

where $B(y,x)$ is as in (5.18). The lemma then follows, since $\partial\Omega$ is a two-dimensional manifold, piecewise smoothly embedded in $\mathbf{R}^3$ so that small distances in $\partial\Omega$ are asymptotic to those in $\mathbf{R}^3$, almost everywhere in $\partial\Omega$. By (4.3) the usual order conditions implicit in the Calderon–Zygmund theorem for $\mathbf{R}^2$ will apply [9, p. 1072; 23, Chapter 9].

To complete the proof of Lemma 1 we note that $\Delta f=0$ and hence

$$\begin{aligned}\|\nabla f\|_2^2=\int_\Omega(\nabla f)^2dV&=\int_{\partial\Omega} f\frac{\partial f}{\partial n}\,dS\\ &\leqslant \|f\|_{2,\partial}\left\|\frac{\partial f}{\partial n}\right\|_{2,\partial}\end{aligned} \tag{5.21}$$

by the Schwarz inequality, where the subscript ∂ on the norm indicates integration over $\partial\Omega$. By Lemma 2, and (5.12), both f and $\partial f/\partial n$ are represented by integral operators with kernels bounded in $L^2(\partial\Omega)$. Hence

$$\begin{aligned}\|\nabla f\|_2^2 &\leqslant C\left\|\frac{\partial u_\alpha}{\partial n}\right\|_{2,\partial}^2 \\ &= C\int_{\partial\Omega}\sum_{\alpha=1}^{2}\left(\frac{\partial u_\alpha}{\partial n}\right)^2 dS.\end{aligned} \tag{5.22}$$

By a well-known technique, we find the last integral is majorized by

$$C\int_\Omega |D_j u_\alpha|\,\{|D_j D_k u_\alpha|+|D_j u_\alpha|\}\,dV.$$

By Schwarz' inequality, and an estimate of Ladyzhenskaya [19, p. 21], we now have

$$\begin{aligned}\|\nabla f\|_2^2 &\leqslant C\|\nabla u\|_2\{\|\Delta u\|_2+\|\nabla u\|_2\} \\ &\leqslant C\|\nabla u\|_2\{\|\tilde{\Delta} u\|_2+\|\nabla f\|_2+\|\nabla u\|_2\}.\end{aligned} \tag{5.23}$$

This last step follows from (2.18) and the triangle inequality.

But now

$$C\|\nabla u\|_2\|\nabla f\|_2 \leqslant \frac{1}{2}\|\nabla f\|_2^2+K\|\nabla u\|_2^2$$

so we obtain finally the inequality stated in Lemma 1.

Since $\|\nabla u\|_2 \in L^2(0,\infty)$, and $\|\tilde{\Delta} u\|_2 \in L^{2/3}(0,T)$ has been established [4], it follows that the right hand side of the inequality of Lemma 1 is in $L^{1/2}(0,T)$. Hence we have shown

$$\|\nabla f\|_2 \in L^1(0,T). \tag{5.24}$$

Chapter II. The gradient and Stokes operators and their time derivatives

6. An integrability lemma

The main stage of our proof will consist of a sequence of estimates, each requiring the use of an appropriate integrability lemma to contribute its single step to an induction process. In preparation, the necessary lemma, which in its original application is due to Foias, Guillopé and Temam [10], will be proved next.

LEMMA 3. *Let $a>1$, $F(t)\geqslant 0$, $F(t)\in L^p(0,T)$ where $p>0$, $G(t)\geqslant 0$ and for $0\leqslant t\leqslant T$ let*

$$F'(t)+G(t)\leqslant KF(t)^{a+p}. \tag{6.1}$$

Then $G(t) \in L^{p(a+p)^{-1}}(0, T)$.

Proof. Introducing an additional positive unit factor or term on the right side, we have

$$\frac{d}{dt}(1+F(t))+G(t) \leq KF^p(t)\,(1+F(t))^a.$$

Divide by $(1+F(t))^a$ and incorporate this factor within the derivative:

$$\frac{-1}{a-1}\frac{d}{dt}(1+F(t))^{1-a}+\frac{G(t)}{(1+F(t))^a} \leq KF^p(t).$$

Integrating from 0 to T, we have

$$\frac{(1+F(0))^{1-a}}{a-1}+\int_0^T \frac{G(t)\,dt}{(1+F(t))^a} \leq K\int_0^T F^p(t)\,dt+\frac{(1+F(T))^{1-a}}{a-1}.$$

Note that if $F(t) \to \infty$ as $t \to 0$ then the first term on the left will be zero. The second term on the right is majorized by $(a-1)^{-1}$ independently of $F(T)$ while the first term is finite by hypothesis. Consequently the integral on the left is convergent. Now

$$\begin{aligned}\int_0^T G(t)^{\frac{p}{a+p}}dt &= \int_0^T (1+F(t))^{\frac{ap}{a+p}} \frac{G(t)^{\frac{p}{a+p}}dt}{(1+F(t))^{\frac{ap}{a+p}}} \\ &\leq \left(\int_0^T (1+F(t))^p dt\right)^{\frac{a}{a+p}} \left(\int_0^T \frac{G(t)\,dt}{(1+F(t))^a}\right)^{\frac{p}{a+p}}\end{aligned}$$

by Hölder's inequality. Both integrals on the right being convergent, the integral on the left converges, and the lemma is proved.

This result in effect shows that the integrability of $G(t)$ is the same as that of the right hand side, over finite time intervals.

7. A triple squence of estimates

The presence of a boundary, and of boundary conditions on the $u_i(x, t)$, introduces two types of difficulties that stand in the way of a direct estimate such as the one used by Foias, Guillopé and Temam for T^3 [10]. The first of these is the distinction between Δu_i and $\tilde{\Delta} u_i$ together with the presence of a scalar viscosity potential f, which has necessitated the preceding lemmas. The second is the failure of non-tangential space derivatives to satisfy boundary conditions (2.5), which we shall circumvent by an indirect method commencing with the use of time derivatives, which do inherit the boundary conditions and subspace properties. In this section we construct three sequences of estimates formed by differentiating the Navier–Stokes equations r times with respect to time t, and multiplying by $D_t^r u_i$, $D_t^r \tilde{\Delta} u_i$, and $D_t^{r+1} u_i$, respectively, where $r=0,1,2,\ldots$ [15]. We present these as a single interleaved sequence in an order determined by increasing integrability indices—that is, by decreasing integrability over the time variable. An induction over r is then established. We shall present the first two groups, that is, six inequalities, and then the typical rth stage group of three inequalities. Each of the inequalities will be numbered by its reciprocal time exponent or index, which specifies the position in the time index diagram of every term in the estimate. A pair of the same index are marked (a) and (b) in a convenient order. Constants such as C appearing on the right hand side of these estimates may be different at different occurrences.

The first two such estimates are (3.1) and (3.4), which we repeat here labelled for later reference:

$$\frac{d}{dt}\|u\|_2^2+2\nu\|\nabla u\|_2^2=0 \tag{1}$$

$$\frac{d}{dt}\|\nabla u\|_2^2+\nu\|\tilde{\Delta} u\|_2^2\leqslant K\|\nabla u\|_2^6. \tag{3(a)}$$

To obtain the third inequality, multiply (2.1) by $u_{i,t}\equiv D_t u_i$ and integrate, obtaining

$$\begin{aligned}\int_\Omega u_{i,t}u_{i,t}\,dx+\int_\Omega u_{i,t}u_k u_{i,k}\,dx&=-\int_\Omega p_{,i}u_{i,t}\,dx+\nu\int_\Omega u_{i,t}\Delta u_i\,dx\\ &=0-\nu\int_\Omega \nabla u_{i,t}\nabla u_i\,dx\\ &=-\frac{1}{2}\nu\frac{d}{dt}\int_\Omega(\nabla u)^2dt.\end{aligned} \tag{7.1}$$

Thus

$$\begin{aligned}\nu\frac{d}{dt}\|\nabla u\|_2^2+2\|u_t\|_2^2&=-2\int_\Omega u_{i,t}u_k u_{i,k}\,dx\\ &\leqslant 2\,\|u_t\|_2\,\|u\|_6\,\|\nabla u\|_3.\end{aligned} \tag{7.2}$$

Since the right hand side is less than or equal to $\|u_t\|_2^2+\|u\|_6^2\,\|\nabla u\|_3^2$ we obtain

$$
\begin{aligned}
\nu\frac{d}{dt}\|\nabla u\|_2^2+\|u_t\|_2^2 &\leq \|u\|_6^2\,\|\nabla u\|_3^2\\
&\leq C\|\nabla u\|_2^2\,\|\nabla u\|_2\,\|\nabla u\|_6 \qquad\qquad 3\,(\mathrm{b})\\
&\leq C\|\nabla u\|_2^3\,\|\tilde{\Delta} u\|_2
\end{aligned}
$$

where we have used (2.12) and (2.20).

To derive the second group of three inequalities, we differentiate (2.1) once with respect to time t:

$$u_{i,tt}+u_{k,t}\,u_{i,k}+u_k\,u_{i,kt}=-p_{,it}+\nu\Delta u_{i,t}. \tag{7.3}$$

Multiplying by $u_{i,t}$ and integrating, we have

$$
\begin{aligned}
&\int_\Omega u_{i,t}\,u_{i,tt}\,dx+\int_\Omega u_{i,t}\,u_{k,t}\,u_{i,k}\,dx+\int_\Omega u_{i,t}\,u_k\,u_{i,kt}\,dx\\
&\quad=-\int_\Omega u_{i,t}\,p_{,it}\,dx+\nu\int_\Omega u_{i,t}\,\Delta u_{i,t}\,dx.
\end{aligned}
\tag{7.4}
$$

The first of these terms is a time derivative, while the third is

$$\frac{1}{2}\int_\Omega u_k\frac{\partial}{\partial x_k}(u_{i,t})^2dx=-\frac{1}{2}\int_\Omega u_{k,k}(u_{i,t})^2dx=0 \tag{7.5}$$

by (2.2). The first term on the right is zero by orthogonality while we can integrate by parts in the last term. The result is

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\nu\|\nabla u_t\|_2^2 &= -\int_\Omega u_{i,t}\,u_{k,t}\,u_{i,k}\,dx\\
&\leq \|u_t\|_4^2\,\|\nabla u\|_2\\
&\leq \|u_t\|_2^{1/2}\,\|u_t\|_6^{3/2}\,\|\nabla u\|_2\\
&\leq C\|u_t\|_2^{1/2}\,\|\nabla u_t\|_2^{3/2}\,\|\nabla u\|_2
\end{aligned}
\tag{7.6}
$$

Employing Young's inequality with exponents $\frac{4}{3}$ and 4, and cancelling a term $\frac{1}{2}\nu\|\nabla u_t\|_2^2$ on either side, then multiplying by 2, we find

$$
\begin{aligned}
\frac{d}{dt}\|u_t\|_2^2+\nu\|\nabla u_t\|_2^2 &\leq C\|u_t\|_2^2\,\|\nabla u\|_2^4\\
&\leq C\{\|u_t\|_2^{10/3}+\|\nabla u\|_2^{10}\}.
\end{aligned}
\qquad 5(\mathrm{b})
$$

Here Young's inequality has been used again with exponents $\frac{5}{3}$ and $\frac{5}{2}$.

Next multiply (7.3) by $\tilde{\Delta}u_{i,t}$ and integrate finding

$$\begin{aligned}\int_\Omega \tilde{\Delta}u_{i,t}u_{i,tt}\,dx+\int_\Omega \tilde{\Delta}u_{i,t}u_{k,t}u_{i,k}\,dx+\int_\Omega \tilde{\Delta}u_{i,t}u_k u_{i,kt}\,dx \\ =-\int_\Omega \tilde{\Delta}u_{i,t}p_{,it}\,dx+\nu\int_\Omega \tilde{\Delta}u_{i,t}\Delta u_{i,t}\,dx.\end{aligned}\tag{7.7}$$

Since $\nabla_i f_t$ is orthogonal to $u_{i,tt}$, the first integral on the left becomes

$$\begin{aligned}\int_\Omega \Delta u_{i,t}u_{i,tt}\,dx &= -\int_\Omega \nabla u_{i,t}\nabla u_{i,tt}\,dx \\ &= -\frac{1}{2}\frac{d}{dt}\int(\nabla u_{i,t})^2dx \\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_2^2.\end{aligned}\tag{7.8}$$

The first integral on the right is zero by the usual orthogonality property, while the second, for a similar reason, becomes

$$\nu\int_\Omega \tilde{\Delta}u_{i,t}\tilde{\Delta}u_{i,t}\,dx=\nu\|\tilde{\Delta}u_t\|_2^2.$$

We thus have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_2^2+\nu\|\tilde{\Delta}u_t\|_2^2 &= \int_\Omega \tilde{\Delta}u_{i,t}\{u_{k,t}u_{i,k}+u_k u_{i,kt}\}\,dx \\ &\le \|\tilde{\Delta}u_t\|_2\{\max_{x\in\Omega}|D_t u|\,\|\nabla u\|_2+\|u\|_6\|\nabla u_t\|_3\} \\ &\le \|\tilde{\Delta}u_t\|_2\{C\|\nabla u_t\|_2^{1/2}\|\tilde{\Delta}u_t\|_2^{1/2}\|\nabla u\|_2+C\|\nabla u\|_2\|\nabla u_t\|_2^{1/2}\|\tilde{\Delta}u_t\|_2^{1/2}\} \\ &\le C\|\tilde{\Delta}u_t\|_2^{3/2}\|\nabla u_t\|_2^{1/2}\|\nabla u\|_2 \\ &\le \frac{\nu}{4}\|\tilde{\Delta}u_t\|_2^2+C\|\nabla u_t\|_2^2\|\nabla u\|_2^4\end{aligned}\tag{7.9}$$

by Young's inequality with exponents $\frac{4}{3}$ and 4. Cancelling the $\|\tilde{\Delta}u_t\|_2^2$ term, we find

$$\frac{d}{dt}\|\nabla u_t\|_2^2+\frac{3}{2}\nu\|\tilde{\Delta}u_t\|_2^2\le C\|\nabla u_t\|_2^2\|\nabla u\|_2^4\le C\{\|\nabla u_t\|_2^{14/5}+\|\nabla u\|_2^{14}\}\tag{7(a)}$$

the last step by Young's inequality with exponents $\frac{7}{5}$ and $\frac{7}{2}$.

Multiply (7.3) by $u_{i,tt}$ and integrate, obtaining

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\nabla u_t\|_2^2+\|u_{tt}\|_2^2 &= -\int_\Omega u_{i,tt}\{u_{k,t}u_{i,k}+u_k u_{i,kt}\}\,dx \\ &\leq \|u_{tt}\|_2\{\max|D_t u|\,\|\nabla u\|_2+\|u\|_6\|\nabla u_t\|_3\}.\end{aligned} \tag{7.10}$$

By Young's inequality (2.12) and the interpolation inequalities, we find

$$\nu\frac{d}{dt}\|\nabla u_t\|_2^2+\|u_{tt}\|_2^2\leq C\|\nabla u\|_2^2\,\|\nabla u_t\|_2\,\|\tilde{\Delta} u_t\|_2. \tag*{7(b)}$$

These first two groups of inequalities display many major features of the higher order calculations, and serve to initiate the induction on r the order of time derivatives. We now give the inequalities for the typical rth stage, found by differentiating (2.1) r times with respect to t. Henceforth we shall also use D_t to indicate a time derivative. With the help of Leibniz' formula for a higher derivative of a product, we now obtain

$$D_t^r u_{i,t}+\sum_{j=0}^{r}\binom{r}{j}D_t^j u_k D_t^{r-j}u_{i,k}=-D_t^r p_{,i}+\nu D_t^r\Delta u_i. \tag{7.11}$$

Multiplying by $D_t^r u_i$ and integrating, we find after some routine steps,

$$\frac{1}{2}D_t\|D_t^r u\|_2^2+\nu\|D_t^r\nabla u\|_2^2=-\sum_{j=0}^{r}\binom{r}{j}\int_\Omega D_t^r u_i D_t^j u_k D_t^{r-j}u_{i,k}\,dx. \tag{7.12}$$

The term with $j=0$ on the right is

$$\int_\Omega D_t^r u_i\cdot u_k D_t^r u_{i,k}\,dx=\frac{1}{2}\int_\Omega u_k (D_t^r u_i)^2_{,k}\,dx=-\int_\Omega u_{k,k}(D_t^r u_i)^2dx=0 \tag{7.13}$$

by (2.2). For the terms with $0<j<r$ we estimate as follows:

$$\begin{aligned}\int_\Omega D_t^r u_i D_t^j u_k D_t^{r-j}u_{i,k}\,dx &\leq \|D_t^r u\|_6\,\|D_t^j u\|_3\,\|D_t^{r-j}\nabla u\|_2 \\ &\leq C\|D_t^r\nabla u\|_2\,\|D_t^j u\|_2^{1/2}\,\|D_t^j u\|_6^{1/2}\,\|D_t^{r-j}\nabla u\|_2 \\ &\leq C\|D_t^r\nabla u\|_2\,\|D_t^j u\|_2^{1/2}\,\|D_t^j\nabla u\|_2^{1/2}\,\|D_t^{r-j}\nabla u\|_2 \\ &\leq \frac{\nu}{2r}\|D_t^r\nabla u\|_2^2+C\|D_t^j u\|_2\,\|D_t^j\nabla u\|_2\,\|D_t^{r-j}\nabla u\|_2^2.\end{aligned} \tag{7.14}$$

Finally the term with $j=r$ becomes

$$\begin{aligned}\int_\Omega D_t^r u_i D_t^r u_k u_{i,k}\,dx &\leq \|D_t^r u\|_4^2 \|\nabla u\|_2 \\ &\leq C\|D_t^r u\|_2^{1/2} \|D_t^r u\|_6^{3/2} \|\nabla u\|_2 \\ &\leq C\|D_t^r \nabla u\|_2^{3/2} \|D_t^r u\|_2^{1/2} \|\nabla u\|_2 \\ &\leq \frac{\nu}{2r} \|D_t^r \nabla u\|_2^2 + C\|D_t^r u\|_2^2 \|\nabla u\|_2^4\end{aligned} \tag{7.15}$$

by Young's inequality with $p=\frac{4}{3}$, $q=4$. Combining all terms, multiplying by 2, and noting certain cancellations involving $\|D_t^r \nabla u\|_2^2$ terms, we find

$$\begin{aligned}D_t\|D_t^r u\|_2^2 + \nu\|D_t^r \nabla u\|_2^2 &\leq C\left\{\|D_t^r u\|_2^2 \|\nabla u\|_2^4 + \sum_{j=1}^{r-1} \|D_t^j u\|_2 \|D_t^j \nabla u\|_2 \|D_t^{r-j} \nabla u\|_2^2\right\} \\ &\leq C\left\{\sum_{j=1}^{r} \|D_t^j u\|_2^{\frac{8r+2}{4j-1}} + \sum_{j=0}^{r-1} \|D_t^j \nabla u\|_2^{\frac{8r+2}{4j+1}}\right\}.\end{aligned} \tag{4r+1}$$

Here we have used the homogeneous version (2.13) of Young's inequality.

