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The Schrödinger operator on the energy space: boundedness and compactness criteria

by

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1. Introduction

We characterize the class of measurable functions (or, more generally, real- or complex-valued distributions) V such that the Schrödinger operator $H=-\Delta+V$ maps the energy space $\mathring{L}_2^1(\mathbf{R}^n)$ to its dual $L_2^{-1}(\mathbf{R}^n)$. Similar results are obtained for the inhomogeneous Sobolev space $W_2^1(\mathbf{R}^n)$. In other words, we give a complete solution to the problem of the relative form-boundedness of the potential energy operator V with respect to the Laplacian $-\Delta$, which is fundamental to quantum mechanics. Relative compactness criteria for the corresponding quadratic forms are established as well. We also give analogous boundedness and compactness criteria for Sobolev spaces on domains $\Omega\subset\mathbf{R}^n$ under mild restrictions on $\partial\Omega$.

One of the main goals of the present paper is to give necessary and sufficient conditions for the classical inequality

$$\left|\int_{\mathbf{R}^n}|u(x)|^2\,V(x)\,dx\right|\leqslant \text{const}\int_{\mathbf{R}^n}|\nabla u(x)|^2\,dx,\quad u\in C_0^\infty(\mathbf{R}^n), \tag{1.1}$$

to hold. Here the "indefinite weight" V may change sign, or even be a complex-valued distribution on $\mathbf{R}^n$, $n\geqslant 3$. (In the latter case, the left-hand side of (1.1) is understood as $|\langle Vu,u\rangle|$, where $\langle V\cdot\,,\cdot\,\rangle$ is the quadratic form associated with the corresponding multiplication operator V.) We also characterize an analogous inequality for the inhomogeneous Sobolev space $W_2^1(\mathbf{R}^n)$, $n\geqslant 1$:

$$\left|\int_{\mathbf{R}^n}|u(x)|^2\,V(x)\,dx\right|\leqslant \text{const}\int_{\mathbf{R}^n}[|\nabla u(x)|^2+|u(x)|^2]\,dx,\quad u\in C_0^\infty(\mathbf{R}^n). \tag{1.2}$$

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Such inequalities are used extensively in spectral and scattering theory of the Schrödinger operator $H=H_0+V$, where $H_0=-\Delta$ is the Laplacian on $\mathbf{R}^n$, and its higher-order analogues, especially in questions of self-adjointness, resolvent convergence, estimates for the number of bound states, Schrödinger semigroups, etc. (See [Bi], [BiS1], [BiS2], [CZ], [D1], [Fa], [Fe], [RS2], [S1], [Si], and the literature cited there.) In particular, (1.2) is equivalent to the fundamental concept of the relative boundedness of V (potential energy operator) with respect to $H_0=-\Delta$ in the sense of quadratic forms. Its abstract version appears in the so-called KLMN theorem, which is discussed in detail, together with applications to quantum-mechanical Hamiltonian operators, in [RS2, Section X.2].

It follows from the polarization identity that (1.1) can be restated equivalently in terms of the corresponding sesquilinear form:

$$|\langle Vu, v\rangle| \leqslant \mathrm{const}\cdot \|\nabla u\|_{L_2} \|\nabla v\|_{L_2}$$

for all $u, v \in C_0^\infty(\mathbf{R}^n)$. In other words, it is equivalent to the boundedness of the operator $H=H_0+V$,

$$H: \mathring{L}_2^1(\mathbf{R}^n) \to L_2^{-1}(\mathbf{R}^n), \quad n \geqslant 3. \tag{1.3}$$

Here the energy space $\mathring{L}_2^1(\mathbf{R}^n)$ is defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the Dirichlet norm $\|\nabla u\|_{L_2}$, and $L_2^{-1}(\mathbf{R}^n)$ is the dual of $\mathring{L}_2^1(\mathbf{R}^n)$. Similarly, (1.2) means that H is a bounded operator which maps $W_2^1(\mathbf{R}^n)$ to $W_2^{-1}(\mathbf{R}^n)$, $n\geqslant 1$.

The idea of considering H as a bounded operator acting from the energy space to its dual goes back at least to E. Nelson's way to prove that densely defined closed quadratic forms bounded from below on a Hilbert space $\mathcal{H}$ are uniquely associated with a self-adjoint operator on $\mathcal{H}$ [Ne, pp. 98–101] (see also [RS1, pp. 278–279 and Notes to Section VIII.6]). Moreover, Nelson also used this technique to prove the existence of the Friedrichs extension for densely defined, symmetric operators bounded from below ([Ne, pp. 101–102], [RS2, pp. 177–179 and Notes to Section X.2]). A proof of the KLMN theorem using this approach (i.e., scales of Hilbert spaces) can be found, for instance, in [RS2, pp. 167–168].

Thus, from the point of view of perturbation theory, we distinguish a natural class of admissible potentials V such that the mapping properties of $H_0=-\Delta$ are preserved for $H=H_0+V$. It is well-known that, in the opposite situation where H_0 is dominated by V, the properties of the perturbed operator may change in a spectacular way. For instance, under the growth conditions on $V\geqslant 0$ at infinity prescribed by the classical A. Molchanov's criterion [Mo], H has a purely discrete spectrum. (Another proof of the discreteness-of-spectrum criterion was found in [Ma2]; see also [EE], [Ma3]. Generalizations to Schrödinger operators on manifolds and magnetic Schrödinger operators are given in [KoS], [KMS].)

Previously, the case of *nonnegative* V in (1.1) and (1.2) has been studied in a comprehensive way. We refer to [CWW], [Fe], [KeS], [Ma3], [MaV], [RS2], [S3], where different analytic conditions for the so-called trace inequalities of this type can be found. (A recent survey of the vast literature on this subject is given in [Ve].) For general V, only sufficient conditions have been known.

It is worthwhile to observe that the usual "naïve" approach is to decompose V into its positive and negative parts, $V=V_+-V_-$, and to apply the just mentioned results to both V_+ and V_-. However, this procedure drastically diminishes the class of admissible weights V by ignoring a possible cancellation between V_+ and V_-. This cancellation phenomenon is evident for strongly oscillating weights considered below. Examples of this type are known, mostly in relation to quantum mechanics problems [AiS], [CG], [NaS], [Stu].

In §2, we establish a general principle which enables us to solve the problems stated above for arbitrary V. Before stating our main results, we reiterate that we do not impose any a priori assumptions on V, and hence throughout the introduction the left-hand sides of (1.1) and other similar inequalities are defined in terms of the corresponding quadratic forms. Also, we use some expressions involving pseudodifferential operators, e.g. $\nabla\Delta^{-1}V$ or $(-\Delta)^{-1/2}V$, which will be carefully defined in the main body of the paper.

THEOREM I. *Let V be a complex-valued distribution on $\mathbf{R}^n$, $n\geqslant 3$. Then* (1.1) *holds if and only if V is the divergence of a vector field $\vec{\Gamma}\colon \mathbf{R}^n\to\mathbf{C}^n$ such that*

$$\int_{\mathbf{R}^n}|u(x)|^2|\vec{\Gamma}(x)|^2\,dx\leqslant \text{const}\int_{\mathbf{R}^n}|\nabla u(x)|^2\,dx, \tag{1.4}$$

where the constant is independent of $u\in C_0^\infty(\mathbf{R}^n)$. The vector field $\vec{\Gamma}\in\mathbf{L}_{2,\text{loc}}(\mathbf{R}^n)$ can be chosen as $\vec{\Gamma}=\nabla\Delta^{-1}V$.

Equivalently, the Schrödinger operator $H=H_0+V$ acting from $\mathring{L}_2^1(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$ is bounded if and only if (1.4) *holds. Furthermore, the corresponding multiplication operator $V\colon\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n)$ is compact if and only if the embedding*

$$\mathring{L}_2^1(\mathbf{R}^n)\subset L_2(\mathbf{R}^n,|\vec{\Gamma}|^2\,dx)$$

is compact.

We remark that once V is written as $V=\operatorname{div}\vec{\Gamma}$, the implication (1.4) $\Rightarrow$ (1.1) becomes trivial: It follows using integration by parts and the Schwarz inequality. This idea has been known for a long time in mathematical physics (see, e.g., [CG]) and in the theory of Sobolev spaces [MaS].

On the other hand, the converse statement (1.1) $\Rightarrow$ (1.4), where $\vec{\Gamma}=\nabla\Delta^{-1}V$, is quite striking, and its proof is rather delicate. It is based on a special factorization of functions in $\overset{\circ}{L}{}^1_2(\mathbf{R}^n)$ involving powers P_K^δ of the equilibrium potential P_K associated with an arbitrary compact set $K\subset\mathbf{R}^n$ of positive capacity. New sharp estimates for P_K^δ, where ultimately δ is picked so that $1<2\delta<n/(n-2)$, are established in a series of lemmas and propositions in §2. We also make use of the fact that standard Calderòn–Zygmund operators are bounded on $L_2(\mathbf{R}^n)$ with a weight P_K^δ, and the corresponding operator norm bounds do not depend on K [MaV].

Thus, Theorem I makes it possible to reduce the problems of boundedness and compactness for general "indefinite" V to the case of nonnegative weights $|\vec{\Gamma}|^2$, which is by now well understood. In particular, combining Theorem I and the known criteria in the case $V\geqslant 0$ (see Theorems 2.1 and 4.1 below) we arrive at the following theorem.

THEOREM II. *Under the assumptions of Theorem* I, *let* $\vec{\Gamma}=\nabla\Delta^{-1}V\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$. *Then the following statements are equivalent*:

(a) *Inequality* (1.1) *holds.*

(b) *For every compact set* $e\subset\mathbf{R}^n$,

$$\int_e |\vec{\Gamma}(x)|^2\,dx \leqslant \mathrm{const}\cdot\mathrm{cap}(e),$$

where $\mathrm{cap}(e)$ *is the Wiener capacity of* e, *and the constant does not depend on* e.

(c) *The function* $g(x)=(-\Delta)^{-1/2}|\vec{\Gamma}(x)|^2$ *is finite a.e., and*

$$(-\Delta)^{-1/2}g^2(x)\leqslant \mathrm{const}\cdot g(x) \quad a.e.$$

(d) *For every dyadic cube* P_0 *in* $\mathbf{R}^n$,

$$\sum_{P\subseteq P_0}\left[\frac{\int_P|\vec{\Gamma}(x)|^2\,dx}{|P|^{1-1/n}}\right]^2|P|\leqslant \mathrm{const}\int_{P_0}|\vec{\Gamma}(x)|^2\,dx,$$

where the sum is taken over all dyadic cubes P *contained in* P_0, *and the constant does not depend on* P_0.

As a corollary, we obtain a necessary condition for (1.1) in terms of Morrey spaces of negative order.

COROLLARY 1. *If* (1.1) *holds, then for every ball* $B_r(x_0)$ *of radius* r,

$$\int_{B_r(x_0)}|\nabla\Delta^{-1}V(x)|^2\,dx\leqslant \mathrm{const}\cdot r^{n-2},$$

where the constant does not depend on $x_0\in\mathbf{R}^n$ *and* $r>0$.

COROLLARY 2. *In the statements of Theorem* I, *Theorem* II *and Corollary* 1, *one can put the scalar function* $(-\Delta)^{-1/2}V$ *in place of* $\vec{\Gamma}=\nabla\Delta^{-1}V$. *In particular,* (1.4) *is equivalent to the inequality*

$$\int_{\mathbf{R}^n} |u(x)|^2 \, |(-\Delta)^{-1/2} V(x)|^2 \, dx \leqslant \text{const} \int_{\mathbf{R}^n} |\nabla u(x)|^2 \, dx \tag{1.5}$$

for all $u\in C_0^\infty(\mathbf{R}^n)$.

The proof of Corollary 2 uses the boundedness of standard singular integral operators in the space of functions $f\in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} |u(x)|^2 \, |f(x)|^2 \, dx \leqslant \text{const} \int_{\mathbf{R}^n} |\nabla u(x)|^2 \, dx$$

for all $u\in C_0^\infty(\mathbf{R}^n)$; this fact was established earlier in [MaV].

Corollary 2 indicates that an appropriate decomposition into a positive and negative part for (1.1) should involve expressions like $(-\Delta)^{-1/2}V$ rather than V itself. Another important consequence is that the class of weights V satisfying (1.1) is invariant under standard singular integral and maximal operators.

Remark 1. Similar results are valid for inequality (1.2); one only has to replace the operator $(-\Delta)^{-1/2}$ by $(1-\Delta)^{-1/2}$, and the Wiener capacity $\operatorname{cap}(e)$ with the corresponding Bessel capacity. In statement (d) of Theorem II and Corollary 1, it suffices to restrict oneself to cubes or balls whose volumes are less than 1 (see details in §4).

Before proceeding to further results and corollaries of Theorem I and Theorem II, it is instructive to demonstrate the cancellation phenomenon mentioned above by considering an example of a strongly oscillating weight.

Example 1. Let us set

$$V(x) = |x|^{N-2} \sin(|x|^N), \tag{1.6}$$

where $N\geqslant 3$ is an integer, which may be arbitrarily large. Obviously, both V_+ and V_- fail to satisfy (1.1) due to the growth of the amplitude at infinity. However,

$$V(x) = \operatorname{div} \vec{\Gamma}(x) + O(|x|^{-2}), \quad \text{where } \vec{\Gamma}(x) = \frac{-1}{N} \frac{\vec{x}}{|x|^2} \cos(|x|^N). \tag{1.7}$$

By Hardy's inequality in $\mathbf{R}^n$, $n\geqslant 3$ (see, e.g., [D2]),

$$\int_{\mathbf{R}^n} |u(x)|^2 \frac{dx}{|x|^2} \leqslant \frac{4}{(n-2)^2} \int_{\mathbf{R}^n} |\nabla u(x)|^2 \, dx, \quad u\in C_0^\infty(\mathbf{R}^n), \tag{1.8}$$

and hence the term $O(|x|^{-2})$ in (1.7) is harmless, whereas $\vec{\Gamma}$ clearly satisfies (1.4) since $|\vec{\Gamma}(x)|^2 \leqslant |x|^{-2}$. This shows that V is admissible for (1.1), while $|V|$ is obviously not. Similar examples of weights with strong *local* singularities can easily be constructed.

We now discuss some related results in terms of more conventional classes of admissible weights V. The following corollary, which is an immediate consequence of Theorem I and Corollary 2, gives a simpler sufficient condition for (1.1) in terms of Lorentz–Sobolev spaces of negative order.

COROLLARY 3. *Suppose that* $n \geqslant 3$ *and that* V *is a distribution on* $\mathbf{R}^n$ *such that* $(-\Delta)^{-1/2} V \in L_{n,\infty}(\mathbf{R}^n)$, *where* $L_{p,\infty}$ *denotes the usual Lorentz (weak* L_p*) space. Then* (1.1) *holds.*

For the definition and basic properties of Lorentz spaces $L_{p,q}(\mathbf{R}^n)$ we refer to [StW]. In particular, it follows that $(-\Delta)^{-1/2} V \in L_{n,\infty}$ is equivalent to the estimate

$$\int_e |(-\Delta)^{-1/2} V(x)|^2 \, dx \leqslant \text{const} \cdot |e|^{1-2/n}, \tag{1.9}$$

where $|e|$ is the Lebesgue measure of a measurable set $e \subset \mathbf{R}^n$.

