Acta Math., 177 (1996), 75–112

Quasi-isometric rigidity and Diophantine approximation

by

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1. Introduction

1.1. Background

A *quasi-isometry* between metric spaces is a map which only distorts distances by a bounded factor, above a certain fixed scale. (See §2.1 for a precise definition.) Each finitely generated group, equipped with a chosen finite generating set, has a natural *word metric*. In this metric, the group becomes a path metric space. Changing the generating set produces a new metric space, which is quasi-isometric to the original one. Thus, the quasi-isometric, or "large-scale geometric", structure of these metric spaces only depends on the group itself.

There has been considerable interest recently in understanding how the algebraic structure of a group influences its large-scale geometric structure, and vice-versa. (See [Gr] for a detailed survey.) Sometimes it happens that one can recover some, or all, of the algebraic structure of a group from its quasi-isometric properties. Broadly speaking, we call this phenomenon *quasi-isometric rigidity*.

Lattices in Lie groups provide a concrete and interesting family of finitely generated groups. A uniform—that is, co-compact—lattice in a Lie group is always quasi-isometric to the Lie group itself. In the co-compact case, then, the study of quasi-isometries of lattices reduces to the study of quasi-isometries of the ambient Lie group.

At least in the semisimple case, this theory is quite well developed. Quasi-isometries of the real hyperbolic space (plane) are just extensions of quasi-conformal (-symmetric) mappings. A similar, though perhaps less developed, theory holds in complex hyperbolic space. The result of [P] says that all quasi-isometries of quaternionic hyperbolic space (and the Cayley plane) are equivalent to isometries. (See §2.1 for the precise notion

Supported by an N.S.F. Postdoctoral Fellowship.

of equivalence.) The recent results of [KL] say that this is also true for higher-rank symmetric spaces with no rank-one factors. These last two results are examples of quasi-isometric rigidity.

In the case of non-uniform lattices, sometimes a different kind of quasi-isometric rigidity holds. Let X be a symmetric space of non-positive sectional curvature, and let $\mathrm{Is}(X)$ denote the isometry group of X. Let $\Gamma\subset\mathrm{Is}(X)$ be a non-uniform lattice. We let $\mathcal{Q}(\Gamma)$ denote the quasi-isometry group of Γ. (See §2.1 for a precise definition.) Let $\mathrm{Comm}(\Gamma, X)\subset\mathrm{Is}(X)$ denote the commensurator of Γ. This group consists of those isometries which conjugate Γ to itself, up to finite index.

There is a canonical injection $i:\mathrm{Comm}(\Gamma, X)\to\mathcal{Q}(\Gamma)$. We say that (Γ, X) is *quasi-isometrically rigid* if i is an isomorphism. This notion is closely related to the quasi-isometry classification of such lattices. If (Γ_1, X) and (Γ_2, X) are both quasi-isometrically rigid, then Γ_1 and Γ_2 are quasi-isometric to each other if and only if they are commensurable.

The two versions of quasi-isometric rigidity, uniform and non-uniform, both imply and generalize Mostow Rigidity in the relevant cases. (See [M1] and [M2] for details about Mostow Rigidity.)

In [Sch] I proved that (Γ, X) is quasi-isometrically rigid provided that Γ is non-uniform and $X\neq\mathbf{H}^2$ is a rank-one symmetric space. If we restrict our attention to pairs $(\Gamma,\mathbf{H}^3)$, where Γ is arithmetic, we get the following concrete result:

Let $K_1, K_2\supset\mathbf{Q}$ be imaginary quadratic fields and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the corresponding rings of algebraic integers. Then $\mathrm{SL}_2(\mathcal{O}_1)$ and $\mathrm{SL}_2(\mathcal{O}_2)$ are quasi-isometric if and only if K_1 and K_2 are isomorphic fields.

In [FS], B. Farb and I extended the techniques of [Sch] to prove that $(\Gamma,\mathbf{H}^2\times\mathbf{H}^2)$ is quasi-isometrically rigid provided that Γ is non-uniform and irreducible. Such lattices are all arithmetic, and as a corollary, we obtain the same result as above for *real* quadratic fields.

1.2. Statement of results

A *number field* is a finite-degree field extension of $\mathbf{Q}$. To save words, we will always take non-trivial extensions of $\mathbf{Q}$. Given such a number field, K, with corresponding ring $\mathcal{O}$ of algebraic integers, the finitely generated group $\mathrm{PSL}_2(\mathcal{O})$ is an irredicible lattice in $\mathrm{Is}(X_K)$. Here X_K is a product of two- and three-dimensional hyperbolic spaces. $\mathrm{Comm}(\mathrm{PSL}_2(\mathcal{O}), X_K)$ contains $\mathrm{PGL}_2(K)$ as a normal subgroup of finite index. It is the purpose of this paper to prove

THEOREM 1 (Main Theorem). *Let K be a number field, and let $\mathcal{O}$ be the corresponding ring of algebraic integers. Then* $(\mathrm{PSL}_2(\mathcal{O}), X_K)$ *is quasi-isometrically rigid.*

As an immediate corollary, we have the following "computation" of the quasi-isometry group:

COROLLARY 1.2. *Let K be a number field, and let $\mathcal{O}$ be the corresponding ring of algebraic integers. Then* $\mathcal{Q}(\mathrm{SL}_2(\mathcal{O}))$ *contains* $\mathrm{PGL}_2(K)$ *as a normal subgroup of finite index.*

This result allows us to state the common generalization of the two classification results mentioned above:

COROLLARY 1.3. *Let K_1 and K_2 be two number fields. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the corresponding rings of algebraic integers. Then* $\mathrm{SL}_2(\mathcal{O}_1)$ *and* $\mathrm{SL}_2(\mathcal{O}_2)$ *are quasi-isometric if and only if K_1 and K_2 are isomorphic fields.*

The proof of the Main Theorem comes in two parts. (See §3 for a detailed overview.) The first part (§3.1, Boundary Detection Theorem) is a generalization of [FS, Boundary Detection Theorem]. The second part (§3.2, Action Rigidity Theorem) is a kind of rigidity result for the orbit structure of linear Abelian group actions on $\mathbf{Z}^n$. This part is entirely new, and comprises the heart of this paper. The proof of the Action Rigidity Theorem consists mainly in a kind of Diophantine analysis—hence the title of this paper.

At this point the central conjecture([1]) about the quasi-isometric rigidity of irreducible non-uniform lattices is

CONJECTURE 1.4. *Let $X \neq \mathbf{H}^2$ be a symmetric space of non-positive curvature, and let $\Gamma \subset \mathrm{Is}(X)$ be any non-uniform irreducible lattice. Then (Γ, X) is quasi-isometrically rigid.*

Acknowledgments. I would like to thank Benson Farb for originally getting me interested in quasi-isometries. I would also like to thank Alex Eskin, Jonathan Rosenberg, and Garrett Stuck for helpful conversations relating to this paper. Finally, I would like to thank the University of Maryland for their hospitality during the writing of this paper.

2. Background

In this preliminary chapter, we introduce some background material which will be used throughout the text.

([1]) As of this printing (a year and a half after the writing of this paper) Alex Eskin has pretty much proved this conjecture in its entirety. He says that his methods could quite possibly give an independent proof of the Main Theorem in this paper.

2.1. Quasi-isometries

Let (M,d) be a metric space, with metric d. A subset $N\subset M$ is said to be a *K-net* if every point of M is within K of some point of N. A *K-quasi-isometric embedding* of (M,d) into (M',d') is a map $q: N\to M'$ such that

(1) N is a K-net in M,

(2) $d'(q(x),q(y))\in[d(x,y)/K-K, Kd(x,y)+K]$, for $x,y\in N$.

q is said to be a *K-quasi-isometry* if, in addition, $N'=q(N)$ is a K-net in M'. In this case, the two metric spaces (M,d) and (M',d') are said to be *K-quasi-isometric*. When the choice of K is not important, we will drop it.

Two quasi-isometric embeddings (or quasi-isometries) $q_1,q_2: M\to M'$ are said to be *equivalent* if there are constants C_1 and C_2 having the following properties:

(1) Every point of N_1 is within C_1 of N_2, and vice versa.

(2) If $x_j\in N_j$ are such that $d(x_1,x_2)\leqslant C_1$, then $d'(q_1(x_1),q_2(x_2))\leqslant C_2$.

Modulo equivalence, the quasi-isometries of M form a group, which we call the *quasi-isometry group* of M.

We say that a net $N\subset M$ is *sparse* if no two points of N are within 1 unit of each other. We say that a quasi-isometry between metric spaces is sparse if it is given by a bi-Lipschitz bijection between sparse nets. Clearly, every quasi-isometry is equivalent to a sparse quasi-isometry.

2.2. Diameter distortion

For $j=1,2$, let S_j be a set. Let ΣS_j be the collection of finite subsets of S_j. Let δ_j: $\Sigma S_j\to[0,\infty)$ be any function. We say that a mapping $\phi: S_1\to S_2$ is *α-quasi-adapted* to the pair (δ_1,δ_2) if there is some function $\alpha: \mathbf{N}\to\mathbf{N}$ having the properties

$$\delta_1(V)\leqslant k \Rightarrow \delta_2(\phi(V))\leqslant\alpha(k),$$
$$\delta_2(\phi(V))\leqslant k \Rightarrow \delta_1(V)\leqslant\alpha(k).$$

Here $V\subset S$ is any compact subset. We will call α the *distortion function*.

As a special case, suppose that S_j is a metric space, and δ_j is the diameter function. In this case, we will say that ϕ is *uniformly proper* if it has the properties mentioned above. Any quasi-isometric embedding is a uniformly proper map, with a linear distortion function.

Suppose that $S_1=S_2$. We say that δ_1 and δ_2 are *γ-quasi-identical* if the identity map is γ-quasi-adapted to (δ_1,δ_2). Briefly, this means that $\delta_1(V)$ is "small" if and only if $\delta_2(V)$ is "small", for any finite subset V.

2.3. Word metric

Let G be a finitely generated group, and let S be a finite, and symmetric, generating set for G. To say that S is symmetric is to say that $s \in S$ if and only if $s^{-1} \in S$. The *word metric* d_S on G, relative to S, is defined as follows: $d_S(g_1, g_2)$ is the minimum number of generators in S required to write the word $g_1 g_2^{-1}$. It is easy to see that different finite generating sets produce Lipschitz equivalent (and hence quasi-isometric) metric spaces.

The criterion of Milnor–Svarc gives a way to understand the word metric in geometric terms:

PROPOSITION 2.1 (Milnor–Svarc criterion). *Suppose that G is a finitely generated group. Suppose that G acts freely, properly, and faithfully on a path metric space X. Suppose that X/G is compact. Then G and X are canonically quasi-isometric.*

Proof. Let $x \in X$ be any point. The map $g \to g(x)$ is easily seen to be a quasi-isometry between G and X. The equivalence class of quasi-isometry is canonical; it doesn't depend on the choice of x. □

2.4. Linear groups

Let $\Lambda \subset \mathbf{R}^n$ be a lattice. We let $\mathrm{Aut}(\Lambda)$ and $\mathrm{SL}(\Lambda)$ respectively denote the affine and linear automorphism group of Λ. Likewise we let $\mathrm{Aut}(\mathbf{Q}\Lambda)$ and $\mathrm{SL}(\mathbf{Q}\Lambda)$ denote the affine and linear automorphism groups of the $\mathbf{Q}$-linear span $\mathbf{Q}\Lambda$. Note that $\mathrm{SL}(\Lambda) \subset \mathrm{SL}_n(\mathbf{R})$.

Let $G \subset \mathrm{SL}(\Lambda)$ be Abelian. G is said to be *semisimple* if

(1) G is isomorphic to $\mathbf{Z}^d$, for some $d \geqslant 1$,

(2) the elements G are simultaneously diagonalizable over $\mathbf{C}$.

We say that G is *virtually semisimple* if G contains a finite index subgroup which is semisimple. We say, additionally, that G is *minimal* if *no* finite index subgroup of G stabilizes a non-trivial infinite index sublattice of Λ. This is equivalent to the condition that the induced action of G on the torus $\mathbf{R}^n/\Lambda$ is ergodic.

Given any affine map T, we say that the *linear part* of T is the linear transformation T_0 which agrees with T up to translations. The map $T \to T_0$ is clearly a homomorphism. This allows us to speak, as we frequently will, about the linear part $G_0 \subset \mathrm{SL}(\Lambda)$ of a subgroup $G \subset \mathrm{Aut}(\Lambda)$.