Multiplying (7.11) by $D_t^r \tilde{\Delta} u_i$ and integrating, we find

$$\begin{aligned}\frac{1}{2} D_t\|D_t^r \nabla u\|_2^2 + \nu\|D_t^r \tilde{\Delta} u\|_2^2 &= \int_\Omega D_t^r \tilde{\Delta} u_i \sum_{j=0}^{r} \binom{r}{j} D_t^j u_k D_t^{r-j} u_{i,k}\,dx \\ &\leq \|D_t^r \tilde{\Delta} u\|_2 \left\{\sum_{j=0}^{[r/2]} \binom{r}{j} \|D_t^j u\|_6 \|D_t^{r-j} \nabla u\|_3\right. \\ &\quad \left.+ \sum_{j=[r/2]+1}^{r} \binom{r}{j} \max |D_t^j u|\, \|D_t^{r-j} \nabla u\|_2\right\} \\ &\leq C\|D_t^r \tilde{\Delta} u\|_2 \left\{\sum_{j=0}^{[r/2]} \|D_t^j \nabla u\|_2 \|D_t^{r-j} \nabla u\|_2^{1/2} \|D_t^{r-j} \tilde{\Delta} u\|_2^{1/2}\right. \\ &\quad \left.+ \sum_{j=[r/2]+1}^{r} \|D_t^j \nabla u\|_2^{1/2} \|D_t^j \tilde{\Delta} u\|_2^{1/2} \|D_t^{r-j} \nabla u\|_2\right\}.\end{aligned} \tag{7.16}$$

By Young's inequality, and treating the terms for $j=0$ in the first sum and $j=r$ in the second sum separately, we find that, for any real γ with $0<\gamma<2$,

$$\begin{aligned}D_t\|D_t^r \nabla u\|_2^2 + 2\nu\|D_t^r \tilde{\Delta} u\|_2^2 &\leq \frac{1}{2}(2-\gamma)\nu\|D_t^r \tilde{\Delta} u\|_2^2 \\ +C(\gamma)&\left\{\|D_t^r \nabla u\|_2^2 \|\nabla u\|_2^4 + \sum_{j=1}^{[r/2]} \|D_t^j \nabla u\|_2^2 \|D_t^{r-j} \nabla u\|_2 \|D_t^{r-j} \tilde{\Delta} u\|_2\right\}.\end{aligned} \tag{7.17}$$

Again, multiplying (7.11) by $D_t^{r+1}u$ and integrating, we find

$$\begin{aligned}
\frac{\nu}{2}D_t\|D_t^r\nabla u\|_2^2+\|D_t^{r+1}u\|_2^2 &= -\int_\Omega D_t^{r+1}u_i\sum_{j=0}^{r}\binom{r}{j}D_t^j u_k D_t^{r-j}u_{i,k}\,dx\\
&\leq \|D_t^{r+1}u\|_2\left\{\sum_{j=0}^{[r/2]}\binom{r}{j}\|D_t^j u\|_6\|D_t^{r-j}\nabla u\|_3+\sum_{j=[r/2]+1}^{r}\binom{r}{j}\max|D_t^j u|\,\|D_t^{r-j}\nabla u\|_2\right\}\\
&\leq C\|D_t^{r+1}u\|_2\left\{\sum_{j=0}^{[r/2]}\|D_t^j\nabla u\|_2\|D_t^{r-j}\nabla u\|_2^{1/2}\|D_t^{r-j}\tilde\Delta u\|_2^{1/2}\right. \qquad (7.18)\\
&\quad\left.+\sum_{j=[r/2]+1}^{r}\|D_t^j\nabla u\|_2^{1/2}\|D_t^j\tilde\Delta u\|_2^{1/2}\|D_t^{r-j}\nabla u\|_2\right\}\\
&\leq \frac{1}{2}(2-\gamma)\|D_t^{r+1}u\|_2^2+C\sum_{j=0}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2\|D_t^{r-j}\tilde\Delta u\|_2.
\end{aligned}$$

Cancelling the $\|D_t^{r+1}u\|_2^2$ term on the right, we find after multiplying by 2 that

$$\begin{aligned}
\nu D_t\|D_t^r\nabla u\|_2^2+\gamma\|D_t^{r+1}u\|_2^2 &\leq C\sum_{j=0}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2\|D_t^{r-j}\tilde\Delta u\|_2 \qquad (7.19)\\
\leq \frac{\nu}{2}(2-\gamma)\|D_t^r\tilde\Delta u\|_2^2 &+C(\gamma)\left\{\|D_t^r\nabla u\|_2^2\|\nabla u\|_2^4+C\sum_{j=1}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2\|D_t^{r-j}\tilde\Delta u\|_2\right\}.
\end{aligned}$$

Adding the resulting forms of the two preceding inequalities, and cancelling terms on the right containing $\|D_t^r\tilde\Delta u\|_2^2$, we obtain

$$\begin{aligned}
&(1+\nu)D_t\|D_t^r\nabla u\|_2^2+\gamma(\nu\|D_t^r\tilde\Delta u\|_2^2+\|D_t^{r+1}u\|_2^2)\\
&\quad\leq C(\gamma)\left\{\|D_t^r\nabla u\|_2^2\|\nabla u\|_2^4+\sum_{j=1}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2\|D_t^{r-j}\tilde\Delta u\|_2\right\}. \qquad (4r+3)
\end{aligned}$$

8. Deduction of the estimates for $\|D_t^r\nabla u\|_2$, $\|D_t^r\tilde\Delta u\|_2$ and $\|D_t^{r+1}u\|_2$

By putting the preceding inequalities into a form to which Lemma 3 applies we can deduce in succession, and by induction on r, the requisite estimates for u, ∇u and $\tilde\Delta u$ and their time derivatives of order r. Further combinations and calculations are required for this.

The estimate $\|\nabla u\|_2\in L^2(0,\infty)$ has been made from 1 in Section 7. In view of 3(a), $F_3(t)=\|\nabla u\|_2^2\in L^1(0,T)$ satisfies Lemma 3 with $a=2$ and $G_3(t)=\nu\|\tilde\Delta u\|_2^2$. Hence $G_3(t)$ has the same integrability as $F_3(t)^3$, that is, $G_3(t)\in L^{1/3}(0,T)$ so that $\|\tilde\Delta u\|_2\in L^{2/3}(0,T)$.

But this method will not work directly for 3(b) which must be combined with 3(a) to gain the result for $\|u_t\|_2$. Adding 3(a) and 3(b) we have

$$(1+\nu)\frac{d}{dt}\|\nabla u\|_2^2+\nu\|\tilde{\Delta} u\|_2^2+\|u_t\|_2^2\leq C(\|\nabla u\|_2^6+\|\nabla u\|_2^3\,\|\tilde{\Delta} u\|_2). \tag{3}$$

By Young's inequality again,

$$C\|\nabla u\|_2^3\,\|\tilde{\Delta} u\|_2\leq\frac{\nu}{2}\|\tilde{\Delta} u\|_2^2+K\|\nabla u\|_2^6$$

so we find

$$(1+\nu)\frac{d}{dt}\|\nabla u\|_2^2+\frac{\nu}{2}\|\tilde{\Delta} u\|_2^2+\|u_t\|_2^2\leq C_1\|\nabla u\|_2^6. \tag{8.1}$$

With $F_3(t)=(1+\nu)\|\nabla u\|_2^2$ and $G_3(t)=\frac{1}{2}\nu\|\tilde{\Delta} u\|_2^2+\|u_t\|_2^2$ the lemma applies with $p=1$ and $a=2$ so we conclude $G_3(t)\in L^{1/3}(0,T)$. Hence $\|\tilde{\Delta} u\|_2$ and $\|u_t\|_2$ are both in $L^{2/3}(0,T)$.

To estimate $\|\nabla u_t\|_2$, we start with 5(b) which however also has a $\|\nabla u\|_2^{10}$ term on the right side. We therefore add 3(a) multiplied by $3\|\nabla u\|_2^4$ which raises its singular index from 3 to 5 and makes possible the introduction of $F_5(t)=\|u_t\|^2+\|\nabla u\|_2^6$. We obtain using (2.14)

$$\begin{aligned}\frac{d}{dt}[\|u_t\|_2^2+\|\nabla u\|_2^6]+\nu[\|\nabla u_t\|_2^2+3\|\nabla u\|_2^4\,\|\tilde{\Delta} u\|_2^2]&\leq C[\|u_t\|_2^{10/3}+\|\nabla u\|_2^{10}]\\&\leq C[\|u_t\|_2^2+\|\nabla u\|_2^6]^{5/3}.\end{aligned} \tag{8.2}$$

Now Lemma 3 applies with $p=\frac{1}{3}$, $a=\frac{4}{3}$, and $G_5(t)=\|\nabla u_t\|_2^2+3\|\nabla u\|_2^4\,\|\tilde{\Delta} u\|_2^2$. It follows that $G_5(t)\in L^{1/5}(0,T)$ and therefore that $\|\nabla u_t\|_2\in L^{2/5}(0,T)$.

At the next level $\|\tilde{\Delta} u_t\|_2$ and $\|u_{tt}\|_2$ are to be estimated, and we must combine 7(a), 7(b), and 3(a) (or (8.1)) multiplied by $5\|\nabla u\|_2^8$ thus forming an inequality homogeneous of singular index 7:

$$\begin{aligned}&\frac{d}{dt}[(1+\nu)\,\|\nabla u_t\|_2^2+\|\nabla u\|_2^{10}]+\frac{3}{2}\nu\|\tilde{\Delta} u_t\|_2^2+\|u_{tt}\|_2^2+5\nu\|\nabla u\|_2^8\,\|\tilde{\Delta} u\|_2^2\\&\leq C[\|\nabla u_t\|_2^2\,\|\nabla u\|_2^4+\|\nabla u_t\|_2\,\|\tilde{\Delta} u_t\|_2\,\|\nabla u\|_2^2+\|\nabla u\|_2^{14}]\\&\leq C[\|\nabla u_t\|_2^{14/5}+\|\nabla u\|_2^{14}]+\varepsilon\nu\|\tilde{\Delta} u_t\|_2^2.\end{aligned} \tag{8.3}$$

Choosing $\varepsilon=\frac{1}{2}$ and canceling the term in $\|\tilde{\Delta} u_t\|_2^2$ on the right side against part of the corresponding term on the left, we find

$$\begin{aligned}\frac{d}{dt}[(1+\nu)\|\nabla u_t\|_2^2+\|\nabla u\|_2^{10}]+\nu\|\tilde{\Delta}u_t\|_2^2+\|u_{tt}\|_2^2+5\nu\|\nabla u\|_2^8\|\tilde{\Delta}u\|_2^2 \\ \leq C[\|\nabla u_t\|_2^{14/5}+\|\nabla u\|_2^{14}] \\ \leq C[(1+\nu)\|\nabla u_t\|_2^2+\|\nabla u\|_2^{10}]^{7/5}.\end{aligned} \tag{8.4}$$

With

$$F_7(t)=(1+\nu)\|\nabla u_t\|_2^2+\|\nabla u\|_2^{10}\in L^{1/5}(0,T)$$

and

$$G_7(t)=\nu\|\tilde{\Delta}u_t\|_2^2+\|u_{tt}\|_2^2+5\nu\|\nabla u\|_2^8\|\tilde{\Delta}u\|_2^2,$$

Lemma 3 applies with $p=\frac{1}{5}$, $a=\frac{6}{5}$ and we conclude that $G_7(t)\in L^{1/7}(0,T)$. Hence $\|\tilde{\Delta}u_t\|_2$ and $\|u_{tt}\|_2$ are both in $L^{2/7}(0,T)$ as required.

This completes the first two stages of an induction on r; we have shown the result for $\|\nabla u\|_2$, $\|\tilde{\Delta}u\|_2$ and $\|u_t\|_2$ at the initial stage and $\|\nabla u_t\|_2$, $\|\tilde{\Delta}u_t\|_2$ and $\|u_{tt}\|_2$ at the stage with $r=1$. We now assume the result holds for $r-1$, that is for $\|D_t^j\nabla u\|_2$, $\|D_t^j\tilde{\Delta}u\|_2$ and $\|D_t^{j+1}u\|_2$ for $j\leq r-1$, and prove that the result holds for r, that is, for $\|D_t^r\nabla u\|_2$, $\|D_t^r\tilde{\Delta}u\|$ and $\|D_t^{r+1}u\|_2$.

To establish the result for $\|D_t^r\nabla u\|_2$ we must rely primarily on estimate $(4r+1)$ but the presence of various terms on the right hand side means that multiples of lower order estimates must be combined with this one. By building up the necessary combinations recursively it is possible to keep the number of such operations at each stage to a maximum of two. The ensuing definitions for $F_{4r+1}(t)$ and $G_{4r+1}(t)$ also involve $F_{4r-1}(t)$ and $G_{4r-1}(t)$ for which the L^p classes are already established by the earlier stages of the induction.

Thus, for $r=2,3,\ldots$, let

$$F_{4r+1}(t)=\|D_t^r u\|_2^2+\frac{4r-7}{4r-1}F_{4r-1}(t)^{\frac{4r-1}{4r-3}}+F_{4r-3}(t)^{\frac{4r-1}{4r-5}} \tag{8.5}$$

$$G_{4r+1}(t)=\nu\|D_t^r\nabla u\|_2^2+F_{4r-1}^{\frac{2}{4r-3}}(t)\,G_{4r-1}(t)+F_{4r-3}^{\frac{4}{4r-5}}(t)\,G_{4r-3}(t) \tag{8.6}$$

and let

$$F_{4r+3}(t)=(1+\nu)\|D_t^r\nabla u\|_2^2+F_{4r-1}(t)^{\frac{4r+1}{4r-3}} \tag{8.7}$$

$$G_{4r+3}(t)=\nu\|D_t^r\tilde{\Delta}u\|_2^2+\|D_t^{r+1}u\|_2^2+\frac{4r+1}{4r-3}F_{4r-1}^{\frac{4}{4r-3}}(t)\,G_{4r-1}(t). \tag{8.8}$$

Here, initially,

$$F_3(t) = \|\nabla u\|_2^2, \quad G_3(t) = \nu\|\tilde{\Delta} u\|_2^2+\|u_t\|_2^2$$

$$F_5(t) = \|u_t\|_2^2+\|\nabla u\|_6^6 = \|u_t\|_2^2+F_3^3(t)$$

$$G_5(t) = \nu\|\nabla u_t\|_2^2+3\nu\|\nabla u\|_2^4\|\tilde{\Delta} u\|_2^2 \quad \text{or} \quad \nu\|\nabla u_t\|_2^2+3F_3^2(t)\,G_3(t)$$

$$F_7(t) = (1+\nu)\,\|\nabla u_t\|_2^2+\|\nabla u\|_2^{10} = (1+\nu)\,\|\nabla u_t\|_2^2+F_3^5(t)$$

$$G_7(t) = \nu\|\tilde{\Delta} u_t\|_2^2+\|u_{tt}\|_2^2+5\|\nabla u\|_2^8(\nu\|\tilde{\Delta} u\|_2^2+\|u_t\|_2^2) = \nu\|\tilde{\Delta} u_t\|_2^2+\|u_{tt}\|_2^2+5F_3^4\,G_3.$$

The index of $F_q(t)$ is $q-2$ and of $G_q(t)$ is q, for every odd positive integer q, while all $F_q(t)$ and $G_q(t)$ are positive quantities, unless $\|\nabla u\|_2=0$ in which case $u\equiv 0$.

