Acta Math., 186 (2001), 1–84


# Primes represented by $x^3+2y^3$

by

D. R. HEATH-BROWN

*University of Oxford*
*Oxford, England, U.K.*

## Contents



## 1. Introduction

It is conjectured that if $f(X)$ is any irreducible integer polynomial such that $f(1), f(2), \ldots$ tend to infinity and have no common factor greater than 1, then $f(n)$ takes infinitely many prime values. Unfortunately this has only been proved for linear polynomials, in which case the assertion is the famous theorem of Dirichlet. One may seek to formulate a weaker conjecture concerning irreducible binary forms $f(X,Y)$. Here the necessary condition is that the values of $f(m,n)$ for positive integers $m,n$ are unbounded above and have no non-trivial common factor. Again one might hope that such a form attains infinitely many prime values. This is trivial for linear forms, as such a form takes all sufficiently large integer values. For quadratic forms it was proved by Dirichlet, although in certain special cases, such as $f(X,Y)=X^2+Y^2$, the result goes back to Fermat. Dirichlet's result was extended by Iwaniec [14] to quadratic polynomials in two variables. Our goal in the present paper is to make progress in the case of binary cubic forms. We shall prove the following.

THEOREM. *There are infinitely many primes of the form $x^3+2y^3$ with integer $x, y$. More specifically, there is a positive constant $c$ such that, if*

$$\eta=\eta(X)=(\log X)^{-c},$$

*then the number of such primes with $X<x, y\leqslant X(1+\eta)$ is*

$$\sigma_0\frac{\eta^2X^2}{3\log X}\{1+O((\log\log X)^{-1/6})\}$$

*as $X\to\infty$, where*

$$\sigma_0=\prod_p\left(1-\frac{\nu_p-1}{p}\right)$$

*and $\nu_p$ denotes the number of solutions of the congruence $x^3\equiv 2 \pmod{p}$.*

It may be noted that the product $\sigma_0$ is conditionally convergent, but not absolutely convergent.

There is nothing special about the particular range chosen for $x$ and $y$, and a similar theorem could be proved for

$$aX<x\leqslant aX+\eta X, \quad bX<y\leqslant bX+\eta X,$$

for any non-zero $a, b$ such that $a+b\sqrt[3]{2}\neq 0$. Indeed it seems likely that one could do this with sufficient uniformity to deduce a result for an arbitrary bounded set $\mathcal{R}\subseteq\mathbf{R}^2$ with a positive Jordan content. Specifically, for such a set $\mathcal{R}$ one would hope to deduce that

$$\#\{(x,y)\in\mathbf{Z}^2: X^{-1}(x,y)\in\mathcal{R},\, x^3+2y^3 \text{ prime}\}\sim\sigma_0\,\mathrm{meas}(\mathcal{R})\frac{X^2}{3\log X}$$

as $X$ tends to infinity.

Hardy and Littlewood [7, Conjecture N] asked whether there are infinitely many primes which are the sum of three non-negative cubes. Our result shows that this is indeed the case. Hardy and Littlewood went on to give a conjectural asymptotic formula for the number of such representations, but our approach gives no information about this. It is the fact that $x^3+2y^3$ factorizes while $x^3+y^3+z^3$ does not which makes the latter problem more difficult.

It is not hard to prove results on the representation of primes by diagonal cubic forms in four variables, by using the circle method. For general non-singular cubic forms, however, it would appear that such techniques require five or more variables. It seems likely that our method will extend to arbitrary irreducible binary cubic forms, in which case one would be able to tackle irreducible cubic forms in two or more variables, whether they are non-singular or not. One may indeed hope to tackle binary cubic polynomials, providing that they are irreducible over $\mathbf{Q}$ and factor completely over $\overline{\mathbf{Q}}$. There are, however, unpleasant technical difficulties to be dealt with, notably the lack of unique factorization in general cubic fields. None the less, it seems unlikely that these are insurmountable. In particular, one should be able to establish the form of Schinzel's Hypothesis required for the author's work [11] on solutions of diagonal cubic equations in five variables.

Another way in which one might hope to extend the theorem would be to consider incomplete norm forms for fields of higher degree. For example, one might attempt to handle

$$N_{\mathbf{Q}(\sqrt[d]{2})/\mathbf{Q}}(x_1+x_2 2^{1/d}+...+x_n 2^{(n-1)/d})$$

for appropriate $n<d$.

In measuring the quality of any theorem on the representation of primes by an integer polynomial $f(x_1, ..., x_n)$ in several variables, it is useful to consider the exponent $\alpha(f)$, defined as follows. Let $|f|$ denote the polynomial obtained by replacing each coefficient of $f$ by its absolute value, and define $\alpha(f)$ to be the infimum of those real numbers $\alpha$ for which

$$\#\{(x_1, ..., x_n)\in\mathbf{N}^n : |f|(x_1, ..., x_n)\leqslant X\}\ll X^{\alpha}.$$

Thus $\alpha(f)$ measures the frequency of values taken by $f$. If $\alpha\geqslant 1$ we expect $f$ to represent at least $X^{1-\varepsilon}$ of the integers up to $X$, while if $\alpha<1$ we expect around $X^{\alpha}$ such integers to be representable. Thus the smaller the value of $\alpha$, the harder it will be to prove that $f$ represents primes. The two classical theorems of Dirichlet both correspond to $\alpha=1$. For representation by diagonal cubic forms in four variables, as handled by the circle method, one has $\alpha=\frac{4}{3}$. Before the present work there was only one theorem proved in which $\alpha<1$, namely the result of Friedlander and Iwaniec [2] that there are infinitely many primes of the form $x^2+y^4$, for which $\alpha=\frac{3}{4}$. Our theorem corresponds to the still smaller value $\alpha=\frac{2}{3}$, while the conjecture that $x^2+1$ takes infinitely many prime values has $\alpha=\frac{1}{2}$. The groundbreaking work of Friedlander and Iwaniec was the inspiration for the present paper, although the techniques used are quite different.

One needs to be rather careful in formulating conjectures concerning the representation of primes by polynomials in more than one variable, as the example

$$f(x,y)=(y^2+15)\{1-(x^2-23y^2-1)^2\}-5$$

shows. One easily verifies that $f(x,y)$ takes arbitrarily large positive values for $x,y\in\mathbf{Z}$, is absolutely irreducible, and that it takes values coprime to any prescribed integer. However, $f(x,y)$ does not take any positive prime value.

It may be appropriate to mention that one can prove results on the distribution of prime elements in $\mathbf{Z}[\sqrt[3]{2}]$ by the use of Hecke $L$-functions with Grössencharacters. If one has a suitably smooth region $\mathcal{R}\subseteq\mathbf{R}^3\cap[0,X]^3$, with volume at least $X^{3/2+\varepsilon}$ for some positive constant $\varepsilon$, then one can hope to pick out the elements $x+y\sqrt[3]{2}+z\sqrt[3]{4}\in\mathbf{Z}[\sqrt[3]{2}]$ for which $(x,y,z)\in\mathcal{R}$, by using sums of Grössencharacters. In this way one may be able to find infinitely many prime elements, if one assumes the Generalized Riemann Hypothesis, or has suitable zero-density theorems available. Unfortunately the region $[0,X]\times[0,X]\times[0,1)$ is not 'suitably smooth', even though its volume is amply large enough. None the less, the above approach can be used to produce first degree prime elements with $z\ll_{\varepsilon}(|x|+|y|)^{\varepsilon}$, under the Generalized Riemann Hypothesis, for any $\varepsilon>0$.

## 2. A broad outline of the proof

In this section we shall describe the overall plan of attack. The next section will go into greater detail, giving precise statements of various lemmas which together suffice for the proof of our theorem. The later sections will then prove these subsidiary results.

We should mention at the outset that our approach to the sieve procedure has much in common with that given by Friedlander and Iwaniec [3]. They describe a quite general approach to problems involving primes in 'thin' sequences. Unfortunately their condition (R1) is not quite met in our case, so that their work cannot be used as it stands. Although it seems possible that Friedlander and Iwaniec's hypothesis (R1) might be relaxed sufficiently for our application, we have chosen instead to present our own version of the sieve argument. In the light of these remarks, it should be stressed that it is the 'Type II' bound, described below, which is the most novel part of our proof, and not the sieve procedure.

It will be convenient to define the weighted sequence

$$\mathcal{A}=\{x^3+2y^3 : x,y\in(X,X(1+\eta)]\cap\mathbf{N},\ (x,y)=1\},$$

where integers in $\mathcal{A}$ are counted according to the multiplicity of representations. In order to motivate our choice of $\eta$ in the theorem we shall work with an arbitrary $\eta$ in the range

$$\exp\{-(\log X)^{1/3}\}\leqslant\eta\leqslant 1. \tag{2.1}$$

We shall write $\pi(\mathcal{A})$ for the number of primes in $\mathcal{A}$, and prove that

$$\pi(\mathcal{A})=\sigma_0\frac{\eta^2X^2}{3\log X}\{1+O((\log\log X)^{-1/6})\}. \tag{2.2}$$

This clearly suffices for our purposes. To establish (2.2) we shall compare $\pi(\mathcal{A})$ with $\pi(\mathcal{B})$, in which

$$\mathcal{B}=\{N(J)\in(3X^3,3X^3(1+\eta)]\},$$

where $J$ runs over integral ideals of $K=\mathbf{Q}(\sqrt[3]{2})$, and $N$ is the norm from $K$ to $\mathbf{Q}$. The primes in $\mathcal{B}$ therefore correspond to first degree prime ideals. However, the Prime Ideal Theorem may be stated in the form

$$\pi_K(x)=\mathrm{Li}(x)+O\big(x\exp\{-c\sqrt{\log x}\,\}\big) \tag{2.3}$$

for a suitable positive constant $c$, where $\pi_K(x)$ is the number of prime ideals of norm at most $x$. Thus our constraints (2.1) imply that

$$\pi(\mathcal{B})=\frac{3\eta X^3}{3\log X}\left(1+O\left(\frac{1}{\log X}\right)\right).$$

In order to establish (2.2) it therefore suffices to show that

$$\pi(\mathcal{A})=\varkappa\pi(\mathcal{B})+O\left(\frac{\eta^2X^2}{\log X}(\log\log X)^{-1/6}\right), \tag{2.4}$$

where

$$\varkappa=\sigma_0\eta(3X)^{-1}.$$

To compare $\pi(\mathcal{A})$ with $\pi(\mathcal{B})$ we shall perform identical sieve operations on the two sequences, and show that the leading terms correspond. Providing that the error terms are acceptable, this will produce the required asymptotic formula (2.4). This is much easier than trying to evaluate explicitly the leading terms produced by the sequence $\mathcal{A}$ alone, and summing them to produce (2.2).

The argument will require 'Type I' and 'Type II' estimates for the sequences $\mathcal{A}$ and $\mathcal{B}$. The Type I bounds are provided by the following lemmas.

LEMMA 2.1. *For any $q\in\mathbf{N}$ let $\varrho_0(q)$ be the multiplicative function defined by*

$$\varrho_0(p^e)=\frac{\nu_p}{1+p^{-1}},$$

*where $\nu_p$ is the number of first degree prime ideals above $p$. Then if $A$ is any positive integer, there exists $c(A)$ such that*

$$\sum_{Q<q\leqslant 2Q}\tau(q)^A\mu(q)^2\left|\#\mathcal{A}_q-\frac{6\eta^2X^2}{\pi^2}\frac{\varrho_0(q)}{q}\right|\ll(Q+XQ^{1/2}+X^{3/2})(\log QX)^{c(A)}.$$

LEMMA 2.2. *For any $q\in\mathbf{N}$ let $\varrho_1(q)$ be the multiplicative function defined by*

$$\varrho_1(p^e)=p\Big(1-\prod_{P\mid p}\Big(1-\frac{1}{N(P)}\Big)\Big),$$

*where $P$ runs over prime ideals of $K$. Then if $A$ is any positive integer, there exists $c(A)$ such that*

$$\sum_{Q<q\leqslant 2Q}\tau(q)^A\mu(q)^2\left|\#\mathcal{B}_q-\gamma_0\frac{3\eta X^3}{q}\varrho_1(q)\right|\ll X^2Q^{1/3}(\log Q)^{c(A)}.$$

*Here*

$$\gamma_0=\frac{\pi\log\varepsilon_0}{\sqrt{27}}$$

*is the residue of the pole of the Dedekind zeta-function $\zeta_K(s)$ at $s=1$, where $\varepsilon_0=1+\sqrt[3]{2}+\sqrt[3]{4}$ is the fundamental unit of $K$.*

The function $\tau(q)$ occurring here is the ordinary divisor function. Note also that the function $\nu_p$ described in Lemma 2.1 agrees with that defined in the statement of our theorem.

It is appropriate to introduce at this point a notational device which will be used throughout this paper. The letter $c$ will be used to denote a positive absolute constant, though not necessarily the same at each occurrence. Similarly, given a parameter $A$, we use $c(A)$ to denote a 'constant' depending only on $A$, again potentially different at each occurrence. The reader should however be warned that the parameter $A$ may have different meanings in different places. Thus, for example, the exponent $c(A)$ in Lemma 2.1 is a function of the parameter $A$ in Lemma 2.1, rather than that which occurs in Lemma 2.2. In practice, the meaning should be clear from the context. Note that all implied constants are allowed to depend on $A$.

Lemmas 2.1 and 2.2 show in particular that $\mathcal{A}$ and $\mathcal{B}$ have 'level of distribution' $X^{2-\varepsilon}$ and $X^{3-\varepsilon}$ respectively, for any $\varepsilon>0$. The result for $\mathcal{B}$ is unsurprising, but it is certainly worthy of comment that one can prove such a sharp result for $\mathcal{A}$. Estimates of this type are not hard to obtain, and go back to Greaves [5], (see also the recent work of Daniel [1] for an alternative approach). It should be noted that for the ternary form $x_1^3+x_2^3+x_3^3$, only a level of distribution $X^{3/2-\varepsilon}$ has been proved unconditionally. Assuming the Riemann Hypothesis for certain Dedekind zeta-functions, Hooley [12] has extended the range for this latter problem to $X^{2-\varepsilon}$. As remarked in the introduction it is the fact that the form $x^3+2y^3$ factorizes which enables such a strong Type I bound to be established.

The 'Type II' estimate will be more complicated to state, but, roughly speaking, it will allow us to handle sums

$$\sum_{U<a\leqslant 2U}\ \sum_{V<b\leqslant 2V:\,ab\in\mathcal{A}} \phi_a\psi_b$$

when $X^{1+\varepsilon}\ll V\ll X^{3/2-\varepsilon}$.

A standard application of the identity of Vaughan, or the author's generalization of it, shows that a Type I bound with level of distribution $X^{2-\varepsilon}$, together with a Type II bound covering the range $X^{1-\varepsilon}\ll V\ll X^{3/2+\varepsilon}$, suffice for an easy proof that $\mathcal{A}$ contains the expected number of primes. The reader will observe that, by symmetry, if one has a Type II bound for $V_1\ll V\ll V_2$, then one can also cover the range $X^3/V_2\ll V\ll X^3/V_1$. It is thus apparent that we have two small intervals $X^{1-\varepsilon}\ll V\ll X^{1+\varepsilon}$ and $X^{3/2-\varepsilon}\ll V\ll X^{3/2+\varepsilon}$ which we are unable to handle by Vaughan's method. This forces us to resort to a more delicate sieve procedure, in which relatively trivial bounds are applied on these ranges. The two intervals are sufficiently small that their total contribution is negligible. This technique is typical in situations where sieve methods are used to prove asymptotic formulae. The author's work [10] on the asymptotic formula for the number of primes in the interval $(x, x+x^{7/12-\varepsilon}]$ is a good illustration of this, though by no means the first occurrence of the method.

At this point we introduce a new parameter

$$\tau=(\log\log X)^{-\varpi}, \tag{2.5}$$

where $\varpi$ is a positive absolute constant. The parameter $\tau$ will play the rôle of the exponent $\varepsilon$ above, making precise its dependence on $X$. We shall eventually choose $\varpi=\frac{1}{6}$, but we shall motivate this choice by recording at each stage of the argument any constraints that must be imposed on the size of $\varpi$ in order for the proof to proceed.

In order to describe the sieve process in simple terms we shall depart from the analysis that is to be adopted in practice. Thus, what follows is for illustrative purposes, the actual procedure being described in the next section.

We start by observing that

$$\pi(\mathcal{A})=S(\mathcal{A},2X^{3/2}).$$

Buchstab's identity now yields

$$\begin{aligned}
S(\mathcal{A},2X^{3/2}) &= S(\mathcal{A},X^{\tau})-\sum_{X^{\tau}\leqslant p<X^{1-\tau}} S(\mathcal{A}_p,p)-\sum_{X^{1-\tau}\leqslant p<X^{1+\tau}} S(\mathcal{A}_p,p)\\
&\quad -\sum_{X^{1+\tau}\leqslant p<X^{3/2-\tau}} S(\mathcal{A}_p,p)-\sum_{X^{3/2-\tau}\leqslant p<2X^{3/2}} S(\mathcal{A}_p,p)\\
&= S_1(\mathcal{A})-S_2(\mathcal{A})-S_3(\mathcal{A})-S_4(\mathcal{A})-S_5(\mathcal{A}),
\end{aligned}$$

say. Since $\tau$ tends to zero as $X$ goes to infinity, we shall be able to handle $S_1(\mathcal{A})$ by a sieve estimate of 'Fundamental Lemma' type. The sums $S_3(\mathcal{A})$ and $S_5(\mathcal{A})$ run over ranges that cannot be handled by our Type II estimate. They will therefore be bounded below by 0, and above via a crude sieve bound. For the latter we only require that $p$ is smaller than our level of distribution $X^{2-\varepsilon}$. This will produce estimates

$$S_3(\mathcal{A}), S_5(\mathcal{A}) \ll \tau \frac{\eta^2 X^3}{\log X},$$

which are acceptably small. The sum $S_4(\mathcal{A})$ is already in a form close to that required for our Type II estimate. However, $S_2(\mathcal{A})$ requires some further manipulation. We set

$$S^{(n)}(\mathcal{A}) = \sum_{\substack{X^\tau \leqslant p_n < \ldots < p_1 < X^{1-\tau} \\ p_1 \ldots p_n < X^{1+\tau}}} S(\mathcal{A}_{p_1 \ldots p_n}, p_n),$$

so that $S_2(\mathcal{A}) = S^{(1)}(\mathcal{A})$. We now observe that Buchstab's identity yields

$$S^{(n)}(\mathcal{A}) = T^{(n)}(\mathcal{A}) - U^{(n)}(\mathcal{A}) - S^{(n+1)}(\mathcal{A}),$$

where

$$T^{(n)}(\mathcal{A}) = \sum_{\substack{X^\tau \leqslant p_n < \ldots < p_1 < X^{1-\tau} \\ p_1 \ldots p_n < X^{1+\tau}}} S(\mathcal{A}_{p_1 \ldots p_n}, X^\tau)$$

and

$$U^{(n)}(\mathcal{A}) = \sum_{\substack{X^\tau \leqslant p_{n+1} < \ldots < p_1 < X^{1-\tau} \\ p_1 \ldots p_n < X^{1+\tau} \leqslant p_1 \ldots p_{n+1}}} S(\mathcal{A}_{p_1 \ldots p_{n+1}}, p_{n+1}).$$

By iteration this leads to

$$S_2(\mathcal{A}) = \sum_{1 \leqslant n \leqslant n_0} (-1)^{n-1} (T^{(n)}(\mathcal{A}) - U^{(n)}(\mathcal{A})),$$

with

$$n_0 \ll \tau^{-1}, \tag{2.6}$$

since any term of the sum $S^{(n)}(\mathcal{A})$ will vanish for $p_1 \ldots p_n > 4X^3$. We may now attempt to handle $S_2(\mathcal{A})$ by applying a Fundamental Lemma sieve to the terms $T^{(n)}(\mathcal{A})$, and a Type II estimate to the terms $U^{(n)}(\mathcal{A})$. For $T^{(n)}(\mathcal{A})$ we have $p_1 \ldots p_n < X^{1+\tau}$, which is certainly small enough for the available level of distribution. For $U^{(n)}(\mathcal{A})$ we note that

$$X^{1+\tau} \leqslant p_1 \ldots p_{n+1} \leqslant (p_1 \ldots p_n)^{(n+1)/n} < X^{4(1+\tau)/3} \leqslant X^{3/2-\tau}$$

for $n\geqslant 3$. However, $U^{(1)}(\mathcal{A})$ and $U^{(2)}(\mathcal{A})$ have to be decomposed as

$$\begin{aligned}U^{(1)}(\mathcal{A})= &\sum_{\substack{X^\tau\leqslant p_2<p_1<X^{1-\tau}\\ X^{1+\tau}\leqslant p_1p_2\leqslant X^{3/2-\tau}}} S(\mathcal{A}_{p_1p_2},p_2)+\sum_{\substack{X^\tau\leqslant p_2<p_1<X^{1-\tau}\\ X^{3/2-\tau}<p_1p_2<X^{3/2+\tau}}} S(\mathcal{A}_{p_1p_2},p_2)\\ &+\sum_{\substack{X^\tau\leqslant p_2<p_1<X^{1-\tau}\\ p_1p_2\geqslant X^{3/2+\tau}}} S(\mathcal{A}_{p_1p_2},p_2)\\ &=U_1^{(1)}(\mathcal{A})+S_6(\mathcal{A})+U_2^{(1)}(\mathcal{A}),\end{aligned}$$

say, and

$$\begin{aligned}U^{(2)}(\mathcal{A})= &\sum_{\substack{X^\tau\leqslant p_3<p_2<p_1<X^{1-\tau}\\ p_1p_2<X^{1+\tau}\leqslant p_1p_2p_3\leqslant X^{3/2-\tau}}} S(\mathcal{A}_{p_1p_2p_3},p_3)+\sum_{\substack{X^\tau\leqslant p_3<p_2<p_1<X^{1-\tau}\\ p_1p_2<X^{1+\tau},\,p_1p_2p_3>X^{3/2-\tau}}} S(\mathcal{A}_{p_1p_2p_3},p_3)\\ &=U_1^{(2)}(\mathcal{A})+S_7(\mathcal{A}),\end{aligned}$$

say. We shall bound $S_6(\mathcal{A})$ and $S_7(\mathcal{A})$ from below by zero, and from above by a crude sieve bound, in the same way as for $S_3(\mathcal{A})$ and $S_5(\mathcal{A})$. Moreover $U_1^{(1)}(\mathcal{A})$ and $U_1^{(2)}(\mathcal{A})$ are in an appropriate form for our Type II estimate, while for $U_2^{(1)}(\mathcal{A})$ we merely have to note that the integer $a=(x^3+2y^3)p_1^{-1}p_2^{-1}$ lies in the range $X^{1+2\tau}\ll a\ll X^{3/2-\tau}$, which is also suitable.

A precisely analogous sieve decomposition applies to $S(\mathcal{B},2X^{3/2})$. We can then compare leading terms from the two decompositions to establish the asymptotic equality (2.4).

We now discuss the Type II bound. We shall do this in the context of the sums $S_4(\mathcal{A})$ and $S_4(\mathcal{B})$, this being the simplest example. It is clear that we cannot get any cancellation from the two sums individually, since they are composed of non-negative terms. We wish, however, to avoid a Type II estimate for the difference $S_4(\mathcal{A})-\varkappa S_4(\mathcal{B})$, which would involve the two sequences simultaneously. We therefore plan to remove a leading term from $S_4(\mathcal{A})$. This latter sum is essentially

$$\sum_{X^{1+\tau}\leqslant n<X^{3/2-\tau}}\frac{\Lambda(n)}{\log n}S(\mathcal{A}_n,n).$$

We shall decompose $\Lambda(n)$ as $\Lambda_1(n)+\Lambda_2(n)$, where

$$\Lambda_1(n)=\sum_{d\mid n:\,d<L}\mu(d)\log\frac{L}{d}$$

and $L=X^{\tau/2}$. This type of splitting (with a slightly different function $\Lambda_1(n)$) seems to have been introduced by the author [9]. The precise form given above was first used in this type of context by Goldston [4]. The function $\Lambda_1(n)$ is so constructed as to mimic the distribution of $\Lambda(n)$ over residue classes. Thus the average of $\Lambda_2(n)$ in residue classes will be small. Moreover the function $\Lambda_1(n)$ is easily handled if $L$ is small, and its contribution will be shown to match $S_4(\mathcal{B})$ closely. In fact the sum $S_4(\mathcal{B})$ can be estimated directly, as one can give asymptotic formulae for the individual terms

$$S(\mathcal{B}_p, p).$$

The outcome of the above discussion is that we require a Type II bound for a sum

$$\sum_{U<a\leqslant 2U}\ \sum_{V<b\leqslant 2V:\, ab\in\mathcal{A}} \phi_a\psi_b$$

where $\psi_b$ comes from the function $\Lambda_2(n)$. We may therefore assume that the average of $\psi_b$ over arithmetic progressions is small. Thus it is no longer necessary to demonstrate cancellation between the two sequences $\mathcal{A}$ and $\mathcal{B}$. Instead the saving will come from sign changes in $\psi_b$.

The treatment of the above Type II sum forms the core of the paper. Eventually the estimation is made to depend on a large sieve inequality, but there is much preparatory work, which the reader will find described in the relevant sections.

## 3. Outline of the proof—further details

Although the description in the previous section was given purely in terms of the arithmetic of $\mathbf{Z}$ it is more natural to consider also the corresponding sieve problem for ideals of the field $K=\mathbf{Q}(\sqrt[3]{2})$. We therefore set

$$\mathcal{A}^{(K)} = \{(x+y\sqrt[3]{2}) : x, y\in (X, X(1+\eta)]\cap\mathbf{N},\ (x,y)=1\}$$

and

$$\mathcal{B}^{(K)} = \{J : N(J)\in (3X^3, 3X^3(1+\eta)]\}.$$

The superscript $(K)$ is intended to remind the reader that we are working over the field $K$. It should be observed at this point that if $\eta$ is small enough, no two values of $x+y\sqrt[3]{2}$ are associates, so that $\mathcal{A}^{(K)}$ contains distinct ideals. The following elementary fact, which will be proved in the next section, will also be used repeatedly.

LEMMA 3.1. *No prime ideal of degree greater than 1 can divide an element of $\mathcal{A}^{(K)}$, nor can a product of two distinct first degree prime ideals of the same norm. Thus if a square-free ideal $R$ divides an element of $\mathcal{A}^{(K)}$, then $N(R)$ must be square-free.*

The Type I bounds for $\mathcal{A}^{(K)}$ and $\mathcal{B}^{(K)}$ are the following.

LEMMA 3.2. *Let $\varrho_2(R)$ be the multiplicative function on ideals defined by*

$$\varrho_2(P^e)=(1+N(P)^{-1})^{-1}.$$

*Then for any positive integer $A$ there exists a corresponding $c(A)$ such that*

$$\sum_{\substack{Q<N(R)\leqslant 2Q\\ R\in\mathcal{R}}} \tau(R)^A\left|\#\mathcal{A}_R^{(K)}-\frac{6\eta^2X^2}{\pi^2N(R)}\varrho_2(R)\right| \ll (Q+XQ^{1/2}+X^{3/2})(\log QX)^{c(A)}.$$

*Here $\mathcal{R}$ is the set of ideals $R$ for which $N(R)$ is square-free.*

LEMMA 3.3. *For any positive integer $A$ there exists a corresponding $c(A)$ such that*

$$\sum_{Q<N(R)\leqslant 2Q} \tau(R)^A\left|\#\mathcal{B}_R^{(K)}-\gamma_0\frac{3\eta X^3}{N(R)}\right| \ll X^2Q^{1/3}(\log Q)^{c(A)}.$$

We shall use the Buchstab identity over the field $K$, sieving $x+y\sqrt[3]{2}$ by prime ideals. If every prime ideal factor $P$ of $x+y\sqrt[3]{2}$ has $N(P)\geqslant 2X^{3/2}$ then $(x+y\sqrt[3]{2})$ will be a prime ideal, whence $x^3+2y^3$ is a prime, by Lemma 3.1.

