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L^p Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations

by

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1. Introduction

The purpose of this paper is twofold. First we prove a delicate Carleman inequality, involving nonconvex weights, for the operator $i\partial_t+\Delta_x$ acting on functions on $\mathbf{R}^n\times[-T,T]$. Then we use this inequality to study uniqueness properties of solutions of nonlinear Schrödinger equations of the form

$$(i\partial_t+\Delta_x)u=Vu+F(u), \tag{1.1}$$

where V is a potential and F is a nonlinear term. We are concerned with the following type of question:

Question Q. Assume that u_1 and u_2 are solutions in $\mathbf{R}^n\times[0,1]$ to (1.1) (in a suitable function space) with the property that for some domain $D\subseteq\mathbf{R}^n$ we have $u_1(x,0)=u_2(x,0)$ and $u_1(x,1)=u_2(x,1)$ for a.e. $x\in D$. Can we then conclude that $u_1\equiv u_2$ in $D'\times[0,1]$ for some domain D'?

In our theorems the domain D will be a half-space.(1) Under suitable assumptions on the potential V, the function F and the solutions u_1 and u_2, we answer Question Q in the affirmative, with the domain D' equal to the entire $\mathbf{R}^n$.

This type of uniqueness question seems to originate in control theory. Zhang [21] used inverse scattering theory to answer Question Q in the affirmative in the special

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(1) We are not aware of any positive results for domains D that do not contain a half-space.

case $n=1$, $V\equiv 0$, $F=\alpha|u|^2u$, $\alpha\in\mathbf{R}$, $u_2\equiv 0$, $D=(a,\infty)$, with $D'=\mathbf{R}$. Bourgain [1] proved uniqueness under analyticity assumptions on the nonlinear term $F=F(u,\bar{u})$, with $u_2\equiv 0$, $V\equiv 0$ and the stronger assumption that u_1 is compactly supported for all $t\in[0,1]$. Kenig, Ponce and Vega [9] answered Question Q in the affirmative for sufficiently smooth functions u_1 and u_2, when the domain D is the complement of a convex cone, $V\equiv 0$ and $F=F(u,\bar{u})$ satisfies bounds of the form

$$|\nabla F(u,\bar{u})|\leqslant C(|u|^{p_1-1}+|u|^{p_2-1}),\quad p_1,p_2>1.$$

We remark that the Carleman estimates of [9] could also have been used to include potentials $V\in L^\infty_{\mathrm{loc}}(\mathbf{R}^n\times[0,1])\cap L^1_tL^\infty_x(\mathbf{R}^n\times[0,1])$, with

$$\lim_{R\to\infty}\|V\|_{L^1_tL^\infty_x(\{x:|x|\geqslant R\})}=0.$$

See also the remark following Theorem 2.1. Local unique continuation theorems were proved by Isakov [6].

A question similar to Question Q was considered in the setting of the generalized Korteweg–de Vries equation on $\mathbf{R}\times\mathbf{R}$,

$$(\partial_t+\partial_x^3)u+F(x,t,u,\partial_x u,\partial_x^2u)=0.$$

Zhang [20] proved uniqueness if $u_2\equiv 0$, in the cases $F=u\partial_x u$ and $F=-6u^2\partial_x u$. This was extended by Kenig, Ponce and Vega [8], [10] to include a large family of functions F, as well as two nonzero solutions u_1 and u_2. Bourgain [1] proved uniqueness of solutions for the more general nonlinear equation

$$(i^{s-1}\partial_t+\partial_x^s)u+F(u,\partial_x u,...,\partial_x^{s-2}u)=0,\quad s\geqslant 2,$$

under analyticity assumptions on F, with $u_2\equiv 0$ and u_1 compactly supported at each time $t\in[0,1]$. This last equation was also considered by Kenig, Ponce and Vega [11], who proved uniqueness under more general assumptions on F, as well as for two nonzero solutions u_1 and u_2. Local unique continuation theorems were proved by Saut and Scheurer [18].

Our results in this paper (and the methods used) mostly resemble those of Kenig, Ponce and Vega [9]. However, we prove theorems under weaker regularity assumptions on the potential V and the function F; in particular, we allow locally unbounded potentials V. We also improve the space of solutions u for which we have uniqueness, and reduce the domain D on which we require the solutions to agree. To explain our theorems, consider the simplest assumption on the potential V and the function F, namely

$V \in L^{(n+2)/2}(\mathbf{R}^n \times [0,1])$ and $F \equiv 0$. Let H denote the operator $i\partial_t + \Delta_x$. The relevant Carleman inequality to use in this case is

$$\begin{aligned}\|e^{\beta\varphi_\lambda(x_1)}u\|_{L^{(2n+4)/n}(\mathbf{R}^n\times[0,1])} &\leqslant C\|e^{\beta\varphi_\lambda(x_1)}Hu\|_{L^{(2n+4)/(n+4)}(\mathbf{R}^n\times[0,1])} \\ &\quad + C[\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,0)\|_{L^2(\mathbf{R}^n)} + \|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,1)\|_{L^2(\mathbf{R}^n)}].\end{aligned} \tag{1.2}$$

In Theorem 2.1 we prove a stronger estimate for functions $u \in C([0,1]:L^2(\mathbf{R}^n))$ with $Hu \in L^{(2n+4)/(n+4)}(\mathbf{R}^n \times [0,1])$, any $\beta \geqslant 0$ and any $\lambda \geqslant \Lambda(\beta)$. The function φ_λ is defined by $\varphi_\lambda(r) = \lambda\varphi(r/\lambda)$, where φ is a fixed smooth function on $\mathbf{R}$ with the properties $\varphi(0)=0$, φ' nonincreasing, $\varphi'(r)=1$ if $r \leqslant 1$, and $\varphi'(r)=0$ if $r \geqslant 2$. The main point of this Carleman inequality is uniformity: the constant C should not depend on β, λ or the function u.

We would like to apply the inequality (1.2) to the function $u = u_1 - u_2$, where u_1 and u_2 are the two solutions in Question Q, and let $\beta, \lambda \to \infty$. For the Carleman argument to go through (i.e. to be able to absorb the main term in the right-hand side) we need to have

$$\|e^{\beta\varphi_\lambda(x_1)}u\|_{L^{(2n+4)/n}(\mathbf{R}^n\times[0,1])} < \infty. \tag{1.3}$$

This condition explains why it is important to prove a Carleman inequality like (1.2) with a bounded weight $e^{\beta\varphi_\lambda(x_1)}$. Such a Carleman inequality can be applied to all solutions u_1 and u_2 in $C([0,1]:L^2(\mathbf{R}^n))$ with $Hu_1, Hu_2 \in L^{(2n+4)/(n+4)}(\mathbf{R}^n \times [0,1])$. For contrast, by (1.3), the easier Carleman inequality with the weight $e^{\beta\varphi_\lambda(x_1)}$ replaced by the exponential weight $e^{\beta x_1}$ can only be applied to solutions u_1 and u_2 that have faster-than-exponential decay at infinity. This was already noticed by Kenig, Ponce and Vega [9], who proved L^2 Carleman inequalities with the bounded weight $e^{\beta\varphi_\lambda(x_1)}$. It is also similar to the situation in the proof of Ionescu and Jerison [4] of the absence of positive eigenvalues of Schrödinger operators $-\Delta + V$ on $\mathbf{R}^n$: to eliminate the possibility of all L^2 solutions one needs a Carleman inequality with nonconvex weights.

Assume as before that $V \in L^{(n+2)/2}(\mathbf{R}^n \times [0,1])$ and $F \equiv 0$. Assume that $u_1, u_2 \in C([0,1]:L^2(\mathbf{R}^n))$ are solutions to (1.1), with $Hu_1, Hu_2 \in L^{(2n+4)/(n+4)}(\mathbf{R}^n \times [0,1])$. For any $w_0 \in \mathbf{R}^n$, $|w_0|=1$, let $D(w_0) = \{x : x \cdot w_0 > 0\}$ denote a half-space. The Carleman inequality (1.2) and an additional local argument can be used to prove that

$$\text{if } u_1 \equiv u_2 \text{ in } D(w_0) \times \{0,1\} \quad \text{then } u_1 \equiv u_2 \text{ in } \mathbf{R}^n \times [0,1].$$

In Theorems 2.4 and 2.5 we prove uniqueness statements of this type under more general assumptions on the potential V and the function F. We also have an existence theorem: If $u(\,\cdot\,,0) \in L^2(\mathbf{R}^n)$, $V \in L^{(n+2)/2}(\mathbf{R}^n \times [0,1])$ and $F \equiv 0$, the equation (1.1) admits a unique solution $u \in C([0,1]:L^2(\mathbf{R}^n))$, with $Hu \in L^{(2n+4)/(n+4)}(\mathbf{R}^n \times [0,1])$ (see Theorem 2.7).

The rest of the paper is organized as follows: In §2 we set up the notation and state the main theorems. The first of our main theorems is the Carleman inequality in Theorem 2.1. We prove this inequality in §§3–8: First we construct suitable parametrices of the conjugated operator $e^{\beta\varphi_\lambda(x_1)}(i\partial_t+\Delta_x)e^{-\beta\varphi_\lambda(x_1)}$ (§3). To construct the parametrices at a given frequency, we think of the equation as either an evolution in time, or a reverse evolution in time, or an evolution in the variable x_1. Then we prove that these parametrices are represented by operators which are bounded between Strichartz spaces (§§4–7). The key technical ingredient we need is a theorem of Keel and Tao [7] that gives a simple criterion for checking this boundedness. In §8 we prove that the remainder terms in the parametrices are small. In §9 we apply the Carleman inequality to prove the uniqueness theorems described above.

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2. The main theorems

We define the set $\mathcal{A}$ of *acceptable* Strichartz exponents (p,q) by the conditions

$$\frac{2}{p}+\frac{n}{q}=\frac{n+4}{2}, \quad p\in[1,2], \quad q\in[1,2], \quad (p,q)\neq(2,1). \tag{2.1}$$

For any $(p,q)\in\mathcal{A}$ let (p',q') denote the dual exponent, i.e. $1/p+1/p'=1/q+1/q'=1$. Clearly $2/p'+n/q'=n/2$, $p'\in[2,\infty]$, $q'\in[2,\infty]$ and $(p',q')\neq(2,\infty)$; let $\mathcal{A}'$ denote the set of such exponents (p',q'). The basic Strichartz spaces we will work with are

$$L_t^pL_x^q=L_t^pL_x^q(\mathbf{R}^n\times\mathbf{R})=\{f\in L^1_{\mathrm{loc}}(\mathbf{R}^n\times\mathbf{R}):\|f\|_{L_t^pL_x^q}<\infty\},$$

where $(p,q)\in\mathcal{A}$ or $(p,q)\in\mathcal{A}'$.

We define two Banach spaces of functions X and X' on $\mathbf{R}^n\times\mathbf{R}$: if $n=1$ then $X=L_t^1L_x^2+L_t^{4/3}L_x^1$ and $X'=L_t^\infty L_x^2\cap L_t^4L_x^\infty$, i.e.

$$\|f\|_X=\inf_{f_1+f_2=f}[\|f_1\|_{L_t^1L_x^2(\mathbf{R}^n\times\mathbf{R})}+\|f_2\|_{L_t^{4/3}L_x^1(\mathbf{R}^n\times\mathbf{R})}]$$

and

$$\|f\|_{X'}=\max\{\|f\|_{L_t^\infty L_x^2(\mathbf{R}^n\times\mathbf{R})},\|f\|_{L_t^4L_x^\infty(\mathbf{R}^n\times\mathbf{R})}\}.$$

If $n\geqslant3$ we define $X=L_t^1L_x^2+L_t^2L_x^{2n/(n+2)}$ and $X'=L_t^\infty L_x^2\cap L_t^2L_x^{2n/(n-2)}$. In dimension $n=2$ we have to exclude the endpoint spaces $L_t^2L_x^1$ and $L_t^2L_x^\infty$ for which the Strichartz estimates fail (cf. [16]). For this purpose we fix an acceptable pair (p_0,q_0), $1\leqslant p_0<2$, and define $X=X_{p_0}=L_t^1L_x^2+L_t^{p_0}L_x^{q_0}$ and $X'=X'_{p_0}=L_t^\infty L_x^2\cap L_t^{p_0'}L_x^{q_0'}$. Spaces of this type were used in recent work by Koch and Tataru [14], [15]. They are often more suitable for Carleman inequalities than the spaces $L_t^p L_x^q$, since they allow better control of the error terms. We notice that

$$L_t^p L_x^q \subseteq X \quad \text{and} \quad L_t^{p'} L_x^{q'} \supseteq X'$$

if $(p,q)\in\mathcal{A}$ (and $p\leqslant p_0$ if $n=2$), and

$$\int_{\mathbf{R}^n\times\mathbf{R}} fg\,dx\,dt \leqslant \|f\|_X \|g\|_{X'}$$

for any locally integrable functions f and g.

For any interval $[a,b]$, we define the Banach space $X([a,b])$ as the space of locally integrable functions $f\colon\mathbf{R}^n\times[a,b]\to\mathbf{C}$ with

$$\|f\|_{X([a,b])} := \|\tilde{f}\|_X < \infty, \tag{2.2}$$

where $\tilde{f}(x,t)=f(x,t)$ if $t\in[a,b]$ and $\tilde{f}(x,t)=0$ if $t\notin[a,b]$. We define the space $X'([a,b])$ in a similar way. For a distribution $f\in\mathcal{S}'(\mathbf{R}^n)$, we set, by a slight abuse of notation,

$$\|f\|_{X([a,b])} := \sup_\eta \|\eta(t)f\|_{X([a,b])}, \tag{2.3}$$

where the supremum is taken over all smooth functions $\eta\colon\mathbf{R}\to[0,1]$ supported in (a,b). Clearly, $\|f\|_{X([a,b])}$ can be finite only if f agrees with a locally integrable function in $\mathbf{R}^n\times(a,b)$. Also, the definitions (2.2) and (2.3) clearly agree for functions $f\in X([a,b])$.