The basic inequality for the previous stage of the induction is

$$F'_{4r-3}(t)+G_{4r-3}(t) \leq C\,F_{4r-3}(t)^{\frac{4r-3}{4r-5}}. \tag{8.9}$$

Assuming as the induction hypothesis that (8.9) holds, we multiply it by

$$\frac{4r-1}{4r-5}F_{4r-3}^{\frac{4}{4r-5}}(t)$$

and add to $(4r+1)$. Likewise, we multiply the second inequality (assumed by hypothesis as established at the second part of the preceding stage of induction on r):

$$F'_{4r-1}(t)+\frac{4r-3}{4r-7}G_{4r-1}(t) \leq C\,F_{4r-1}(t)^{\frac{4r-1}{4r-3}} \tag{8.10}$$

by

$$\frac{4r-7}{4r-3}F_{4r-1}(t)^{\frac{2}{4r-3}}$$

and add to the preceding relation. Thus we obtain

$$\begin{aligned} &D_t\left[\|D_t^r u\|_2^2+\frac{4r-7}{4r-1}F_{4r-1}(t)^{\frac{4r-1}{4r-3}}+F_{4r-3}(t)^{\frac{4r-1}{4r-5}}\right] \\ &\quad +\nu\|D_t^r\nabla u\|_2^2+F_{4r-1}(t)^{\frac{2}{4r-3}}G_{4r-1}(t)+\frac{4r-1}{4r-5}F_{4r-3}^{\frac{4}{4r-5}}(t)\,G_{4r-3}(t) \\ &\leq C\left[\sum_{j=1}^{r}\|D_t^j u\|_2^{\frac{8r+2}{4j-1}}+\sum_{j=0}^{r-1}\|D_t^j\nabla u\|_2^{\frac{8r+2}{4j+1}}+F_{4r-3}(t)^{\frac{4r+1}{4r-5}}+F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right]. \end{aligned} \tag{8.11}$$

However

$$\begin{aligned}\|D_t^j u\|_2^{\frac{8r+2}{4j-1}} &\leqslant F_{4j+1}(t)^{\frac{4r+1}{4j-1}}\\ &\leqslant F_{4j+5}(t)^{\frac{4r+1}{4j+3}}\\ &\leqslant F_{4j+9}(t)^{\frac{4r+1}{4j+7}}\\ &\;\vdots\\ &\leqslant F_{4r-3}(t)^{\frac{4r+1}{4r-5}}, \quad j=1,\dots,r-1.\end{aligned} \tag{8.12}$$

$$\begin{aligned}\|D_t^j \nabla u\|_2^{\frac{8r+2}{4j+1}} &\leqslant F_{4j+3}(t)^{\frac{4r+1}{4j+1}}\\ &\leqslant F_{4j+7}(t)^{\frac{4r+1}{4j+5}}\\ &\leqslant F_{4j+11}(t)^{\frac{4r+1}{4j+9}}\\ &\;\vdots\\ &\leqslant F_{4r-1}(t)^{\frac{4r+1}{4r-3}}, \quad j=0,\dots,r-1.\end{aligned} \tag{8.13}$$

Hence by (2.14) the right hand side of our inequality (8.11) is bounded by

$$\begin{aligned}&C\left[\|D_t^r u\|_2^{\frac{8r+2}{4r-1}}+F_{4r-3}(t)^{\frac{4r+1}{4r-5}}+\frac{4r-7}{4r-1}F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right]\\ &\quad\leqslant C\left[\|D_t^r u\|_2^2+F_{4r-3}(t)^{\frac{4r-1}{4r-5}}+\frac{4r-7}{4r-1}F_{4r-1}(t)^{\frac{4r-1}{4r-3}}\right]^{\frac{4r+1}{4r-1}}\\ &\quad= C\,F_{4r+1}(t)^{\frac{4r+1}{4r-1}}.\end{aligned} \tag{8.14}$$

This establishes the basic inequality (8.9) for $F_{4r+1}(t)$ in the form

$$F'_{4r+1}(t)+G_{4r+1}(t)\leqslant C\,F_{4r+1}(t)^{\frac{4r+1}{4r-1}}. \tag{8.15}$$

In Lemma 3, assuming as induction hypothesis $F_{4r+1}(t)\in L^{1/(4r-1)}(0,T)$, we have $p=1/(4r-1)$, $a+p=(4r+1)/(4r-1)=1+2/(4r-1)$ so that $a=1+1/(4r-1)$. Hence $G_{4r+1}(t)\in L^{1/(4r+1)}(0,T)$ by Lemma 3. It follows that $\|D_t^r\nabla u\|_2\in L^{2/(4r+1)}(0,T)$.

For the second stage of the induction proof, we therefore have as hypotheses the integrability results up to $\|D_t^{r-1}\tilde{\Delta}u\|_2$, $\|D_t^r u\|_2$ and $\|D_t^r\nabla u\|_2$ and wish to establish the result for $\|D_t^r\tilde{\Delta}u\|_2$ and $\|D_t^{r+1}u\|_2$.

The basic inequality for the previous stage of the induction is taken to be

$$F'_{4r-1}(t)+\frac{4r-3}{4r-7}G_{4r-1}(t)\leq C\,F_{4r-1}(t)^{\frac{4r-1}{4r-3}} \tag{8.16}$$

for $r\geq 2$.

Assuming (8.16) as a part of the induction hypothesis, multiply it by

$$\frac{4r+1}{4r-3}F_{4r-1}(t)^{\frac{4}{4r-3}}$$

and add to $(4r+3)$, wherein also γ is set equal to $(4r+1)/(4r-3)$. Then

$$\begin{aligned}
&D_t\left[(1+\nu)\|D_t^r\nabla u\|_2^2+F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right]\\
&\quad+\frac{4r+1}{4r-3}\{\nu\|D_t^r\tilde{\Delta}u\|_2^2+\|D_t^{r+1}u\|_2^2\}+\frac{4r+1}{4r-7}F_{4r-1}(t)^{\frac{4}{4r-3}}G_{4r-1}(t)\\
&\quad\leq C\left[\|D_t^r\nabla u\|_2^2\|\nabla u\|_2^4+\sum_{j=1}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2\|D_t^{r-j}\tilde{\Delta}u\|_2+F_{4r-1}(t)^{\frac{4r+3}{4r-3}}\right].
\end{aligned} \tag{8.17}$$

To apply Young's inequality to the product terms on the right hand side, particularly those containing $\|D_t^r\tilde{\Delta}u\|_2$, we write the right hand side of (8.17) in the form

$$\begin{aligned}
&\leq C\left[\|D_t^r\nabla u\|_2^{\frac{8r+6}{4r+1}}+\|\nabla u\|_2^{8r+6}+F_{4r-1}(t)^{\frac{4r+3}{4r-3}}\right.\\
&\qquad\left.+\sum_{j=1}^{[r/2]}\|D_t^j\nabla u\|_2^2\|D_t^{r-j}\nabla u\|_2^{\frac{4r-8j+1}{4(r-j)+1}}\|D_t^{r-j}\nabla u\|_2^{\frac{4j}{4(r-j)+1}}\|D_t^{r-j}\tilde{\Delta}u\|_2\right]\\
&\leq C\left[\|D_t^r\nabla u\|_2^{\frac{8r+6}{4r+1}}+\|\nabla u\|_2^{8r+6}+F_{4r-1}(t)^{\frac{4r+3}{4r-3}}\right.\\
&\qquad\left.+\sum_{j=1}^{[r/2]}(\|D_t^j\nabla u\|_2^{\frac{8r+6}{4j+1}}+\|D_t^{r-j}\nabla u\|_2^{\frac{8r+6}{4(r-j)+1}})\right]\\
&\qquad+\varepsilon\nu\sum_{j=1}^{[r/2]}\|D_t^{r-j}\nabla u\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde{\Delta}u\|_2^2
\end{aligned} \tag{8.18}$$

where ε is a positive number not exceeding $1/4r^3$. We also note that $2j\leq r$ in the sums above so all exponents used are positive.

Again, by the definition given in (8.7),

$$\begin{aligned}
\|D_t^j \nabla u\|_2^{\frac{8r+6}{4j+1}} &\leq F_{4j+3}(t)^{\frac{4r+3}{4j+1}} \\
&\leq F_{4j+7}(t)^{\frac{4r+3}{4j+5}} \\
&\leq F_{4j+11}(t)^{\frac{4r+3}{4j+9}} \\
&\;\vdots \\
&\leq F_{4r-1}(t)^{\frac{4r+3}{4r-3}}, \quad j=0,\dots,r-1.
\end{aligned} \tag{8.19}$$

Also, using the definition of $G_{4r+3}(t)$ successively with r replaced by $r-j+1$, $r-j+2,\dots,r-1$ and r, we find by (8.7) and (8.13)

$$\begin{aligned}
\nu\|D_t^{r-j}\nabla u\|_2^{\frac{8j}{4(r-j)+1}}\|D_t^{r-j}\tilde{\Delta} u\|_2^2 &\leq \|D_t^{r-j}\nabla u\|_2^{\frac{8(j-1)}{4(r-j)+1}} F_{4(r-j)+3}(t)^{\frac{4}{4(r-j)+1}} G_{4(r-j)+3}(t) \\
&\leq \|D_t^{r-j}\nabla u\|_2^{\frac{8(j-2)}{4(r-j)+1}} F_{4(r-j)+7}(t)^{\frac{4}{4(r-j)+5}} G_{4(r-j)+7}(t) \\
&\leq \|D_t^{r-j}\nabla u\|_2^{\frac{8(j-3)}{4(r-j)+1}} F_{4(r-j)+11}(t)^{\frac{4}{4(r-j)+9}} G_{4(r-j)+11}(t) \\
&\;\vdots \\
&\leq \|D_t^{r-j}\nabla u\|_2^{\frac{8}{4(r-j)+1}} F_{4r-5}^{\frac{4}{4r-7}}(t)\cdot G_{4r-5}(t) \\
&\leq F_{4r-1}(t)^{\frac{4}{4r-3}} G_{4r-1}(t).
\end{aligned} \tag{8.20}$$

The terms containing $\|D_t^{r-j}\tilde{\Delta} u\|_2$ on the right hand side of the inequality (8.18) can now be replaced by small multiples of $F_{4r-1}(t)^{4/(4r-3)}G_{4r-1}(t)$ and thus cancelled against a small part of the corresponding term on the left side in (8.17). Assuming that the sum of all such multiples for $j=1,\dots,[r/2]$ does not exceed

$$\left(\frac{4}{4r-3}\right)^2\left(\frac{4r+1}{4r-7}\right)=\frac{4r+1}{4r-7}-\left(\frac{4r+1}{4r-3}\right)^2$$

we find

$$\begin{aligned}
&D_t\left[(1+\nu)\|D_t^r\nabla u\|_2^2+F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right] \\
&\quad+\frac{4r+1}{4r-3}\left\{\nu\|D_t^r\tilde{\Delta} u\|_2^2+\|D_t^{r+1}u\|_2^2+\frac{4r+1}{4r-3}F_{4r-1}^{\frac{4}{4r-3}}(t)\,G_{4r-1}(t)\right\}
\end{aligned} \tag{8.21}$$

$$\leqslant C\left[\|D_t^r \nabla u\|_2^{\frac{8r+6}{4r+1}} + F_{4r-1}(t)^{\frac{4r+3}{4r-3}}\right]$$
$$\leqslant C\left[(1+\nu)\,\|D_t^r \nabla u\|_2^2 + F_{4r-1}(t)^{\frac{4r+1}{4r-3}}\right]^{\frac{4r+3}{4r+1}}.$$

This establishes the basic inequality for the next induction step in the form

$$F'_{4r+3}(t) + \frac{4r+1}{4r-3} G_{4r+3}(t) \leqslant C F_{4r+3}(t)^{\frac{4r+3}{4r+1}}. \tag{8.22}$$

Since $F_{4r+3}(t) \in L^{1/(4r+1)}(0, T)$ by the induction hypothesis, we have $p = 1/(4r+1)$ and $a + p = 1 + 2/(4r+1)$ in Lemma 3. Hence $a = 1 + 1/(4r+1) > 1$ and it now follows that $G_{4r+3}(t) \in L^{1/(4r+3)}(0, T)$. From this we now easily obtain both

$$\|D_t^r \tilde{\Delta} u\|_2 \quad \text{and} \quad \|D_t^{r+1} u\|_2 \in L^{\frac{2}{4r+3}}(0, T).$$

This completes the induction proof for $\|D_t^r \nabla u\|_2$, $\|D_t^r \tilde{\Delta} u\|_2$, $\|D_t^{r+1} u\|_2$ for $r = 0, 1, 2, \ldots$.

9. Deduction of the estimates for $s = 0, 1, 2$

The results of the preceding section show directly that for $D_t^r u_i$ and $\nabla D_t^r u_i$ the estimates (4.1) stated in the theorem are established. For the second order space derivatives we must also use Lemma 1 and its analogue for higher time derivatives.

Differentiating (5.12) r times with respect to time t, which is a parameter in the above representation, we obtain

$$D_t^r f = \int_{\partial\Omega} K_{,\alpha} \frac{\partial D_t^r u_\alpha}{\partial n} \, dS. \tag{9.1}$$

Repeating the calculations of Lemma 1, we find

$$\|D_t^r \nabla f\|_2^2 \leqslant C \|D_t^r \nabla u\|_2 (\|D_t^r \tilde{\Delta} u\|_2 + C_1 \|D_t^r \nabla u\|_2). \tag{9.2}$$

The first factor on the right lies in $L^{2/(4r+1)}(0, T)$ and the second in $L^{2/(4r+3)}(0, T)$.

LEMMA 1r. *$\|D_t^r \nabla f\|_2$ lies in $L^{1/(2r+1)}(0, T)$.*

By the estimate of Ladyzhenskaya [19, p. 21], we have

$$\begin{aligned} \|D_t^r D_j D_k u\|_2^2 &\leqslant C(\|D_t^r \Delta u\|_2^2 + \|D_t^r \nabla u\|_2^2) \quad (D_j = \partial/\partial x_j,\ j = 1, 2, 3) \\ &\leqslant C(\|D_t^r \tilde{\Delta} u\|_2^2 + \|D_t^r \nabla f\|_2^2 + \|D_t^r \nabla u\|_2^2) \end{aligned} \tag{9.3}$$

and the right side expression lies in $L^{1/(4r+3)}(0,T)$. This establishes the result (4.1) of the theorem for the second order space derivatives and their time derivatives of every order: now (4.1) of the theorem is established for $s=0,1,2$ and $r=0,1,2,\ldots$.

From (2.20) now follow

$$\|D_t^r \nabla u\|_6 \leq C\|D_t^r \tilde{\Delta} u\|_2 \in L^{\frac{2}{4r+3}}(0,T)$$

and

$$\|D_t^r u\|_6 \leq C\|D_t^r \nabla u\|_2 \in L^{\frac{2}{4r+1}}(0,T).$$

Hence by (2.16), $\max|D_t^r u| \in L^{1/(2r+1)}(0,T)$. This establishes the result (4.2) of the theorem for $s=0$.

Chapter III. Estimates for tangential derivatives

10. Tangential coordinate systems

To estimate higher order space derivatives, we shall work with tangential derivatives that vanish on the boundary. Thus we introduce tangential coordinate systems in which the equation of the boundary has the form $x_n \equiv x_3 = 0$. We shall work locally and do not enter explicitly into any aspects of the theory of integration on manifolds that may be required to define the integrals used.

To construct a tangential coordinate system, choose an interior point $P_0 \in \Omega - \partial\Omega$ and an ε-sphere $S(P_0,\varepsilon) \subset \Omega - \partial\Omega$ with centre P_0. Let $w(P)$ be the harmonic function with boundary values zero on $\partial\Omega$ and unity on $\partial S(P_0,\varepsilon)$. Then the level surfaces $w=$constant and the sphere $r=$constant $\leq \varepsilon$ where $r=r(P,P_0)=|P-P_0|$ form a family of surfaces filling Ω. By a smooth deformation in an outer neighbourhood of $r=\varepsilon$ we can ensure that the modified family of surfaces is C^∞ embedded in $\mathbf{R}^3$ throughout the interior of Ω. Now let $x_3=w$ in $\Omega - S(P_0,\varepsilon)$ and $x_3 = 1+\varepsilon-r$ in $S(P_0,\varepsilon)$ so that the new family of surfaces become coordinate surfaces $x_3=$constant.

On each surface $x_3=c$ let isothermic coordinates x_1 and x_2 be introduced, with line element

$$ds^2 = h_1^2(dx_1^2+dx_2^2)+h_3^2\,dx_3^2 \tag{10.1}$$

where h_1 and h_3 are smooth functions. Within $S(P_0,\varepsilon)$, we choose stereographic coordinates $x_1=\xi_1$, $x_2=\xi_2$, $x_3=r$ where [38, p. 6]

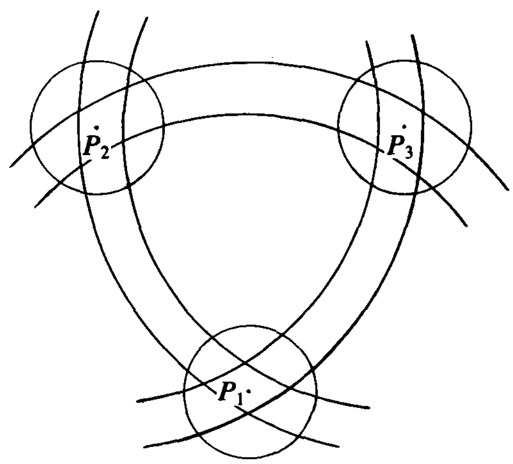

Fig. 2

$$ds^2 = \frac{4r^2}{(1+\xi_1^2+\xi_2^2)^2}(d\xi_1^2+d\xi_2^2)+dr^2. \tag{10.2}$$

By considering two hemispheres we may restrict to bounded values of ξ_1 and ξ_2 but this is not strictly necessary.

In curvilinear coordinates the gradient operator has components $\nabla_i = h_i^{-1} D_i$, where $D_i \equiv \partial/\partial x_i$; thus [24, p. 32]

$$(\nabla u)^2 = \sum_{i=1}^{3} (\nabla_i u)^2 = \sum_{i=1}^{3} \frac{1}{h_i^2} u_{x_i}^2 \tag{10.3}$$

In particular, within the sphere $S(P_0, \varepsilon)$

$$\nabla_\alpha u = \frac{1+\xi_1^2+\xi_2^2}{2r} D_\alpha u. \tag{10.4}$$

For our isothermal coordinates $h_2 = h_1$ and these factors are non-zero except at $r=0$. Thus $\nabla_i u$ and $D_i u$ are locally equivalent, in the sense of integral norms, except at $P_0: r=0$. In the sequel we shall estimate norms of $D_\alpha u$, $\alpha=1,2$, and higher derivatives $D_\alpha^s u$, $s=1,2,3,\ldots$. By our next two lemmas it will be shown that $\nabla_\alpha^s u$ is majorized by a sum of $D_\alpha^s u$ norms based on three tangential coordinate systems with distinct poles P_1, P_2, P_3, situated at the vertices of an equilateral triangle as shown in Figure 2, where all tangential coordinate surfaces shown can be chosen as three families of concentric spheres.