For a set $\mathcal{I}$ of integral ideals of $K$, and any integral ideal $E$, we write

$$\mathcal{I}_E=\{I\in\mathcal{I}:E\,|\,I\}.$$

We also set

$$S_K(\mathcal{I},z)=\#\{I\in\mathcal{I}:P|I\Rightarrow N(P)\geqslant z\},$$

for any real $z>1$. The subscript $K$ is again intended to remind the reader that we are working over the field $K$. With this obvious extension of the standard notation we see that

$$\pi(\mathcal{A})=S_K(\mathcal{A}^{(K)},2X^{3/2}).$$

Buchstab's identity now yields

$$\begin{aligned} S_K(\mathcal{A}^{(K)},2X^{3/2}) = S_K(\mathcal{A}^{(K)},X^\tau) &- \sum_{X^\tau\leqslant N(P)<X^{1-\tau}} S_K(\mathcal{A}_P^{(K)},N(P)) \\ &- \sum_{X^{1-\tau}\leqslant N(P)<X^{1+\tau}} S_K(\mathcal{A}_P^{(K)},N(P)) \\ &- \sum_{X^{1+\tau}\leqslant N(P)<X^{3/2-\tau}} S_K(\mathcal{A}_P^{(K)},N(P)) \\ &- \sum_{X^{3/2-\tau}\leqslant N(P)<2X^{3/2}} S_K(\mathcal{A}_P^{(K)},N(P)) \\ = S_1(\mathcal{A})-S_2(\mathcal{A})-S_3(\mathcal{A})-S_4(\mathcal{A})-S_5(\mathcal{A}), \end{aligned}$$

as in the previous section. Similarly we set

$$\begin{aligned} S^{(n)}(\mathcal{A}) &= \sum_{\substack{X^\tau\leqslant N(P_n)<\ldots<N(P_1)<X^{1-\tau} \\ N(P_1\ldots P_n)<X^{1+\tau}}} S_K(\mathcal{A}_{P_1\ldots P_n}^{(K)},N(P_n)), \\ T^{(n)}(\mathcal{A}) &= \sum_{\substack{X^\tau\leqslant N(P_n)<\ldots<N(P_1)<X^{1-\tau} \\ N(P_1\ldots P_n)<X^{1+\tau}}} S_K(\mathcal{A}_{P_1\ldots P_n}^{(K)},X^\tau) \end{aligned} \tag{3.1}$$

and

$$U^{(n)}(\mathcal{A}) = \sum_{\substack{X^\tau\leqslant N(P_{n+1})<\ldots<N(P_1)<X^{1-\tau} \\ N(P_1\ldots P_n)<X^{1+\tau}\leqslant N(P_1\ldots P_{n+1})}} S_K(\mathcal{A}_{P_1\ldots P_{n+1}}^{(K)},N(P_{n+1})).$$

We should note here that the various prime ideals $P_i$ which occur when Buchstab's identity is applied must have distinct norms, by Lemma 3.1. It now follows that

$$S_2(\mathcal{A}) = \sum_{1\leqslant n\leqslant n_0} (-1)^{n-1}(T^{(n)}(\mathcal{A})-U^{(n)}(\mathcal{A})),$$

with $n_0\ll\tau^{-1}$. As before we note that

$$X^{1+\tau}\leqslant N(P_1\ldots P_{n+1})\leqslant N(P_1\ldots P_n)^{(n+1)/n}<X^{4(1+\tau)/3}\leqslant X^{3/2-\tau} \tag{3.2}$$

for $n\geqslant 3$, so that $U^{(n)}(\mathcal{A})$ can be handled as a Type II sum for $n\geqslant 3$.

Following the procedure of the previous section we also put

$$U_1^{(1)}(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_2)<N(P_1)<X^{1-\tau}\\ X^{1+\tau}\leqslant N(P_1P_2)\leqslant X^{3/2-\tau}}} S_K(\mathcal{A}^{(K)}_{P_1P_2},N(P_2)),$$

$$U_2^{(1)}(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_2)<N(P_1)<X^{1-\tau}\\ N(P_1P_2)\geqslant X^{3/2+\tau}}} S_K(\mathcal{A}^{(K)}_{P_1P_2},N(P_2)),$$

$$U_1^{(2)}(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_3)<N(P_2)<N(P_1)<X^{1-\tau}\\ N(P_1P_2)<X^{1+\tau}\leqslant N(P_1P_2P_3)\leqslant X^{3/2-\tau}}} S_K(\mathcal{A}^{(K)}_{P_1P_2P_3},N(P_3)),$$

$$S_6(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_2)<N(P_1)<X^{1-\tau}\\ X^{3/2-\tau}<N(P_1P_2)<X^{3/2+\tau}}} S_K(\mathcal{A}^{(K)}_{P_1P_2},N(P_2))$$

and

$$S_7(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_3)<N(P_2)<N(P_1)<X^{1-\tau}\\ N(P_1P_2)<X^{1+\tau}\\ N(P_1P_2P_3)>X^{3/2-\tau}}} S_K(\mathcal{A}^{(K)}_{P_1P_2P_3},N(P_3)).$$

We then have

$$U^{(1)}(\mathcal{A})=U_1^{(1)}(\mathcal{A})+S_6(\mathcal{A})+U_2^{(1)}(\mathcal{A})$$

and

$$U^{(2)}(\mathcal{A})=U_1^{(2)}(\mathcal{A})+S_7(\mathcal{A}).$$

A precisely analogous sieve decomposition applies to $S_K(\mathcal{B}^{(K)},2X^{3/2})$, where

$$\mathcal{B}^{(K)}=\{A:N(A)\in(3X^3,3X^3(1+\eta)]\}.$$

In this case, however, the various prime ideals $P_i$ that arise need not have distinct norms, although the ideals themselves must be distinct. A further difference is that $J\in\mathcal{B}^{(K)}$ can be a prime ideal without $N(J)$ being prime. Thus

$$\pi(\mathcal{B})=S_K(\mathcal{B}^{(K)},2X^{3/2})+O(X^2).$$

We can extend the definition (3.1) to the case $n=0$ in the natural way, so that $S_1(\mathcal{A})=T^{(0)}(\mathcal{A})$, and similarly for $\mathcal{B}$. This gives us the following basic sieve decomposition, in the obvious notation.

LEMMA 3.4. *We have*

$$\begin{aligned}\pi(\mathcal{A})-\varkappa\pi(\mathcal{B}) \ll X^{3/2} &+ \sum_{0\leqslant n\leqslant n_0} |T^{(n)}(\mathcal{A})-\varkappa T^{(n)}(\mathcal{B})| \\ &+|U_1^{(1)}(\mathcal{A})-\varkappa U_1^{(1)}(\mathcal{B})|+|U_1^{(2)}(\mathcal{A})-\varkappa U_2^{(1)}(\mathcal{B})| \\ &+|U_1^{(2)}(\mathcal{A})-\varkappa U_1^{(2)}(\mathcal{B})|+\sum_{3\leqslant n\leqslant n_0} |U^{(n)}(\mathcal{A})-\varkappa U^{(n)}(\mathcal{B})| \\ &+\sum_{j=3,5,6,7} (S_j(\mathcal{A})+\varkappa S_j(\mathcal{B}))+|S_4(\mathcal{A})-\varkappa S_4(\mathcal{B})|.\end{aligned}$$

For this expression we shall establish the following bound via the Fundamental Lemma sieve.

LEMMA 3.5. *We have*

$$\sum_{0\leqslant n\leqslant n_0} |T^{(n)}(\mathcal{A})-\varkappa T^{(n)}(\mathcal{B})| \ll \tau\frac{\eta^2X^2}{\log X}.$$

Moreover an upper bound sieve will yield the following estimate.

LEMMA 3.6. *We have*

$$S_j(\mathcal{A})+\varkappa S_j(\mathcal{B}) \ll \tau\frac{\eta^2X^2}{\log X}$$

*for* $j=3,5,6$ *or* $7$.

We now prepare the terms $U_j^{(n)}(\mathcal{A})$, $U^{(n)}(\mathcal{A})$ and $S_4(\mathcal{A})$ for the Type II estimate. Our goal is to approximate each of these by a combination of sums

$$\sum_R c_R \sum_{S:\,RS\in\mathcal{A}^{(K)}} d_S. \tag{3.3}$$

The coefficients $c_R$ will take only the values 1 or 0, and will be supported on ideals $R\in\mathcal{R}$ all of whose prime factors $P$ satisfy $N(P)\geqslant X^\tau$. Similarly the coefficients $d_S$ will be supported on ideals

$$S=\prod_{i=1}^{n+1} P_i,$$

where $P_i$ are first degree prime ideals with $N(P_i)\in J(m_i)$. Here the intervals $J(m)$ take the form $[X^{m\xi}, X^{(m+1)\xi})$, where

$$\xi=(\log\log X)^{-\varpi_0} \tag{3.4}$$

for some constant $\varpi_0\in[\varpi,1)$. Moreover we shall require that

$$m_1>...>m_{n+1}\geqslant\tau\xi^{-1}, \tag{3.5}$$

whence the intervals $J(m_i)$ are disjoint, and the ideals $S$ are square-free. In addition we shall need

$$\sum_{i=1}^{n+1} m_i\geqslant(1+\tau)\xi^{-1} \tag{3.6}$$

and

$$\sum_{i=1}^{n+1}(m_i+1)\leqslant\left(\tfrac{3}{2}-\tau\right)\xi^{-1}, \tag{3.7}$$

so that

$$X^{1+\tau}\leqslant N(S)<X^{3/2-\tau}. \tag{3.8}$$

Finally, for $S$ as above, the coefficient $d_S$ will take the form

$$d_S=d_S(\mathbf{m})=\prod_{i=1}^{n+1}\frac{\log N(P_i)}{m_i\xi\log X},$$

where $\mathbf{m}=(m_1,...,m_{n+1})$.

In order to show how this is achieved, we begin by discussing $U^{(n)}(\mathcal{A})$ for $n\geqslant3$. Here we shall define

$$\widehat{U}^{(\mathbf{m},n)}(\mathcal{A})=\sum_{N(P_i)\in J(m_i)} S_K^{(*)}(\mathcal{A}_S^{(K)},X^{m_{n+1}\xi})\prod_{i=1}^{n+1}\frac{\log N(P_i)}{m_i\xi\log X},$$

where $\mathbf{m}$ satisfies both

$$\sum_{i=1}^{n}(m_i+1)\leqslant(1+\tau)\xi^{-1}$$

and $m_1+1\leqslant(1-\tau)\xi^{-1}$, in addition to the constraints (3.5), (3.6) and (3.7). The notation $S_K^{(*)}$ indicates that only elements $RS\in\mathcal{A}^{(K)}$ for which $N(R)$ is square-free are to be counted. It is now clear that $\widehat{U}^{(\mathbf{m},n)}(\mathcal{A})$ takes the required form (3.3), where $c_R=1$ precisely for those $R\in\mathcal{R}$ which have no prime ideal factor $P$ with $N(P)<X^{m_{n+1}\xi}$. Moreover we set

$$\widehat{U}^{(n)}(\mathcal{A})=\sum_{\mathbf{m}}\widehat{U}^{(\mathbf{m},n)}(\mathcal{A}),$$

which will be the required approximation to $U^{(n)}(\mathcal{A})$.

We can handle $U_1^{(n)}(\mathcal{A})$ in exactly the same way for $n=1$ and 2, to produce approximations $\widehat{U}_1^{(n)}(\mathcal{A})$. We may also treat $S_4(\mathcal{A})$ along the same lines. Here we set

$$\widehat{S}_4^{(m)}(\mathcal{A})=\sum_{N(P)\in J(m)} S_K^{(*)}(\mathcal{A}_P^{(K)},X^{m\xi})\frac{\log N(P)}{m\xi\log X},$$

where

$$(1+\tau)\xi^{-1} \leqslant m \leqslant \left(\tfrac{3}{2}-\tau\right)\xi^{-1}-1.$$

We then take

$$\widehat{S}_4(\mathcal{A}) = \sum_m \widehat{S}_4^{(m)}(\mathcal{A})$$

as our approximation to $S_4(\mathcal{A})$.

In the case of $U_2^{(1)}(\mathcal{A})$ the rôles of $R$ and $S$ are reversed. We confine $N(P_1)$ and $N(P_2)$ to intervals $J(n_1)$ and $J(n_2)$ respectively, where

$$\tau\xi^{-1} \leqslant n_2 < n_1 \leqslant (1-\tau)\xi^{-1}-1$$

and

$$n_1+n_2 \geqslant \left(\tfrac{3}{2}+\tau\right)\xi^{-1},$$

and we replace

$$S_K(\mathcal{A}_{P_1P_2}^{(K)}, N(P_2))$$

by

$$S_K(\mathcal{A}_{P_1P_2}^{(K)}, X^{n_2\xi}).$$

This counts products

$$P_1P_2Q_1Q_2 \ldots Q_{n+1}$$

of prime ideals in which

$$N(Q_1) \geqslant \ldots \geqslant N(Q_{n+1}) \geqslant X^{n_2\xi}.$$

We therefore introduce intervals $J(m_i)$ as before, with

$$m_1 > \ldots > m_{n+1} \geqslant n_2,$$

and require that $N(Q_i) \in J(m_i)$. If we now define

$$\widehat{U}_2^{(\mathbf{n},\mathbf{m},1)} = \sum_{R=P_1P_2} \sum_{\substack{S=Q_1\ldots Q_{n+1} \\ RS\in\mathcal{A}^{(K)}}} \prod_{i=1}^{n+1} \frac{\log N(Q_i)}{m_i\xi\log X},$$

we may approximate $U_2^{(1)}(\mathcal{A})$ satisfactorily by

$$\widehat{U}_2^{(1)}(\mathcal{A}) = \sum_{\mathbf{n},\mathbf{m}} \widehat{U}_2^{(\mathbf{n},\mathbf{m},1)},$$

where we sum over appropriate vectors $\mathbf{m}$, regardless of their length. We may note, however, that $n \ll \tau^{-1}$, just as before.

The corresponding approximations with $\mathcal{A}$ replaced by $\mathcal{B}$ are defined analogously, though we must bear in mind that only prime ideals of first degree may divide $S$. The following lemma, whose proof uses simple sieve upper bounds, estimates the errors involved in all these approximations. Anticipating the form of the result, it is natural to set

$$\xi = \tau^5, \tag{3.9}$$

which we now do. Thus $\varpi_0 = 5\varpi$, and we therefore require that

$$0 < \varpi < \tfrac{1}{5} \tag{3.10}$$

in order to ensure that $\varpi_0 < 1$.

LEMMA 3.7. *We have*

$$\sum_{n \geqslant 3} |U^{(n)}(\mathcal{A}) - \widehat{U}^{(n)}(\mathcal{A})| \ll \frac{\eta^2 X^2}{\log X} \xi \tau^{-4},$$

$$|U_1^{(n)}(\mathcal{A}) - \widehat{U}_1^{(n)}(\mathcal{A})| \ll \frac{\eta^2 X^2}{\log X} \xi \tau^{-3} \quad \textit{for } n = 1 \textit{ and } 2,$$

$$|S_4(\mathcal{A}) - \widehat{S}_4(\mathcal{A})| \ll \frac{\eta^2 X^2}{\log X} \xi \tau^{-3}$$

*and*

$$|U_2^{(1)}(\mathcal{A}) - \widehat{U}_2^{(1)}(\mathcal{A})| \ll \frac{\eta^2 X^2}{\log X} \xi \tau^{-4}.$$

*Similarly we have*

$$\sum_{n \geqslant 3} |U^{(n)}(\mathcal{B}) - \widehat{U}^{(n)}(\mathcal{B})| \ll \frac{\eta X^3}{\log X} \xi \tau^{-4},$$

$$|U_1^{(n)}(\mathcal{B}) - \widehat{U}_1^{(n)}(\mathcal{B})| \ll \frac{\eta X^3}{\log X} \xi \tau^{-3} \quad \textit{for } n = 1 \textit{ and } 2,$$

$$|S_4(\mathcal{B}) - \widehat{S}_4(\mathcal{B})| \ll \frac{\eta X^3}{\log X} \xi \tau^{-3}$$

*and*

$$|U_2^{(1)}(\mathcal{B}) - \widehat{U}_2^{(1)}(\mathcal{B})| \ll \frac{\eta X^3}{\log X} \xi \tau^{-4}.$$

We have now to consider sums of the form

$$\sum_{R\in\mathcal{R}} c_R \sum_{S:\,RS\in\mathcal{A}^{(K)}} d_S = U(\mathcal{A}),$$

say. It is clear that we cannot get any cancellation from $U(\mathcal{A})$ and $U(\mathcal{B})$ individually, but only from the difference

$$U(\mathcal{A}) - \varkappa U(\mathcal{B}).$$

In order to avoid a Type II estimate involving the two sequences $\mathcal{A}$ and $\mathcal{B}$ simultaneously, we remove a leading term from the first sum, by writing

$$d_S = e_S + f_S, \tag{3.11}$$

where the 'leading part' $e_S$ is given by

$$e_S = \frac{w'(N(S))}{\prod_{i=1}^{n+1}(m_i\xi\log X)} \sum_{J\mid S:\,N(J)<L} \mu(J)\log\frac{L}{N(J)}. \tag{3.12}$$

Here

$$L = X^{\tau/2} \tag{3.13}$$

and

$$w(t) = w(t,\mathbf{m}) = \text{meas}\Big\{\mathbf{x}\in\mathbf{R}^{n+1} : x_i\in J(m_i),\ \prod x_i \leqslant t\Big\}.$$

Note that for the case $n=0$, the function $w(t)$ is only piecewise continuously differentiable, in which case we define the derivative $w'(t)$ to be the right-hand derivative, for precision. The function $w(t)$ which occurs here has been constructed so that

$$\frac{w(t)}{\prod_{i=1}^{n+1}(m_i\xi\log X)}$$

is an approximation to

$$\sum_{N(S)\leqslant t} d_S,$$

according to the Prime Ideal Theorem. We shall see that $e_S$ is easily handled if $L$ is small, and is so constructed as to mimic the distribution of $d_S$ over residue classes. Thus the average of $f_S$ in residue classes will be small. The following lemma makes this precise.

LEMMA 3.8. *Let $C\subseteq\mathbf{R}^3$ be a cube of side $S_0\geqslant L^2$ and edges parallel to the coordinate axes. Suppose that for every vector $(x,y,z)\in C$ we have $x,y,z\ll V^{1/3}$ and*

$$x^3+2y^3+4z^3-6xyz \gg V.$$

*For each* $\beta=a+b\sqrt[3]{2}+c\sqrt[3]{4}\in K$ *let* $\hat{\beta}$ *be the vector* $(a,b,c)$. *Let a constant* $A>0$ *be given. Then for any integer* $\alpha\in\mathbf{Z}[\sqrt[3]{2}]$ *we have*

$$\sum_{\substack{\beta\equiv\alpha \,(\mathrm{mod}\, q)\\ \hat{\beta}\in\mathcal{C}}} f_{(\beta)} \ll V\exp\{-c\sqrt{\log L}\}$$

*uniformly for* $q\leqslant(\log X)^A$.

The reader should note that the implied constant is ineffective, as a result of potential problems with Siegel zeros. The reader should also observe that Lemma 3.8 does not require $\alpha$ to be coprime to $q$.

We can now decompose our sums as $U=U_e+U_f$, with

$$U_e(\mathcal{A})=\sum_R c_R \sum_{S:RS\in\mathcal{A}^{(K)}} e_S$$

and

$$U_f(\mathcal{A})=\sum_R c_R \sum_{S:RS\in\mathcal{A}^{(K)}} f_S.$$

The parameter $L$ has been chosen so that $N(JR)\ll X^{2-\tau/2}$, as we shall see. This is sufficiently small that $U_e(\mathcal{A})$ can be readily handled via Lemma 3.2. On the other hand, $U(\mathcal{B})$ can be estimated directly by the Prime Ideal Theorem. This leads to the following bound.

LEMMA 3.9. *There is an absolute constant c such that*

$$U_e(\mathcal{A})-\varkappa U(\mathcal{B})\ll M^{-1}\eta^{5/2}X^2(\log X)^c,$$

*where*

$$M=\prod_{i=1}^{n+1} m_i.$$

*Moreover, in the obvious notation, each of*

$$\sum_{n\geqslant 3}|\widehat{U}_e^{(n)}(\mathcal{A})-\varkappa\widehat{U}^{(n)}(\mathcal{B})|,$$

$$\widehat{U}_{1,e}^{(n)}(\mathcal{A})-\varkappa\widehat{U}_1^{(n)}(\mathcal{B}) \quad \textit{for } n=1 \textit{ and } 2,$$

$$\widehat{S}_{4,e}(\mathcal{A})-\varkappa\widehat{S}_4(\mathcal{B})$$

*and*

$$\widehat{U}_{2,e}^{(1)}(\mathcal{A})-\varkappa\widehat{U}_2^{(1)}(\mathcal{B})$$

*is* $O(\eta^{5/2}X^2(\log X)^c)$.

Having removed the leading terms from $U(\mathcal{A})$ we can proceed to estimate the remaining parts $U_f(\mathcal{A})$ individually. It is no longer necessary to demonstrate cancellation between the two sequences $\mathcal{A}$ and $\mathcal{B}$. Instead the cancellation will come from $f_S$. The following result shows how $U_f(\mathcal{A})$ can be bounded in terms of averages of $f_S$.

LEMMA 3.10. *Suppose that we have a bound of the form*

$$\sum_{\substack{\beta\equiv\alpha\,(\mathrm{mod}\,q)\\ \beta\in\mathcal{C}}} f_{(\beta)} \ll V\exp\{-c\sqrt{\log L}\,\}, \tag{3.14}$$

*subject to the conditions of Lemma* 3.8, *uniformly in a range*

$$q\leqslant Q_1\leqslant \exp\{\sqrt[3]{\log X}\,\}.$$

*Then there exists an absolute positive constant c such that*

$$\sum_{\substack{RS\in\mathcal{A}^{(K)}\\ V<N(S)\leqslant 2V}} c_R f_S \ll X^2 Q_1^{-1/160}(\log X)^c,$$

*for* $X^{1+\tau}\ll V\ll X^{3/2-\tau}$.

This result, which is our Type II estimate, is the most novel part of our entire argument, and it is here that the structure of the form $x^3+2y^3$ is most crucially used.

One readily sees that each term $U_f(\mathcal{A})$ may be written as a sum of $O(\log X)$ sums of the form considered in Lemma 3.10. Moreover, since $n\ll\tau^{-1}$ and $m_i\ll\xi^{-1}$, the number of possibilities for $n,\mathbf{m}$ is

$$\ll\tau^{-1}(c\xi^{-1})^{c\tau^{-1}}\ll\log X,$$

by (2.5), (3.4) and (3.10). It follows, in the obvious notation, that each of

$$\sum_{n\geqslant 3}|\widehat{U}_f^{(n)}(\mathcal{A})|,$$

$$\widehat{U}_{1,f}^{(n)}(\mathcal{A})\quad\text{for } n=1 \text{ and } 2,$$

$$\widehat{S}_{4,f}(\mathcal{A})$$

and

$$\widehat{U}_{2,f}^{(1)}(\mathcal{A}),$$

is $O(X^2Q_1^{-1/160}(\log X)^c)$.

We can now combine this with the estimates of Lemmas 3.5, 3.6, 3.7 and 3.9, to deduce from Lemma 3.4 that

$$\pi(\mathcal{A})-\varkappa\pi(\mathcal{B})\ll\tau\frac{\eta^2X^2}{\log X}+\xi\tau^{-4}\frac{\eta^2X^2}{\log X}+(\eta^{5/2}+Q_1^{-1/160})X^2(\log X)^c. \tag{3.15}$$

We have already chosen $\xi=\tau^5$ in (3.9). In order to specify our choices for $\eta$ and $Q_1$ we suppose that (3.15) holds with the constant $c=c_0$, say. We then take

$$\eta=(\log X)^{-2c_0-2}$$

and

$$Q_1=(\log X)^{160(c_0+1)}\eta^{-320}=(\log X)^{600(c_0+1)}.$$

These choices are consistent with (2.1) and Lemma 3.8, and lead to

$$\pi(\mathcal{A})-\varkappa\pi(\mathcal{B})\ll\tau\frac{\eta^2X^2}{\log X}.$$

We may then choose $\varpi=\frac{1}{6}$ to produce (2.4), and the theorem follows.

## 4. Preliminaries

In this section we establish Lemma 3.1, and prove a number of results about divisor sums over the ring $\mathbf{Z}[\sqrt[3]{2}]$. For most stages in the proof of our theorem a loss of an arbitrary power of $\log X$ will be acceptable, while a loss of $\exp(\log X/\log\log X)$ is not. Thus it is important to have estimates for divisor sums which only lose powers of $\log X$. We shall also give sundry other results, including some elementary facts from the geometry of numbers, and a tool for counting points 'near' a non-singular hypersurface.

We begin with Lemma 3.1. Let $P$ be a prime ideal factor of $x+y\sqrt[3]{2}$. If $P|y$ then $P|x$ so that $(x,y)\neq 1$. Otherwise $\sqrt[3]{2}\equiv -xy^{-1}\pmod{P}$, so that any element of $\mathbf{Z}[\sqrt[3]{2}]$ is congruent to a rational integer. It follows that the residue field modulo $P$ has $p$ elements, where $p$ is the rational prime above $P$. We then have $N(P)=p$, so that $P$ has degree 1. If $p$ is any rational prime then, according to Dedekind's theorem, the first degree prime ideals above $p$ take the form $(p,n-\sqrt[3]{2})$, where $n$ runs over the distinct solutions of the congruence $n^3\equiv 2\pmod{p}$. Thus distinct first degree primes $P_1,P_2$ above $p$ correspond to distinct values of $n$. If $P_1,P_2|x+y\sqrt[3]{2}$ this leads to a contradiction, on taking $n\equiv -xy^{-1}\pmod{p}$. This proves Lemma 3.1.

We next record the following estimate, which goes back to Weber.

LEMMA 4.1. *The number of integral ideals of $K=\mathbf{Q}(\sqrt[3]{2})$ with norm at most $x$ is*

$$\gamma_0 x+O(x^{2/3}),$$

*where $\gamma_0$ is as in Lemma 2.2.*

We now move on to the divisor function estimates. We shall use the notation $\tau(\cdot)$ both for the divisor function in $\mathbf{Z}$ and for the divisor function in $\mathbf{Z}[\sqrt[3]{2}]$. The meaning will always be clear from the context. We begin with the following bounds.

LEMMA 4.2. *For any integer $A>0$ there is a constant $c(A)$ such that*

$$\sum_{n\leqslant x}\tau(n)^A \ll x(\log x)^{c(A)}$$

*and*

$$\sum_{N(I)\leqslant x}\tau(I)^A \ll x(\log x)^{c(A)}.$$

*Indeed there is a positive constant $\delta=\delta(A)$ such that*

$$\sum_{x<n\leqslant x+y}\tau(n)^A \ll y(\log x)^{c(A)}$$

*and*

$$\sum_{x<N(I)\leqslant x+y}\tau(I)^A \ll y(\log x)^{c(A)}$$

*for $x^{1-\delta}\leqslant y\leqslant x$.*

The estimates for $\sum\tau(n)^A$ are well known, and indeed one may take the constant $c(A)$ to be $2^A-1$. For the bounds for $\sum\tau(I)^A$ one may note that there are at most $\tau(n)^2$ ideals $I$ with $N(I)=n$, and that $\tau(I)\leqslant\tau(n)^3$ for each. Thus

$$\sum_{N(I)=n}\tau(I)^A\leqslant\tau(n)^{3A+2},$$

so that the required results follow from the estimates for $\sum\tau(n)^A$.

We shall make frequent use of the following elementary fact, without further comment.

LEMMA 4.3. *We have $\tau(IJ)\leqslant\tau(I)\tau(J)$ for any two non-zero integral ideals.*

Since the divisor function is multiplicative, it suffices to prove this when $I$ and $J$ are powers of the same prime ideal $P$. The lemma is then a consequence of the inequality

$$\tau(P^{e+f})=e+f+1\leqslant(e+1)(f+1)=\tau(P^e)\tau(P^f).$$

Our next result will be used in an auxiliary capacity, to establish the main estimates of this section.

LEMMA 4.4. *Let $n$ be a positive integer. For any number field $k$ and any non-zero integral ideal $I$ of $k$ there is an ideal $J|I$ with*

$$N(J)\leqslant N(I)^{1/n} \quad \text{and} \quad \tau(I)\leqslant 2^{n-1}\tau(J)^{2n-1}.$$

To prove this write $I=I_1I_2$, where $I_1$ is the product of all prime ideal divisors of $I$ with norm at most $N(I)^{1/n}$. Then $I_2$ is a product of at most $n-1$ primes, whence $\tau(I_2)\leqslant 2^{n-1}$. We can write $I_1$ as a product $J_1\dots J_t$ with $N(J_i)\leqslant N(I)^{1/n}$ and $N(J_rJ_s)> N(I)^{1/n}$ for $r\neq s$. It follows that $t\leqslant 2n-1$. We therefore deduce that

$$\tau(I)\leqslant 2^{n-1}\tau(I_1)\leqslant 2^{n-1}\prod_{i=1}^{t}\tau(J_i)\leqslant 2^{n-1}(\max\tau(J_i))^t\leqslant 2^{n-1}(\max\tau(J_i))^{2n-1},$$

which suffices for the lemma.

We can now give our first main result.

LEMMA 4.5. *Let $\mathcal{C}=(a_1,a_1+S_0]\times(a_2,a_2+S_0]\times(a_3,a_3+S_0]$ be a cube of side $S_0$, and suppose that $\max|a_i|\leqslant S_0^A$ for some positive constant $A$. For any $\beta=x+y\sqrt[3]{2}+z\sqrt[3]{4}\in K$ write $\hat{\beta}=(x,y,z)$. Then there is a constant $c(A)$ such that*

$$\sum_{\hat{\beta}\in\mathcal{C}}\tau(\beta)^2\ll S_0^3(\log S_0)^{c(A)}.$$

For the proof we apply Lemma 4.4, with $n>(3A)^{-1}$, to show that

$$\tau(\beta)^2\ll\max\{\tau(I)^{c(A)}: I|\beta,\, N(I)\ll S_0\}\leqslant\sum_{\substack{I|\beta\\ N(I)\ll S_0}}\tau(I)^{c(A)}.$$

It follows that

$$\begin{aligned}\sum_{\hat{\beta}\in\mathcal{C}}\tau(\beta)^2&\ll\sum_{N(I)\ll S_0}\tau(I)^{c(A)}\#\{\hat{\beta}\in\mathcal{C}: I|\beta\}\\&\ll\sum_{N(I)\ll S_0}\tau(I)^{c(A)}S_0^3N(I)^{-1}\ll S_0^3(\log S_0)^{c(A)},\end{aligned}$$

by Lemma 4.2. Here we have used the fact that if $\mathcal{C}'$ is a cube of side $N(I)$, then there are $O(N(I)^2)$ values of $\hat{\beta}\in\mathcal{C}'$ for which $I|\beta$. This completes the proof of Lemma 4.5.

Our second main result is the following.