Remark 2. Using duality and the Sobolev embedding theorem for $L_{p,1}(\mathbf{R}^n)$-spaces one can show that the class of potentials V such that $(-\Delta)^{-1/2} V \in L_{n,\infty}(\mathbf{R}^n)$ is wider than the well-known class $V \in L_{n/2,\infty}(\mathbf{R}^n)$.

Remark 3. Corollary 3 demonstrates that $(-\Delta)^{-1/2} V \in L_{n,\infty}(\mathbf{R}^n)$, $n \geqslant 3$, is sufficient for V to be relatively form-bounded with respect to $-\Delta$. For $n \geqslant 5$, this condition is enough for V to be even $(-\Delta)$-bounded, according to the terminology of Reed and Simon; see [RS2, pp. 162–172].

A sharper version of Corollary 3 can be stated in terms of Morrey spaces of negative order. We recall that a measurable function W lies in the Fefferman–Phong class, introduced in [Fe], if for every ball $B_r(x_0)$ of radius r in $\mathbf{R}^n$, the inequality

$$\int_{B_r(x_0)} |W(x)|^p \, dx \leqslant \text{const} \cdot r^{n-2p} \tag{1.10}$$

holds for some $p > 1$, where the constant does not depend on x_0 and r.

It is easy to see that (1.10) holds for every $1 < p < \frac{1}{2} n$ if $W \in L_{n/2,\infty}(\mathbf{R}^n)$. As was shown in [Fe], (1.10) with $p > 1$ is sufficient for W to be relatively form-bounded with respect to $-\Delta$.

The following corollary of Theorem I is applicable to distributions V, and encompasses a class of weights which is broader than the Fefferman–Phong class even in the case where V is a nonnegative measurable function.

COROLLARY 4. *Let V be a distribution on $\mathbf{R}^n$ which satisfies, for some $p>1$, the inequality*

$$\int_{B_r(x_0)} |(-\Delta)^{-1/2}V(x)|^{2p}\,dx \leqslant \mathrm{const}\cdot r^{n-2p} \tag{1.11}$$

for every ball $B_r(x_0)$ in $\mathbf{R}^n$. Then (1.1) *holds.*

Note that by Corollary 1 the preceding inequality with $p=1$ is necessary in order that (1.1) hold.

Remark 4. A refinement of (1.11) in terms of the Dini-type conditions established by Chang, Wilson and Wolff [CWW] is readily available by combining them with our Theorem I.

To clarify the multi-dimensional characterizations for "indefinite weights" V presented above, we state an elementary analogue of Theorem I for the Sturm–Liouville operator $H=-d^2/dx^2+V$ on the half-line.

THEOREM III. *The inequality*

$$\left|\int_{\mathbf{R}_+} |u(x)|^2 V(x)\,dx\right| \leqslant \mathrm{const}\int_{\mathbf{R}_+} |u'(x)|^2\,dx \tag{1.12}$$

holds for all $u\in C_0^\infty(\mathbf{R}_+)$ if and only if

$$\sup_{a>0} a\int_a^\infty \left|\int_x^\infty V(t)\,dt\right|^2 dx < \infty, \tag{1.13}$$

where $\Gamma(x)=\int_x^\infty V(t)\,dt$ is understood in terms of distributions.

Equivalently, $H\colon \mathring{L}_2^1(\mathbf{R}_+)\to L_2^{-1}(\mathbf{R}_+)$ is bounded if and only if (1.13) *holds. Moreover, the corresponding multiplication operator V is compact if and only if*

$$a\int_a^\infty |\Gamma(x)|^2\,dx = o(1), \quad \textit{where } a\to 0^+ \textit{ and } a\to+\infty. \tag{1.14}$$

For nonnegative V, condition (1.13) is easily seen to be equivalent to the standard Hille condition [Hi]:

$$\sup_{a>0} a\int_a^\infty |V(x)|\,dx < \infty. \tag{1.15}$$

A similar statement is true for the compactness criterion (1.14).

The gap between (1.13) and (1.15) is evident from the following example which is of interest to spectral and scattering theory.

Example 2. Let $V(x)=\sin(x)/x^p$, $p>0$, where $x\geqslant 1$, and $V(x)=0$ for $0<x<1$. Then the operator $H=-d^2/dx^2+V\colon \mathring{L}_2^1(\mathbf{R}_+)\to L_2^{-1}(\mathbf{R}_+)$ is bounded if and only if $p\geqslant 1$. Moreover, by (1.14), V is compact for $p>1$. However, (1.15) is applicable only when $p>2$.

We observe that Theorem III, in spite of its simplicity, seems to be new for experts in spectral theory. Its proof will be given elsewhere in a more general framework.

We now briefly outline the contents of the paper. In §2, we define the Schrödinger operator on the energy space $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$, and characterize the basic inequality (1.1). The compactness problem is treated in §3. Analogous results for the Sobolev space $W_2^1(\mathbf{R}^n)$ are obtained in §4, while §5 is devoted to similar problems on a domain $\Omega\subset\mathbf{R}^n$ for a broad class of Ω, including those with Lipschitz boundaries.

In this paper, we restrict ourselves to the Hilbert case $p=2$, and the second-order operator $H_0=-\Delta$. However, our boundedness and compactness criteria can be carried over to Sobolev spaces $\overset{\circ}{L}{}_p^m(\mathbf{R}^n)$ and $W_p^m(\mathbf{R}^n)$, where $1<p<\infty$ and $m>0$, and higher-order operators like $H=(-\Delta)^m+V$. The proofs of the necessity statements for $p\neq 2$ and $m\neq 1$ are technically more complicated, and will be presented separately. The corresponding L_p-inequalities have applications to certain nonlinear problems (see, e.g., [HMV], [KaV]).

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2. The Schrödinger operator on $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$

We start with some prerequisites for our main results. Let $\mathcal{D}(\mathbf{R}^n)=C_0^\infty(\mathbf{R}^n)$ be the class of all infinitely differentiable, compactly supported complex-valued functions, and let $\mathcal{D}'(\mathbf{R}^n)$ denote the corresponding space of (complex-valued) distributions. In this section, we assume that $n\geqslant 3$, since for the homogeneous space $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ our results become vacuous if $n=1$ and $n=2$: they hold only for Schrödinger operators with zero potential. (Analogous results for inhomogeneous Sobolev spaces $W_2^1(\mathbf{R}^n)$ are valid for all $n\geqslant 1$; see §4 and §5 below.)

For $V\in\mathcal{D}'(\mathbf{R}^n)$, consider the multiplication operator on $\mathcal{D}(\mathbf{R}^n)$ defined by

$$\langle Vu, v\rangle := \langle V, \bar{u}v\rangle, \quad u, v\in\mathcal{D}(\mathbf{R}^n), \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ represents the usual pairing between $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$.

The space $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ is defined as the completion of $\mathcal{D}(\mathbf{R}^n)$ in the Dirichlet norm $\|\nabla u\|_{L_2(\mathbf{R}^n)}$. Elements of $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$, for $n\geqslant 3$, are weakly differentiable functions $u\in L_{2n/(n-2)}(\mathbf{R}^n)$ whose first-order weak derivatives lie in $L_2(\mathbf{R}^n)$. By Hardy's inequality, an equivalent norm on $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ is given by

$$\|u\|_{\overset{\circ}{L}{}_2^1(\mathbf{R}^n)} = \left[\int_{\mathbf{R}^n} (|x|^{-2}|u(x)|^2 + |\nabla u(x)|^2)\,dx\right]^{1/2}.$$

If the sesquilinear form $\langle V\cdot,\cdot\rangle$ is bounded on $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\times\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$:

$$|\langle Vu,v\rangle| \leqslant c\|\nabla u\|_{L_2(\mathbf{R}^n)}\|\nabla v\|_{L_2(\mathbf{R}^n)}, \quad u,v\in\mathcal{D}(\mathbf{R}^n), \tag{2.2}$$

where the constant c is independent of u,v, then $Vu\in L_2^{-1}(\mathbf{R}^n)$, and the multiplication operator can be extended by continuity to all of the energy space $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. As usual, this extension is also denoted by V.

We denote the class of multipliers V such that the corresponding operator from $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$ is bounded by

$$M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n)).$$

Note that the least constant c in (2.2) is equal to the multiplier norm:

$$\|V\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} = \sup\{\|Vu\|_{L_2^{-1}(\mathbf{R}^n)} : \|u\|_{\overset{\circ}{L}{}_2^1(\mathbf{R}^n)}\leqslant 1,\, u\in\mathcal{D}(\mathbf{R}^n)\}.$$

For $V\in M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$, we will extend the form $\langle V,\bar u v\rangle$ defined by the right-hand side of (2.1) to the case where both u and v are in $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. This can be done by letting

$$\langle Vu,v\rangle := \lim_{N\to\infty}\langle Vu_N, v_N\rangle,$$

where $u=\lim_{N\to\infty}u_N$ and $v=\lim_{N\to\infty}v_N$ in $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$, with $u_N,v_N\in\mathcal{D}(\mathbf{R}^n)$. It is known that this extension is independent of the choice of u_N and v_N.

We now define the Schrödinger operator $H=H_0+V$, where $H_0=-\Delta$, on the energy space $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. Since $H_0\colon\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n)$ is bounded, it follows that H is a bounded operator acting from $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$ if and only if $V\in M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$. By the polarization identity, (2.2) is equivalent to the boundedness of the corresponding quadratic form:

$$|\langle Vu,u\rangle| = |\langle V,|u|^2\rangle| \leqslant c\|\nabla u\|^2_{L_2(\mathbf{R}^n)}, \quad u\in\mathcal{D}(\mathbf{R}^n), \tag{2.2$'$}$$

where the constant c is independent of u. If V is a (complex-valued) Borel measure on $\mathbf{R}^n$, then (2.2′) can be recast in the form (see the Introduction)

$$\left|\int_{\mathbf{R}^n}|u(x)|^2\,dV(x)\right| \leqslant c\|\nabla u\|^2_{L_2(\mathbf{R}^n)}, \quad u\in\mathcal{D}(\mathbf{R}^n). \tag{2.3}$$

For *positive* distributions (measures) V, this inequality is well studied. We collect several equivalent characterizations of (2.3) for this case in Theorem 2.1 below.

For a *compact* set $e\subset\mathbf{R}^n$, define the Wiener capacity by

$$\operatorname{cap}(e)=\inf\{\|\nabla u\|^2_{L_2(\mathbf{R}^n)}: u\in\mathcal{D}(\mathbf{R}^n),\ u(x)\geqslant 1 \text{ on } e\}. \tag{2.4}$$

Let V be a positive Borel measure on $\mathbf{R}^n$. By $I_1V=(-\Delta)^{-1/2}V$, we denote the Riesz potential of order 1:

$$I_1V(x)=c(n)\int_{\mathbf{R}^n}\frac{dV(t)}{|x-t|^{n-1}},$$

where $c(n)=\Gamma\big(\tfrac12(n-1)\big)/2\pi^{(n+1)/2}$. More generally, the Riesz potential of order $\alpha\in(0,n)$ is defined by

$$I_\alpha V(x)=c(n,\alpha)\int_{\mathbf{R}^n}\frac{dV(t)}{|x-t|^{n-\alpha}},$$

where $c(n,\alpha)=\Gamma\big(\tfrac12(n-\alpha)\big)/2^\alpha\pi^{n/2}\Gamma(\alpha)$. In particular, for $\alpha=2$ we get the Newtonian potential $I_2=(-\Delta)^{-1}$.

THEOREM 2.1. *Let V be a locally finite positive measure on $\mathbf{R}^n$. Then the following statements are equivalent*:

(i) *The trace inequality*

$$\int_{\mathbf{R}^n}|u(x)|^2\,dV(x)\leqslant c_1\|\nabla u\|^2_{L_2(\mathbf{R}^n)},\quad u\in\mathcal{D}(\mathbf{R}^n), \tag{2.5}$$

holds, where c_1 does not depend on u.

(ii) *For every compact set $e\subset\mathbf{R}^n$,*

$$V(e)\leqslant c_2\operatorname{cap}(e), \tag{2.6}$$

where c_2 does not depend on e.

(iii) *For every ball B in $\mathbf{R}^n$,*

$$\int_B(I_1V_B)^2\,dx\leqslant c_3V(B), \tag{2.7}$$

where $dV_B=\chi_B\,dV$, and c_3 does not depend on B.

(iv) *The pointwise inequality*

$$I_1(I_1V)^2(x)\leqslant c_4I_1V(x)<\infty\quad a.e. \tag{2.8}$$

holds, where c_4 does not depend on $x\in\mathbf{R}^n$.

(v) *For every compact set* $e\subset\mathbf{R}^n$,

$$\int_e (I_1V)^2\,dx \leqslant c_5^2\,\mathrm{cap}(e), \tag{2.9}$$

where c_5 *does not depend on* e.

(vi) *For every dyadic cube* P_0 *in* $\mathbf{R}^n$,

$$\sum_{P\subseteq P_0}\left[\frac{V(P)}{|P|^{1-1/n}}\right]^2|P| \leqslant c_6V(P_0), \tag{2.10}$$

where the sum is taken over all dyadic cubes P *contained in* P_0, *and* c_6 *does not depend on* P_0.

The equivalence (i)$\Leftrightarrow$(ii) is due to Maz'ya [Ma1], and (i)$\Leftrightarrow$(iii) to Kerman and Sawyer [KeS]; (i)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) was obtained in [MaV]; (i)$\Leftrightarrow$(vi) is discussed in [Ve], where a survey of trace inequalities of this type in L_p-spaces is given.

Remark 1. The least constants in the inequalities (2.5)–(2.10) are equivalent in the sense that the quotients c_i/c_j $(i,j=1,...,6)$ are bounded from above and below by positive constants which may depend only on n. Moreover,

$$c_2\leqslant c_1\leqslant 4c_2,$$

where both the lower and the upper estimates are sharp (see [Ma1], [Ma3]).

We now state our main result for arbitrary (complex-valued) distributions V. By $\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)=L_{2,\mathrm{loc}}(\mathbf{R}^n)\otimes\mathbf{C}^n$ we denote the space of vector functions $\vec{\Gamma}=(\Gamma_1,...,\Gamma_n)$ such that $\Gamma_i\in L_{2,\mathrm{loc}}(\mathbf{R}^n)$, $i=1,...,n$.