2.5. Adapted inner products

Let $G \subset \mathrm{SL}_n(\mathbf{R})$ be an Abelian semisimple subgroup. Let $E_1, \ldots, E_m$ be a maximal collection of simultaneous eigenspaces for elements of T. By maximal, we mean that the eigenspaces on this list do not further decompose into products of lower-dimensional eigenspaces. In particular, $\dim(E_j)\leqslant 2$ and $\dim(E_j)=2$ if and only if some $T\in G$ has a non-real eigenvalue corresponding to E_j.

LEMMA 2.2. *Let E be one of the simultaneous eigenspaces of G. There is an inner product on E such that $G|_E$ acts by similarities.*

Proof. The one-dimensional case is obvious. The two-dimensional case is a standard exercise in linear algebra, so we remain a bit sketchy. Passing to a finite index subgroup, we may assume that G is free Abelian. If the lemma was false, then some element $g\in G$ would act hyperbolically on E. The two distinct eigenspaces of g, in E, would also be eigenspaces for the other elements in G, since G is free Abelian. This contradicts maximality. □

We say that an inner product $\mathcal{G}$ is *adapted to G* if

(1) $G|_{E_i}$ acts by similarities relative to $\mathcal{G}|_{E_i}$,

(2) distinct eigenspaces are orthogonal with respect to $\mathcal{G}$.

Such an inner product is easy to construct by piecing together the inner products guaranteed by the preceding lemma. Whenever a G-action is present, we will make metric constructions relative to an adapted inner product.

2.6. Horoballs and symmetric spaces

Let X be a symmetric space of non-positive curvature. We say that a *horoball* of X is the Hausdorff limit of unboundedly large metric balls, provided that this limit exists, and is not all of X. We say that two horoballs are *parallel* if they arise as limits of concentric balls.

We say that a *horosphere* is the boundary of a horoball. Each horosphere in X has a Riemannian metric induced from X by restriction. We will always equip horospheres with this metric. Given this metric, a horosphere is a homogeneous space, though it need not have non-positive curvature.

As a special case, let

$$X_{p,q}=\mathbf{H}^2\times\ldots\times\mathbf{H}^2\times\mathbf{H}^3\times\ldots\times\mathbf{H}^3$$

be the product of p hyperbolic planes and q hyperbolic 3-spaces. Define

$$\partial_0 X_{p,q}=\partial\mathbf{H}^2\times\ldots\times\partial\mathbf{H}^2\times\partial\mathbf{H}^3\times\ldots\times\partial\mathbf{H}^3.$$

This space is sometimes called the *Furstenberg boundary.*

If we use the upper half-space models, then we have

$$\partial_0 X_{p,q} = \mathbf{R}^{p+2q} \cup \infty \cup S_{p,q},$$

where $\infty=(\infty, ..., \infty)$, and $S_{p,q}$ is a union of lower-dimensional Euclidean spaces. $\mathrm{Is}(X_{p,q})$ acts transitively on $\partial_0 X_{p,q}$. The stabilizer of ∞ acts on $\mathbf{R}^{p+2q}$ by linear transformations.

We will use the coordinates (x_i, t_i) and (ξ_j, τ_j) respectively on $\mathbf{H}^2$ and $\mathbf{H}^3$. Here $x_i \in \mathbf{R}$, $\xi_j \in \mathbf{R}^2$ and $t_i, \tau_j \in (0, \infty)$. Let $h_\infty \subset X_{p,q}$ denote the subset given by

$$\prod_{i=1}^{p} \prod_{j=1}^{q} t_i \tau_j^2 \geqslant 1.$$

h_∞ is a horoball in $X_{p,q}$. Indeed, it is not hard to check that

$$h_\infty = \lim_{r \to \infty} B_{r\sqrt{p+4q}}(p_r),$$

where

$$p_r = ((0, e^r), ..., (0, e^r), (0, e^{2r}), ..., (0, e^{2r})).$$

We say that a *diagonal horoball* of $X_{p,q}$ is any image of h_∞ under an element of $\mathrm{Is}(X_{p,q})$. The diagonal horoball $h=T(h_\infty)$ uniquely determines the point

$$\partial_0 h = T(\infty) \in \partial_0(X_{p,q}).$$

We will say that h is *based* at $\partial_0 h$. More generally, given any collection H of diagonal horoballs, we define

$$\partial_0 H = \{\partial_0 h \mid h \in H\}.$$

Note that parallel diagonal horoballs are based at the same point of $\partial_0 X_{p,q}$. We say that two diagonal horoballs h_1 and h_2 are *totally distinct* if $\partial_0 h_1$ and $\partial_0 h_2$ do not agree at any coordinate. The following facts are easy to check.

(1) Let h_1 and h_2 be totally distinct diagonal horoballs. Then $\partial h_1 - h_2$ is path connected, as long as $(p, q) \neq (1, 0)$.

(2) Let h_1 and h_2 be totally distinct diagonal horoballs. Then there are disjoint horoballs h_1' and h_2' which are parallel respectively to h_1 and h_2.

(3) The set of points in $X_{p,q}$ within n units from three pairwise disjoint, and totally distinct, diagonal horoballs has diameter at most n'. The function $n \to n'$ can be taken independently of the choice of horoballs.

2.7. Modular groups

Let $K \neq \mathbf{Q}$ be a number field. To avoid certain exceptional cases, we will assume that K is not an imaginary quadratic field. Let $\mathcal{O}$ be the subset of K whose elements are solutions of monic integer polynomials. $\mathcal{O}$ is called the *ring of algebraic integers*.

In general, there are p distinct Galois embeddings $\varrho_1, \ldots, \varrho_p \colon K \to \mathbf{R}$, and $2q$ distinct Galois embeddings $\sigma_1, \ldots, \sigma_{2q} \colon K \to \mathbf{C}$. The image $\sigma_j(K)$ is not contained in $\mathbf{R}$. The complex embeddings come in pairs. Each has a conjugate embedding. We shall order these embeddings so that no two complex embeddings in the first half of the sequence are conjugate to each other. We define the product embedding

$$\Pi = \varrho_1 \times \ldots \times \varrho_p \times \sigma_1 \times \ldots \times \sigma_q.$$

We let $\Lambda, \mathbf{Q}\Lambda \subset \mathbf{R}^n$ be the images respectively of $\mathcal{O}$ and K under Π. The notation here reflects the fact that $\mathbf{Q}\Lambda$ is the $\mathbf{Q}$-linear span of Λ.

Given any element $M \in \mathrm{PSL}_2(\mathcal{O})$, we define $\varrho_j(M)$ and $\sigma_j(M)$ in the obvious, entry-by-entry, way. Thus, this extended version of Π gives us a faithful representation of $\mathrm{PSL}_2(\mathcal{O})$ onto a subgroup $\Gamma \subset \mathrm{Is}(X_{p,q})$. When $q=0$, such a group is called a *Hilbert modular group*.

It is easy to see that Γ is discrete. It is well known that Γ is an irreducible non-uniform lattice. (See [Se] in general, and [Ge, pp. 1–11] for a concise proof in the totally real case.)

The *cusps* or fixed points of Γ in $\partial_0 X_{p,q}$ are clearly bijective with $Q\Lambda \cup \infty$. The stabilizer Γ_∞ of ∞ consists exactly of images of upper triangular matrices of the form

$$T = \begin{bmatrix} u & o \\ 0 & 1/u \end{bmatrix},$$

where u is a unit in $\mathcal{O}$ and $o \in \mathcal{O}$ is arbitrary. To say that u is a unit is to say that the product of Galois conjugates of u is ± 1. From this, an easy calculation shows that Γ_∞ preserves the diagonal horoballs based at ∞.

Clearly, $\Gamma_\infty \subset A(\Lambda)$. The linear part L of Γ_∞ consists exactly of the diagonal matrices. L is clearly Abelian. The elements of L are obviously simultaneously diagonalizable over $\mathbf{C}$. Dirichlet's unit theorem [O, p. 118] says that L is virtually semisimple.

LEMMA 2.3. *L is minimal.*

Proof. If L is not minimal, then we can find a free Abelian finite index subgroup $L' \subset L$ which preserves some infinite index sublattice $\Lambda' \subset \Lambda$. Since Λ' has infinite index, it has lower rank. Furthermore, Λ' must be contained in a product of eigenspaces of L'. Putting both statements together, we see that Λ' is contained in a proper sub-product of eigenspaces of L'. Now, the eigenspaces of L' coincide with those of L. These eigenspaces are all coordinate hyperplanes, of various dimensions. Putting everything together, we see that some Galois conjugate of any non-trivial element of Λ' is zero. This is a contradiction. $\square$

It is easy to verify, that the commensurator $\mathrm{Comm}(\Gamma, X_{p,q})$ contains $\mathrm{PGL}_2(K)$ as a subgroup of finite index. In particular, the commensurator acts transitively on the cusps. Hence, the stabilizer subgroup $\Gamma_p \subset \Gamma$ of a point $p \in \mathbf{Q}\Lambda \cup \infty$ preserves the diagonal horospheres based at p, and acts on each of these with compact quotient.

Let H be a maximal Γ-invariant family of diagonal horoballs. The maximality condition implies that

$$\partial_0 H = \mathbf{Q}\Lambda \cup \infty.$$

(Recall that this first set is the collection of basepoints of horoballs.) From the definition of $\mathbf{Q}\Lambda$, and from the fact that Γ is a lattice, we have:

(1) the horoballs in H are pairwise totally distinct,

(2) H/Γ is a finite set,

(3) Γ acts with compact quotient on ∂h, for each $h \in H$.

These three properties imply that we can replace the horoballs in H by parallel horoballs (in an equivariant way) so that the horoballs in H are pairwise disjoint.

3. Overview of the proof

3.1. Detecting boundary components

Let X be a symmetric space of non-positive curvature. Let H be a non-empty family of disjoint horoballs in X. Let $\Omega(H) \subset X$ denote the closure, in X, of the complement of horoballs in H. Note that $\partial\Omega(H)$ consists of a collection of disjoint codimension-one boundary components, one per member of H. We call $\Omega(H)$ a *neutered space*. We equip Ω with the path metric induced from the ambient Riemannian metric on X.

We say that $\Omega(H)$ has *unobtrusive boundaries* if the following condition holds: For any element $h \in H$ and any positive real number r the closure of $\Omega(H) - T_r(h)$ is path connected. Here $T_r(h)$ is the r-tubular neighborhood of $h \in H$, defined relative to the Riemannian metric on X.

Example. Let $X = X_{p,q}$ be the space discussed in §2.6. Let $\Omega(H) \subset X$ be a neutered space, where H consists of pairwise totally distinct diagonal horoballs. If $(p,q) \neq (1,0)$ then $\Omega(H)$ has unobtrusive boundaries. This follows almost immediately from property (1), listed in §2.6.

We say that a quasi-isometry $q:\Omega(H_1)\to\Omega(H_2)$ *pairs boundaries* if there is a bijection $\phi: H_1\to H_2$ having the following property: If $x\in\Omega(H_j)$ is within m units of $h\in H_j$, and if q is defined on x, then $q(x)\in\Omega(H_{j+1})$ is within m' units of $\phi(h)$. Here, indices have been taken modulo 2, and ϕ has deliberately been confused with its inverse.

In §4 and §5 we will prove

THEOREM 3.1 (Boundary Detection Theorem). *Let X be a symmetric space of non-positive curvature. For $j=1,2$, let $\Omega(H_j)\subset X$ be a neutered space with unobtrusive boundaries. Any quasi-isometry $q:\Omega(H_1)\to\Omega(H_2)$ pairs boundaries.*

3.2. Action rigidity

Let $\Lambda\subset\mathbf{R}^n$ be a lattice, and let $G\subset\mathrm{Aut}(\Lambda)$ be a subgroup. For any compact subset $S\subset\mathbf{R}^n$, we define

$$\delta_G(S)=\inf_{T\in G}\mathrm{diam}(T(S)).$$

Here, diameter is defined relative to an adapted inner product. (For our coarse purposes, any inner product would do.) Note that $\delta_G=\delta_{G_0}$, where G_0 is the linear part of G.