LEMMA 4.6. *Let $x \geqslant y \geqslant 2$ be given, and let $\alpha, \beta$ be coprime integers of $K$. Suppose that there is an integer $r$ such that*

$$|\alpha^{(j)}|, |\beta^{(j)}| \leqslant x^r$$

*for each conjugate. Then for any positive integer $A$ there exists a positive constant $c(A,r)$ such that*

$$\sum_{\substack{|m|\leqslant x, |n|\leqslant y\\ n\neq 0}} \tau(m\alpha+n\beta)^A \ll xy(\log x)^{c(A,r)}.$$

We begin the proof by observing that the terms of our sum will have

$$0 < |N(m\alpha+n\beta)| \ll x^{3r+3}.$$

According to Lemma 4.4 we have

$$\tau(m\alpha+n\beta) \ll \tau(I)^{6r+5}$$

for some ideal $I|m\alpha+n\beta$ such that $N(I) \leqslant N(m\alpha+n\beta)^{1/(3r+3)}$. It follows that

$$\sum_{|m|\leqslant x, |n|\leqslant y} \tau(m\alpha+n\beta)^A \ll \sum_{N(I)\ll x} \tau(I)^{(6r+5)A} \#\{|m|\leqslant x, |n|\leqslant y : I|m\alpha+n\beta, n\neq 0\}.$$

We put $(I,\alpha)=I_1$ and $I=I_1I_2$. Since $(\alpha,\beta)=1$ we see that $I_1|n$. We now write $\nu(J)$ for the smallest rational multiple of the ideal $J$, whence $\nu(I_1)|n$. As $n$ cannot be zero there are $O(y/\nu(I_1))$ possible values for $n$. Moreover each such value of $n$ will determine $m$ to modulus $I_2$. Since

$$\nu(I_2) \leqslant N(I_2) \leqslant N(I) \ll x,$$

it follows that there are $O(x\nu(I_2)^{-1})$ possible values of $m$ corresponding to each $n$. For any $\Delta$ in the range $(0,1)$, we now find that

$$\sum_{\substack{|m|\leqslant x, |n|\leqslant y\\ n\neq 0}} \tau(m\alpha+n\beta)^A \ll xy \sum_{N(I)\ll x} \tau(I)^{(6r+5)A} \sum_{I=I_1I_2} \nu(I_1)^{-1}\nu(I_2)^{-1} \ll x^{1+\Delta} y f(\Delta) \quad (4.1)$$

uniformly in $\Delta$, where $f(\sigma)$ is the Dirichlet series

$$f(\sigma) = \sum_I \frac{\tau(I)^{(6r+5)A}}{N(I)^\sigma} \sum_{I=I_1I_2} \nu(I_1)^{-1}\nu(I_2)^{-1}.$$

The function $f(\sigma)$ has an Euler product, with factors

$$1 + \sum_{m=1}^{\infty} e_{m,p} p^{-m\sigma}$$

where

$$e_{m,p}=\sum_{N(I_1I_2)=p^m}\frac{\tau(I_1I_2)^{(6r+5)A}}{\nu(I_1)\nu(I_2)}.$$

We note that there are at most $(m+1)^5$ pairs $I_1,I_2$, and that $\tau(I_1I_2)\leqslant(m+1)^3$ for each pair. In order to give a lower bound for $\nu(I_1)\nu(I_2)$ we note that $I_1|\nu(I_1)$, whence, on taking norms, we have $N(I_1)|\nu(I_1)^3$. Since $N(I_2)|\nu(I_2)^3$ similarly we deduce that $p^m=N(I)|\nu(I_1)^3\nu(I_2)^3$. It follows that $\nu(I_1)\nu(I_2)\geqslant p^{m/3}$. We also have $\nu(I_1)\nu(I_2)\geqslant p$ for $m\geqslant1$. It therefore follows that

$$e_{m,p}\leqslant(m+1)^{5+(18r+15)A}\min(p^{-m/3},p^{-1}),$$

whence

$$1+\sum_{m=1}^{\infty}e_{m,p}p^{-m\Delta}\leqslant 1+c(A,r)p^{-1-\Delta}\leqslant(1+p^{-1-\Delta})^{c(A,r)},$$

for a suitable constant $c(A,r)$. We therefore deduce that

$$f(\Delta)\leqslant\zeta(1+\Delta)^{c(A,r)}\ll\Delta^{-c(A,r)}.$$

If we choose $\Delta=(\log x)^{-1}$ the lemma then follows from (4.1).

Our final result on divisor function sums is a corollary of Lemma 4.6.

LEMMA 4.7. *Let $x,y\geqslant2$ be given. Then for any positive integer $A$ there exists a positive constant $c(A)$ such that*

$$\sum_{\substack{|m|\leqslant x,\,|n|\leqslant y\\ mn\neq0}}\tau\big(m+n\sqrt[3]{2}\big)^A\ll xy(\log xy)^{c(A)}.$$

We turn now to the following result from the geometry of numbers.

LEMMA 4.8. *Let $\mathbf{w}\in\mathbf{Z}^3$ be a primitive integer vector. Then the set of $\mathbf{x}\in\mathbf{Z}^3$ for which $\mathbf{w}\cdot\mathbf{x}=0$ forms a 2-dimensional lattice of determinant $|\mathbf{w}|$. If $\mathbf{z}_1,\mathbf{z}_2$ is any basis of this lattice, then $\mathbf{z}_1\wedge\mathbf{z}_2=\pm\mathbf{w}$. The basis can be chosen in such a way that $|\mathbf{z}_1|\leqslant|\mathbf{z}_2|$ and*

$$|\mathbf{w}|\ll|\mathbf{z}_1|\cdot|\mathbf{z}_2|\ll|\mathbf{w}|,$$

*and with the property that*

$$|\lambda\mathbf{z}_1+\mu\mathbf{z}_2|\gg|\lambda|\cdot|\mathbf{z}_1|+|\mu|\cdot|\mathbf{z}_2|$$

*for any scalars $\lambda,\mu$.*

We note that an integer vector is said to be 'primitive' if its coordinates have no non-trivial common factor. If we let $\Lambda$ be the set of integer multiples of $\mathbf{w}$ then the set of vectors $\mathbf{x}$ described in the lemma will be the 'dual lattice' $\Lambda^*$, as described in the author's paper [8, §2]. The first assertion of the lemma is therefore an immediate consequence of [8, Lemma 1]. It follows that

$$|\mathbf{z}_1 \wedge \mathbf{z}_2| = d(\Lambda^*) = d(\Lambda) = |\mathbf{w}|,$$

in the notation of [8]. However, $\mathbf{z}_1 \wedge \mathbf{z}_2$ is orthogonal to $\mathbf{z}_1$ and $\mathbf{z}_2$, and so lies in $\Lambda^{**}=\Lambda$. It therefore follows that $\mathbf{z}_1 \wedge \mathbf{z}_2 = \pm\mathbf{w}$ as asserted. The remainder of Lemma 4.8 follows from the argument used to prove [8, Lemma 2].

Our next result allows us to count non-singular points near to a hypersurface.

LEMMA 4.9. *Let $\mathcal{C}_i \subseteq \mathbf{R}^n$ be disjoint hypercubes with parallel edges of length $S_0$, and contained in a ball of radius $R$, centred on the origin. Let $F$ be a real cubic form in $n$ variables, and let $F_0$ be a real constant. Suppose that each hypercube contains a point $\mathbf{x}$ for which $F(\mathbf{x}) = F_0 + O(R^2 S_0)$ and $|\nabla F(\mathbf{x})| \gg R^2$. Then the number of hypercubes $\mathcal{C}_i$ contained in any ball of radius $R_0$ is $\ll_F 1 + (R_0/S_0)^{n-1}$.*

For the proof we may clearly suppose that $S_0 \leqslant c_0 R_0$ with a suitably small absolute constant $c_0$, since the result is trivial otherwise. It follows that each vertex $\mathbf{v}$ of every hypercube $\mathcal{C}_i$ satisfies both $F(\mathbf{v}) = F_0 + O(R^2 S_0)$ and $|\nabla F(\mathbf{v})| \gg R^2$. We divide the vertices into sets $\mathcal{B}_j$, not necessarily disjoint, for which $|\partial F/\partial v_j| \gg R^2$ for any $\mathbf{v}$ in $\mathcal{B}_j$. We shall examine the case $j=1$, the other cases being similar. For a given choice of $\mathbf{u} = (v_2, \ldots, v_n)$ let

$$\mathcal{B}_1(\mathbf{u}) = \{v_1 : (v_1, \mathbf{u}) \in \mathcal{B}_1\}.$$

Now if $v_1$ and $v_1' = v_1 + \delta$ are any two elements of $\mathcal{B}_1(\mathbf{u})$ we will have

$$F(v_1', \mathbf{u}) = F_1(v_1, \mathbf{u}) + \delta \frac{\partial F}{\partial x_1}(v_1, \mathbf{u}) + O(R\delta^2).$$

However,

$$F(v_1', \mathbf{u}),\, F_1(v_1, \mathbf{u}) = F_0 + O(R^2 S_0),$$

whence

$$\delta \frac{\partial F}{\partial x_1}(v_1, \mathbf{u}) \ll R^2 S_0 + R\delta^2.$$

It therefore follows that $\delta \ll S_0 + R^{-1}\delta^2$. We deduce that either $\delta \ll S_0$ or $|\delta| \gg R$. Since this holds for any two elements of $\mathcal{B}_1(\mathbf{u})$, it follows that

$$\#\mathcal{B}_1(\mathbf{u}) \ll 1,$$

and therefore that $\#\mathcal{B}_1 \ll (R_0/S_0)^{n-1}$. The lemma then follows.

Finally we have the following corollary of the Prime Ideal Theorem.

LEMMA 4.10. *Let $J_1, \dots, J_m$ be intervals of the form $J_i=[a_i, \varrho a_i)$, with $\varrho>1$ and $a_i \geqslant A>1$ for each $i \leqslant m$. Let $Y \geqslant 1$ be given and define $\mathcal{J}(Y,m) \subseteq \mathbf{R}^m$ as the set of $(x_1, \dots, x_m)$ with $x_i \in J_i$ and $\prod x_i \leqslant Y$. Then there are positive absolute constants $c_1$ and $c_2$ such that*

$$\sum_{(N(P_1),\dots,N(P_m)) \in \mathcal{J}(Y,m)} \prod_{i=1}^{m} \log N(P_i) = \operatorname{meas}(\mathcal{J}(Y,m)) + O(mY(c_1+\log\varrho)^{m-1}\exp\{-c_2(\log A)^{1/2}\}),$$

*uniformly in $m$.*

For the proof we use induction on $m$. For $m=1$ the result is an immediate consequence of the Prime Ideal Theorem, in the form given by (2.3). For the induction step we shall write $c_3$ for the constant implied by the $O(\cdot)$-notation. When we have $m+1$ variables $P_i$ we fix the first $m$, so that the final prime ideal has $N(P_{m+1}) \in J_{m+1}$ and

$$N(P_{m+1}) \leqslant \frac{Y}{\prod_{i=1}^{m} N(P_i)}.$$

The contribution from the factor $\log N(P_{m+1})$ is thus

$$\int_{t \in J_{m+1},\, t \leqslant Y/\prod N(P_i)} dt + O\left(\frac{Y}{\prod N(P_i)} \exp\{-c_4(\log A)^{1/2}\}\right),$$

by the Prime Ideal Theorem. We write $c_5$ for the implicit constant. The contribution from the error term is then at most

$$\begin{aligned} c_5 Y \exp\{-c_4(\log A)^{1/2}\} &\sum_{N(P_i)\in J_i} \prod_{i=1}^{m} \frac{\log N(P_i)}{N(P_i)} \\ &\leqslant c_5 Y \exp\{-c_4(\log A)^{1/2}\} \prod_{i=1}^{m} (c_6+\log a_i\varrho - \log a_i) \qquad (4.2) \\ &= c_5 Y \exp\{-c_4(\log A)^{1/2}\}(c_6+\log\varrho)^m. \end{aligned}$$

The main term produces

$$\int_{t\in J_{m+1}} \sum_{(N(P_1),\dots,N(P_m)) \in \mathcal{J}(Y/t,m)} \prod_{i=1}^{m} \log N(P_i)\, dt.$$

According to our induction hypothesis this differs from

$$\int_{t\in J_{m+1}} \operatorname{meas}(\mathcal{J}(Y/t,m))\, dt$$

by at most

$$c_3 \int_{t\in J_{m+1}} m\frac{Y}{t}(c_1+\log\varrho)^{m-1}\exp\{-c_2(\log A)^{1/2}\}\,dt \\ \leqslant c_3 mY(c_1+\log\varrho)^m\exp\{-c_2(\log A)^{1/2}\}. \tag{4.3}$$

We may note that

$$\int_{t\in J_{m+1}} \mathrm{meas}(\mathcal{J}(Y/t,m))\,dt = \mathrm{meas}(\mathcal{J}(Y,m+1)),$$

which produces the required main term. Moreover the two error terms (4.2) and (4.3) will produce a total at most

$$c_3(m+1)Y(c_1+\log\varrho)^m\exp\{-c_2(\log A)^{1/2}\}$$

providing that

$$c_3\geqslant c_5,\quad c_1\geqslant c_6\quad\text{and}\quad c_2\leqslant c_4.$$

Since we may clearly choose $c_3$, $c_1$ and $c_2$ in this way, the induction step is complete. This proves Lemma 4.10.

## 5. The Type I estimates—Lemmas 2.1, 2.2, 3.2 and 3.3

We begin this section by examining

$$\#\{x,y\in(X,X(1+\eta)] : R|x+y\sqrt[3]{2}\} = S(R;X) = S(R),$$

say. We shall establish the following estimate.

LEMMA 5.1. *If $A$ is any positive integer, there exists $c(A)$ such that*

$$\sum_{\substack{Q<N(R)\leqslant 2Q\\ R\in\mathcal{R}}} \tau(R)^A\left|S(R)-\frac{\eta^2X^2}{N(R)}\right| \ll (X+Q)(\log Q)^{c(A)}, \tag{5.1}$$

*for $X\geqslant 1$.*

We begin the proof of Lemma 5.1 by splitting the vectors $(x,y)$ into congruence classes modulo $N(R)$, whence

$$S(R) = \sum_{\substack{u,v\,(\mathrm{mod}\,N(R))\\ R|u+v\sqrt[3]{2}}} \#\{x,y\in(X,X(1+\eta)] : x\equiv u,\ y\equiv v\ (\mathrm{mod}\,N(R))\}.$$

Using the notation $e_q(x)=\exp(2\pi ix/q)$, this becomes

$$\begin{aligned} N(R)^{-2} \sum_{\substack{u,v\,(\mathrm{mod}\,N(R))\\ R\mid u+v\sqrt[3]{2}}} \sum_{a,b\,(\mathrm{mod}\,N(R))} \sum_{X<x,y\leqslant X(1+\eta)} e_{N(R)}(a(u-x)+b(v-y)) \\ = N(R)^{-2} \sum_{a,b\,(\mathrm{mod}\,N(R))} S_0(R,a,b) \sum_{X<x,y\leqslant X(1+\eta)} e_{N(R)}(-ax-by)) \end{aligned}$$

where

$$S_0(R,a,b)=\sum_{\substack{u,v\,(\mathrm{mod}\,N(R))\\ R\mid u+v\sqrt[3]{2}}} e_{N(R)}(au+bv).$$

To evaluate the sum $S_0(R,a,b)$ we note that there is a multiplicative property

$$S_0(R_1R_2,a,b)=S_1(R_1,a,b)S_0(R_2,a,b),$$

for $R_1R_2\in\mathcal{R}$, so that it suffices to investigate the case in which $R$ is a prime. When $a=b=0$ we note that the number of pairs $u,v$ modulo $N(R)$, for which $R\mid u+v\sqrt[3]{2}$, will be $N(R)$. We therefore see in general that $S_0(R,0,0)=N(R)$, whence

$$S(R)=\frac{\eta^2X^2+O(X)}{N(R)}+O\Bigg(\sum_{\substack{|a|,|b|\leqslant N(R)/2\\ (a,b)\neq(0,0)}} \frac{|S_0(R,a,b)|}{N(R)^2}\min\Big\{X,\frac{N(R)}{|a|}\Big\}\min\Big\{X,\frac{N(R)}{|b|}\Big\}\Bigg). \tag{5.2}$$

The total contribution from the first error term on the right is

$$\ll X\sum_{N(R)\leqslant 2Q} N(R)^{-1}\tau(R)^A \ll X(\log Q)^{c(A)},$$

in view of Lemma 4.2. This is satisfactory for Lemma 5.1.

To handle $S_0(R,a,b)$ when $(a,b)\neq(0,0)$ we first examine the case in which $N(R)$ is a prime $p$. For any integer $t$ coprime to $p$, the pairs $tu,tv$ run over the residues modulo $p$ when $u,v$ do. Hence

$$S_0(R,a,b)=\sum_{\substack{u,v\,(\mathrm{mod}\,p)\\ R\mid tu+tv\sqrt[3]{2}}} e_p(atu+btv)=\sum_{\substack{u,v\,(\mathrm{mod}\,p)\\ R\mid u+v\sqrt[3]{2}}} e_p(atu+btv).$$

It follows that

$$\begin{aligned} (p-1)S_0(R,a,b) &= \sum_{t=1}^{p-1} \sum_{\substack{u,v\,(\mathrm{mod}\,p)\\ R\mid u+v\sqrt[3]{2}}} e_p(atu+btv) = \sum_{\substack{u,v\,(\mathrm{mod}\,p)\\ R\mid u+v\sqrt[3]{2}}} \sum_{t=1}^{p-1} e_p(atu+btv) \\ &= p\#\{u,v\,(\mathrm{mod}\ p): R\mid u+v\sqrt[3]{2},\, p\mid au+bv\} \\ &\quad -\#\{u,v\,(\mathrm{mod}\ p): R\mid u+v\sqrt[3]{2}\}. \end{aligned}$$

If $p\nmid(a,b)$ then the condition $p|au+bv$ shows that we have $u\equiv\lambda b \pmod p$, $v\equiv-\lambda a \pmod p$, for some integer $\lambda$. If we also have $R|u+v\sqrt[3]{2}$ then either $p|\lambda$ or $R|b-a\sqrt[3]{2}$. Thus $S_0(R,a,b)=0$ for $R\nmid b-a\sqrt[3]{2}$ and $p\nmid(a,b)$. The final condition is clearly superfluous. It follows for a general $R$ that $S_0(R,a,b)$ vanishes if $R\nmid b-a\sqrt[3]{2}$, while if $R|b-a\sqrt[3]{2}$ we have the trivial bound

$$|S_0(R,a,b)|\leqslant N(R)\ll Q.$$

We proceed to estimate the contribution to (5.1) arising from terms in (5.2) for which $a,b$ are both non-zero. This is

$$\ll Q\sum_{0<|a|,|b|\leqslant Q}|ab|^{-1}\sum_{\substack{Q<N(R)\leqslant 2Q\\ R|b-a\sqrt[3]{2}}}\tau(R)^A\ll Q\sum_{0<|a|,|b|\leqslant Q}|ab|^{-1}\tau\big(b-a\sqrt[3]{2}\big)^{c(A)}=\Sigma_1,$$

say. We split the available $a,b$ into ranges $M\leqslant|a|<2M$, $N\leqslant|b|<2N$, where $M,N$ run over powers of 2. There will be $O((\log Q)^2)$ such pairs $M,N$. We use Lemma 4.7 for each range, whence

$$\Sigma_1\ll Q\sum_{M,N}(MN)^{-1}MN(\log MN)^{c(A)}\ll Q(\log Q)^{c(A)},$$

which is satisfactory.

We turn now to the terms of (5.2) in which $a$, say, is zero. By the same argument as before we find that the corresponding contribution to (5.1) is

$$\ll X\sum_{0<|b|\leqslant Q}|b|^{-1}\sum_{\substack{Q<N(R)\leqslant 2Q\\ R|b}}\tau(R)^A\ll X\sum_{0<|b|\leqslant Q}|b|^{-1}\tau(|b|)^{c(A)}\ll X(\log Q)^{c(A)}$$

by Lemma 4.2. Again this is satisfactory for Lemma 5.1. An entirely analogous argument applies for terms with $b=0$.

We may now deduce Lemma 3.2. We have

$$\#\mathcal{A}_R^{(K)}=\sum_{d=1}^{\infty}\mu(d)\#\{x,y\in(X,X(1+\eta)]:d|x,y,\ R|x+y\sqrt[3]{2}\}.$$

Writing $x=dx'$, $y=dy'$ we find that

$$\begin{aligned}\#\mathcal{A}_R^{(K)}&=\sum_{d=1}^{\infty}\mu(d)\#\left\{x',y'\in\left(\frac{X}{d},\frac{X}{d}(1+\eta)\right]:\frac{R}{(R,d)}\,\middle|\,x'+y'\sqrt[3]{2}\right\}\\&=\sum_{d=1}^{\infty}\mu(d)\,S\left(\frac{R}{(R,d)};\frac{X}{d}\right).\end{aligned}\tag{5.3}$$

Moreover, for $R\in\mathcal{R}$ we have

$$\begin{aligned}\sum_{d=1}^{\infty}\mu(d)\frac{\eta^2X^2}{d^2}N(R/(R,d))^{-1} &= \frac{\eta^2X^2}{N(R)}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}N((R,d)) \\ &= \frac{\eta^2X^2}{N(R)}\prod_{p\nmid N(R)}\left(1-\frac{1}{p^2}\right)\prod_{p\mid N(R)}\left(1-\frac{1}{p}\right) \\ &= \frac{6\eta^2X^2}{\pi^2N(R)}\prod_{p\mid N(R)}\left(1+\frac{1}{p}\right)^{-1},\end{aligned} \tag{5.4}$$

which produces the leading terms in Lemma 3.2.

We shall split the sums (5.3) and (5.4) at $d=\Delta$, where $1\leqslant\Delta\leqslant X$ will be specified below. Terms in (5.4) for which $d>\Delta$ contribute a total

$$\ll \sum_{\substack{Q<N(R)\leqslant 2Q\\ R\in\mathcal{R}}}\frac{\eta^2X^2}{N(R)}\tau(R)^A\sum_{d>\Delta}d^{-2}N((R,d))$$

in Lemma 3.2. We put $(R,d)=S$ and $R=ST$. Thus $N(S)|d$, and on setting $d=N(S)e$, the above becomes

$$\begin{aligned}&\ll \eta^2X^2\sum_{Q<N(ST)\leqslant 2Q}\tau(S)^A\tau(T)^AN(S)^{-2}N(T)^{-1}\sum_{e>\Delta/N(S)}e^{-2} \\ &\ll \eta^2X^2\sum_{N(S),N(T)\leqslant 2Q}\tau(S)^A\tau(T)^AN(S)^{-2}N(T)^{-1}\min\{1,N(S)/\Delta\} \\ &\ll \eta^2X^2\Delta^{-1}(\log Q)^{c(A)}\end{aligned} \tag{5.5}$$

by Lemma 4.2.

Similarly, the contribution from the terms of (5.3) in which $d>\Delta$ is

$$\begin{aligned}&\ll \sum_{Q<N(R)\leqslant 2Q}\sum_{d>\Delta}\tau(R)^AS\left(\frac{R}{(R,d)};\frac{X}{d}\right) \\ &\ll \sum_{d>\Delta}\sum_{N(S)|d}\tau(S)^{c(A)}\sum_{Q/N(S)<N(T)\leqslant 2Q/N(S)}\tau(T)^{c(A)}S\left(T;\frac{X}{d}\right) \\ &\ll \sum_{d>\Delta}\tau(d)^{c(A)}\sum_{X/d<x,y\leqslant(1+\eta)X/d}\tau\big(x+y\sqrt[3]{2}\big)^{c(A)}.\end{aligned}$$

Here we shall use Lemma 4.7 again, so that the above expression is

$$\ll \sum_{\Delta<d\ll X}\tau(d)^{c(A)}\left(\frac{X}{d}\right)^2(\log X)^{c(A)}\ll X^2(\log X)^{c(A)}\Delta^{-1} \tag{5.6}$$

by Lemma 4.2.

If we now write $S=(R,d)$ and $R=ST$ once more, it follows via Lemma 5.1 that the overall contribution from terms of (5.3) and (5.4) with $d\leqslant\Delta$ is

$$\begin{aligned}
&\ll \sum_{\substack{Q<N(R)\leqslant 2Q\\ R\in\mathcal{R}}} \tau(R)^A \sum_{d\leqslant\Delta}\left|S\Big(\frac{R}{(R,d)};\frac{X}{d}\Big)-\frac{\eta^2(X/d)^2}{N(R/(R,d))}\right|\\
&\ll \sum_{d\leqslant\Delta}\sum_{N(S)|d}\tau(S)^A \sum_{\substack{Q/N(S)<N(T)\leqslant 2Q/N(S)\\ T\in\mathcal{R}}}\tau(T)^A\left|S\Big(T;\frac{X}{d}\Big)-\frac{\eta^2(X/d)^2}{N(T)}\right|\\
&\ll \sum_{d\leqslant\Delta}\sum_{N(S)|d}\tau(S)^A\Big(\frac{X}{d}+\frac{Q}{N(S)}\Big)(\log Q)^{c(A)}\\
&\ll (X+Q)(\log Q)^{c(A)}\sum_{d\leqslant\Delta}\tau(d)\ll (X+Q)(\log Q)^{c(A)}\Delta\log X.
\end{aligned}$$

On comparing this with (5.5) and (5.6) we find that the sum in Lemma 3.2 is

$$\ll \frac{X^2}{\Delta}(\log XQ)^{c(A)}+(X+Q)(\log Q)^{c(A)}\Delta\log X.$$

The choice $\Delta=1+\min\{X^{1/2},XQ^{-1/2}\}$, which is essentially optimal, then yields a bound

$$\ll (Q+XQ^{1/2}+X^{3/2})(\log QX)^{c(A)},$$

for a suitable constant $c(A)$, thus completing the proof of Lemma 3.2.

The proof of Lemma 3.3 is, by contrast, almost trivial. We have

$$\#\mathcal{B}_R^{(K)}=\#\Big\{I:N(I)\in\Big(\frac{3X^3}{N(R)},\frac{3X^3}{N(R)}(1+\eta)\Big]\Big\}.$$

According to Lemma 4.1 we deduce that

$$\#\mathcal{B}_R^{(K)}=3\gamma_0X^3\eta N(R)^{-1}+O(X^2N(R)^{-2/3}). \tag{5.7}$$

The sum in Lemma 3.3 is thus

$$\ll \sum_{Q<N(R)\leqslant 2Q}\tau(R)^A\frac{X^2}{N(R)^{2/3}}\ll X^2Q^{1/3}(\log Q)^{c(A)}$$

by Lemma 4.2.

It remains to deduce Lemmas 2.1 and 2.2. For any rational prime $p$ we have

$$-\sum_{R|p,\,R|J,\,R\neq(1)}\mu(R)=\begin{cases}1, & p|N(J),\\ 0, & p\nmid N(J),\end{cases} \tag{5.8}$$

where $R$ runs over ideals. We may rewrite the condition $R\neq(1)$ as $p|N(R)$. It follows that

$$\#\mathcal{A}_q=\mu(q)\sum_{R|q,\,q|N(R)}\mu(R)\#\mathcal{A}_R^{(K)},$$

if $q$ is square-free. By Lemma 3.1 we have $\#\mathcal{A}_R^{(K)}=0$ unless $R\in\mathcal{R}$, in which case we must have $q=N(R)$. We use this to substitute in Lemma 2.1, so that Lemma 3.2 may be applied. The contribution to $\#\mathcal{A}_q$ arising from the main term in Lemma 3.2 is

$$\mu(q)\sum_{N(R)=q}\frac{\mu(R)}{N(R)}\varrho_2(R).$$

This is multiplicative in $q$, and for $q=p$, prime, it reduces to $\varrho_0(p)/p$. Thus

$$\begin{aligned}
&\sum_{Q<q\leqslant 2Q}\tau(q)^A\mu(q)^2\left|\#\mathcal{A}_q-\frac{6\eta^2X^2}{\pi^2}\frac{\varrho_0(q)}{q}\right| \\
&\qquad\ll\sum_{Q<q\leqslant 2Q}\tau(q)^A\sum_{\substack{N(R)=q\\ R\in\mathcal{R}}}\left|\#\mathcal{A}_R^{(K)}-\frac{6\eta^2X^2}{\pi^2N(R)}\varrho_2(R)\right| \\
&\qquad\ll\sum_{\substack{Q<N(R)\leqslant 2Q\\ R\in\mathcal{R}}}\tau(N(R))^A\left|\#\mathcal{A}_R^{(K)}-\frac{6\eta^2X^2}{\pi^2N(R)}\varrho_2(R)\right| \\
&\qquad\ll(Q+XQ^{1/2}+X^{3/2})(\log QX)^{c(A)},
\end{aligned}$$

since $\tau(N(R))=\tau(R)$ for $R\in\mathcal{R}$. This completes our treatment of Lemma 2.1.

Finally, to handle Lemma 2.2, we proceed as above using (5.8). We find that

$$\#\mathcal{B}_q=\mu(q)\sum_{R|q,\,q|N(R)}\mu(R)\#\mathcal{B}_R^{(K)}.$$

It follows from (5.7) that

$$\#\mathcal{B}_q=3\gamma_0\eta X^3\mu(q)\sum_{R|q,\,q|N(R)}\frac{\mu(R)}{N(R)}+O\Bigl(X^2\sum_{R|q,\,q|N(R)}N(R)^{-2/3}\Bigr).$$

We readily find that the main term is

$$\gamma_0\frac{3\eta X^3}{q}\varrho_1(q).$$

Moreover the contribution of the error term to the sum in Lemma 2.2 is

$$\begin{aligned}
&\ll X^2\sum_{Q<q\leqslant 2Q}\tau(q)^A\sum_{R|q,\,q|N(R)}N(R)^{-2/3} \\
&\ll X^2\sum_{Q<q\leqslant 2Q}\tau(q)^A\tau(q)^3q^{-2/3}\ll X^2Q^{1/3}(\log Q)^{c(A)},
\end{aligned}$$

as required.