Let H denote the operator $i\partial_t+\Delta_x$ acting (in the sense of distributions) on functions in $L^2(\mathbf{R}^n\times\mathbf{R})$. For any interval $[a,b]$, we define the space $Z([a,b])$ as the space of locally integrable functions $u\colon\mathbf{R}^n\times[a,b]\to\mathbf{C}$ with the properties

$$u\in C([a,b]:L^2(\mathbf{R}^n)) \quad \text{and} \quad \|Hu\|_{X([a,b])}<\infty. \tag{2.4}$$

The meaning of the first condition is that u is a continuous mapping from the interval $[a,b]$ to $L^2(\mathbf{R}^n)$. The second condition is to be interpreted as in (2.3). Corollary 1.4 in [7] shows that

$$\|u\|_{X'([a,b])} \leqslant C\|Hu\|_{X([a,b])}+C\|u(\,\cdot\,,a)\|_{L^2(\mathbf{R}^n)}$$

if $u\in Z([a,b])$. In particular, $Z([a,b])\subseteq X'([a,b])$.

Let φ denote a fixed smooth function on $\mathbf{R}$ with the following properties: $\varphi(0)=0$, φ' nonincreasing, $\varphi'(r)=1$ if $r\leqslant 1$, and $\varphi'(r)=0$ if $r\geqslant 2$. For any $\lambda\geqslant 1$ let $\varphi_\lambda(r)=\lambda\varphi(r/\lambda)$. Clearly $\varphi_\lambda(r)=r$ if $r\leqslant\lambda$, and the function $r\mapsto\varphi_\lambda(r)$ is increasing and bounded.

In this section and in the rest of the paper, we will use the letters C and c to denote constants that may depend only on the dimension n if $n\neq 2$, and on the exponent p_0 if $n=2$. For any set E, χ_E will denote its characteristic function. Our first main theorem is a Carleman inequality.

THEOREM 2.1. *There is an increasing function* $\Lambda\colon[0,\infty)\to[0,\infty)$ *and a constant* C *such that*

$$\begin{aligned}\|e^{\beta\varphi_\lambda(x_1)}u\|_{X'([-T,T])}\leqslant C\|e^{\beta\varphi_\lambda(x_1)}Hu\|_{X([-T,T])}&\\ +C[\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,-T)\|_{L^2(\mathbf{R}^n)}+\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,T)\|_{L^2(\mathbf{R}^n)}]&\end{aligned}\tag{2.5}$$

for any $u\in Z([-T,T])$, *any* $\beta\in[0,\infty)$ *and any* $\lambda\geqslant T^{1/2}\Lambda(T^{1/2}\beta)$.

The norm $\|e^{\beta\varphi_\lambda(x_1)}Hu\|_{X([-T,T])}$ is defined as in (2.3). A weaker form of the Carleman inequality (2.5) was implicitly proved by Kenig, Ponce and Vega [9] by the use of energy methods. This implicit result in [9] corresponds to the inequality (2.5) with the spaces X and X' replaced by $L^1_tL^2_x$ and $L^\infty_tL^2_x$, respectively. Most likely, however, the energy methods of [9] cannot be used to prove the L^p estimates in Theorem 2.1. Our proof of Theorem 2.1 is based on constructing suitable parametrices.

The estimates in §8 show that we may take

$$\Lambda(\beta)=C(1+\beta)^6$$

for some large constant C. In our applications it is important to have a Carleman inequality like (2.5) with bounded weights $e^{\beta\varphi_\lambda(x_1)}$ (which is equivalent to $\lambda<\infty$). Such a Carleman inequality can be applied to a large class of functions u, not just to those u that have faster-than-exponential decay. We remark that a Carleman inequality like (2.5) with nonconvex weights can only hold for functions u with bounded support in t, i.e. if $T<\infty$. Without this support restriction it is only possible to prove a Carleman inequality with linear weights.

COROLLARY 2.2. *For any* $u\in C_0^\infty(\mathbf{R}^n\times\mathbf{R})$ *and any* $\beta\in\mathbf{R}$,

$$\|e^{\beta x_1}u\|_{X'}\leqslant C\|e^{\beta x_1}Hu\|_X.$$

Corollary 2.2 follows from Theorem 2.1 with T larger than the time support of u and $\lambda\to\infty$. Since $L^{2(n+2)/(n+4)}_{x,t}\subseteq X$ and $L^{2(n+2)/n}_{x,t}\supseteq X'$, this improves the Carleman inequality of Kenig and Sogge [13]. Notice that the case $\beta=0$ is equivalent to the Strichartz estimates for the Schrödinger operator, including the endpoint estimate of Keel and Tao [7]. Such estimates have a long history, starting with the fundamental paper of Strichartz [19]; for more references on the development of Strichartz-type estimates for the wave equation and the Schrödinger equation, we refer the reader to the recent work of Keel and Tao [7], where a nontrivial endpoint estimate is proved.

Our main applications concern quantitative and qualitative properties of solutions of nonlinear Schrödinger equations of the form

$$Hu=Vu+F(u),\tag{2.6}$$

where V is a potential and $F:\mathbf{C}\to\mathbf{C}$ is a continuous function. We define the Banach space Y in such a way that

$$\|Vu\|_X \leqslant \|V\|_Y \|u\|_{X'}. \tag{2.7}$$

Thus, if $n=1$ then $Y=L^1_tL^\infty_x+L^2_tL^1_x$. If $n\geqslant 3$ then $Y=L^1_tL^\infty_x+L^\infty_tL^{n/2}_x$. If $n=2$ then $Y=L^1_tL^\infty_x+L^{p_0/(2-p_0)}_tL^{q_0/(2-q_0)}_x$. Notice that $L^{(n+2)/2}_{x,t}\subseteq Y$ in any dimension n, if p_0 is sufficiently close to 2. For any interval $[a,b]$, we also define the space $Y[a,b]$ as the Banach space of functions $V:\mathbf{R}^n\times[a,b]\to\mathbf{C}$, with $\|V\|_{Y[a,b]}=\|\widetilde{V}\|_Y$, $\widetilde{V}\equiv V$ if $t\in[a,b]$, and $\widetilde{V}\equiv 0$ if $t\notin[a,b]$.

Let $\overline{C}$ denote the constant in Theorem 2.1 and $\bar{c}=1/2\overline{C}$. We have the following quantitative estimate:

THEOREM 2.3. *Assume that $V:\mathbf{R}^n\times[0,1]\to\mathbf{C}$ is a potential with the property that*

$$\|V\|_{Y([0,1])}\leqslant \bar{c}. \tag{2.8}$$

Assume that $u\in Z([0,1])$ and

$$Hu=Vu \quad \textit{in } X([0,1]).$$

Then

$$\sup_{t\in[0,1]} \|e^{\beta x_1}u(\,\cdot\,,t)\|_{L^2(\mathbf{R}^n)} \leqslant C[\|e^{\beta x_1}u(\,\cdot\,,0)\|_{L^2(\mathbf{R}^n)}+\|e^{\beta x_1}u(\,\cdot\,,1)\|_{L^2(\mathbf{R}^n)}]$$

uniformly in $\beta\in\mathbf{R}$.

We consider now uniqueness properties of solutions of the Schrödinger equation (2.6). We are concerned with the following question: Assume that $u_1,u_2\in C([0,1]:L^2(\mathbf{R}^n))$ are (weak) solutions to (2.6) with the property that for some domain $D\subseteq\mathbf{R}^n$ we have $u_1=u_2$ in $D\times\{0,1\}$. Under what assumptions on F, V, u_1 and u_2 can we then conclude that $u_1\equiv u_2$ (or $u_1=u_2$ in $D'\times[0,1]$ for some domain D')?

For the nonlinear term F, we make the assumption that there is a function $G:\mathbf{C}\to[0,\infty)$ with the property that

$$|F(z_1)-F(z_2)|\leqslant |z_1-z_2|(G(z_1)+G(z_2)) \tag{2.9}$$

for any $z_1,z_2\in\mathbf{C}$. For any unit vector w_0, let $D(w_0)=\{x:x\cdot w_0>0\}$ denote a half-space.

THEOREM 2.4. *Assume that $u_1,u_2\in C([0,1]:L^2(\mathbf{R}^n))\cap X'([0,1])$ are (weak) solutions of the nonlinear Schrödinger equation*

$$Hu=Vu+F(u) \quad \textit{in } \mathcal{S}'(\mathbf{R}^n\times(0,1)).$$

Let $W=|V|+G(u_1)+G(u_2)$; *assume that*

$$W\in Y([0,1]) \quad \text{and} \quad \|W\chi_{bw_0+D(w_0)}(x)\|_{Y([0,1])}\leqslant \bar{c} \quad \text{for some } b\in\mathbf{R}. \tag{2.10}$$

If $u_1=u_2$ *in* $[bw_0+D(w_0)]\times\{0,1\}$, *then* $u_1\equiv u_2$ *in* $[bw_0+D(w_0)]\times[0,1]$.

We notice that $|F(u)|\leqslant|F(0)|+|u|(G(u)+G(0))$. Since $u_1,u_2\in X'([0,1])$, it follows from (2.10) that $Vu_1+F(u_1), Vu_2+F(u_2)\in X([0,1])+L^\infty(\mathbf{R}^n\times[0,1])\subseteq\mathcal{S}'(\mathbf{R}^n\times\mathbf{R})$.

Using a local unique continuation argument we also prove a global vanishing theorem. Our local unique continuation argument is sharper than the one used by Kenig, Ponce and Vega [9], who assumed that the functions u_1 and u_2 agree in the complement of a convex cone at times 0 and 1.

THEOREM 2.5. *Assume that* $u_1,u_2\in C([0,1]:L^2(\mathbf{R}^n))\cap X'([0,1])$ *are (weak) solutions of the nonlinear Schrödinger equation*

$$Hu=Vu+F(u) \quad \text{in } \mathcal{S}'(\mathbf{R}^n\times(0,1)).$$

Let $W=|V|+G(u_1)+G(u_2)$; *assume that*

$$W\in L_t^{p_1}L_x^{q_1}(\mathbf{R}^n\times[0,1])+L_t^{p_2}L_x^{q_2}(\mathbf{R}^n\times[0,1]) \tag{2.11}$$

for some $p_1,q_1,p_2,q_2\in[1,\infty)$ *with* $2/p_1+n/q_1\leqslant 2$ *and* $2/p_2+n/q_2\leqslant 2$. *If* $u_1=u_2$ *in* $[bw_0+D(w_0)]\times\{0,1\}$ *for some* $b\in\mathbf{R}$, *then* $u_1\equiv u_2$ *in* $\mathbf{R}^n\times[0,1]$.

We remark that (2.10) and (2.11) are, in fact, conditions on the potential V, the function F, *and* the space of solutions u_1 and u_2. For technical reasons, (2.11) is slightly more restrictive than (2.10). In fact, Theorem 2.5 holds if the assumption (2.11) is replaced by the less restrictive assumptions (2.10) and (9.3), see the proof in §9.

Example 2.6. Assume that $V\in L_t^{p_1}L_x^{q_1}$ with p_1 and q_1 as in (2.11), that $G(z)=C(|z|^{a_1}+|z|^{a_2})$, $a_1,a_2\in(0,\infty)$, and that $u_1,u_2\in C([0,1]:L^2\cap L^\infty)$. Then (2.11) holds (with p_2 and q_2 large) and Theorem 2.5 applies (compare with [9, Theorem 1.1]).

We conclude with a theorem concerning well-posedness in $Z([0,T])$.

THEOREM 2.7. *Assume that* $V:\mathbf{R}^n\times[0,T]$ *is a potential with the property that there is* $\varepsilon>0$ *such that*

$$\|V\|_{Y([a,a+\varepsilon])}\leqslant\bar{c} \quad \text{for any } a\in[0,T-\varepsilon]. \tag{2.12}$$

Then the initial value problem

$$\begin{cases} (i\partial_t+\Delta_x)u=Vu,\\ u(\,\cdot\,,0)=u_0,\end{cases}$$

$u_0 \in L^2(\mathbf{R}^n)$, admits a unique solution $u \in Z([0,T])$ with

$$\|u\|_{X'([0,T])} \leqslant C(T)\|u_0\|_{L^2}.$$

The proof of this theorem is routine and probably known: it follows from the Strichartz estimates, Duhamel's formula and a fixed-point argument (for details, see [5]). The counterexample in [5, §3] shows that the space of potentials Y (see (2.12)) is optimal for local well-posedness.

3. Proof of Theorem 2.1: construction of parametrices

Notice first that it suffices to prove the following simplified version of Theorem 2.1:

LEMMA 3.1. *With the same notation as in Theorem 2.1 we have*

$$\|e^{\beta\varphi_\lambda(x_1)}u(x,t)\|_{X'} \leqslant C\|e^{\beta\varphi_\lambda(x_1)}(Hu)(x,t)\|_X \tag{3.1}$$

for any function $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ supported in $\mathbf{R}^n \times [-T,T]$, any $\beta \in [0,\infty)$ and any $\lambda \geqslant T^{1/2}\Lambda(T^{1/2}\beta)$.