We say that a subset $S\subset\mathbf{Q}\Lambda$ has *bounded height* if $S\subset(1/d)\Lambda$. We say that a bijection $\phi:\mathbf{Q}\Lambda\to\mathbf{Q}\Lambda$ is *quasi-integral* if both ϕ and ϕ^{-1} take sets of bounded height to sets of bounded height. We say that a bijection $\phi:\mathbf{Q}\Lambda\to\mathbf{Q}\Lambda$ is *quasi-compatible with G* if

(1) ϕ is quasi-integral,

(2) for any $S\subset\mathbf{Q}\Lambda$ having bounded height, $\phi|_S$ and $\phi^{-1}|_S$ are quasi-adapted to δ_G (see §2.2 for a definition).

Let $\mathcal{QC}(G,\Lambda)$ denote the set of such mappings.

The following result, which we prove in §§ 6–8, is at the heart of this paper.

THEOREM 3.2 (Action Rigidity Theorem). *Supose that $\Lambda\subset\mathbf{R}^n$ is a lattice. Let $G\subset\mathrm{Aut}(\Lambda)$ be a subgroup whose linear part is Abelian, minimal and virtually semisimple. Then every element of $\mathcal{QC}(G,\Lambda)$ is the restriction of an affine map.*

Remark. Our notation $\mathcal{QC}$ is a bit of a pun. This set of mappings (formally) plays the same role here as the set of quasi-conformal mappings plays in [Sch].

3.3. Sufficient properties

In this section, we list the properties of the groups $\mathrm{PSL}_2(\mathcal{O})$ which we actually use in the proof of the Main Theorem. The facts here are distillations of the discussion in §2.6 and §2.7.

Let X be a (non-trivial) product of hyperbolic spaces, and let $\Gamma \subset \mathrm{Is}(X)$ be a non-uniform lattice. We normalize so that ∞ is a cusp of Γ. Let $\Gamma_\infty \subset \Gamma$ be the stabilizer of this cusp.

PROPERTY 3.3. *Γ preserves a neutered space $\Omega(H) \subset X$, and $\Omega(H)/\Gamma$ is compact. Moreover, $\Omega(H)$ has unobtrusive boundaries.*

PROPERTY 3.4. *There is a lattice $\Lambda \subset \mathbf{R}^n$ such that $\partial_0 H = \mathbf{Q}\Lambda \cup \infty$. Furthermore, $\Gamma_\infty \subset \mathrm{Aut}(\Lambda)$ has linear part which is Abelian, virtually semisimple, and minimal.*

PROPERTY 3.5. *The commensurator $\mathrm{Comm}(\Gamma, X)$ acts transitively on $\partial_0 H$. The stabilizer subgroup $\mathrm{Comm}_\infty(\Gamma)$ of ∞ contains $\mathrm{Aut}(\mathbf{Q}\Lambda)$.*

PROPERTY 3.6. *Let $m \in \mathbf{N}$ be arbitrary. The set of points in X which are at most m units from three distinct horoballs of $\Omega(H)$ has diameter at most m'. The function $m \to m'$ does not depend on the choice of horoballs.*

3.4. Compatibility

Let $h_\infty \in H$ be the horoball corresponding to ∞. Let $\partial_0^d H \subset \mathbf{Q}\Lambda$ denote those points which correspond to horoballs which are metrically no more than d units from h_∞. Also define $\Lambda_d = (1/d)\Lambda$.

Given any finite subset $S \subset \mathbf{Q}\Lambda$, define $\delta_H(S)$ to be the diameter of the metric ball of minimal size which intersects all horoballs of H which are based at points of S. We would like to compare this function with the function δ_{Γ_∞}, associated to the group action $\Gamma_\infty \subset \mathrm{SL}(\Lambda)$. For ease of notation, we will abbreviate this latter function to δ_Γ.

LEMMA 3.7. *For every $d \in \mathbf{N}$,*

(1) *$\partial_0^d H \subset \Lambda_{d'}$,*

(2) *$\Lambda_d \subset \partial_0^{d'} H$,*

(3) *the restrictions of δ_Γ and δ_H to Λ_d are quasi-identical.*

Proof. Property 3.3 says that Γ_∞ acts on ∂h_∞ with compact quotient. Hence, modulo Γ_∞, there are only finitely many horoballs of H within d units of h_∞. Similarly, the same compactness implies that Λ_d/Γ_∞ is a finite set. Statements (1) and (2) follow immediately. We now prove statement (3). Suppose first that $\delta_H(S)$ is small for some collection $S \subset \Lambda_d$. This means that there is a point x close—say, within K—to all of the horospheres whose basepoints belong to S. We may apply an isometry of the symmetric space to guanantee that x is some prechosen origin in the symmetric space. By compactness/discreteness, there are only finitely many horoballs, within K of x. Hence, modulo

$\mathrm{Aut}(\Lambda)$, there are only finitely many choices for S. This bounds $\delta_\Gamma(S)$. Suppose conversely, that $\delta_\Gamma(S)$ is small. Then, by compactness/discreteness, there are only finitely many possible choices of $S\subset\Lambda^d$ modulo the affine group $\mathrm{Aut}(\Lambda)$. This immediately gives the bound on $\delta_H(S)$. □

3.5. Putting everything together

We will now prove the Main Theorem for any pair (Γ, X) which satisfies the properties listed in §3.3. Note that lattices in $X=\mathbf{H}^3$ do not satisfy Property 3.4. However, these lattices were treated in [Sch].

Let $q:\Gamma\to\Gamma$ be a quasi-isometry of Γ.

Step 1. Let $\Omega(H)$ be the neutered space given in Property 3.3. Since $\Omega(H)/\Gamma$ is compact, it follows from the criterion of Milnor–Svarc that Γ and $\Omega(H)$ are canonically quasi-isometric. In this way, we replace q by the induced quasi-isometry of $\Omega(H)$, which we give the same name.

Step 2. Property 3.3 says that $\Omega(H)$ has unobtrusive boundaries. The Boundary Detection Theorem now says that q pairs boundaries. Consider the bijection $\partial_0 q: \partial_0 H\to\partial_0 H$ induced by q. Property 3.5 says that, after composing with elements of $\mathrm{Comm}(\Gamma)$, we can assume that $\partial_0 q(\infty)=\infty$.

Step 3. Since q is a quasi-isometry, it follows from Lemma 3.7 that $\partial_0 q\in\mathcal{QC}(\Gamma_\infty,\Lambda)$. From Property 3.4, and the Action Rigidity Theorem, $\partial_0 q$ is the restriction of some affine map T_q. Since T_q permutes the elements of $\mathbf{Q}\Lambda$, we have $T_q\in\mathrm{Aut}(\mathbf{Q}\Lambda)$.

Step 4. From Property 3.5 we can assume, after composing with an element of $\mathrm{Comm}_\infty(\Gamma)$, that $T_q=\mathrm{Id}$. This is to say that q preserves each and every boundary component of $\Omega(H)$. It now follows from Property 3.6 that q is equivalent to the identity.

This completes the proof of the Main Theorem.

3.6. Corollaries

The first corollary of the Main Theorem is immediate. We now prove the second corollary.

Suppose that K_1 and K_2 are number fields, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are the corresponding rings of algebraic integers. Suppose that $\mathrm{SL}_2(\mathcal{O}_1)$ and $\mathrm{SL}_2(\mathcal{O}_2)$ are quasi-isometric. Then $\mathcal{Q}(\mathrm{SL}_2(\mathcal{O}_1))$ and $\mathcal{Q}(\mathrm{SL}_2(\mathcal{O}_2))$ are isomorphic groups. From the Main Theorem, and our explicit knowledge of the commensurators, we see that $\mathrm{PGL}_2(K_1)$ and $\mathrm{PGL}_2(K_2)$ are finite index normal subgroups of isomorphic groups. Since $\mathrm{PGL}_2(K_1)$ and $\mathrm{PGL}_2(K_2)$

have no finite index normal subgroups, we see that $\mathrm{PGL}_2(K_1)$ and $\mathrm{PGL}_2(K_2)$ are themselves isomorphic groups. It is an easy exercise to show that this implies that K_1 and K_2 are isomorphic fields.

4. Boundary Detection Theorem

In this chapter, we prove Theorem 3.1 (Boundary Detection Theorem), modulo the central technical result, the Coarse Separation Theorem. We will prove the Coarse Separation Theorem in §5. The core material of §4 and §5 also appears in [FS]. However, the presentation here is substantially more general in places.

4.1. Coarse separation

We say that a subset $S\subset Y$, not necessarily connected, is *deep* if it contains arbitrarily large metric balls. We say that two subsets $S_1, S_2\subset Y$ are *disconnected from each other* if it *never* happens that $S_1'\cup S_2'$ is a connected set, for non-empty $S_j'\subset S_j$. Finally, we say that a subset $Z\subset Y$ *coarsely separates in* Y provided that, for sufficiently large r, the complement $Y-T_r(Z)$ contains at least two deep subsets which are disconnected from each other. In §5 we will prove

THEOREM 4.1 (Coarse Separation Theorem). *Suppose that X and Y are symmetric spaces of non-positive curvature. Suppose that J is a horosphere in X. Suppose that $q\colon J\to Y$ is a uniformly proper map. Then $q(J)$ coarsely separates in Y.*

4.2. Coarsely derived maps

Let X be a symmetric space, as above. Let $\Omega=\Omega(H)\subset X$ be a neutered space. If $h\in H$ is a horoball, then the horosphere ∂h is a boundary component of Ω.

Suppose that $\Omega_j=\Omega(H_j)$ are two neutered spaces in X, and $q\colon \Omega_1\to\Omega_2$ is a sparse K-quasi-isometry. This is to say that q is a K-bi-Lipschitz bijection between sparse K-nets $N_j\subset\Omega_j$. Let $h\subset H_i$ be a horoball.

We say that a map $\phi\colon \partial h\to X$ is *K-coarsely derived from q* if there is a constant K having the following property: For each point $x\in\partial h$ there is a point $x'\in N_j$ such that $d_X(x,x')\leqslant C$, and $\phi(x)=q(x')$.

LEMMA 4.2. *Suppose that $\phi\colon \partial h\to X$ is coarsely derived from q. Then $\phi(\partial h)$ coarsely separates in X.*

Proof. It follows from convexity that the inclusion $\partial h \to \Omega_i$ is an isometry. Hence, the map $x \to x'$ is uniformly proper. Here $x' \in \Omega_i$ is the point for which $q(x')=\phi(x)$. It follows from compactness and symmetry that the inclusion $\Omega_{i+1} \to X$ is a uniformly proper map. Since ϕ is the composition of uniformly proper maps, it is also uniformly proper. Thus $\phi: \partial h \to X$ is a uniformly proper map. Now apply the Coarse Separation Theorem. ☐

4.3. Images of horospheres

Assume now that $\Omega_j = \Omega(H_j)$ is a neutered space with unobtrusive boundaries.

LEMMA 4.3 (Tubular Neighborhood Lemma). *Let $q: \Omega_1 \to \Omega_2$ be as above. Let $h_1 \in H_1$ be a horoball. Let $\phi: \partial h_1 \to X$ be a map which is K-coarsely derived from q. Then there is a horoball $h_2 \in H_2$ such that $\phi(\partial h_1)$ is contained in a tubular neighborhood of h_2. The size of this tubular neighborhood is independent of the choice of h_1.*

Proof. As a first step, we prove "one half" of the desired result. Namely,

SUBLEMMA 4.4. *There exists a constant C and a horoball $h_2 \in H_2$ such that every point of h_2 is within C units of some point of $\phi(\partial h_1)$.*

Proof. If this lemma is false, then for each $h \in H_2$, and each constant R, there are points $p \in \partial h$ which remain at least R units from $\phi(\partial h_1)$. We call this property $(*)$. We will use property $(*)$ to show the following. Let N be any constant. Then there is a constant U such that two points in X which avoid $\phi(\partial h_1)$ by at least U units can be joined together by a path which avoids $\phi(\partial h_1)$ by at least N units. This contradicts the fact that $\phi(\partial h_1)$ coarsely separates in X.

Let T_N be the N-tubular neighborhood of $\phi(\partial h_1)$. Provided that $U > N$, the points a and b do not belong to T_N. Suppose first that $a \notin \Omega_2$. Then a is contained in some horoball $h \in H_2$. From property $(*)$ we may find a point $a' \subset \partial h$ and a path joining a to a' which avoids T_N, provided that U is taken large enough. Thus, we may assume that a and b both belong to Ω_2. Also, if we take U large enough, then we can assume that a and b actually belong to the net N_2.