## 6. The Fundamental Lemma sieve bounds

The first object of this section is to derive an asymptotic formula for

$$T^{(n)}(\mathcal{A})=\sum_{\substack{X^\tau\leqslant N(P_n)<\ldots<N(P_1)<X^{1-\tau}\\ N(P_1\ldots P_n)<X^{1+\tau}}} S_K(\mathcal{A}^{(K)}_{P_1\ldots P_n},X^\tau).$$

This will be done via a 'Fundamental Lemma'. We could obtain versions of the classical Fundamental Lemma appropriate to the field $K$, but it seems simpler to relate our sieve functions to ones over the rationals. We shall think of $S_1(\mathcal{A})$ as $T^{(0)}(\mathcal{A})$ in what follows.

We therefore proceed to show that

$$T^{(n)}(\mathcal{A})=\sum_{\substack{X^\tau\leqslant p_n<\ldots<p_1<X^{1-\tau}\\ p_1\ldots p_n<X^{1+\tau}}} S(\mathcal{A}_{p_1\ldots p_n},X^\tau), \tag{6.1}$$

by demonstrating that

$$S(\mathcal{A}_{p_1\ldots p_n},z)=\sum_{N(P_i)=p_i} S(\mathcal{A}^{(K)}_{P_1\ldots P_n},z), \tag{6.2}$$

if $p_i\geqslant z$. To this end we observe that if $x^3+2y^3$ is counted by $\mathcal{A}$, and $p|x^3+2y^3$, then the ideal $(x+y\sqrt[3]{2}\,,p)$ will be a first degree prime, $P$, say. Thus, for each relevant pair $x,y$, every prime $p_i$ determines a unique first degree prime ideal $P_i$ with $N(P_i)=p_i$. Conversely, if $P|x+y\sqrt[3]{2}$, then $P$ will be a first degree prime ideal. Thus each $P_i$ gives rise to a corresponding prime $p_i$. This suffices for the proof of (6.2), and hence of (6.1).

We proceed to estimate

$$S(\mathcal{A}_{p_1\ldots p_n},X^\tau)$$

via a classical 'Fundamental Lemma', in the form given by Theorem 7.1 of Halberstam and Richert [6]. We apply this with '$\omega(p)$'$=\varrho_0(p)$, '$X$'$=6\eta^2X^2/\pi^2$, '$\xi$'$=X^{1/6}$ and '$z$'$=X^\tau$. It then follows that

$$S(\mathcal{A}_q,X^\tau)=M(q)\{1+O(\exp(-\tau^{-1}))\}+O(E(q)),$$

where

$$M(q)=\frac{\varrho_0(q)}{q}\frac{6\eta^2X^2}{\pi^2}\prod_{p<X^\tau}\left(1-\frac{\varrho_0(p)}{p}\right)$$

and

$$E(q)=\sum_{\substack{d<X^{1/3}\\ p|d\,\Rightarrow\,p<X^\tau}}\mu(d)^2\tau(d)^2\left|\#\mathcal{A}_{qd}-\frac{6\eta^2X^2}{\pi^2qd}\varrho_0(qd)\right|.$$

Taking $q=p_1\dots p_n$, the error term $E(q)$ contributes to $T^{(n)}(\mathcal{A})$ a total

$$\ll \sum_{\substack{X^\tau\leqslant p_n<\dots<p_1<X^{1-\tau}\\ p_1\dots p_n<X^{1+\tau}}} \sum_{\substack{d<X^{1/3}\\ p|d\Rightarrow p<X^\tau}} \mu(d)^2\tau(d)^2\left|\#\mathcal{A}_{qd}-\frac{6\eta^2X^2}{\pi^2qd}\varrho_0(qd)\right|$$

$$\ll \sum_{r\leqslant X^{3/2}} \mu(r)^2\tau(r)^2\left|\#\mathcal{A}_r-\frac{6\eta^2X^2}{\pi^2 r}\varrho_0(r)\right| \ll X^{7/4}(\log X)^c$$

by Lemma 2.1. Note here that $qd$ is square-free. We now find that

$$T^{(n)}(\mathcal{A})=\frac{6\eta^2X^2}{\pi^2}\prod_{p<X^\tau}\left(1-\frac{\varrho_0(p)}{p}\right)\Sigma_0\{1+O(\exp(-\tau^{-1}))\}+O(X^{7/4}(\log X)^c), \quad (6.3)$$

where

$$\Sigma_0=\sum_{\substack{X^\tau\leqslant p_n<\dots<p_1<X^{1-\tau}\\ p_1\dots p_n<X^{1+\tau}}}\frac{\varrho_0(p_1\dots p_n)}{p_1\dots p_n}.$$

The above procedure may be repeated with the sequence $\mathcal{A}$ replaced by $\mathcal{B}$. We begin by showing that

$$S(\mathcal{B}_q,z)=\sum_{N(Q)=q}S(\mathcal{B}_Q^{(K)},z)+O(\tau(q)^7q^{-1}X^3z^{-1/2}(\log X)^c), \quad (6.4)$$

if $q=p_1\dots p_n$ is square-free, with $p_i\geqslant z$. If $N(J)$ is counted on the left-hand side, and $N(J)$ has no factor $p_i^2$, then $Q=(q,J)$ must have $N(Q)=q$, so that $J$ is counted on the right-hand side. Clearly any $J$ appearing on the right also contributes on the left, unless $P|J$ for some second degree prime ideal with $N(P)=p^2\in[z,z^2)$, or for some inert prime ideal $P$ with $N(P)=p^3\in[z,z^3)$. Moreover, again assuming that $N(J)$ has no factor $p_i^2$, there cannot be distinct divisors $Q,Q'$ of $J$ with $N(Q)=N(Q')=q$. Since there are at most $\tau(n)^3$ possible ideals of norm $n$, it follows that

$$\begin{aligned}S(\mathcal{B}_q,z)-\sum_{N(Q)=q}S(\mathcal{B}_Q^{(K)},z)&\ll\tau(q)^3\Big\{\sum_{p|q}\#\mathcal{B}_{pq}+\sum_{z^{1/2}\leqslant p<z}\#\mathcal{B}_{p^2q}+\sum_{z^{1/3}\leqslant p<z}\#\mathcal{B}_{p^3q}\Big\}\\ &\ll\tau(q)^3\Big\{\sum_{p|q}\sum_{\substack{n\ll X^3\\ pq|n}}\tau(n)^3+\sum_{z^{1/2}\leqslant p<z}\sum_{\substack{n\ll X^3\\ p^2q|n}}\tau(n)^3\\ &\qquad+\sum_{z^{1/3}\leqslant p<z}\sum_{\substack{n\ll X^3\\ p^3q|n}}\tau(n)^3\Big\}\end{aligned}$$

$$\ll \tau(q)^3 \Big\{ \sum_{p\mid q} (pq)^{-1}\tau(q)^3 + \sum_{z^{1/2}\leqslant p<z} (p^2q)^{-1}\tau(q)^3$$

$$+ \sum_{z^{1/3}\leqslant p<z} (p^3q)^{-1}\tau(q)^3 \Big\} X^3(\log X)^c$$

$$\ll \tau(q)^7 q^{-1} X^3 z^{-1/2}(\log X)^c,$$

as required for (6.4).

We now write

$$T_0^{(n)}(\mathcal{B}) = \sum_{\substack{X^\tau \leqslant p_n < \ldots < p_1 < X^{1-\tau} \\ p_1\ldots p_n < X^{1+\tau}}} S(\mathcal{B}_{p_1\ldots p_n}, X^\tau),$$

and proceed to compare $T_0^{(n)}(\mathcal{B})$ with $T^{(n)}(\mathcal{B})$. We shall do this in two stages, passing via

$$T_1^{(n)}(\mathcal{B}) = \sum_{\substack{X^\tau \leqslant N(P_n) < \ldots < N(P_1) < X^{1-\tau} \\ N(P_1\ldots P_n) < X^{1+\tau}}}^{(1)} S(\mathcal{B}^{(K)}_{P_1\ldots P_n}, X^\tau),$$

in which $\sum^{(1)}$ indicates that $N(P_1 \ldots P_n)$ must be square-free. According to (6.4) and Lemma 4.2, we have

$$T_0^{(n)}(\mathcal{B}) - T_1^{(n)}(\mathcal{B}) \ll X^{3-\tau/2}(\log X)^c \sum_{q<X^{1+\tau}} \tau(q)^7 q^{-1} \ll X^{3-\tau/2}(\log X)^c. \tag{6.5}$$

Moreover

$$T^{(n)}(\mathcal{B}) - T_1^{(n)}(\mathcal{B}) \ll \sum_{\substack{X^\tau \leqslant N(P_n) \leqslant \ldots \leqslant N(P_1) < X^{1-\tau} \\ N(P_1\ldots P_n) < X^{1+\tau}}}^{(2)} \#\mathcal{B}^{(K)}_{P_1\ldots P_n},$$

where $\sum^{(2)}$ indicates that the ideals $P_i$ are distinct, and that $N(P_1 \ldots P_n)$ is not square-free. In view of Lemma 4.1, together with the fact that $q = N(Q)$ has at most $\tau(q)^3$ solutions $Q$, we conclude that

$$T^{(n)}(\mathcal{B}) - T_1^{(n)}(\mathcal{B}) \ll X^3 \sum_{p \geqslant X^{\tau/3}} \sum_{\substack{q \leqslant X^{1+\tau} \\ p^2 \mid q}} \tau(q)^3 q^{-1}$$

$$\ll X^3 \sum_{p \geqslant X^{\tau/3}} (\log X)^c p^{-2} \ll X^3 (\log X)^c X^{-\tau/3}.$$

When we compare this with (6.5) we conclude that

$$T^{(n)}(\mathcal{B}) - T_0^{(n)}(\mathcal{B}) \ll X^{3-\tau/3}(\log X)^c.$$

We may now proceed as before to deduce that

$$T^{(n)}(\mathcal{B})=3\gamma_0\eta X^3\prod_{p<X^\tau}\left(1-\frac{\varrho_1(p)}{p}\right)\Sigma_1\{1+O(\exp(-\tau^{-1}))\}+O(X^{3-\tau/4}), \tag{6.6}$$

where

$$\Sigma_1=\sum_{\substack{X^\tau\leqslant p_n<\ldots<p_1<X^{1-\tau}\\ p_1\ldots p_n<X^{1+\tau}}}\frac{\varrho_1(p_1\ldots p_n)}{p_1\ldots p_n}.$$

We must now compare the main terms in (6.3) and (6.6). We look first at the product involving the function $\varrho_0(p)$. Since

$$\sum_{Y<p\leqslant Z}\frac{\nu_p-1}{p}=\sum_{Y<N(P)\leqslant Z}\frac{1}{N(P)}-\sum_{Y<p\leqslant Z}\frac{1}{p}+O(Y^{-1/3})\ll(\log Y)^{-2},$$

by the Prime Number Theorem and the Prime Ideal Theorem, it follows that the infinite product

$$\prod_{p\geqslant Y}\left(1-\frac{\nu_p-1}{p}\right)$$

is convergent, and is $1+O((\log Y)^{-2})$. This shows that

$$\begin{aligned}\prod_{p<z}\left(1-\frac{\varrho_0(p)}{p}\right)&=\prod_{p<z}\left(1-\frac{\nu_p-1}{p}\right)\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}\\ &=\sigma_0\frac{\pi^2}{6}(1+O((\log z)^{-2}))\prod_{p<z}\left(1-\frac{1}{p}\right).\end{aligned} \tag{6.7}$$

For the product in (6.6) we begin by observing that

$$\prod_{p<z}\left(1-\frac{\varrho_1(p)}{p}\right)=\prod_{p<z}\prod_{P|p}\left(1-\frac{1}{N(P)}\right)=\prod_{p<z}\left(1-\frac{1}{p}\right)\cdot\prod_{p<z}\prod_{P|p}\left(1-\frac{1}{N(P)}\right)\left(1-\frac{1}{p}\right)^{-1}.$$

On the other hand, for any $\sigma>1$ we have

$$\sum_{Y<p\leqslant Z}\sum_{P|p}\log\left(1-\frac{1}{N(P)^\sigma}\right)-\log\left(1-\frac{1}{p^\sigma}\right)=\sum_{Y<p\leqslant Z}\frac{1}{p^\sigma}-\sum_{Y<N(P)\leqslant Z}\frac{1}{N(P)^\sigma}+O(Y^{-1/2}),$$

after consideration of the contribution from prime ideals of degree 2. Partial summation, using the Prime Number Theorem and the Prime Ideal Theorem, reveals that

$$\sum_{Y<p\leqslant Z}\frac{1}{p^\sigma}=\sum_{Y<N(P)\leqslant Z}\frac{1}{N(P)^\sigma}+O((\log Y)^{-2}),$$

uniformly in $\sigma$. Thus, taking $Y=z$ and letting $Z\to\infty$ we conclude that

$$\prod_{p<z}\prod_{P|p}\left(1-\frac{1}{N(P)^\sigma}\right)\left(1-\frac{1}{p^\sigma}\right)^{-1}=\frac{\zeta(\sigma)}{\zeta_K(\sigma)}\prod_{p>z}\prod_{P|p}\left(1-\frac{1}{N(P)^\sigma}\right)^{-1}\left(1-\frac{1}{p^\sigma}\right)$$
$$=\frac{\zeta(\sigma)}{\zeta_K(\sigma)}(1+O((\log z)^{-2})).$$

We may now let $\sigma$ tend to 1 to deduce that

$$\prod_{p<z}\prod_{P|p}\left(1-\frac{1}{N(P)}\right)\left(1-\frac{1}{p}\right)^{-1}=\gamma_0^{-1}(1+O((\log z)^{-2})). \tag{6.8}$$

It follows that

$$\prod_{p<z}\left(1-\frac{\varrho_1(p)}{p}\right)=\gamma_0^{-1}(1+O((\log z)^{-2}))\prod_{p<z}\left(1-\frac{1}{p}\right),$$

whence (6.7) yields

$$\prod_{p<z}\left(1-\frac{\varrho_1(p)}{p}\right)=\gamma_0^{-1}\sigma_0^{-1}\frac{6}{\pi^2}(1+O((\log z)^{-2}))\prod_{p<z}\left(1-\frac{\varrho_0(p)}{p}\right). \tag{6.9}$$

Moreover we may note that

$$\prod_{p<z}\left(1-\frac{\varrho_0(p)}{p}\right)\ll(\log z)^{-1}, \tag{6.10}$$

again via (6.7).

We have also to compare the sum

$$\Sigma_0=\sum_{\substack{X^\tau\leqslant p_n<\dots<p_1<X^{1-\tau}\\ p_1\dots p_n<X^{1+\tau}}}\frac{\varrho_0(p_1\dots p_n)}{p_1\dots p_n}$$

with the corresponding sum $\Sigma_1$, in which the function $\varrho_0$ is replaced by $\varrho_1$. At this point we observe that $\varrho_0(p)=\nu_p+O(p^{-1})$, and similarly $\varrho_1(p)=\nu_p+O(p^{-1})$. Thus

$$\varrho_1(q)=\varrho_0(q)\{1+O(X^{-\tau})\}^n=\varrho_0(q)\{1+O((\log X^\tau)^{-2})\},$$

by (2.5) and (2.6), unless $q$ is divisible by an inert prime $p$, say. In the latter case $\varrho_0(q)=0$ and

$$\varrho_1(q)\leqslant 3^n p^{-1}\ll X^{-\tau/2},$$

again by (2.5) and (2.6). We may now compare our various estimates to show that

$$T^{(n)}(\mathcal{A})-\varkappa T^{(n)}(\mathcal{B}) \ll \frac{\eta^2X^2}{\log X^\tau}(\Sigma_0\{\exp(-\tau^{-1})+(\log X^\tau)^{-2}\}+\Sigma_0' X^{-\tau/2})+X^{2-\tau/4}$$
$$\ll \frac{\eta^2X^2}{\log X^\tau}\{\Sigma_0\exp(-\tau^{-1})+\Sigma_0' X^{-\tau/2}\}+X^{2-\tau/4},$$

where

$$\Sigma_0' = \sum_{\substack{X^\tau\leqslant p_n<\ldots<p_1<X^{1-\tau}\\ p_1\ldots p_n<X^{1+\tau}}} \frac{1}{p_1\ldots p_n}.$$

Moreover, since $\varrho_0(p)\leqslant\nu_p$, we have

$$\Sigma_0 \ll \sum_{\substack{X^\tau\leqslant p_n<\ldots<p_1<X^{1-\tau}\\ p_1\ldots p_n<X^{1+\tau}}} \frac{\nu_{p_1}\ldots\nu_{p_n}}{p_1\ldots p_n} \ll \frac{1}{n!}\Bigg(\sum_{X^\tau\leqslant p<X}\frac{\nu_p}{p}\Bigg)^n \ll \frac{1}{n!}\left(\log\frac{1}{\tau}+O(1)\right)^n,$$

and similarly

$$\Sigma_0' \ll \frac{1}{n!}\left(\log\frac{1}{\tau}+O(1)\right)^n.$$

When we sum over $n$ we therefore deduce that

$$\sum_n |T^{(n)}(\mathcal{A})-\varkappa T^{(n)}(\mathcal{B})| \ll \frac{\eta^2X^2}{\log X^\tau}\{\exp(-\tau^{-1})+X^{-\tau/2}\}$$
$$\times\exp\left(\log\frac{1}{\tau}+O(1)\right)+\tau^{-1}X^{2-\tau/4}$$
$$\ll \frac{\tau\eta^2X^2}{\log X},$$

by (2.5). This proves Lemma 3.5.

## 7. Upper bound sieve results

This section is devoted to the proof of Lemmas 3.6 and 3.7. We begin by establishing the following result, which we shall use repeatedly.

LEMMA 7.1. *Let $\mathcal{Q}$ be a set of square-free integers $q$ with $N<q\leqslant 2N$. Suppose that $z\gg X^\tau$ and $N\ll X^{2-\tau}$. Then*

$$\sum_{N(Q)\in\mathcal{Q}} S_K(\mathcal{A}_Q^{(K)},z) \ll \sum_{q\in\mathcal{Q}}\frac{\eta^2X^2}{q\log\min(z,X^{2-\tau}/N)}+X^{2-\tau/5}$$

*and*

$$\sum_{N(Q)\in\mathcal{Q}} S_K(\mathcal{B}_Q^{(K)},z) \ll \sum_{q\in\mathcal{Q}}\frac{\eta X^3}{q\log\min(z,X^{2-\tau}/N)}+X^{3-\tau/5}.$$

For the proof we begin by converting our problem into one which involves only rational numbers. For the sequence $\mathcal{A}$ we may use (6.2) to show that

$$\sum_{N(Q)\in\mathcal{Q}} S_K(\mathcal{A}_Q^{(K)}, z) = \sum_{q\in\mathcal{Q}} S(\mathcal{A}_q, z),$$

while for the sequence $\mathcal{B}$, we find that

$$\begin{aligned}\sum_{N(Q)\in\mathcal{Q}} S_K(\mathcal{B}_Q^{(K)}, z) &= \sum_{q\in\mathcal{Q}} S(\mathcal{B}_q, z) + O\Big(X^3 z^{-1/2}(\log X)^c \sum_{q\in\mathcal{Q}} \tau(q)^7 q^{-1}\Big)\\ &= \sum_{q\in\mathcal{Q}} S(\mathcal{B}_q, z) + O(X^3 z^{-1/2}(\log X)^c)\end{aligned}$$

by (6.4) and Lemma 4.2.

We now apply the form of Selberg's upper bound sieve given by Halberstam and Richert [6, Theorem 4.1]. We set

$$z_0 = \min(z^{1/2}, N^{-1/2}X^{1-\tau/2}),$$

and for $\mathcal{A}$ we take '$z$'$=z_0$, '$\omega(p)$'$=\varrho_0(p)$ and '$X$'$=6\eta^2X^2/\pi^2 q$. Similarly for $\mathcal{B}$ we take '$z$'$=z_0$, '$\omega(p)$'$=\varrho_1(p)$ and '$X$'$=3\gamma_0\eta X^3/q$. We then deduce that

$$S(\mathcal{A}_q, z) \leqslant S(\mathcal{A}_q, z_0) \ll \frac{\eta^2X^2}{q}(\log z_0)^{-1} + \sum_{d\leqslant z_0^2} \tau(d)^2\mu(d)^2|R_{dq}(\mathcal{A})|$$

and

$$S(\mathcal{B}_q, z) \leqslant S(\mathcal{B}_q, z_0) \ll \frac{\eta X^3}{q}(\log z_0)^{-1} + \sum_{d\leqslant z_0^2} \tau(d)^2\mu(d)^2|R_{dq}(\mathcal{B})|,$$

where

$$R_m(\mathcal{A}) = \#\mathcal{A}_m - \frac{6\eta^2X^2}{\pi^2}\frac{\varrho_0(m)}{m},$$

and similarly for $R_m(\mathcal{B})$. Note that we have used (6.9) and (6.10) to bound the products '$W(z)$' (in the notation of Halberstam and Richert). Clearly we may suppose that every prime factor $p$ of an element $q\in\mathcal{Q}$ satisfies $p\geqslant z$, since otherwise $S(\mathcal{A}_q, z)$ vanishes, and similarly for $S(\mathcal{B}_q, z)$. Thus we may suppose that $dq$ is square-free for $d\leqslant z_0^2$. We may now sum for $q\in\mathcal{Q}$, and use Lemmas 3.2 and 3.3 to bound the error terms. Since $Nz_0^2 \ll X^{2-\tau/2}$, by choice of $z_0$, we deduce that

$$\sum_{q\in\mathcal{Q}} S(\mathcal{A}_q, z) \ll \frac{\eta^2X^2}{\log z_0}\sum_{q\in\mathcal{Q}} q^{-1} + O(X^{2-\tau/4}(\log X)^c),$$

and similarly that

$$\sum_{q\in\mathcal{Q}} S(\mathcal{B}_q, z) \ll \frac{\eta X^3}{\log z_0} \sum_{q\in\mathcal{Q}} q^{-1} + O(X^{3-\tau/4}(\log X)^c).$$

The lemma then follows.

It is now a straightforward matter to establish Lemma 3.6. For $S_3(\mathcal{A})$ we have

$$S_3(\mathcal{A}) = \sum_{X^{1-\tau}\leqslant N(P)<X^{1+\tau}} S_K(\mathcal{A}_P^{(K)}, N(P)) \leqslant \sum_{X^{1-\tau}\leqslant N(P)<X^{1+\tau}} S_K(\mathcal{A}_P^{(K)}, X^{1/2}).$$

By Lemma 3.1, we may assume that $N(P)$ is prime. Thus Lemma 7.1 yields

$$S_3(\mathcal{A}) \ll \sum_{X^{1-\tau}\leqslant p<X^{1+\tau}} \frac{\eta^2 X^2}{p\log X} + X^{2-\tau/5} \ll \frac{\tau\eta^2 X^2}{\log X} + X^{2-\tau/5} \ll \frac{\tau\eta^2 X^2}{\log X}$$

by (2.1) and (2.5). This is satisfactory for Lemma 3.6. One may handle $S_3(\mathcal{B})$ in much the same way. We no longer know that $N(P)$ is prime. The contribution from prime ideals $P$ of degree 2, however, is

$$\ll \sum_{N(P)\geqslant X^{1/2}} \#\mathcal{B}_P^{(K)} \ll \sum_{N(P)\geqslant X^{1/2}} X^3/N(P) \ll X^{11/4},$$

the sum being over such primes. Inert primes may be handled similarly.

The treatment of $S_5$ is entirely analogous to that used for $S_3$. For $S_6(\mathcal{A})$ we have to observe that

$$\begin{aligned}\sum_{\substack{X^{\tau}\leqslant p_2<p_1<X^{1-\tau}\\ X^{3/2-\tau}<p_1p_2<X^{3/2+\tau}}} (p_1p_2)^{-1} &\leqslant \sum_{X^{1/2}<p_2<X^{1-\tau}} p_2^{-1} \sum_{X^{3/2-\tau}/p_2<p_1<X^{3/2+\tau}/p_2} p_1^{-1}\\ &\ll \sum_{X^{1/2}<p_2<X^{1-\tau}} p_2^{-1}\tau \ll \tau.\end{aligned}$$

Similarly we note that the summation conditions for $S_7$ imply that

$$N(P_3) = \frac{N(P_1P_2P_3)}{N(P_1P_2)} > X^{1/2-2\tau}$$

and

$$N(P_3)^2 < N(P_1P_2) < X^{1+\tau}.$$

We thus have a sum over $(p_1p_2p_3)^{-1}$ in which $X^{1/2-\tau/2}<p_2, p_1<X^{1-\tau}$ and $X^{1/2-2\tau}<p_3<X^{1/2+\tau/2}$. This therefore produces a total $O(\tau)$ as for $S_6$. For $S_6(\mathcal{B})$ and $S_7(\mathcal{B})$ we

again have to note that prime ideals of degree greater than 1 may occur. As for $S_3(\mathcal{B})$ these contribute $O(X^{3-1/5})$, say, which is negligible. This completes our discussion of Lemma 3.6.

For our treatment of Lemma 3.7 we note at the outset that in every application of Lemma 7.1 we will have $z \gg X^{\tau}$ and $N \ll X^{2-2\tau}$, so that $\min(z, X^{2-\tau}/N) \gg X^{\tau}$. We begin by examining $U^{(n)}(\mathcal{A})$ for $n \geqslant 3$. We shall record at the outset two estimates which we shall use repeatedly. If $z \geqslant X^{\tau}$ we have

$$\sum_{z \leqslant N(P) \leqslant zX^{\xi}} N(P)^{-1} \ll \xi\tau^{-1}. \tag{7.1}$$

Moreover we have

$$\begin{aligned}\sum_{X^{\tau} \leqslant N(P) \leqslant X^{1-\tau}} N(P)^{-1} &\leqslant \log(\tau^{-1}-1) + O\left(\frac{1}{\log X^{\tau}}\right) \\ &\leqslant \log(\tau^{-1}) - \tau + O\left(\frac{1}{\log X^{\tau}}\right) \leqslant \log(\tau^{-1}).\end{aligned} \tag{7.2}$$

In each case we use partial summation, based on the Prime Ideal Theorem, together with (2.5).

The contribution to $U^{(n)}(\mathcal{A})$ arising from terms in which

$$X^{\tau} \leqslant N(P_{n+1}) < X^{\tau+\xi}$$

may now be estimated via Lemma 7.1 as

$$\ll \sum_{N(Q) \in \mathcal{Q}} S_K(\mathcal{A}_Q^{(K)}, X^{\tau}) \ll \sum_{q \in \mathcal{Q}} \frac{\eta^2 X^2}{q \log X^{\tau}} + X^{2-\tau/5}, \tag{7.3}$$

for an appropriate set $\mathcal{Q}$. Moreover

$$\begin{aligned}\sum_{N(Q) \in \mathcal{Q}} \frac{1}{N(Q)} &\leqslant \sum_{X^{\tau} \leqslant N(P_{n+1}) < X^{\tau+\xi}} \frac{1}{N(P_{n+1})} \frac{1}{n!} \left( \sum_{X^{\tau} \leqslant N(P) < X^{1-\tau}} \frac{1}{N(P)} \right)^n \\ &\ll \xi\tau^{-1} \frac{1}{n!} (\log \tau^{-1})^n,\end{aligned} \tag{7.4}$$

by (7.1) and (7.2). In view of (7.3), the total error when we sum over $n$ is

$$\ll \frac{\eta^2 X^2}{\log X^{\tau}} \xi\tau^{-1} \exp\{\log \tau^{-1}\} \ll \frac{\eta^2 X^2}{\log X} \xi\tau^{-3},$$

which is satisfactory. The term $X^{2-\tau/5}$ in (7.3) contributes $O(\tau^{-1}X^{2-\tau/5})$ after summing over $n$, which is also satisfactory.

Those terms of $U^{(n)}(\mathcal{A})$ for which $X^{1-\tau-\xi}\leqslant N(P_1)<X^{1-\tau}$ may be estimated in a similar fashion. We must also handle the cases in which

$$X^{1+\tau-\xi}\leqslant N(P_1\dots P_n)<X^{1+\tau},$$

as well as those for which

$$X^{1+\tau}\leqslant N(P_1\dots P_{n+1})<X^{1+\tau+\xi}.$$

In each case we have

$$X^{\tau}<N(P_{n+1})<N(P_n)<\dots<N(P_2)<X^{1-\tau}$$

and

$$Y\leqslant N(P_1)<YX^{\xi}.$$

Here $Y$ may depend on $N(P_i)$ for $i\geqslant 2$, and satisfies $Y\geqslant X^{\tau}$. Thus the total contribution, after summing over $n$, is again $O((\eta^2X^2/\log X)\xi\tau^{-3})$.

Finally we shall estimate the terms for which there are primes $P_{i-1},P_i$ with

$$N(P_i)<N(P_{i-1})\leqslant X^{\xi}N(P_i).$$

To estimate

$$\sum_{N(Q)\in\mathcal{Q}}N(Q)^{-1}$$

in this case, we fix $i$, so that the sum over $P_{i-1}$ produces $O(\xi\tau^{-1})$, by (7.1). The remaining prime ideals produce a factor $O((\log\tau^{-1})^n/n!)$ as before. We therefore obtain a contribution

$$\ll\frac{\eta^2X^2}{\log X^{\tau}}\xi\tau^{-2}\frac{1}{n!}(\log\tau)^n,$$

on allowing for the various indices $i\leqslant n$, which produces $O((\eta^2X^2/\log X)\xi\tau^{-4})$ after summing over $n$.