To deduce Theorem 2.1 from Lemma 3.1 we show first that for any $\varepsilon \in \left(0, \frac{1}{10}T\right]$ the bound (3.1) holds uniformly for any $v \in Z([-T,T])$ supported in $\mathbf{R}^n \times [-T+\varepsilon, T-\varepsilon]$. Let $\psi: \mathbf{R}^n \times \mathbf{R} \to [0,\infty)$ denote a smooth function supported in the set $\{(x,t): |x|, |t| \leqslant 1\}$ with $\int_{\mathbf{R}^n \times \mathbf{R}} \psi(x,t)\,dx\,dt = 1$, and for $0 < \delta < \min\left\{\frac{1}{10}\varepsilon^{1/2}, 1\right\}$ let $\psi_\delta(x,t) = \delta^{-(n+2)}\psi(x/\delta, t/\delta^2)$. Let $\tilde{\psi}: \mathbf{R}^n \to [0,1]$ denote a smooth function equal to 1 in the set $\{x: |x| \leqslant 1\}$ and equal to 0 in the set $\{x: |x| \geqslant 2\}$, and for $R \geqslant 1$ let $\tilde{\psi}_R(x) = \tilde{\psi}(x/R)$. We apply (3.1) to the smooth, compactly supported function

$$u(x,t) = (v * \psi_\delta)(x,t)\tilde{\psi}_R(x).$$

Then

$$\begin{aligned} \|e^{\beta\varphi_\lambda(x_1)}(v*\psi_\delta)(x,t)\tilde{\psi}_R(x)\|_{X'} \leqslant C\|e^{\beta\varphi_\lambda(x_1)}&(Hv*\psi_\delta)(x,t)\tilde{\psi}_R(x)\|_X \\ +C\|e^{\beta\varphi_\lambda(x_1)}&[|\nabla_x(v*\psi_\delta)(x,t)|\cdot|\nabla_x\tilde{\psi}_R(x)| \\ &+|(v*\psi_\delta)(x,t)|\cdot|\Delta_x\tilde{\psi}_R(x)|]\|_{L_t^1 L_x^2}. \end{aligned} \tag{3.2}$$

For the term in the second line of (3.2) notice that $|\nabla_x(v*\psi_\delta)(x,t)| \leqslant C\delta^{-1}(|v|*\chi_\delta)(x,t)$, where χ is the characteristic function of the set $\{(x,t): |x|, |t| \leqslant 1\}$ and $\chi_\delta(x,t) = \delta^{-(n+2)}\chi(x/\delta, t/\delta^2)$. Also, for the term in the third line, $|(v*\psi_\delta)(x,t)| \leqslant C(|v|*\chi_\delta)(x,t)$.

Since $v\in C([-T,T]:L^2(\mathbf{R}^n))$ and the weight $e^{\beta\varphi_\lambda(x_1)}$ is bounded, the term in the second and third lines of (3.2) converges to 0 as $R\to\infty$. Then we let $\delta\to 0$ to conclude that

$$\|e^{\beta\varphi_\lambda(x_1)}v\|_{X'}\leqslant C\|e^{\beta\varphi_\lambda(x_1)}Hv\|_X, \tag{3.3}$$

if $v\in Z([-T,T])$ is supported in $\mathbf{R}^n\times[-T+\varepsilon,T-\varepsilon]$.

To deduce Theorem 2.1 apply the inequality (3.3) to the function

$$v(x,t)=u(x,t)\eta_\varepsilon(t),$$

where $\varepsilon\leqslant\frac{1}{10}T$, and the smooth cutoff functions $\eta_\varepsilon:[-T,T]\to[0,1]$ have the properties $\eta_\varepsilon(t)=1$ if $t\in[-T+2\varepsilon,T-2\varepsilon]$, $\eta_\varepsilon(t)=0$ if $t\notin[-T+\varepsilon,T-\varepsilon]$, and $\int_{\mathbf{R}}|\eta'_\varepsilon(t)|=2$. Clearly

$$Hv(x,t)=Hu(x,t)\eta_\varepsilon(t)+iu(x,t)\partial_t\eta_\varepsilon(t).$$

By (3.3),

$$\|e^{\beta\varphi_\lambda(x_1)}u(x,t)\eta_\varepsilon(t)\|_{X'}\leqslant C\|e^{\beta\varphi_\lambda(x_1)}Hu(x,t)\eta_\varepsilon(t)\|_X+C\|e^{\beta\varphi_\lambda(x_1)}u(x,t)\eta'_\varepsilon(t)\|_{L^1_tL^2_x}.$$

Recall that the weight $e^{\beta\varphi_\lambda(x_1)}$ is bounded. By (2.4) we may let ε tend to 0; the Carleman inequality (2.5) follows.

We turn now to the proof of Lemma 3.1. By rescaling (using the anisotropic dilations $(x,t)\mapsto(\delta x,\delta^2 t)$) we can assume that $T=1$. Let

$$f=(i\partial_t+\Delta_x)u. \tag{3.4}$$

We have to prove that

$$\|e^{\beta\varphi_\lambda(x_1)}u\|_{X'}\leqslant C\|e^{\beta\varphi_\lambda(x_1)}f\|_X \tag{3.5}$$

for $\lambda\geqslant\Lambda(\beta)$ and $u\in C_0^\infty(\mathbf{R}^n\times(-1,1))$. We will assume from now on that $\lambda\geqslant(\beta+1)^2$. Let $U=e^{\beta\varphi_\lambda(x_1)}u$ and $F=e^{\beta\varphi_\lambda(x_1)}f$. The estimate (3.5) is equivalent to

$$\|U\|_{X'}\leqslant C\|F\|_X. \tag{3.6}$$

The equation (3.4) is equivalent to

$$(i\partial_t+\Delta_x-a_{\beta,\lambda}(x_1)\partial_{x_1}+b_{\beta,\lambda}(x_1))U=F, \tag{3.7}$$

where $a_{\beta,\lambda}=2\beta\varphi'_\lambda$ and $b_{\beta,\lambda}=\beta^2[\varphi'_\lambda]^2-\beta\varphi''_\lambda$. We have $a_{\beta,\lambda}(x_1)\in[0,2\beta]$; more importantly, for any integer $j\geqslant 0$ and $x_1\in[\lambda,2\lambda]$,

$$(\beta+1)^{-1}|\partial^j a_{\beta,\lambda}(x_1)|+(\beta+1)^{-2}|\partial^j b_{\beta,\lambda}(x_1)|\leqslant C_j\lambda^{-j}. \tag{3.8}$$

The term in the left-hand side of (3.8) vanishes if $x_1 \notin [\lambda, 2\lambda]$ and $j \geqslant 1$.

Let $\psi: \mathbf{R} \to [0,1]$ denote a smooth, even cutoff function supported in the interval $[-2,2]$ and equal to 1 in the interval $[-1,1]$. Let χ_+, χ_- and χ_1 denote the characteristic functions of the intervals $[0,\infty)$, $(-\infty,0]$ and $[-1,1]$, respectively. For numbers $\gamma \geqslant 1$ let $\psi_\gamma(r) = \psi(r/\gamma)$. We fix $\gamma = C(\beta+1)$, where C is a large constant. We define the operators A_+, A_-, $\tilde{A}$ and B (acting on Schwartz functions on $\mathbf{R}^n \times \mathbf{R}$) by Fourier multipliers:

A_- defined by the Fourier multiplier $\chi_-(\xi_1)\psi_\gamma(\xi_1)$,

A_+ defined by the Fourier multiplier $\chi_+(\xi_1)\psi_\gamma(\xi_1)$,

$\tilde{A}$ defined by the Fourier multiplier $[1-\psi_\gamma(\xi_1)][1-\psi(10(\tau+|\xi|^2)/\xi_1^2)]$,

B defined by the Fourier multiplier $[1-\psi_\gamma(\xi_1)]\psi(10(\tau+|\xi|^2)/\xi_1^2)$.

The variables τ, ξ_1, etc., are the dual variables to t, x_1, etc., and clearly

$$A_- + A_+ + \tilde{A} + B = \mathrm{Id}.$$

For $\varepsilon > 0$, let P_ε denote the operator defined by the Fourier multiplier $(\xi,\tau) \mapsto e^{-\varepsilon^2|\xi|^2}$, and Q_ε the operator defined by the Fourier multiplier $(\xi,\tau) \mapsto e^{-\varepsilon^2(\tau+|\xi|^2)^2}$. We will prove the estimates

$$\|\chi_1(t)P_\varepsilon A_-(U)\|_{X'} \leqslant C\|F\|_X + C_1(\gamma,\lambda)\|U\|_{L_t^\infty L_x^2}, \tag{3.9}$$

$$\|\chi_1(t)P_\varepsilon A_+(U)\|_{X'} \leqslant C\|F\|_X + C_1(\gamma,\lambda)\|U\|_{L_t^\infty L_x^2}, \tag{3.10}$$

$$\|\chi_1(t)Q_\varepsilon P_\varepsilon \tilde{A}(U)\|_{X'} \leqslant C\|F\|_X + C_1(\gamma,\lambda)\|U\|_{L_t^\infty L_x^2} \tag{3.11}$$

$$\|\chi_1(t)P_\varepsilon B(U)\|_{X'} \leqslant C\|F\|_X. \tag{3.12}$$

The constant $C_1(\gamma,\lambda)$ is small if λ is sufficiently large compared to β. Thus the estimates (3.9)–(3.12) would suffice to prove (3.6).

The parametrices for A_- and A_+. In this case the variable ξ_1 is much smaller than λ. We construct the parametrices starting from the equation (3.7), as if the functions $a_{\beta,\lambda}$ and $b_{\beta,\lambda}$ were constant. Consider the integral

$$\begin{aligned} I_-(F)(x,t) = \int_{\mathbf{R}^n}\int_{-\infty}^{t} F(y,s) \int_{\mathbf{R}^n} & e^{i(x-y)\cdot\xi} e^{-i(t-s)|\xi|^2} \\ & \times \psi_\gamma(\xi_1)\chi_-(\xi_1) e^{-\varepsilon^2|\xi|^2} e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)} e^{ib_{\beta,\lambda}(y_1)(t-s)}\, d\xi\, ds\, dy. \end{aligned}$$

Recall that $F(y,s) = (i\partial_s + D_y)U(y,s)$, where $D_y = \Delta_y - a_{\beta,\lambda}(y_1)\partial_{y_1} + b_{\beta,\lambda}(y_1)$. We substitute this into the formula of $I_-(F)(x,t)$ and integrate by parts in s and y. The result is

$$\begin{aligned} I_-(F)(x,t) &= \int_{\mathbf{R}^n} iU(y,t) \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}\psi_\gamma(\xi_1)\chi_-(\xi_1)e^{-\varepsilon^2|\xi|^2}\,d\xi\,dy \\ &\quad + \int_{\mathbf{R}^n}\int_{-\infty}^{t} U(y,s)[-i\partial_s+D_y^*] \\ &\quad\quad \times \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}\psi_\gamma(\xi_1)\chi_-(\xi_1)e^{-\varepsilon^2|\xi|^2} \\ &\quad\quad \times e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)}e^{ib_{\beta,\lambda}(y_1)(t-s)}\,d\xi\,ds\,dy \\ &= cP_\varepsilon A_-(U)(x,t)+c\widetilde{R}_1(U)(x,t), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \widetilde{R}_1(U)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} U(y,s)\chi_+(t-s) \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2} \\ \times\psi_\gamma(\xi_1)\chi_-(\xi_1)e^{-\varepsilon^2|\xi|^2}e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)}e^{ib_{\beta,\lambda}(y_1)(t-s)}q_1(y_1,\xi_1,t,s)\,d\xi\,ds\,dy. \end{aligned}$$

The function $q_1(y_1,\xi_1,t,s)$ can be written explicitly by inspecting the identity above; the important fact is that when we compute $-i\partial_s+D_y^*$, all the terms that are not small cancel out. The remaining terms have either a derivative of $a_{\beta,\lambda}$ or a derivative of $b_{\beta,\lambda}$. By (3.8), if $|t-s|\leqslant 2$ and $1+|\xi_1|\leqslant C\gamma$, we have

$$|q_1(y_1,\xi_1,t,s)|+\gamma|\partial_{\xi_1}q_1(y_1,\xi_1,t,s)|+\lambda|\partial_{y_1}q_1(y_1,\xi_1,t,s)|\leqslant C\frac{\gamma^3}{\lambda} \tag{3.14}$$

for $y_1\in[\lambda,2\lambda]$, and the left-hand side of (3.14) vanishes if $y_1\notin[\lambda,2\lambda]$. From (3.13) we get

$$P_\varepsilon A_-(U) = cI_-(F)+c\widetilde{R}_1(U). \tag{3.15}$$

To summarize, for (3.9), we have to prove first that the operator

$$T_1(g)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_1(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from X to X', where

$$\mu_1(y_1,\xi_1,t,s)=\chi_1(t)\chi_1(s)\chi_+(t-s)\psi_\gamma(\xi_1)\chi_-(\xi_1)e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)}e^{ib_{\beta,\lambda}(y_1)(t-s)}. \tag{3.16}$$

In addition we have to prove that the operator

$$R_1(g)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}s_1(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from $L_s^\infty L_y^2$ to X' with small norm, where

$$\begin{aligned} s_1(y_1,\xi_1,t,s) = {} & \chi_1(t)\chi_1(s)\chi_+(t-s)\psi_\gamma(\xi_1)\chi_-(\xi_1) \\ & \times e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)} e^{ib_{\beta,\lambda}(y_1)(t-s)} q_1(y_1,\xi_1,t,s). \end{aligned} \tag{3.17}$$

(Note that the role of the various signs is clear: because of the exponential term we need that $a_{\beta,\lambda}(y_1)\xi_1(t-s)\leqslant 0$, and this is achieved because $t-s\geqslant 0$, $\xi_1\leqslant 0$ and $a_{\beta,\lambda}(y_1)\geqslant 0$.)

The construction for A_+ is similar; the only changes are to replace the function μ_1 with

$$\mu_2(y_1,\xi_1,t,s) = \chi_1(t)\chi_1(s)\chi_-(t-s)\psi_\gamma(\xi_1)\chi_+(\xi_1) e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)} e^{ib_{\beta,\lambda}(y_1)(t-s)}, \tag{3.18}$$

and the error function s_1 with

$$\begin{aligned} s_2(y_1,\xi_1,t,s) = {} & \chi_1(t)\chi_1(s)\chi_-(t-s)\psi_\gamma(\xi_1)\chi_+(\xi_1) \\ & \times e^{a_{\beta,\lambda}(y_1)\xi_1(t-s)} e^{ib_{\beta,\lambda}(y_1)(t-s)} q_1(y_1,\xi_1,t,s). \end{aligned} \tag{3.19}$$

We then construct the operators T_2 and R_2 in the same way as the operators T_1 and R_1.