LEMMA 4. *There is a constant K depending only on P_1, P_2, P_3, ε and Ω such that*

$$\|\nabla_\alpha^{(1)}u\|_2^2 \leq K\sum_{j=1}^{3}\|D_\alpha^{(j)}u\|_2^2. \tag{10.5}$$

Here the parenthesized superscripts refer to one of the three selected tangential coordinate systems. A similar result holds for $\nabla_\alpha^{(i)}u, i=1,2,3$.

Proof. We have

$$\begin{aligned}\|\nabla_\alpha^{(1)}u\|_2^2 &= \int_{\Omega-S(P_1,\varepsilon)}|\nabla_\alpha^{(i)}u|^2\,dx+\int_{S(P_1,\varepsilon)}|\nabla_\alpha^{(i)}u|^2\,dx\\ &\leq K_1\int_{\Omega-S(P_1,\varepsilon)}|D_\alpha^{(i)}u|^2\,dx+\int_{S(P_1,\varepsilon)}|\nabla u|^2\,dx\end{aligned} \tag{10.6}$$

where K_1 is determined by the minima of the h_i outside $S(P_1,\varepsilon)$, and $|\nabla u|^2$ is the Euclidean squared gradient.

Since the tangential coordinate systems centred on P_2 and P_3 create a nonsingular three dimensional coordinate system in $S(P_1,\varepsilon)$ with one redundant coordinate among the four available tangential coordinates, we may express the Cartesian derivatives of u as smooth bounded linear combinations of the available tangential derivatives in these two coordinate systems:

$$\frac{\partial u}{\partial x_i}=\lambda_i\frac{\partial u}{\partial x_{i_1}^{(1)}}+\mu_i\frac{\partial u}{\partial x_{i_2}^{(1)}}+\nu_i\frac{\partial u}{\partial x_{i_1}^{(2)}} \tag{10.7}$$

and hence we may write

$$(\nabla u)^2 \leq K_2((D_\alpha^{(2)}u)^2+(D_\alpha^{(3)}u)^2). \tag{10.8}$$

Integrating this inequality over $S(P_1,\varepsilon)$ we finally obtain

$$\|\nabla_\alpha^{(1)}u\|_2^2 \leq K_1\|D_\alpha^{(1)}u\|_2^2+K_2(\|D_\alpha^{(2)}u\|_2^2+\|D_\alpha^{(3)}u\|_2^2) \tag{10.9}$$

whence the lemma follows with $K=\max(K_1,K_2)$.

When Ω is not simply connected, a tangential coordinate system will have other singular points. The above reasoning can still be applied to such systems if it is assumed that, of the three tangential coordinate systems employed, no point is singular for more than one, while at such a point the surfaces x_3=constant for the remaining two systems are not tangent. Assuming finite connectivity and smoothness, such conditions can be satisfied through C^∞ deformations.

For higher tangential derivatives we shall use a tangential multi-index j, so that

$$\|D_\alpha^j u\|_2^2 = \sum_{j_1+j_2=j} \|D_1^{j_1} D_2^{j_2} u\|_2^2 \tag{10.10}$$

and similarly for the higher gradients.

LEMMA 5. *There is a constant K_s depending only on Ω, P_1, P_2, P_3, ε amd s such that*

$$\|\nabla_\alpha^{(1)s} u\|_2^2 \leq K_s \sum_{\substack{1\leq j\leq s \\ 1\leq i\leq 3}} \|D_\alpha^{(i)j} u\|_2^2. \tag{10.11}$$

We shall demonstrate the result by induction on $s=1,2,3,\ldots$, and at each stage will also show that a similar inequality holds for the norms of all the mixed gradient-derivative quantities $\nabla_\alpha D_\beta \ldots u$ of s factors, any number of them being ∇ or D in any order.

For $s=2$ we may suppose $D_\alpha u$ and $D_\alpha D_\beta u$ given of finite norm. By Lemma 4, $\nabla_\alpha D_\beta u$ and $\nabla_\alpha u$ have finite norm. Now

$$D_\alpha \nabla_\beta u = \begin{cases} \nabla_\beta D_\alpha u - \dfrac{h_{1\alpha}}{h_1^2} D_\beta u & \text{in} \quad \Omega - S(P_1, \varepsilon) \\ \nabla_\beta D_\alpha u + \dfrac{2\xi_\alpha}{1+\xi_1^2+\xi_2^2} \nabla_\beta u & \text{in} \quad S(P_1, \varepsilon) \end{cases} \tag{10.12}$$

The first of these clearly has finite norm while the second is bounded by the norms of $\nabla_\beta D_\alpha u$ and $\nabla_\beta u$. Finally, boundedness for $\nabla_\alpha \nabla_\beta u$ now follows from that of $D_\alpha \nabla_\beta u$ by Lemma 4.

Proceeding by induction we suppose that the lemma and all boundedness results for mixed products hold for $j=s-1$. As additional hypothesis at the level s we have the boundedness of $D_\alpha D_\beta \ldots D_\lambda u$ with s D factors. By Lemma 4 we find $\nabla_\alpha D_\beta \ldots D_\lambda u$ is also bounded. Now in $S(P_1, \varepsilon)$,

$$D_\beta \nabla_\alpha \ldots D_\lambda u = \nabla_\alpha D_\beta \ldots D_\lambda u + \frac{\xi_\beta}{r} D_\alpha \ldots D_\lambda u;$$

the first term is bounded and the second term with $s-1$ D factors is majorized by $\nabla_\alpha D_\gamma \ldots D_\lambda u$ which having also $s-1$ factors is bounded by hypothesis. We continue to move the ∇_α factor to the right by permuting it with successive D factors. At a typical step we have

$$D_\beta \dots D_\gamma \nabla_\alpha D_\varkappa \dots D_\lambda u = D_\beta \dots \nabla_\alpha D_\gamma D_\varkappa \dots D_\lambda u + D_\beta \dots \frac{\xi_\gamma}{r} D_\alpha D_\varkappa \dots D_\lambda u.$$

Boundedness of the first factor has been established at the previous step while the second gives rise to

$$\frac{\xi_\gamma}{r} D_\beta \dots D_\alpha D_\varkappa \dots D_\lambda u$$

which as above is majorized by the bounded $s-1$ factor term $\nabla_\gamma D_\beta \dots D_\alpha D_\varkappa \dots D_\lambda u$, and to several further differentiated terms of the form $(1/r)\, \delta_{\beta\gamma} D_\alpha D_\varkappa \dots D_\lambda u$ with $s-2$ factors including the factor D_α. Such a term is majorized by the bounded $s-2$ factor term $\delta_{\beta\gamma} \nabla_\alpha D_\varkappa \dots D_\lambda u$. When ∇_α has been shifted to the rightmost position of a ∇ factor in our general term, we introduce another ∇_β factor replacing D_β on the left, by Lemma 4, and shift it step by step to the right as far as necessary, by the same process. Proceeding in this way we can, using only terms bounded by hypothesis or previous calculation, arrive at the conclusion of boundedness for any desired product of s ∇ or D factors in $S(P_1, \varepsilon)$. The corresponding calculation for the complementary region $\Omega - S(P_1, \varepsilon)$ is even more straightforward and will be left to the reader.

Thus we have reached the necessary conclusion for stage s of the induction: boundedness of all mixed ∇D products with s factors. In view of the given data and the induction process, which must be carried through for all components in all three tangential coordinate systems, Lemma 5 follows.

In our estimates of $D_\alpha^s u$ to follow, we shall establish that these quantities have L^2 norms that lie in $L^p(0, T)$ where p is a decreasing function of the order of derivatives. Since these estimates hold for an arbitrary choice of the pole P_1, it will follow that the tangential gradients $\nabla_\alpha^s u$ have L^2 norms in the same $L^p(0, T)$, as is desired to prove for the main theorem. That is, it will suffice from now on to consider only estimates for the $D_\alpha^s u$ to obtain those for $\nabla_\alpha^s u$.

Because P_1 is arbitrary, and in view of the proof of Lemma 4 of this section, these results actually show that locally, in the interior of Ω, all space derivatives of u can be thus estimated. Since however all tangential coordinate surfaces must become parallel as $\partial\Omega$ is approached, such a result does not yield global $L^2(\Omega)$ estimates for the normal derivatives. These will be found in Section 15 below by another method.

Certain formulas and commutation relations will be needed for the estimates in tangential coordinates. In the metric (10.1) the Laplacian operator is given by [24, p. 51] with $h_2=h_1$

$$\begin{aligned}\Delta v &= \frac{1}{h_1^2 h_3}\left\{\frac{\partial}{\partial x_1}\left(h_3\frac{\partial v}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(h_3\frac{\partial v}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1^2}{h_3}\frac{\partial v}{\partial x_3}\right)\right\}\\ &= a(x)\,(D_1^2+D_2^2)\,v+b^1(x)\,D_1\,v+b^2(x)\,D_2\,v+c(x)\,D_3^2\,v+e(x)\,D_3\,v\end{aligned}\tag{10.13}$$

the coefficients a, b_1, b_2, c and e being thus defined. We note that $c(x)=h_3^{-2}\equiv 1$ in $S(P_0,\varepsilon)$ and $c(x)\neq 0$ in Ω. Also $a(x)=h_1^{-2}=h_2^{-2}=(1+\xi_1^2+\xi_2^2)^2/4r^2$, $b^1(x)=b^2(x)=0$ and $e(x)=2r^{-1}$ in $S(P_0,\varepsilon)$. For brevity their dependence on coordinates will not be shown explicitly, and we shall denote their derivatives with respect to x_1 or x_2 with subscripts. Differentiating and using (10.13) to eliminate the term in D_3^2u, we find

$$\begin{aligned}D_1\,\Delta v &= \Delta D_1 v+\frac{c_1}{c}\,\Delta v+\left(a_1-\frac{c_1}{c}\,a\right)(D_1^2\,v+D_2^2\,v)\\ &\quad+\left(b_1^1-\frac{c_1}{c}\,b^1\right)D_1\,v+\left(b_1^2-\frac{c_1}{c}\,b^2\right)D_2\,v+\left(e_1-\frac{c_1}{c}\,e\right)D_3\,v.\end{aligned}\tag{10.14}$$

Differentiating repeatedly, eliminating D_3^2u at each stage using (10.13), and replacing terms such as $D_1\,\Delta v,\ldots,D_1^k\,\Delta v$ using (10.14) and its successors, we find a commutation relation of the form

$$\begin{aligned}D_\alpha^n\,\Delta v &= \Delta D_\alpha^n\,v+\sum_{\beta<\alpha}A_\beta^\alpha\,\Delta D_\beta^k\,v+\sum_{\beta<\alpha}B_\beta^\alpha\,D_3\,D_\beta^k\,v\\ &\quad+\sum_{\beta\leqslant\alpha}C_\beta^\alpha\,D_\beta^k\,v+\sum_{\beta<\alpha}E_\beta^\alpha\,D_\beta^k(D_1^2+D_2^2)\,v\end{aligned}\tag{10.15}$$

where the A_β^α, B_β^α, C_β^α and E_β^α are smooth coefficients but with singularities at P_0, as described below. The two component multi-index β with $\beta_1+\beta_2=k$ has non-negative components and $\beta<\alpha$ means β is subordinate to α, i.e. $\beta_1\leqslant\alpha_1$, $\beta_2\leqslant\alpha_2$ and $\beta_1+\beta_2<\alpha_1+\alpha_2=n$. Here $\beta\leqslant\alpha$ should be interpreted as equivalence of order: $\beta_1+\beta_2\leqslant\alpha_1+\alpha_2$ only. The behaviour of these coefficients in $S(P_0,\varepsilon)$ is

$$A_\beta^\alpha\equiv 0,\quad B_\beta^\alpha=O\left(\frac{1}{r}\right),\quad C_\beta^\alpha=0,\quad E_\beta^\alpha=O\left(\frac{(1+\xi_1^2+\xi_2^2)^2}{r^2}\right).\tag{10.16}$$

We shall also use a second commutation formula obtained in a similar way but replacing terms $\Delta D_1\,v,\ldots,\Delta D_\beta^k\,v$ at intermediate stages using the intermediate preceding results. This has the form

$$
\begin{aligned}
\Delta D^n_\alpha v = D^n_\alpha \Delta v + \sum_{\beta<\alpha} \bar{A}^\alpha_\beta D^k_\beta \Delta v + \sum_{\beta<\alpha} \bar{B}^\alpha_\beta D_3 D^k_\beta v \\
+ \sum_{\beta<\alpha} \bar{C}^\alpha_\beta D^k_\beta v + \sum_{\beta<\alpha} \bar{E}^\alpha_\beta D^k_\beta (D^2_1 + D^2_2)\, v
\end{aligned}
\tag{10.17}
$$

with smooth coefficients that satisfy conditions (10.16) also.

Similarly, we may establish the commutation formula

$$D^n_\alpha \nabla_i u = \sum_{|\beta|=0}^{n} G^\alpha_{i\beta} \nabla_i D^j_\beta u \tag{10.18}$$

where $G^\alpha_{in} \equiv 1$, $G^\alpha_{in-1} = nD_\alpha(\log h_i)$, ... and $G^\alpha_{i\beta}$ is bounded, smooth and independent of r in $S(P_0, \varepsilon)$. The dual formula

$$\nabla_i D^n_\alpha u = \sum_{|\beta|=0}^{n} \bar{G}^\alpha_{i\beta} D^j_\beta \nabla_i u \tag{10.19}$$

with $\bar{G}^\alpha_{in} \equiv 1$, and similar properties, shows that $D^n_\alpha u_i$ is solenoidal when u_i is solenoidal.

We conclude this section with a lemma that enables us to improve the order of estimates arising from the highest order terms in the last sum on the right in (10.15) or (10.17). If $\beta<\alpha$ and $|\beta| = k = |\alpha| - 1 = s-1$ then two equivalent cases can arise: $\alpha = \beta + (1,0)$ or $\alpha = \beta + (0,1)$. In the lemma, we choose the first case for definiteness; the second case is exactly similar.

LEMMA 6. *Let E be a smooth function on $\Omega - P_0$ satisfying the fourth condition of (10.16) and let $\alpha = \gamma + (1,0)$. Then there is a constant C independent of v such that*

$$\left| \int_\Omega D^s_\alpha v \cdot E \cdot D^k_\gamma (D^2_1 + D^2_2)\, v \, dx \right| \leq C \|\nabla D^{s-1}_\gamma v\|^2_2. \tag{10.20}$$

Proof. Consider the portion of the given integral taken over $S(P_0, \varepsilon)$ the central sphere of the tangential coordinates defining D_α. We note that

$$dx = r^2 dr\, d\Omega = r^2 dr \frac{4 d\xi_1\, d\xi_2}{(1+\xi^2_1+\xi^2_2)^2}$$

and write

$$
\begin{aligned}
I_s &= \int_{S(P_0,\varepsilon)} D^s_\alpha v \cdot E_1 \frac{(1+\xi^2_1+\xi^2_2)^2}{4r^2} D^k_\gamma (D^2_1 + D^2_2)\, v \cdot \frac{r^2 dr\, 4 d\xi_1\, d\xi_2}{(1+\xi^2_1+\xi^2_2)^2} \\
&= \int_0^\varepsilon dr \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_1 D^s_\alpha v \cdot D^k_\gamma (D^2_1 + D^2_2)\, v \, d\xi_1\, d\xi_2.
\end{aligned}
$$

Here E_i denotes a bounded smooth function for $i=1,2,\dots$. Integrating by parts over the closed sphere surface we have

$$
\begin{aligned}
I_s &= \int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_2(D_1 D_\alpha^s v D_1 D_\gamma^k v + D_2 D_\alpha^s v D_2 D_\gamma^k v)\, d\xi_1\, d\xi_2 \\
&\quad - \int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (E_3(D_\alpha^s v D_1 D_\gamma^k v + E_4 D_\alpha^s v D_2 D_\gamma^k v)\, d\xi_1\, d\xi_2 \\
&= -\int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_2(\nabla_1 D_\alpha^s v\cdot\nabla_1 D_\gamma^k v + \nabla_2 D_\alpha^s v\cdot\nabla_2 D_\gamma^k v)\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2} \\
&\quad - \int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_3(\nabla_1 D_\alpha^{s-1} v)^2 + E_4(\nabla_1 D_\gamma^{s-1} v\cdot\nabla_2 D_\gamma^{s-1} v))\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2}
\end{aligned}
\tag{10.21}
$$

in view of (10.4) and the condition $\alpha=\gamma+(1,0)$. The second integral is clearly of the desired type and so need not be further considered. In the first integrand we use the relation

$$
D_1 \nabla_\alpha v = D_1 \frac{1+\xi_1^2+\xi_2^2}{2r} D_\alpha v = \nabla_\alpha D_1 v + \frac{\xi_1}{r} D_\alpha v
$$

where $\alpha=1$ or 2. The second term is majorized by $\nabla_\alpha v$, in view of (10.4). Writing

$$
\nabla_1 D_\alpha^s v = \nabla_1 D_1 D_\gamma^{s-1} v = D_1 \nabla_1 D_\gamma^{s-1} v - \frac{\xi_1}{r} D_1 D_\gamma^{s-1} v
$$

we find the first integral in (10.20) becomes

$$
\begin{aligned}
&-\int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_2 D_1(\nabla_1 D_\gamma^{s-1} v\cdot\nabla_1 D_\gamma^{s-1} v + D_1\nabla_2 D_\gamma^{s-1} v\cdot\nabla_2 D_\gamma^{s-1} v)\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2} \\
&+\int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_2 \frac{\xi_1}{r}(D_1(D_\gamma^{s-1} v)\cdot\nabla_1 D_\gamma^{s-1} v + D_2(D_\gamma^{s-1} v)\cdot\nabla_2 D_\gamma^{s-1} v)\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2}.
\end{aligned}
$$

Again the second integral is seen to be dominated by an integral of the desired form and can thus be dropped from further consideration. The new first term can be expressed as

$$
-\int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_2 \frac{1}{2} D_1\{(\nabla_1 D_\gamma^{s-1} v)^2 + (\nabla_2 D_\gamma^{s-1} v)^2\}\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2}.
$$

and then integrated by parts with respect to ξ_1 around the closed sphere, yielding

$$
\frac{1}{2}\int_0^\varepsilon dr \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_5\{(\nabla_1 D_\gamma^{s-1} v)^2 + (\nabla_2 D_\gamma^{s-1} v)^2\}\frac{4r^2 d\xi_1\, d\xi_2}{(1+\xi_1^2+\xi_2^2)^2}. \tag{10.22}
$$

where

$$E_5 = D_1 E_2 - \frac{4\xi_1}{1+\xi_1^2+\xi_2^2} E_2$$

is also a smooth bounded function. This last integral having the desired form, we see that

$$|I_s| \leqslant C \int_{S(P_0, \varepsilon)} (\nabla D_\gamma^{s-1} v)^2 \, dx. \tag{10.23}$$

The remaining portion of the integral over $\Omega - S(P_0, \varepsilon)$, can be treated similarly since $D_\alpha = h_\alpha \nabla_\alpha$ is bounded by a multiple of ∇_α at all ordinary points. Also the integration by parts with respect to tangential coordinates is taken over closed (or infinite) level surfaces of the normal coordinate, and so does not introduce integrated terms. Details are left to the reader. This completes the proof of Lemma 6.