Below, the constants $U_1, U_2, \ldots$ tend to ∞ with U. The points $q^{-1}(a)$ and $q^{-1}(b)$ belong to N_1. If U is taken sufficiently large, then these two points will be at least U_1 from h_1. Since Ω_1 has unobtrusive boundaries, there is a path γ connecting these two points and remaining at least U_2 from h_1. Using approximation and geodesic interpolation, it makes sense to talk about the path $q(\gamma)$ which connects a to b. Since q is a quasi-isometry, $q(\gamma)$ will remain U_3 from $\phi(\partial h_1)$. Once $U_3 > N$, we have a path connecting a to b but avoiding T_N. ☐

We define a map $\psi: \partial h_2 \to \partial h_1$ as follows. Let $\psi(x) \in \partial h_1$ be any point such that $\phi(\psi(x))$ is metrically closest to x. From Lemma 4.4, it follows that ψ is coarsely derived from q^{-1}. It follows that $\psi(\partial h_2)$ coarsely separates in X. Since $\psi(\partial h_2) \subset \partial h_1$, this separation property implies that $\psi(\partial h_2)$ is C'-dense in ∂h_1, for some constant C'. This is to say that every point of ∂h_1 is within C' of some point which maps, via ϕ, to a point within C of ∂h_2. If we modify the constants, we get the Tubular Neighborhood Lemma from this statement.

Since we are working in a symmetric space, and all the horoballs are isometric, the constants in the proof above can be taken independent of the horoball in question. □

Applying the Tubular Neighborhood Lemma to both q and q^{-1}, we get the Boundary Detection Theorem.

5. Coarse separation

5.1. Uniform contractibility

A finite-dimensional Riemannian manifold M is said to be *uniformly contractible* if there is a function $\alpha: \mathbf{N} \to \mathbf{N}$ having the following property: If a continuous map of a finite simplicial complex $\Delta \to M$ is contained in an r-ball, then it is contractible in an $\alpha(r)$-ball.

The following lemma is obvious.

LEMMA 5.1. *Let X be a symmetric space of non-positive curvature. X is uniformly contractible.*

5.2. Unpinched spheres

Let M be a smooth Riemannian $(n+1)$-manifold. Let S^n denote the standard unit sphere of Euclidean space $\mathbf{E}^{n+1}$. Fix, once and for all, some small constant $\delta_0 = \frac{1}{100}\pi$. (The precise value of δ_0 is rather arbitrary.) We say that a smoothly embedded sphere $\Sigma \subset M$ is *d-unpinched* if there is a diffeomorphism $h: \Sigma \to S^n$ such that the spherical distance from $h(x)$ to $h(y)$ is less than δ_0 provided that $\mathrm{dist}_M(x,y) < d$. Note that dist_M is *not* the path metric on Σ, but rather the ambient metric in M. We say that M has *unpinched spheres* if M contains a d-unpinched sphere for every positive integer d.

At the end of this chapter, we will prove

LEMMA 5.2. *Let X be a symmetric space of non-positive curvature. Horospheres in X have unpinched spheres.*

Remark. Having unpinched spheres seems to be a mild condition. I suspect that all uniformly contractible manifolds, diffeomorphic to $\mathbf{R}^n$, have unpinched spheres. However, I have no idea how to prove this.

5.3. General Coarse Separation Theorem

In this chapter we will prove

THEOREM 5.3 (General Coarse Separation Theorem). *Suppose that J and Y are smooth Riemannian manifolds, diffeomorphic to $\mathbf{R}^n$ and $\mathbf{R}^{n+1}$ respectively. Suppose that*

(1) *$q: J \to Y$ is a uniformly proper map,*

(2) *J is a path metric space, and has unpinched spheres,*

(3) *Y is uniformly contractible.*

Then $q(J)$ coarsely separates in Y.

The Coarse Separation Theorem of §4.1 follows from this general result, together with Lemma 5.1 and Lemma 5.2.

5.4. Continuous extension

Let T be a locally finite triangulation of J. Subdividing, we can guarantee that there is an upper bound to the size of simplices in T. (Note, however, that this triangulation need not have uniform geometry, in any sense.) Let $T^{(k)}$ be the k-skeleton of T. By altering q trivially, we can assume that $q=q_0$ is defined on the zero skeleton $T^{(0)}$. Inductively, we extend q_k from $T^{(k)}$ to $T^{(k+1)}$. The fact that the simplices of T have uniformly bounded diameter, and that Y is uniformly contractible, implies that the extension q_{k+1} is still uniformly proper. By induction, then, the final map q_n is continuous and defined on all of J, and is still a uniformly proper map. Also, we have $T_r(q(J)) \supset q_n(J)$ for some $r>0$. Thus, it is sufficient to prove the Separation Theorem for q_n. Replacing q by q_n, we henceforth assume that q is defined on all of J, and continuous.

5.5. Images of unpinched spheres

For any space M, let $Z_q(M)$ denote singular q-cycles of M. Let $H_q(M)=H_q(M;\mathbf{Z})$ be the singular homology classes of M. Suppose that $p+q=n-1$. Given two cycles $z_p \in Z_p(\mathbf{R}^n)$ and $z_q \in Z_q(\mathbf{R}^n)$, they have an *algebraic linking number*, called $\operatorname{link}(z_p, z_q)$. This is defined in the usual way: Pick a generic coboundary b_{p+1} such that $\partial b_{p+1} = z_p$, and count the algebraic intersection number $b_{p+1} \cap z_q$.

LEMMA 5.4 (Linking Lemma). *For any constant k the following is true: There is a closed curve $\gamma \subset Y$, and a smoothly embedded sphere $\Sigma \subset J$ such that*

(1) *every point of γ remains at least k from $q(\Sigma)$,*

(2) $\mathrm{link}(q(\Sigma), \gamma) \neq 0$.

Proof. Let Σ be any embedded sphere. By choosing a triangulation of Σ, we consider $q(\Sigma)$ as a singular homology class in $H_n(T)$, where $T \subset Y$ is any subset containing $q(\Sigma)$.

SUBLEMMA 5.5. *Let Σ be an unpinched sphere in J. Let ϱ be a fixed constant. Let T be a compact subset such that*

$$q(\Sigma) \subset T \subset T_\varrho(q(\Sigma)).$$

Then there is a second constant r such that $q(\Sigma)$ is an infinite element of $H_n(T)$ provided that Σ is r-unpinched.

Proof. Let α denote the distortion function of q. We choose some number $r > \max(1, \alpha(0), \alpha(2k+1))$. If Σ is a torsion element of $H_n(T)$, then there is some $(n+1)$-chain β, with underlying simplicial complex $b = |\beta|$, having the following properties:

(1) $\beta(b) \subset T$.

(2) ∂b consists of $d > 0$ copies of Σ. Call these copies $c_1, \dots, c_d$.

(3) $\beta|_{c_j} = q|_\Sigma$.

Recall that there is a diffeomorphism $h\colon \Sigma \to S^n$, which takes J-metric r-balls into sets having $\delta_0 = \frac{1}{100}\pi$ spherical diameter. Below, we will construct a continuous map $\eta\colon b \to S^n$ such that $h^{-1} \circ \eta|_{\partial b}$ has positive degree. This is a contradiction.

By subdivision, we may assume that b has the following property: If two vertices $v_1, v_2 \in b$ belong to the same simplex, then the points $\beta(v_1)$ and $\beta(v_2)$ are at most one unit apart in Y. We first construct the map η from the vertices of b into S^n. Here is the formula:

(1) Choose a vertex $v \in b$.

(2) Let $v' \in \Sigma$ be any point such that $q(v')$ is as close to $\beta(v)$ as possible.

(3) Let $\eta(v) = h(v')$.

For two vertices v_1 and v_2 belonging to the same simplex, the points $\beta(v_1)$ and $\beta(v_2)$ are at most 1 apart. Since $\beta(b) \subset T$, the points $q(v_1')$ and $q(v_2')$ are at most $2k+1$ apart. Hence, the points v_1' and v_2' belong to the same $\alpha(2k+1)$-metric ball in J. Since $r > \alpha(2k+1)$, we have that $\eta(v_1)$ and $\eta(v_2)$ have spherical distance at most $\frac{1}{100}\pi$ in S^n. This property allows one to extend η, skeleta by skeleta, using spherical totally geodesic interpolations. The resulting map is defined and continuous on all of b.

We now show that $h^{-1} \circ \eta|_{\partial b}$ has positive degree. To show this, it is sufficient to show that η_{c_j} has degree one.

By subdivision, we can assume that c_j is trangulated by simplices which have diameter at most 1. Recall that c_j is actually a copy of Σ. Let $x \subset c_j = \Sigma$, be a vertex of b. By construction, $q(x)=q(x')$. Hence x and x' are at most $\alpha(0)$ units away in J. Since $r > \alpha(0)$, we have that $\eta(x)$ lies at most $\frac{1}{100}\pi$ from $h(x)$. Suppose that $y \in c_j$ is arbitrary. Let x be a vertex of a simplex containing y. Then, since $r>1$, the spherical distance between $h(x)$ and $h(y)$ is at most $\frac{1}{100}\pi$. By constuction, the spherical distance from $\eta(x)$ to $\eta(y)$ is at most $\frac{1}{100}\pi$. Putting these bounds together, we see that $\eta(y)$ and $h(y)$ lie at most $\frac{3}{100}\pi$ from each other. This easily implies that $\eta|_{c_j}$ has degree one. □

The following lemma is a standard result from algebraic topology.

SUBLEMMA 5.6 (Alexander duality). *Suppose that* $p+q=n-1$. *Suppose that* $A \subset \mathbf{R}^n$ *is a smooth compact manifold-with-boundary. Suppose that* $\xi_p \in H_p(A)$ *is an infinite element. Then there is some* $\xi_q \in H_q(\mathbf{R}^n - A)$ *such that* $\mathrm{link}(\xi_p, \xi_q) \neq 0$.

Let Σ be an r-unpinched sphere. For any constant k, we can find a smooth compact manifold-with-boundary T such that

$$T_k(q(\Sigma)) \subset T \subset T_{k+1}(q(\Sigma)).$$

If we choose r sufficiently large, then Lemma 5.5 says that $q(\Sigma)$ is not torsion in $H_n(T)$. It follows from Lemma 5.6 that there is some element of $H_1(Y-T)$ which links $q(\Sigma)$ algebraically, but avoids T. This element is represented by a cycle consisting of finitely many closed curves. One of these curves must also link $q(\Sigma)$. □

5.6. Curve modification

Let γ, Σ and k be as in the Linking Lemma. We define $I_\gamma \subset J$ to be the set of points $x \in J$ such that $q(x) \in \gamma$. In other words I_γ is the inverse image of the intersection $\gamma \cap q(J)$. Let B be the ball bounded by Σ. Clearly, the set I_γ does not intersect Σ. Say that a point of I_γ is *interior* if it belongs to B, and *exterior* otherwise.

If the constant k is sufficiently large, then for any $p \in \gamma \cap q(J)$, the *set* $q^{-1}(p)$ will either be entirely interior or entirely exterior. We make such a choice of k. Thus, we may unambiguously say that a point $p \in \gamma \cap q(J)$ is interior or exterior, depending on its inverse images under q.

LEMMA 5.7 (Curve Modification Lemma). *Suppose that* $q(J)$ *does* not *coarsely separate in* Y. *Then we can find a curve* δ *having the properties*

(1) $\mathrm{link}(q(\Sigma), \delta) \neq 0$,

(2) I_δ *either has only interior points, or only exterior points.*

Proof. Say that a *traversing arc* is a subarc of γ whose one endpoint is an interior point, and whose other endpoint is an exterior point.

SUBLEMMA 5.8. *Let ξ be any constant. Let $\gamma' \subset \gamma$ be a traversing arc. If k is sufficiently large, then there is a point $x \in \gamma'$ which is at least ξ from $q(J)$.*

Proof. The constants $\xi_1, \xi_2, \ldots$ have the desired dependence. Choose successive points on γ', called $a = x_0, x_1, \ldots, x_n = b$, having the following properties:

(1) The distance in Y from x_i to x_{i+1} is between $\frac{1}{2}$ and 1.

(2) Every point of γ' is within 1 of some x_j.