The parametrix for $\tilde{A}$. We start from the integral

$$\begin{aligned} \tilde{I}(F)(x,t) = {} & \int_{\mathbf{R}^n}\int_{\mathbf{R}} F(y,s) \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i(x-y)\cdot\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2} e^{-\varepsilon^2(\tau+|\xi|^2)^2} \\ & \times \frac{[1-\psi_\gamma(\xi_1)][1-\psi(10(\tau+|\xi|^2)/\xi_1^2)]}{-\tau-|\xi|^2-i\xi_1 a_{\beta,\lambda}(y_1)+b_{\beta,\lambda}(y_1)}\, d\tau\, d\xi\, ds\, dy. \end{aligned} \tag{3.20}$$

We substitute the formula (3.7) and integrate by parts. Let

$$q_3(y_1,\xi,\tau) = \frac{1}{-\tau-|\xi|^2-i\xi_1 a_{\beta,\lambda}(y_1)+b_{\beta,\lambda}(y_1)}.$$

The result is

$$\begin{aligned} \tilde{I}(F)(x,t) = {} & c\int_{\mathbf{R}^n}\int_{\mathbf{R}} U(y,s) \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i(x-y)\cdot\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2} \\ & \quad\times e^{-\varepsilon^2(\tau+|\xi|^2)^2} [1-\psi_\gamma(\xi_1)][1-\psi(10(\tau+|\xi|^2)/\xi_1^2)]\, d\tau\, d\xi\, ds\, dy \\ & + c\int_{\mathbf{R}^n}\int_{\mathbf{R}} U(y,s) \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i(x-y)\cdot\xi} e^{i(t-s)\tau} e^{-\varepsilon^2|\xi|^2} e^{-\varepsilon^2(\tau+|\xi|^2)^2} \\ & \quad\times [1-\psi_\gamma(\xi_1)][1-\psi(10(\tau+|\xi|^2)/\xi_1^2)][\partial_{y_1} a_{\beta,\lambda}(y_1) q_3(y_1,\xi,\tau) \\ & \qquad + (a_{\beta,\lambda}(y_1)-2i\xi_1)\partial_{y_1} q_3(y_1,\xi,\tau) + \partial_{y_1}^2 q_3(y_1,\xi,\tau)]\, d\tau\, d\xi\, ds\, dy \\ = {} & cQ_\varepsilon P_\varepsilon \tilde{A}(U)(x,t) + c\widetilde{R}_3(U)(x,t). \end{aligned} \tag{3.21}$$

Thus, for (3.11), we have to prove first that the operator

$$T_3(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_3(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from X to X', where

$$\begin{aligned}\mu_3(y_1,\xi_1,t,s)&=\chi_1(t)\chi_1(s)[1-\psi_\gamma(\xi_1)]\\&\quad\times\int_{\mathbf{R}} e^{i(t-s)\tau}e^{-\varepsilon^2\tau^2}\frac{1-\psi(10\tau/\xi_1^2)}{-\tau-i\xi_1 a_{\beta,\lambda}(y_1)+b_{\beta,\lambda}(y_1)}\,d\tau.\end{aligned}\tag{3.22}$$

In addition, we have to prove that the operator

$$R_3(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}s_3(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from $L^\infty_s L^2_y$ to X' with small norm, where

$$\begin{aligned}s_3(y_1,\xi_1,t,s)&=\chi_1(t)\chi_1(s)[1-\psi_\gamma(\xi_1)]\int_{\mathbf{R}} e^{i(t-s)\tau}e^{-\varepsilon^2\tau^2}[1-\psi(10\tau/\xi_1^2)]\\&\quad\times[a'_{\beta,\lambda}(y_1)\tilde q_3(y_1,\xi_1,\tau)+(a_{\beta,\lambda}(y_1)-2i\xi_1)\tilde q_3'(y_1,\xi_1,\tau)+\tilde q_3''(y_1,\xi_1,\tau)]\,d\tau.\end{aligned}\tag{3.23}$$

The notation in (3.23) is $\tilde q_3(y_1,\xi_1,\tau)=[-\tau-i\xi_1 a_{\beta,\lambda}(y_1)+b_{\beta,\lambda}(y_1)]^{-1}$, and the primes denote differentiation with respect to y_1.

The parametrix for B. This is the more delicate case. We think of the equation as an evolution in x_1 rather than t, and start from (3.4) rather than (3.7). Let $\tilde u(x_1,\xi',\tau)$, $\tilde f(x_1,\xi',\tau)$, etc., denote the partial Fourier transforms of the functions u, f, etc., in the variables x' and t. By taking this partial Fourier transform the equation (3.4) becomes

$$[\partial^2_{x_1}-(\tau+|\xi'|^2)]\tilde u(x_1,\xi',\tau)=\tilde f(x_1,\xi',\tau).$$

By using this equation and integrating by parts we have

$$\int_{z_1}^\infty \tilde f(y_1,\xi',\tau)\frac{\sin\big[(z_1-y_1)\sqrt{-(\tau+|\xi'|^2)}\,\big]}{\sqrt{-(\tau+|\xi'|^2)}}\,dy_1=-\tilde u(z_1,\xi',\tau)\tag{3.24}$$

whenever $\tau+|\xi'|^2\leqslant 0$. Let

$$L\big(z_1-y_1,\sqrt{-(\tau+|\xi'|^2)}\,\big)=\chi_+(y_1-z_1)\frac{\sin\big[(z_1-y_1)\sqrt{-(\tau+|\xi'|^2)}\,\big]}{\sqrt{-(\tau+|\xi'|^2)}}.\tag{3.25}$$

We multiply the equation (3.24) by $e^{\beta\varphi_\lambda(z_1)}$ to obtain

$$\widetilde U(z_1,\xi',\tau)=-\int_{\mathbf{R}}\widetilde F(y_1,\xi',\tau)e^{\beta\varphi_\lambda(z_1)-\beta\varphi_\lambda(y_1)}L\big(z_1-y_1,\sqrt{-(\tau+|\xi'|^2)}\,\big)\,dy_1,$$

and take the Fourier transform in z_1 to obtain

$$\widehat{U}(\xi_1,\xi',\tau)=-\int_{\mathbf{R}}\int_{\mathbf{R}} e^{-iz_1\xi_1}\widetilde{F}(y_1,\xi',\tau)e^{\beta\varphi_\lambda(z_1)-\beta\varphi_\lambda(y_1)}L\big(z_1-y_1,\sqrt{-(\tau+|\xi'|^2)}\,\big)\,dz_1\,dy_1.$$

We multiply this by $[1-\psi_\gamma(\xi_1)]\,\psi(10(\tau+|\xi|^2)/\xi_1^2)\,e^{-\varepsilon^2|\xi|^2}$ and notice that

$$\psi(10(\tau+|\xi|^2)/\xi_1^2)=0$$

unless $\tau+|\xi'|^2\in\left[-\frac{6}{5}\xi_1^2,-\frac{4}{5}\xi_1^2\right]$. We use the fact that

$$\widetilde{F}(y_1,\xi',\tau)=\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} F(y_1,y',s)\,e^{-i(y'\cdot\xi'+s\tau)}\,ds\,dy'$$

and take the inverse Fourier transform. The result is

$$P_\varepsilon B(U)(x_1,x',t)=c\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} F(y_1,y',s)K(x_1,y_1,x',y',t,s)\,dy_1\,dy'ds, \tag{3.26}$$

where

$$\begin{aligned}K(x_1,y_1,x',y',t,s)=\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}} & e^{i(x_1-z_1)\xi_1}e^{i(x'-y')\cdot\xi'}e^{i(t-s)\tau}\\ &\times e^{\beta\varphi_\lambda(z_1)-\beta\varphi_\lambda(y_1)}[1-\psi_\gamma(\xi_1)]\,\psi(10(\tau+|\xi|^2)/\xi_1^2)\\ &\times e^{-\varepsilon^2|\xi|^2}L\big(z_1-y_1,\sqrt{-(\tau+|\xi'|^2)}\,\big)\,d\xi'\,d\tau\,d\xi_1\,dz_1.\end{aligned}$$

We make the change of variables $z_1=y_1-\alpha$ and $\tau=-w-|\xi'|^2$. The integral for K becomes

$$\begin{aligned}K(x_1,y_1,x',y',t,s)=\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}} & e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi'|^2}[1-\psi_\gamma(\xi_1)]\,e^{-\varepsilon^2|\xi|^2}\\ &\times\int_{\mathbf{R}}\int_{\mathbf{R}} e^{i\alpha\xi_1}e^{-i(t-s)w}e^{\beta\varphi_\lambda(y_1-\alpha)-\beta\varphi_\lambda(y_1)}\\ &\times\psi(10(\xi_1^2-w)/\xi_1^2)\,L\big(-\alpha,\sqrt{w}\,\big)\,dw\,d\alpha\,d\xi'\,d\xi_1.\end{aligned}$$

The change of variable $w=\xi_1^2r^2$ in the inner integral together with the fact that $L(-\alpha,r)=-\chi_+(\alpha)\sin(\alpha r)/r$ shows that

$$\begin{aligned}K(x_1,y_1,x',y',t,s)=-2\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}} & e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi'|^2}[1-\psi_\gamma(\xi_1)]\,e^{-\varepsilon^2|\xi|^2}\\ &\times\int_0^\infty\int_0^\infty e^{i\alpha\xi_1}e^{-i(t-s)\xi_1^2r^2}e^{\beta\varphi_\lambda(y_1-\alpha)-\beta\varphi_\lambda(y_1)}\\ &\times\psi(10(1-r^2))\sin(\xi_1\alpha r)\,\xi_1\,dr\,d\alpha\,d\xi'\,d\xi_1.\end{aligned}$$

For $r\geqslant 0$ let $\tilde{\psi}(r)=\psi(10(1-r^2))$; clearly $\tilde{\psi}$ is smooth and supported in the interval $\left[\left(\frac{4}{5}\right)^{1/2},\left(\frac{6}{5}\right)^{1/2}\right]$. The formula for K becomes

$$\begin{aligned}K(x_1,y_1,x',y',t,s)=c\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}}&e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi'|^2}[1-\psi_\gamma(\xi_1)]e^{-\varepsilon^2|\xi|^2}\\&\times\int_0^\infty\int_{\mathbf{R}}e^{i\alpha\xi_1}e^{-i(t-s)\xi_1^2r^2}e^{\beta\varphi_\lambda(y_1-\alpha)-\beta\varphi_\lambda(y_1)}\\&\times\tilde{\psi}(r)\sin(\xi_1\alpha r)\xi_1\,dr\,d\alpha\,d\xi'\,d\xi_1.\end{aligned}$$

By (3.26) it is clear that (3.12) follows if we can prove that the operator

$$T_4(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}}g(y,s)\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_4(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from X to X', where

$$\begin{aligned}\mu_4(y_1,\xi_1,t,s)&=\chi_1(t)\chi_1(s)[1-\psi_\gamma(\xi_1)]\\&\quad\times\int_0^\infty\int_{\mathbf{R}}e^{i\alpha\xi_1}e^{-i(t-s)\xi_1^2(r^2-1)}e^{\beta\varphi_\lambda(y_1-\alpha)-\beta\varphi_\lambda(y_1)}\tilde{\psi}(r)\sin(\xi_1\alpha r)\xi_1\,dr\,d\alpha.\end{aligned}\tag{3.27}$$

To summarize, it remains to prove that the operators T_j, $j=1,2,3,4$, are bounded from X to X', and that the operators R_j, $j=1,2,3$, are bounded from $L_s^\infty L_y^2$ to X' with small norm. The estimates for the operators T_j are proved in §§5–7, and the estimates for the operators R_j are proved in §8. We first prove some preliminary symbol-type estimates for the multiplier μ_4 and the associated kernel.

4. Preliminary estimates

We start by defining two spaces of symbols on $\mathbf{R}$. For functions $m\in C^1(\mathbf{R})$ we define the bounded-variation norm

$$\|m\|_{\mathrm{BV}}=\sup_{\eta\in\mathbf{R}}|m(\eta)|+\int_{\mathbf{R}}|m'(\eta)|\,d\eta,\tag{4.1}$$

and define the space $\mathrm{BV}(\mathbf{R})=\{m\in C^1(\mathbf{R}):\|m\|_{\mathrm{BV}}<\infty\}$. Also, for $b\in\mathbf{R}$ and functions $m\in C^1(\mathbf{R}\setminus\{b\})$ we define the Hörmander–Mikhlin norm

$$\|m\|_{\mathrm{HM}^b}=\sup_{\eta\in\mathbf{R}\setminus\{b\}}|m(\eta)|+\sup_{\eta\in\mathbf{R}\setminus\{b\}}|(\eta-b)m'(\eta)|,\tag{4.2}$$

and define the space $\mathrm{HM}^b(\mathbf{R})=\{m\in C^1(\mathbf{R}\setminus\{b\}):\|m\|_{\mathrm{HM}^b}<\infty\}$. Notice that

$$\begin{aligned}\|\eta\mapsto m(a\eta)\|_{\mathrm{BV}}&=\|m\|_{\mathrm{BV}},\\\|\eta\mapsto m(b+\eta)\|_{\mathrm{BV}}&=\|m\|_{\mathrm{BV}}\end{aligned}\tag{4.3}$$

for any $a\in(0,\infty)$ and $b\in\mathbf{R}$, and

$$\begin{gathered}\|\eta\mapsto m(a\eta)\|_{\mathrm{HM}^0}=\|m\|_{\mathrm{HM}^0},\\ \|\eta\mapsto m(b_1+\eta)\|_{\mathrm{HM}^{b_2}}=\|m\|_{\mathrm{HM}^{b_1+b_2}}\end{gathered}\tag{4.4}$$

for any $a\in(0,\infty)$ and $b_1,b_2\in\mathbf{R}$. Also we have

$$\|m_1m_2\|_{\mathrm{BV}}\leqslant 3\|m_1\|_{\mathrm{BV}}\|m_2\|_{\mathrm{BV}},\tag{4.5}$$

$$\|m_1m_2\|_{\mathrm{HM}^b}\leqslant 3\|m_1\|_{\mathrm{HM}^b}\|m_2\|_{\mathrm{HM}^b}\tag{4.6}$$

for any $b\in\mathbf{R}$.