The net effect of these estimates is that we may restrict the prime ideals $P_i$ so that $N(P_i)\in J(m_i)$, with integers $m_i$ satisfying (3.5), (3.6) and the other relevant conditions, providing that we allow for an error $O((\eta^2X^2/\log X)\xi\tau^{-4})$. Our next step is to replace

$$S_K(\mathcal{A}^{(K)}_{P_1\dots P_{n+1}},N(P_{n+1}))$$

by

$$S_K(\mathcal{A}^{(K)}_{P_1\dots P_{n+1}},X^{m_{n+1}\xi}).$$

According to Buchstab's formula this introduces an error

$$\sum_{X^{m_{n+1}\xi}\leqslant N(P)<N(P_{n+1})} S_K(\mathcal{A}^{(K)}_{P_1\dots P_{n+1}P}, N(P)).$$

We sum over the various prime ideals $P_1, ..., P_{n+1}$, using Lemma 7.1, along with (7.1) and (7.4), to show that the error is

$$\ll \frac{\eta^2X^2}{\log X^\tau}\xi\tau^{-1}\frac{1}{(n+1)!}(\log\tau^{-1})^{n+1}.$$

When this is summed over $n$ we get a total $O((\eta^2X^2/\log X)\xi\tau^{-3})$, which is satisfactory.

We proceed to introduce the factor

$$d_S = \prod_{i=1}^{n+1}\frac{\log N(P_i)}{m_i\xi\log X}.$$

Since $N(P_i)\in J(m_i)$ we find that

$$1\leqslant\frac{\log N(P_i)}{m_i\xi\log X}\leqslant 1+m_i^{-1}\leqslant 1+\xi\tau^{-1},$$

by (3.5). Thus (2.6), (3.4) and (3.9) yield

$$1\leqslant d_S\leqslant 1+O(\xi\tau^{-2}).$$

We must therefore allow for an error $O(\xi\tau^{-2}U^{(n)}(\mathcal{A}))$. Since Lemma 7.1 shows that

$$U^{(n)}(\mathcal{A})\ll\frac{\eta^2X^2}{\log X^\tau}\frac{1}{(n+1)!}(\log\tau^{-1})^{n+1},$$

by (7.2), the total contribution to Lemma 3.7 is

$$\ll\frac{\eta^2X^2}{\log X^\tau}\xi\tau^{-2}\exp\{\log\tau^{-1}\} = \frac{\eta^2X^2}{\log X}\xi\tau^{-4},$$

which is again satisfactory.

To complete the proof of Lemma 3.7 for $U^{(n)}(\mathcal{A})$, it remains to replace

$$S_K(\mathcal{A}^{(K)}_{P_1\dots P_{n+1}}, X^{m_{n+1}\xi})$$

by

$$S_K^{(*)}(\mathcal{A}^{(K)}_{P_1\dots P_{n+1}}, X^{m_{n+1}\xi}).$$

The difference between these is at most

$$\sum_{N(P)\geqslant X^\tau} \#\mathcal{A}^{(K)}_{P^2S},$$

where $S=P_1 \dots P_{n+1}$. Since we have $N(S)\geqslant X^{1+\tau}$, from (3.8), we deduce that $N(P)\ll X^{1-\tau/2}$. The total contribution, when we sum over all admissible ideals $S$, regardless of the value of $n$, is thus

$$\sum_{X^\tau\leqslant N(P)\ll X^{1-\tau/2}} \; \sum_{I:\, P^2I\in\mathcal{A}^{(K)}} \#\{S:S|I\}.$$

$I$ can, however, have $O(\tau^{-1})$ prime ideal factors $P_i$ with $N(P_i)\geqslant X^\tau$, whence

$$\#\{S:S|I\}\ll \exp\{c\tau^{-1}\}.$$

It follows that the error under consideration is

$$\ll \exp\{c\tau^{-1}\} \sum_{X^\tau\leqslant N(P)\ll X^{1-\tau/2}} \#\mathcal{A}^{(K)}_{P^2}. \tag{7.5}$$

Now if $P$ is a first degree prime ideal then

$$\#\{x\in\mathbf{Z}: X<x\leqslant X(1+\eta),\, P^2|x+y\sqrt[3]{2}\,\}\leqslant 1+\frac{\eta X}{N(P)^2}$$

for every integer $y$. Thus

$$\#\mathcal{A}^{(K)}_{P^2}\ll X+X^2N(P)^{-2}, \tag{7.6}$$

whence (7.5) is

$$\ll \exp\{c\tau^{-1}\} \sum_{X^\tau\leqslant N(P)\ll X^{1-\tau/2}} \left(X+\frac{X^2}{N(P)^2}\right) \ll \exp\{c\tau^{-1}\}X^{2-\tau/2}\ll X^{2-\tau/4},$$

say. This is satisfactory for Lemma 3.7, and completes the treatment of $U^{(n)}(\mathcal{A})$.

In order to deal with the sequence $\mathcal{B}$, it will be convenient to record the estimate

$$\#\mathcal{B}_J\ll X^3N(J)^{-1}, \tag{7.7}$$

which follows from Lemma 4.1. To handle $U^{(n)}(\mathcal{B})$ we shall first remove those terms in which some $P_i$ (call it $P_0$) has degree 2. The total effect of this, after summing over $n$, is an error at most

$$\sum_{X^\tau\leqslant N(P_0)\ll X^{1-\tau}} \; \sum_{I:\, P_0I\in\mathcal{B}} \#\{S:S|I\},$$

where $S$ runs over all products of distinct prime ideals $P$, subject to $N(P)\geqslant X^{\tau}$. Just as in the analysis of the previous paragraph, we may bound this as

$$\ll \exp\{c\tau^{-1}\}\sum_{X^{\tau}\leqslant N(P_0)\ll X^{1-\tau}} \#\mathcal{B}_{P_0}.$$

However, $\#\mathcal{B}_{P_0}\ll X^3 N(P_0)^{-1}$, by (7.7). Since $P_0$ is restricted to be of degree 2 the total error is thus

$$\ll X^3\exp\{c\tau^{-1}\}\sum_{X^{\tau}\leqslant N(P_0)\ll X^{1-\tau}} N(P_0)^{-1}\ll X^3\exp\{c\tau^{-1}\}X^{-\tau/2}\ll X^{3-\tau/4},$$

say, which is satisfactory, in view of (2.5). Primes of degree 3 may be handled similarly.

In the same way we may remove those terms in which there are two prime ideals $P_i, P_{i+1}$ with the same norm. The analysis is much as above, save that we use the bound

$$\sum_{X^{\tau}\leqslant N(P_i)=N(P_{i+1})<X^{1-\tau}} \#\mathcal{B}_{P_iP_{i+1}}\ll X^3 \sum_{X^{\tau}\leqslant N(P_i)=N(P_{i+1})<X^{1-\tau}} N(P_iP_{i+1})^{-1}\ll X^{3-\tau}.$$

This having been done, we proceed to estimate the effect of confining the primes $P_i$ so that $N(P_i)\in J(m_i)$, with the $m_i$ satisfying (3.5), (3.6) and the other relevant constraints. The analysis mimics that used for $U^{(n)}(\mathcal{A})$ precisely. Similarly we can bound the error caused by introducing the factor $d_S$, by the same argument as previously. Finally, when we replace

$$S_K(\mathcal{B}^{(K)}_{P_1\dots P_{n+1}}, X^{m_{n+1}\xi})$$

by

$$S_K^{(*)}(\mathcal{B}^{(K)}_{P_1\dots P_{n+1}}, X^{m_{n+1}\xi}),$$

we again copy the argument used before, using (7.7) instead of (7.6). This completes our discussion of Lemma 3.7 as far as $U^{(n)}(\mathcal{B})$ is concerned.

The treatment of $U_1^{(1)}$ and $U_1^{(2)}$, and also of $S_4$, follows the lines given above, both for $\mathcal{A}$ and for $\mathcal{B}$. In fact we get errors which are $O((\eta^2X^2/\log X)\xi\tau^{-3})$, since we are able to use a bound $n\ll 1$ instead of $n\ll\tau^{-1}$.

There remain the terms $U_2^{(1)}(\mathcal{A})$ and $U_2^{(1)}(\mathcal{B})$. Here too we follow the same argument as used for $U^{(n)}$. We first restrict each of $P_1$ and $P_2$ to have its norm in the relevant interval $J(n_i)$, and replace

$$S_K(\mathcal{A}^{(K)}_{P_1P_2}, N(P_2))$$

by

$$S_K(\mathcal{A}^{(K)}_{P_1P_2}, X^{n_2\xi}).$$

The errors here are estimated just as before. Note that, in applying Lemma 7.1, we have $N(P_1P_2)\leqslant X^{2-2\tau}$. Then, when we introduce the ideals $Q_1,...,Q_{n+1}$ all the errors in the subsequent manœuvres can still be estimated via Lemma 7.1, in view of the bound

$$\#\{P_1,P_2:P_1P_2\in\mathcal{A}_{Q_1...Q_{n+1}}\}\leqslant S_K(\mathcal{A}^{(K)}_{Q_1...Q_{n+1}},X^\tau).$$

Since we still have $n\ll\tau^{-1}$ in this new situation, the rest of the argument proceeds just as with $U^{(n)}$.

## 8. Proof of Lemma 3.8—the contribution from the terms $e_{(\beta)}$

We shall begin our treatment of Lemma 3.8 by considering the function $e_S$. Here we shall prove the following result.

LEMMA 8.1. *Let $\mathcal{C}\subseteq\mathbf{R}^3$ be as in Lemma 3.8. Define*

$$N((x,y,z))=x^3+2y^3+4z^3-6xyz$$

*and*

$$\mathcal{I}=\int_{\mathcal{C}} w'(N(\mathbf{x}))\,dx\,dy\,dz.$$

*Then for any positive integer $q\leqslant L^{1/6}$ and any integer $\alpha\in\mathbf{Z}[\sqrt[3]{2}]$ we have*

$$\sum_{\substack{\beta\equiv\alpha\,(\mathrm{mod}\,q)\\ \hat{\beta}\in\mathcal{C}}} e_{(\beta)}=\gamma_0^{-1}M^{-1}(\xi\log X)^{-n-1}\mathcal{I}\frac{\varepsilon(\alpha,q)}{\phi_K(q)}+O(S_0^3M^{-1}\tau(q)^c\exp\{-c\sqrt{\log L}\}),$$

*where $\varepsilon(\alpha,q)=1$ if $\alpha$ and $q$ are coprime, and $\varepsilon(\alpha,q)=0$ otherwise. Moreover we have defined*

$$M=\prod_{i=1}^{n+1}m_i,$$

*and we have written $\phi_K$ for the Euler function over the field $K$.*

According to the definition (3.12) we have

$$\sum_{\substack{\beta\equiv\alpha\,(\mathrm{mod}\,q)\\ \hat{\beta}\in\mathcal{C}}} e_{(\beta)}=M^{-1}(\xi\log X)^{-n-1}\sum_{N(J)<L}\mu(J)\log\frac{L}{N(J)}\sum_{\substack{\hat{\beta}\in\mathcal{C},\,J\mid\beta\\ \beta\equiv\alpha\,(\mathrm{mod}\,q)}} w'(N(\beta)). \tag{8.1}$$

The two conditions $J\mid\beta$ and $\beta\equiv\alpha\,(\mathrm{mod}\,q)$ are compatible only when $(J,q)\mid\alpha$, and in this latter case they define a unique residue class for $\beta$ modulo the lowest common multiple $[J,q]$.

We therefore investigate the sum

$$\sum_{\substack{\beta\in C\\ \beta\equiv\gamma\,(\mathrm{mod}\,r)}} w'(N(\beta)), \tag{8.2}$$

where $r$ is a rational integer multiple of $[J,q]$. To be specific, we shall take $r=N([J,q])$. We begin by considering the case $n\geqslant 1$. We write, temporarily, $J(m_n)=[a,b)$ and $J(m_{n+1})=[c,d)$. Then

$$w'(t)=\int\frac{dx_1\dots dx_n}{x_1\dots x_n},$$

where the integration is subject to $x_i\in J(m_i)$ for $1\leqslant i\leqslant n$ and

$$\frac{t}{d}\leqslant\prod_{i=1}^{n}x_i\leqslant\frac{t}{c}.$$

It follows that

$$w'(t)=\int(I_c(t)-I_d(t))\,\frac{dx_1\dots dx_{n-1}}{x_1\dots x_{n-1}},$$

where

$$I_v(t)=\int_0^{t/v\Pi}\frac{\chi_{[a,b)}(x)}{x}\,dx$$

and

$$\Pi=\prod_{i=1}^{n-1}x_i.$$

We may therefore deduce that

$$0\leqslant I_v(t+h)-I_v(t)\leqslant\int_{t/v\Pi}^{(t+h)/v\Pi}\frac{dx}{x}\leqslant\frac{h}{t},$$

if $h\geqslant 0$, whence

$$|w'(t+h)-w'(t)|\leqslant\frac{h}{t}\prod_{i=1}^{n-1}\int_{x\in J(m_i)}\frac{dx}{x}\leqslant\frac{h}{t}(\xi\log X)^{n-1}. \tag{8.3}$$

Moreover, we have

$$0\leqslant w'(t)\leqslant(\xi\log X)^n, \tag{8.4}$$

since

$$\int_{x\in J(m)}\frac{dx}{x}=\xi\log X.$$

For each vector $\hat{\beta}$ we take $\mathcal{C}(\beta)$ to be the cube of side $r$, centred at $\hat{\beta}$, and with sides parallel to those of $\mathcal{C}$. Then, since

$$r \leqslant N(J)q^3 \leqslant L^2 \leqslant S_0 \ll V^{1/3},$$

we have

$$N(\mathbf{x}) = N(\beta) + O(V^{2/3}r)$$

for any $\mathbf{x} \in \mathcal{C}(\beta)$. Thus (8.3) shows that

$$w'(N(\mathbf{x})) = w'(N(\beta)) + O(V^{-1/3}r(\xi \log X)^{n-1}),$$

whence

$$w'(N(\beta)) = r^{-3}\int_{\mathcal{C}(\beta)} w'(N(\mathbf{x}))\,dx\,dy\,dz + O(V^{-1/3}r(\xi\log X)^{n-1}).$$

The cubes $\mathcal{C}(\beta)$ for $\beta \equiv \gamma \pmod r$ will be disjoint, except for their boundaries. Moreover as $\hat{\beta}$ runs over $\mathcal{C}$ the union of the cubes $\mathcal{C}(\beta)$ will be a set which differs from $\mathcal{C}$ only at points within a distance $O(r)$ of the boundary. Since $r \leqslant S_0$ it follows that

$$\begin{aligned}
\sum_{\substack{\hat{\beta}\in\mathcal{C} \\ \beta\equiv\gamma \,(\mathrm{mod}\, r)}} w'(N(\beta)) &= r^{-3} \sum_{\substack{\hat{\beta}\in\mathcal{C} \\ \beta\equiv\gamma \,(\mathrm{mod}\, r)}} \int_{\mathcal{C}(\beta)} w'(N(\mathbf{x}))\,dx\,dy\,dz \\
&\qquad + O\left(V^{-1/3}r(\xi\log X)^{n-1}\left(\frac{S_0}{r}\right)^3\right) \\
&= r^{-3}\int_{\mathcal{C}} w'(N(\mathbf{x}))\,dx\,dy\,dz + O(r^{-3}\cdot rS_0^2\cdot(\xi\log X)^n) \\
&\qquad + O\left(V^{-1/3}r(\xi\log X)^{n-1}\left(\frac{S_0}{r}\right)^3\right) \\
&= r^{-3}\mathcal{I} + O(r^{-2}S_0^2(\xi\log X)^n),
\end{aligned} \tag{8.5}$$

by (8.4).

We proceed to derive the analogous estimate in the case $n=0$. Here we find that $w'(t,\mathbf{m})$ is just the characteristic function of $J(m_1)$. (Since we chose the right-hand derivative, this is correct even at the endpoints of the interval.) In particular, (8.4) remains true. If we write, temporarily, $J(m_1)=[a,b)$, we find that

$$w'(N(\beta)) = r^{-3}\int_{\mathcal{C}(\beta)} w'(N(\mathbf{x}))\,dx\,dy\,dz \tag{8.6}$$

unless $N(\mathbf{x})=a$ or $b$, for some $\mathbf{x}\in\mathcal{C}(\beta)$. Since

$$\mathbf{x}\cdot\nabla N(\mathbf{x}) = 3N(\mathbf{x}) \gg V,$$

we have $|\nabla N(\mathbf{x})| \gg V^{2/3}$, so that Lemma 4.9 may be applied with '$R$'$\ll V^{1/3}$, '$S_0$'$=r$ and '$R_0$'$\ll S_0$. The number of cubes for which (8.6) fails is therefore $O(S_0^2 r^{-2})$, whence we may deduce as before that

$$\begin{aligned}\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \beta\equiv\gamma\,(\mathrm{mod}\,r)}} w'(N(\beta)) &= r^{-3} \sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \beta\equiv\gamma\,(\mathrm{mod}\,r)}} \int_{\mathcal{C}(\beta)} w'(N(\mathbf{x}))\,dx\,dy\,dz + O(S_0^2 r^{-2})\\ &= r^{-3}\mathcal{I} + O(S_0^2 r^{-2}).\end{aligned}$$

Thus (8.5) holds for $n=0$ too.

We now observe that

$$\sum_{\substack{\hat{\beta}\in\mathcal{C},\, J|\beta\\ \beta\equiv\alpha\,(\mathrm{mod}\,q)}} w'(N(\beta))$$

is composed of $r^3/N([J,q])$ subsums of the form (8.2), whence

$$\sum_{\substack{\hat{\beta}\in\mathcal{C},\, J|\beta\\ \beta\equiv\alpha\,(\mathrm{mod}\,q)}} w'(N(\beta)) = N([J,q])^{-1}\mathcal{I} + O(S_0^2(\xi\log X)^n),$$

providing that $(J,q)|\alpha$. The error term clearly contributes

$$\ll S_0^2 M^{-1}(\xi\log X)^{-1} L\log L \ll S_0^2 M^{-1}\xi^{-1} L \ll S_0^3 M^{-1}\exp\{-c\sqrt{\log L}\}$$

to (8.1), by (3.13), (2.5) and (3.4). The main term of (8.1) may be written in the form

$$M^{-1}(\xi\log X)^{-n-1} q^{-3}\mathcal{I} \sum_{\substack{N(J)<L\\ (J,q)|\alpha}} \frac{\mu(J)}{N(J)} N((J,q))\log\frac{L}{N(J)}.$$

We write $I=(J,q)$ so that

$$\begin{aligned}\sum_{\substack{N(J)<L\\ (J,q)|\alpha}} \frac{\mu(J)}{N(J)} N((J,q))\log\frac{L}{N(J)} &= \sum_{I|q,\alpha} N(I) \sum_{\substack{N(J)<L\\ I|J}} \Big\{\sum_{A|JI^{-1},qI^{-1}} \mu(A)\Big\} \frac{\mu(J)}{N(J)}\log\frac{L}{N(J)}\\ &= \sum_{I|q,\alpha} N(I) \sum_{A|qI^{-1}} \mu(A)\frac{\mu(IA)}{N(IA)} \qquad (8.7)\\ &\quad\times \sum_{\substack{N(B)<L/N(IA)\\ (B,IA)=1}} \frac{\mu(B)}{N(B)}\log\frac{L/N(IA)}{N(B)}.\end{aligned}$$

To handle the innermost sum we therefore investigate

$$\Sigma = \sum_{\substack{N(B)<x \\ (B,C)=1}} \frac{\mu(B)}{N(B)} \log \frac{x}{N(B)},$$

using the Dirichlet series

$$f(s) = \sum_{B:(B,C)=1} \frac{\mu(B)}{N(B)^s} = \zeta_K(s)^{-1} \prod_{P|C} (1-N(P)^{-s})^{-1}.$$

The Perron formula shows that

$$\Sigma = \int_{1-i\infty}^{1+i\infty} f(s+1) x^s \frac{ds}{s^2}.$$

Now if $\mathrm{Re}(s) \geqslant \frac{3}{4}$ then

$$\begin{aligned}\prod_{P|C} (1-N(P)^{-s})^{-1} &\ll \exp\Big\{\sum_{P|C} N(P)^{-3/4}\Big\} \ll \exp\Big\{c \sum_{n \leqslant \omega(C)} n^{-3/4}\Big\} \\ &\ll \exp\{c(\log N(C))^{1/4}\}.\end{aligned}$$

Using the standard zero-free region for $\zeta_K(s)$ we may therefore change the path of integration in the usual way to obtain

$$\Sigma = \mathrm{res}\{f(s+1)x^s s^{-2} : s=1\} + O\big(\exp\{-c\sqrt{\log x}\,\}\big)$$

for a suitable constant $c$, whenever $N(C) \leqslant x$. The residue is easily found to be $\gamma_0^{-1} N(C)/\phi_K(C)$. Moreover, since $q \leqslant L^{1/6}$, we will have $N(IA) \leqslant L/N(IA)$ in (8.7), and $L/N(IA) \geqslant L^{1/2}$, so that it becomes

$$\gamma_0^{-1} \sum_{I|q,\alpha} N(I) \sum_{A|qI^{-1}} \mu(A) \frac{\mu(IA)}{\phi_K(IA)} + O\Big(\exp\{-c\sqrt{\log L}\,\} \sum_{IA|q} N(A)^{-1}\Big).$$

Using multiplicativity, the main term is readily evaluated as $\gamma_0^{-1} N(q)/\phi_K(q)$, if $q$ and $\alpha$ are coprime, and zero otherwise. The error term is also easily estimated as $O\big(\tau(q)^c \exp\{-c\sqrt{\log L}\,\}\big)$. Lemma 8.1 then follows, since (8.4) yields

$$\mathcal{I} \ll S_0^3 (\xi \log X)^n.$$

## 9. Proof of Lemma 3.8—the contribution from the terms $d_{(\beta)}$

We now turn to the analysis of $d_S$. We begin by disposing of the trivial case, in which $(\alpha,q)\neq 1$.

LEMMA 9.1. *Let $\mathcal{C}\subseteq\mathbf{R}^3$ be as in Lemma 3.8. Then for any $q\leqslant L^{1/6}$ and any integer $\alpha\in\mathbf{Z}[\sqrt[3]{2}]$ we have*

$$\sum_{\substack{\beta\equiv\alpha \,(\mathrm{mod}\, q)\\ \hat{\beta}\in\mathcal{C}}} d_{(\beta)}=0$$

*whenever $\alpha$ and $q$ have a common factor.*

For the proof we merely note that $(\beta)$ will be a product of prime ideals $P_i$ with $N(P_i)\geqslant X^{\tau}\geqslant L>N(q)$, whence $\beta$ and $q$ must be coprime.

For the remaining case we shall prove the following estimate.

LEMMA 9.2. *Let $\mathcal{C}\subseteq\mathbf{R}^3$ be as in Lemma 3.8, and let a positive integer $A$ be given. Then for any natural number $q\leqslant(\log L)^A$ and any integer $\alpha\in\mathbf{Z}[\sqrt[3]{2}]$ coprime to $q$ we have*

$$\sum_{\substack{\beta\equiv\alpha \,(\mathrm{mod}\, q)\\ \hat{\beta}\in\mathcal{C}}} d_{(\beta)}=\gamma_0^{-1}M^{-1}\phi_K(q)^{-1}(\xi\log X)^{-n-1}\mathcal{I}+O_A\big(V\exp\{-c\sqrt{\log L}\,\}\big),$$

*where $\mathcal{I}$ is as in Lemma 8.1.*

We remark that the implied constant is ineffective, because of problems with Siegel zeros.

A comparison of Lemmas 8.1, 9.1 and 9.2 immediately yields Lemma 3.8.

In order to establish Lemma 9.2 we begin by using characters to modulus $q$ to pick out the condition $\beta\equiv\alpha \pmod q$. Thus

$$\sum_{\substack{\beta\equiv\alpha \,(\mathrm{mod}\, q)\\ \hat{\beta}\in\mathcal{C}}} d_{(\beta)}=\phi_K(q)^{-1}\sum_{\chi \,(\mathrm{mod}\, q)}\bar{\chi}(\alpha)\sum_{\hat{\beta}\in\mathcal{C}} d_{(\beta)}\chi(\beta). \tag{9.1}$$

Here we stress that $\chi$ runs over characters of the multiplicative group for $\mathbf{Z}[\sqrt[3]{2}]$ modulo $q$. In order to handle the condition $\hat{\beta}\in\mathcal{C}$ we shall use Hecke Grössencharacters. For any non-zero $\beta=a+b\sqrt[3]{2}+c\sqrt[3]{4}\in\mathbf{Z}[\sqrt[3]{2}]$ we shall write

$$\beta'=a+b\omega\sqrt[3]{2}+c\omega^2\sqrt[3]{4},$$

where $\omega=\frac{1}{2}(-1+\sqrt{-3})$. We then set

$$\chi(-1)=(-1)^s,\quad \chi(\varepsilon_0)=e^{it},\quad \frac{\varepsilon_0'}{|\varepsilon_0'|}=e^{iu},\quad \log\varepsilon_0=v.$$

Here we shall choose $s=0$ or 1, $0\leqslant t,u<2\pi$ and $v\in\mathbf{R}$. We now define

$$\nu_0(\beta)=\chi(\beta)\left(\frac{\beta}{|\beta|}\right)^s\exp\{-itv^{-1}\log|\beta|\},\tag{9.2}$$

$$\nu_1(\beta)=\frac{\beta\beta'}{|\beta\beta'|}\exp\{-iuv^{-1}\log|\beta|\}$$

and

$$\nu_2(\beta)=\exp\{-2\pi iv^{-1}\log|\beta|\}.$$

Then, for each index $i$, the function $\nu_i(\beta)$ is completely multiplicative, and has modulus 1 (or possibly 0 when $i=0$). Moreover, $\nu_i(\beta_1)=\nu_i(\beta_2)$ whenever $\beta_1$ and $\beta_2$ are associates. If $S$ is an integral ideal generated by $\beta$, we may then define $\nu_i(S)=\nu_i(\beta)$. For any $\mathbf{x}\in\mathbf{R}^3$ such that $N(\mathbf{x})\neq 0$, we shall write

$$\beta(\mathbf{x})=x_1+x_2\sqrt[3]{2}+x_3\sqrt[3]{4}\quad\text{and}\quad\beta'(\mathbf{x})=x_1+x_2\omega\sqrt[3]{2}+x_3\omega^2\sqrt[3]{4}.$$

We then set

$$\nu_1(\mathbf{x})=\frac{\beta(\mathbf{x})\beta'(\mathbf{x})}{|\beta(\mathbf{x})\beta'(\mathbf{x})|}\exp\{-iuv^{-1}\log|\beta(\mathbf{x})|\}$$

and

$$\nu_2(\mathbf{x})=\exp\{-2\pi iv^{-1}\log|\beta(\mathbf{x})|\}.$$

We define

$$M=\begin{pmatrix}1&2&2\\1&1&2\\1&1&1\end{pmatrix},$$

and note that $\beta(M\mathbf{x})=\varepsilon_0\beta(\mathbf{x})$ and $\beta'(M\mathbf{x})=\varepsilon_0'\beta'(\mathbf{x})$. Thus, if we say that two vectors $\mathbf{x}$ and $\mathbf{x}'$ are associates when $\mathbf{x}'=\pm M^n\mathbf{x}$ for some $n\in\mathbf{Z}$, then we will have $\nu_i(\mathbf{x})=\nu_i(\mathbf{x}')$ $(i=1,2)$ whenever $\mathbf{x}$ and $\mathbf{x}'$ are associates.

We proceed to introduce a weight function $W(S;\Delta,\mathbf{x})$, defined for positive $\Delta<\frac{1}{2}$ by

$$W(S;\Delta,\mathbf{x})=h\left(\arg\left(\frac{\nu_1(S)}{\nu_1(\mathbf{x})}\right)\right)h\left(\arg\left(\frac{\nu_2(S)}{\nu_2(\mathbf{x})}\right)\right),$$

where

$$h(x)=\begin{cases}1-\Delta^{-1}\|x/2\pi\|, & \|x/2\pi\|\leqslant\Delta,\\ 0, & \|x/2\pi\|\geqslant\Delta,\end{cases}$$

and

$$\|t\|=\min_{n\in\mathbf{Z}}|t-n|,$$

as usual. We note that

$$h(x)=\Delta\sum_{-\infty}^{\infty}\left(\frac{\sin(\pi n\Delta)}{\pi n\Delta}\right)^{2}e^{inx}.$$

We proceed to study

$$\Sigma(\mathbf{x})=\sum_{N(\mathbf{x})<N(S)\leqslant N(\mathbf{x})+\Delta V} d_S\nu_0(S)W(S;\Delta,\mathbf{x}),$$

where $\mathbf{x}\in\mathcal{C}$, so that $V\ll N(\mathbf{x})\ll V$.

Our goal is the following result.