Define $\phi(x_j) \subset J$ to be any point of J whose image under q is as close as possible to x_i. Assuming that γ' remains within ξ from $q(J)$, the distance between $q(\phi(x_j))$ and x_j is at most ξ. Hence, the distance from $\phi(x_j)$ and $\phi(x_{j+1})$ is at most ξ_1. Hence, for some j, we have that two successive points $\phi(x_j)$ and $\phi(x_{j+1})$ are on either side of Σ and within ξ_2 of each other. Since J is a path metric space, both of these points are within ξ_3 of Σ. Hence (in particular) $q(\phi(x_j))$ is within ξ_4 of $q(\Sigma)$. This implies that x_j is within ξ_5 of $q(\Sigma)$. Taking $k > \xi_5$ does the trick. □

Assuming that $q(J)$ does not coarsely separate in Y, there is a constant N having the following property: If $x, y \in Y$ are at least N away from $q(J)$, then there is a path $\alpha(x, y) \subset Y - q(J)$ joining them.

Choose a maximal sequence of points $p_1, \ldots, p_l \subset \gamma$ such that $\overline{p_j p_{j+1}}$ is a traversing arc. (Here, the indices are taken cyclically.) Choose the constant k above so large that, according to Sublemma 5.8, there is a point $x_j \subset \overline{p_j p_{j+1}}$ which avoids $q(J)$ by at least N.

By assumption, then, there are paths $\alpha_j \subset Y - q(J)$ which connect x_j to x_{j+1}. Also, let $\beta_j = \overline{x_j x_{j+1}} \subset \gamma$. The closed loops $\delta_j = \alpha_j \cup (-\beta_j)$ are 1-cycles. If none of these cycles linked $q(\Sigma)$, then by homology addition, the cycle $\alpha_1 \cup \ldots \cup \alpha_n$ would link $q(\Sigma)$ but would not intersect $q(J)$. This contradicts the definition of linking number. So, some δ_l links $q(\Sigma)$ non-trivially. By the maximality assumption on the original arcs, the arc $\delta = \delta_l$ has the desired intersection properties. □

5.7. Homological contradiction

We will suppose that $q(J)$ does not coarsely separate in Y, and derive a contradiction. Let δ and Σ be as in the Curve Modification Lemma. Let B be the ball bounding Σ. If I_δ has no interior points, then $q(B)$ does not intersect δ. This contradicts the fact that δ and $q(\Sigma)$ are linked. Assume on the other hand that I_δ has no exterior points. Since J is diffeomorphic to $\mathbf{R}^n$, the sphere Σ is cobordant to a sphere $\Sigma' \supset \Sigma$, whose distance from Σ can be taken to be as large as we like. Since there are no exterior points of I_γ, the big sphere $q(\Sigma')$ still links δ. However, this implies that $q(\Sigma')$ must intersect each spanning surface of δ. In particular, points of $q(\Sigma')$ must remain fairly close to δ, no matter how big Σ' is. This contradicts the fact that q is uniformly proper.

5.8. Existence of unpinched spheres

Let X be a symmetric space of non-positive curvature. As usual, we equip horospheres in X with their induced Riemannian metric.

We say that two horospheres are parallel if they arise as the limit of concentric spheres. Given two such horospheres σ_1 and σ_2, we write $\sigma_1<\sigma_2$ if σ_1 is contained in the interior of the horoball bounded by σ_2.

Let $\tau_r<\sigma$ be the horosphere parallel to σ, and exactly r units away from σ. By first considering the finite case of concentric spheres, it is easy to construct a geodesic α which intersects τ_r at a right angle for every $r\geqslant 0$. Let $p_r=\alpha\cap\tau_r$. Let UT_r denote the unit tangent bundle to X at the point p_r. Let $C_r\subset UT_r$ denote those unit vectors which make an angle of $\frac{1}{4}\pi$ with the outward normal to τ_r at p_r. The metric on X allows us to canonically identify C_r with the standard Euclidean sphere.

For each $v\in C_r$, let $\gamma(v,r)$ denote the oriented geodesic ray through v. We define

$$\Sigma_r=\bigcup_{v\in C_r}\gamma(v,r)\cap\sigma.$$

LEMMA 5.9. *Suppose that σ_1 and σ_2 are parallel horopsheres, and $\sigma_1<\sigma_2$. Suppose that γ is a geodesic which intersects σ_1 transversely. Then γ intersects σ_2 transversely as well.*

Proof. If this was false, then there would exist a horosphere τ such that

(1) τ is parallel to σ_j, and $\sigma_1<\tau$,

(2) γ is tangent to τ.

By perturbation, we produce a finite sphere T and a geodesic g such that g is tangent to T, and also intersects the interior of the ball bounded by T. This is impossible in non-positive curvature. $\square$

By the preceding lemma, all intersections are transverse (and unique). Hence we have a canonical diffeomorphism

$$h_r\colon\Sigma_r\to C_r$$

defined by the formula

$$h_r(\gamma(v,r)\cap\sigma)=v.$$

Let d be any fixed positive integer. We assert that Σ_r is d-unpinched, for sufficiently large r. Let $q\in\Sigma_r$ be any point, and let $B_\sigma(d,q)\subset\sigma$ be the ball of radius d about p, as measured in the path metric on σ.

We just have to show that we can make the diameter of $h_r(B_\sigma(d,q))$ tend to zero as $r\to\infty$, independent of $q\in\Sigma_r$. Let $B_X(d,q)\subset X$ be the ball of radius d about the point q, as measured in the metric on X. Then, clearly,

$$B_\sigma(d,q)\subset B_X(d,q).$$

Since X has non-positive curvature, the visual measure $B_X(d,q)$ from the point p_r is at most $d/(r-d)$. This follows from the usual comparison theorems. Choosing r sufficiently large, we can make the visual measure of $B_X(d,q)$ as small as we like. Hence we can make $h_r(B_\sigma(d,q))$ have arbitrarily small size, independent of $q\in\Sigma_r$.

This completes the proof of Lemma 5.2.

6. Action rigidity

We now begin the proof of Theorem 3.2 (Action Rigidity Theorem). We will use the notation developed there. We would like to show that our map $\phi\in\mathcal{QC}(G,\Lambda)$ is (the restriction of) an affine map. Let G_0 be the linear part of $G\in\mathrm{Aut}(\Lambda)$. Since $\delta_G=\delta_{G_0}$, we may assume that $G=G_0$. There is some linear map taking Λ to $\mathbf{Z}^n$. Conjugating by this linear map, we may assume that $\Lambda=\mathbf{Z}^n$. Hence $\mathrm{SL}(\Lambda)=\mathrm{SL}_n(\mathbf{Z})$ and $\mathbf{Q}\Lambda=\mathbf{Q}^n$. Passing to a finite index subgroup only makes a Lipschitz change on the function δ_G. Hence, we may assume that the G is Abelian and semisimple. Thus, we are working with semisimple and minimal $\mathbf{Z}^d$-actions on $\mathbf{Z}^n$.

6.1. Logarithmic coordinates

Recall that all our metric constructions on $\mathbf{R}^n$ will be done relative to an adapted inner product $\mathcal{G}$. (See §2.5 for the definition.) Let $E_1,\dots,E_m$ be the simultaneous eigenspaces of G, and let $\pi_j:\mathbf{R}^n\to E_j$ denote projection.

LEMMA 6.1. *Let $x\in\mathbf{Q}^n$ be any non-zero element. Then, for all $j=1,\dots,m$, we have $\pi_j(x)\neq 0$.*

Proof. Suppose, for some j, that $\pi_j(x)=0$. By rescaling, we can assume that $x\in\mathbf{Z}^n$. Let $W_j\subset\mathbf{R}^n$ be the orthogonal complement to E_j. Note that $x\in W_j$. Therefore, $\Lambda_j=W_j\cap\mathbf{Z}^n$ is a non-trivial infinite index sublattice of $\mathbf{Z}^n$, preserved by G. This contradicts the minimality of G. $\square$

Let $x\in\mathbf{Q}^n$ be non-zero. If E_j is one-dimensional, we define $\log_j(x)=\log(|\pi_j(x)|)$. In case E_j is two-dimensional, we choose, once and for all, some identification of E_j with $\mathbf{C}$, and then define $\log_j(x)$ to be the complex log of $\pi_j(x)$, taken so that the imaginary part lies in the circle $\mathbf{R}/2\pi\mathbf{Z}$. Finally, we define

$$\log_G(x)=(\log_1(x),...,\log_m(x)).$$

$\log_G$ gives us a mapping from $\mathbf{Q}^n-0\subset\mathbf{R}^n$ to $\mathbf{L}=E^m\times T^q$, obtained by separating out the real and imaginary parts of the coordinates. Here T^q is a q-dimensional torus. This compact piece is more or less irrelevant to our arguments. We equip $\mathbf{L}$ with a flat Riemannian metric in such a way that the projection $\mathbf{L}\to\mathbf{E}^m$ is an isometry up to an additive constant.

Given any $T\in G$, define $\log_G(T)\in\mathbf{L}$ in the unique way such that

$$\log_G(\mathrm{Id})=0,\quad \log_G(T(x))=\log_G(T)+\log_G(x).$$

Since G is semisimple and free Abelian, $\log_G$ is a homomorphism from G into the translation group of $\mathbf{L}$. Any element of G in the kernel of $\log_G$ would be an isometry with respect to $\mathcal{G}$ and hence would have finite order. Since we are assuming G is free, $\log_G$ is an injection on G. We write $G_l=\log_G(G)$.

6.2. Images of parallelograms

Let

$$P=\begin{bmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix},\quad p_{ij}\subset\mathbf{Q}^n.$$

We say that P is a *parallelogram* if $p_{11}+p_{22}=p_{12}+p_{21}$. We define

$$\mathrm{vec}_G(P)=\log_G(p_{21}-p_{11})-\log_G(p_{12}-p_{11})\in\mathbf{L},$$
$$\mathrm{per}_G(P)=\delta_G(p_{11}\cup p_{21})+\delta_G(p_{11}\cup p_{12}).$$

These two notions have the following invariance properties: For any $x\in\mathbf{Q}^n$ and any $T\in G$, we have

$$\mathrm{per}_G(T(P)+x)=\mathrm{per}_G(P),\quad \mathrm{vec}_G(T(P)+x)=\mathrm{vec}_G(P).$$

per_G is a kind of G-invariant perimeter, and vec_G is a kind of G-invariant $\mathbf{L}$-valued shape.

Define, for any $k\in\mathbf{N}$,

$$S_k=\left\{x\subset\frac{1}{k}\mathbf{Z}^n\,\middle|\,\delta(0\cup x)\leqslant k\right\}.$$

It follows from compactness that S_k/G is a finite set. Define

$$\Gamma_k = \{\log_G(a) \pm \log_G(b) \mid a, b \in S_k\}.$$

Since S_k/G is a finite set, and G_l acts on $\mathbf{L}$ by translations, Γ_k is a finite union of orbits of G_l. By finiteness and equivariance, Γ_k is contained in a C_k-tubular neighborhood of the orbit $G_l(0)$. The constant C_k tends to ∞ with k. Since G is Abelian semisimple, and minimal, $G_l(0)$ is not contained in any of the coordinate hyperplanes through the origin in $\mathbf{E}^m$. Hence, there is a *hyperbolic* element $h \in G$. This is to say that no eigenvalue of h has norm 1. We shall fix this element h once and for all, for the remainder of this paper.

Let $x \in \mathbf{L}$ be any point. We define $E_j(x) \subset \mathbf{L}$ to be the space of points which have the same jth coordinate as x. In case E_j is two-dimensional, we take this to mean that both real and imaginary parts coincide. $E_j(x)$ either has codimension one or two. Recall that Γ_k is a finite collection of G_l-orbits. Let O_k denote a complete collection of orbit representatives for Γ_k/G_l. We define

$$H_k = \bigcup_{j=1}^{m} \bigcup_{x \in O_k} E_j(x).$$

For any $k \in (1, \infty)$, and any (non-compact) set $S \subset E^m$, define

$$\Omega_k(S) = \{x \mid \exp(-k\|x\|) \leqslant d(x, S) \leqslant \exp(-\|x\|/k)\}.$$

$\Omega(S)$ is the region between two exponentially decaying tubular neighborhoods of S.