THEOREM 2.2. *Let* $V\in\mathcal{D}'(\mathbf{R}^n)$. *Then* $V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$, *i.e., the inequality*

$$|\langle Vu,v\rangle|\leqslant c\|u\|_{\mathring{L}_2^1(\mathbf{R}^n)}\|v\|_{\mathring{L}_2^1(\mathbf{R}^n)} \tag{2.11}$$

holds for all $u,v\in\mathcal{D}(\mathbf{R}^n)$, *if and only if there is a vector field* $\vec{\Gamma}\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$ *such that* $V=\operatorname{div}\vec{\Gamma}$ *and*

$$\int_{\mathbf{R}^n}|u(x)|^2|\vec{\Gamma}(x)|^2\,dx\leqslant C\int_{\mathbf{R}^n}|\nabla u(x)|^2\,dx \tag{2.12}$$

for all $u\in\mathcal{D}(\mathbf{R}^n)$. *The vector field* $\vec{\Gamma}$ *can be chosen in the form* $\vec{\Gamma}=\nabla\Delta^{-1}V$.

Remark 2. For $\vec{\Gamma}=\nabla\Delta^{-1}V$, the least constant C in the inequality (2.12) is equivalent to $\|V\|^2_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}$.

Proof of Theorem 2.2. Suppose that $V=\operatorname{div}\vec{\Gamma}$, where $\vec{\Gamma}$ satisfies (2.12). Then using integration by parts and the Schwarz inequality we obtain:

$$\begin{aligned}|\langle Vu, v\rangle| &= |\langle V, \bar{u}v\rangle| = |\langle \vec{\Gamma}, v\nabla\bar{u}\rangle + \langle \vec{\Gamma}, \bar{u}\nabla v\rangle| \\ &\leqslant \|\vec{\Gamma}v\|_{\mathbf{L}_2(\mathbf{R}^n)}\|\nabla\bar{u}\|_{L_2(\mathbf{R}^n)} + \|\vec{\Gamma}u\|_{\mathbf{L}_2(\mathbf{R}^n)}\|\nabla v\|_{L_2(\mathbf{R}^n)} \\ &\leqslant 2\sqrt{C}\,\|\nabla u\|_{L_2(\mathbf{R}^n)}\|\nabla v\|_{L_2(\mathbf{R}^n)},\end{aligned}$$

where C is the constant in (2.12). This completes the proof of the "if" part of Theorem 2.2.

The proof of the "only if" part of Theorem 2.2 is based on several lemmas and propositions.

In the next lemma, we show that $\vec{\Gamma}=\nabla\Delta^{-1}V\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$, and give a crude preliminary estimate of the rate of its decay at ∞. Denote by $B_R=B_R(x_0)$ a Euclidean ball of radius R centered at $x_0\in\mathbf{R}^n$.

LEMMA 2.3. *Suppose that*

$$V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n)). \tag{2.13}$$

Then $\vec{\Gamma}=\nabla\Delta^{-1}V\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$ *and* $V=\operatorname{div}\vec{\Gamma}$ *in* $\mathcal{D}'$. *Moreover, for any ball* $B_R(x_0)$ $(R>0)$ *and* $\varepsilon>0$,

$$\int_{B_R(x_0)}|\vec{\Gamma}(x)|^2\,dx\leqslant C(n,\varepsilon)R^{n-2+\varepsilon}\|V\|^2_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}, \tag{2.14}$$

where $R\geqslant\max\{1,|x_0|\}$.

Proof of Lemma 2.3. Suppose that $V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$. Define the vector field $\vec{\Gamma}\in\mathcal{D}'$ by

$$\langle\vec{\Gamma},\vec{\phi}\rangle=-\langle V,\Delta^{-1}\operatorname{div}\vec{\phi}\rangle \tag{2.15}$$

for every $\vec{\phi}\in\mathcal{D}\otimes\mathbf{C}^n$. In particular,

$$\langle\vec{\Gamma},\nabla\psi\rangle=-\langle V,\psi\rangle,\quad \psi\in\mathcal{D}, \tag{2.15$'$}$$

i.e., $V=\operatorname{div}\vec{\Gamma}$ in $\mathcal{D}'$.

We first have to check that the right-hand side of (2.15) is well-defined, which a priori is not obvious. For $\vec{\phi}\in\mathcal{D}\otimes\mathbf{C}^n$, let $w=\Delta^{-1}\operatorname{div}\vec{\phi}$, where $-\Delta^{-1}f=I_2f$ is the Newtonian potential of $f\in\mathcal{D}$. Clearly,

$$w(x)=O(|x|^{1-n})\quad\text{and}\quad|\nabla w(x)|=O(|x|^{-n})\quad\text{as }|x|\to\infty,$$

and hence

$$w = \Delta^{-1} \operatorname{div} \vec{\phi} \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n).$$

We will show below that $w = uv$, where u is real-valued, and both u and v are in $\overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$. Then, since $V \in M(\overset{\circ}{L}{}^1_2(\mathbf{R}^n) \to L^{-1}_2(\mathbf{R}^n))$, it follows that $\langle V, w\rangle = \langle Vu, v\rangle$ is defined through the extension of the multiplication operator V as explained above.

For our purposes, it is important to note that this extension of $\langle V, w\rangle$ to the case where $w = \bar{u}v$, and $u, v \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$, is independent of the choice of factors u and v. To demonstrate this, we define a real-valued cut-off function $\eta_N(x) = \eta(N^{-1}|x|)$, where $\eta \in C^\infty(\mathbf{R}_+)$, so that $\eta(t) = 1$ for $0 \leqslant t \leqslant 1$ and $\eta(t) = 0$ for $t \geqslant 2$. Note that $\nabla \eta_N$ is supported in the annulus $N \leqslant |x| \leqslant 2N$, and $|\nabla \eta_N(x)| \leqslant c|x|^{-1}$. It follows easily (for instance, from Hardy's inequality) that

$$\lim_{N\to\infty} \|\eta_N u - u\|_{\overset{\circ}{L}{}^1_2(\mathbf{R}^n)} = 0, \quad u \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n).$$

Then letting $u_N = \eta_N u$ and $v_N = \eta_N v$, so that $\bar{u}_N v_N = \eta_N^2 w$, we define $\langle V, w\rangle$ explicitly by setting

$$\langle V, w\rangle := \lim_{N\to\infty} \langle V u_N, v_N\rangle = \lim_{N\to\infty} \langle V, \eta_N^2 w\rangle.$$

This definition is independent of the choice of η, and the factors u, v. Moreover,

$$|\langle V, w\rangle| \leqslant C \inf\bigl\{ \|u\|_{\overset{\circ}{L}{}^1_2(\mathbf{R}^n)} \|v\|_{\overset{\circ}{L}{}^1_2(\mathbf{R}^n)} : w = \bar{u}v;\ u, v \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n) \bigr\},$$

where $C = \|V\|_{M(\overset{\circ}{L}{}^1_2(\mathbf{R}^n) \to L^{-1}_2(\mathbf{R}^n))}$.

Now we fix $\varepsilon > 0$ and factorize: $w(x) = \Delta^{-1} \operatorname{div} \vec{\phi}(x) = u(x)v(x)$, where

$$u(x) = (1+|x|^2)^{-(n-2+\varepsilon)/4} \quad \text{and} \quad v(x) = (1+|x|^2)^{(n-2+\varepsilon)/4} \Delta^{-1} \operatorname{div} \vec{\phi}(x). \tag{2.16}$$

Obviously, $u \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ and

$$\|u\|_{\overset{\circ}{L}{}^1_2(\mathbf{R}^n)} = c(n, \varepsilon) < \infty.$$

It is easy to see that $v \in \overset{\circ}{L}{}^1_2(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ as well. Furthermore, the following statement holds.

PROPOSITION 2.4. *Suppose that $\vec{\phi} \in C^\infty(\mathbf{R}^n)$ and $\operatorname{supp} \vec{\phi} \subset B_R(x_0)$. Let v be defined by* (2.16) *where $0 < \varepsilon < 2$. Then*

$$\|v\|_{\overset{\circ}{L}{}^1_2(\mathbf{R}^n)} \leqslant c(n, \varepsilon) R^{(n-2+\varepsilon)/2} \|\vec{\phi}\|_{L_2(\mathbf{R}^n)} \tag{2.17}$$

for $R \geqslant \max\{1, |x_0|\}$.

Proof of Proposition 2.4. Since $\vec{\phi}$ is compactly supported, it follows that

$$|\Delta^{-1} \operatorname{div} \vec{\phi}(x)| \leqslant c(n) I_1 |\vec{\phi}|(x), \quad x \in \mathbf{R}^n.$$

Hence

$$c(n,\varepsilon)\|v\|_{\mathring{L}_2^1(\mathbf{R}^n)} \leqslant \|(1+|x|^2)^{(n-2+\varepsilon)/4}\nabla\Delta^{-1}\operatorname{div}\vec{\phi}(x)\|_{L_2(\mathbf{R}^n)} + \|(1+|x|^2)^{(n-4+\varepsilon)/4} I_1|\vec{\phi}|(x)\|_{L_2(\mathbf{R}^n)}.$$

Note that $\nabla\Delta^{-1}\operatorname{div}$ is a Calderon–Zygmund operator, and that the weight $w(x)=(1+|x|^2)^{(n-2+\varepsilon)/2}$ belongs to the Muckenhoupt class $A_2(\mathbf{R}^n)$ if $0<\varepsilon<2$ (see [CF]). Applying the corresponding weighted norm inequality, we have

$$\begin{aligned}\|(1+|x|^2)^{(n-2+\varepsilon)/4}\nabla\Delta^{-1}\operatorname{div}\vec{\phi}(x)\|_{L_2(\mathbf{R}^n)}& \\ \leqslant c(n,\varepsilon)\|(1+|x|^2)^{(n-2+\varepsilon)/4}|\vec{\phi}(x)|\,\|_{L_2(\mathbf{R}^n)}&.\end{aligned} \tag{2.18}$$

The other term is estimated by the weighted Hardy inequality (see, e.g., [Ma3]):

$$\int_{\mathbf{R}^n}(I_1|\vec{\phi}|(x))^2(1+|x|^2)^{(n-4+\varepsilon)/2}\,dx \leqslant c(n,\varepsilon)\int_{\mathbf{R}^n}|\vec{\phi}(x)|^2(1+|x|^2)^{(n-2+\varepsilon)/2}\,dx. \tag{2.19}$$

Clearly,

$$\|(1+|x|^2)^{(n-2+\varepsilon)/4}\vec{\phi}(x)\|_{L_2(\mathbf{R}^n)} \leqslant c(n,\varepsilon)R^{(n-2+\varepsilon)/2}\|\vec{\phi}\|_{L_2(\mathbf{R}^n)}.$$

Hence, combining (2.18), (2.19) and the preceding estimate, we obtain the desired inequality (2.17). The proof of Proposition 2.4 is complete.

Now let us prove (2.14). Suppose that $\vec{\phi}\in C^\infty(\mathbf{R}^n)\otimes\mathbf{C}^n$ and $\operatorname{supp}\vec{\phi}\subset B_R(x_0)$. Then by (2.15) and Proposition 2.4,

$$\begin{aligned}|\langle\vec{\Gamma},\vec{\phi}\rangle| = |\langle V,uv\rangle| &\leqslant \|V\|_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\|u\|_{\mathring{L}_2^1(\mathbf{R}^n)}\|v\|_{\mathring{L}_2^1(\mathbf{R}^n)} \\ &\leqslant C(n,\varepsilon)R^{(n-2+\varepsilon)/2}\|V\|_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\|\vec{\phi}\|_{L_2(\mathbf{R}^n)}.\end{aligned} \tag{2.20}$$

Taking the supremum over all ϕ supported in $B_R(x_0)$ with unit L_2-norm, we arrive at (2.14). The proof of Lemma 2.3 is complete.

It remains to prove the main estimate (2.12) of Theorem 2.2. By Theorem 2.1, it suffices to establish the inequality

$$\int_e|\vec{\Gamma}(x)|^2\,dx \leqslant c(n)\|V\|^2_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\operatorname{cap}(e) \tag{2.21}$$

for every compact set $e\subset\mathbf{R}^n$. Notice that in the special case $e=\overline{B_R(x_0)}$, the preceding estimate gives a sharper version of (2.14):

$$\int_{B_R(x_0)}|\vec{\Gamma}(x)|^2\,dx \leqslant C(n)R^{n-2}\|V\|^2_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))},\quad x_0\in\mathbf{R}^n,\ R>0.$$

Without loss of generality we assume that $\operatorname{cap}(e)>0$; otherwise $|e|=0$, and (2.21) holds. Denote by $P(x)=P_e(x)$ the *equilibrium potential* on e (see [AdH], [Ma3]). It is well known that P is the Newtonian potential of a positive measure which gives a solution to several variational problems. This measure ν_e is called the *equilibrium measure* for e.

We list some standard properties of ν_e and its potential $P_e(x)=I_2\nu_e(x)$ which will be used below (essentially due to O. Frostman):

$$\begin{aligned}&\text{(a)}\ \ \operatorname{supp}\nu_e\subset e;\\&\text{(b)}\ \ P_e(x)=1\ \ d\nu_e\text{-a.e.};\\&\text{(c)}\ \ \nu_e(e)=\operatorname{cap}(e)>0;\\&\text{(d)}\ \ \|\nabla P_e\|^2_{L_2(\mathbf{R}^n)}=\operatorname{cap}(e);\\&\text{(e)}\ \ \sup_{x\in\mathbf{R}^n}P_e(x)\leqslant 1.\end{aligned}\tag{2.22}$$

The rest of the proof of Theorem 2.2 is based on some inequalities involving the powers $P_e(x)^\delta$ which are established below.

PROPOSITION 2.5. *Let $\delta>\frac{1}{2}$ and let $P=P_e$ be the equilibrium potential of a compact set e of positive capacity. Then*

$$\|\nabla P^\delta\|_{L_2(\mathbf{R}^n)}=\frac{\delta}{\sqrt{2\delta-1}}\sqrt{\operatorname{cap}(e)}\,.\tag{2.23}$$

Remark 3. For $\delta\leqslant\frac{1}{2}$, it is easy to see that $\nabla P^\delta\notin L_2(\mathbf{R}^n)$.

Proof of Proposition 2.5. Clearly,

$$\int_{\mathbf{R}^n}|\nabla P(x)^\delta|^2\,dx=\delta^2\int_{\mathbf{R}^n}|\nabla P(x)|^2P(x)^{2\delta-2}\,dx.\tag{2.24}$$

Using integration by parts, together with the properties $-\Delta P=\nu_e$ (understood in the distributional sense) and $P(x)=1$ $d\nu_e$-a.e., we have

$$\begin{aligned}\int_{\mathbf{R}^n}|\nabla P(x)|^2P(x)^{2\delta-2}\,dx&=\int_{\mathbf{R}^n}\nabla P(x)\cdot\nabla P(x)\,P(x)^{2\delta-2}\,dx\\&=\int_{\mathbf{R}^n}P(x)^{2\delta-1}\,d\nu_e-(2\delta-2)\int_{\mathbf{R}^n}|\nabla P(x)|^2P(x)^{2\delta-2}\,dx\\&=\operatorname{cap}(e)-(2\delta-2)\int_{\mathbf{R}^n}|\nabla P(x)|^2P(x)^{2\delta-2}\,dx.\end{aligned}$$

The integration by parts above is easily justified for $\delta>\frac{1}{2}$ by examining the behavior of the potential and its gradient at infinity:

$$c_1|x|^{2-n}\leqslant P(x)\leqslant c_2|x|^{2-n},\quad |\nabla P(x)|=O(|x|^{1-n})\ \text{ as } |x|\to\infty.\tag{2.25}$$

From these calculations it follows that

$$(2\delta-1)\int_{\mathbf{R}^n}|\nabla P(x)|^2 P(x)^{2\delta-2}\,dx=\operatorname{cap}(e).$$

Combining this with (2.24) yields (2.23). The proof of Proposition 2.5 is complete.