LEMMA 9.3. *Let $\mathbf{x}\in\mathcal{C}$ and suppose that $q\leqslant(\log L)^A$. Then*

$$\Sigma(\mathbf{x})=\varepsilon(\chi)m(\mathbf{x})\frac{\Delta^2}{M}(\xi\log X)^{-n-1}+O_A(\Delta^{-2}M^{-1}V\exp\{-c\sqrt{\log L}\,\}),\tag{9.3}$$

*where*

$$m(\mathbf{x})=w(N(\mathbf{x})+\Delta V)-w(N(\mathbf{x}))$$

*and $\varepsilon(\chi)=1$ or $0$ depending on whether $\chi$ is trivial or not.*

We begin by observing that

$$\Sigma(\mathbf{x})=\Delta^2\sum_{j,k=-\infty}^{\infty}\left(\frac{\sin(\pi j\Delta)}{\pi j\Delta}\right)^{2}\left(\frac{\sin(\pi k\Delta)}{\pi k\Delta}\right)^{2}\nu_1(\mathbf{x})^{-j}\nu_2(\mathbf{x})^{-k}\Sigma_{j,k},\tag{9.4}$$

where

$$\Sigma_{j,k}=\sum_{N(\mathbf{x})<N(S)\leqslant N(\mathbf{x})+\Delta V} d_S\nu^{(j,k)}(S)$$

and

$$\nu^{(j,k)}(S)=\nu(S)=\nu_0(S)\nu_1(S)^j\nu_2(S)^k.$$

We shall say that the character $\nu(S)$ is trivial if it takes the value 1 whenever $S$ is coprime to $q$. This corresponds to having the trivial character $\chi$ modulo $q$, letting $s=t=0$ in the definition of $\nu_0$, and taking $j=k=0$. In our situation the condition that $(S,q)=1$ is redundant, since, if $d_S$ is non-zero, then $S$ and $q$ are automatically coprime, as in the proof of Lemma 9.1.

For the case in which $\nu$ is trivial we now apply Lemma 4.10. Since $J(m_i)=[X^{m_i\xi},X^{(1+m_i)\xi})$ with $X^{m_i\xi}\geqslant X^{\tau}\geqslant L$, we may take $\varrho=X^{\xi}$ and $A=L$. Moreover we note that $m(\mathbf{x})$ is the measure of the set of $(n+1)$-tuples $(t_1,\dots,t_{n+1})$ with $t_i\in J(m_i)$ and $N(\mathbf{x})<\prod t_i\leqslant N(\mathbf{x})+\Delta V$. In view of the bound $n\ll\tau^{-1}$, the lemma then yields

$$\sum_{P_1,\dots,P_{n+1}}\prod_{i=1}^{n+1}\log N(P_i)=m(\mathbf{x})+O\left(\frac{V}{\tau}(\xi\log X)^n\exp\{-c(\log L)^{1/2}\}\right),$$

since

$$\left(1+\frac{c}{\xi\log X}\right)^n \leqslant \exp\left(\frac{cn}{\xi\log X}\right) \leqslant \exp\left(\frac{c\tau^{-1}}{\xi\log X}\right) \leqslant \exp\{O(1)\} \tag{9.5}$$

by (2.5) and (3.4). We therefore conclude that

$$\Sigma_{j,k} = m(\mathbf{x})M^{-1}(\xi\log X)^{-n-1} + O(VM^{-1}\exp\{-c(\log L)^{1/2}\}), \tag{9.6}$$

for the trivial character, in view of (2.5) and (3.4) again.

In order to bound $\Sigma_{j,k}$ in the remaining cases, we shall use a version of the Prime Number Theorem with Grössencharacters, due to Mitsui [15, Lemma 5], which yields the following.

LEMMA 9.4. *If the character $\nu^{(j,k)}$ is non-trivial then, for any positive constant $A$, we have*

$$\sum_{N(P)\leqslant z} \log N(P)\nu^{(j,k)}(P) \ll_A z\exp\{-c\sqrt{\log z}\}$$

*uniformly for*

$$|j|,|k| \ll \exp\{\sqrt{\log z}\}$$

*and* $q \leqslant (\log z)^A$.

Note that Mitsui imposes his bound on the modulus ($q$ in our notation) on [15, p. 11]. It does not appear explicitly in his statement of the result. For non-quadratic characters one may in fact allow $q \leqslant \exp\{\sqrt{\log z}\}$. As usual, however, quadratic characters are a potential problem, and Mitsui's treatment employs the familiar arguments concerning Siegel zeros. In particular, one should note that the implied constant in Lemma 9.4 is ineffective.

We now write

$$\Sigma_{j,k} = (m_1\xi\log X)^{-1}\sum_J g_J\nu(J)\sum_P \nu(P)\log N(P),$$

where $J$ runs over products $P_2 \dots P_{n+1}$ with $N(P_i) \in J(m_i)$, and

$$g_J = \prod_{i=2}^{n+1} \frac{\log N(P_i)}{m_i\xi\log X}.$$

Moreover the sum over $P$ is for

$$N(P) \in J(m_1) \quad \text{and} \quad \frac{N(\mathbf{x})}{\Pi} < N(P) \leqslant \frac{N(\mathbf{x})+\Delta V}{\Pi}, \tag{9.7}$$

where
$$\Pi = \prod_{i=2}^{n+1} N(P_i).$$
The sum $\Sigma_{j,k}$ therefore vanishes unless $V/\Pi \gg X^\tau \gg L$. Lemma 9.4 now yields
$$\Sigma_{j,k} \ll_A m_1^{-1} \sum_J g_J \frac{V}{\Pi} \exp\{-c\sqrt{\log L}\}$$
for $|j|, |k| \leqslant \exp\{c\sqrt{\log L}\}$ and $q \ll (\log L)^A$. Since
$$\sum_{N(P)\in J(m)} \frac{\log N(P)}{N(P) m\xi \log X} \ll m^{-1} \tag{9.8}$$
we deduce that
$$\Sigma_{j,k} \ll_A M^{-1} V e^{cn} \exp\{-c\sqrt{\log L}\}.$$
In view of the bounds (2.5) and (2.6) we conclude that
$$\Sigma_{j,k} \ll_A M^{-1} V \exp\{-c\sqrt{\log L}\}. \tag{9.9}$$

We also have the trivial bound
$$\Sigma_{j,k} \ll (m_1 \xi \log X)^{-1} \sum_J g_J \sum_P \log N(P).$$
The inner sum, which is subject to (9.7), is $O(V/\Pi)$. Thus if we use (9.8) for $m = m_2, \ldots, m_{n+1}$ we see that
$$\Sigma_{j,k} \ll VM^{-1}(\xi \log X)^{-1} e^{cn} \ll VM^{-1}, \tag{9.10}$$
in view of (2.5), (2.6) and (3.4). To complete the proof of Lemma 9.3, we insert the estimates (9.6) and (9.9) into (9.4) when $|j|, |k| \leqslant \exp\{c\sqrt{\log L}\}$, and use the bound (9.10) otherwise.

Our next task is to investigate the relationship between values of $\hat{\beta}$ and $\mathbf{x}$ for which $S = (\beta)$ is counted by $\Sigma(\mathbf{x})$.

LEMMA 9.5. *Let $V \ll N(S)$, $N(\mathbf{x}) \ll V$, and suppose that $W(S; \Delta, \mathbf{x}) \neq 0$ and that $N(\mathbf{x}) < N(S) \leqslant N(\mathbf{x}) + \Delta V$. Then $S$ has a generator $\beta$ for which*
$$\hat{\beta} = (1 + O(\Delta))\mathbf{x}. \tag{9.11}$$
*Similarly, if $\beta$ is any generator of $S$, then $\mathbf{x}$ has an associate $\mathbf{x}'$ for which*
$$\hat{\beta} = (1 + O(\Delta))\mathbf{x}'.$$

We begin by noting that $\beta(\mathbf{x})|\beta'(\mathbf{x})|^2=N(\mathbf{x})$ is positive, whence $\beta(\mathbf{x})$ must also be positive. Now, if $S$ and $\mathbf{x}$ are as above, and $S=(\beta)$, then

$$\left\|\frac{\arg(\nu_2(\beta))-\arg(\nu_2(\beta(\mathbf{x})))}{2\pi}\right\|\leqslant\Delta.$$

Thus

$$\left\|\frac{\log\beta(\mathbf{x})-\log|\beta|}{v}\right\|\leqslant\Delta,$$

where $v=\log\varepsilon_0$. We may therefore replace $\beta$ by a suitable associate so that $\beta>0$ and

$$\log\frac{\beta}{\beta(\mathbf{x})}\ll\Delta.$$

This latter condition implies that

$$\beta=(1+O(\Delta))\beta(\mathbf{x}). \tag{9.12}$$

Since $S=(\beta)$ is counted by $\Sigma(\mathbf{x})$ we also have

$$\left\|\frac{\arg(\nu_1(\beta))-\arg(\nu_1(\beta(\mathbf{x})))}{2\pi}\right\|\leqslant\Delta.$$

Now, since $\log\beta=\log\beta(\mathbf{x})+O(\Delta)$, we conclude that

$$\arg(\beta')=\arg(\beta'(\mathbf{x}))+O(\Delta). \tag{9.13}$$

Finally, since $N(\mathbf{x})<N(\beta)\leqslant N(\mathbf{x})+\Delta V$ we have

$$\beta|\beta'|^2=(1+O(\Delta))\beta(\mathbf{x})|\beta'(\mathbf{x})|^2,$$

so that (9.12) yields

$$|\beta'|=(1+O(\Delta))|\beta'(\mathbf{x})|. \tag{9.14}$$

A comparison of (9.13) and (9.14) shows that $\beta'=(1+O(\Delta))\beta'(\mathbf{x})$, whence (9.12) yields $\hat{\beta}=(1+O(\Delta))\mathbf{x}$, as required for (9.11). The second assertion of Lemma 9.5 follows similarly.

If $\mathbf{x}\in\mathcal{C}$ then Lemma 9.5 shows that $\hat{\beta}=\mathbf{x}+O(V^{1/3}\Delta)$. Taking the implied constant to be $c$, say, we therefore define $\mathcal{C}'$ as the set of vectors $\mathbf{t}$ for which there is at least one $\mathbf{x}\in\mathcal{C}$ with $|\mathbf{t}-\mathbf{x}|\leqslant cV^{1/3}\Delta$. Thus, if $S$ is counted by $\Sigma(\mathbf{x})$ then $S=(\beta)$ for some $\beta$ with $\hat{\beta}\in\mathcal{C}'$. Moreover, if $\Delta$ and $S_0V^{-1/3}$ are small enough, as we now assume, there is at most one such $\beta$. In view of (9.2) we may also note that we will have

$$\nu_0(\beta)=(1+O(\Delta))\chi(\beta)\exp(-itv^{-1}\log\beta(\mathbf{x})).$$

We therefore set

$$\Sigma'(\mathbf{x}) = \exp(itv^{-1}\log\beta(\mathbf{x}))\,\Sigma(\mathbf{x}) = \sum_{\substack{\beta\in C' \\ N(\mathbf{x})<N(\beta)\leqslant N(\mathbf{x})+\Delta V}} d_{(\beta)}\chi(\beta)W((\beta);\Delta,\mathbf{x})\{1+O(\Delta)\}. \tag{9.15}$$

In view of the definition of $\varepsilon(\chi)$ it then follows from (9.3) that

$$\Sigma'(\mathbf{x}) = \varepsilon(\chi)m(\mathbf{x})\frac{\Delta^2}{M}(\xi\log X)^{-n-1} + O_A\big(\Delta^{-2}M^{-1}V\exp\{-c\sqrt{\log L}\,\}\big). \tag{9.16}$$

We proceed to investigate

$$\int_{\mathcal{C}} \Sigma'(\mathbf{x})\,dx\,dy\,dz = \mathcal{J},$$

say. On the one hand, the estimate (9.16) shows that

$$\mathcal{J} = \varepsilon(\chi)\frac{\Delta^2}{M}(\xi\log X)^{-n-1}\int_{\mathcal{C}} m(\mathbf{x})\,dx\,dy\,dz + O_A\big(\Delta^{-2}M^{-1}VS_0^3\exp\{-c\sqrt{\log L}\,\}\big). \tag{9.17}$$

When $n\geqslant 1$ we may estimate $m(\mathbf{x})$ via the Mean Value Theorem, in conjunction with (8.3). Thus there is a real number $\lambda\in(0,\Delta V)$ such that

$$\begin{aligned} m(\mathbf{x}) &= w(N(\mathbf{x})+\Delta V) - w(N(\mathbf{x})) = \Delta V w'(N(\mathbf{x})+\lambda) \\ &= \Delta V\{w'(N(\mathbf{x})) + O(\Delta V N(\mathbf{x})^{-1}(\xi\log X)^{n-1})\} \\ &= \Delta V w'(N(\mathbf{x})) + O(\Delta^2 V(\xi\log X)^{n-1}). \end{aligned}$$

In this case the integral in (9.17) is

$$\Delta V\int_{\mathcal{C}} w'(N(\mathbf{x}))\,dx\,dy\,dz + O(\Delta^2 VS_0^3(\xi\log X)^{n-1}) = \Delta V\mathcal{I} + O(\Delta^2 VS_0^3(\xi\log X)^{n-1}). \tag{9.18}$$

When $n=0$ we observe that $0\leqslant m(\mathbf{x})\leqslant \Delta V$ for all $\mathbf{x}$, and that if $J(m_1)=[a,b)$, say, then

$$m(\mathbf{x}) = \begin{cases} \Delta V, & a<N(\mathbf{x})<b-\Delta V, \\ 0, & N(\mathbf{x})<a-\Delta V \text{ or } N(\mathbf{x})>b. \end{cases}$$

When $\mathbf{x}$ is confined to the cube $\mathcal{C}$, the set for which $|N(\mathbf{x})-a|\leqslant\Delta V$ has measure $O(\Delta V)$, and similarly for $|N(\mathbf{x})-b|\leqslant\Delta V$. Since $w'(t)$ is the characteristic function of $J(m_1)$, as was noted in the previous section, it follows that

$$\int_{\mathcal{C}} m(\mathbf{x})\,dx\,dy\,dz = \Delta V\int_{\mathcal{C}} w'(N(\mathbf{x}))\,dx\,dy\,dz + O(\Delta^2V^2). \tag{9.19}$$

We may now compare the bounds (9.18), for the case $n\geqslant 1$, or (9.19), for the case $n=0$, with (9.17), to deduce that

$$\mathcal{J}=\varepsilon(\chi)\frac{\Delta^3 V}{M}(\xi\log X)^{-n-1}\mathcal{I}+O(\Delta^4V^2M^{-1})+O_A\big(\Delta^{-2}M^{-1}VS_0^3\exp\{-c\sqrt{\log L}\,\}\big). \tag{9.20}$$

On the other hand, (9.15) shows that

$$\mathcal{J}=\sum_{\beta\in C'}d_{(\beta)}\chi(\beta)\{1+O(\Delta)\}\int_{\mathbf{x}\in\mathcal{C},\,N(\mathbf{x})<N(\beta)\leqslant N(\mathbf{x})+\Delta V}W((\beta);\Delta,\mathbf{x})\,dx\,dy\,dz.$$

At this point it will be convenient to assume that $\mathcal{C}$ is inside some appropriate 'fundamental domain'

$$\mathcal{F}=\{\mathbf{x}\in\mathbf{R}^3:\lambda<\beta(\mathbf{x})\leqslant\varepsilon_0\lambda\}.$$

This is certainly the case if $S_0\leqslant cV^{1/3}$ with a sufficiently small absolute constant $c$, and it is clearly enough to prove Lemma 9.2 under such an assumption. The set $\mathcal{F}$ has the property that each non-zero $\mathbf{x}$ has a unique associate in $\mathcal{F}$.

Now suppose that $\hat{\beta}\in\mathcal{C}$ and that $|\hat{\beta}-\mathbf{t}|>c'\Delta V^{1/3}$ for all $\mathbf{t}$ on the boundary of $\mathcal{C}$, where $c'$ is a suitably chosen large absolute constant. Then, according to Lemma 9.5, if $W((\beta);\Delta,\mathbf{x})\neq 0$ there is some associate $\mathbf{x}'$ of $\mathbf{x}$ for which $|\hat{\beta}-\mathbf{x}'|\leqslant c'\Delta V^{1/3}$, whence $\mathbf{x}'\in\mathcal{C}$. In particular, if $\mathbf{x}\in\mathcal{F}$ it follows that $\mathbf{x}'=\mathbf{x}$, so that $\mathbf{x}\in\mathcal{C}$. For such $\beta$ we may therefore deduce that

$$\int_{\mathbf{x}\in\mathcal{C},\,N(\mathbf{x})<N(\beta)\leqslant N(\mathbf{x})+\Delta V}W((\beta);\Delta,\mathbf{x})\,dx\,dy\,dz$$
$$=\int_{\mathbf{x}\in\mathcal{F},\,N(\mathbf{x})<N(\beta)\leqslant N(\mathbf{x})+\Delta V}W((\beta);\Delta,\mathbf{x})\,dx\,dy\,dz=I(\beta),$$

say. We now conclude that

$$\mathcal{J}=\sum_{\hat{\beta}\in\mathcal{C}}d_{(\beta)}\chi(\beta)\{1+O(\Delta)\}I(\beta)+O\Big(\sum_{\beta}{}^{*}d_{(\beta)}I(\beta)\Big), \tag{9.21}$$

where $\sum^*$ counts those $\beta$ for which $|\hat{\beta}-\mathbf{t}|\ll\Delta V^{1/3}$ for some $\mathbf{t}$ on the boundary of $\mathcal{C}$.

We now examine $I(\beta)$ more closely. We make a change of variables by setting

$$\beta(\mathbf{x})=y,\quad \beta'(\mathbf{x})=re^{i\theta}.$$

A straightforward computation shows that the Jacobian of this transformation is $r/\sqrt{27}$. Thus, if $\theta_i=\arg(\nu_i(\beta))$, then

$$I(\beta)=\frac{1}{\sqrt{27}}\int_{\lambda}^{\varepsilon_0\lambda}h\Big(\theta_2-\frac{2\pi}{v}\log y\Big)I_1(y)I_2(y)\,dy,$$

where

$$I_1(y)=\int_{N(\beta)-\Delta V\leqslant r^2y<N(\beta)} r\,dr=\frac{\Delta V}{2y}$$

and

$$I_2(y)=\int_0^{2\pi} h\Big(\theta_1-\frac{u}{v}\log y-\theta\Big)\,d\theta=2\pi\Delta.$$

An easy calculation now reveals that

$$I(\beta)=\frac{\pi\log\varepsilon_0}{\sqrt{27}}\Delta^3V=\gamma_0\Delta^3V.$$

To estimate the error terms in (9.21) we use the trivial bound

$$d_S\leqslant\prod_{i=1}^{n+1}\frac{m_i+1}{m_i}\leqslant n+2\ll\tau^{-1}\ll\log X, \tag{9.22}$$

whence (9.21) produces

$$\mathcal{J}=\gamma_0\Delta^3V\sum_{\hat\beta\in\mathcal{C}}d_{(\beta)}\chi(\beta)+O(\Delta^4V^2\log X).$$

We compare this estimate with (9.20) to deduce that

$$\begin{aligned}\sum_{\hat\beta\in\mathcal{C}}d_{(\beta)}\chi(\beta)&=\frac{\varepsilon(\chi)}{\gamma_0M}(\xi\log X)^{-n-1}\mathcal{I}+O_A\big(\Delta^{-5}M^{-1}S_0^3\exp\{-c\sqrt{\log L}\,\}\big)+O(\Delta V\log X)\\&=\frac{\varepsilon(\chi)}{\gamma_0M}(\xi\log X)^{-n-1}\mathcal{I}+O_A\big(\Delta^{-5}V\exp\{-c\sqrt{\log L}\,\}\big)+O(\Delta V\log X).\end{aligned}$$

We may now choose

$$\Delta=\exp\{-c'\sqrt{\log L}\,\},$$

with an appropriate constant $c'$, to conclude that

$$\sum_{\hat\beta\in\mathcal{C}}d_{(\beta)}\chi(\beta)=\frac{\varepsilon(\chi)}{\gamma_0M}(\xi\log X)^{-n-1}\mathcal{I}+O_A\big(V\exp\{-c\sqrt{\log L}\,\}\big).$$

This may now be fed into (9.1) to deduce Lemma 9.2.

## 10. Proof of Lemma 3.9

To handle $U_e(\mathcal{A})$ we begin by replacing $N(S)$ by $3X^3/N(R)$, where it occurs in $w'(N(S))$. We shall first suppose that $n\geqslant 1$. Since

$$N(S)=\frac{3X^3}{N(R)}(1+O(\eta)),$$

we conclude from (8.3) that

$$\frac{w'(N(S))}{M(\xi\log X)^{n+1}}=\frac{w'(3X^3/N(R))}{M(\xi\log X)^{n+1}}+O\Big(\frac{\eta}{M}(\xi\log X)^{-2}\Big).$$

The total contribution of the error term to $U_e(\mathcal{A})$ is thus

$$\begin{aligned}&\ll\frac{\eta}{M}\sum_{RS\in\mathcal{A}^{(K)}}c_R\sum_{\substack{J\mid S\\ N(J)<L}}\log\frac{L}{N(J)}\ll\frac{\eta}{M}(\log X)\sum_{I\in\mathcal{A}^{(K)}}\tau(I)^2\\ &\ll\frac{\eta}{M}(\log X)\{\#\mathcal{A}^{(K)}\}^{3/4}\Big\{\sum_{X\ll m,n\ll X}\tau\big(m+n\sqrt[3]{2}\big)^8\Big\}^{1/4}\ll\frac{\eta^{5/2}X^2}{M}(\log X)^c\end{aligned}\tag{10.1}$$

in view of (3.12) and Lemma 4.7.

We turn now to the case $n=0$. Here we note, as in §8, that $w'(t)$ is just the characteristic function of $J(m_1)$. If we write, temporarily, $E$ to denote the error on replacing $N(S)$ by $3X^3/N(R)$, we see that

$$E\leqslant(m_1\xi\log X)^{-1}\sum_{RS\in\mathcal{A}^{(K)}}c_R\Bigg|\sum_{\substack{J\mid S\\ N(J)<L}}\mu(J)\log\frac{L}{N(J)}\Bigg|,$$

where the outer sum is restricted to values for which exactly one out of $N(S)$ and $3X^3/N(R)$ belongs to $J(m_1)$. On setting $J(m_1)=[a_1,a_2)$, the above condition requires that $N(R)=3X^3a_i^{-1}\{1+O(\eta)\}$ for $i=1$ or 2. It follows that

$$E\ll(M\xi)^{-1}\sum_{R,J}c_R\mu(J)^2\#\mathcal{A}_{RJ}^{(K)},$$

with the sum restricted to such ideals $R$. We note that $c_R$ and $\mu(J)^2$ are supported on square-free ideals. Moreover all prime ideal factors of $R$ have $N(P)\geqslant X^\tau$, while $N(J)\leqslant L=X^{\tau/2}$. Thus $R$ and $J$ are coprime, whence $RJ$ may be assumed to be square-free. We also have $N(RJ)\ll X^{2-\tau/2}$, in view of (3.8). We are therefore in a position to apply Lemma 3.2. In conjunction with Lemma 4.2, this yields

$$\begin{aligned}\sum_{R,J}c_R\mu(J)^2\#\mathcal{A}_{RJ}^{(K)}&\ll\sum_{R,J}\frac{\eta^2X^2}{N(RJ)}+X^{2-\tau/4}(\log X)^c\\ &\ll\eta^2X^2\cdot\eta(\log X)^c+X^{2-\tau/4}(\log X)^c\ll\eta^3X^2(\log X)^c\end{aligned}\tag{10.2}$$

in view of the restriction on $N(R)$. Here we have used the fact that

$$X^{-c_1\tau}(\log X)^{c_2}\ll\eta^{c_3}\tag{10.3}$$

for any positive constants $c_i$, as one sees from (2.1) and (2.5).

In view of (10.1) and (10.2) we deduce that

$$U_e(\mathcal{A})=\sum_{R,J} C_{R,J}\#\mathcal{A}_{RJ}^{(K)}+O\Big(\frac{\eta^{5/2}X^2}{M}(\log X)^c\Big),$$

where

$$C_{R,J}=c_R\frac{w'(3X^3/N(R))}{M(\xi\log X)^{n+1}}\mu(J)\log\frac{L}{N(J)}.$$

As above, $RJ$ may be assumed to have norm at most $X^{2-\tau/2}$. We may therefore apply Lemma 3.2, which yields

$$\begin{aligned}U_e(\mathcal{A})=&\sum_{R,J:RJ\in\mathcal{R}} C_{R,J}\frac{6\eta^2X^2}{\pi^2N(RJ)}\varrho_2(RJ)+O(M^{-1}X^{2-\tau/4}(\log X)^c)\\ &+O(M^{-1}\eta^{5/2}X^2(\log X)^c),\end{aligned}$$

by (8.4). The second error term dominates the first, by (10.3). The main term above is

$$\frac{6\eta^2X^2}{\pi^2}\sum_R c_R\frac{w'(3X^3/N(R))}{M(\xi\log X)^{n+1}}N(R)^{-1}\varrho_2(R)\Sigma_1,$$

where

$$\Sigma_1=\sum_{\substack{N(J)\leqslant L\\ J\in\mathcal{R}}}\frac{\mu(J)}{N(J)}\varrho_2(J)\log\frac{L}{N(J)}.$$

To estimate $\Sigma_1$ we set $N(J)=q$, and observe that $\mu(J)=\mu(q)$ and

$$\varrho_2(J)=\prod_{p|q}\Big(1+\frac{1}{p}\Big)^{-1}.$$

Moreover a given value of $q$ will arise from $\prod_{p|q}\nu_p$ different ideals $J\in\mathcal{R}$, in the notation of Lemma 2.1. Thus

$$\Sigma_1=\sum_{q\leqslant L}\frac{\varrho_0(q)}{q}\mu(q)\log\frac{L}{q}.$$

We therefore define a Dirichlet series

$$f(s)=\sum_{q=1}^{\infty}\varrho_0(q)\mu(q)q^{-s},$$

and conclude from Perron's formula that

$$\Sigma_1=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}f(s+1)\frac{L^s}{s^2}\,ds.$$

However, $f(s)=\zeta_K(s)^{-1}f_0(s)$, where $f_0(s)$ has an Euler product which converges absolutely and uniformly for $\mathrm{Re}(s)\geqslant\frac{3}{4}$, say. The function $f_0(s)$ is therefore uniformly bounded in this latter region. We now move the line of integration to a path joining the points

$$-i\infty,\quad -iT,\quad -iT-\frac{c}{\log T},\quad iT-\frac{c}{\log T},\quad iT,\quad i\infty.$$

Here the constant $c\in(0,1)$ is suitably chosen, using standard results on the zero-free region, so that one has $\zeta_K(s+1)^{-1}\ll\log(2+|s|)$ to the right of the path. On choosing $T=\exp((\log L)^{1/2})$ we deduce via (6.8) that

$$\begin{aligned}
\Sigma_1 &= \mathrm{Res}\{f(s+1)L^s s^{-2}: s=0\}+O(\exp\{-c(\log L)^{1/2}\})\\
&= \gamma_0^{-1}f_0(1)+O(\exp\{-c(\log L)^{1/2}\})\\
&= \gamma_0^{-1}\prod_p\left\{\left(1-\frac{\nu_p}{p+1}\right)\prod_{P|p}\left(1-\frac{1}{N(P)}\right)^{-1}\right\}+O(\exp\{-c(\log L)^{1/2}\})\\
&= \prod_p\left\{\left(1-\frac{\nu_p}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1}\right\}+O(\exp\{-c(\log L)^{1/2}\})\\
&= \prod_p\left\{\left(1-\frac{\nu_p-1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}\right\}+O(\exp\{-c(\log L)^{1/2}\})\\
&= \tfrac{1}{6}\pi^2\sigma_0+O(\exp\{-c(\log L)^{1/2}\}),
\end{aligned}$$

where $\gamma_0$ is the residue of $\zeta_K(s)$ at $s=1$, as usual.

We may now conclude that

$$U_e(\mathcal{A})=\sigma_0\eta^2X^2\Sigma_2\{1+O(\exp\{-c(\log L)^{1/2}\})\}+O(M^{-1}\eta^{5/2}X^2(\log X)^c),$$

where

$$\Sigma_2=\sum_R c_R\frac{w'(3X^3/N(R))}{M(\xi\log X)^{n+1}}N(R)^{-1}\varrho_2(R).$$

We note that

$$\varrho_2(R)=\prod_{P|R}(1+N(P)^{-1})^{-1}.$$

Since $N(P)\geqslant X^\tau$, there are $O(\tau^{-1})$ factors, so that

$$\varrho_2(R)=1+O(\tau^{-1}X^{-\tau})=1+O(\exp\{-c(\log L)^{1/2}\}),$$

by (2.5). Hence

$$U_e(\mathcal{A})=\sigma_0\eta^2X^2\Sigma_3\{1+O(\exp\{-c(\log L)^{1/2}\})\}+O(M^{-1}\eta^{5/2}X^2(\log X)^c),$$

where

$$\Sigma_3=\sum_R c_R \frac{w'(3X^3/N(R))}{M(\xi\log X)^{n+1}}N(R)^{-1}.$$

We now deduce the bound

$$\Sigma_3 \ll (M\xi\log X)^{-1}\sum_R \frac{c_R}{N(R)} \ll M^{-1}\log X,$$

using (8.4), so that

$$\begin{aligned}U_e(\mathcal{A}) &= \sigma_0\eta^2X^2\Sigma_3+O(M^{-1}\eta^2X^2\log X\exp\{-c(\log L)^{1/2}\})\\ &\quad+O(M^{-1}\eta^{5/2}X^2(\log X)^c)\\ &= \sigma_0\eta^2X^2\Sigma_3+O(M^{-1}\eta^{5/2}X^2(\log X)^c),\end{aligned}\tag{10.4}$$

by (2.1) and (2.5).