Let $\mathcal{H}(C, s)$ be the family of parallelograms P having the properties

(1) $P \subset (1/C)\mathbf{Z}^n$,

(2) $\mathrm{per}_G(P) \leqslant C$,

(3) $\mathrm{vec}_G(P)$ lies within $\sqrt{s}$ units of $s \log_G(h)$,

(4) $\phi(P)$ is not a parallelogram.

In §7 we will prove the following estimate.

LEMMA 6.2 (Diophantine Approximation Lemma). *Suppose that $P \in \mathcal{H}(C_1, s)$. If s is sufficiently large, then there is a point $z(P)$ having the properties*

(1) $z(P) \in \Gamma_{C_2}$,

(2) $d(z(P), 0) \in [s/C_2, C_2 s]$,

(3) $z(P) \in \Omega_{C_2}(H_{C_2})$.

C_2 only depends on C_1 and on the distortion function of ϕ.

We remark that the only feature we use of the function $s \to \sqrt{s}$ is that it is sublinear and tends to infinity.

On the other hand, as a special case of Theorem 8.1 (Subspace Approximation Theorem) we have

LEMMA 6.3. *Let $C\in\mathbf{N}$ be arbitrary. Then, for infinitely many values $s\in\mathbf{N}$, there is* no *intersection point $p\in\Gamma_C\cap\Omega_C(H_C)$ such that $\|p\|\in[s/C, Cs]$.*

Combining the two preceding results, we see that $\mathcal{H}(C,s_j)=\varnothing$ for an infinite increasing sequence $s_1, s_2, \ldots\in\mathbf{N}$.

We say that $W\subset\mathbf{L}$ is *G-fat* if, for every positive integer N, there is an element $g\in G_l$ such that the ball of radius N about $g(0)$ is contained in W. We define

$$W(C,\phi)=\bigcup_{j=1}^{\infty} B_{\sqrt{s_j}}(s_j \log_G(h)).$$

(Here B_s is the ball of radius s.) Summarizing the discussion of this section, we have

LEMMA 6.4 (Parallelogram Lemma). *Let $\phi\in\mathcal{QC}(G,\mathbf{Z}^n)$ and let $C\in\mathbf{N}$. Then there exists a G-fat subset $W(\phi,C)$ having the property: Suppose that P is a parallelogram such that*

(1) $P\subset(1/C)\mathbf{Z}^n$,
(2) $\mathrm{per}_G(P)\leqslant C$,
(3) $\mathrm{vec}_G(P)\in W(\phi,C)$.

Then $\phi(P)$ is a parallelogram.

The Parallelogram Lemma has a simple independent proof, in the special case of hyperbolic $\mathbf{Z}$-actions on $\mathbf{Z}^2$. We sketch this proof at the end of this chapter.

6.3. Proof of the Action Rigidity Theorem

Let $q\in\mathbf{N}$ be fixed. Choose a finite generating set S for $(1/q)\mathbf{Z}^n$. For convenience, assume that S is symmetric: $s\in S$ if and only if $-s\in S$. Define

$$C=\max(q, 2\max_{s\in S} d_G(0\cup s)).$$

Let $W=W(\phi,C)$ denote the set guaranteed by the Parallelogram Lemma.

Let $G(S)$ denote the orbit of S under G. Given $x,y\in(1/q)\mathbf{Z}^n$, we say that (x,y) is a *distinguished pair* if $x-y\in G(S)$. Given two distinguished pairs (x_1,y_1) and (x_2,y_2), we write $(x_1,y_1)\rightarrow_W(x_2,y_2)$ if quadrilateral

$$P=\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

is a parallelogram, and

(1) $P\subset(1/q)\mathbf{Z}^n$,
(2) $\mathrm{per}_G(P)\leqslant r$,
(3) $\mathrm{vec}_G(P)\in W$.

By the Parallelogram Lemma, $\phi(P)$ is a parallelogram. In other words,

$$(x_1, y_1) \to_W (x_2, y_2) \;\Rightarrow\; \phi(x_1)-\phi(y_1) = \phi(x_2)-\phi(y_2).$$

For two distinguished pairs, (x, y) and $(\bar{x}, \bar{y})$, we write $(x, y) \longrightarrow_W (\bar{x}, \bar{y})$ if there is a finite sequence of distinguished pairs,

$$(x, y) = (x_1, y_1) \to_W \dots \to_W (x_n, y_n) = (\bar{x}, \bar{y}).$$

By induction, it follows that

$$(x, y) \longrightarrow_W (\bar{x}, \bar{y}) \;\Rightarrow\; \phi(x)-\phi(y) = \phi(\bar{x})-\phi(\bar{y}). \tag{$*$}$$

Here is where we use the fact that W is G-fat.

LEMMA 6.5. *Let $a \in (1/q)\mathbf{Z}^n$ be arbitrary. Let (u, v) be an arbitrary distinguished pair. Then $(u, v) \longrightarrow_W (u+a, v+a)$.*

Proof. For any $y \in G(S)$, consider the parallelogram

$$P_y = \begin{bmatrix} u & v \\ u+y & v+y \end{bmatrix}.$$

Note that $\operatorname{per}_G(P_y) \leqslant r$. Let $Y = Y(W, u, v) \subset G(S)$ denote those points $y \in G(S)$ such that $\operatorname{vec}_G(P_y) \in W$. Finally, let $\Sigma Y \subset (1/q)\mathbf{Z}^n$ denote the sublattice generated by Y.

We assert that $(u, v) \longrightarrow_W (u+x, v+x)$ for any $x \in \Sigma Y$. By induction, we set $x = x'+y$ where $(u, v) \longrightarrow_W (u+x', v+x')$ and $y \in Y$. Consider the parallelogram

$$P = \begin{bmatrix} u+x' & v+x' \\ u+x'+y & v+x'+y \end{bmatrix} = \begin{bmatrix} u+x' & v+x' \\ u+x & v+x \end{bmatrix}.$$

Clearly, $\operatorname{vec}_G(P) = \operatorname{vec}_G(P_y) \in W$. Also,

$$\delta_G(u+x' \cup u+x) = \delta_G(0 \cup y) \leqslant \tfrac{1}{2}r.$$

Likewise,

$$\delta_G(u+x' \cup v+x') = \delta_G(u \cup v) \leqslant \tfrac{1}{2}r.$$

Hence $\operatorname{per}_G(P) \leqslant r$. Therefore, $(u+x', v+x') \to_W (u+x, v+x)$. The assertion above follows by concatenating the arrows.

To complete the proof, we show that $\Sigma Y = (1/q)\mathbf{Z}^n$. Note that

$$\operatorname{vec}_G(P_y) = \log_G(y) - \log_G(u-v).$$

If we set $\omega = \log_G(u-v)$, then Y consists of those points $y \in G(S)$ such that $\log_G(y) \in W+\omega$. Since W is G-fat, so is the translate $W+\omega$.

Since $\log_G(S)$ is finite, and $W+\omega$ is G-fat, there is some $g \in G_l$ such that $g(\log_G(S)) \subset W+\omega$. This is to say that, for some $T \in G$, we have $\log_G(T(S)) \subset W+\omega$. This is to say that $T(S) \subset Y$. By equivariance, $T(S)$ generates $(1/q)\mathbf{Z}^n$, and hence so does Y. $\square$

Let $x, y \in (1/q)\mathbf{Z}^n$ be arbitrary. Let $s \in S$ be a generator. Consider the distinguished pairs $(x, x+s)$ and $(y, y+s)$. From the preceding lemma,

$$(x, x+s) \longrightarrow_W (y, y+s).$$

Hence, from $(*)$, we have

$$\phi(x+s) - \phi(x) = \phi(y+s) - \phi(y), \quad \text{for all } x, y \in \frac{1}{q}\mathbf{Z}^n, \text{ and for all } s \in S.$$

Since S generates $(1/q)\mathbf{Z}^n$, we have

$$\phi(x+z) - \phi(x) = \phi(y+z) - \phi(y), \quad \text{for all } x, y, z \in \frac{1}{q}\mathbf{Z}^n.$$

This last equation tells us that ϕ is affine on $(1/q)\mathbf{Z}^n$. Thus ϕ coincides, at least on $(1/q)\mathbf{Z}^n$, with some $T_q \in \mathrm{Aut}(\mathbf{Q}^n)$.

Let $q_1, q_2 \in \mathbf{N}$. Clearly, $T_{q_1 q_2}$ agrees with T_{q_j} on $(1/q_j)\mathbf{Z}^n$. Hence $T_{q_1} = T_{q_2}$. Let T be this common element. Then, clearly, T and ϕ agree on all of $\mathbf{Q}^n$.

6.4. Parallelogram Lemma: quadratic case

In this section, we sketch a self-contained proof of the Paralellogram Lemma in the simplest case. In the case at hand, the group $G \subset \mathrm{SL}_2(\mathbf{Z})$ is generated by a single hyperbolic automorphism T. In this case, $\mathbf{L} = \mathbf{E}^2$.

Suppose that $P_1, P_2, \ldots \subset (1/C)\mathbf{Z}^2$ is a sequence of parallelograms such that

(1) $\mathrm{per}_G(P_s) \leqslant C$,

(2) $\|\mathrm{vec}_G(P_s)\| \geqslant s$.

We will show that $\phi(P_s)$ must be a parallelogram once $s \geqslant s_0$. Our G-fat set is then just the complement of a large ball about the origin in $\mathbf{E}^2$.

We write

$$P_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix}.$$

For convenience, we drop explicit mention of s. For simplicity, we normalize so that $a=0$ and $\phi(a)=0$. For any object X, we define $X' = \phi(X)$. We define

$$e' = b' + c'.$$

We would like to show that $e'=d'$ once s is large.

We may postcompose ϕ by an element $g\in G$ so that $\|b'\|$ and $\|c'\|$ agree up to a multiplicative constant which does not depend on s. If we take $x=d'$ or $x=e'$, we have

$$\delta_G(x\cup b')<C, \quad \delta_G(x\cup c')<C,$$

for some constant C independent of s. However, the set of points satisfying these properties lies in the intersection of uniformly thin hyperbolas which are centered far from each other if s is large. The axes of these hyperbolas are the two eigenspaces of G. Hence the intersection of the hyperbolas consists of two disks whose diameters tend to 0 as s tends to ∞. One of the disks contains the point 0 and the other will eventually only contain one point of $(1/C)\mathbf{Z}^2$. Thus, for large s, $e'=d'$, as desired.

7. Diophantine Approximation Lemma

In this chapter, we prove Lemma 6.2 (Diophantine Approximation Lemma). We will prove it at the end, after making some estimates. The constants $C_1, C_2, \ldots$ and the constants $K_1, K_2, \ldots$ will only depend on C and on the distortion function of α. We will reset these constants from time to time. We will use the notation of §6.2.

7.1. Normalizations

If we compose ϕ by translations in $\mathrm{Aut}(\mathbf{Q}^n)$ having bounded height (independent of s), we can assume without loss of generality that $p_{11}=0$ and $\phi(p_{11})=0$. Replacing ϕ by $g\circ\phi$ for some $g\in G$, we can assume that $\phi(e_1)$ is a uniformly short vector. Here $e_1=(1,0,\ldots,0)$.

For the sake of reference, we collect together facts which we will use below. For convenience, we set $a_1=p_{12}$, $a_2=p_{21}$ and $b=p_{22}$. Also, for any object X, we define $X'=\phi(X)$.

(1) $\delta_G(0\cup a_i)<C_1$,
(2) $\delta_G(0\cup a_i')<C_1$,
(3) $\delta_G(b\cup a_i)<C_1$,
(4) $\delta_G(b'\cup a_i')<C_1$,
(5) a_i, b, a_i' and b' all belong to $(1/C_1)\mathbf{Z}^n$,
(6) P' is not a parallelogram,
(7) the jth coordinate of $\log_G(a_1)-\log_G(a_2)$ has norm at least s/C_1, for all j,
(8) $\|e_1'\|\leqslant C_1$.

The only parameter that varies here is s, which can be arbitrarily large. Items (2) and (4) follow from quasi-compatibility and (respectively) from items (1) and (3).

Item (5) follows from quasi-integrality. Item (7) follows from the fact that h is a hyperbolic element, and that $\mathrm{vec}_G(P)$ lies within $\sqrt{s}$ of $s\log_G(h)$.

7.2. Growth principle

Before embarking on our estimates, we introduce a principle concerning growth of sequences of small height.