In the next lemma we demonstrate that $\|\nabla v\|_{L_2(\mathbf{R}^n)}$ is equivalent to the weighted norm $\|P^{-\delta}\nabla(vP^\delta)\|_{L_2(\mathbf{R}^n)}$.

LEMMA 2.6. *Let $\delta>0$ and let $v\in \mathring{L}^1_2(\mathbf{R}^n)$. Then*

$$\|\nabla v\|^2_{L_2(\mathbf{R}^n)}\leqslant \int_{\mathbf{R}^n}|\nabla(vP^\delta)(x)|^2\,\frac{dx}{P(x)^{2\delta}}\leqslant(\delta+1)(4\delta+1)\|\nabla v\|^2_{L_2(\mathbf{R}^n)}. \tag{2.26}$$

In what follows only the lower estimate will be used, together with the fact that $\|P^{-\delta}\nabla(vP^\delta)\|_{L_2(\mathbf{R}^n)}<\infty$ for every $v\in\mathring{L}^1_2(\mathbf{R}^n)$.

Proof of Lemma 2.6. Without loss of generality we may assume that v is real-valued. We first prove (2.26) for $v\in\mathcal{D}(\mathbf{R}^n)$. The general case will follow using an approximation argument. Clearly,

$$\begin{aligned}\int_{\mathbf{R}^n}|\nabla(vP^\delta)(x)|^2\,\frac{dx}{P^{2\delta}(x)}&=\int_{\mathbf{R}^n}|\nabla v(x)+\delta v(x)\nabla P(x)P(x)^{-1}|^2\,dx\\&=\int_{\mathbf{R}^n}|\nabla v(x)|^2\,dx+\delta^2\int_{\mathbf{R}^n}v(x)^2\,\frac{|\nabla P(x)|^2}{P(x)^2}\,dx\\&\quad+2\delta\int_{\mathbf{R}^n}\nabla v\cdot\nabla P(x)\,\frac{v(x)}{P(x)}\,dx.\end{aligned}$$

Integration by parts and the equation $-\Delta P=\nu_e$ (understood in the distributional sense) give

$$2\int_{\mathbf{R}^n}\nabla v\cdot\nabla P(x)\,\frac{v(x)}{P(x)}\,dx=\int_{\mathbf{R}^n}v(x)^2\,\frac{d\nu_e(x)}{P(x)}\,dx+\int_{\mathbf{R}^n}v(x)^2\,\frac{|\nabla P(x)|^2}{P(x)^2}\,dx.$$

Using this identity, we rewrite the preceding equation in the form

$$\begin{aligned}\int_{\mathbf{R}^n}|\nabla(vP^\delta)(x)|^2\,\frac{dx}{P^{2\delta}(x)}&=\int_{\mathbf{R}^n}|\nabla v(x)|^2\,dx+\delta(\delta+1)\int_{\mathbf{R}^n}v(x)^2\,\frac{|\nabla P(x)|^2}{P(x)^2}\,dx\\&\quad+\delta\int_{\mathbf{R}^n}v(x)^2\,\frac{d\nu_e(x)}{P(x)}.\end{aligned}\tag{2.27}$$

The lower estimate in (2.26) is now obvious provided the last two terms on the right-hand side of the preceding equation are finite. They are estimated in the following proposition, which holds for Newtonian potentials of arbitrary (not necessarily equilibrium) positive measures.

PROPOSITION 2.7. *Let ω be a positive Borel measure on $\mathbf{R}^n$ such that $P(x) = I_2\omega(x) \not\equiv \infty$. Then the following inequalities hold*:

$$\int_{\mathbf{R}^n} v(x)^2 \frac{|\nabla P(x)|^2}{P(x)^2}\, dx \leqslant 4\, \|\nabla v\|^2_{L_2(\mathbf{R}^n)}, \quad v \in \mathcal{D}(\mathbf{R}^n), \tag{2.28}$$

and

$$\int_{\mathbf{R}^n} v(x)^2 \frac{d\omega(x)}{P(x)} \leqslant \|\nabla v\|^2_{L_2(\mathbf{R}^n)}, \quad v \in \mathcal{D}(\mathbf{R}^n). \tag{2.29}$$

Remark 4. The constants 4 and 1 respectively in (2.28) and (2.29) are sharp.

Indeed, if ω is a point mass at $x=0$, it follows that $P(x)=c(n)|x|^{2-n}$. Hence, (2.28) boils down to the classical Hardy inequality (1.8) with the best constant $4/(n-2)^2$. To show that the constant in (2.29) is sharp, it suffices to let $\omega=\nu_e$ for a compact set e of positive capacity, so that $P(x)=1$ $d\omega$-a.e. and $\nu_e(e)=\mathrm{cap}(e)$, and minimize the right-hand side over all $v \geqslant 1$ on e, where $v \in \mathcal{D}(\mathbf{R}^n)$.

Remark 5. An inequality more general than (2.29), for Riesz potentials and L_p-norms (with nonlinear Wolff potential in place of $P(x)$), but with a different constant, is proved in [Ve].

Proof of Proposition 2.7. Suppose $v \in \mathcal{D}(\mathbf{R}^n)$. Then $A = \operatorname{supp} v$ is a compact set, and obviously $\inf_{x\in A} P(x) > 0$. Without loss of generality we may assume that $\nabla P \in \mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$, and hence the left-hand side of (2.28) is finite. (Otherwise we replace ω by its convolution with a compactly supported mollifier $\omega_t = \omega * \varepsilon_t$, and complete the proof by applying the estimates given below to $P(x) = I_2\omega_t(x)$, and then passing to the limit as $t \to \infty$.)

Using integration by parts together with the equation $-\Delta P = \omega$ as above, and applying the Schwarz inequality, we get

$$\begin{aligned}
\int_{\mathbf{R}^n} v(x)^2 \frac{|\nabla P(x)|^2}{P(x)^2}\, dx + \int_{\mathbf{R}^n} v(x)^2 \frac{d\omega(x)}{P(x)} \\
&= 2 \int_{\mathbf{R}^n} \nabla v(x) \cdot \nabla P(x) \frac{v(x)}{P(x)}\, dx \\
&\leqslant 2 \left(\int_{\mathbf{R}^n} v(x)^2 \frac{|\nabla P(x)|^2}{P(x)^2}\, dx \right)^{1/2} \left(\int_{\mathbf{R}^n} |\nabla v(x)|^2\, dx \right)^{1/2}
\end{aligned}$$

for all $v \in \mathcal{D}(\mathbf{R}^n)$. The preceding inequality obviously yields both (2.28) and (2.29). This completes the proof of Proposition 2.7.

We now complete the proof of Lemma 2.6. Combining (2.27) with (2.28) and (2.29) (with ν_e in place of ω), we arrive at the estimate

$$\|\nabla v\|^2_{L_2(\mathbf{R}^n)} \leqslant \int_{\mathbf{R}^n} |\nabla(vP^\delta)(x)|^2 \frac{dx}{P(x)^{2\delta}} \leqslant (\delta+1)(4\delta+1)\|\nabla v\|^2_{L_2(\mathbf{R}^n)}$$

for all $v\in\mathcal{D}(\mathbf{R}^n)$.

To verify this inequality for arbitrary v in $\mathring{L}^1_2(\mathbf{R}^n)$, let $v=\lim_{N\to\infty} v_N$ both in $\mathring{L}^1_2(\mathbf{R}^n)$ and dx-a.e. for $v_N\in\mathcal{D}(\mathbf{R}^n)$. Now put v_N in place of v in (2.28) and let $N\to\infty$. Using Fatou's lemma we see that (2.28) holds for all $v\in\mathring{L}^1_2(\mathbf{R}^n)$. Hence

$$\lim_{N\to\infty}\int_{\mathbf{R}^n} |v_N(x)-v(x)|^2 \frac{|\nabla P(x)|^2}{P(x)^2}\, dx = 0,$$

and consequently

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathbf{R}^n} |\nabla(v_N P^\delta)(x)|^2 \frac{dx}{P^{2\delta}(x)} &= \lim_{N\to\infty}\int_{\mathbf{R}^n} \left|\nabla v_N(x)+\delta v_N(x)\frac{\nabla P(x)}{P(x)}\right|^2 dx\\ &= \int_{\mathbf{R}^n} \left|\nabla v(x)+\delta v(x)\frac{\nabla P(x)}{P(x)}\right|^2 dx\\ &= \int_{\mathbf{R}^n} |\nabla(vP^\delta)(x)|^2 \frac{dx}{P^{2\delta}(x)}.\end{aligned}$$

Thus, the proof of the general case is completed by putting v_N in place of v in (2.26), and letting $N\to\infty$. The proof of Lemma 2.6 is complete.

In the next proposition, we extend the equation $\langle V, w\rangle = -\langle \Gamma, \nabla w\rangle$ to the case where $w=uv$, where both u and v lie in $\mathring{L}^1_2(\mathbf{R}^n)$, are locally bounded, and have a certain decay at infinity.

PROPOSITION 2.8. *Suppose that $V\in M(\mathring{L}^1_2(\mathbf{R}^n)\to L^{-1}_2(\mathbf{R}^n))$, and that $\vec{\Gamma}=\nabla\Delta^{-1}V\in \mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$ is defined as in Lemma 2.3. Suppose that $w=uv$, where $u,v\in\mathring{L}^1_2(\mathbf{R}^n)$, and*

$$|u(x)|\leqslant C(1+|x|^2)^{-\beta/2}, \quad |v(x)|\leqslant C(1+|x|^2)^{-\beta/2}, \quad x\in\mathbf{R}^n, \tag{2.30}$$

for some $\beta>\frac12(n-2)$. Then $\vec{\Gamma}\cdot\nabla\bar{w}$ is summable, and

$$\langle V, w\rangle = -\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\nabla\bar{w}(x)\, dx. \tag{2.31}$$

Proof of Proposition 2.8. Clearly,

$$\begin{aligned}\int_{\mathbf{R}^n} |\vec{\Gamma}\cdot\nabla\bar{w}(x)|\, dx \leqslant &\left(\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2|u(x)|^2\, dx\right)^{1/2}\left(\int_{\mathbf{R}^n}|\nabla v(x)|^2\, dx\right)^{1/2}\\ &+\left(\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2|v(x)|^2\, dx\right)^{1/2}\left(\int_{\mathbf{R}^n}|\nabla u(x)|^2\, dx\right)^{1/2}.\end{aligned}$$

To show that the right-hand side is finite, note that, for every $\varepsilon>0$ and $R\geqslant 1$,

$$\int_{|x|\leqslant R} |\vec{\Gamma}(x)|^2\,dx \leqslant CR^{n-2+\varepsilon}, \tag{2.32}$$

by Lemma 2.3. It is easy to see that the preceding estimate yields

$$\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2(1+|x|^2)^{-\beta}\,dx < \infty \tag{2.33}$$

for $\beta>\frac{1}{2}(n-2)$. Indeed, pick $\varepsilon\in(0,2\beta-n+2)$, and estimate

$$\begin{aligned}\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2(1+|x|^2)^{-\beta}\,dx &\leqslant \int_{|x|\leqslant 1} |\vec{\Gamma}(x)|^2\,dx + \int_{|x|>1} |\vec{\Gamma}(x)|^2|x|^{-2\beta}\,dx \\ &\leqslant c_1+c_2\int_1^\infty \left(\int_{|x|\leqslant r} |\vec{\Gamma}(x)|^2\,dx\right) r^{-2\beta-1}\,dx \\ &\leqslant c_1+c_2\int_1^\infty r^{n-3-2\beta}\,dx<\infty.\end{aligned}$$

From this and (2.30) it follows that

$$\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2|u(x)|^2\,dx<\infty, \quad \int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2|v(x)|^2\,dx<\infty.$$

Thus $\vec{\Gamma}\cdot\nabla\bar{w}$ is summable.

To prove (2.31), we first assume that both u and v lie in $\mathring{L}_2^1(\mathbf{R}^n)\cap C^\infty(\mathbf{R}^n)$, and satisfy (2.30). Let $\eta_N(x)$ be a smooth cut-off function as in the proof of Lemma 2.3. Let $u_N=\eta_N u$ and $v_N=\eta_N v$. Then by (2.15′),

$$\begin{aligned}\langle V, u_N v_N\rangle &= -\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\nabla(\bar{u}_N\bar{v}_N)(x)\,dx \\ &= -\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\nabla\bar{u}_N(x)\,\bar{v}_N(x)\,dx - \int_{\mathbf{R}^n} \vec{\Gamma}\cdot\nabla\bar{v}_N(x)\,\bar{u}_N(x)\,dx.\end{aligned}$$

Note that $0\leqslant\eta_N(x)\leqslant 1$ and $|\nabla\eta_N(x)|\leqslant C|x|^{-1}$, which gives

$$\begin{aligned}&|\vec{\Gamma}\cdot\nabla\bar{u}_N(x)\,\bar{v}_N(x)|+|\vec{\Gamma}\cdot\nabla\bar{u}_N(x)\,\bar{v}_N(x)| \\ &\qquad\leqslant C|\vec{\Gamma}(x)|(|u(x)|\,|v(x)|\,|x|^{-1}+|\nabla u(x)|\,|v(x)|+|\nabla v(x)|\,|u(x)|).\end{aligned}$$

Since $v\in\mathring{L}_2^1(\mathbf{R}^n)$, it follows from Hardy's inequality (or directly from (2.30)) that $|v(x)|\,|x|^{-1}\in L_2(\mathbf{R}^n)$. Applying (2.33) and the Schwarz inequality, we conclude that the right-hand side of the preceding inequality is summable. Thus (2.31) follows from the dominated convergence theorem in this case.

It remains to show that the C^∞-restriction on u and v can be dropped. We set $u_r = u*\phi_r$, $v_r=v*\phi_r$, where $\phi_r(x)=r^{-n}\phi(x/r)$. Here $\phi\in C_0^\infty(\mathbf{R}^n)$ is a C^∞-mollifier supported in $B(0,1)$ such that $0\leqslant\phi(x)\leqslant 1$. It is not difficult to verify that u_r and v_r satisfy estimates (2.30). We use the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{0<r<\infty} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,dy, \quad x\in\mathbf{R}^n.$$

Obviously, $|u_r(x)|=|u*\phi_r(x)|\leqslant Mu(x)$. We can suppose without loss of generality that $\frac{1}{2}(n-2)<\beta<n$ in (2.30). Notice that, for $0<\beta<n$,

$$M(1+|x|^2)^{-\beta/2}\leqslant C(1+|x|^2)^{-\beta/2}, \quad x\in\mathbf{R}^n.$$

Hence,

$$|u_r(x)|\leqslant Mu(x)\leqslant C(1+|x|^2)^{-\beta/2}, \quad x\in\mathbf{R}^n, \tag{2.34}$$

where C does not depend on r, and a similar estimate holds for v.