We turn now to the analysis of $U(\mathcal{B})$. We begin by considering

$$\sum_{S\in\mathcal{B}_R^{(K)}} d_S = M^{-1}(\xi\log X)^{-n-1}\sum_{P_1,\dots,P_{n+1}}\prod_{i=1}^{n+1}\log N(P_i),$$

where $N(P_i)\in J(m_i)$ and

$$\frac{3X^3}{N(R)} < \prod_{i=1}^{n+1} N(P_i) \leqslant \frac{3X^3}{N(R)}(1+\eta).$$

We now apply Lemma 4.10, as in the previous section. Since

$$J(m_i)=[X^{m_i\xi},X^{(1+m_i)\xi})$$

with $X^{m_i\xi}\geqslant X^\tau\geqslant L$, we may take $\varrho=X^\xi$ and $A=L$. In view of the bound $n\ll\tau^{-1}$, this yields

$$\begin{aligned}\sum_{P_1,\dots,P_{n+1}}\prod_{i=1}^{n+1}\log N(P_i) &= w\Big(\frac{3X^3}{N(R)}(1+\eta)\Big)-w\Big(\frac{3X^3}{N(R)}\Big)\\ &\quad+O\Big(\frac{X^3}{\tau N(R)}(\xi\log X)^n\exp\{-c(\log L)^{1/2}\}\Big),\end{aligned}$$

by (9.5). The error term above contributes to $\sum d_S$ a total

$$\ll \frac{X^3}{MN(R)}\exp\{-c(\log L)^{1/2}\},$$

by (2.5), which produces a contribution

$$\ll \frac{X^3}{M}\exp\{-c(\log L)^{1/2}\}\sum_R \frac{c_R}{N(R)} \ll \frac{X^3}{M}\exp\{-c(\log L)^{1/2}\}$$

to $U(\mathcal{B})$.

When $n\geqslant 1$ the Mean Value Theorem shows that

$$w\Big(\frac{3X^3}{N(R)}(1+\eta)\Big)-w\Big(\frac{3X^3}{N(R)}\Big)=\eta\frac{3X^3}{N(R)}w'(\lambda)$$

for some $\lambda$ in the range

$$\frac{3X^3}{N(R)}<\lambda<\frac{3X^3}{N(R)}(1+\eta).$$

We may then use (8.3) to deduce that

$$w'(\lambda)=w'\Big(\frac{3X^3}{N(R)}\Big)+O(\eta(\xi\log X)^{n-1}).$$

Thus

$$\begin{aligned}U(\mathcal{B})&=\frac{3\eta X^3}{M}(\xi\log X)^{-n-1}\sum_R\frac{c_R}{N(R)}w'\Big(\frac{3X^3}{N(R)}\Big)\\&\quad+O\Big(\frac{\eta^2X^3}{M(\xi\log X)^2}\sum_R\frac{c_R}{N(R)}\Big)+O\Big(\frac{X^3}{M}\exp\{-c(\log L)^{1/2}\}\Big)\qquad(10.5)\\&=3\eta X^3\Sigma_3+O\Big(\frac{X^3}{M}(\eta^2+\exp\{-c(\log L)^{1/2}\})\Big)=3\eta X^3\Sigma_3+O\Big(\frac{\eta^2X^3}{M}\Big),\end{aligned}$$

by (2.1). A comparison of (10.4) and (10.5) then establishes the first part of Lemma 3.9 for $n\geqslant 1$.

When $n=0$ we recall that $w'(t)$ is the characteristic function of $J(m_1)$. A little thought then reveals that

$$w\Big(\frac{3X^3}{N(R)}(1+\eta)\Big)-w\Big(\frac{3X^3}{N(R)}\Big)=\eta\frac{3X^3}{N(R)}w'\Big(\frac{3X^3}{N(R)}\Big)$$

unless one of the endpoints of $J(m_1)$ lies in the interval between $3X^3/N(R)$ and $3X^3(1+\eta)/N(R)$. In the latter case we have

$$w\Big(\frac{3X^3}{N(R)}(1+\eta)\Big)-w\Big(\frac{3X^3}{N(R)}\Big)\ll\eta\frac{3X^3}{N(R)}.$$

It therefore follows that

$$U(\mathcal{B})=\frac{3\eta X^3}{M\xi\log X}\sum_R \frac{c_R}{N(R)}w'\left(\frac{3X^3}{N(R)}\right)$$
$$+O\left(\frac{\eta X^3}{M\xi\log X}\sum_R\frac{c_R}{N(R)}\right)+O\left(\frac{X^3}{M}\exp\{-c(\log L)^{1/2}\}\right),$$

where the sum over $R$ in the error term is for

$$N(R)=\frac{3X^3}{X^{m_1\xi}}\{1+O(\eta)\}\quad\text{or}\quad\frac{3X^3}{X^{(1+m_1)\xi}}\{1+O(\eta)\}.$$

We deduce from Lemma 4.1 that the corresponding sum is $O(\eta)$, whence

$$U(\mathcal{B})=\sum_R\frac{c_R}{M\xi\log X}\eta\frac{3X^3}{N(R)}w'\left(\frac{3X^3}{N(R)}\right)+O\left(\frac{\eta^2X^3}{M}\right),$$

and the first part of Lemma 3.9 follows as in the case $n\geqslant 1$.

In order to complete the proof of the lemma we have to sum the error term $O(M^{-1}\eta^{5/2}X^2(\log X)^c)$ over the various possibilities for $n$ and $m_1,\ldots,m_{n+1}$. We note that $m_i\ll\xi^{-1}$ and that the $m_i$ are distinct. Thus

$$\begin{aligned}\sum_n\sum_{m_1,\ldots,m_{n+1}}(m_1\ldots m_{n+1})^{-1}&\leqslant\sum_n\frac{1}{(n+1)!}\left\{\sum_{m\ll\xi^{-1}}m^{-1}\right\}^{n+1}\\&\leqslant\sum_n\frac{1}{(n+1)!}\{\log\xi^{-1}+O(1)\}^{n+1}\\&\leqslant\exp\{\log\xi^{-1}+O(1)\}\ll\xi^{-1}.\end{aligned}$$

The final part of Lemma 3.9 then follows.

## 11. The proof of Lemma 3.10—first steps

In this section we shall begin our treatment of Lemma 3.10. We write

$$\sum_{\substack{RS\in\mathcal{A}^{(K)}\\ V<N(S)\leqslant 2V}}c_Rf_S=S_V,$$

where

$$X^{1+\tau}\ll V\ll X^{3/2-\tau}. \tag{11.1}$$

If $\phi(\mathbf{x})=x+y\sqrt[3]{2}$ for $\mathbf{x}=(x,y)$, and

$$W(\mathbf{x})=\begin{cases}1, & X<x,y\leqslant X(1+\eta),\\ 0, & \text{otherwise},\end{cases}$$

we will have

$$S_V=\sum_R c_R \sum_{V<N(S)\leqslant 2V} f_S \sum_{\substack{\mathbf{x}\in\mathbf{Z}^2\\ (\phi(\mathbf{x}))=RS}} W(\mathbf{x}),$$

where $\mathbf{x}$ is restricted to run over primitive integer vectors. We shall call an integer of $K$ 'primitive' if it has no rational prime factor, and we shall write $\mathcal{P}$ for the set of ideals generated by primitive integers. Thus $R$ and $S$ may be taken to belong to $\mathcal{P}$, in the above sum. In the case of $R$ this is automatic, since $c_R$ is supported on $\mathcal{R}$.

We proceed to remove the condition that $\mathbf{x}$ is a primitive vector, by writing

$$S_V=\sum_{d\ll X}\mu(d)\sum_{R\in\mathcal{P}} c_R \sum_{\substack{V<N(S)\leqslant 2V\\ S\in\mathcal{P}}} f_S \sum_{\substack{d\mid\mathbf{x}\\ (\phi(\mathbf{x}))=RS}} W(\mathbf{x}).$$

If $d>1$ then $(d,R)\neq 1$, whence there is a prime ideal $P$ dividing $d$ for which $N(P)\geqslant X^\tau$. It follows that $d\geqslant X^{\tau/2}$. We may therefore conclude that

$$S_V=\sum_{R\in\mathcal{P}} c_R \sum_{\substack{V<N(S)\leqslant 2V\\ S\in\mathcal{P}}} f_S \sum_{\substack{\mathbf{x}\in\mathbf{Z}^2\\ (\phi(\mathbf{x}))=RS}} W(\mathbf{x})+E_V,$$

where the error term $E_V$ satisfies

$$E_V\ll \sum_{X^{\tau/2}\leqslant d\ll X}\sum_R \sum_{\substack{V<N(S)\leqslant 2V\\ V\in\mathcal{P}}} |f_S| \sum_{\substack{d\mid\mathbf{x}\\ (\phi(\mathbf{x}))=RS}} W(\mathbf{x}).$$

From (3.12) and (8.4) we see that $e_S\ll\tau(S)\log X$, whence $f_S\ll\tau(S)\log X$, by (3.11) and (9.22). Thus

$$\begin{aligned} E_V &\ll \sum_{X^{\tau/2}\leqslant d\ll X}\sum_R \sum_{V<N(S)\leqslant 2V} \tau(S)\log X \sum_{\substack{d\mid\mathbf{x}\\ (\phi(\mathbf{x}))=RS}} W(\mathbf{x})\\ &\ll \log X \sum_{X^{\tau/2}\leqslant d\ll X}\sum_{d\mid\mathbf{x}} W(\mathbf{x})\tau\big(x+y\sqrt[3]{2}\big)^2\\ &\ll \log X \sum_{X^{\tau/2}\leqslant d\ll X}\tau(d)^c\sum_{m,n\ll X/d}\tau\big(m+n\sqrt[3]{2}\big)^2\\ &\ll \log X \sum_{X^{\tau/2}\leqslant d\ll X}\tau(d)^c\left(\frac{X}{d}\right)^2(\log X)^c\ll X^{2-\tau/2}(\log X)^c, \end{aligned} \tag{11.2}$$

by Lemmas 4.7 and 4.2. This will be satisfactory for our purposes.

We now replace $R$ and $S$ by their generators $\alpha$ and $\beta$, say, and write $\mathcal{Q}$ for the set of primitive integers of $K$. If we take $\beta$ to run over a suitable set $\mathcal{Q}'$ of non-associated primitive integers of $K$, and require that $\phi(\mathbf{x})=\alpha\beta$, then we will obtain exactly one value of $\alpha$ from each relevant set of associates. It therefore follows that

$$S_V=\sum_{\alpha\in\mathcal{Q}} c_{(\alpha)} \sum_{V<|N(\beta)|\leqslant 2V} F_\beta \sum_{\substack{\mathbf{x}\in\mathbf{Z}^2\\ \phi(\mathbf{x})=\alpha\beta}} W(\mathbf{x})+O(X^{2-\tau/2}(\log X)^c),$$

where $F_\beta=f_{(\beta)}$ if $\beta\in\mathcal{Q}'$, and $F_\beta=0$ otherwise. In order to specify a suitable set of non-associated integers $\beta$ we take $\beta>0$ and require that

$$N(\beta)^{1/3}\varepsilon_0^{-1/2}<\beta\leqslant N(\beta)^{1/3}\varepsilon_0^{1/2}, \tag{11.3}$$

where $\varepsilon_0=1+\sqrt[3]{2}+\sqrt[3]{4}$ is the fundamental unit of $K$.

Since $N(\alpha)\ll X^3/V$, an application of Cauchy's inequality yields

$$\sum_{\alpha\in\mathcal{Q}} c_{(\alpha)} \sum_{V<|N(\beta)|\leqslant 2V} F_\beta \sum_{\substack{\mathbf{x}\in\mathbf{Z}^2\\ \phi(\mathbf{x})=\alpha\beta}} W(\mathbf{x})\ll\Big(\sum_{N(\alpha)\ll X^3/V} 1\Big)^{1/2} S^{1/2}\ll (X^3/V)^{1/2}S^{1/2},$$

where

$$S=\sum_{\alpha\in\mathcal{Q}}\Bigg|\sum_{V<|N(\beta)|\leqslant 2V} F_\beta \sum_{\substack{\mathbf{x}\in\mathbf{Z}^2\\ \phi(\mathbf{x})=\alpha\beta}} W(\mathbf{x})\Bigg|^2.$$

We proceed to expand the square of the sum over $\beta$ to obtain

$$S=\sum_{\beta_1,\beta_2} F_{\beta_1}F_{\beta_2}\sum_{\mathbf{x}_1,\mathbf{x}_2} W(\mathbf{x}_1)W(\mathbf{x}_2)\delta,$$

where

$$\delta=\#\{\alpha\in\mathcal{Q}:\alpha=\phi(\mathbf{x}_1)/\beta_1=\phi(\mathbf{x}_2)/\beta_2\},$$

so that $\delta=1$ or 0.

We now split $S$ as $S_1+S_2$, where $S_1$ consists of the terms for which $\beta_1=\beta_2$, and $S_2$ consists of the terms for which $\beta_1\neq\beta_2$. Since $F_\beta\ll\tau(\beta)\log X$, we find that

$$S_1\ll\sum_\beta F_\beta^2\sum_{\mathbf{x}:\beta|\phi(\mathbf{x})} W(\mathbf{x})\ll(\log X)^2\sum_{x,y\ll X}\tau\big(x+y\sqrt[3]{2}\big)^3\ll X^2(\log X)^c,$$

by Lemma 4.7.

To handle the off-diagonal terms we write

$$\alpha = r + s\sqrt[3]{2} + t\sqrt[3]{4}$$

and

$$\beta_i = u_i + v_i\sqrt[3]{2} + w_i\sqrt[3]{4}.$$

It will also be convenient to set

$$\widehat{\alpha} = (r,s,t), \quad \hat{\beta}_i = (u_i, v_i, w_i).$$

The conditions $\alpha\beta_i = x_i + y_i\sqrt[3]{2}$ for $i=1,2$ then yield $rw_i + sv_i + tu_i = 0$. Thus, unless $\beta_1 = \beta_2$, we see that the primitive vector $\widehat{\alpha} \in \mathbf{Z}^3$ must be given by

$$\widehat{\alpha} = \pm D^{-1}(v_1u_2 - u_1v_2, u_1w_2 - w_1u_2, w_1v_2 - v_1w_2), \tag{11.4}$$

where

$$D = \text{h.c.f.}(v_1u_2 - u_1v_2, u_1w_2 - w_1u_2, w_1v_2 - v_1w_2).$$

We may note at once that the condition $V < |N(\beta_i)| \leqslant 2V$, along with the requirement (11.3), leads to the constraints

$$V^{1/3} \ll |\hat{\beta}_i| \ll V^{1/3}. \tag{11.5}$$

Similarly we observe that

$$r, s, t \ll \max_j |\alpha^{(j)}|,$$

where $\alpha^{(j)}$ denotes the $j$th conjugate. However, $\alpha^{(j)}\beta^{(j)} \ll X$ for any conjugate, and $|\beta^{(j)}| \gg V^{1/3}$, by (11.5). It follows that

$$XV^{-1/3} \ll |\widehat{\alpha}| \ll XV^{-1/3}. \tag{11.6}$$

In view of (11.4) we therefore deduce that $V^{2/3}D^{-1} \gg XV^{-1/3}$, whence

$$D \ll VX^{-1}. \tag{11.7}$$

Our next task in this section is to show that values of $D$ which are appreciably smaller than $VX^{-1}$ make a negligible contribution. To be more precise, we shall introduce a new parameter $Y = Y(X)$ such that

$$1 \ll Y(X) \ll X^{\tau/3}, \tag{11.8}$$

and we shall deal with the case $D \leqslant VX^{-1}Y^{-1}$. We shall specify $Y$ later, see (13.7).

We begin by observing that

$$|F_{\beta_1}F_{\beta_2}| \leqslant \tfrac{1}{2}(|F_{\beta_1}|^2+|F_{\beta_2}|^2),$$

so that it suffices to estimate

$$S_3 = \sum_{\beta_1,\beta_2} \tau(\beta_1)^2 \sum_{\mathbf{x}_1,\mathbf{x}_2} W(\mathbf{x}_1)W(\mathbf{x}_2)\delta,$$

where the sum is subject to the condition $D \leqslant VX^{-1}Y^{-1}$.

It will be convenient to argue in slightly greater generality than we actually need at this point. We begin by proving the following estimate.

LEMMA 11.1. *Let $C_1, C_2$ be cubes of side $S_0$, not necessarily containing the origin. Suppose that $C_1$ and $C_2$ are included in a sphere, centred on the origin, of radius $S_0^A$ for some positive constant $A$. Then, if the vectors $\hat{\beta}_i$ are restricted to be primitive, we will have*

$$\sum_{\substack{\hat{\beta}_i \in C_i \\ D|\hat{\beta}_1\wedge\hat{\beta}_2}} \tau(\beta_1)^2 \ll S_0^6 D^{-2}(\log S_0)^{c(A)}$$

*for some constant $c(A)$, providing that $D \ll S_0$.*

For the proof of the lemma we begin by observing that if $D|\hat{\beta}_1\wedge\hat{\beta}_2$ then for each prime power $p^e||D$ there is a corresponding integer $\lambda$, depending on $\hat{\beta}_1$ and $\hat{\beta}_2$, such that

$$\hat{\beta}_2 \equiv \lambda\hat{\beta}_1 \pmod{p^e}.$$

Here we use the fact that $p\nmid\hat{\beta}_1$, since $\hat{\beta}_1$ is primitive. The Chinese Remainder Theorem then shows that $\hat{\beta}_2 \equiv \lambda\hat{\beta}_1 \pmod{D}$ for some integer $\lambda$. If $\hat{\beta}_1$ is given there are therefore at most $D$ possible residue classes modulo $D$ in which $\hat{\beta}_2$ may lie. Since $D \ll S_0$ it follows that there are $O(S_0^3 D^{-2})$ values of $\hat{\beta}_2$ corresponding to each $\hat{\beta}_1$. The sum in Lemma 11.1 is therefore

$$\ll S_0^3 D^{-2} \sum_{\hat{\beta}_1 \in C_1} \tau(\beta_1)^2 \ll S_0^3 D^{-2} \cdot S_0^3 (\log S_0)^{c(A)},$$

by Lemma 4.5. This completes the proof of Lemma 11.1.

We are now ready to estimate $S_3$. Here we observe that

$$\hat{\beta}_1\wedge\hat{\beta}_2 = \pm D\widehat{\alpha},$$

whence

$$|\hat{\beta}_1\wedge\hat{\beta}_2| \ll DXV^{-1/3},$$

by (11.6). In view of (11.5) we therefore see that $\hat{\beta}_2$ is confined to a circular cylinder of radius $O(DXV^{-2/3})$ and length $O(V^{1/3})$, whose axis is parallel to the vector $\hat{\beta}_1$. Since $D\ll VX^{-1}$ we see that $DXV^{-2/3}\ll V^{1/3}$. We now decompose the available region for $\hat{\beta}_1$ into

$$\ll\left(\frac{V^{1/3}}{DXV^{-2/3}}\right)^3$$

cubes $\mathcal{C}_1$ with side $S_0$ of order $DXV^{-2/3}$. For $\hat{\beta}_1$ in a given cube $\mathcal{C}_1$ the available region for $\hat{\beta}_2$ may be covered by

$$\ll\frac{V^{1/3}}{DXV^{-2/3}}$$

cubes $\mathcal{C}_2$ with side $S_0$, giving $O(V^4D^{-4}X^{-4})$ pairs $\mathcal{C}_1,\mathcal{C}_2$ in total. If $D\geqslant V^{3/4}X^{-1}$ we have $S_0\gg V^{1/12}$, and Lemma 11.1 shows that each hypercube contributes

$$\ll S_0^6D^{-2}(\log X)^c\ll D^4X^6V^{-4}(\log X)^c$$

to $S_3$, making a total $O(X^2(\log X)^c)$. In the alternative case $D\leqslant V^{3/4}X^{-1}$ we ignore the condition $D|\hat{\beta}_1\wedge\hat{\beta}_2$, and merely note that the cylindrical region for $\hat{\beta}_2$ described above contains $O(V^{1/3}(DXV^{-2/3})^2)$ points, since

$$DXV^{-2/3}\geqslant XV^{-2/3}\geqslant 1.$$

Each such $D$ therefore contributes

$$\ll V^{-1}D^2X^2\sum_{\hat{\beta}_1\ll V^{1/3}}\tau(\beta_1)^2\ll V^{-1}D^2X^2\cdot V(\log X)^c=D^2X^2(\log X)^c$$

to $S_3$. We may now sum over all $D\leqslant VX^{-1}Y^{-1}$ to find that

$$\begin{aligned}S_3&\ll X^2(\log X)^c\cdot VX^{-1}Y^{-1}+(V^{3/4}X^{-1})^3X^2(\log X)^c\\&\ll(VXY^{-1}+V^{9/4}X^{-1})(\log X)^c\ll VXY^{-1}(\log X)^c,\end{aligned}$$

by (11.8), since $V\leqslant X^{3/2}$.

We may now summarize our conclusions thus far in the following lemma.

LEMMA 11.2. *There exists an absolute constant $c$ such that*

$$S_V\ll X^2Y^{-1/2}(\log X)^c+X^{3/2}V^{-1/2}S_4^{1/2},$$

*where*

$$S_4=\sum_{\beta_1,\beta_2}F_{\beta_1}F_{\beta_2}\sum_{\mathbf{x}_1,\mathbf{x}_2}W(\mathbf{x}_1)W(\mathbf{x}_2)\delta,$$

*subject to the condition $D>VX^{-1}Y^{-1}$.*

Note that the first term in the above estimate dominates (11.2), by virtue of the bound (11.8).

## 12. The proof of Lemma 3.10—separation of the variables

In this section we shall convert the sum $S_4$ into one in which the variables $\beta_1$ and $\beta_2$ are independent. This will enable us to put the sum into a form suitable for a large sieve estimate.

Since $\alpha\beta_i = x_i + y_i\sqrt[3]{2}$ we see from (11.4) that $x_i$ and $y_i$ may be expressed in terms of $\beta_1$ and $\beta_2$ as

$$(x_i, y_i) = \pm D^{-1}\mathbf{p}_i(\beta_1, \beta_2) = \pm D^{-1}(p_i(\beta_1, \beta_2), q_i(\beta_1, \beta_2)), \quad i = 1, 2,$$

say. It then follows that

$$S_4 = \sum_{\beta_1 \neq \beta_2} F_{\beta_1} F_{\beta_2} \sum_{\pm} W(\pm D^{-1}\mathbf{p}_1(\beta_1, \beta_2))\, W(\pm D^{-1}\mathbf{p}_2(\beta_1, \beta_2)).$$

Since $\mathbf{p}_1(\beta_1, \beta_2) = -\mathbf{p}_2(\beta_2, \beta_1)$, our expression for $S_4$ may be reduced to

$$S_4 = 2 \sum_{\beta_1 \neq \beta_2} F_{\beta_1} F_{\beta_2} W(D^{-1}\mathbf{p}_1(\beta_1, \beta_2))\, W(D^{-1}\mathbf{p}_2(\beta_1, \beta_2)).$$

We shall write

$$\mathbf{v} = (v_1 u_2 - u_1 v_2, u_1 w_2 - w_1 u_2, w_1 v_2 - v_1 w_2).$$

The conditions on the variables $\beta_i$ require that $D$ is the highest common factor of the entries of $\mathbf{v}$. We shall write this as $D = \text{h.c.f.}(\mathbf{v})$. The remaining constraints may be written in the form $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{R}_D$, where $\mathcal{R}_D \subseteq \mathbf{R}^6$ is defined by the inequalities

$$V < |N(\beta_i)| \leqslant 2V,$$
$$N(\beta_i)^{1/3}\varepsilon_0^{-1/2} < \beta_i \leqslant N(\beta_i)^{1/3}\varepsilon_0^{1/2}$$

and

$$XD < p_i(\beta_1, \beta_2),\ q_i(\beta_1, \beta_2),\ p_i(\beta_2, \beta_1),\ q_i(\beta_2, \beta_1) \leqslant XD(1+\eta).$$

Note that $\mathbf{p}_i(\beta, \beta) = 0$, so that the condition $\beta_1 \neq \beta_2$ is redundant. In order to remove the dependence of the region $\mathcal{R}_D$ on the modulus $D$, we shall decompose the range for $D$ firstly into intervals $\Delta < D \leqslant 2\Delta$, and then into subintervals

$$D \in I_m = \left(\frac{m-1}{N}\Delta, \frac{m}{N}\Delta\right], \quad N < m \leqslant 2N. \tag{12.1}$$

Here

$$N \ll X^{2\tau/3} \tag{12.2}$$

is a large integer parameter which we shall specify later (see (12.7)). We may note that $\Delta \ll VX^{-1}$, in view of (11.7). It follows that there is at least one pair of values $\Delta, m$ for which

$$S_4 \ll N(\log X)S_5,$$

where

$$S_5 = \sum_{D\in I_m} \Bigg| \sum_{\substack{(\hat\beta_1,\hat\beta_2)\in\mathcal{R}_D \\ D=\mathrm{h.c.f.}(\mathbf{v})}} F_{\beta_1}F_{\beta_2} \Bigg|.$$

We now proceed to cover the region of summation by means of hypercubes $\mathcal{C}(n_1,n_2,n_3)\times\mathcal{C}(n_4,n_5,n_6)$, where

$$\mathcal{C}(n_i,n_j,n_k) = I(n_i)\times I(n_j)\times I(n_k), \tag{12.3}$$

with

$$I(n) = \left(V^{1/3}\frac{n-1}{N}, V^{1/3}\frac{n}{N}\right]. \tag{12.4}$$

Clearly we may suppose that $n_i \ll N$, whence there are $O(N^6)$ possible hypercubes $\mathcal{C}_1\times\mathcal{C}_2$ to consider. We shall say that a hypercube is of Class I if it lies completely inside $\mathcal{R}_D$ for each $D\in I_m$, and of Class II if there is at least one $D\in I_m$ for which

$$\mathcal{R}_D \cap (\mathcal{C}_1\times\mathcal{C}_2) \neq \varnothing \quad \text{and} \quad \mathcal{C}_1\times\mathcal{C}_2 \not\subseteq \mathcal{R}_D.$$

Hypercubes which are neither of Class I nor of Class II clearly make no contribution to $S_5$, so that we may write $S_5 = S^{(\mathrm{I})} + S^{(\mathrm{II})}$, with the obvious notation.

Our next task is to make a trivial estimate for $S^{(\mathrm{II})}$. To do this we shall begin by bounding the number of Class II hypercubes, using Lemma 4.9. Each Class II hypercube contains a point for which one of the equations

$$\begin{aligned} N(\beta_i) &= V \text{ or } 2V, \quad i=1 \text{ or } 2, \\ \beta_i^3 &= N(\beta_i)\varepsilon_0^{-3/2} \text{ or } N(\beta_i)\varepsilon_0^{3/2}, \quad i=1 \text{ or } 2, \\ p_i(\beta_1,\beta_2) &= XD \text{ or } XD(1+\eta), \quad i=1 \text{ or } 2, \end{aligned} \tag{12.5}$$

or

$$q_i(\beta_1,\beta_2) = XD \text{ or } XD(1+\eta), \quad i=1 \text{ or } 2, \tag{12.6}$$

holds. We may write each equation in the form $F_i(\hat\beta_1,\hat\beta_2)=H_i$, for some positive integer $i\leqslant 16$, where $F_i$ is homogeneous of degree 3. In the case of the equations (12.5) and (12.6) we may use (12.1) and (11.7) to replace $H_i = XD$ or $XD(1+\eta)$ by $H_i'+O(V/N)$, where

$$H_i' = X\frac{m}{N}\Delta \quad \text{or} \quad X\frac{m}{N}(1+\eta).$$

This produces a value independent of $D$. In each case we therefore find, in the notation of (12.3), that the vertices of the hypercube satisfy an equation of the form

$$F_i(n_1, ..., n_6) = H_i'' + O(N^2),$$

with $H_i''$ fixed.

We now observe that the polynomials $F_i$ are non-singular in the relevant region. This is a straightforward calculation, and we shall give the details only for the case $i=1$ of (12.6). Here we find that

$$q_1(\beta_1, \beta_2) = \pm D(v_1 r + u_1 s + 2w_1 t),$$

with $(r, s, t)$ given by (11.4). It follows that

$$F(x_1, ..., x_6) = x_2(x_2x_4 - x_1x_5) + x_1(x_1x_6 - x_3x_4) + 2x_3(x_3x_5 - x_2x_6).$$

We may then calculate

$$\frac{\partial F}{\partial x_4} = x_2^2 - x_1x_3, \quad \frac{\partial F}{\partial x_5} = 2x_3^2 - x_1x_2, \quad \frac{\partial F}{\partial x_6} = x_1^2 - 2x_2x_3.$$

Now the hypercube under consideration contains a point of $\mathcal{R}_D$, for some $D$. This point therefore satisfies $|N(\beta_1)| \gg V$, whence

$$n_1^3 + 2n_2^3 + 4n_3^3 - 6n_1n_2n_3 \gg N^3,$$

if $N$ is large enough. Since

$$x_1^3 + 2x_2^3 + 4x_3^3 - 6x_1x_2x_3 = x_1\frac{\partial F}{\partial x_6} + 2x_2\frac{\partial F}{\partial x_4} + 2x_3\frac{\partial F}{\partial x_5},$$

we deduce that $|\nabla F(\mathbf{n})| \gg N^2$, as required.

We may therefore apply Lemma 4.9 with $S_0 = 1$ and $R = R_0 \ll N$, to show that there are $O(N^5)$ Class II hypercubes. This allows us to deduce as follows.