LEMMA 4.1. *Assume that $\|m\|_{\mathrm{BV}}\leqslant 1$. Then we have the uniform bound*

$$\left|\int_{\mathbf{R}}e^{i\delta\xi^2}e^{ia\xi}e^{-\varepsilon^2(\xi-b)^2}m(\xi)\,d\xi\right|\leqslant C|\delta|^{-1/2}$$

for any $\delta\in\mathbf{R}\setminus\{0\}$, $a,b\in\mathbf{R}$ and $\varepsilon\in(0,\infty)$.

Proof. By a linear change of variable using (4.3) we can assume that $\delta=\pm1$ and $a=0$. Then we break up the integral into two parts, corresponding to $|\xi|$ small and $|\xi|$ large, and integrate by parts when $|\xi|\geqslant 1$. The estimate follows easily. □

LEMMA 4.2. *Assume that $a_1,...,a_k$ are real numbers and that the functions $m_j\in C^1(\mathbf{R}\setminus\{a_j\})$ have the property that $\|m_j\|_{\mathrm{HM}^{a_j}}\leqslant 1$ for $j=1,2,...,k$. Then we have the uniform bound*

$$\left|\int_{\mathbf{R}}e^{i\delta\xi^2}e^{ia\xi}e^{-\varepsilon^2(\xi-b)^2}m_1(\xi)\,...\,m_k(\xi)\,d\xi\right|\leqslant C_k|\delta|^{-1/2}\tag{4.7}$$

for any $\delta\in\mathbf{R}\setminus\{0\}$, $a,b\in\mathbf{R}$ and $\varepsilon\in(0,\infty)$.

Proof. By a linear change of variable using (4.4) we can assume that $\delta=\pm1$ and $a=0$. Let $\widetilde{m}(\xi)=e^{-\varepsilon^2(\xi-b)^2}m_1(\xi)\,...\,m_k(\xi)$ and B denote the set of numbers $b,a_1,...,a_k$. Clearly $\widetilde{m}\in L^1(\mathbf{R})$ and

$$|\widetilde{m}(\xi)|+\operatorname{dist}(\xi,B)|\widetilde{m}'(\xi)|\leqslant C_k$$

for any $\xi\in\mathbf{R}\setminus B$. By breaking up the integral in (4.7) into at most $2k+2$ integrals we see that it suffices to prove that

$$\left|\int_A^{\tilde{A}}e^{i\delta\xi^2}m(\xi)\,d\xi\right|\leqslant C$$

uniformly in $A,\tilde{A}\in\mathbf{R}$, provided that $\delta=\pm1$ and

$$|m(\xi)|+|(\xi-A)m'(\xi)|\leqslant 1.$$

This follows by a routine integration-by-parts argument. □

The first main lemma in this section concerns the multiplier μ_4:

LEMMA 4.3. *The multiplier μ_4 in (3.27) satisfies the bound*

$$\|\mu_4(\,\cdot\,,\xi_1,t,s)\|_{\mathrm{BV}_{y_1}} \leqslant C \tag{4.8}$$

uniformly in ξ_1, t and s.

Proof. By taking limits we can assume that $t\neq s$. We will assume that $t-s>0$ (the case $t-s<0$ then follows since $\mu_4(y_1,\xi_1,t,s)=\overline{\mu_4(y_1,-\xi_1,-t,-s)}$). Let $A=2(t-s)^{1/2}$. Thus $A\in(0,\sqrt{8}\,]$. In the integral in (3.27) that defines the multiplier μ_4 we make the change of variable $\alpha=2(t-s)^{1/2}\theta=A\theta$. We then have

$$\mu_4(y_1,\xi_1,t,s)=2\chi_1(t)\chi_1(s)[1-\psi_\gamma(\xi_1)]\,I(y_1,(t-s)^{1/2}\xi_1), \tag{4.9}$$

where

$$I(y_1,\eta_1)=\int_0^\infty\!\!\int_{\mathbf{R}} e^{2i\eta_1\theta}e^{-i\eta_1^2(r^2-1)}e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)}\tilde{\psi}(r)\sin(2\eta_1\theta r)\,\eta_1\,dr\,d\theta. \tag{4.10}$$

It suffices to prove that the function I has bounded variation in y_1, i.e.

$$\|I(\,\cdot\,,\eta_1)\|_{\mathrm{BV}_{y_1}} \leqslant C$$

for any $\eta_1\in\mathbf{R}$, provided that $A\in(0,C]$. Assume first that $|\eta_1|\leqslant 2$. In this case we write the integral for the function I in the form

$$I(y_1,\eta_1)=\int_{\mathbf{R}} \chi_+(\theta)e^{2i\eta_1\theta}e^{i\eta_1^2}e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)}H(\eta_1,\theta)\,d\theta, \tag{4.11}$$

where

$$H(\eta_1,\theta)=\int_{\mathbf{R}} \eta_1 e^{-i\eta_1^2r^2}\sin(2\eta_1\theta r)\tilde{\psi}(r)\,dr. \tag{4.12}$$

Notice that

$$|H(\eta_1,\theta)|\leqslant C|\eta_1|(1+|\eta_1\theta|)^{-2}$$

if $|\eta_1|\leqslant 2$. Thus

$$\begin{aligned}\|I(\,\cdot\,,\eta_1)\|_{\mathrm{BV}_{y_1}} &\leqslant C\int_{\mathbf{R}} \chi_+(\theta)\,\|y_1\mapsto e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)}\|_{\mathrm{BV}_{y_1}}|H(\eta_1,\theta)|\,d\theta\\ &\leqslant C\int_{\mathbf{R}} |\eta_1|(1+|\eta_1\theta|)^{-2}\,d\theta\leqslant C,\end{aligned}$$

as desired (we used the fact that the function $y_1\mapsto e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)}$ takes values in the interval $[0,1]$ for any $A\theta\geqslant 0$ and is nondecreasing in y_1, and thus has bounded variation).

It remains to prove the same estimate in the case $|\eta_1|\geqslant 2$. We start from (4.11) and (4.12). Recall that the function $\tilde{\psi}$ is smooth and supported in the interval $\left[\left(\frac{4}{5}\right)^{1/2}, \left(\frac{6}{5}\right)^{1/2}\right]$. Let $\psi_1: \mathbf{R}\to[0,1]$ be a smooth function supported in the set $\{\eta: |\eta|\in\left[\frac{4}{5}, \frac{6}{5}\right]\}$, and equal to 1 in the set $\{\eta: |\eta|\in\left[\left(\frac{4}{5}\right)^{3/4}, \left(\frac{6}{5}\right)^{3/4}\right]\}$. We have

$$\begin{aligned} H(\eta_1,\theta) &= \frac{1}{2i} e^{i\theta^2}\int_{\mathbf{R}} \eta_1 e^{-i\theta^2} e^{-i\eta_1^2 r^2}[e^{2i\eta_1 r\theta}-e^{-2i\eta_1 r\theta}]\tilde{\psi}(r)\,dr \\ &= \frac{1}{2i} e^{i\theta^2}\int_{\mathbf{R}} \eta_1 [e^{-i(\eta_1 r-\theta)^2}-e^{-i(\eta_1 r+\theta)^2}]\tilde{\psi}(r)\,dr \\ &= \frac{1}{2i} e^{i\theta^2}\int_{\mathbf{R}} e^{-ir^2}[\tilde{\psi}((r+\theta)/\eta_1)-\tilde{\psi}((r-\theta)/\eta_1)]\,dr \\ &= e^{i\theta^2}(H_0(\eta_1,\theta)+H_1(\eta_1,\theta)), \end{aligned} \tag{4.13}$$

where

$$H_0(\eta_1,\theta) = [1-\psi_1(\theta/\eta_1)]\frac{1}{2i}\int_{\mathbf{R}} e^{-ir^2}[\tilde{\psi}((r+\theta)/\eta_1)-\tilde{\psi}((r-\theta)/\eta_1)]\,dr$$

and

$$H_1(\eta_1,\theta) = \psi_1(\theta/\eta_1)\frac{1}{2i}\int_{\mathbf{R}} e^{-ir^2}[\tilde{\psi}((r+\theta)/\eta_1)-\tilde{\psi}((r-\theta)/\eta_1)]\,dr.$$

By the support properties of the functions $\tilde{\psi}$ and ψ_1 we can integrate by parts in the integral defining $H_0(\eta_1,\theta)$ to obtain

$$|H_0(\eta_1,\theta)| \leqslant C(1+|\theta|)^{-2} \tag{4.14}$$

if $|\eta_1|\geqslant 1$. Also, the function $H_1(\eta_1,\theta)$ is supported in the set $\{(\eta_1,\theta): |\theta/\eta_1|\in\left[\frac{4}{5}, \frac{6}{5}\right]\}$. We substitute the formula (4.13) into the definition (4.11) of the function I, and decompose $I(y_1,\eta_1)=I_0(y_1,\eta_1)+I_1(y_1,\eta_1)$ corresponding to the terms $e^{i\theta^2}H_0$ and $e^{i\theta^2}H_1$. By (4.14) and an argument similar to the one used in the case $|\eta_1|\leqslant 2$ we have

$$\|I_0(\cdot,\eta_1)\|_{\mathrm{BV}_{y_1}} \leqslant C.$$

It remains to prove a similar estimate for the function I_1. We have

$$\begin{aligned} I_1(y_1,\eta_1) &= \int_{\mathbf{R}} \chi_+(\theta) e^{2i\eta_1\theta} e^{i\eta_1^2} e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)} e^{i\theta^2} H_1(\eta_1,\theta)\,d\theta \\ &= \int_{\mathbf{R}} \chi_+(\alpha-\eta_1) e^{i\alpha^2} e^{\beta\varphi_\lambda(y_1-A(\alpha-\eta_1))-\beta\varphi_\lambda(y_1)} H_1(\eta_1,\alpha-\eta_1)\,d\alpha. \end{aligned}$$

We consider two cases depending on the sign of η_1. It is somewhat harder to prove estimates if η_1 is negative, so we will concentrate on this case. Since $|\eta_1|\geqslant 2$ we can assume that $\eta_1 \leqslant -2$. By the support property of the function H_1 and because of the factor $\chi_+(\alpha-\eta_1)$, the variable α in the integral representing I_1 runs over the interval $\alpha \in \left[-\frac{1}{5}|\eta_1|, \frac{1}{5}|\eta_1|\right]$. Thus

$$I_1(y_1,\eta_1) = \int_{\mathbf{R}} e^{i\alpha^2} e^{\beta\varphi_\lambda(y_1 - A(\alpha-\eta_1)) - \beta\varphi_\lambda(y_1)} \tilde{\psi}_1(\alpha/|\eta_1|) H_1(\eta_1, \alpha-\eta_1)\, d\alpha,$$

where $\tilde{\psi}_1$ is a smooth function supported in the interval $\left[-\frac{2}{9}, \frac{2}{9}\right]$ and equal to 1 in the interval $\left[-\frac{1}{5}, \frac{1}{5}\right]$. Let $\delta_0, \delta: \mathbf{R} \to [0,1]$ denote two smooth functions with the property that

$$1 = \delta_0(\alpha) + \sum_{j \geqslant 1} \delta(2^{-j}\alpha)$$

for any $\alpha \in \mathbf{R}$. We can also assume that δ_0 is supported in the interval $[-2,2]$ and δ is supported in the set $\left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$. We insert this partition of unity into the integral formula defining I_1; the result is

$$I_1(y_1,\eta_1) = \sum_{j \geqslant 0} I_1^j(y_1,\eta_1),$$

where, with $\delta_j(\alpha) = \delta(2^{-j}\alpha)$ for any $j \geqslant 1$,

$$I_1^j(y_1,\eta_1) = \int_{\mathbf{R}} e^{i\alpha^2} e^{\beta\varphi_\lambda(y_1 - A(\alpha-\eta_1)) - \beta\varphi_\lambda(y_1)} \delta_j(\alpha) \tilde{\psi}_1(\alpha/|\eta_1|) H_1(\eta_1, \alpha-\eta_1)\, d\alpha.$$

The main estimate we will prove is

$$\|I_1^j(\cdot\,,\eta_1)\|_{\mathrm{BV}_{y_1}} \leqslant C 2^{-j} \tag{4.15}$$

for any integer $j \geqslant 0$. Notice that for $j=0$ this follows by the same argument as in the case $|\eta_1| \leqslant 2$. We only need to notice that by Lemma 4.1,

$$|H_1(\eta_1, \alpha-\eta_1)| \leqslant C$$

uniformly in η_1 and α.