We will also need the estimate

$$|\nabla u_r(x)| = |\nabla u*\phi_r(x)|\leqslant M|\nabla u|(x). \tag{2.35}$$

As was shown above,

$$\langle V, u_r v_r\rangle = -\int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{u}_r(x)\,\bar{v}_r(x)\,dx - \int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{v}_r(x)\,\bar{u}_r(x)\,dx.$$

Moreover, by (2.34) and (2.35) we have

$$|\vec{\Gamma}\cdot\nabla\bar{u}_r(x)\,\bar{v}_r(x)| + |\vec{\Gamma}\cdot\nabla\bar{v}_r(x)\,\bar{u}_r(x)|\leqslant C|\vec{\Gamma}(x)|(1+|x|^2)^{-\beta/2}(M|\nabla u|(x)+M|\nabla v|(x)).$$

Since $u, v\in \mathring{L}_2^1(\mathbf{R}^n)$, and M is a bounded operator on $L_2(\mathbf{R}^n)$, it follows that $M|\nabla u|$ and $M|\nabla v|$ lie in $L_2(\mathbf{R}^n)$. Applying (2.33) again, we see that the right-hand side of the preceding inequality is summable. Thus, letting $r\to 0$, and using the dominated convergence theorem, we obtain

$$\langle V, w\rangle = \lim_{r\to 0}\langle V, u_r v_r\rangle = -\int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{w}(x)\,dx,$$

which completes the proof of Proposition 2.8.

We now continue the proof of (2.21). Suppose that $V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$, i.e., the inequality

$$|\langle Vu, v\rangle|\leqslant \|V\|_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\|u\|_{\mathring{L}_2^1(\mathbf{R}^n)}\|v\|_{\mathring{L}_2^1(\mathbf{R}^n)}$$

holds, where $u, v \in \overset{\circ}{L}{}_2^1(\mathbf{R}^n)$.

Let $\vec{\phi}=(\phi_1, ..., \phi_n)$ be an arbitrary vector field in $\mathcal{D}\otimes\mathbf{C}^n$, and let

$$w=\Delta^{-1}\operatorname{div}\vec{\phi}=-I_2\operatorname{div}\vec{\phi}, \tag{2.36}$$

so that

$$\vec{\phi}=\nabla w+\vec{s}, \quad \operatorname{div}\vec{s}=0.$$

Note that $w\in \overset{\circ}{L}{}_2^1(\mathbf{R}^n)\cap C^\infty(\mathbf{R}^n)$, since

$$w(x)=O(|x|^{1-n}) \quad \text{and} \quad |\nabla w(x)|=O(|x|^{-n}) \quad \text{as } |x|\to\infty. \tag{2.37}$$

Now set

$$u(x)=P(x)^\delta \quad \text{and} \quad v(x)=\frac{w(x)}{P(x)^\delta}, \tag{2.38}$$

where $P(x)$ is the equilibrium potential of a compact set $e\subset\mathbf{R}^n$, and $1<2\delta<n/(n-2)$.

By (2.22) and (2.25) we have $0\leqslant P(x)\leqslant 1$ for all $x\in\mathbf{R}^n$, and $P(x)\leqslant c|x|^{2-n}$ for $|x|$ large. Hence $|P(x)|^\delta\leqslant C(1+|x|^2)^{-\delta(n-2)/2}$. Since $\beta=\delta(n-2)>\frac{1}{2}(n-2)$, it follows that u satisfies (2.30).

To verify that (2.30) holds for $v=wP^{-\delta}$, note that $\inf_K P(x)>0$ for every compact set K, and hence by (2.25), $P(x)^{-\delta}\leqslant C(1+|x|^2)^{\delta(n-2)/2}$. Combining this estimate with (2.37) we conclude that

$$|v(x)|\leqslant C(1+|x|^2)^{-\beta/2},$$

where $\beta=-\delta(n-2)+n-1>\frac{1}{2}(n-2)$.

By Proposition 2.5 and Lemma 2.6 both u and v lie in $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. Now applying Proposition 2.8 we obtain

$$\langle Vu, v\rangle=\langle V, w\rangle=-\int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{w}(x)\,dx.$$

Hence,

$$\left|\int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{w}(x)\,dx\right|\leqslant\|V\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\|\nabla u\|_{L_2(\mathbf{R}^n)}\|\nabla v\|_{L_2(\mathbf{R}^n)}.$$

By Lemma 2.6,

$$\|\nabla v\|^2_{L_2(\mathbf{R}^n)}\leqslant\int_{\mathbf{R}^n}|\nabla(vP^\delta)(x)|^2\frac{dx}{P(x)^{2\delta}}=\int_{\mathbf{R}^n}|\nabla w(x)|^2\frac{dx}{P(x)^{2\delta}}<\infty.$$

Applying this together with Proposition 2.5, we estimate

$$\begin{aligned}\left|\int_{\mathbf{R}^n}\vec{\Gamma}\cdot\nabla\bar{w}(x)\,dx\right|&\leqslant C(\delta)\|V\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}\operatorname{cap}(e)^{1/2}\\&\quad\times\left(\int_{\mathbf{R}^n}|\nabla w(x)|^2\frac{dx}{P(x)^{2\delta}}\right)^{1/2}.\end{aligned} \tag{2.39}$$

To complete the proof of Theorem 2.1, we need one more estimate which involves powers of equilibrium potentials.

PROPOSITION 2.9. *Let w be defined by* (2.36) *with $\vec{\phi}\in\mathcal{D}\otimes\mathbf{C}^n$. Suppose that* $1<2\delta<n/(n-2)$. *Then*

$$\int_{\mathbf{R}^n} |\nabla w(x)|^2 \frac{dx}{P(x)^{2\delta}} \leqslant C(n,\delta) \int_{\mathbf{R}^n} |\vec{\phi}(x)|^2 \frac{dx}{P(x)^{2\delta}}. \tag{2.40}$$

Proof of Proposition 2.9. Note that ∇w is related to $\vec{\phi}$ through the Riesz transforms R_j, $j=1,...,n$ ([St1]):

$$\nabla w = \left\{\sum_{k=1}^n R_j R_k \phi_k\right\}, \quad j=1,...,n.$$

Since R_j are bounded operators on $L_2(\mathbf{R}^n,\varrho)$ with a weight ϱ in the Muckenhoupt class $A_2(\mathbf{R}^n)$ ([CF], [St2]), we have

$$\|\nabla w\|_{L_2(\mathbf{R}^n,\varrho)} \leqslant C\|\vec{\phi}\|_{L_2(\mathbf{R}^n,\varrho)},$$

where the constant C depends only on the Muckenhoupt constant of the weight.

Let $\varrho(x)=P(x)^{-2\delta}$. It is easily seen that $\inf_{x\in K} P(x)>0$ for every compact set K, and hence $P(x)^{-2\delta}\in L_{1,\mathrm{loc}}(\mathbf{R}^n)$. In our earlier work, it was proved that $P(x)^{2\delta}$ is an A_2-weight, provided $1<2\delta<n/(n-2)$. Moreover, its Muckenhoupt constant depends only on n and δ, but not on the compact set e. (See [MaV, p. 95, the proof of Lemma 2.1 in the case $p=2$].) Clearly, the same is true for $\varrho(x)=P(x)^{-2\delta}$. This completes the proof of Proposition 2.9.

We are now in a position to complete the proof of Theorem 2.2. Recall that from (2.15′) and Proposition 2.8 it follows that

$$\langle V,w\rangle = -\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\nabla\bar{w}(x)\,dx = -\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\bar{\vec{\phi}}(x)\,dx.$$

Using (2.39) and Proposition 2.9 we obtain

$$\left|\int_{\mathbf{R}^n} \vec{\Gamma}\cdot\bar{\vec{\phi}}(x)\,dx\right| \leqslant C(n,\delta)\|V\|_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \operatorname{cap}(e)^{1/2}\left(\int_{\mathbf{R}^n} \frac{|\vec{\phi}(x)|^2}{P(x)^{2\delta}}\,dx\right)^{1/2}$$

for all $\vec{\phi}\in\mathcal{D}(\mathbf{R}^n)\otimes\mathbf{C}^n$, and hence for all $\vec{\phi}\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$.

Now pick $R>0$ so that $e\subset B(0,R)$. Letting $\vec{\phi}=\chi_{B(0,R)}P^{2\delta}\vec{\Gamma}$ in the preceding inequality, we conclude that

$$\left(\int_{B(0,R)} |\vec{\Gamma}(x)|^2 P(x)^{2\delta}(x)\,dx\right)^{1/2} \leqslant C(n,\delta)\|V\|_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \operatorname{cap}(e)^{1/2}.$$

Since $P(x)\geqslant 1$ dx-a.e. on e (actually $P(x)=1$ on $e\backslash E$, where E is a polar set, i.e., $\operatorname{cap}(E)=0$) it follows that

$$\int_e |\vec{\Gamma}(x)|^2\,dx \leqslant C(n,\delta)^2 \|V\|^2_{M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \operatorname{cap}(e).$$

Thus, (2.21) holds for every compact set $e\subset\mathbf{R}^n$, and by Theorem 2.1 this yields (2.12). The proof of Theorem 2.2 is complete.

We now prove an analogue of Theorem 2.2 formulated in terms of $(-\Delta)^{-1/2}V$, which is stated as Corollary 2 in the Introduction.

THEOREM 2.10. *Under the assumptions of Theorem* 2.2, *it follows that*

$$V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$$

if and only if $(-\Delta)^{-1/2}V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))$.

Proof. By Theorem 2.2, $\nabla\Delta^{-1}V\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$ is well defined in terms of distributions. We now have to show that $(-\Delta)^{-1/2}V$ is well defined as well.

Let $\mathcal{M}$ be the function space which consists of $f\in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} |u(x)|^2|f(x)|^2\,dx \leqslant \mathrm{const}\int_{\mathbf{R}^n}|\nabla u(x)|^2\,dx$$

for every $u\in\mathcal{D}(\mathbf{R}^n)$. By Theorem 2.2, $\nabla\Delta^{-1}V$ lies in $\mathcal{M}\otimes\mathbf{C}^n$. It follows from Corollary 3.2 in [MaV] that the Riesz transforms R_j $(j=1,...,n)$ are bounded operators on $\mathcal{M}$. Hence $(-\Delta)^{-1/2}\nabla=\{R_j\}_{1\leqslant j\leqslant n}$ is a bounded operator from $\mathcal{M}$ to $\mathcal{M}\otimes\mathbf{C}^n$. Then $(-\Delta)^{-1/2}V$ can be defined by

$$(-\Delta)^{-1/2}V=(-\Delta)^{-1/2}\nabla\cdot\nabla\Delta^{-1}V$$

as an element of $\mathcal{M}$. By Theorem 2.1, $(-\Delta)^{-1/2}V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))$. The proof of Theorem 2.10 is complete.

The following corollary is immediate from Theorem 2.10.

COROLLARY 2.11. *Let V be a complex-valued distribution on $\mathbf{R}^n$, $n\geqslant 3$. Then the Schrödinger operator $H=-\Delta+V$, originally defined on $\mathcal{D}(\mathbf{R}^n)$, can be extended to a bounded operator from $\mathring{L}_2^1(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$ if and only if*

$$(-\Delta)^{-1/2}V\in M(\mathring{L}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n)).$$

Equivalently, any one of the conditions (ii)–(vi) *of Theorem* 2.1 *holds with $|(-\Delta)^{-1/2}V|^2$ in place of V.*

3. A compactness criterion

In this section we give a compactness criterion for $V\in M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$. Denote by $\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$ the class of compact multiplication operators acting from $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$. Obviously,

$$\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))\subset M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n)),$$

where the latter class was characterized in the preceding section.

THEOREM 3.1. *Let $V\in\mathcal{D}'(\mathbf{R}^n)$, $n\geqslant 3$. Then $V\in\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))$ if and only if*

$$V=\operatorname{div}\vec{\Gamma}, \tag{3.1}$$

where $\vec{\Gamma}=(\Gamma_1,\ldots,\Gamma_n)$ is a vector field such that $\Gamma_i\in\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))$ $(i=1,\ldots,n)$. Moreover, $\vec{\Gamma}$ can be represented in the form $\nabla\Delta^{-1}V$, as in Theorem 2.2.

Remark 1. The compactness of the multipliers $\Gamma_i:\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n)$, where $i=1,\ldots,n$, is obviously equivalent to the compactness of the embedding

$$\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\subset L_2(\mathbf{R}^n,|\vec{\Gamma}|^2\,dx). \tag{3.2}$$

Different characterizations of the compactness of such embeddings are known (see [AdH], [Ma3], [MaS]).

Proof. Let V be given by (3.1), and let u belong to the unit ball $\mathcal{B}$ in $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. Then

$$Vu=\operatorname{div}(u\vec{\Gamma})-\vec{\Gamma}\cdot\nabla u. \tag{3.3}$$

The set $\{\operatorname{div}(u\vec{\Gamma}):u\in\mathcal{B}\}$ is compact in $L_2^{-1}(\mathbf{R}^n)$ because the set $\{u\vec{\Gamma}:u\in\mathcal{B}\}$ is compact in $\mathbf{L}_2^{-1}(\mathbf{R}^n)$. The set $\{\vec{\Gamma}\cdot\nabla u:u\in\mathcal{B}\}$ is also compact in $L_2^{-1}(\mathbf{R}^n)$ since the set $\{|\nabla u|:u\in\mathcal{B}\}$ is bounded in $L_2(\mathbf{R}^n)$, and the multiplier operators $\bar{\Gamma}_i$, being adjoint to Γ_i $(i=1,\ldots,n)$, are compact from $L_2(\mathbf{R}^n)$ to $L_2^{-1}(\mathbf{R}^n)$. This completes the proof of the sufficiency of (3.2).

We now prove the necessity. Pick $F\in C^\infty(\mathbf{R}_+)$, where $F(t)=1$ for $t\leqslant 1$ and $F(t)=0$ for $t\geqslant 2$. For $x_0\in\mathbf{R}^n$, $\delta>0$ and $R>0$, define the cut-off functions

$$\varkappa_{\delta,x_0}(x)=F(\delta^{-1}|x-x_0|)\quad\text{and}\quad\xi_R(x)=1-F(R^{-1}|x|).$$

LEMMA 3.2. *If* $f \in L_2^{-1}(\mathbf{R}^n)$, *then*

$$\lim_{\delta\to 0} \sup_{x_0\in\mathbf{R}^n} \|\varkappa_{\delta,x_0} f\|_{L_2^{-1}(\mathbf{R}^n)} = 0 \tag{3.4}$$

and

$$\lim_{R\to\infty} \|\xi_R f\|_{L_2^{-1}(\mathbf{R}^n)} = 0. \tag{3.5}$$

Proof of Lemma 3.2. Let us prove (3.4). The distribution f has the form $f = \operatorname{div}\vec{\phi}$, where $\vec{\phi} = (\phi_1, ..., \phi_n) \in \mathbf{L}_2(\mathbf{R}^n)$. Hence,

$$\varkappa_{\delta,x_0} f = \operatorname{div}(\varkappa_{\delta,x_0}\vec{\phi}) - \vec{\phi}\,\nabla\varkappa_{\delta,x_0}.$$

Clearly,

$$\|\varkappa_{\delta,x_0} f\|_{L_2^{-1}(\mathbf{R}^n)} \leqslant \|\varkappa_{\delta,x_0}|\vec{\phi}|\,\|_{L_2(\mathbf{R}^n)} + c\delta\,\|\nabla\varkappa_{\delta,x_0}\cdot\vec{\phi}\|_{L_2(\mathbf{R}^n)} \leqslant c\|\,|\vec{\phi}|\,\|_{L_2(B_{2\delta}(x_0))}.$$

This proves (3.4). Since (3.5) is derived in a similar way, the proof of Lemma 3.3 is complete.