11. Derivative estimates for the pressure

Since the tangential coordinate systems are curvilinear, and not in general Cartesian, the pressure term does not disappear from the estimates and inequalities for higher order tangential derivatives. We therefore set down here the necessary estimates for the pressure terms which, as in Lemma 1, will be given in terms of u derivative norms.

From (2.6) and (2.7) it follows that

$$p = b + \nu f \tag{11.1}$$

where f is the harmonic scalar viscosity potential (5.9) satisfying boundary condition (5.2). The "volume pressure" term $b(x, t)$ satisfies in view of (2.2)

$$\begin{aligned} \Delta b &= -u_{i,k} u_{k,i} = -(u_k u_{i,k})_{,i} \\ &= -(u_i u_k)_{,ik} \end{aligned} \tag{11.2}$$

with

$$\frac{\partial b}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{11.3}$$

In terms of the Neumann function we have

$$\begin{aligned} b(x,t) &= \int_\Omega N(x,y)\, u_{i,k}(y,t)\, u_{k,i}(y,t)\, dy \\ &= -\int_\Omega \frac{\partial N(x,y)}{\partial y_k} u_i(y,t)\, u_{k,i}(y,t)\, dy \\ &= \int_\Omega \frac{\partial^2 N(x,y)}{\partial y_i \partial y_k} u_i(y,t)\, u_k(y,t)\, dy. \end{aligned} \tag{11.4}$$

Since $\partial^2 N/\partial y_i \partial y_k$ has the singularity of a Calderon-Zygmund operator in $\mathbf{R}^3$, [9, 23], we may expect the singular or integrability behaviour of $b(x,t)$ to be equivalent to that of $u^2(x,t)$.

Turning first to the viscosity potential f, we have by (5.12)

$$f(x,t) = \int_{\partial\Omega} K_{,\alpha}(x,y) \frac{\partial u_\alpha(y,t)}{\partial n_y} dS_y \tag{11.5}$$

where by (5.7) and (5.14),

$$K(x,y) = \frac{1}{2\pi r'(x,y)} + k(x,y) \tag{11.6}$$

with $k(x,y)=n(x,y)-g(x,y)$ regular nonsingular on the boundary. Hence, for a tangential derivative D_β,

$$\begin{aligned} D_\beta f &= \int_{\partial\Omega} \frac{\partial^2}{\partial \xi_\beta \partial y_\alpha} \left[\frac{1}{2\pi r'(x,y)} + k(x,y) \right] \frac{\partial u_\alpha}{\partial n_y}(y,t)\, dS_y \\ &= \int_{\partial\Omega} \left[-\frac{\partial^2}{\partial y_\beta \partial y_\alpha} \left\{ \frac{1}{2\pi r'} \right\} + \frac{\partial^2 k(x,y)}{\partial \xi_\beta \partial y_\alpha} \right] \frac{\partial u_\alpha}{\partial n_y} dS_y \\ &= \int_{\partial\Omega} \left[\frac{\partial(1/2\pi r')}{\partial y_\alpha} \frac{\partial^2 u_\alpha}{\partial y_\beta \partial n_y} + \frac{\partial^2 k(x,y)}{\partial \xi_\beta \partial y_\alpha} \frac{\partial u_\alpha}{\partial n_y} \right] dS_y. \end{aligned} \tag{11.7}$$

We note that ξ_β is a tangential coordinate with $\beta=1$ or 2 and y_β the corresponding tangential y coordinate. Since $\partial\Omega$ is a closed manifold, the integration by parts brings in no integrated terms. As the singularity of $\partial(r'^{-1})/\partial y_\alpha$ is Calderon–Zygmund on $\partial\Omega$, we find

$$\|D_\beta f\|_{2,\partial} \le C\left(\left\| D_\beta \frac{\partial u_\alpha}{\partial n} \right\|_{2,\partial} + \left\| \frac{\partial u_\alpha}{\partial n} \right\|_{2,\partial} \right).$$

Iterating this calculation for $j=2,3,\ldots$ we find

$$\|D_\beta^j f\|_{2,\partial} \leq C \sum_{h=0}^{j} \left\| D_\beta^h \frac{\partial u_\alpha}{\partial n} \right\|_{2,\partial} \tag{11.8}$$

and by (5.18) a similar estimate is obtained for $\|D_\beta^j \partial f/\partial n\|_{2,\partial}$.

By Green's formula

$$\|\nabla D_\beta^s f\|_2^2 = \int_{\partial\Omega} D_\beta^s f \frac{\partial D_\beta^s f}{\partial n} dS - \int_\Omega D_\beta^s f \Delta D_\beta^s f\, dx. \tag{11.9}$$

The first of these integrals is bounded above by

$$\|D_\beta^s f\|_{2,\partial} \left\| D_\beta^s \frac{\partial f}{\partial n} \right\|_{2,\partial} \leq C \sum_{h=0}^{s} \left\| D_\beta^h \frac{\partial u_\alpha}{\partial n} \right\|_{2,\partial}^2. \tag{11.10}$$

As in the proof of Lemma 1 we may estimate the right hand side in (11.10) by the volume norm expression

$$E_s \equiv C \sum_{h=0}^{s} \|\nabla D_\beta^h u\|_2 \{\|\tilde{\Delta} D_\beta^h u\|_2 + \|\nabla D_\beta^h u\|_2\}. \tag{11.11}$$

The second integral on the right side of (11.9) can be estimated by means of the commutator expression (10.17) for $\Delta D_\beta^s f$, and we obtain

$$\begin{aligned} -\int_\Omega D_\beta^s f \Bigg(D_\beta^s \Delta f + \sum_{\gamma<\beta} \bar{A}_\gamma^\beta D_\gamma^k \Delta f + \sum_{\gamma<\beta} \bar{B}_\gamma^\beta D_3 D_\gamma^k f \\ + \sum_{\gamma\leq\beta} \bar{C}_\gamma^\beta D_\gamma^k f + \sum_{\gamma<\beta} \bar{E}_\gamma^\beta D_\gamma^k (D_1^2 + D_2^2) f \Bigg) dx. \end{aligned} \tag{11.12}$$

The first term, and the first sum, vanish in view of (5.1). A typical term of the second sum is

$$\int_\Omega D_\beta^s f \cdot \bar{B}_\gamma^\beta D_3 D_\gamma^k f\, dx = \int_\Omega D_\beta^s f \cdot \bar{\bar{B}}_\gamma^\beta \nabla_3 D_\gamma^k f\, dx$$

in view of (10.4) and (10.16), where $\bar{\bar{B}}_\gamma^B$ is bounded over Ω including $S(P_0, \varepsilon)$. The second sum in (11.12) is therefore bounded by

$$C \|D_\beta^s f\|_2 \sum_{\gamma<\beta} \|\nabla D_\gamma^k f\|_2. \tag{11.13}$$

A typical term of the third sum is

$$\int_\Omega D^s_\beta f\cdot \bar{C}^\beta_\gamma D^k_\gamma f\,dx=\int_\Omega D^s_\beta f\bar{\bar{C}}^\beta_\gamma \nabla D^{k-1}_\gamma f\,dx$$

where we may note from (10.16) that $\bar{C}^\beta_\gamma$ vanishes in $S(P_0,\varepsilon)$. Since $k-1<s$ it follows that the third sum in (11.12) is also bounded by an expression of the form (11.13).

To analyze the last sum in (11.12) we note that Lemma 6 provides a bound $C\|\nabla D^{s-1}_\gamma f\|_2^2$ for the highest order terms. A similar calculation but ending at a point corresponding to (10.21) will yield the same order of estimate for the second highest terms, and so on for lower terms, so that a bound

$$C\sum_{h=0}^{s-1}\|\nabla D^h_\gamma f\|_2^2 \tag{11.14}$$

is obtained for the whole sum. Noting $\|D^s_\beta f\|_2\leq C\|\nabla_1 D^{s-1}_\gamma f\|_2$ where $\beta=\gamma+(1,0)$, we obtain from (11.13) and (11.14) the inequality based on (11.9) in the form

$$\begin{aligned}\|\nabla D^s_\beta f\|_2^2&\leq 2E_s+C\|\nabla D^{s-1}_\beta f\|_2\sum_{h=0}^{s-1}\|\nabla D^h_\beta f\|_2\\ &\leq 2E_s+C\sum_{h=0}^{s-1}\|\nabla D^h_\beta f\|_2^2.\end{aligned}\tag{11.15}$$

Thus we have again used the property that the gradient operator dominates a first order tangential derivative operator. The case $s=0$ of this inequality (11.15) is equivalent to Lemma 1. We may substitute this result on the right hand side, and so obtain the result for $s=1$. Upon substitution successively for $s=2,3,\ldots$ the result now follows by induction on s.

LEMMA 7. *For* $s=0,1,2,\ldots,$

$$\|\nabla D^s_\beta f\|_2^2\leq C\sum_{h=0}^{s}\|\nabla D^h_\beta u\|_2\{\|\bar{\Delta} D^h_\beta u\|_2+\|\nabla D^h_\beta u\|_2\} \tag{11.16}$$

where the constant C depends on s and on the choice of tangential coordinates but not on u.

A similar estimate, which we need not state explicitly here, holds for time derivatives of f in terms of the time derivatives of u of the same order.

The necessary estimate for b can also be found by means of the commutator formula (10.17). Since D^s_α is tangential it will commute on $\partial\Omega$ with $\partial/\partial n$. Thus

$$\|\nabla D_\alpha^s b\|_2^2=\int_{\partial\Omega} D_\alpha^s b\cdot\frac{\partial D_\alpha^s}{\partial n}b\,dS-\int_\Omega D_\alpha^s b\cdot\Delta D_\alpha^s b\,dx.$$

Since the normal derivative term vanishes on $\partial\Omega$, the first integral is zero. The second integral on the right hand side becomes, in view of (11.2),

$$-\int_\Omega D_\alpha^s b\left[\sum_{j=0}^{s}\bar{A}_\beta^\alpha D_\beta^j\,\Delta b+\sum_{\beta<\alpha}\bar{B}_\beta^\alpha D_3 D_\beta^k b+\sum_{\beta\leqslant\alpha}\bar{C}_\beta^\alpha D_\beta^k b+\sum_{\beta<\alpha}\bar{E}_\beta^\alpha D_\beta^k(D_1^2+D_2^2)\,b\right]dx$$

$$\leqslant C\|D_\alpha^s b\|_6\sum_{j=0}^{s}\|D_\alpha^j(\nabla u)^2\|_{6/5}+C\sum_{h=0}^{s-1}\|\nabla D_\alpha^h b\|_2^2$$

$$\leqslant C\|\nabla D_\alpha^s b\|_2\sum_{j=0}^{s}\sum_{l=0}^{[j/2]}\|\nabla D_\alpha^l u\|_2\|\nabla D_\alpha^{j-l}u\|_3+C\sum_{l=0}^{s-1}\|\nabla D_\alpha^l b\|_2^2$$

$$\leqslant\frac{1}{2}\|\nabla D_\alpha^s b\|_2^2+C\sum_{j=0}^{s}\sum_{l=0}^{[j/2]}\|\nabla D_\alpha^l u\|_2^2\|\nabla D_\alpha^{j-l}u\|_2\|\tilde{\Delta}D_\alpha^{j-l}u\|_2+C\sum_{l=0}^{s-1}\|\nabla D_\alpha^l b\|_2^2.$$

To estimate the $\bar{B}_\beta^\alpha$, $\bar{C}_\beta^\alpha$ and $\bar{E}_\beta^\alpha$ type terms here, we have proceeded exactly as in the proof of Lemma 7. Note that in the estimation of the nonlinear term, the L^2 norm is applied to the factor with the lower singularity.

It now follows that

$$\|\nabla D_\alpha^s b\|_2^2\leqslant C\sum_{j=0}^{s}\sum_{l=0}^{[j/2]}\|\nabla D_\alpha^l u\|_2^2\|\nabla D_\alpha^{j-l}u\|_2\|\tilde{\Delta}D_\alpha^{j-l}u\|_2+C\sum_{l=0}^{s-1}\|\nabla D_\alpha^l b\|_2^2.$$

For $s=0$ this relation is

$$\|\nabla b\|_2^2\leqslant C\|\nabla u\|_2^3\|\tilde{\Delta}u\|_2,$$

which can easily be obtained directly. Substituting this result on the right hand side we obtain the result for $s=1$, and so on by successive substitution. Thus we have established

LEMMA 8. *For $s=0,1,2,\ldots$ we have*

$$\|\nabla D_\alpha^s b\|_2^2\leqslant C\sum_{j=0}^{s}\sum_{l=0}^{[j/2]}\|\nabla D_\alpha^l u\|_2^2\|\nabla D_\alpha^{j-l}u\|_2\|\tilde{\Delta}D_\alpha^{j-l}u\|_2$$

where C depends on s and on the tangential coordinate system, but not on u.

For completeness we state here the modified form of this lemma needed to estimate time derivatives $D_t^r\nabla D_\alpha^s b$.

LEMMA 8*r*. *For* $r, s=0, 1, 2, \ldots$ *we have*

$$\|D_t^r \nabla D_\alpha^s b\|_2^2 \leqslant C \sum_{h=0}^{r} \sum_{j=0}^{s} \sum_{l=0,\, l\leqslant s}^{[j/2+r-2h]} \|D_t^h \nabla D_\alpha^l u\|_2^2 \, \|D_t^{r-h} \nabla D_\alpha^{j-l} u\|_2 \, \|D_t^{r-h} \tilde{\Delta} D_\alpha^{j-l} u\|_2$$

where C depends on r and s but not on u.

The demonstration is similar using Leibniz' formula for the time derivatives of $(\nabla u)^2$ and again applying the L^2 norm to the factor with lowest overall singularity.

12. Tangential derivative inequalities

To establish estimates of the tangential derivatives of u, ∇u, $\tilde{\Delta} u$ and their time derivatives, we differentiate the Navier–Stokes equations with respect to tangential variables and then conduct calculations similar to those of Sections 7 and 8 above. For each value of $r=0, 1, 2, \ldots$, an induction on the order $s=s_1+s_2$ of tangential derivatives is necessary, so the details of the induction are now somewhat different. Moreover the normal coordinate derivatives are not included in our summations at this stage, so that full tensorial invariance is not preserved [31, p. 36]. Thus the divergence property (2.2) cannot be used, as it was earlier, to annihilate the pressure terms; instead the estimates of the preceding Section 11 will be employed.

We denote by $D_\alpha^s u = D_1^{s_1} D_2^{s_2} u$ a typical tangential derivative of order $s_1+s_2=s$ and denote by $\|D_\alpha^s u\|_2^2$ the sum of squares of the $L^2(\Omega)$ norms of all such tangential derivatives of order s. For reasons of homogeneity in the orders of integrability, we consider at the same time the space derivatives $\nabla D_\alpha^s u$ and $\Delta D_\alpha^{s-1} u$, the latter being replaced in inequalities by $\tilde{\Delta} D_\alpha^{s-1} u$. Thus we will obtain estimates of $D_\alpha^{s+1} u$, $D_\alpha^s D_3 u$ and $D_\alpha^{s-1} D_3^2 u$ simultaneously at a given stage of induction on s; and this structure also applies to all orders r of time derivatives.