We turn to the proof of (4.15) in the case $j \geqslant 1$. By a change of variable, the integral for I_1^j becomes

$$I_1^j(y_1,\eta_1) = 2^j \int_{\mathbf{R}} e^{i2^{2j}\alpha^2 + \beta\varphi_\lambda(y_1 - A(2^j\alpha - \eta_1)) - \beta\varphi_\lambda(y_1)} \tilde{\delta}(\alpha,\eta_1)\, d\alpha, \tag{4.16}$$

where $\tilde{\delta}(\alpha,\eta_1) = \delta(\alpha)\tilde{\psi}_1(2^j\alpha/|\eta_1|) H_1(\eta_1, 2^j\alpha - \eta_1)$. The function $\tilde{\delta}(\alpha,\eta_1)$ is smooth and supported in the set $\left\{\alpha : |\alpha| \in \left[\frac{1}{2}, 2\right]\right\}$. By integrating by parts in the formula of H_1 it is easy to see that

$$|H_1(\eta_1,\theta)| + |\eta_1 \partial_\theta H_1(\eta_1,\theta)| \leqslant C$$

if $\eta_1 \leqslant -1$. Thus if $9\cdot 2^{j-2} \leqslant |\eta_1|$ and $|\alpha| \in \left[\frac{1}{2}, 2\right]$, we have

$$|\tilde{\delta}(\alpha,\eta_1)|+|\partial_\alpha \tilde{\delta}(\alpha,\eta_1)| \leqslant C. \tag{4.17}$$

Clearly $\tilde{\delta}(\alpha,\eta_1)\equiv 0$ if $9\cdot 2^{j-2} > |\eta_1|$.

In (4.16) we integrate by parts in α to obtain

$$\begin{aligned} I_1^j(y_1,\eta_1) = -2^j \int_{\mathbf{R}} & e^{i2^{2j}\alpha^2+\beta\varphi_\lambda(y_1-A(2^j\alpha-\eta_1))-\beta\varphi_\lambda(y_1)} \\ & \times \partial_\alpha \frac{\tilde{\delta}(\alpha,\eta_1)}{i2^{2j+1}\alpha-\beta A 2^j \varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))}\, d\alpha. \end{aligned} \tag{4.18}$$

Since $A \leqslant C$ and $|\varphi_\lambda''(r)| \leqslant C/\lambda \leqslant C/\beta$ for any $r \in \mathbf{R}$, we have by (4.17),

$$\left| \partial_\alpha \frac{\tilde{\delta}(\alpha,\eta_1)}{i2^{2j+1}\alpha-\beta A 2^j \varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))} \right| \leqslant C 2^{-2j}. \tag{4.19}$$

Thus

$$|I_1^j(y_1,\eta_1)| \leqslant C 2^{-j} \tag{4.20}$$

uniformly in y_1 and η_1, as desired.

By taking the y_1-derivative in (4.18) we have

$$\begin{aligned} |\partial_{y_1} I_1^j(y_1)| \leqslant 2^j \int_{|\alpha|\in[1/2,2]} & [\beta\varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))-\beta\varphi_\lambda'(y_1)] \\ & \times e^{\beta\varphi_\lambda(y_1-A(2^j\alpha-\eta_1))-\beta\varphi_\lambda(y_1)} \\ & \times \left| \partial_\alpha \frac{\tilde{\delta}(\alpha,\eta_1)}{i2^{2j+1}\alpha-\beta A 2^j \varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))} \right| d\alpha \\ +2^j \int_{|\alpha|\in[1/2,2]} & e^{\beta\varphi_\lambda(y_1-A(2^j\alpha-\eta_1))-\beta\varphi_\lambda(y_1)} \\ & \times \left| \partial_\alpha \partial_{y_1} \frac{\tilde{\delta}(\alpha,\eta_1)}{i2^{2j+1}\alpha-\beta A 2^j \varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))} \right| d\alpha \\ = J_1(y_1)+J_2(y_1). & \end{aligned} \tag{4.21}$$

By (4.19)

$$\|J_1\|_{L^1_{y_1}} \leqslant C 2^{-j}, \tag{4.22}$$

as desired. For $J_2(y_1)$ we estimate the $\partial_\alpha \partial_{y_1}$-derivative. By (4.17),

$$\begin{aligned} & \left| \partial_\alpha \partial_{y_1} \frac{\tilde{\delta}(\alpha,\eta_1)}{i2^{2j+1}\alpha-\beta A 2^j \varphi_\lambda'(y_1-A(2^j\alpha-\eta_1))} \right| \\ & \quad \leqslant C 2^{-3j}\beta A |\varphi_\lambda''(y_1-A(2^j\alpha-\eta_1))| + C 2^{-2j}\beta A^2 |\varphi_\lambda'''(y_1-A(2^j\alpha-\eta_1))|. \end{aligned} \tag{4.23}$$

Notice that the second term in the right-hand side of (4.23) is dominated by

$$C2^{-2j}\beta\lambda^{-2}\chi_{[\lambda,2\lambda]}(y_1-A(2^j\alpha-\eta_1)).$$

Since $\beta\leqslant\lambda$ this suffices to control the second term. For the first term we recall that $9\cdot 2^{j-2}\leqslant|\eta_1|$ and $\eta_1\leqslant-1$. Thus $2^j\alpha-\eta_1\geqslant c2^j$ if $|\alpha|\in\left[\frac{1}{2},2\right]$, and so, to prove that $\|J_2\|_{L^1_{y_1}}\leqslant C2^{-j}$, it suffices to prove that

$$2^{-j}\beta A\int_{\mathbf{R}}|\varphi''_\lambda(y_1)|e^{\beta\varphi_\lambda(y_1)-\beta\varphi_\lambda(y_1+cA2^j)}\,dy_1\leqslant C. \tag{4.24}$$

The function φ'_λ is nonincreasing and nonnegative. Thus

$$\beta\varphi_\lambda(y_1)-\beta\varphi_\lambda(y_1+cA2^j)\leqslant -c\beta A2^j\varphi'_\lambda(y_1+cA2^j).$$

Therefore, the expression in the left-hand side of (4.24) can be dominated by

$$\begin{aligned}2^{-j}\beta A\int_{\mathbf{R}}-\varphi''_\lambda(y_1+cA2^j)e^{-c\beta A2^j\varphi'_\lambda(y_1+cA2^j)}\,dy_1&\\ +2^{-j}\beta A\int_{\mathbf{R}}|\varphi''_\lambda(y_1)-\varphi''_\lambda(y_1+cA2^j)|\,dy_1.&\end{aligned} \tag{4.25}$$

The first term in (4.25) can be dominated by $C2^{-2j}$, and the second term can be dominated by $C\beta A^2\lambda^{-1}\leqslant C$. Thus (4.24) follows. The main estimate (4.15) follows by (4.20) and (4.22). This completes the proof of the lemma. □

We will now prove an estimate for the kernel of the operator T_4. Recall that the operators T_j are of the form

$$T_j(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}}g(y,s)\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_j(y_1,\xi_1,t,s)\,d\xi\,ds\,dy,$$

where the multipliers μ_j are defined in (3.16), (3.18), (3.22) and (3.27). Let

$$K_j(x,y,t,s)=\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_j(y_1,\xi_1,t,s)\,d\xi \tag{4.26}$$

and

$$k_j(x_1,y_1,t,s)=\int_{\mathbf{R}}e^{i(x_1-y_1)\xi_1}e^{-i(t-s)\xi_1^2}e^{-\varepsilon^2\xi_1^2}\mu_j(y_1,\xi_1,t,s)\,d\xi_1. \tag{4.27}$$

Note that the integral representing K_j splits as a product of n integrals, the first of which is the integral representing k_j. In this section we prove estimates for the kernel k_4.

Assume $t-s>0$ and, as in Lemma 4.3, let $A=2(t-s)^{1/2}$. By (4.9), (4.27) and the change of variable $\xi_1=2\eta_1/A$, we have

$$k_4(x_1,y_1,t,s)=\chi_1(t)\chi_1(s)\frac{C}{A}\int_{\mathbf{R}} e^{i2\eta_1(x_1-y_1)/A}e^{-i\eta_1^2}e^{-\varepsilon_1^2\eta_1^2}[1-\psi_\gamma(2\eta_1/A)]I(y_1,\eta_1)\,d\eta_1$$

with $\varepsilon_1=2\varepsilon/A$. For the function I we use the integral formula (4.10). Then

$$\begin{aligned}k_4(x_1,y_1,t,s)=\chi_1(t)\chi_1(s)\frac{C}{A}\int_0^\infty\int_{\mathbf{R}}\tilde{\psi}(r)e^{\beta\varphi_\lambda(y_1-A\theta)-\beta\varphi_\lambda(y_1)}\\ \times\int_{\mathbf{R}} e^{-i\eta_1^2r^2}e^{i2\eta_1[(x_1-y_1)/A+\theta]}\sin(2\eta_1\theta r)\eta_1 e^{-\varepsilon_1^2\eta_1^2}[1-\psi_\gamma(2\eta_1/A)]\,d\eta_1\,dr\,d\theta.\end{aligned}\tag{4.28}$$

To compute the η_1-integral notice that

$$\int_{\mathbf{R}} e^{-a\eta_1^2+b\eta_1}\,d\eta_1=Ca^{-1/2}e^{b^2/4a}\tag{4.29}$$

for any $a,b\in\mathbf{R}$, $a>0$. By taking a derivative with respect to b we have

$$\int_{\mathbf{R}} e^{-a\eta_1^2+b\eta_1}\eta_1\,d\eta_1=Ca^{-3/2}be^{b^2/4a}\tag{4.30}$$

for any $a,b\in\mathbf{R}$, $a>0$. By analytic continuation, (4.30) holds for any $a,b\in\mathbf{C}$ with $\operatorname{Re}a>0$. Let $H_A=H_{A,\varepsilon,\gamma}$ denote the inverse Fourier transform of the function

$$\eta_1\longmapsto e^{-\varepsilon_1^2\eta_1^2}[1-\psi_\gamma(2\eta_1/A)],$$

so that

$$e^{-\varepsilon_1^2\eta_1^2}[1-\psi_\gamma(2\eta_1/A)]=C\int_{\mathbf{R}} H_A(\alpha)e^{-i\eta_1\alpha}\,d\alpha.$$

We also have $\|H_A\|_{L^1}\leqslant C$ uniformly. By (4.30) the η_1-integral in (4.28) is equal to

$$\begin{aligned}Cr^{-3}\int_{\mathbf{R}} H_A(\alpha)\big(e^{i[(x_1-y_1)/A+\theta+\theta r-\alpha/2]^2/r^2}\big[(x_1-y_1)/A+\theta+\theta r-\tfrac{1}{2}\alpha\big]\\ -e^{i[(x_1-y_1)/A+\theta-\theta r-\alpha/2]^2/r^2}\big[(x_1-y_1)/A+\theta-\theta r-\tfrac{1}{2}\alpha\big]\big)\,d\alpha.\end{aligned}$$

To rewrite the integral in (4.28) let

$$\begin{aligned}F_{A,\pm}(\tilde{x}_1,\tilde{y}_1)=\int_0^\infty\int_{\mathbf{R}}\tilde{\psi}(r)r^{-3}e^{\beta\varphi_\lambda(A\tilde{y}_1-A\theta)-\beta\varphi_\lambda(A\tilde{y}_1)}\\ \times e^{i(\tilde{x}_1-\tilde{y}_1+\theta\pm\theta r)^2/r^2}(\tilde{x}_1-\tilde{y}_1+\theta\pm\theta r)\,dr\,d\theta.\end{aligned}$$

Then, by (4.28)

$$\begin{aligned}k_4(x_1,y_1,t,s)=\chi_1(t)\chi_1(s)\frac{C}{A}\int_{\mathbf{R}} H_A(\alpha)\\ \times\big[F_{A,+}(x_1/A-\tfrac{1}{2}\alpha,y_1/A)-F_{A,-}(x_1/A-\tfrac{1}{2}\alpha,y_1/A)\big]\,d\alpha.\end{aligned}\tag{4.31}$$

LEMMA 4.4. *We have*

$$F_{A,\pm}(\tilde{x}_1,\tilde{y}_1)=e^{i(\tilde{x}_1-\tilde{y}_1)^2}m_{A,\pm}(\tilde{x}_1,\tilde{y}_1)+J_{A,\pm}(\tilde{x}_1,\tilde{y}_1), \tag{4.32}$$

where

$$\|m_{A,\pm}(\tilde{x}_1,\cdot\,)\|_{\mathrm{BV}_{\tilde{y}_1}}+\|m_{A,\pm}(\,\cdot\,,\tilde{y}_1)\|_{\mathrm{BV}_{\tilde{x}_1}}\leqslant C \tag{4.33}$$

uniformly in $\tilde{x}_1$ *and* $\tilde{y}_1$, *and*

$$(1+|\tilde{x}_1-\tilde{y}_1|)\,|J_{A,\pm}(\tilde{x}_1,\tilde{y}_1)|\leqslant C. \tag{4.34}$$

Proof. By a change of variable we have

$$F_{A,\pm}(\tilde{x}_1,\tilde{y}_1)=\int_0^\infty e^{\beta\varphi_\lambda(A\tilde{y}_1-A\theta)-\beta\varphi_\lambda(A\tilde{y}_1)}G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)\,d\theta, \tag{4.35}$$

where

$$G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)=\int_{\mathbf{R}}\psi_0(r)\,e^{i[(\tilde{x}_1-\tilde{y}_1+\theta)r\pm\theta]^2}[(\tilde{x}_1-\tilde{y}_1+\theta)r\pm\theta]\,dr. \tag{4.36}$$

In (4.36), $\psi_0(r)=\tilde{\psi}(1/r)$ is a smooth function supported in the interval $\left[\frac{5}{6},\frac{5}{4}\right]$. Recall that we fixed $\psi:\mathbf{R}\to[0,1]$, a smooth cutoff function supported in the interval $[-2,2]$ and equal to 1 in the interval $[-1,1]$. Let $\widetilde{\chi}_-:\mathbf{R}\to[0,1]$ denote a smooth function supported in the interval $(-\infty,-10]$ and equal to 1 in the interval $(-\infty,-20]$. Let

$$\begin{aligned}m_{A,\pm}(\tilde{x}_1,\tilde{y}_1)&=e^{-i(\tilde{x}_1-\tilde{y}_1)^2}\widetilde{\chi}_-(\tilde{x}_1-\tilde{y}_1)\\&\quad\times\int_0^\infty\psi(\tilde{x}_1-\tilde{y}_1+\theta)\,e^{\beta\varphi_\lambda(A\tilde{y}_1-A\theta)-\beta\varphi_\lambda(A\tilde{y}_1)}G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)\,d\theta\end{aligned}$$

and

$$\begin{aligned}J_{A,\pm}(\tilde{x}_1,\tilde{y}_1)&=[1-\widetilde{\chi}_-(\tilde{x}_1-\tilde{y}_1)]\int_0^\infty e^{\beta\varphi_\lambda(A\tilde{y}_1-A\theta)-\beta\varphi_\lambda(A\tilde{y}_1)}G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)\,d\theta\\&\quad+\widetilde{\chi}_-(\tilde{x}_1-\tilde{y}_1)\int_0^\infty[1-\psi(\tilde{x}_1-\tilde{y}_1+\theta)]\,e^{\beta\varphi_\lambda(A\tilde{y}_1-A\theta)-\beta\varphi_\lambda(A\tilde{y}_1)}G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)\,d\theta.\end{aligned}$$

The identity (4.32) is clear; it remains to prove the bounds (4.33) and (4.34).