LEMMA 3.3. *If* $V \in \overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n) \to L_2^{-1}(\mathbf{R}^n))$, *then*

$$\lim_{\delta\to 0} \sup_{x_0\in\mathbf{R}^n} \|\varkappa_{\delta,x_0} V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} = 0 \tag{3.6}$$

and

$$\lim_{R\to\infty} \|\xi_R V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} = 0. \tag{3.7}$$

Proof of Lemma 3.3. Fix $\varepsilon > 0$, and pick a finite number of $f_k \in L_2^{-1}(\mathbf{R}^n)$ such that

$$\|Vu - f_k\|_{L_2^{-1}(\mathbf{R}^n)} < \varepsilon$$

for $k = 1, ..., N(\varepsilon)$, and for all $u \in \mathcal{B}$, where $\mathcal{B}$ is the unit ball in $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$. Note that by Hardy's inequality

$$\sup_{x_0\in\mathbf{R}^n,\,\delta>0} \|\varkappa_{\delta,x_0}\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \leqslant c < \infty.$$

Next,

$$\begin{aligned}\|\varkappa_{\delta,x_0} Vu\|_{L_2^{-1}(\mathbf{R}^n)} &\leqslant \|\varkappa_{\delta,x_0}(Vu - f_k)\|_{L_2^{-1}(\mathbf{R}^n)} + \|\varkappa_{\delta,x_0} f_k\|_{L_2^{-1}(\mathbf{R}^n)} \\ &\leqslant c\varepsilon + \|\varkappa_{\delta,x_0} f_k\|_{L_2^{-1}(\mathbf{R}^n)}.\end{aligned}$$

Hence,

$$\|\varkappa_{\delta,x_0}\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \leqslant c\varepsilon + \|\varkappa_{\delta,x_0} f_k\|_{L_2^{-1}(\mathbf{R}^n)}.$$

By Lemma 3.2, this gives (3.6), and the proof of (3.7) is quite similar. The proof of Lemma 3.3 is complete.

We can now complete the proof of the necessity part of Theorem 3.1. Suppose that $V \in \overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n) \to L_2^{-1}(\mathbf{R}^n))$. By Theorem 2.2,

$$\|\nabla\Delta^{-1}(\xi_R V)\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))} \leqslant c\|\xi_R V\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))}.$$

By the preceding estimate and (3.7),

$$\lim_{R\to\infty} \|\nabla\Delta^{-1}(\xi_R V)\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))} = 0.$$

Hence we can assume without loss of generality that V is compactly supported, e.g., $\operatorname{supp} V \subset B_1(0)$. To show that

$$\vec{\Gamma} = \nabla\Delta^{-1} V \in \overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n) \to L_2(\mathbf{R}^n)),$$

consider a covering of the closed unit ball $\overline{B_1(0)}$ by open balls B_k $(k=1, ..., n)$ of radius $\sqrt{n}\,\delta$ centered at the nodes x_k of the lattice with mesh size δ. We introduce a partition of unity ϕ_k subordinate to this covering and satisfying the estimate $|\nabla\phi_k| \leqslant c\delta^{-1}$, so that $\operatorname{supp}\phi_k \subset B_k^*$, where B_k^* is a ball of radius $2\sqrt{n}\,\delta$ concentric to B_k. Also, pick $\psi_k \in C_0^\infty(B_k^*)$, where $\phi_k\psi_k = \phi_k$ and $|\nabla\psi_k| \leqslant c\delta^{-1}$.

We have

$$\nabla\Delta V = \sum_{k=1}^{N(\delta)} \nabla\Delta(\phi_k V) = \sum_{k=1}^{N(\delta)} \nabla\Delta(\phi_k\psi_k V) = \sum_{k=1}^{N(\delta)} \psi_k \nabla\Delta(\phi_k V) + \sum_{k=1}^{N(\delta)} [\nabla\Delta, \psi_k]\phi_k V,$$

where $[A, B] = AB - BA$ is the commutator of the operators A and B. We estimate

$$\left\| \sum_{k=1}^{N(\delta)} \psi_k \nabla\Delta(\phi_k V) \right\|_{M(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))} \leqslant c(n) \sup_{1\leqslant k\leqslant N(\delta)} \|\nabla\Delta(\phi_k V)\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))},$$

since the multiplicity of the covering $\bigcup_{k=1}^{N(\delta)} B_k$ depends only on n. The last supremum is bounded by $c\|\phi_k V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))}$, which is made smaller than any $\varepsilon > 0$ by choosing $\delta = \delta(\varepsilon)$ small enough.

It remains to check that each function $\Phi_k := [\nabla\Delta, \psi_k]\phi_k V$ is a compact multiplier from $\overset{\circ}{L}{}_2^1(\mathbf{R}^n)$ to $L_2(\mathbf{R}^n)$, $k=1, ..., n$. Indeed, the kernel of the operator $V \to [\nabla\Delta, \psi_k]\phi_k V$ is smooth, and hence,

$$\begin{aligned} |\Phi_k(x)| &= |([\nabla\Delta, \psi_k]\phi_k V)(x)| \leqslant c_k(1+|x|)^{1-n} \|\phi_k V\|_{L_2^{-1}(\mathbf{R}^n)} \\ &\leqslant c_k(1+|x|)^{1-n} \|V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\mathbf{R}^n)\to L_2^{-1}(\mathbf{R}^n))} \|\phi_k\|_{\overset{\circ}{L}{}_2^1(\mathbf{R}^n)} \leqslant C_k(1+|x|)^{1-n}, \end{aligned}$$

where the constant C_k does not depend on x. Since $n > 2$, this means that the multiplier operator $\Phi_k : \overset{\circ}{L}{}_2^1(\mathbf{R}^n) \to L_2(\mathbf{R}^n)$ is compact. The proof of Theorem 3.1 is complete.

4. The space $M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$

In this section, we characterize the class of multipliers $V\colon W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n)$ for $n\geqslant 1$. Here $W_2^{-1}(\mathbf{R}^n)=W_2^1(\mathbf{R}^n)^*$, where $W_2^1(\mathbf{R}^n)=H^1$ is the classical Sobolev space of weakly differentiable functions $u\in L_2(\mathbf{R}^n)$ such that $\nabla u\in L_2(\mathbf{R}^n)$ with norm

$$\|u\|_{W_2^1(\mathbf{R}^n)}=\left[\int_{\mathbf{R}^n}(|u(x)|^2+|\nabla u(x)|^2)\,dx\right]^{1/2}. \tag{4.1}$$

Let $J_\alpha=(I-\Delta)^{-\alpha/2}$ $(0<\alpha<+\infty)$ denote the Bessel potential of order α. (Here I stands for the identity operator.) Every $u\in W_2^1(\mathbf{R}^n)$ can be represented in the form $u=J_1 g$ where

$$c_1\|g\|_{L_2(\mathbf{R}^n)}\,\|u\|_{W_2^1(\mathbf{R}^n)}\leqslant c_2\|g\|_{L_2(\mathbf{R}^n)}.$$

(See [St1].)

Let $\mathcal{S}'(\mathbf{R}^n)$ denote the space of tempered distributions on $\mathbf{R}^n$. We say that $V\in \mathcal{S}'(\mathbf{R}^n)$ is a multiplier from $W_2^1(\mathbf{R}^n)$ to $W_2^{-1}(\mathbf{R}^n)$ if the sesquilinear form defined by $\langle Vu,v\rangle:=\langle V,\bar{u}v\rangle$ is bounded on $W_2^1(\mathbf{R}^n)\times W_2^1(\mathbf{R}^n)$:

$$|\langle Vu,v\rangle|\leqslant c\|u\|_{W_2^1(\mathbf{R}^n)}\|v\|_{W_2^1(\mathbf{R}^n)},\quad u,v\in\mathcal{S}(\mathbf{R}^n), \tag{4.2}$$

where the constant c is independent of u and v in Schwartz space $\mathcal{S}(\mathbf{R}^n)$. As in the case of homogeneous spaces, the preceding inequality is equivalent to the boundedness of the corresponding quadratic form; i.e., it suffices to verify (4.2) for $u=v$.

If (4.2) holds, then V defines a bounded multiplier operator from $W_2^1(\mathbf{R}^n)$ to $W_2^{-1}(\mathbf{R}^n)$. (Originally, it is defined on $\mathcal{S}(\mathbf{R}^n)$, but by continuity is extended to $W_2^1(\mathbf{R}^n)$.) The corresponding class of multipliers is denoted by $M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$.

We observe that $I-\Delta\colon W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n)$ is a bounded operator (see [St1]). Hence, $V\in M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$ if and only if the operator $(I-\Delta)+V\colon W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n)$ is bounded.

If V is a locally finite complex-valued measure on $\mathbf{R}^n$, then (4.2) can be rewritten in the form

$$\left|\int_{\mathbf{R}^n}u(x)\,\overline{v(x)}\,dV(x)\right|\leqslant c\|u\|_{W_2^1(\mathbf{R}^n)}\|v\|_{W_2^1(\mathbf{R}^n)}, \tag{4.3}$$

where $u,v\in\mathcal{S}(\mathbf{R}^n)$.

For positive measures V, this inequality is characterized as above (cf. Theorem 2.1), with Bessel potentials J_1 in place of Riesz potentials I_1, and with the Riesz capacity cap replaced by the Bessel capacity

$$\operatorname{cap}(e,W_2^1)=\inf\{\|u\|^2_{W_2^1(\mathbf{R}^n)}: u\in\mathcal{S}(\mathbf{R}^n),\,u(x)\geqslant 1 \text{ on } e\}. \tag{4.4}$$

For convenience, we state several equivalent characterizations below (see [KeS], [Ma3], [MaS], [MaV]).

THEOREM 4.1. *Let V be a locally finite positive measure on $\mathbf{R}^n$. Then the following statements are equivalent:*

(i) *The trace inequality*

$$\int_{\mathbf{R}^n} |u(x)|^2 \, dV(x) \leqslant c_1 \|u\|^2_{W_2^1(\mathbf{R}^n)} \tag{4.5}$$

holds, where c_1 does not depend on $u \in \mathcal{S}(\mathbf{R}^n)$.

(ii) *For every compact set $e \subset \mathbf{R}^n$,*

$$V(e) \leqslant c_2 \operatorname{cap}(e, W_2^1), \tag{4.6}$$

where c_2 does not depend on e.

(iii) *For every open ball B in $\mathbf{R}^n$,*

$$\int_B (J_1 V_B)^2 \, dx \leqslant c_3 V(B), \tag{4.7}$$

where $dV_B = \chi_B \, dV$, and c_3 does not depend on B.

(iv) *The pointwise inequality*

$$J_1 (J_1 V)^2(x) \leqslant c_4 J_1 V(x) < \infty \quad a.e. \tag{4.8}$$

holds, where c_4 does not depend on $x \in \mathbf{R}^n$.

(v) *For every compact set $e \subset \mathbf{R}^n$,*

$$\int_e (J_1 V)^2 \, dx \leqslant c_5^2 \operatorname{cap}(e, W_2^1), \tag{4.9}$$

where c_5 does not depend on e.

(vi) *For every dyadic cube P_0 in $\mathbf{R}^n$ of sidelength $l(P_0) \leqslant 1$,*

$$\sum_{P \subseteq P_0} \left[\frac{V(P)}{|P|^{1-1/n}} \right]^2 |P| \leqslant c_6 V(P_0), \tag{4.10}$$

where the sum is taken over all dyadic cubes P contained in P_0, and c_6 does not depend on P_0.

The least constants $c_1, ..., c_6$ in the inequalities (4.5)–(4.10) *are equivalent.*

Remark 1. It suffices to verify (4.6) and (4.9) for compact sets $e \subset \mathbf{R}^n$ such that $\operatorname{diam} e \leqslant 1$. In this case, the capacity $\operatorname{cap}(e, W_2^1)$ is equivalent to the Riesz capacity $\operatorname{cap}(e)$ provided $n \geqslant 3$.

Remark 2. For $n=1$, the Bessel capacity of a single point set is positive, and hence $\operatorname{cap}(e, W_2^1)$, for sets e such that $\operatorname{diam} e \leqslant 1$, can be replaced by a constant independent of e. Thus, in this case (4.5) holds if and only if

$$\sup_{x \in \mathbf{R}} V(B_1(x)) < \infty. \tag{4.10}$$

We now characterize (4.3) in the general case of distributions V.

THEOREM 4.2. *Let $V\in\mathcal{S}'(\mathbf{R}^n)$. Then $V\in M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$ if and only if there exist a vector field $\vec{\Gamma}=\{\Gamma_1,...,\Gamma_n\}\in\mathbf{L}_{2,\mathrm{loc}}(\mathbf{R}^n)$ and $\Gamma_0\in L_{2,\mathrm{loc}}(\mathbf{R}^n)$ such that*

$$V=\operatorname{div}\vec{\Gamma}+\Gamma_0 \tag{4.11}$$

and

$$\int_{\mathbf{R}^n}|u(x)|^2\,|\Gamma_i(x)|^2\,dx\leqslant C\|u\|^2_{W_2^1(\mathbf{R}^n)},\quad i=0,1,...,n, \tag{4.12}$$

where C does not depend on $u\in\mathcal{S}(\mathbf{R}^n)$.

In (4.11), *one can set*

$$\vec{\Gamma}=-\nabla(I-\Delta)^{-1}V\quad\textit{and}\quad\Gamma_0=(I-\Delta)^{-1}V. \tag{4.13}$$

Remark 3. It is easy to see that in the sufficiency part of Theorem 4.2 the restriction on the "lower-order" term Γ_0 in (4.12) can be relaxed. It is enough to assume that $\Gamma_0\in L_{1,\mathrm{loc}}(\mathbf{R}^n)$ is such that

$$\int_{\mathbf{R}^n}|u(x)|^2\,|\Gamma_0(x)|\,dx\leqslant C\|u\|^2_{W_2^1(\mathbf{R}^n)}. \tag{4.14}$$

Proof of Theorem 4.2. Suppose that V is represented in the form (4.11), and (4.12) holds. Then using integration by parts and the Schwarz inequality, we have

$$\begin{aligned}|\langle V,\bar{u}v\rangle|&=|\langle\vec{\Gamma},v\nabla\bar{u}\rangle+\langle\vec{\Gamma},\bar{u}\nabla v\rangle+\langle\Gamma_0,\bar{u}v\rangle|\\&\leqslant\|\vec{\Gamma}v\|_{\mathbf{L}_2(\mathbf{R}^n)}\|\nabla u\|_{L_2(\mathbf{R}^n)}+\|\vec{\Gamma}u\|_{\mathbf{L}_2(\mathbf{R}^n)}\|\nabla v\|_{L_2(\mathbf{R}^n)}+\|\Gamma_0u\|_{L_2(\mathbf{R}^n)}\|v\|_{L_2(\mathbf{R}^n)}\\&\leqslant3\sqrt{C}\,\|u\|_{W_2^1(\mathbf{R}^n)}\|v\|_{W_2^1(\mathbf{R}^n)},\end{aligned}$$

where C is the constant in (4.11). This proves the "if" part of Theorem 4.2.