Applying the operator $D_t^r D_\alpha^s$ to (2.1) we find, using Leibniz' formula, and (10.15),

$$\begin{aligned} D_t^{r+1} D_\alpha^s u_i + \sum_{h=0}^{r} \sum_{j=0}^{s} \binom{r}{h} \binom{s}{j} D_t^h D_\alpha^j u_k D_t^{r-h} D_\alpha^{s-j} u_{i,k} \\ = -D_t^r D_\alpha^s (b_{,i} + \nu f_{,i}) + \nu D_t^r D_\alpha^s \Delta u_i \\ = -D_t^r D_\alpha^s (b_{,i} + \nu f_{,i}) + \nu D_t^r \Delta D_\alpha^s u_i + \nu \sum_{l=0}^{s-1} A_\beta^\alpha D_t^r \Delta D_\beta^l u_i \\ + \sum_{l=0}^{s-1} B_\beta^\alpha D_t^r D_3 D_\beta^l u_i + \sum_{l=0}^{s} C_\beta^\alpha D_t^r D_\beta^l u_i + \sum_{\beta<\alpha}^{s-1} E_\beta^\alpha D_t^r D_\beta^l (D_1^2 + D_2^2) u_i. \end{aligned} \tag{12.1}$$

Multiplying by $D_t^r D_\alpha^s u_i$ and integrating by parts in the leading viscosity term on the right hand side, we find

$$\begin{aligned}\frac{1}{2}D_t\|D_t^r D_\alpha^s u\|_2^2+\nu\|D_t^r \nabla D_\alpha^s u\|_2^2 = &-\int_\Omega D_t^r D_\alpha^s u_i \sum_{h,j=0}^{r,s}\binom{r}{h}\binom{s}{j} D_t^h D_\alpha^j u_k D_t^{r-h} D_\alpha^{s-j} u_{i,k}\,dx\\ &-\int_\Omega D_t^r D_\alpha^s u_i\{D_t^r D_\alpha^s b_{,i}+\nu D_t^r D_\alpha^s f_{,i}\}\,dx\\ &+\nu\int_\Omega D_t^r D_\alpha^s u_i \sum_{l=0}^{s-1} A_\beta^\alpha D_t^r \Delta D_\beta^l u_i\,dx\\ &+\nu\int_\Omega D_t^r D_\alpha^s u_i \sum_{l=0}^{s-1} B_\beta^\alpha D_t^r D_3 D_\beta^l u_i\,dx\\ &+\nu\int_\Omega D_t^r D_\alpha^s u_i \sum_{l=0}^{s} C_\beta^\alpha D_t^r D_\beta^l u_i\,dx\\ &+\nu\int_\Omega D_t^r D_\alpha^s u_i \sum_{l=0}^{s-1} E_\beta^\alpha D_t^r D_\beta^l (D_1^2+D_2^2)\, u_i\,dx.\end{aligned} \tag{12.2}$$

Let us denote the integrals on the right side by $I_1, I_2+I_3, I_4, I_5, I_6$ and I_7 respectively. By (10.18) we have

$$I_1=-\int_\Omega D_t^r D_\alpha^s u_i \sum_{h,j=0}^{r,s}\binom{r}{h}\binom{s}{j} D_t^h D_\alpha^j u_k D_t^{r-h}\sum_{l=0}^{s,j} G_{kl}^\alpha \nabla_k D_\alpha^l u_i\,dx.$$

We observe that the term in the sum with $h=0, j=0, l=s$ gives an integral similar to (7.5) that vanishes after integration by parts and use of the divergence condition (2.2).

Indicating omission of this term from the sum by a prime, we integrate by parts with respect to ∇_k and use (10.19), obtaining

$$\begin{aligned}I_1=&\int_\Omega D_t^r \nabla_k D_\alpha^s u_i \sum_{h,j,l=0}^{r,s,s-j}{}' \binom{r}{h}\binom{s}{j} D_t^h D_\alpha^j u_k G_{kl}^\alpha D_t^{r-h} D_\alpha^l u_i\,dx\\ &-\int_\Omega D_t^r D_\alpha^s u_i \sum_{h,j,l=0}^{r,s,s-j}{}' \binom{r}{h}\binom{s}{j} D_t^h D_\alpha^j u_k (\nabla_k G_{kl}^\alpha) D_t^{r-h} D_\alpha^l u_i\,dx.\end{aligned}$$

Hence

$$|I_1|\le \frac{\nu}{16}\|D_t^r \nabla D_\alpha^s u\|_2^2 + C\|D_t^r D_\alpha^s u\|_2^2 + C\sum_{h,j,l=0}^{r,s,s-j}{}' \|D_t^h D_\alpha^j u_k D_t^{r-h} D_\alpha^l u_i\|_2^2.$$

In this last sum the opposite corner term with $h=r$, $j=s$, $l=0$ may be estimated by

$$C\|D_t^r D_\alpha^s u_k\cdot u_i\|_2^2 \leq C\|D_t^r D_\alpha^s u\|_3^2\|u\|_6^2$$
$$\leq C\|D_t^r\nabla D_\alpha^s u\|_2\|D_t^r D_\alpha^s u\|_2\|\nabla u\|_2^2$$
$$\leq \frac{\nu}{16}\|D_t^r\nabla D_\alpha^s u\|_2^2 + C\|D_t^r D_\alpha^s u\|_2^2\|\nabla u\|_2^4.$$

Further terms of the triple sum with $2h+j/2 \geq r+l/2$ will be estimated by

$$\|D_t^h D_\alpha^j u_k D_t^{r-h} D_\alpha^l u_i\|_2^2 \leq \|D_t^h D_\alpha^j u\|_3^2\|D_t^{r-h} D_\alpha^l u\|_6^2$$
$$\leq C\|D_t^h D_\alpha^j u\|_2\|D_t^h\nabla D_\alpha^j u\|_2\|D_t^{r-h}\nabla D_\alpha^l u\|_2^2$$

while if $2h+j/2 < r+l/2$ we write

$$\|D_t^h D_\alpha^j u_k D_t^{r-h} D_\alpha^l u_i\|_2^2 \leq \|D_t^h D_\alpha^j u\|_6^2\|D_t^{r-h} D_\alpha^l u\|_3^2$$
$$\leq C\|D_t^h\nabla D_\alpha^j u\|_2^2\|D_t^{r-h} D_\alpha^l u\|_2\|D_t^{r-h}\nabla D_\alpha^l u\|_2.$$

The involution exchanging h with $r-h$, and j with l, shows that the latter set are included among the former, so that only the sum with $2h+j \geq r+l/2$ need be included if the constant C is adjusted.

Now by integration by parts,

$$|I_2| = \left|\int_\Omega D_t^r D_\alpha^s u_i D_t^r D_\alpha^s b_{,i}\,dx\right|$$
$$= \left|-\int_\Omega D_t^r\nabla_i D_\alpha^s u_i \sum_{l=0}^{s} G_{il}^\alpha D_t^r D_\alpha^l b\,dx\right| \tag{12.6}$$
$$\leq C\|D_t^r\nabla D_\alpha^s u\|_2\sum_{l=0}^{s}\|D_t^r D_\alpha^l b\|_2$$
$$\leq \frac{\nu}{32}\|D_t^r\nabla D_\alpha^s u\|_2^2 + C\left(\sum_{l=0}^{s-1}\|D_t^r\nabla D_\alpha^l b\|_2^2 + \|D_t^r b\|_2^2\right)$$

while

$$|I_3| = \left|\int_\Omega D_t^r D_\alpha^s u_i D_t^r D_\alpha^s f_{,i}\,dx\right|$$
$$= \left|-\int_\Omega D_t^r\nabla D_\alpha^s u_i \sum_{l=0}^{s} G_{il}^\alpha D_t^r D_\alpha^l f\,dx\right| \tag{12.7}$$

$$\leqslant C\|D_t^r\nabla D_\alpha^s u\|_2\sum_{l=0}^{s}\|D_t^r D_\alpha^l f\|_2$$

$$\leqslant \frac{\nu}{32}\|D_t^r\nabla D_\alpha^s u\|_2^2+C\left(\sum_{l=0}^{s-1}\|D_t^r\nabla D_{,\alpha}^l f\|_2^2+\|D_t^r f\|_2^2\right).$$

We may integrate by parts in I_4, since $D_\alpha^s u=0$ on $\partial\Omega$:

$$\begin{aligned}|I_4| &= \nu\left|\int_\Omega D_t^r\nabla D_\alpha^s u_i\sum_{l=0}^{s-1}A_\beta^\alpha D_t^r\nabla D_\beta^l u_i\,dx\right|\\ &\leqslant C\|D_t^r\nabla D_\alpha^s u\|_2\sum_{l=0}^{s-1}\|D_t^r\nabla D_\beta^l u\|_2 \qquad (12.8)\\ &\leqslant \frac{\nu}{16}\|D_t^r\nabla D_\alpha^s u\|_2^2+C\sum_{l=0}^{s-1}\|D_t^r\nabla D_\beta^l u\|_2^2.\end{aligned}$$

In I_5 we integrate by parts with respect to D_3, noting that the surface integral vanishes:

$$\begin{aligned}|I_5| &= \nu\left|\int_\Omega D_t^r D_3 D_\alpha^s u_i\sum_{l=1}^{s-1}B_\beta^\alpha D_t^r D_\beta^l u_i\,dx\right|\\ &\leqslant C\|D_t^r\nabla D_\alpha^s u\|_2\sum_{l=0}^{s-2}\|D_t^r\nabla D_\beta^l u\|_2 \qquad (12.9)\\ &\leqslant \frac{\nu}{16}\|D_t^r\nabla D_\alpha^s u\|_2^2+C\sum_{l=0}^{s-2}\|D_t^r\nabla D_\beta^l u\|_2^2\end{aligned}$$

where we have taken account of (10.4) and (10.16) to convert one D_α factor to a gradient. Then in I_6 we obtain

$$\begin{aligned}|I_6| &\leqslant \nu\left|\int_\Omega D_t^r D_\alpha^s u\sum_{l=0}^{s}C_\beta^\alpha D_t^r D_\beta^l u\,dx\right|\\ &\leqslant C\|D_t^r D_\alpha^s u\|_2\sum_{l=0}^{s}\|D_t^r D_\beta^l u\|_2 \qquad (12.10)\\ &\leqslant C\sum_{l=0}^{s}\|D_t^r D_\beta^l u\|_2^2.\end{aligned}$$

For the integral I_7 we can proceed exactly as in (11.12) – (11.14) with $D_t^r u_i$ in place of f, and so obtain the estimate

$$I_7\leqslant C\sum_{l=0}^{s-1}\|D_t^r\nabla D_\beta^l u_i\|_2^2. \qquad (12.11)$$

The foregoing results will be combined with a second inequality obtained by changing s into $s-1$ in (12.1), multiplying by $D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i$ and integrating over Ω. This process yields, after an integration by parts similar to (3.2), and noting that $D_\alpha^{s-1}u_i$ is solenoidal, by (10.19),

$$
\begin{aligned}
&\frac{1}{2}D_t\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2+\nu\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2\\
&\quad=+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{h,l=0}^{r,s-1}\binom{r}{h}\binom{s-1}{l}D_t^hD_\alpha^l u_kD_t^{r-h}D_\alpha^{s-l-1}u_{i,k}\,dx\\
&\quad+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\{D_t^rD_\alpha^{s-1}b_{,i}+\nu D_t^rD_\alpha^{s-1}f_{,i}\}\,dx\\
&\quad+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{l=0}^{s-2}A_\beta^\alpha D_t^r\Delta D_\beta^l u_i\,dx\\
&\quad+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{l=0}^{s-2}B_\beta^\alpha D_t^rD_3D_\beta^l u_i\,dx\\
&\quad+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{l=1}^{s-1}C_\beta^\alpha D_t^rD_\beta^l u_i\,dx\\
&\quad+\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{l=0}^{s-2}E_\beta^\alpha D_t^rD_\beta^l(D_1^2+D_2^2)u_i\,dx.
\end{aligned}
\tag{12.12}
$$

We shall denote the integrals on the right side of this second relation by $J_1, J_2+J_3, J_4, J_5, J_6$ and J_7 respectively.

In the estimation of J_1 by Hölder's inequality it is necessary to use an $L^2(\Omega)$ norm for the first factor, so that an $L^6(\Omega)$ and an $L^3(\Omega)$ norm or an $L^\infty(\Omega)$ and an $L^2(\Omega)$ norm, could be available for the remaining two factors.

We apply the $L^6(\Omega)$ norm to the factor $D_t^hD_\alpha^l u$ when it has the lesser total order of singularity. Otherwise we apply the $L^\infty(\Omega)$ norm to this factor. We have

$$
\begin{aligned}
|J_1|&=\left|\int_\Omega D_t^r\tilde{\Delta}D_\alpha^{s-1}u_i\sum_{h,l=0}^{r,s-1}\binom{r}{h}\binom{s-1}{l}D_t^hD_\alpha^l u_k\sum_{j=0}^{s-l-1}G_{kl}^\alpha D_t^{r-h}\nabla_kD_\alpha^{s-j}u_i\,dx\right|\\
&\leqslant C\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2\Bigg(\sum_{\substack{h,l,j=0\\ 2h+l/2<r+(j-1)/2}}^{r,s-1,s-l-1}\|D_t^hD_\alpha^l u\|_6\|D_t^{r-h}\nabla D_\alpha^j u\|_3\\
&\quad+\sum_{\substack{h,l,j=0\\ 2h+l/2\geqslant r+(j-1)/2}}^{r,s-1,s-l-1}\|D_t^hD_\alpha^l u\|_\infty\|D_t^{r-h}\nabla D_\alpha^j u\|_2\Bigg)
\end{aligned}
\tag{12.13}
$$

$$\leq C\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2\left(\sum_{\substack{h,l,j=0\\ 2h+l/2<r+(j-1)/2}}^{r,s-1,s-l-1}\|D_t^h\nabla D_\alpha^l u\|_2\|D_t^{r-h}\nabla D_\alpha^j u\|_2^{1/2}\|D_t^{r-h}\tilde{\Delta}D_\alpha^j u\|_2^{1/2}\right.$$
$$\left.+\sum_{\substack{h,l,j=0\\ 2h+l/2\geq r+(j-1)/2}}^{r,s-1,s-l-1}\|D_t^h\nabla D_\alpha^l u\|_2^{1/2}\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2^{1/2}\|D_t^{r-h}\nabla D_\alpha^j u\|_2\right)$$

where (2.16), (2.20) and (2.21) have been invoked as appropriate. There is an involutory correspondence $h\to r-h$, $l\to j$ between the terms of the two sums which therefore can be combined into one such sum. Employing Young's inequality at the same time we now find the estimate

$$|J_1|\leq\frac{\nu}{16}\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2+C\sum_{\substack{h,l,j=0\\ 2h+l/2\geq r+(j-1)/2}}^{r,s-1,s-l-1}\|D_t^h\nabla D_\alpha^l u\|_2\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2\|D_t^{r-h}\nabla D_\alpha^j u\|_2^2. \tag{12.14}$$

The term with $h=r$ and $l=s-1, j=0$ in this sum also contains $\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2$ so that we apply Young's inequality once more and obtain

$$|J_1|\leq\frac{\nu}{16}\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2+C\left\{\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2\|\nabla u\|_2^4\right.$$
$$\left.+\sideset{}{'}\sum_{\substack{h,l,j=0\\ 2h+l/2\geq r+(j-1)/2}}^{r,s-1,s-1-l}\|D_t^h\nabla D_\alpha^l u\|_2\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2\|D_t^{r-h}\nabla D_\alpha^j u\|_2^2\right\} \tag{12.15}$$

where the prime on the summation sign denotes omission of the term $h=r$, $l=s-1, j=0$. Noting the lemmas of the preceding section we write the expressions for J_2 and J_3 in the form

$$|J_2|+|J_3|\leq C\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2\sum_{l=0}^{s-1}\{\|D_t^r\nabla D_\alpha^l b\|_2+\nu\|D_t^r\nabla D_\alpha^l f\|_2\}$$
$$\leq\frac{\nu}{16}\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2+C\sum_{l=0}^{s-1}\{\|D_t^r\nabla D_\alpha^l b\|_2^2+\|D_t^r\nabla D_\alpha^l f\|_2^2\}. \tag{12.16}$$

For the integral J_4 we use the orthogonal decomposition $\Delta v_i=\tilde{\Delta}v_i+\nabla_i f_\alpha$, where $v_i=D_t^r D_\alpha^l u_i$, and so write, noting $(\tilde{\Delta}w_i, \nabla_i f_\alpha)=0$,

$$
\begin{aligned}
|J_4| &\leq C\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \sum_{l=0}^{s-2} \|D_t^r \tilde{\Delta} D_\alpha^l u\|_2 \\
&\leq \frac{\nu}{16} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C \sum_{l=0}^{s-2} \|D_t^r \tilde{\Delta} D_\alpha^l u\|_2^2
\end{aligned} \tag{12.17}
$$

by Young's inequality.

In J_5, we see from the second condition of (10.16) that one D_α factor should be converted to a ∇_α to absorb the r^{-1} factor. Since D_3 corresponds to ∇_3 we obtain

$$
|J_5| \leq C\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \sum_{l=0}^{s-2} \|D_t^r \nabla_3 \nabla_\alpha D_\beta^l u\|_2.
$$

The latter factor may be estimated as in [19, p. 21] by the formula

$$
\|D_t^r \nabla\nabla D_\beta^l u\|_2 \leq C(\|D_t^r \tilde{\Delta} D_\beta^l u\|_2 + \|D_t^r \nabla D_\beta^l u\|_2)
$$

so that we obtain by Young's formula

$$
|J_5| \leq \frac{\nu}{16} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C\left(\sum_{l=0}^{s-2} \|D_t^r \tilde{\Delta} D_\beta^l u\|_2^2 + \|D_t^r \nabla D_\beta^l u\|_2^2\right). \tag{12.18}
$$

By the third condition of (10.16), we see that D_α and ∇_α are equivalent in J_6, and so obtain

$$
\begin{aligned}
|J_6| &\leq C\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \sum_{l=0}^{s-1} \|D_t^r D_\beta^l u\|_2 \\
&\leq C\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \left(\sum_{l=0}^{s-2} \|D_t^r \nabla D_\beta^l u\|_2 + \|D_t^r u\|_2\right) \\
&\leq \frac{\nu}{16} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C\left(\sum_{l=0}^{s-2} \|D_t^r \nabla D_\beta^l u\|_2^2 + \|D_t^r u\|_2\right).
\end{aligned} \tag{12.19}
$$

From the fourth condition of (10.16) it is evident that two D_α factors in J_7 should be converted to ∇_α operators, and these are best chosen as the factors $D_1^2 + D_2^2$. The integral over $S(P_0, \varepsilon)$ has the form

$$
\begin{aligned}
\int_S w \frac{(1+\xi_1^2+\xi_2^2)^2}{4r^2} (D_1^2 + D_2^2) v \, dx &= \int_0^\varepsilon dr \int_{-\infty}^\infty \int_{-\infty}^\infty w(D_1^2 + D_2^2) v \cdot d\xi_1 \, d\xi_2 \\
&= -\int_0^\varepsilon dr \int_{-\infty}^\infty \int_{-\infty}^\infty (D_1 w \cdot D_1 v + D_2 w \cdot D_2 v) \, d\xi_1 \, d\xi_2
\end{aligned} \tag{12.20}
$$

where w and v are smooth functions. Supplying the factors $(1+\xi_1^2+\xi_2^2)^2/4r^2$ in numerator and denominator we find the expression

$$-\int_0^\varepsilon dr \int_\omega (\nabla_1 w\cdot\nabla_1 v+\nabla_2 w\cdot\nabla_2 v)\, r^2 d\omega$$

where ω denotes solid angle.