LEMMA 12.1. *We have*

$$S_4 \ll N(\log X) S_5,$$

*where*

$$S_5 = \sum_{D \in I_m} \Bigg| \sum_{\substack{(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{R}_D \\ D = \text{h.c.f.}(\mathbf{v})}} F_{\beta_1} F_{\beta_2} \Bigg|.$$

*Moreover, there are cubes* $\mathcal{C}_1=\mathcal{C}(m_1,m_2,m_3)$, $\mathcal{C}_2=\mathcal{C}(m_4,m_5,m_6)$, $\mathcal{C}_1'=\mathcal{C}(n_1,n_2,n_3)$ *and* $\mathcal{C}_2'=\mathcal{C}(n_4,n_5,n_6)$, *given by* (12.3) *and* (12.4), *such that*

$$S_5 \ll N^6 S_6 + N^5 (\log X)^2 S_7,$$

*with*

$$S_6 = \sum_{D\in I_m} \Bigg| \sum_{\substack{\beta_i\in\mathcal{C}_i \\ D=\text{h.c.f.}(\mathbf{v})}} F_{\beta_1}F_{\beta_2} \Bigg|$$

*and*

$$S_7 = \sum_{D\in I_m} \sum_{\substack{\beta_i\in\mathcal{C}_i' \\ D\mid\mathbf{v}}} \tau(\beta_1)\tau(\beta_2).$$

*The hypercube* $\mathcal{C}_1\times\mathcal{C}_2$ *is of Class* I, *so that* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are distinct, and therefore disjoint.*

We proceed to estimate $S_7$, using Lemma 11.1, along with the bound

$$\tau(\beta_1)\tau(\beta_2) \ll \tau(\beta_1)^2+\tau(\beta_2)^2.$$

Note that the condition $D\ll S_0$ follows from (11.1), (11.7) and (12.2). We deduce that

$$S_7 \ll \sum_{D\in I_m} V^2N^{-6}D^{-2}(\log X)^c \ll \Delta^{-1}N^{-7}V^2(\log X)^c \ll VXYN^{-7}(\log X)^c,$$

since $\Delta\gg VX^{-1}Y^{-1}$. This yields

$$S_4 \ll VXYN^{-1}(\log X)^c + N^7(\log X)S_6.$$

We therefore define

$$N = Y^2, \tag{12.7}$$

so that (12.2) follows from (11.8). We now see that Lemma 11.2 yields

$$S_V \ll X^2Y^{-1/2}(\log X)^c + X^{3/2}V^{-1/2}Y^7S_6^{1/2}(\log X)^c. \tag{12.8}$$

We turn now to the sum $S_6$. Since $D=\text{h.c.f.}(\mathbf{v})$ we have

$$S_6 = \sum_{D\in I_m} \Bigg| \sum_{d=1}^{\infty} \mu(d) \sum_{\substack{\beta_i\in\mathcal{C}_i \\ Dd\mid\mathbf{v}}} F_{\beta_1}F_{\beta_2} \Bigg|.$$

Our remaining goal in this section is to show that values $dD>d_0$, where

$$d_0 = Y^{15}VX^{-1}+V^{1/6}, \tag{12.9}$$

make a negligible contribution. To handle the range $H<dD\leqslant 2H$, say, we write $\mathbf{v}=r\mathbf{w}$ for some primitive $\mathbf{w}$. Note that $\mathbf{v}\neq\mathbf{0}$, as $\beta_1$ and $\beta_2$ are positive, primitive and unequal. Since $dD|r$ we have $|\mathbf{w}|\ll V^{2/3}H^{-1}$. We now denote the contribution to $S_6$ arising from each pair $H,\mathbf{w}$ by $S_6(H,\mathbf{w})$. We observe that $\mathbf{w}\cdot\hat{\beta}_i=0$, so that each $\hat{\beta}_i$ lies on a certain 2-dimensional lattice $\Lambda$. It then follows from Lemma 4.8 that there exist $\mathbf{z}_1,\mathbf{z}_2$ with $|\mathbf{z}_1|\leqslant|\mathbf{z}_2|$ such that

$$\hat{\beta}_i=\lambda_i\mathbf{z}_1+\mu_i\mathbf{z}_2$$

for appropriate integers $\lambda_i,\mu_i$ with

$$\lambda_i\ll\frac{V^{1/3}}{|\mathbf{z}_1|},\quad \mu_i\ll\frac{V^{1/3}}{|\mathbf{z}_2|}.$$

Moreover we have

$$\hat{\beta}_1\wedge\hat{\beta}_2=(\lambda_1\mu_2-\mu_1\lambda_2)\mathbf{z}_1\wedge\mathbf{z}_2=\pm(\lambda_1\mu_2-\mu_1\lambda_2)\mathbf{w},$$

by Lemma 4.8, whence $r=\pm(\lambda_1\mu_2-\mu_1\lambda_2)$. It therefore follows that

$$S_6(H,\mathbf{w})\ll(\log X)^2\sum_{\hat{\beta}_1,\hat{\beta}_2\in\Lambda}\tau(\beta_1)\tau(\beta_2)\tau(\lambda_1\mu_2-\mu_1\lambda_2)^2.$$

We note that if $\mu_1=\mu_2=0$ then $\hat{\beta}_i$ are primitive and scalar multiples of each other. Thus $\hat{\beta}_1=\pm\hat{\beta}_2$, contradicting the fact that the hypercube $\mathcal{C}_1\times\mathcal{C}_2$ is of Class I. We may therefore assume that $|\mathbf{z}_2|\ll V^{1/3}$.

Since

$$ABC\leqslant\tfrac{1}{3}(A^3+B^3+C^3)$$

for any positive numbers $A,B,C$, we deduce that

$$S_6(H,\mathbf{w})\ll(\log X)^2\sum_{\hat{\beta}_1,\hat{\beta}_2\in\Lambda}\tau(\gamma)^6,$$

where $\gamma$ is $\beta_1$, $\beta_2$ or $\lambda_1\mu_2-\mu_1\lambda_2$. We may suppose that we have $\gamma=\beta_1$, say, or $\lambda_1\mu_2-\mu_1\lambda_2$. We then write $\gamma$ as $\gamma=m\phi+n\psi$, where $m=\lambda_1$ and $n=\mu_1$, and $\phi,\psi$ may depend on $\mathbf{z}_1$, $\mathbf{z}_2$, $\lambda_2$ and $\mu_2$. Since $\phi^{(j)},\psi^{(j)}\ll V^{1/3}$ for any conjugate, we may apply Lemma 4.6 with

$$x=\frac{V^{1/3}}{|\mathbf{z}_1|}+V^{1/12}$$

and $y=2+V^{1/3}|\mathbf{z}_2|^{-1}$. This yields

$$\sum_{\substack{\hat{\beta}_1\in\Lambda\\ \mu_1\neq0}}\tau(\gamma)^6\ll xy(\log X)^c.$$

However, since $\hat{\beta}_1$ must be primitive there are at most two terms with $\mu_1=0$, and $\tau(\gamma)^6 \ll V^{1/12} \ll x$ for each of these. It follows that

$$\sum_{\hat{\beta}_1\in\Lambda} \tau(\gamma)^6 \ll xy(\log X)^c \ll \left\{\frac{V^{2/3}}{|\mathbf{z}_1|\cdot|\mathbf{z}_2|}+\frac{V^{5/12}}{|\mathbf{z}_2|}\right\}(\log X)^c \ll \left\{\frac{V^{2/3}}{|\mathbf{w}|}+\frac{V^{5/12}}{|\mathbf{w}|^{1/2}}\right\}(\log X)^c.$$

Since the number of values taken by $\hat{\beta}_2$ is

$$\ll \left(1+\frac{V^{1/3}}{|\mathbf{z}_1|}\right)\left(1+\frac{V^{1/3}}{|\mathbf{z}_2|}\right) \ll \frac{V^{2/3}}{|\mathbf{z}_1|\cdot|\mathbf{z}_2|} \ll \frac{V^{2/3}}{|\mathbf{w}|},$$

we deduce that

$$\begin{aligned}\sum_{|\mathbf{w}|\ll V^{2/3}H^{-1}} |S_6(H,\mathbf{w})| &\ll \sum_{\mathbf{w}}\left(\frac{V^{4/3}}{|\mathbf{w}|^2}+\frac{V^{13/12}}{|\mathbf{w}|^{3/2}}\right)(\log X)^c \\ &\ll \{V^{4/3}\cdot V^{2/3}H^{-1}+V^{13/12}(V^{2/3}H^{-1})^{3/2}\}(\log X)^c \\ &\ll \frac{V^2}{H}(\log X)^c,\end{aligned}$$

for $H\geqslant d_0$. We may now sum up over the available ranges $(H,2H]$ with $H\geqslant d_0$ to get a bound $O(VXY^{-15}(\log X)^c)$, so that

$$S_6 \ll S_8+VXY^{-15}(\log X)^c,$$

where

$$S_8 = \sum_{D\in I_m}\sum_{d}\left|\sum_{\substack{\hat{\beta}_i\in C_i\\ Dd|\mathbf{v}}} F_{\beta_1}F_{\beta_2}\right|, \tag{12.10}$$

with $d$ restricted by the inequality $dD\leqslant d_0$.

On combining this with (12.8) we finally deduce as follows.

LEMMA 12.2. *There is an absolute constant $c>0$ such that*

$$S_V \ll X^2Y^{-1/2}(\log X)^c+X^{3/2}V^{-1/2}Y^7S_8^{1/2}(\log X)^c,$$

*where $S_8$ is given by* (12.10).

As we shall see in the next section $S_8$ is ready for a large sieve estimation.

## 13. Proof of Lemma 3.10—a large sieve estimate

In this section we shall express the condition $dD|\mathbf{v}$ in $S_8$ by means of additive characters, and use a 3-dimensional large sieve to complete the proof of Lemma 3.10.

We have already remarked in §11 that the condition $D|\mathbf{v}$ may be rewritten as $\hat{\beta}_2 \equiv \lambda\hat{\beta}_1 \pmod{D}$ for some integer $\lambda$, which is necessarily coprime to $D$. We apply this remark with $D$ replaced by $dD$. If we introduce the exponential sum

$$S(\mathbf{a}) = S(\mathbf{a}, \mathcal{C}) = \sum_{\hat{\beta}\in\mathcal{C}} \exp\{2\pi i\mathbf{a}\cdot\hat{\beta}\}G_{\hat{\beta}}, \tag{13.1}$$

where $G_{\hat{\beta}} = F_\beta$, we find that

$$\sum_{\substack{\hat{\beta}_i\in\mathcal{C}_i\\ Dd|\mathbf{v}}} F_{\beta_1}F_{\beta_2} = (Dd)^{-3} \sum_{\substack{\lambda \,(\mathrm{mod}\, dD)\\ (\lambda,dD)=1}} \sum_{\mathbf{a}\,(\mathrm{mod}\, dD)} S((dD)^{-1}\lambda\mathbf{a}, \mathcal{C}_1)\,\overline{S((dD)^{-1}\mathbf{a}, \mathcal{C}_2)}.$$

It follows that

$$|S_8| \leqslant \sum_{D,d,\lambda,\mathbf{a}} (Dd)^{-3}|S((dD)^{-1}\lambda\mathbf{a}, \mathcal{C}_1)\,S((dD)^{-1}\mathbf{a}, \mathcal{C}_2)|,$$

where $D$ runs over $I_m$, $d$ runs over positive integers $d \leqslant d_0/D$, $\lambda$ runs over positive integers less than and coprime to $dD$, and $\mathbf{a}$ runs modulo $dD$. Cauchy's inequality then yields

$$|S_8| \leqslant \sum_{D\in I_m} \sum_{d} (Dd)^{-2} \sum_{\mathbf{a}\,(\mathrm{mod}\, dD)} |S((dD)^{-1}\mathbf{a}, \mathcal{C})|^2,$$

for $\mathcal{C} = \mathcal{C}_1$ or $\mathcal{C}_2$. Here we have used the observation that

$$\sum_{\mathbf{a}\,(\mathrm{mod}\, dD)} |S((dD)^{-1}\lambda\mathbf{a}, \mathcal{C})|^2 = \sum_{\mathbf{a}\,(\mathrm{mod}\, dD)} |S((dD)^{-1}\mathbf{a}, \mathcal{C})|^2$$

whenever $\lambda$ is coprime to $dD$. Finally we reduce the fractions $(dD)^{-1}\mathbf{a}$ to lowest terms. A given vector $q^{-1}\mathbf{b}$ with h.c.f.$(q, b_1, b_2, b_3) = 1$ will occur with weight at most

$$\sum_{D\geqslant VX^{-1}Y^{-1}} \sum_{d:q|dD} (Dd)^{-2} \ll \sum_{\substack{v\geqslant VX^{-1}Y^{-1}\\ q|v}} \tau(v)v^{-2} \ll \frac{\tau(q)}{q} XY \frac{\log V}{V}.$$

Moreover, only values $q \leqslant d_0$ will arise. We therefore conclude that

$$S_8 \ll XY\frac{\log V}{V} \sum_{q\leqslant d_0} \frac{\tau(q)}{q} \sum_{\mathbf{b}\,(\mathrm{mod}\, q)}^{*} |S(q^{-1}\mathbf{b})|^2, \tag{13.2}$$

where $\sum^*$ denotes summation for h.c.f.$(q, b_1, b_2, b_3) = 1$.

Our principal tool in handling the above sum will be the following large sieve bound.

Lemma 13.1. *Let $S(\mathbf{a})$ be given by* (13.1), *with $\mathcal{C}$ a cube of side $S_0$. Then*

$$\sum_{Q<q\leqslant 2Q} \sum_{\mathbf{b}\,(\mathrm{mod}\, q)}^{*} |S(q^{-1}\mathbf{b})|^2 \ll (S_0^3+Q^2S_0^2+Q^4)\sum_{\hat{\beta}\in\mathcal{C}} |G_{\hat{\beta}}|^2.$$

Various multi-dimensional forms of the large sieve appear in the literature, but none seem quite suited to our purpose. In particular, the estimate of Huxley [13] would have the factor $S_0^3+Q^6$ when applied to our situation, and this is too large.

To prove Lemma 13.1 we observe at the outset that

$$|S(\mathbf{a},\mathcal{C})| = \left| \sum_{\hat{\beta}\in\mathcal{C}-\mathbf{x}} \exp\{2\pi i\mathbf{a}\cdot\hat{\beta}\} G_{\hat{\beta}+\mathbf{x}} \right|,$$

for any translation $\mathcal{C}-\mathbf{x}$ of the cube $\mathcal{C}$ by an integral vector $\mathbf{x}$. It therefore suffices to prove Lemma 13.1 for

$$\mathcal{C} = \{\mathbf{b} : 0 \leqslant b_i \leqslant S_0\}.$$

We now start our estimations with the Sobolev–Gallagher inequality, which states that

$$|f(0)| \leqslant (2\delta)^{-1}\int_{-\delta}^{\delta} |f(t)|\,dt + 2^{-1}\int_{-\delta}^{\delta} |f'(t)|\,dt.$$

By iterating this we deduce that

$$|f(\mathbf{0})| \leqslant \frac{1}{8}\sum_{\mathcal{I}\subseteq\{1,2,3\}} \delta^{\#\mathcal{I}-3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f^{(\mathcal{I})}(\mathbf{t})|\,dt_1\,dt_2\,dt_3,$$

where $f^{(\mathcal{I})}$ denotes the partial derivative, of order $\#\mathcal{I}$, with respect to each variable $t_i$ for $i\in\mathcal{I}$. An application of Cauchy's inequality now produces

$$|f(\mathbf{0})|^2 \leqslant \sum_{\mathcal{I}\subseteq\{1,2,3\}} \delta^{2\#\mathcal{I}-3}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f^{(\mathcal{I})}(\mathbf{t})|^2\,dt_1\,dt_2\,dt_3.$$

We employ this, with

$$f(\mathbf{t}) = S(q^{-1}\mathbf{b}+\mathbf{t})$$

and $\delta = S_0^{-1}$. It then follows that

$$\sum_{Q<q\leqslant 2Q} \sum_{\mathbf{b}\,(\mathrm{mod}\, q)}^{*} |S(q^{-1}\mathbf{b})|^2 \leqslant \sum_{\mathcal{I}\subseteq\{1,2,3\}} S_0^{3-2\#\mathcal{I}}\int_{-1}^{2}\int_{-1}^{2}\int_{-1}^{2} \#\nu(\mathbf{t})|S^{(\mathcal{I})}(\mathbf{t})|^2\,dt_1\,dt_2\,dt_3,$$

where

$$\nu(\mathbf{t}) = \{(q,\mathbf{b}) : |q^{-1}b_i - t_i| \leqslant S_0^{-1},\ \text{for } 1\leqslant i\leqslant 3\}.$$

We set

$$\bar{\nu} = \sup \#\nu(\mathbf{t}),$$

and we note that Parseval's identity gives

$$\int_0^1\int_0^1\int_0^1 |S^{(\mathcal{I})}(\mathbf{t})|^2\,dt_1\,dt_2\,dt_3 \ll S_0^{2\#\mathcal{I}} \sum_{\hat{\beta}\in C} |G_{\hat{\beta}}|^2.$$

It follows that

$$\sum_{Q<q\leqslant 2Q}\ \sideset{}{^*}\sum_{\mathbf{b}\,(\mathrm{mod}\,q)} |S(q^{-1}\mathbf{b})|^2 \ll S_0^3\bar{\nu}\sum_{\hat{\beta}\in C}|G_{\hat{\beta}}|^2.$$

It therefore remains to show that

$$\#\nu(\mathbf{t}) \ll 1+Q^2S_0^{-1}+Q^4S_0^{-3} \tag{13.3}$$

uniformly in $\mathbf{t}$.

Since we may clearly assume that $\nu(\mathbf{t})\neq\varnothing$, we may begin by fixing a particular point $(r,\mathbf{a})\in\nu(\mathbf{t})$. We classify the remaining elements into three types $\nu_j$, not necessarily disjoint, where $q^{-1}b_j\neq r^{-1}a_j$ for each point $(q,\mathbf{b})\in\nu_j$. Since the vectors $q^{-1}\mathbf{b}$ with h.c.f.$(q,b_1,b_2,b_3)=1$ are necessarily distinct, this classification does indeed cover all $(q,\mathbf{b})\in\nu(\mathbf{t})$ apart from $(r,\mathbf{a})$ itself.

We now proceed to examine the contribution from elements of $\nu_1$, the other cases being similar. For a point $(q,\mathbf{b})$ of $\nu_1$ we have

$$0\neq\left|\frac{a_1}{r}-\frac{b_1}{q}\right|\leqslant\frac{2}{S_0}.$$

Thus if $qa_1-rb_1=s$, say, then $0\neq|s|\leqslant 8Q^2/S_0$. Moreover, if h.c.f.$(a_1,r)=d$, then $d|s$. It follows that there are at most $16Q^2/S_0d$ possible values for $s$. (In particular, if $Q^2/S_0d$ is small enough, there are no available integers $s$.) Moreover, once $s$ is also fixed, the congruence

$$q\frac{a_1}{d}\equiv\frac{s}{d}\quad\left(\mathrm{mod}\ \frac{r}{d}\right)$$

determines $q$ modulo $r/d$, since $a_1/d$ and $r/d$ are coprime. It follows that there are at most $O(d)$ values for $q$. Once $a_1$, $r$, $s$ and $q$ are fixed, the value of $b_1$ is also determined. We therefore find that there can be at most $O(Q^2S_0^{-1})$ pairs $b_1,q$. Finally, we see that $b_2$ satisfies

$$\left|b_2-\frac{q}{r}a_2\right|\leqslant\frac{2q}{S_0}\leqslant\frac{4Q}{S_0},$$

so that there are $O(1+QS_0^{-1})$ possible values, and similarly for $b_3$. We therefore conclude that

$$\#\nu(\mathbf{t}) \ll 1+\frac{Q^2}{S_0}\left(1+\frac{Q}{S_0}\right)^2,$$

as required for (13.3). This completes the proof of Lemma 13.1.

We shall now use Lemma 13.1 to handle the contribution to (13.2) arising from terms with $q>Q_0$, say. We cover the range $Q_0<q\leqslant d_0$ with intervals $Q<q\leqslant 2Q$, where $Q$ runs over powers of 2. Since

$$\sum_{\hat\beta\in C}|F_\beta|^2 \ll (\log X)^2\sum_{\hat\beta\in C}\tau(\beta)^2 \ll V(\log X)^c,$$

by Lemma 4.5, we conclude that the range $Q<q\leqslant 2Q$ contributes

$$\ll XY\frac{\log V}{V}Q^{-1}\exp\left\{c\frac{\log Q}{\log\log Q}\right\}(V+Q^2V^{2/3}+Q^4)V(\log X)^c$$

to $S_8$. (This is one point where it suffices to use the simple upper bound for the divisor function.) We sum this over the relevant values of $Q$ to get a total

$$\ll XY\left(VQ_0^{-1}\exp\left\{c\frac{\log Q_0}{\log\log Q_0}\right\}+(V^{2/3}d_0+d_0^3)\exp\left\{c\frac{\log d_0}{\log\log d_0}\right\}\right)(\log X)^c.$$

To handle the exponentials we note that

$$\exp\left\{c\frac{\log Q_0}{\log\log Q_0}\right\} \ll Q_0^{1/2}$$

and

$$\exp\left\{c\frac{\log d_0}{\log\log d_0}\right\} \ll \exp\left\{c\frac{\log X}{\log\log X}\right\} \ll X^{\tau/6},$$

by (12.9) and (2.5). The range $Q_0<q\leqslant d_0$ therefore contributes a total

$$\ll XY(VQ_0^{-1/2}+\{V^{2/3}d_0+d_0^3\}X^{\tau/6})(\log X)^c$$

to $S_8$. In view of (12.9) and (11.1) we see that this is

$$\ll XV(YQ_0^{-1/2}+Y^{46}X^{-\tau/2})(\log X)^c. \tag{13.4}$$

There remains the range $q\leqslant Q_0$, where we shall use the hypothesis (3.14). We begin by observing that

$$S(q^{-1}\mathbf{b}) = \sum_{\mathbf{c}\ (\mathrm{mod}\ q)}\exp\left\{2\pi i\frac{\mathbf{b}\cdot\mathbf{c}}{q}\right\}\sum_{\substack{\hat\beta\in C\\ \hat\beta\equiv\mathbf{c}\ (\mathrm{mod}\ q)}}F_\beta. \tag{13.5}$$

In the terminology of §12, the hypercube $\mathcal{C}_1\times\mathcal{C}_2$ corresponding to the estimate (13.2) for $S_8$ is of Class I. Thus (11.3) holds for any $\hat{\beta}\in\mathcal{C}_i$. Referring to the definition of $F_\beta$ in §11, we therefore see that for $\hat{\beta}\in\mathcal{C}$ in (13.5) we have $F_\beta=f_{(\beta)}$ for primitive $\beta$, and $F_\beta=0$ otherwise. It therefore follows that

$$\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \hat{\beta}\equiv\mathbf{c}\ (\mathrm{mod}\ q)}} F_\beta=\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \hat{\beta}\equiv\mathbf{c}\ (\mathrm{mod}\ q)}} f_{(\beta)}\sum_{d\mid\hat{\beta}}\mu(d)=\sum_{d:(q,d)\mid\mathbf{c}}\mu(d)\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \hat{\beta}\equiv\mathbf{c}\ (\mathrm{mod}\ q)\\ d\mid\hat{\beta}}} f_{(\beta)},$$

whence

$$S(q^{-1}\mathbf{b})\ll\sum_{\mathbf{c}\ (\mathrm{mod}\ q)}\sum_{d}\Bigg|\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ \hat{\beta}\equiv\mathbf{c}\ (\mathrm{mod}\ q)\\ d\mid\hat{\beta}}} f_{(\beta)}\Bigg|. \tag{13.6}$$

The conditions $\hat{\beta}\equiv\mathbf{c}$ (mod $q$) and $d\mid\hat{\beta}$ confine $\hat{\beta}$ to a single residue class modulo $[q,d]$. The inner sum above is therefore $O\big(V\exp\{-c\sqrt{\log L}\,\}\big)$, by the hypothesis of Lemma 3.10, providing that $[q,d]\leqslant Q_1$. Consequently, if $q\leqslant Q_1^{1/2}$, then the contribution to (13.6) corresponding to values $d\leqslant Q_1^{1/2}$ is

$$\ll q^3Q_1^{1/2}V\exp\{-c\sqrt{\log L}\,\}\ll Q_1^2V\exp\{-c\sqrt{\log L}\,\}.$$

For the remaining values of $d$ we use the trivial bound $f_{(\beta)}\ll\tau(\beta)\log X$. This produces a contribution

$$\ll(\log X)\sum_{d>Q_1^{1/2}}\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ d\mid\hat{\beta}}}\tau(\beta)\sum_{\mathbf{c}\equiv\hat{\beta}\ (\mathrm{mod}\ q)}1=(\log X)\sum_{d>Q_1^{1/2}}\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ d\mid\hat{\beta}}}\tau(\beta)$$

to (13.6). According to Lemma 4.5, however, we have

$$\sum_{\substack{\hat{\beta}\in\mathcal{C}\\ d\mid\hat{\beta}}}\tau(\beta)\ll\tau(d)^c\sum_{|\mathbf{x}|\ll V^{1/3}/d}\tau\big(x_1+x_2\sqrt[3]{2}+x_3\sqrt[3]{4}\big)\ll\tau(d)^cVd^{-3}(\log X)^c.$$

It follows that values $d>Q_1^{1/2}$ contribute

$$\ll V(\log X)^c\sum_{d>Q_1^{1/2}}\tau(d)^cd^{-3}\ll VQ_1^{-1}(\log X)^c$$

to (13.6).

We now have a bound

$$S(q^{-1}\mathbf{b})\ll Q_1^2V\exp\{-c\sqrt{\log L}\,\}+VQ_1^{-1}(\log X)^c.$$

The terms with $q \leqslant Q_0$ therefore contribute

$$\ll XYV(Q_1^4 \exp\{-c\sqrt{\log L}\}+Q_1^{-2})(\log X)^c$$

to (13.2), providing that $Q_0 \leqslant Q_1^{1/2}$. Taken in conjunction with our estimate (13.4) for the terms with $Q_0 < q \leqslant d_0$ we therefore deduce that

$$S_8 \ll XV\big(YQ_1^4 \exp\{-c\sqrt{\log L}\}+YQ_1^{-2}+YQ_0^{-1/2}+Y^{46}X^{-\tau/2}\big)(\log X)^c.$$

We now choose

$$Q_0 = Q_1^{1/2},$$

so that

$$S_8 \ll XV\big(YQ_1^4 \exp\{-c\sqrt{\log L}\}+YQ_1^{-1/4}+Y^{46}X^{-\tau/2}\big)(\log X)^c,$$

whence Lemma 12.2 yields

$$S_V \ll X^2\big(Y^{-1/2}+Y^{30}X^{-\tau/4}+Y^8Q_1^{-1/8}+Y^8Q_1^2 \exp\{-c\sqrt{\log L}\}\big)(\log X)^c.$$

We now choose

$$Y = Q_1^{1/80}, \tag{13.7}$$

which is in accordance with the condition (11.8), since

$$Q_1 \leqslant \exp\{\sqrt[3]{\log X}\}.$$

By virtue of this bound for $Q_1$ we finally see that our estimate reduces to

$$S_V \ll X^2 Q_1^{-1/160}(\log X)^c,$$

as required for Lemma 3.10.

*Acknowledgement.* Part of this paper was written while the author was a visitor at the Isaac Newton Institute, Cambridge. The hospitality and financial support of the institute is gratefully acknowledged.

## References

[1] DANIEL, S., On the divisor-sum problem for binary forms. *J. Reine Angew. Math.*, 507 (1999), 107–129.

[2] FRIEDLANDER, J. & IWANIEC, H., The polynomial $X^2+Y^4$ captures its primes. *Ann. of Math.*, 148 (1998), 945–1040.

[3] — Asymptotic sieve for primes. *Ann. of Math.*, 148 (1998), 1041–1065.

[4] GOLDSTON, D. A., On Bombieri and Davenport's theorem concerning small gaps between primes. *Mathematika*, 39 (1992), 10–17.

[5] GREAVES, G. R. H., Large prime factors of binary forms. *J. Number Theory*, 3 (1971), 35–59.

[6] HALBERSTAM, H. & RICHERT, H.-E., *Sieve Methods*. London Math. Soc. Monographs, 4. Academic Press, London–New York, 1974.

[7] HARDY, G. H. & LITTLEWOOD, J. E., Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes. *Acta Math.*, 44 (1923), 1–70.

[8] HEATH-BROWN, D. R., Diophantine approximation with square-free numbers. *Math. Z.*, 187 (1984), 335–344.

[9] — The ternary Goldbach problem. *Rev. Mat. Iberoamericana*, 1 (1985), 45–59.

[10] — The number of primes in a short interval. *J. Reine Angew. Math.*, 389 (1988), 22–63.

[11] — The solubility of diagonal cubic Diophantine equations. *Proc. London Math. Soc. (3)*, 79 (1999), 241–259.

[12] HOOLEY, C., On a problem of Hardy and Littlewood. *Acta Arith.*, 79 (1997), 289–311.

[13] HUXLEY, M. N., The large sieve inequality for algebraic number fields. *Mathematika*, 15 (1968), 178–187.

[14] IWANIEC, H., Primes represented by quadratic polynomials in two variables. *Acta Arith.*, 24 (1974), 435–459.

[15] MITSUI, T., Generalized prime number theorem. *Japan J. Math.*, 26 (1956), 1–42.

D. R. HEATH-BROWN
Mathematical Institute
University of Oxford
24–29 St. Giles'
Oxford OX1 3LB
England, U.K.
rhb@maths.ox.ac.uk

*Received April 29, 1999*