GROWTH LEMMA. *Let $x, y_1, ..., y_n \subset (1/K_1)\mathbf{Z}^n$ be such that $\delta_G(x\cup y_i\cup y_{i+1})<K_2$, for all i. Then $\|y_n-x\|<K_3^n\|y_1-x\|$. The constant K_3 only depends on K_1 and K_2.*

Proof. By induction, we just have to prove the statement for the points x, y_1, y_2. Without loss of generality, we translate so that $x=0$. There is an element $T\in G$ such that $0\cup T(y_1)\cup T(y_2)$ has diameter at most K_2. Since y_1 and y_2 have small height, there are finitely many possible locations for $T(y_1)$ and $T(y_2)$. First of all, this places uniform bounds on $\|T(y_2)\|/\|T(y_1)\|$. Also, this places positive upper and lower bounds on the length of the projection of $T(y_j)$ onto the eigenspaces of T^{-1}. These two facts give us the desired control on $y_j=T^{-1}T(y_j)$. □

7.3. The estimates

Recall that $\mathbf{R}^n$ is equipped with an adapted inner product. However, changing the inner product merely changes the constants in the estimate below.

LEMMA 7.1 (Estimate 1). *Let π_j be the projection onto the j-th eigenspace E_j of G. Let $s\geqslant 0$ be arbitrary. Suppose that $0\neq x\in(1/K_1)\mathbf{Z}^n$. Then,*

$$\log\|x\|\leqslant s \;\Rightarrow\; |\log\|\pi_j(x)\||\leqslant K_2 s.$$

Here K_2 is independent of s.

Proof. The minimality condition on G implies that $\pi_j^{-1}(0)=0$, for all j. Therefore, $\pi_j(x)\neq 0$. By hypothesis, $\|x\|<\exp(s)$. Clearly $\|\pi_j(x)\|<\exp(s)$. Since $G\subset \mathrm{SL}_n(\mathbf{Z})$, the eigenspaces of G are given by linear equations involving algebraic numbers. Hence, the size of $\pi_j(x)$ can decay at most polynomially in the size of x. This says that $\|\pi_j(x)\|>\exp(-K_3 s)$. Taking logs of our two bounds, we get our estimate. □

We define

$$X=\phi^{-1}(b'-a_1').$$

Note that $X=a_2$ if and only if $\phi(P)$ is a parallelogram.

LEMMA 7.2 (Estimate 2). *Let X be as above. Then,*

(1) $\delta_G(0 \cup X) \leqslant C_2$,

(2) $\delta_G(b \cup X) \leqslant C_2$,

(3) $X(P) \neq a_i$, *for* $s > C_2$.

Proof. We compute

$$\delta_G(X' \cup 0) = \delta_G(b' - a_1' \cup 0) = \delta_G(b' \cup a_1') \leqslant K_2,$$
$$\delta_G(X' \cup b') = \delta_G(b' - a_1' \cup b') = \delta_G(-a_1' \cup 0) = \delta_G(a_1' \cup 0) \leqslant K_3.$$

The last two inequalities are just items (2) and (4) of §7.1.

Since P' is not a parallelogram, we see immediately that $X \neq a_2$. Suppose that $X = a_1$. This is to say that $a_1' = b' - a_1'$, or, equivalently, $b' = 2a_1'$. Hence,

$$\delta_G(0 \cup b') \leqslant 2\delta_G(0, a_1') \leqslant K_4.$$

By quasi-compatibility, we get a universal bound on $\delta_G(0 \cup a_1 + a_2)$, independent of s. By translation and symmetry,

$$\delta_G(a_1 \cup a_2) = \delta_G(-a_1 \cup a_2) = \delta_G(0 \cup a_1 + a_2) \leqslant K_5.$$

By definition of δ_G, we can replace P by $g(P)$ for some $g \in G$ so that $\|a_1 - a_2\| < K_5$. From our normalization, the triangle inequality, item (7) of §7.1, and compactness, we see that $\min(\|a_1\|, \|a_2\|)$ tends to infinity with s. In particular, for large s, we have

$$\|\pi_v(a_i)\| > 100, \quad \text{for some } v.$$

Also, from our normalization,

$$\|\pi_j(a_1) - \pi_j(a_2)\| < K_5, \quad \text{for all } j.$$

Since $\log_G$ is contracting above scale 1, we have

$$\|\log(\pi_v(a_1)) - \log(\pi_v(a_2))\| < K_6,$$

independent of s. This contradicts item (7) of §7.1. □

We introduce the quantity

$$\mathrm{twist}_j(\phi, P) = \log_G(X) - \log_G(a_j).$$

For any two elements $g, g' \in G$, we have

$$\begin{aligned}\mathrm{twist}_j(g' \circ \phi \circ g, g^{-1}(P)) &= \mathrm{twist}_j(\phi \circ g, g^{-1}(P)) \\ &= \log_G(g^{-1} \circ \phi^{-1}(\phi \circ g(g^{-1}(b)) - \phi \circ g(g^{-1}(a_1)))) - \log_G(g^{-1}(a_j)) \\ &= (\log_G(X) - \log_G(g)) - (\log_G(a_j) - \log_G(g)) \\ &= \mathrm{twist}_j(\phi, P).\end{aligned}$$

We will speak of this below the G-invariance of twist. When the choices of P and ϕ are implicit, we will omit them from our terminology.

LEMMA 7.3 (Estimate 3). *For* $i=1,2$,

$$\|\text{twist}_i(P)\| \leqslant C_3 s.$$

Proof. Since twist is G-invariant, we may replace P by $g(P)$ and ϕ by $g'\circ\phi\circ g^{-1}$ for any g, g' in G. Since $a_i\in(1/C)\mathbf{Z}^n$, there is a constant C_1 such that $\log_G(a_u\pm a_v)\in\Gamma_{C_1}$. Since Γ_{C_1}/G_l is finite, there is some $g\in G$ so that

$$\|\log_G(a_1)+\log_G(a_2)\| \leqslant K_0, \tag{$*$}$$

independent of s. We then choose g' so that item (8) of §7.1 remains true.

Consider points in Γ_C which are closest to the straight line in $\mathbf{L}$ connecting $\log_G(a_1)$ to $\log_G(a_2)$. From $(*)$, this line will come close to the origin. Since $\|\text{vec}_G\|<K_1 s$, this line will have length at most $K_2 s$. Since a_1 and a_2 have small height, this line will lie in a small tubular neighborhood of Γ_C. Choosing such points appropriately, and then exponentiating, we see that there are points $a_2=x_1, ..., x_s=a_1\in(1/K_3)\mathbf{Z}^n$ such that $\delta_G(0\cup x_i\cup x_{i+1})<K_3$, and $x_v=e_1$ for some index v. Applying item (8) and the Growth Lemma to the images of these points under ϕ, we see that $\|a_j'\|<\exp(K_4 s)$.

Next, since P is a parallelogram, we may by symmetry find points $0=y_1, ..., y_s=b\in(1/K_3)\mathbf{Z}^n$ such that $\delta_G(a_1\cup y_i\cup y_{i+1})<K_3$. Again, we consider the images of these points under ϕ. Since $\|a_1'-y_1'\|=\|a_1'\|<\exp(K_4 s)$, we conclude from the Growth Lemma that $\|b'-a_1'\|<\exp(K_5 s)$. Note that $X'=b'-a_1'$.

By Estimate 2 and quasi-compatibility, $\delta_G((b'-a_1')\cup 0)<K_5$. Once again looking at the picture in $\mathbf{L}$, we may find points $e_1'=z_1', ..., z_s'=b'-a_1'\in(1/K_6)\mathbf{Z}^n$ such that $\delta_G(0\cup z_i'\cup z_{i+1}')<K_6$. Applying the Growth Lemma to the inverse images of these points, we get an exponential bound on $\|X\|$. Also, from item (7) of §7.1 and from $(*)$, we get an exponential bound on $\|a_i\|$. Our Estimate now follows, after taking logs, from Estimate 1 and the triangle inequality. $\square$

For our last two estimates, it is useful to introduce

$$\Delta_G(x)=\prod\|\pi_j(x)\|, \quad \Delta_G(x,y)=\Delta_G(x-y).$$

The product is taken over distinct eigenspaces of G, and ε^j is the dimension of the eigenspace E_j. Since elements of G are volume preserving, Δ_G is a G-invariant notion.

LEMMA 7.4 (Estimate 4). *If s is sufficiently large, then*

$$\|\text{twist}_j(P)\| > s/C_4.$$

Proof. We will carry out the proof for $\mathrm{twist}_1(P)$. The other case is quite similar.

Since twist is G-invariant, we can replace P by $g^{-1}(P)$, and ϕ by $g'\circ\phi\circ g$, as in the proof of Estimate 3. This time, however, we choose g such that

(1) $\|a_1\| \leqslant K_1$, independent of s.

Using the fact that $a_1 \in (1/C_1)\mathbf{Z}^n$, we get $\|\log_G(a_1)\| \leqslant K_2$. By the triangle inequality,

(2) $\|\mathrm{twist}_1(P)\| \in [\|\log_G(X)\| - K_2, \|\log_G(X)\| + K_2]$.

We will suppose that $\|\mathrm{twist}_1(P)\| \leqslant s/\Theta$, and derive a contradiction for sufficiently large Θ. Using equation (2) and exponentiating, we get

(3) $\|X\| \leqslant \exp(K_3 s/\Theta)$.

Note that X has small height. Applying Estimate 1 to the point $X - a_1$, and plugging in equations (1) and (3), we get

(4) $\Delta_G(a_1, X) \geqslant \exp(-K_6 s/\Theta)$.

SUBLEMMA 7.5. *For sufficiently large Θ, there is some $\omega \in \{1, ..., m\}$ such that*

$$\|\pi_\omega(X) - \pi_\omega(a_1)\| \leqslant \exp(-s/K_7).$$

Proof. Since $\log_G(a_1)$ is small, $\log_G(a_2) - \mathrm{vec}_G(P)$ is small independent of s. Recall that $\mathrm{vec}_G(P)$ is within $\sqrt{s}$ of $s\log_G(h)$. Since h is hyperbolic, we get a lower bound on each norm $\|\log_i(a_2)\|$ which is linear in s. Exponentiating, we get

(5) $\|\pi_i(a_2)\| \notin [\exp(-s/K_8), \exp(s/K_8)]$, for all $i = 1, ..., m$.

Define

$$\xi = \{i \mid \|\pi_i(a_2)\| < 1\} \subset \{1, ..., m\}.$$

Since $b = a_1 + a_2$, Equations (1) and (5) say that

(6a) $\|\pi_i(a_1) - \pi_i(b)\| \leqslant \exp(-s/K_9)$, for all $i \in \xi$.

(6b) $\|\pi_i(b)\| \geqslant \exp(s/K_{10})$, for all $i \notin \xi$.

Equation (3) implies

(7) $\|\pi_i(X)\| \leqslant \exp(K_{11} s/\Theta)$, for all i.

Combining this with equation (6b) we have

(8) $\|\pi_i(X) - \pi_i(b)\| \geqslant \exp(s/K_{12})$, for all $i \notin \xi$,

provided that Θ is chosen large enough.

Since X and b have small height, and $\delta_G(X \cup b)$ is small, compactness gives us

$$\Delta_G(X, b) \leqslant K_{14}.$$

Combining this with equation (8), we have

$$\prod_{i \in \xi} \|\pi_i(b) - \pi_i(X)\|^{e_i} \leqslant \exp(-s/K_{15}).$$

From this, we clearly have

(9) $\|\pi_\omega(b)-\pi_\omega(X)\| \leqslant \exp(-s/K_{16})$, for some $\omega \in \xi$.

Our sublemma follows, by the triangle inequality, from quations (6a) and (9). $\square$

Combining equations (1) and (3), we see that

(10) $\|\pi_i(a_1)-\pi_i(X)\| \leqslant \exp(K_{17}s/\Theta)$, for all i.

Combining equation (10) with Sublemma 7.5, we see that

$$\Delta_G(X, a_1) \leqslant \exp(-s/K_{18}),$$

provided that Θ is chosen large enough. This last equation contradicts equation (4) if Θ is too large. $\square$

LEMMA 7.6 (Estimate 5). *Assume that* $\|\log_G(a_1)+\log_G(a_2)\| < C_5$. *First,* $\|X\| \leqslant \exp(C_6 s)$. *Second, for at least one* $i \in \{1,2\}$ *and at least one* $j \in \{1, ..., m\}$ *it is true that*

(1) $\|\pi_j(X)-\pi_j(a_i)\| \leqslant 1$,

(2) $\|\pi_j(X)\| \geqslant \exp(s/C_6)$.