We may now regard ∇_α as the gradient operator with respect to arc length s_α along the appropriate parametric curve on S_r, and, since the isothermic curves form an orthogonal net, write

$$dS = r^2\, d\omega = ds_1\, ds_2.$$

Integrating by parts in the reverse direction we now find

$$\begin{aligned}\int_0^\varepsilon dr \int_{S_r} w(\nabla_1(\nabla_1 v)+\nabla_2(\nabla_2 v))\, ds_1\, ds_2 &= \int_0^\varepsilon dr \int_{S_r} w(\nabla_1^2 v+\nabla_2^2 v)\, dS_r \\ &= \int_{S(P_0,\,\varepsilon)} w(\nabla_1^2 v+\nabla_2^2 v)\, dx.\end{aligned} \tag{12.21}$$

The corresponding integral over $\Omega - S(P_0, \varepsilon)$ can be shown to have a similar form.

Consequently in view of [19, p. 21], and Young's inequality,

$$\begin{aligned}|J_7| &= \left| -\int_\Omega D_t^r \tilde{\Delta} D_\alpha^{s-1} u_i \sum_{l=0}^{s-2} E_\beta^\alpha D_t^r D_\beta^l (D_1^2+D_2^2)\, u_i\, dx \right| \\ &\leqslant C\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \sum_{l=0}^{s-2} (\|D_t^r \tilde{\Delta} D_\beta^l u\|_2 + \|D_t^r \nabla D_\beta^l u\|_2) \\ &\leqslant \frac{\nu}{16} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C \sum_{l=0}^{s-2} (\|D_t^r \tilde{\Delta} D_\beta^l u\|_2^2 + \|D_t^r \nabla D_\beta^l u\|_2^2)\end{aligned} \tag{12.22}$$

Combining all the foregoing inequalities and multiplying by 2, we find after certain cancellations

$$D_t\{\|D_t^r D_\alpha^s u\|_2^2 + \|D_t^r \nabla D_\alpha^{s-1} u\|_2^2\} + \frac{3\nu}{2} \{\|D_t^r \nabla D_\alpha^s u\|_2^2 + \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2\}$$

$$\leqslant C\Big((\|D_t^r D_\alpha^s u\|_2^2 + \|D_t^r \nabla D_\alpha^{s-1} u\|_2^2)\, \|\nabla u\|_2^4$$

$$
\begin{aligned}
&+\sum_{h,l,j=0,\,2h+j/2\geqslant r+l/2}^{r,s,s-l}{}' \|D_t^h D_\alpha^j u\|_2 \|D_t^h \nabla D_\alpha^j u\|_2 \|D_t^{r-h} \nabla D_\alpha^l u\|_2^2 \\
&+\sum_{h,l,j=0,\,2h+l/2\geqslant r+j/2}^{r,s-1,s-l-1}{}' \|D_t^h \nabla D_\alpha^l u\|_2 \|D_t^h \tilde{\Delta} D_\alpha^l u\|_2 \|D_t^{r-h} \nabla D_\alpha^j u\|_2^2 \\
&+\sum_{h=0}^{r}\sum_{j=0}^{s-1}\sum_{l=0,\,l\leqslant s}^{[j/2+r-2h]} \|D_t^h \nabla D_\alpha^l u\|_2^2 \|D_t^{r-h} \nabla D_\alpha^{j-l} u\|_2 \|D_t^{r-h} \tilde{\Delta} D_\alpha^{j-l} u\|_2 \\
&+\sum_{j=0}^{s-1} \|D_t^r \nabla D_\alpha^j u\|_2 \{\|D_t^r \tilde{\Delta} D_\alpha^j u\|_2 + \|D_t^r \nabla D_\alpha^j u\|_2\} \\
&+\sum_{l=0}^{s-2} \|D_t^r \tilde{\Delta} D_\alpha^l u\|_2^2 + \sum_{l=0}^{s-1} \|D_t^r \nabla D_\alpha^l u\|_2^2 + \sum_{l=0}^{s} \|D_t^r D_\alpha^l u\|_2^2 \Bigg).
\end{aligned}
\tag{12.23}
$$

Denote the seven summations on the right hand side of (12.23) by $\Sigma_1, \ldots, \Sigma_7$ in the order that they appear there.

We shall cancel off against the left hand side any terms containing $\|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2$, in the following way: In the triple sum Σ_3 such a term appears for $h=0$, $l=0$, $j=s-1$, namely

$$
\begin{aligned}
&\|\nabla u\|_2^2 \|D_t^r \nabla D_\alpha^{s-1} u\|_2 \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2 \\
&\leqslant \frac{\nu}{32} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C\left(\|D_t^r \nabla D_\alpha^{s-1} u\|_2^{\frac{8r+4s+2}{4r+2s-1}} + \|\nabla u\|_2^{8r+4s+2}\right).
\end{aligned}
\tag{12.24}
$$

The first term with $j=s-1$ in Σ_4 can be majorized by

$$
\frac{\nu}{32} \|D_t^r \tilde{\Delta} D_\alpha^{s-1} u\|_2^2 + C \|D_t^r \nabla D_\alpha^{s-1} u\|_2^2 \tag{12.25}
$$

and the first of these terms will cancel against part of a term on the left hand side. The gradient products in Σ_4 are identical with those in Σ_5 and Σ_6, so that Σ_4 reduces in effect to

$$
\sum_{j=0}^{s-2} \|D_t^r \nabla D_\alpha^j u\|_2 \|D_t^r \tilde{\Delta} D_\alpha^j u\|_2 \leqslant \frac{1}{2} \sum_{j=0}^{s-2} \|D_t^r \tilde{\Delta} D_\alpha^j u\|_2^2 + \frac{1}{2} \sum_{j=0}^{s-2} \|D_t^r \nabla D_\alpha^j u\|_2^2. \tag{12.26}
$$

As these sums are included in Σ_5 and Σ_6 it is clear that, adjusting C (as necessary), we can drop Σ_4 altogether.

13. Tangential derivative estimates

We now apply Young's inequality to the product terms on the right hand side of (12.23), noting that the maximum singular index is $4r+2s+1$. As in Section 8, certain gradient and Stokesian terms must be kept together, so that the necessary lower order estimates can later be adjoined consistently with the integrability lemma. These terms will also be given a sufficiently small coefficient to make possible cancellation against left hand side terms. The result of these calculations is the inequality

$$
\begin{aligned}
&D_t\{\|D_t^r D_\alpha^s u\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2\}+\frac{5\nu}{4}\{\|D_t^r\nabla D_\alpha^s u\|_2^2+\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2\}\\
&\leq C_{r,s}\Bigg\{\|D_t^r D_\alpha^s u\|_2^{2\frac{4r+2s+1}{4r+2s-1}}+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^{2\frac{4r+2s+1}{4r+2s-1}}+\|\nabla u\|_2^{8r+4s+2}+\sum_{h,l=0}^{r,s}\|D_t^h D_\alpha^l u\|_2^{2\frac{4r+2s+1}{4h+2l-1}}\\
&\quad+\sum_{h,l=0}^{r,s-1}{}'\|D_t^h\nabla D_\alpha^l u\|_2^{2\frac{4r+2s+1}{4h+2l+1}}+\nu\sum_{h,l=0}^{r,s-1}\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2^2\Bigg\}\\
&\quad+\frac{\nu}{8}\sum_{h,l=0}^{r,s-1}{}'\|D_t^h\nabla D_\alpha^l u\|_2^{2\frac{4(r-h)+2(s-l-1)}{4h+2l+1}}\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2^2.
\end{aligned}
\tag{13.1}
$$

Here the prime on the first two summations over r, s indicates that the "upper right corner" $h=r$ and $l=s$ is omitted. On the last sums the prime indicates omission of $h=r$, $l=s-1$. In view of (3.12), we have supposed that the singular degree of every term except those in Σ_5 has been raised to the maximum $4r+2s+1$ by multiplying by a term $K\|\nabla u\|^a\geq 1$. This means that when we apply (2.13) with appropriate exponents, only terms with singular index $4r+2s+1$ will appear, including terms in $\|\nabla u\|_2^{8r+4s+2}$. Σ_5 is exempted from this process because terms containing the Stokes operator $\tilde{\Delta}$ will occur only in $G_{h,j}(t)$ so that a different disposition must be made for these terms.

The constant C on the right hand side of (13.1) will be definitively denoted by $C_{r,s}$ and will not be increased hereafter.

We now define, first for $r=0$, $s=0, 1, 2, \ldots$ then for r, $s=0, 1, 2, \ldots$ and so on for all positive integers (r, s):

$$
F_{r,s}(t)=\|D_t^r D_\alpha^s u\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2+\frac{5}{4}\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}''F_{h,j}(t)^{\frac{4r+2s-1}{4h+2j-1}}+\frac{5}{2}C_{r,s}\sum_{h,l=0}^{r,s-1}F_{h,l}(t)+1 \tag{13.2}
$$

$$
\begin{aligned}
G_{r,s}(t)&=\nu(\|D_t^r\nabla D_\alpha^s u\|_2^2+\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2)\\
&\quad+\sum_{h=0}^{r}\sum_{j=0}^{s+1}{}''\frac{4r+2s-1}{4h+2j-1}F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}}G_{h,j}(t)+C_{r,s}\sum_{h,l=0}^{r,s-1}G_{h,l}(t).
\end{aligned}
\tag{13.3}
$$

Again the double prime denotes omission of the terms $h=r$, $j=s$ and $h=r$, $j=s+1$. Terms with $4r+2s-1<0$ or $4h+2j-1<0$ are also omitted from the sums. The terms on the right side of (13.1) evidently satisfy

$$\begin{aligned} \|D_t^h D_\alpha^l u\|_2^{2\frac{4r+2s+1}{4h+2l-1}} &\leqslant F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}} \\ \|D_t^h \nabla D_\alpha^l u\|_2^{2\frac{4r+2s+1}{4h+2l+1}} &\leqslant F_{h,l+1}(t)^{\frac{4r+2s+1}{4h+2(l+1)-1}} \\ \nu\|D_t^h \tilde{\Delta} D_\alpha^l u\|_2^2 &\leqslant G_{h,l+1}(t) \end{aligned} \tag{13.4}$$

and

$$\nu\|D_t^h \nabla D_\alpha^l u\|_2^{2\frac{4(r-h)+2(s-l-1)}{4h+2l+1}} \|D_t^h \tilde{\Delta} D_\alpha^l u\|_2^2 \leqslant F_{h,l+1}(t)^{\frac{4(r-h)+2(s-l-1)}{4h+2(l+1)-1}} G_{h,l+1}(t).$$

We also note the inequalities, where $h\leqslant r, l\leqslant s$:

$$\begin{aligned} F_{h,l}(t)^{\frac{1}{4h+2l-1}} &\leqslant F_{r,s}(t)^{\frac{1}{4r+2s-1}}, \\ \sum_{h,l=0}^{r,s}{}' F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}} \leqslant \sum_{h,l=0}^{r,s+1}{}'' F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}} &\leqslant CF_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}, \\ \sum_{h,l=0}^{r,s}{}' F_{h,l+1}(t)^{\frac{4r+2s+1}{4h+2l+1}} &\leqslant F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}} + \sum_{h,l=0}^{r,s}{}'' F_{h,l+1}(t)^{\frac{4r+2s+1}{4h+2l+1}} \\ &\leqslant CF_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}, \\ \|\nabla u\|_2^{8r+4s+2} \leqslant F_{0,1}(t)^{4r+2s+1} &< F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}, \end{aligned} \tag{13.5}$$

$$\begin{aligned} \sum_{h,l=0}^{r,s}{}' F_{h,l+1}(t)^{\frac{4(r-h)+2(s-l-1)}{4h+2(l+1)-1}} G_{h,l+1}(t) &\leqslant G_{r,s}(t) + \sum_{h,j=0}^{r,s+1}{}'' \frac{4r+2s-1}{4h+2j-1} F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}} G_{h,j}(t) \\ &\leqslant 2G_{r,s}(t), \end{aligned}$$

$$\sum_{h,l=0}^{r,s+1}{}'' \frac{4r+2s-1}{4h+2l-1} F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}} < (4r+2s+1)(r+1)(s+2) F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}$$

and

$$C_{r,s} \sum_{h,l=0}^{r,s}{}' C'_{h,l} F_{h,l}(t)^{\frac{4h+2l+1}{4h+2l-1}} < CF_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}.$$

The general inequality with which we shall work has the form

$$\frac{d}{dt}F_{r,s}(t)+G_{r,s}(t)\leqslant C'_{r,s}F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}. \tag{13.6}$$

We shall proceed by induction with $F_{r,s}(t)\in L^{1/(4r+2s-1)}(0,T)$ and thus prove that $G_{r,s}(t)\in L^{1/(4r+2s+1)}(0,T)$. However we must first show that (13.1) can be transformed into the form (13.6). For this purpose we multiply (13.6) (where r, s are replaced by h, j) by

$$\frac{5}{4}\cdot\frac{4r+2s-1}{4h+2j-1}F_{h,j}(t)^{\frac{4(r-h)+2(s-j)}{4h+2j-1}}$$

and add the resulting inequalities for $h=0,1,\ldots,r$, $j=0,1,\ldots,s+1$, but omitting $h=r$, $j=s$ and $j=s+1$. Also for $h=0,1,\ldots,r$ and $l=0,\ldots,s$ but omitting $h=r$, $l=s$ we multiply (13.6), with r, s replaced by h, l, by $\frac{5}{2}C_{r,s}$ and add on.

As induction hypothesis we assume that these preceding inequalities are valid. We now obtain from (13.1)

$$\begin{aligned}
&D_tF_{r,s}(t)+\frac{5}{4}G_{r,s}(t)+C_{r,s}\sum_{h,l=0}^{r,s}{}' G_{h,l}(t)\\
&\leqslant C_{r,s}\Bigg\{\|D_t^rD_\alpha^s u\|_2^{2\frac{4r+2s+1}{4r+2s-1}}+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^{2\frac{4r+2s+1}{4r+2s-1}}\\
&\quad+C\sum_{h,l=0}^{r,s+1}{}'' F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}}+\sum_{h,l=0}^{r,s-1}F_{h,l+1}(t)^{\frac{4r+2s+1}{4h+2(l+1)-1}}\\
&\quad+\nu\sum_{h,l=0}^{r,s-1}{}'\|D_t^h\tilde{\Delta}D_\alpha^l u\|_2^2\Bigg\}+\frac{1}{8}\sum_{h,l=0}^{r,s-1}{}' F_{h,l+1}(t)^{\frac{4(r-h)+2(s-l-1)}{4h+2(l+1)-1}}G_{h,l+1}(t)\\
&\quad+\frac{5}{4}\sum_{h,l=0}^{r,s+1}{}''\frac{4r+2s+1}{4h+2l-1}F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-11}}+\frac{5}{2}C_{r,s}\sum_{h,l=0}^{r,s}{}' C'_{h,l}F_{h,l}(t)^{\frac{4h+2l+1}{4h+2l-1}}.
\end{aligned} \tag{13.7}$$

We note that the Stokes operator terms are together majorized by the sum of G function terms on the left hand side.