For the bound (4.33) we may assume that $\tilde{y}_1-\tilde{x}_1\geqslant 10$ and make the change of variable $\theta=\tilde{y}_1-\tilde{x}_1+u/(\tilde{y}_1-\tilde{x}_1)$. The formula (4.36) shows that

$$G_\pm(\tilde{x}_1-\tilde{y}_1,\tilde{y}_1-\tilde{x}_1+u/(\tilde{y}_1-\tilde{x}_1))=e^{i(\tilde{y}_1-\tilde{x}_1)^2}(\tilde{y}_1-\tilde{x}_1)H_\pm(u,\tilde{y}_1-\tilde{x}_1), \tag{4.37}$$

where

$$H_\pm(u,\eta)=\int_{\mathbf{R}}\psi_0(r)\,e^{i(\pm 2u(r\pm 1)+u^2(r\pm 1)^2/\eta^2)}[\pm 1+u(r\pm 1)/\eta^2]\,dr.$$

A routine integration-by-parts argument shows that

$$|H_\pm(u,\eta)|+|\partial_\eta H_\pm(u,\eta)|\cdot|\eta|^3\leqslant C(1+|u|)^{-2} \tag{4.38}$$

if $|\eta|\geqslant 1$ and $|u|\leqslant 2|\eta|$. We substitute the formula (4.37) into the definition of the function $m_{A,\pm}$. Thus

$$\begin{aligned} m_{A,\pm}(\tilde{x}_1,\tilde{y}_1) &= \widetilde{\chi}_-(\tilde{x}_1-\tilde{y}_1) \\ &\quad\times\int_{\mathbf{R}}\psi(u/(\tilde{y}_1-\tilde{x}_1))\,H_\pm(u,\tilde{y}_1-\tilde{x}_1)\,e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}\,du. \end{aligned} \tag{4.39}$$

Recall that the function φ_λ is nondecreasing and that $|H_\pm(u,\eta)|\leqslant C(1+|u|)^{-2}$. Thus $|m_{A,\pm}(\tilde{x}_1,\tilde{y}_1)|\leqslant C$. To estimate the derivatives of $m_{A,\pm}$ notice that

$$\begin{aligned} |\partial_{\tilde{x}_1} m_{A,\pm}(\tilde{x}_1,\tilde{y}_1)| &\leqslant C(1+|\tilde{y}_1-\tilde{x}_1|)^{-2}+C\chi_+(\tilde{y}_1-\tilde{x}_1-10) \\ &\quad\times\int_{|u|\leqslant 2(\tilde{y}_1-\tilde{x}_1)}(1+|u|)^{-2}\,|\partial_{\tilde{x}_1}e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}|\,du \end{aligned}$$

and

$$\begin{aligned} |\partial_{\tilde{y}_1} m_{A,\pm}(\tilde{x}_1,\tilde{y}_1)| &\leqslant C(1+|\tilde{y}_1-\tilde{x}_1|)^{-2}+C\chi_+(\tilde{y}_1-\tilde{x}_1-10) \\ &\quad\times\int_{|u|\leqslant 2(\tilde{y}_1-\tilde{x}_1)}(1+|u|)^{-2}\,|\partial_{\tilde{y}_1}e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}|\,du. \end{aligned}$$

These estimates follow easily by inspecting the formula (4.39) and using (4.38). The term $(1+|\tilde{y}_1-\tilde{x}_1|)^{-2}$ in these two estimates is integrable, thus harmless. For (4.33) it remains to prove that for any $u\in\mathbf{R}$,

$$\|\chi_+(\tilde{y}_1-\tilde{x}_1-10)\,\chi_+(2(\tilde{y}_1-\tilde{x}_1)-|u|)\,\partial_{\tilde{x}_1}[e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}]\|_{L^1_{\tilde{x}_1}}\leqslant C \tag{4.40}$$

and

$$\begin{aligned} \|\chi_+(\tilde{y}_1-\tilde{x}_1-10)\,\chi_+(2(\tilde{y}_1-\tilde{x}_1)-|u|)\,\partial_{\tilde{y}_1}[e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}]\|_{L^1_{\tilde{y}_1}} \\ \leqslant C(1+|u|)^{1/2}. \end{aligned} \tag{4.41}$$

For (4.40) we notice that the function $\tilde{x}_1\mapsto A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1)$ is increasing in the interval $\tilde{x}_1\in\left(-\infty,\min\{\tilde{y}_1-10,\tilde{y}_1-\frac{1}{2}|u|\}\right]$. Since φ_λ is a nondecreasing function it follows that $\tilde{x}_1\mapsto e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}$ is a nondecreasing function in the relevant interval, which proves (4.40). To prove (4.41) we notice that if $\tilde{y}_1-\tilde{x}_1\geqslant\max\{10,\frac{1}{2}|u|\}$ and $A\leqslant C$, then

$$\begin{aligned}
&|\partial_{\tilde{y}_1}e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}|\\
&\quad\leqslant e^{\beta\varphi_\lambda(A\tilde{x}_1-Au/(\tilde{y}_1-\tilde{x}_1))-\beta\varphi_\lambda(A\tilde{y}_1)}\left[\beta A\varphi'_\lambda(A\tilde{y}_1)+\frac{\beta A|u|}{(\tilde{y}_1-\tilde{x}_1)^2}\,\varphi'_\lambda\left(A\tilde{x}_1-\frac{Au}{\tilde{y}_1-\tilde{x}_1}\right)\right]\\
&\quad\leqslant 2e^{\beta\varphi_\lambda(A\tilde{x}_1+2A)-\beta\varphi_\lambda(A\tilde{y}_1)}\\
&\qquad\times\left[\beta A\varphi'_\lambda(A\tilde{y}_1)+\frac{\beta A|u|}{(\tilde{y}_1-\tilde{x}_1)^2}\left[\varphi'_\lambda\left(A\tilde{x}_1-\frac{Au}{\tilde{y}_1-\tilde{x}_1}\right)-\varphi'_\lambda(A\tilde{y}_1)\right]\right]\\
&\quad\leqslant Ce^{\beta\varphi_\lambda(A\tilde{x}_1+2A)-\beta\varphi_\lambda(A\tilde{y}_1)}\beta A\varphi'_\lambda(A\tilde{y}_1)+\frac{C\beta|u|}{(\tilde{y}_1-\tilde{x}_1)^2}\min\left\{1,\frac{\tilde{y}_1-\tilde{x}_1}{\lambda}\right\}.
\end{aligned}$$

The estimate (4.41) follows easily by integrating the two terms in the last line of the above estimate and recalling that $1+\beta^2\leqslant\lambda$ (the first term is equal to the derivative of a nonincreasing function). This completes the proof of (4.33).

To estimate the function $J_{A,\pm}$ notice first that

$$|G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)|\leqslant C(1+|\tilde{x}_1-\tilde{y}_1+\theta|)^{-2}$$

if $\tilde{x}_1-\tilde{y}_1\geqslant-20$ and $\theta\geqslant 0$. These estimates follow easily by integrating by parts in (4.36) and using standard bounds for oscillatory integrals. The estimate (4.34) for the functions $J_{A,\pm}$ follows in the range $\tilde{x}_1-\tilde{y}_1\geqslant-20$. In the range $\tilde{x}_1-\tilde{y}_1\leqslant-20$, only the integral in the second line of the formula of $J_{A,\pm}$ does not vanish. If, in addition, $|\tilde{x}_1-\tilde{y}_1+\theta|\geqslant 1$ then we integrate by parts in (4.36). Recall that the function ψ_0 in (4.36) is supported in a small interval around 1. By checking the cases $\theta\leqslant\frac{9}{10}|\tilde{x}_1-\tilde{y}_1|$, $\theta\in[\frac{9}{10}|\tilde{x}_1-\tilde{y}_1|,\frac{11}{10}|\tilde{x}_1-\tilde{y}_1|]$ and $\theta\geqslant\frac{11}{10}|\tilde{x}_1-\tilde{y}_1|$, it is not hard to see that

$$|G_\pm(\tilde{x}_1-\tilde{y}_1,\theta)|\leqslant C(\theta+|\tilde{x}_1-\tilde{y}_1|)^{-2}$$

if $\tilde{x}_1-\tilde{y}_1\leqslant-20$ and $|\tilde{x}_1-\tilde{y}_1+\theta|\geqslant 1$. The estimate (4.34) in the range $\tilde{x}_1-\tilde{y}_1\leqslant-20$ follows. □

This completes our analysis in the case $t-s>0$. If $t-s<0$ then we let $A=2(s-t)^{1/2}$ and argue as before. Notice also that the function $(2/A)H_A(2\alpha/A)=H(\alpha)$ does not depend on A. By rewriting (4.31) and using Lemma 4.4 we have

$$\begin{aligned}
&k_4(x_1,y_1,t,s)\\
&\quad=\frac{1}{|t-s|^{1/2}}\chi_1(t)\chi_1(s)\int_{\mathbf{R}}H(\alpha)e^{i(x_1-\alpha-y_1)^2/4(t-s)}m_4(t,s,x_1-\alpha,y_1)\,d\alpha\\
&\qquad+\frac{1}{|t-s|^{1/2}}\chi_1(t)\chi_1(s)\int_{\mathbf{R}}H(\alpha)J_4(t,s,(x_1-\alpha)/(2|t-s|^{1/2}),y_1/(2|t-s|^{1/2}))\,d\alpha\\
&\quad=k_4^1(x_1,y_1,t,s)+k_4^2(x_1,y_1,t,s),
\end{aligned}\tag{4.42}$$

where $\|H\|_{L^1(\mathbf{R})} \leqslant C$,

$$\|m_4(t,s,\tilde{x}_1,\cdot\,)\|_{\mathrm{BV}_{\tilde{y}_1}} + \|m_4(t,s,\cdot\,,\tilde{y}_1)\|_{\mathrm{BV}_{\tilde{x}_1}} \leqslant C \tag{4.43}$$

uniformly in t, s, $\tilde{x}_1$ and $\tilde{y}_1$, and

$$(1+|\tilde{x}_1-\tilde{y}_1|)|J_4(t,s,\tilde{x}_1,\tilde{y}_1)| \leqslant C. \tag{4.44}$$

5. Boundedness of the operators T_j, I

In this section we start proving that the operators T_1, T_2, T_3 and T_4 are bounded from X to X'. To cover all dimensions fix an acceptable pair (p,q), with $p\leqslant\frac{4}{3}$ if $n=1$, $p\leqslant p_0$ if $n=2$, and $p\leqslant 2$ if $n\geqslant 3$. Clearly an operator is bounded from X to X' if it is bounded from $L^1_sL^2_y$ to $L^\infty_tL^2_x$, from $L^p_sL^q_y$ to $L^\infty_tL^2_x$, from $L^1_sL^2_y$ to $L^{p'}_tL^{q'}_x$, and from $L^p_sL^q_y$ to $L^{p'}_tL^{q'}_x$, with bounds that depend only on the dimension n (or p_0 if $n=2$). Recall that the operators T_j are of the form

$$T_j(g)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_j(y_1,\xi_1,t,s)\,d\xi\,ds\,dy,$$

where the multipliers μ_j are defined in (3.16), (3.18), (3.22) and (3.27).

Proposition 5.1. *The operators T_j, $j=1,2,3,4$, are bounded from $L^1_sL^2_y$ to $L^\infty_tL^2_x$.*

Proof. A simple condition for $L^1_sL^2_y \to L^\infty_tL^2_x$ boundedness of an operator of the form

$$T(g)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is that the operator

$$S_{t,s}(h)(x) = \int_{\mathbf{R}^n} h(y)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu(y_1,\xi_1,t,s)\,d\xi\,dy$$

is bounded on $L^2(\mathbf{R}^n)$ uniformly in t and s. The fact that this condition is sufficient follows easily by the Minkowski inequality for integrals. By Plancherel's theorem it suffices to prove that for any $h\in\mathcal{S}(\mathbf{R})$,

$$\left\|\int_{\mathbf{R}} h(y_1)\mu(y_1,\xi_1,t,s)e^{-iy_1\xi_1}\,dy_1\right\|_{L^2_{\xi_1}} \leqslant C\|h\|_{L^2} \tag{5.1}$$

uniformly in t and s. A simple criterion for this to hold is that μ has bounded variation in y_1:

$$\|\mu(\,\cdot\,,\xi_1,t,s)\|_{\mathrm{BV}_{y_1}} \leqslant C \tag{5.2}$$

uniformly in ξ_1, t and s. The BV-norm was defined in (4.1). To see that (5.2) implies (5.1) we can use Carleson's theorem [2]: the operator

$$\mathcal{C}(h)(\xi_1) = \sup_N \left| \int_{-\infty}^{N} h(y_1) e^{-iy_1\xi_1} \, dy_1 \right|$$

is bounded from $L^2_{y_1}$ to $L^2_{\xi_1}$. Thus, for any ξ_1 we have

$$\begin{aligned}\left| \int_{\mathbf{R}} h(y_1)\mu(y_1,\xi_1,t,s)e^{-iy_1\xi_1}\,dy_1 \right| &= \left| \int_{\mathbf{R}} \left[\int_{-\infty}^{y_1} h(z)e^{-iz\xi_1}\,dz \right]' \mu(y_1,\xi_1,t,s)\,dy_1 \right| \\ &\leqslant |\mu(\infty,\xi_1,t,s)| \cdot |\hat{h}(\xi_1)| \\ &\quad + \left| \int_{\mathbf{R}} \mathcal{C}(h)(\xi_1)\,|\mu'(y_1,\xi_1,t,s)|\,dy_1 \right| \\ &\leqslant \mathcal{C}(h)(\xi_1)\,\|\mu(\,\cdot\,,\xi_1,t,s)\|_{\mathrm{BV}_{y_1}}.\end{aligned}$$

By Carleson's theorem this proves (5.1).