To prove the "only if" part, define $\vec{\Gamma}=\{\Gamma_1,...,\Gamma_n\}$ and Γ_0 by (4.13). Then, for every $j=0,1,...,n$, it follows that $\Gamma_j\in L_{2,\mathrm{loc}}(\mathbf{R}^n)$, and the following crude estimates hold:

$$\int_{B_R(x_0)}|\Gamma_j(x)|^2\,dx\leqslant C(n,\varepsilon)\,R^{n-2+\varepsilon}\,\|V\|^2_{M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))}, \tag{4.15}$$

where $R\geqslant\max\{1,|x_0|\}$. The proof uses the same argument as in the proof of Lemma 2.3 in the homogeneous case.

Now fix a compact set $e\subset\mathbf{R}^n$ such that $\mathrm{diam}(e)\leqslant1$ and $\mathrm{cap}(e,W_2^1)>0$. Denote by $P(x)=P_e(x)$ the equilibrium potential of e which corresponds to the Bessel capacity (4.4). Letting

$$u(x)=P(x)^\delta\quad\text{and}\quad v(x)=\frac{w(x)}{P(x)^\delta},$$

where $1<2\delta<n/(n-2)$ and $w\in\mathcal{S}(\mathbf{R}^n)$, we have

$$|\langle V, w\rangle| \leqslant \|V\|_{M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))} \|P^\delta\|_{W_2^1(\mathbf{R}^n)} \|\nabla v\|_{W_2^1(\mathbf{R}^n)}.$$

Calculations analogous to those of Propositions 2.5–2.9 yield

$$\|P^\delta\|_{W_2^1(\mathbf{R}^n)} \leqslant C(n,\delta)\operatorname{cap}(e, W_2^1)^{1/2}$$

and

$$\|\nabla v\|_{W_2^1(\mathbf{R}^n)} \leqslant C(n,\delta)\left[\int_{\mathbf{R}^n} (|w(x)|^2+|\nabla w(x)|^2)\,\frac{dx}{P(x)^{2\delta}}\right]^{1/2}.$$

Combining the preceding inequalities, we obtain

$$|\langle V, w\rangle| \leqslant C(n,\delta)\|V\|_{M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))}\operatorname{cap}(e, W_2^1)^{1/2} \times\left[\int_{\mathbf{R}^n} (|w(x)|^2+|\nabla w(x)|^2)\,\frac{dx}{P(x)^{2\delta}}\right]^{1/2}.$$

Set $w=(1-\Delta)^{-1}\operatorname{div}\vec{\phi}$, where $\vec{\phi}$ is an arbitrary vector field with components in $\mathcal{S}(\mathbf{R}^n)$. Then the preceding estimate can be restated in the form

$$|\langle\vec{\Gamma}, \vec{\phi}\rangle| \leqslant C(n,\delta)\operatorname{cap}(e, W_2^1)^{1/2}\left[\int_{\mathbf{R}^n} (|w(x)|^2+|\nabla w(x)|^2)\,\frac{dx}{P(x)^{2\delta}}\right]^{1/2}. \tag{4.16}$$

Unlike in the homogeneous case, for Bessel potentials, $P(x)^{-2\delta}$ is not a Muckenhoupt weight. To proceed, we will need a localized version of the estimates used in §3.

LEMMA 4.3. *Let $P(x)=P_e(x)$ be the equilibrium potential of a compact set e of positive Bessel capacity, and such that $e\subset B$, where $B=B_1(x_0)$ is a ball of radius 1 centered at $x_0\in\mathbf{R}^n$. Let $w=(I-\Delta)^{-1}\nabla\psi$, where $\psi\in C^\infty(\mathbf{R}^n)$ and $\operatorname{supp}\psi\subset B$. Suppose that $1<2\delta<n/(n-2)$. Then*

$$\int_{\mathbf{R}^n} (|w(x)|^2+|\nabla w(x)|^2)\,\frac{dx}{P(x)^{2\delta}} \leqslant C(n,\delta)\int_{\mathbf{R}^n} |\psi(x)|^2\,\frac{dx}{P(x)^{2\delta}}. \tag{4.17}$$

Proof. Let $\nu=\nu_e$ be the equilibrium measure of the compact set e in the sense of Bessel capacities, so that $P(x)=J_2\nu(x)$ (see [AdH], [Ma3]). Suppose first that $n\geqslant 3$. Since both $\operatorname{supp}\nu$ and $\operatorname{supp}\psi$ are contained in B, it follows that

$$P(x)=J_2\nu(x)\asymp I_2\nu(x)=c(n)\int_B \frac{d\nu(y)}{|x-y|^{n-2}}, \quad x\in 2B, \tag{4.18}$$

where $2B$ is a concentric ball of radius 2.

We set $\varrho(x)=I_2\nu(x)^{-2\delta}$. Then $\varrho(x)\asymp P(x)^{-2\delta}$ on $2B$, and $\varrho(x)$ is an A_2-weight (see the proof of Proposition 2.8). Note that $\nabla w=\nabla^2(I-\Delta)^{-1}\psi$, where

$$\nabla^2(I-\Delta)^{-1}=\{-R_jR_k\Delta(I-\Delta)^{-1}\},\quad j,k=1,...,n.$$

Here R_j, $j=1,...,n$, are the Riesz transforms which are bounded operators on $L_2(\mathbf{R}^n,\varrho)$ (see [St2]).

Since $\Delta(I-\Delta)^{-1}=I-(I-\Delta)^{-1}$, we have to show that $J_2=(I-\Delta)^{-1}$ is a bounded operator on $L_2(\mathbf{R}^n,\varrho)$, and its norm is bounded by a constant which depends only on the Muckenhoupt constant of ϱ. It is not difficult to see that the same is true for more general operators $J_\alpha=(I-\Delta)^{-\alpha/2}$, where $\alpha>0$.

Indeed, denote by $G_\alpha(x)$ the kernel of the Bessel potential J_α. Then clearly,

$$|J_\alpha f(x)|=|G_\alpha*f(x)|\leqslant c(n,\alpha)Mf(x)\sum_{k=-\infty}^{\infty}2^{kn}\max_{2^k\leqslant|t|\leqslant 2^{k+1}}G_\alpha(t),$$

where $Mf(x)$ is the Hardy–Littlewood maximal function defined by

$$Mf(x)=c(n)\sup_{0<r<\infty}r^{-n}\int_{B_r(x)}|f(y)|\,dy.$$

Standard estimates of Bessel kernels $G_\alpha(x)$ (see, e.g., [AdH, Sections 1.2.4 and 1.2.5]) show that

$$\sum_{k=-\infty}^{\infty}2^{kn}\max_{2^k\leqslant|t|\leqslant 2^{k+1}}G_\alpha(t)<\infty$$

for every $\alpha>0$. Since M is bounded on $L_2(\mathbf{R}^n,\varrho)$ (see [St2]), it follows that

$$\|J_\alpha f\|_{L_2(\mathbf{R}^n,\varrho)}\leqslant C\|f\|_{L_2(\mathbf{R}^n,\varrho)},\tag{4.19}$$

where C depends only on n, α and the Muckenhoupt constant of ϱ.

Applying (4.19) with $\alpha=2$, we get

$$\int_{2B}|\nabla w(x)|^2\frac{dx}{P(x)^{2\delta}}\leqslant C(n,\delta)\int_{\mathbf{R}^n}|\psi(x)|^2\varrho(x)\,dx\leqslant C(n,\delta)\int_{\mathbf{R}^n}|\psi(x)|^2\frac{dx}{P(x)^{2\delta}}.$$

Similarly,

$$|w(x)|=|\nabla(I-\Delta)^{-1}\psi(x)|\leqslant CJ_1|\psi|(x),$$

and by (4.19) with $\alpha=1$,

$$\begin{aligned}\int_{2B}|w(x)|^2\frac{dx}{P(x)^{2\delta}}&\leqslant C\int_{2B}(J_1|\psi|(x))^2\varrho(x)\,dx\\&\leqslant C(n,\delta)\int_{\mathbf{R}^n}|\psi(x)|^2\varrho(x)\,dx\leqslant C(n,\delta)\int_{\mathbf{R}^n}|\psi(x)|^2\frac{dx}{P(x)^{2\delta}}.\end{aligned}$$

Now suppose $x \in (2B)^c$. Then, by standard estimates of the Bessel kernel as $|x| \to \infty$ ([AdH, Sections 1.2.4 and 1.2.5]),

$$|\nabla w(x)| = |\nabla^2 J_2 \psi(x)| \leqslant C(n) |x|^{(1-n)/2} e^{-|x|} \int_B |\psi(y)|\, dy$$

and

$$|w(x)| \leqslant C(n) |\nabla J_2 \psi(x)| \leqslant C |x|^{-n/2} e^{-|x|} \int_B |\psi(y)|\, dy.$$

Also, for $x \in (2B)^c$,

$$P(x) = J_2 \nu(x) \asymp |x|^{(1-n)/2} e^{-|x|} \nu(e), \quad |x| \to \infty,$$

where $\nu(e) = \operatorname{cap}(e, W_2^1) > 0$.

Now pick δ so that $1 < 2\delta < \min[2, n/(n-2)]$. Using the above estimates of $w(x)$, $\nabla w(x)$ and $P(x)$, and the inequality $2\delta < 2$, we get

$$\int_{(2B)^c} (|w(x)|^2 + |\nabla w(x)|^2) \frac{dx}{P(x)^{2\delta}} \leqslant C(n, \delta)\, \nu(e)^{-2\delta} \left(\int_B |\psi(y)|\, dy \right)^2.$$

By the Schwarz inequality,

$$\left(\int_B |\psi(y)|\, dy \right)^2 \leqslant \int_B |\psi(y)|^2 \frac{dy}{P(y)^{2\delta}} \int_B P(x)^{2\delta}\, dx.$$

Applying Minkowski's integral inequality and the fact that $2\delta < n/(n-2)$, we obtain

$$\int_B P(x)^{2\delta}\, dx \leqslant \int_B (I_2 \nu)^{2\delta}\, dx \leqslant C(n, \delta)\, \nu(e)^{2\delta}.$$

Thus,

$$\int_{(2B)^c} (|w(x)|^2 + |\nabla w(x)|^2) \frac{dx}{P(x)^{2\delta}} \leqslant C(n, \delta) \int_{\mathbf{R}^n} |\psi(x)|^2 \frac{dx}{P(x)^{2\delta}}.$$

This completes the proof of (4.17) for $n \geqslant 3$. The cases $n = 1, 2$ are treated in a similar way with obvious modifications. The proof of Lemma 4.3 is complete.

Let $w = (I - \Delta)^{-1} \operatorname{div} \vec{\phi}$, where $\vec{\phi} = \{\phi_k\} \in \mathcal{S}(\mathbf{R}^n)$. Applying Lemma 4.3 with $\psi = \phi_k$, $k = 1, \ldots, n$, we obtain

$$\int_{\mathbf{R}^n} (|w(x)|^2 + |\nabla w(x)|^2) \frac{dx}{P(x)^{2\delta}} \leqslant C(n, \delta) \int_{\mathbf{R}^n} |\vec{\phi}(x)|^2 \frac{dx}{P(x)^{2\delta}}.$$

This and (4.16) yield

$$|\langle \vec{\Gamma}, \vec{\phi} \rangle| \leqslant C(n, \delta) \operatorname{cap}(e, W_2^1)^{1/2} \left[\int_{\mathbf{R}^n} |\vec{\phi}(x)|^2 \frac{dx}{P(x)^{2\delta}} \right]^{1/2}.$$

By duality, the preceding inequality is equivalent to

$$\int_{\mathbf{R}^n} |\vec{\Gamma}(x)|^2 P(x)^{2\delta}\, dx \leqslant C(n,\delta)\|V\|^2_{M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))} \operatorname{cap}(e, W_2^1).$$

Since $P(x)\geqslant 1$ a.e. on e, we obtain the desired estimate

$$\int_e |\vec{\Gamma}(x)|^2\, dx \leqslant C(n,\delta)\|V\|^2_{M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))} \operatorname{cap}(e, W_2^1).$$

The corresponding inequality with Γ_0 in place of $\vec{\Gamma}$ is verified in a similar way. By Theorem 4.1 these inequalities are equivalent to (4.12). The proof of Theorem 4.2 is complete.

Finally, we state a compactness criterion in the case of the space $W_2^1(\mathbf{R}^n)$ analogous to that of Theorem 3.1.

THEOREM 4.4. *Let $V\in\mathcal{S}'(\mathbf{R}^n)$, $n\geqslant 1$. Then $V\in \mathring{M}(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$ if and only if*

$$V=\operatorname{div}\vec{\Gamma}+\Gamma_0,$$

where $\vec{\Gamma}=(\Gamma_1,\dots,\Gamma_n)$, and $\Gamma_i\in\mathring{M}(W_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))$ $(i=0,\dots,n)$. Moreover, one can set $\vec{\Gamma}=-\nabla(I-\Delta)^{-1}V$ and $\Gamma_0=(I-\Delta)^{-1}V$, as in Theorem 4.2.

The proof of Theorem 4.4 requires only minor modifications outlined in the proof of Theorem 4.2, and is omitted here.

5. The space $M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))$

Using dilation and the description of the space $M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$ given in the preceding section, we arrive at the following auxiliary statement.

COROLLARY 5.1. *Let $V\in M(W_2^1(\mathbf{R}^n)\to W_2^{-1}(\mathbf{R}^n))$. Suppose that there exists a number $d>0$ such that*

$$|\langle V,|u|^2\rangle|\leqslant c(\|\nabla u\|^2_{L^2(\mathbf{R}^n)}+d^{-2}\|u\|^2_{L^2(\mathbf{R}^n)}), \tag{5.1}$$

where c does not depend on $u\in C_0^\infty(\mathbf{R}^n)$. Then V can be represented as

$$V=\operatorname{div}\vec{\Gamma}+d^{-1}\Gamma_0, \tag{5.2}$$

where Γ_0 *and* $\vec{\Gamma}=(\Gamma_1, ..., \Gamma_n)$ *are in* $M(W_2^1(\mathbf{R}^n)\to L_2(\mathbf{R}^n))$, *and*

$$\int_{\mathbf{R}^n} |\Gamma_i u(x)|^2\,dx \leqslant C(\|\nabla u\|^2_{L^2(\mathbf{R}^n)} + d^{-2}\|u\|^2_{L^2(\mathbf{R}^n)}) \tag{5.3}$$

for all $i=0,1,...,n$.