Also the sum containing the products of F and G terms is less, term by term, than the corresponding sum in (13.3) plus $G_{r,s}(t)$. Hence this term cancels against $\frac{1}{4}G_{r,s}(t)$ on the left side. Inequality (13.6) now implies, by (13.5),

$$D_tF_{r,s}(t)+G_{r,s}(t)\leqslant C\Bigg\{F_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}+2\sum_{h,l=0}^{r,s-1}F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}}$$

$$+\frac{9}{4}\sum_{h,l=0}^{r,s}{}' F_{h,l+1}(t)^{\frac{4r+2s+1}{4h+2(l+1)-1}}+\sum_{h,l=0}^{r,s-1}F_{h,l}(t)^{\frac{4r+2s+1}{4h+2l-1}}\Bigg\} \tag{13.8}$$

$$\leqslant CF_{r,s}(t)^{\frac{4r+2s+1}{4r+2s-1}}.$$

It now follows from Lemma 3 that

$$G_{r,s}(t)\in L^{\frac{1}{4r+2s+1}}(0,T) \tag{13.9}$$

and hence

$$\|D_t^r\nabla D_\alpha^s u\|_2 \quad\text{and}\quad \|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2\in L^{\frac{2}{4r+2s+1}}(0,T). \tag{13.10}$$

14. Deduction of the estimates for $s_3=0,1,2$ and r,s_1,s_2 arbitrary

At a given stage in the induction proof of the preeding section we establish (13.10) and at that stage corresponding results for all lower order tangential and time derivatives are also known. By Lemma 7 and its analogues for general values of r, we can then conclude $\|D_t^r\nabla D_\alpha^{s-1}f\|_2\in L^{1/(2r+s)}(0,T)$. Noting that the tangential derivatives of u vanish on $\partial\Omega$, we may again apply the estimate [19, p. 21] and obtain

$$\|D_t^rD_jD_kD_\alpha^{s-1}u\|_2^2\leqslant C\{\|D_t^r\Delta D_\alpha^{s-1}u\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2\}$$

$$\leqslant C\{\|D_t^r\tilde{\Delta}D_\alpha^{s-1}u\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}f\|_2^2+\|D_t^r\nabla D_\alpha^{s-1}u\|_2^2\} \tag{14.1}$$

$$\in L^{\frac{1}{4r+2s+1}}(0,T).$$

We are now in a postition to undertake the main proof by mathematical induction on s, and then on r. The results of Section 7, 8 and 9 assure us that $F_{0,0}(t)$, $F_{0,1}(t)$ and $F_{0,2}(t)$ satisfy the induction hypothesis on s. Consequently $G_{0,0}(t)$, $G_{0,1}(t)$ and $G_{0,2}(t)$ satisfy the conclusion of these induction steps. At any step s of the sequence with $r=0$, the new induction hypothesis requires $\|D_\alpha^s u\|_2$ and $\|\nabla D_\alpha^{s-1}u\|_2\in L^{2/(2s-1)}(0,T)$. But the result of the preceding induction step $s-1$ assures us that $\|\nabla D_\alpha^{s-1}u\|_2$ (and also $\|\tilde{\Delta}D_\alpha^{s-2}u\|_2$ and $\|\Delta D_\alpha^{s-2}u\|_2$) satisfy precisely this same condition, which since

$\|D_\alpha^s u\|_2 < C\|\nabla D_\alpha^{s-1} u\|_2$ ensures the necessary induction hypothesis for the next step. This proves the induction for $r=0, s=0,1,2,\dots,n,\dots$

Considering now the induction over r, we assume the results proved for all s and for values of r up to and including $r-1$. Then, by (13.2), the new hypothesis at any s step is $\|D_t^r D_\alpha^s u\|_2$ and $\|D_t^r \nabla D_\alpha^{s-1} u\|_2 \in L^{2/(4r+2s-1)}$. But the result of the preceding $(s-1)$st step of the s-induction is that $\|D_t^r \nabla D_\alpha^{s-1} u\|_2$ and also $\|D_t^r \tilde{\Delta} D_\alpha^{s-2} u\|_2$ and $\|D_t^r \Delta D_\alpha^{s-2} u\|_2$ satisfy exactly this same condition. Since again $\|D_t^r D_\alpha^s u\|_2 < C\|D_t^r \nabla D_\alpha^{s-1} u\|_2$ the induction hypothesis for each step from $s-1$ to s does hold. Now by our earlier results the induction hypothesis can be seen to hold for $F_{r,0}(t)$, $F_{r,1}(t)$ and $F_{r,2}(t)$. By (13.2) this requires the results of the earlier induction steps up to $r-1$, as well as the results of Sections 7, 8 and 9, but these results are all available at the rth stage of the induction. Hence the induction over $s=0,1,2,3,\dots,n,\dots$ holds at the rth stage, and this establishes the general step of the induction over r. Letting $r=0,1,2,\dots,m,\dots$ we now easily complete the induction over r and with it the proof that (14.1) holds in general for $r, s=0,1,2,\dots,n,\dots$.

A further conclusion we can now draw is that, by (2.22)

$$\begin{aligned}\max_{x\in\Omega} |D_t^r D_\alpha^s u| &\equiv \| D_t^r D_\alpha^s u\|_\infty \\ &\leq C\|D_t^r \nabla D_\alpha^s u\|_2^{1/2} \, \| D_t^r \tilde{\Delta} D_\alpha^s u\|_2^{1/2} \\ &\in L^{\frac{1}{2r+s+1}}(0,T).\end{aligned} \tag{14.2}$$

Note that the s_3 component is still necessarily zero in (14.2) at this stage of proof.

15. Normal derivatives of third and higher order

Remaining to be estimated are the partial derivatives containing three or more normal derivations. These we shall estimate inductively on the normal order, by considering each component separately and by introducing the incompressibility and vorticity relations.

The continuity equation $\operatorname{div} u \equiv u_{i,i} = 0$ yields in tangential cordinates

$$D_3 u_3 = -D_\alpha u_\alpha + g_i u_i, \quad \alpha = 1,2 \tag{15.1}$$

so that

$$D_t^r D_3^{s_3} D_2^{s_2} D_1^{s_1} u_3 = -D_t^r D_3^{s_3-1} D_2^{s_2} D_1^{s_1} D_\alpha u_\alpha + \text{lower order.} \tag{15.2}$$

The right side contains only derivatives of lower normal order so suggesting an induction on s_3. In fact by iterating (15.2) we can express the derivative on the left side as a finite linear combination of derivatives of normal order two or less, each of which has L^2 norm with the appropriate property of integrability by the preceding section, namely inclusion in $L^{2/(4r+2s_1+2s_2+2s_3-1)}(0, T)$.

For the tangential components u_1 and u_2 we make use of the corresponding tangential components of the vorticity equation

$$\omega_{i,t}+u_k\omega_{i,k}=\omega_k u_{i,k}+\nu\Delta\omega_i \tag{15.3}$$

with $i, k=1,2,3$ and

$$\omega_i=\text{curl } u_i=\frac{1}{2}\varepsilon^{ijk}\left(\frac{\partial u_k}{\partial x_j}-\frac{\partial u_j}{\partial x_k}\right) \tag{15.4}$$

where ε^{ijk} is the permutation symbol. Solving (15.3) for the Laplacian term we have, with $\alpha=1,2$

$$\nu\Delta\omega_\alpha=\omega_{\alpha,t}+u_k\omega_{\alpha,k}-\omega_k u_{\alpha,k}. \tag{15.5}$$

Hence, with bounded coefficients a_1, a_2

$$\begin{aligned} D_3^3 u_1 &= a_1\Delta D_3 u_1-a_2(D_1^2+D_2^2)D_3u_1+\text{lower order}\\ &= a_1\Delta(\omega_2-D_1u_3)-a_2(D_1^2+D_2^2)D_3u_1+\text{ lower order}\\ &=\frac{a_1}{\nu}(\omega_{2,t}+u_k\omega_{2,k}-\omega_k u_{2,k})-a_1\Delta D_1u_3-a_2(D_1^2+D_2^2)D_3u_1+\text{lower order}\end{aligned} \tag{15.6}$$

is now expressed by third and lower space derivatives of components of u_i containing at most two normal derivatives, together with a time derivative and products of the form

$$u_k\omega_{2,k}=u_k\frac{\partial}{\partial x_k}\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right), \tag{15.7}$$

and

$$\omega_k u_{2,k}=\left(\frac{\partial u_m}{\partial x_n}-\frac{\partial u_n}{\partial x_m}\right)\frac{\partial u_2}{\partial x_k}. \tag{15.8}$$

As in all our preceding calculations with the hypoelliptic heat flow operator, we see that the time derivative takes the place of two space derivatives, and that the L^2 norms of the time derivative term and all the third order derivative terms on the right side of (15.6) lie in $L^{2/5}(0,T)$. The same is true of terms such as (15.7), for

$$\|u_k\,\omega_{2,k}\|_2 \leq \max|u_k|\cdot\|D_j^2 u\|_2 \tag{15.9}$$

and these two factors are known to be in $L^1(0,T)$ and $L^{2/3}(0,T)$, respectively. Likewise for (15.8) we have

$$\begin{aligned}\|\omega_k u_{2,k}\|_2 &\leq \|\omega\|_6\,\|\nabla u\|_3\\ &\leq \|\nabla u\|_6\cdot\|\nabla u\|_6^{1/2}\,\|\nabla u\|_2^{1/2}\\ &\leq C\|\tilde{\Delta} u\|_2^{3/2}\,\|\nabla u\|_2^{1/2}\end{aligned}$$

and this product has singular index $(\frac{3}{2})^2+(\frac{1}{2})^2=\frac{5}{2}$. Hence $\|D_3 u_1\|_2$ and also $\|D_3 u_2\|_2\in L^{2/5}[0,T]$.

Consider now a typical derivative of (15.6) with respect to time and tangential derivatives. We may neglect all lower derivative terms arising from variability of the coefficients a_1 and a_2 which will have bounded tangential derivatives at every order. Thus

$$D_t^r D_\alpha^s D_3^3 u_1 = \frac{a_1}{\nu} D_t^r D_\alpha^s(\omega_{2,t} - \Delta D_1 u_3 + u_k\,\omega_{2,k} - \omega_k u_{2,k}) - a_2 D_t^r D_\alpha^s(D_1^2+D_2^2) D_3 u_1 + \ldots. \tag{15.10}$$

Evidently each of the four linear terms in ω or u will have the singularity index $\frac{1}{2}(4r+2s+5)$, by the results of Section 14. To calculate the product terms we again use Leibniz' rule:

$$D_t^r D_\alpha^s(u_k\,\omega_{2,k}) = \sum_{j,l=0}^{r,s}\binom{r}{j}\binom{s}{l} D_t^j D_\alpha^l u_k\cdot D_t^{r-j} D_\alpha^{s-l}\omega_{2,k}. \tag{15.11}$$

Again we may use the results of Section 14, namely that the product in the (j,l) term has singular index determined by

$$\|D_t^j D_\alpha^l u_k\cdot D_t^{r-j} D_\alpha^{s-l}\omega_{2,k}\|_2 \leq \max_\Omega|D_t^j D_\alpha^l u|\cdot\|D_t^{r-j} D_\alpha^{s-l}\omega_{2,k}\|_2. \tag{15.12}$$

The factors of this expression have singular indices

$$2j+l+1 \quad\text{and}\quad \frac{1}{2}(4(r-j)+2(s-l)+3)$$

by (14.2) and (14.1) respectively. Thus the product, and hence (15.11), has singular index $\frac{1}{2}(4r+2s+5)$ also. Similarly, the derivative

$$D_t^r D_a^s(\omega_k u_{2,k}) = \sum_{j,l=0}^{r,s} \binom{r}{j}\binom{s}{l} D_t^j D_a^l \omega_k \cdot D_t^{r-j} D_a^{s-l} u_{2,k} \tag{15.13}$$

is a sum of terms with typical $L^2(\Omega)$ norm

$$\begin{aligned} \|D_t^j D_a^l \omega_k \cdot D_t^{r-j} D_a^{s-l} u_{2,k}\|_2 &\leqslant \|D_t^j D_a^l \omega\|_6 \cdot \|D_t^{r-j} D_a^{s-l} u_{2,k}\|_3 \\ &\leqslant C\|D_t^j \nabla D_a^l \omega\|_2 \sum_{m=0}^{s-l} \|D_t^{r-j} \nabla D_a^m u\|_6^{1/2} \|D_t^{r-j} \nabla D_a^m u\|_2^{1/2} \\ &\leqslant C\|D_t^j \tilde{\Delta} D_a^l u\|_2 \sum_{m=0}^{s-l} \|D_t^{r-j} \tilde{\Delta} D_a^m u\|_2^{1/2} \cdot \|D_t^{r-j} \nabla D_a^m u\|_2^{1/2}. \end{aligned} \tag{15.14}$$

By (14.1) the singularity indices of these factors do not exceed

$$\frac{1}{2}(4j+2l+3), \quad \frac{1}{4}(4(r-j)+2(s-l)+3) \quad \text{and} \quad \frac{1}{4}(4(r-j)+2(s-l)+1)$$

the sum again being at most $\frac{1}{2}(4r+2s+5)$ in every case.

An entirely similar proof works for the other tangential component u_2, and the result is known to be true for u_3. This proves the desired result for the partial derivatives of the form (15.10) and sets the stage for an induction on the normal order s_3. However we should note at this point that (14.2) can now be extended to first normal derivatives as follows (see also [1, p. 718]).

$$\begin{aligned} \max_{x\in\Omega}|D_t^r D_a^s D_3 u| &= \|D_t^r D_a^s D_3 u\|_\infty \\ &\leqslant C\|D_t^r \nabla D_a^s D_3 u\|_2^{1/2} (\|D_t^r \nabla\nabla D_a^s D_3 u\|_2^{1/2} + \|D_t^r \nabla D_a^s D_3 u\|_2^{1/2}) \\ &\in L^{\frac{1}{2r+s+2}}(0, T). \end{aligned} \tag{15.15}$$

Suppose, therefore, as induction hypothesis that the first or $L^2(\Omega)$ result of the main theorem has been established for all partial derivatives with respect to x_3 of order less than or equal to s_3-1, where $s_3 \geqslant 4$. Suppose also that the second, or maximum norm, result of the main theorem has been shown for all partial derivatives with respect to x_3 of order less than or equal to s_3-3. We wish to establish the first result to order s_3 and the second to order s_3-2.

Observe in view of (15.10) that

$$D_t^r D_a^s D_3^{s_3} u_1 = \frac{a_1}{\nu} D_t^r D_a^s D_3^{s_3-3}(\omega_{2,t} - \Delta D_1 u_3 + u_k \omega_{2,k} - \omega_k u_{2,k}) - a_2 D_t^r D_a^s (D_1^2 + D_2^2) D_3^{s_3-2} u_1 + \dots \tag{15.16}$$

where now the omitted lower order derivative terms may contain terms with coefficients differentiated with respect to x_3 which near a singular point is the radial coordinate.

To estimate such lower order terms with coefficients singular at the centre P_0 of a tangential coordinate system, we first omit the sphere $S(P_0, \varepsilon)$ from the domain of integration. By the induction hypothesis on s_3, the integrability property for this reduced domain is easily established. By means of a second tangential coordinate system and corresponding domain covering $S(P_0, \varepsilon)$, and by recalling that all lower order derivatives in this second system have already been estimated, we can however obtain the result for $S(P_0, \varepsilon)$. This establishes the integrability of $D_t^r D_\alpha^s D_3^{s_3} u$, as required for the next stage of the induction on s_3.

The terms on the right in (15.16) have normal order at most s_3-1. The linear terms among them have therefore been estimated as required, by the induction hypothesis. The product terms can be calculated exactly as in (15.11)–(15.14); indeed we need only read the two-dimensional derivative symbol D_α^s as including a third component $D_3^{s_3-3}$ with appropriate Leibniz rule factors. Thus the calculation based on (15.12) goes through as before since the highest index for the D_3 factor in the maximum norm is s_3-3, and the maxima up to this order inclusive have been estimated and are a part of the induction hypothesis. Likewise the calculation based on (15.14) goes through since the highest order normal term is the Stokes operator term with normal order at most $s_3-3+2=s_3-1$; and these terms have been estimated and are included in the induction hypothesis. Therefore the first estimate of the main theorem holds for the derivative on the left side in (15.16). Finally, since all $L^2(\Omega)$ norms of derivatives of normal order s_3 have been estimated as in the main theorem, the estimate for the maximum or $L^\infty(\Omega)$ norms of all derivatives of normal order s_3-2 now follows from the known imbedding inequalities [1, p. 718]. This completes the proof of the induction with respect to s_3. Therefore all derivatives of all normal orders, and consequently, all partial derivatives of all orders in the tangential or Cartesian coordinates, satisfy the conclusions of the theorem. This completes the proof of the main theorem.

16. Concluding comments

The main theorem and the Integrability lemma will be extended in another paper to cover the presence on the right hand side of the Navier–Stokes equations of a forcing term. If however such a forcing term were only finitely differentiable, then the estimates may be valid only up to the corresponding finite stage.

Consider solutions of the form

$$u_i(x,t) = (T-t)^{-\beta} U_i\left(\frac{x_j}{(T-t)^{\alpha}}\right) H(T-t) + u_{2i}(x,t), \tag{16.1}$$

where T, α, β are positive numbers, $H(T-t)$ denotes the Heaviside unit function, $U_i(X_j)$ is C^∞ and $u_{2i}(x,t)$ is C^∞ at least near $(0,T)$. Then for $t<T$,

$$\|D_t^r D_x^s u\|_2 = O\left((T-t)^{-\beta-r-\alpha s+\frac{3\alpha}{2}}\right) \tag{16.2}$$

while

$$\max_{x\in\Omega} |D_t^r D_x^s u| = O((T-t)^{-\beta-r-\alpha s}). \tag{16.3}$$

From the main theorem we may now conclude using (16.2) that

$$\beta+r+\alpha s-\frac{3\alpha}{2} < 2r+s-\frac{1}{2}$$

holds for all permitted sets of values of r and s. Thus $\alpha<1$ follows if $r=0$, s is large, while from the case $r=0$, $s=1$ we conclude $\beta<\frac{1}{2}(1+\alpha)<1$. Equivalent results can be deduced from (16.3). From the $1/p_2$ intercept of Serrin's critical line [33] we may infer $\beta \geq \frac{1}{2}$. The condition $\|\nabla u\|_2 \in L^4(0,T)$ is known to be sufficient for regularity [21, p. 227], and this can also be seen from our estimates of every order in succession, if the last terms on the left are dropped and the estimates are treated as first order linear differential inequalities. This condition implies $\frac{1}{2}\alpha+\frac{1}{4}\leq\beta$. Thus finally,

$$\max\left(\frac{1}{2}, \frac{\alpha}{2}+\frac{1}{4}\right) \leq \beta < \frac{\alpha}{2}+\frac{1}{2} < 1. \tag{16.4}$$

Perhaps the most natural parameter values in the range thus indicated are $\alpha=\beta=\frac{1}{2}$ which were considered by Leray [21, p. 225].

While the asymptotic expansion of a solution with a point singularity need not, in general, have a form as simple as (16.1), it is evident from our results that the singular behaviour must fall within a well defined and non-trivial range of algebraic behaviour. One point singularities with $\alpha=\beta=\frac{1}{2}$ might occur in sequences or condensations leading to a more complicated higher order asymptotic behaviour. Our estimates are a consequence of the underlying algebraic and differential structure in $\mathbf{R}^3$ of the nonlinear convective terms, which seem to permit exceptions to smoothness but only with well-defined limitations. In higher dimensions of space any such limitations would necessarily be very much weaker, as comparable integrability estimates are not available.

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