Proof. The first statement follows from the proof of Estimate 3. Now for the second statement: By exponentiating, we have

(1) $\min(\|\pi_j(a_1)\|, \|\pi_j(a_2)\|) \leqslant K_1$, for all j.

Item (7) of §7.1 and our hypothesis say that every coordinate $\log_G(a_j)$ has norm at least s/K_2. Therefore

(2) $\|\pi_j(a_1+a_2)\| \geqslant \exp(s/K_3)$, for all j.

From Estimate 2, we have $\delta_G(X \cup b) \leqslant K_4$. By quasi-integrality, X has small height. So does b. Hence, there are only finitely many choices for $X-b$, mod G. Therefore,

$$\Delta_G(X, b) \leqslant K_5.$$

Hence, for some $j \in \{1, ..., m\}$ we have

(3) $\|\pi_j(X)-\pi_j(a_1+a_2)\| \leqslant K_5$.

For this value of j, choose a_i such that $\pi_j(a_i) > K_1$. From equation (1), we have $\pi_j(a_{3-i}) \leqslant K_1$. The triangle inequality and equation (3) now imply statement (1). To get statement (2) observe that $\pi_j(a_i)$ is exponentially large, by equation (2) and the triangle inequality. Statement (2) now follows from the triangle inequality. $\square$

7.4. Putting it together

In this section, we prove the Diophantine Approximation Lemma. For convenience, we restate it here (with the names of the constants changed).

LEMMA 7.7 (Diophantine Approximation Lemma). *Suppose that* $P\in\mathcal{H}(C,s)$. *If* s *is sufficiently large, then there is a point* $z(P)$ *having the properties*

(1) $z(P)\in\Gamma_{C'}$,

(2) $d(z(P),0)\in[s/C',C's]$,

(3) $z(P)\in\Omega_{C'}(H_{C'})$.

C' *only depends on* C *and on the distortion function of* ϕ.

Proof. By replacing P by $g(P)$ and ϕ by $g'\circ\phi\circ g$, as we had done above several times, we can assume that the hypothesis of Estimate 5 is satisfied. Let i and j be as in Estimate 5. We define $\operatorname{twist}(P)=\operatorname{twist}_i(P)$ for this choice of i. We will also set $\pi=\pi_j$ and $a=a_i$. Since X and a have small height, we have $\log_G(X)\in\Gamma_{K_1}$ and $\log_G(a)\in\Gamma_{K_1}$. Let $\alpha\in O_{K_1}$ be an orbit representative for $\log_G(a)$. This means that $u(\alpha)=\log_G(a)$ for some $u\in G_l$. We define

$$z(P)=\alpha+\operatorname{twist}(P).$$

Since u is a translation, we have

$$u(z(P))=u(\alpha)+\operatorname{twist}(P)=\log_G(X).$$

Since $\log_G(X)\in\Gamma_{K_1}$ so is $u^{-1}(\log_G(X))=z(P)$. This is statement (1). From Estimates 3 and 4, and the fact that $\|\alpha\|<K_2$, we have $\|z(P)\|\in[s/K_3,K_3s]$. This is statement (2).

Now for statement (3). We know from item (7) of §7.1, Estimate 5 and the triangle inequality that

(1) $\|X\|,\|a\|,\|\pi(X)\|,\|\pi(a)\|\in[\exp(s/K_4),\exp(K_4s)]$,

(2) $\|\pi(a)-\pi(X)\|\leqslant K_5$.

Since the logarithm is exponentially contracting at exponential distances from the origin, equations (1) and (2) imply that

(3) $\|\log(\pi(a))-\log(\pi(X))\|\leqslant\exp(-s/K_7)$.

On the other hand, from Estimates 1, 2, 5 and equation (1),

(4) $\|\pi(a)-\pi(X)\|\geqslant\exp(-K_8s)$.

Equations (1) and (4), together with properties of the logarithm, say that

(5) $\|\log(\pi(a))-\log(\pi(X))\|\geqslant\exp(-K_9s)$.

Equations (3) and (5) say that the jth coordinate of $\operatorname{twist}(P)$ has norm belonging to $[\exp(-s/K_{10}),\exp(-K_{10}s)]$. This is to say that

$$z(P)\in\Omega_{K_{11}}(E_j(\alpha)).$$

This is statement (3). □

8. Points near subspaces

The goal of this chapter is to prove a general result about lattice points near subspaces.

8.1. Subspace Approximation Theorem

Let $\mathbf{L}$ be a product of a flat torus and Euclidean space. By a *flat* in L, we shall mean a lower-dimensional geodesically embedded flat manifold, which is the product of a Euclidean space and a flat torus.

For any non-compact set S, define

$$\Omega_k(S) = \{x \mid \exp(-k\|x\|) \leqslant d(x,S) \leqslant \exp(-\|x\|/k)\}.$$

Let $H \subset \mathbf{L}$ be a finite union of flats. Let $L = \mathbf{Z}^r$ be any discrete isometric group action of $\mathbf{L}$, and let Λ be a finite union of L-orbits. We define

$$\Psi(\Lambda, H, K) = \{\|x\| \mid x \in \Lambda \cap \Omega_K(H)\}.$$

The goal of this chapter is to prove

THEOREM 8.1 (Subspace Approximation Theorem). *There is an infinite sequence* $s_1, s_2, ... \in \mathbf{N}$ *such that*

$$\Psi(\Lambda, H, K) \cap [s/K, Ks] = \varnothing.$$

For ease of terminology, we will assume that $\mathbf{L} = \mathbf{E}^d$. In this case, flats are just lower-dimensional Euclidean spaces. The case with a non-trivial torus factor can be obtained by passing to the universal cover (or by modifying the terminology of the proof a bit).

Also, we will assume that L does not have full rank. This is to say that $r < d$. To reduce the full rank case to this case, we simply embed everything into a higher-dimensional Euclidean space $\mathbf{L}'$. (When we redefine $\Omega_k(H)$ relative to the larger ambient space, we only add points.)

8.2. Controlled sequences

We say that an infinite sequence $X = \{x_j\}$ of positive real numbers is *loosely exponential* if there are positive integers j_0, N such that

$$x_j \in [N^{j+j_0}, N^{j+j_0+1}], \quad \text{for all } j.$$

We say that a sequence $y_1, ..., y_n \subset (0, \infty)$ is (S, ξ)*-controlled* if

$$q_1 \geqslant S; \quad q_{j+1} \in [S, S^{\xi}]q_j, \quad \text{for all } j.$$

LEMMA 8.2. *Let X be a loosely exponential sequence. Let $f: X \to \{1, ..., t\}$ be any function. Let $D, S \in \mathbf{N}$ be given. Then there is a finite subsequence $X' \subset X$ such that*

(1) *X' has D elements,*

(2) *f is constant on X',*

(3) *X' is (S, ξ)-controlled.*

The constant ξ does not depend on S.

Proof. Let α be a fixed but unspecified positive integer. Let $y_j = x_{\alpha(j+1)}$ and let $Y = \{y_j\}$. By the Pidgeonhole Principle, every subsequence of $Y' \subset Y$ having length tD must have a smaller D-element subsequence Y'' on which f is constant. If we take Y' to be a string of consecutive elements in Y, then Y'' will be (S_α, ξ)-controlled. Clearly, S_α grows unboundedly with α, and the constant ξ only depends on t, D and X, but not on α. □

Recall that d is the dimension of our Euclidean space. Let $D = 2^d$. As an immediate application of the preceding lemma, we have

COROLLARY 8.3. *Suppose that the Subspace Approximation Lemma is false. Then there is a single Euclidean subspace Π, a single orbit O of L, and a fixed constant ξ_0 such that $\Psi(O, \Pi, K)$ has an (S, ξ_0)-controlled sequence of length D, for arbitrarily large values of S.*

Since the constant ξ_0 is fixed, we will omit it from our notation. We say that a sequence of points $p_1, ..., p_v$ is S-controlled if their norms $\|p_1\|, ..., \|p_v\|$ form an S-controlled sequence. By translating, and adjusting constants if necessary, we can assume—for ease of notation mainly—that Π contains the origin. To prove the Subspace Approximation Lemma, it is sufficient to prove

LEMMA 8.4 (Exclusion Lemma). *Let $k \in \mathbf{N}$ be given. For sufficiently large S, there does not exist an S-controlled sequence $p_1, ..., p_D \subset O \cap \Omega_k(\Pi)$.*

Since the rank of L is less than d, the convex hull of any collection of points in O has dimension at most $d-1$. To prove the Exclusion Lemma, we will prove

LEMMA 8.5. *Suppose that $j \leqslant d$ and $J = 2^j$. Suppose that $p_1, ..., p_J \in O \cap \Pi_k$ form an S-controlled sequence. Then $\dim(h(p_1 \cup ... \cup p_J)) = j$, provided that S is sufficiently large.*

Proof. We must find $j+1$ general position points of our sequence. The result is obviously true for $j = 0, 1$. By induction, there are j points, $a_1, ..., a_j$, in the first half of this sequence, which are in general position, provided that S is large enough. Likewise, there are j points, $b_1, ..., b_j$ in the second half of this sequence, which are in general position. Let A and B be the $(j-1)$-dimensional subspaces respectively spanned by these points. To prove the induction step it is sufficient to show that $A \neq B$ provided that S is chosen sufficiently large. Let θ_a and θ_b respectively denote the angles which A and B make with Π. We will estimate these angles in turn, and show that $\sin(\theta_a) > \sin(\theta_b)$ provided that S is chosen large enough.

A helpful characterization. Let π denote projection onto Π. Given $x \in \mathbf{E}^d$, we will write

$$x^h = \pi(x), \quad x^v = x - \pi(x).$$

Provided that $\|x\|$ is sufficiently large,

$$x \in \Omega_k(\Pi) \Rightarrow \|x^v\| \in [\exp(-K_3\|x^h\|), \exp(-\|x^h\|/K_3)].$$

This constant $\underline{K_3}$ is important, so we underline it in what follows. Another important constant will be

$$m = \|a_j^h\|.$$

Estimate from below. Let σ be the "slope", relative to Π, of the line joining a_j to a_1,

$$\sigma = \frac{\|a_j^v - a_1^v\|}{\|a_j^h - a_1^h\|}.$$

From trigonometry,

$$\sin(\theta_a) \geqslant \tfrac{1}{2}\tan(\theta_a) \geqslant \tfrac{1}{2}|\sigma|.$$

The first inequality holds for sufficiently small θ_a. If we choose S sufficiently large, then we can guarantee that

$$\|a_i^v\| \geqslant 2\|a_{i+1}^v\|, \quad \text{for all } i < j.$$

Hence,

$$\|a_j^v - a_1^v\| \geqslant \tfrac{1}{2}\|a_1^v\| \geqslant \tfrac{1}{2}\exp(-m\underline{K_3}/S).$$

Also, trivially, we have

$$\|a_j^h - a_1^h\| \leqslant 2m.$$

Putting everything together, we have

$$\sin(\theta_a) \geqslant \exp(-m),$$

provided that S is sufficiently large.

Estimate from above. Let h_b denote the convex hull of the points $b_1, ..., b_j$. We have the estimates

$$\mathrm{vol}_j(h_b) \geqslant 1/K_1, \quad \mathrm{diam}(h_b) \leqslant mS^{K_4}, \quad \min\|b_i^h\| \geqslant mS.$$

The first estimate follows from the determinant formula for volumes of points in a lattice. This is where we use the structure of the group $L=\mathbf{Z}^r$. The second and third estimates follow from the definition of S-controlled sequences.

Using trigonometry, we see that that the distance, in B, from any point b_j to $\Pi\cap B$ is at most

$$\frac{\exp(-mS/\underline{K}_3)}{\sin(\theta_b)}.$$

It follows from the basic geometry ("bast times height") that

$$\mathrm{vol}_j(h_b) \leqslant (mS)^{K_5} \frac{\exp(-mS/\underline{K}_3)}{\sin(\theta_b)}.$$

Comparing the two estimates for volume, we get

$$\sin(\theta_b) \leqslant \frac{1}{K_6}(mS)^{K_5} \exp(-mS/\underline{K}_3).$$

For sufficiently large S, this expression is less than $\exp(-2m)$. □

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Received June 8, 1996