Now let Ω be an open set in $\mathbf{R}^n$ such that, for all $u\in\mathcal{D}(\Omega)$, Hardy's inequality holds:

$$\int_\Omega |u(x)|^2 \frac{dx}{d_{\partial\Omega}(x)^2} \leqslant \text{const} \int_\Omega |\nabla u(x)|^2\,dx. \tag{5.4}$$

Here $d_{\partial\Omega}(x)=\operatorname{dist}(x,\partial\Omega)$. It is well-known that (5.4) holds for a wide class of domains including those with Lipschitz and NTA boundaries. (See [An], [D2], [Le], [MMP] for a discussion of Hardy's inequality and related questions, including best constants, on domains Ω in $\mathbf{R}^n$.)

Let Q_j be the cubes with sidelength d_j forming Whitney's covering of Ω (see [St1, Section 5.1]). Denote by Q_j^* the open cube obtained from Q by dilation with coefficient $\frac{9}{8}d_j$. The cubes Q_j^* form an open covering of Ω of finite multiplicity which depends only on n. By $\{\eta_j\}$ ($\eta_j\in C_0^\infty(Q_j^*)$) we denote a smooth partition of unity subordinate to the covering $\{Q_j\}$ and such that $|\nabla\eta_j(x)|\leqslant c d_j^{-1}$. In the proof of the following theorem we also will need the functions $\zeta_j\in C_0^\infty(Q_j^*)$ such that

$$\zeta_j(x)\eta_j(x)=\eta_j(x) \quad \text{and} \quad |\nabla\zeta_j(x)|\leqslant c d_j^{-1}. \tag{5.5}$$

Now we give a characterization of the space $M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))$.

THEOREM 5.2. (i) *Let* $d_{\partial\Omega}(x)=\operatorname{dist}(x,\partial\Omega)$, *and let*

$$V=\operatorname{div}\vec{\Gamma}+d_{\partial\Omega}^{-1}\Gamma_0,$$

where $\vec{\Gamma}=\{\Gamma_1,...,\Gamma_n\}$ *and* $\Gamma_i\in M(\mathring{L}_2^1(\Omega)\to L_2(\Omega))$ *for* $i=0,1,...,n$. *Suppose that* (5.4) *holds. Then* $V\in M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))$ *and*

$$\|V\|_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))} \leqslant c \sum_{0\leqslant i\leqslant n} \|\Gamma_i\|_{M(\mathring{L}_2^1(\Omega)\to L_2(\Omega))}. \tag{5.6}$$

(ii) *Conversely, if* $V\in M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))$, *then there exist* $\vec{\Gamma}=(\Gamma_1,...,\Gamma_n)$ *and* Γ_0 *such that* $\Gamma_i\in M(\mathring{L}_2^1(\Omega)\to L_2(\Omega))$ *for* $i=0,1,...,n$, *and* $V=\operatorname{div}\vec{\Gamma}+d_{\partial\Omega}^{-1}\Gamma_0$. *Moreover,*

$$\sum_{0\leqslant i\leqslant n} \|\Gamma_i\|_{M(\mathring{L}_2^1(\Omega)\to L_2(\Omega))} \leqslant C\|V\|_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))}. \tag{5.7}$$

Proof. The proof of statement (i) is straightforward (see, e.g., the proof of Theorem 4.2 above). To prove (ii), note that, for all $u, v \in C_0^\infty(\Omega)$, and the functions ζ_j satisfying the properties (5.5), we have

$$|\langle V\eta_j, uv\rangle| = |\langle V\eta_j, \zeta_j u \zeta_j v\rangle| \leqslant \|V\eta_j\|_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))}(\|\nabla u\|_{L^2(\mathbf{R}^n)} + d_j^{-1}\|u\|_{L^2(\mathbf{R}^n)})$$
$$\times (\|\nabla v\|_{L^2(\mathbf{R}^n)} + d_j^{-1}\|v\|_{L^2(\mathbf{R}^n)}).$$

Hence by Corollary 5.1,

$$V\eta_j = \operatorname{div} \vec{\Gamma}^{(j)} + d_j^{-1}\Gamma_0^{(j)}, \tag{5.8}$$

where $\vec{\Gamma}^{(j)}$ and $\Gamma_0^{(j)}$ satisfy the inequality

$$\int_{\mathbf{R}^n} |\Gamma_i^{(j)} u(x)|^2\,dx \leqslant C\|V\eta_j\|^2_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))}(\|\nabla u\|^2_{L^2(\mathbf{R}^n)} + d_j^{-2}\|u\|^2_{L^2(\mathbf{R}^n)}) \tag{5.9}$$

for all $i=0,1,\dots,n$. Multiplying (5.8) by ζ_j we obtain

$$V\eta_j = \operatorname{div}(\zeta_j \vec{\Gamma}^{(j)}) + d_j^{-1}\Gamma_0^{(j)} - \vec{\Gamma}^{(j)}\nabla\zeta_j.$$

We set

$$\vec{\Gamma} = \sum_j \zeta_j \vec{\Gamma}^{(j)} \quad \text{and} \quad \Gamma_0 = \sum_j (d_j\Gamma_0^{(j)} - \vec{\Gamma}^{(j)}\nabla\zeta_j).$$

If $u \in C_0^\infty(\Omega)$, then

$$\int_\Omega |(|\vec{\Gamma}| + |\Gamma_0|)u|^2\,dx \leqslant c\sum_j \left(\int_\Omega |\vec{\Gamma}^{(j)}\zeta_j u|^2\,dx + d_j^{-2}\int_\Omega |(d_j\Gamma_0^{(j)}\zeta_j - \vec{\Gamma}^{(j)}\nabla\zeta_j)\varkappa_j u|^2\,dx\right),$$

where $\varkappa_j \in C_0^\infty(Q_j^*)$ and $\varkappa_j = 1$ on $\operatorname{supp}\zeta_j$. By (5.9), the last sum does not exceed

$$\sup_j \|V\eta_j\|^2_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))} \sum_j \int_\Omega (|\nabla(\varkappa_j u)|^2 + d_j^{-2}|\varkappa_j u|^2)\,dx.$$

By Hardy's inequality (5.4), this is bounded by

$$c\|V\|^2_{M(\mathring{L}_2^1(\Omega)\to L_2^{-1}(\Omega))}\int_\Omega |\nabla u|^2\,dx.$$

The proof of Theorem 5.2 is complete.

Remark. In Theorem 5.2, one can replace

$$\sum_{0\leqslant i\leqslant n} \|\Gamma_i\|_{M(\mathring{L}_2^1(\Omega)\to L_2(\Omega))}$$

with the equivalent norm

$$\sup_j \sup_{e\subset Q_j} \frac{\|(|\vec{\Gamma}| + |\Gamma_0|)\|_{L_2(e)}}{(\operatorname{cap}(e, \mathring{L}_2^1(Q_j^*)))^{1/2}}. \tag{5.10}$$

In the case $n>2$, one can use Wiener's capacity in place of $\operatorname{cap}(\cdot, \mathring{L}_2^1(Q_j^*))$ (see [MaS, Section 5.7.2]).

We now characterize the class of compact multipliers, $\mathring{M}(\mathring{L}_2^1(\Omega) \to L_2^{-1}(\Omega))$. We use the same notation as in the previous section.

THEOREM 5.3. *Under the assumptions of Theorem 5.2, a distribution V is in* $\overset{\circ}{M}(\overset{\circ}{L}{}^1_2(\Omega)\to L_2^{-1}(\Omega))$ *if and only if*

$$V=\operatorname{div}\vec{\Gamma}+d_{\partial\Omega}^{-1}\Gamma_0, \tag{5.11}$$

where $\Gamma_i\in\overset{\circ}{M}(\overset{\circ}{L}{}^1_2(\Omega)\to L_2(\Omega))$ *for* $i=0,1,\ldots,n$.

Proof. Suppose that V is given by (5.11). Let u be an arbitrary function in the unit ball $\mathcal{B}$ of $\overset{\circ}{L}{}^1_2(\Omega)$. Then

$$Vu=\operatorname{div}(u\vec{\Gamma})-\vec{\Gamma}+d_{\partial\Omega}^{-1}u\Gamma_0.$$

The set $\{\operatorname{div}(u\vec{\Gamma}):u\in\mathcal{B}\}$ is compact in $L_2^{-1}(\Omega)$ since the set $\{u\vec{\Gamma}:u\in\mathcal{B}\}$ is compact in $\mathbf{L}_2(\Omega)$. The sets $\{\nabla u\cdot\vec{\Gamma}:u\in\mathcal{B}\}$ and $\{d_{\partial\Omega}^{-1}\Gamma_0 u:u\in\mathcal{B}\}$ are also compact in $L_2^{-1}(\Omega)$ since the sets $\{|\nabla u|:u\in\mathcal{B}\}$ and $\{d_{\partial\Omega}^{-1}u:u\in\mathcal{B}\}$ are bounded in $L_2(\Omega)$, and the multiplier operators $\bar{\Gamma}_i: L_2(\Omega)\to L_2^{-1}(\Omega)$, $i=1,\ldots,n$, are compact, being adjoint to Γ_i. This completes the proof of the "if" part of Theorem 5.3.

To prove the "only if" part let us assume that the origin $O\in\mathbf{R}^n\setminus\Omega$. Then, for any $x\in\Omega$, it follows that $|x|\geqslant d_{\partial\Omega}(x)$, and the inequality

$$\int_\Omega\frac{|u(x)|^2}{|x|^2}\,dx\leqslant c\int_\Omega|\nabla u(x)|^2\,dx \tag{5.12}$$

follows from (5.4).

As in the previous section, we introduce the cut-off functions

$$\varkappa_\delta(x)=F(d_{\partial\Omega}/\delta)$$

and

$$\xi_R(x)=1-F(|x|/R),$$

where $F\in C^\infty(\mathbf{R}_+)$ so that $F(t)=1$ for $t\leqslant1$ and $F(t)=0$ for $t\geqslant2$.

The proofs of the following two lemmas are similar to those of Lemma 3.2 and Lemma 3.3.

LEMMA 5.4. *If* $f\in L_2^{-1}(\Omega)$, *then*

$$\lim_{\delta\to0}\|\varkappa_\delta f\|_{L_2^{-1}(\Omega)}=0 \tag{5.13}$$

and

$$\lim_{R\to\infty}\|\xi_R f\|_{L_2^{-1}(\Omega)}=0. \tag{5.14}$$

LEMMA 5.5. *If* $V \in \overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\Omega) \to L_2^{-1}(\Omega))$, *then*

$$\lim_{\delta\to 0} \|\varkappa_\delta V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\Omega)\to L_2^{-1}(\Omega))} = 0 \tag{5.15}$$

and

$$\lim_{R\to\infty} \|\xi_R V\|_{\overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\Omega)\to L_2^{-1}(\Omega))} = 0. \tag{5.16}$$

We now complete the proof of the "only if" part of Theorem 5.3. Write V in the form

$$V = \varkappa_\delta V + \xi_R V + (1 - \varkappa_\delta - \xi_R) V.$$

By Theorem 5.2 (ii), there exist $\vec{\Gamma}_\delta$ and $\Gamma^{(0)}$ such that

$$\varkappa_\delta V = \operatorname{div} \vec{\Gamma}_\delta + d_{\partial\Omega}^{-1} \Gamma_\delta^{(0)},$$

where

$$\sum_{0\leqslant i\leqslant n} \|\Gamma_\delta^{(i)}\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2(\Omega))} \leqslant C \|\varkappa_\delta V\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2^{-1}(\Omega))}.$$

Analogously,

$$\xi_R V = \operatorname{div} \vec{\Gamma}_{(R)} + |x|^{-1} \Gamma_{(R)}^{(0)},$$

where

$$\sum_{0\leqslant i\leqslant n} \|\Gamma_{(R)}^{(i)}\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2(\Omega))} \leqslant C \|\xi_R V\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2^{-1}(\Omega))}.$$

Hence, by Lemma 5.5,

$$\lim_{\delta\to 0} \sum_{0\leqslant i\leqslant n} \|\Gamma_\delta^{(i)}\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2(\Omega))} = 0$$

and

$$\lim_{R\to\infty} \sum_{0\leqslant i\leqslant n} \|\Gamma_{(R)}^{(i)}\|_{M(\overset{\circ}{L}{}_2^1(\Omega)\to L_2(\Omega))} = 0.$$

Now we estimate the multiplier

$$V_{\delta,R} := (1 - \varkappa_\delta - \xi_R) V.$$

Note that $V_{\delta,R} \in \overset{\circ}{M}(\overset{\circ}{L}{}_2^1(\Omega) \to L_2^{-1}(\Omega))$. Since its support is separated from ∞ and from $\partial\Omega$, it follows that

$$V_{\delta,R} \in \overset{\circ}{M}(W_2^1(\mathbf{R}^n) \to W_2^{-1}(\mathbf{R}^n)).$$

By Theorem 4.4,

$$V_{\delta,R} = \operatorname{div} \vec{\Gamma}_{\delta,R} + \Psi_{\delta,R}, \tag{5.17}$$

where each component of $\vec{\Gamma}_{\delta,R}$, together with $\Psi_{\delta,R}$, are in $\overset{\circ}{M}(W_2^1(\mathbf{R}^n) \to L_2(\mathbf{R}^n))$.

Multiplying, if necessary, both sides of (5.17) by a cut-off function as before, we may assume that the supports of $|\vec{\Gamma}_{\delta,R}|$ and $\Psi_{\delta,R}$ are in Ω, and are both separated from ∞, and from $\partial\Omega$. Hence, the components of $\vec{\Gamma}_{\delta,R}$, as well as $d_{\partial\Omega}\Psi_{\delta,R}$, are in $\mathring{M}(\mathring{L}_2^1(\Omega)\to L_2(\Omega))$. Finally,

$$V = \operatorname{div}\vec{\Gamma} + d_{\partial\Omega}^{-1}\Gamma^{(0)},$$

where

$$\vec{\Gamma} = \vec{\Gamma}_\delta + \vec{\Gamma}_{(R)} + \vec{\Gamma}_{\delta,R}$$

and

$$\Gamma^{(0)} = \Gamma_\delta^{(0)} + |x|^{-1} d_{\partial\Omega}\Gamma_{(R)}^{(0)} + d_{\partial\Omega}\Gamma_{\delta,R}^{(0)}.$$

It remains to note that $\vec{\Gamma}_\delta$, $\vec{\Gamma}_{(R)}$, $\Gamma_\delta^{(0)}$ and $|x|^{-1}d_{\partial\Omega}\Gamma_{(R)}^{(0)}$ are small in the corresponding operator norms, while $\vec{\Gamma}_{\delta,R}$ and $\Gamma_{\delta,R}^{(0)}$ are compact. This completes the proof of Theorem 5.3.

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