For the multiplier μ_1 in (3.16) notice first that the factor $e^{ib_{\beta,\lambda}(y_1)(t-s)}$ is bounded and depends only on y_1 (and not on ξ_1), so it can be incorporated into h. In addition, the function $e^{-\delta a_{\beta,\lambda}(y_1)}$ is nondecreasing and bounded for any $\delta \geqslant 0$. Thus the bounded-variation condition (5.2) is clearly verified. The same argument applies for the multiplier μ_2 in (3.18).

For the multiplier μ_3 in (3.22) we make the change of variable $\tau = \xi_1^2 u$ and write

$$\begin{aligned}\mu_3(y_1,\xi_1,t,s) &= \chi_1(t)\chi_1(s)[1-\psi_\gamma(\xi_1)] \\ &\quad \times \int_{\mathbf{R}} e^{i(t-s)\xi_1^4 u} e^{-\varepsilon^2\xi_1^2 u^2} \frac{1-\psi(10u)}{-u - ia_{\beta,\lambda}(y_1)/\xi_1 + b_{\beta,\lambda}(y_1)/\xi_1^2}\,du.\end{aligned} \tag{5.3}$$

Notice that the variable u in the integral has the property $|u| \geqslant \frac{1}{10}$ and $|\xi_1| \geqslant \gamma \geqslant C(1+\beta)$. Therefore the integral in (5.3) is the inverse Fourier transform of a Hörmander–Mikhlin multiplier evaluated at $(t-s)\xi_1^2$, and is thus bounded. By differentiating with respect to y_1 we have

$$|\partial_{y_1}\mu_3(y_1,\xi_1,t,s)| \leqslant C[1-\psi_\gamma(\xi_1)] \left(\frac{|a'_{\beta,\lambda}(y_1)|}{|\xi_1|} + \frac{|b'_{\beta,\lambda}(y_1)|}{|\xi_1|^2} \right).$$

By (3.8) this suffices to prove the estimate (5.2) for the multiplier μ_3. Finally, for the multiplier μ_4 the condition (5.2) is proved in Lemma 4.3. □

Using the decomposition $k_4 = k_4^1 + k_4^2$ in (4.42), we decompose the kernel K_4 into $K_4^1 + K_4^2$, where

$$K_4^m(x,y,t,s) = k_4^m(x_1,y_1,t,s) \int_{\mathbf{R}^{n-1}} e^{i(x'-y')\cdot\xi'} e^{-i(t-s)|\xi'|^2} e^{-\varepsilon^2|\xi'|^2}\,d\xi', \quad m=1,2, \tag{5.4}$$

and then decompose the operator T_4 as $T_4^1+T_4^2$. We can use (4.43) and the criterion (5.2) to prove that the kernel $k_4^1(x_1,y_1,t,s)$ defines a bounded operator on $L^2(\mathbf{R})$: since the function H in (4.42) is in $L^1(\mathbf{R})$, it suffices to prove that

$$\left\| \frac{1}{|t-s|^{1/2}} \int_{\mathbf{R}} h(y_1)\, e^{i(x_1-\alpha-y_1)^2/4(t-s)} m_4(t,s,x_1-\alpha,y_1)\, dy_1 \right\|_{L^2_{x_1}} \leqslant C\|h\|_{L^2_{y_1}}.$$

This follows from (5.2) and scaling. By Proposition 5.1 we know that the kernel $k_4(x_1,y_1,t,s)$ defines a bounded operator from $L^2_{y_1}(\mathbf{R})$ to $L^2_{x_1}(\mathbf{R})$. Thus the kernel $k_4^2(x_1,y_1,t,s)$ defines a bounded operator from $L^2_{y_1}(\mathbf{R})$ to $L^2_{x_1}(\mathbf{R})$ as well. To summarize, both kernels $k_4^1(x_1,y_1,t,s)$ and $k_4^2(x_1,y_1,t,s)$ define bounded operators on $L^2(\mathbf{R})$, both kernels $K_4^1(x,y,t,s)$ and $K_4^2(x,y,t,s)$ define bounded operators from $L^2_y(\mathbf{R}^n)$ to $L^2_x(\mathbf{R}^n)$, and both operators T_4^1 and T_4^2 are bounded from $L^1_sL^2_y$ to $L^\infty_tL^2_x$.

6. Boundedness of the operators T_j, II

In this section we prove that the operators T_j are bounded from $L^p_sL^q_y$ to $L^\infty_tL^2_x$ and from $L^1_sL^2_y$ to $L^{p'}_tL^{q'}_x$. For this, we use a theorem of Keel and Tao [7, Theorem 1.2]:

LEMMA 6.1. (Keel and Tao [7]) *Assume that $U(t)\colon L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$ denotes a family of operators indexed over $t\in\mathbf{R}$ with the properties*

$$\|U(t)f\|_{L^2(\mathbf{R}^n)} \leqslant C\|f\|_{L^2(\mathbf{R}^n)}$$

for any $t\in\mathbf{R}$ and $f\in L^2(\mathbf{R}^n)$, and

$$\|U(s)U(t)^*f\|_{L^\infty(\mathbf{R}^n)} \leqslant C|t-s|^{-n/2}\|f\|_{L^1(\mathbf{R}^n)}$$

for any $t,s\in\mathbf{R}^n$ and $f\in S(\mathbf{R}^n)$. Then

$$\|U(t)f\|_{L^{p'}_tL^{q'}_x} \leqslant C\|f\|_{L^2}.$$

PROPOSITION 6.2. *The operators T_1, T_2, T_3 and T_4^1 are bounded from $L^p_sL^q_y$ to $L^\infty_tL^2_x$.*

Proof. An operator of the form

$$T(g)(x,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)K(x,y,t,s)\, ds\, dy$$

is bounded from $L^p_sL^q_y$ to $L^\infty_tL^2_x$ if the operators

$$S_{t_0,\pm}(g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\, e^{-(\varepsilon')^2|x|^2} K(x,y,t_0,s)\chi_\pm(t_0-s)\, ds\, dy$$

are bounded from $L^p_s L^q_y$ to L^2_x uniformly in t_0 and $\varepsilon'>0$. This is equivalent to the fact that the operators $S^*_{t_0,\pm}$ are bounded from L^2_x to $L^{p'}_s L^{q'}_y$ uniformly in t_0 and $\varepsilon'>0$. For this we apply Lemma 6.1. The L^2-condition was already verified in Proposition 5.1 (for the operators T_1, T_2 and T_3) and the remark at the end of §5 (for the operator T^1_4): the kernels $\overline{K}_1(x,y,t,s)$, $\overline{K}_2(x,y,t,s)$, $\overline{K}_3(x,y,t,s)$ and $\overline{K}^1_4(x,y,t,s)$ define bounded operators from L^2_x to L^2_y uniformly in t and s. It remains to check the $L^1\to L^\infty$ bound, i.e.

$$\left|\int_{\mathbf{R}^n} e^{-(\varepsilon')^2|z|^2}K(z,y,t_0,s)\chi_\pm(t_0-s)e^{-(\varepsilon')^2|z|^2}\overline{K}(z,x,t_0,t)\chi_\pm(t_0-t)\,dz\right|\leqslant C|t-s|^{-n/2} \tag{6.1}$$

uniformly in x, y, t and s, where K stands for K_1, K_2, K_3 or K^1_4. For the kernels K_j, $j=1,2,3$, we substitute the formula (4.26) and integrate first the variable z. Notice that all the integrals converge absolutely because of the exponentially decaying factors. It remains to prove that for any $v=(v_1,...,v_n)\in\mathbf{R}^n$ the absolute value of the integral

$$\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{i(t_0-t)2v\cdot\xi}e^{-\varepsilon^2(|\xi|^2+|\xi+v|^2)}\mu_j(y_1,\xi_1,t_0,s)\bar{\mu}_j(x_1,\xi_1+v_1,t_0,t)\,d\xi \tag{6.2}$$

is dominated by $C|t-s|^{-n/2}$, provided that $(t_0-s)(t_0-t)>0$ and $j=1,2,3$. For this we use Lemma 4.2. Notice that the integral in (6.2) splits as a product of n integrals in $\xi_1,\xi_2,...,\xi_n$. The integrals in $\xi_2,...,\xi_n$ are each bounded by $C|t-s|^{-1/2}$ by Lemma 4.2. It remains to prove the same bound for the integral in ξ_1. Let $w_1=x_1-y_1+2v_1(t_0-t)$. We need to prove that

$$\left|\int_{\mathbf{R}} e^{iw_1\xi_1}e^{-i(t-s)\xi_1^2}e^{-\varepsilon^2(\xi_1^2+(\xi_1+v_1)^2)}\mu_j(y_1,\xi_1,t_0,s)\bar{\mu}_j(x_1,\xi_1+v_1,t_0,t)\,d\xi_1\right|\leqslant C|t-s|^{-1/2} \tag{6.3}$$

uniformly in all the variables, where $j=1,2,3$.

The estimate (6.3) for $j=1,2,3$ would follow from Lemma 4.2 with $k=2$, provided that we could verify that the multipliers μ_j, $j=1,2,3$, belong to $\mathrm{HM}^0_{\xi_1}$. This is clear if $j=1$ or $j=2$, simply by inspecting the formulas (3.16) and (3.18) and noticing that the functions $\chi_+(\xi_1)e^{-\delta\xi_1}$ and $\chi_-(\xi_1)e^{\delta\xi_1}$ belong to $\mathrm{HM}^0_{\xi_1}$ uniformly in $\delta\geqslant0$. If $j=3$, we examine the formula (5.3). We already noticed in the proof of Lemma 5.1 that the function μ_3 is bounded; an elementary estimate using the fact that $|\xi_1|\geqslant\gamma\geqslant C(1+\beta)$ shows that it is actually in the symbol class $\mathrm{HM}^0_{\xi_1}$ (see (6.22) below for a more precise estimate).

To prove (6.1) for the kernel K^1_4 we substitute the formula (5.4) into (6.1) and notice that the integral splits as a product of n integrals. By the same argument as before, the integrals in $z_2,...,z_n$ are each bounded by $C|t-s|^{-1/2}$. It remains to prove that

$$\left|\int_{\mathbf{R}} e^{-(\varepsilon')^2z_1^2}k^1_4(z_1,y_1,t_0,s)\chi_\pm(t_0-s)e^{-(\varepsilon')^2z_1^2}\bar{k}^1_4(z_1,x_1,t_0,t)\chi_\pm(t_0-t)\,dz_1\right|\leqslant C|t-s|^{-1/2}$$

uniformly in all the variables. For this we substitute the formula (4.42) and integrate the variable z_1 first. The estimate follows from Lemma 4.1 with $\delta=(s-t)/4(t_0-s)(t_0-t)$. $\square$

PROPOSITION 6.3. *The operator T_4^2 is bounded from $L_s^pL_y^q$ to $L_t^\infty L_x^2$.*

Proof. With the same notation as in Proposition 6.2, we have to prove that the operators $S_{t_0,\pm}$ are bounded from $L_s^pL_y^q$ to L_x^2 uniformly in t_0. This is equivalent to the fact that the operators $S_{t_0,\pm}^*S_{t_0,\pm}$ are bounded from $L_s^pL_y^q$ to $L_t^{p'}L_x^{q'}$. The kernels of the operators $S_{t_0,\pm}^*S_{t_0,\pm}$ are

$$\begin{aligned}&L_{4,t_0,\pm}^2(x,y,t,s)\\&\quad=\int_{\mathbf{R}^n}e^{-(\varepsilon')^2|z|^2}K_4^2(z,y,t_0,s)\,\chi_\pm(t_0-s)\,e^{-(\varepsilon')^2|z|^2}\overline{K}_4^2(z,x,t_0,t)\,\chi_\pm(t_0-t)\,dz.\end{aligned}$$

Let

$$U_{4,t_0,\pm,t,s}^2(h)(x)=\int_{\mathbf{R}^n}L_{4,t_0,\pm}^2(x,y,t,s)h(y)\,dy.$$

We claim that

$$\|U_{4,t_0,\pm,t,s}^2\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)}\leqslant C \tag{6.4}$$

and

$$\begin{aligned}&\|U_{4,t_0,\pm,t,s}^2\|_{L^1(\mathbf{R}^n)\to L^\infty(\mathbf{R}^n)}\\&\qquad\leqslant C|t-s|^{-(n-1)/2}(|t-t_0|+|s-t_0|)^{-1/2}\log\frac{|t-t_0|^2+|s-t_0|^2}{|t-t_0|\cdot|s-t_0|}.\end{aligned} \tag{6.5}$$

uniformly in t, s and t_0. Assuming (6.4) and (6.5) we would have by interpolation

$$\begin{aligned}&\|U_{4,t_0,\pm,t,s}^2\|_{L^q(\mathbf{R}^n)\to L^{q'}(\mathbf{R}^n)}\\&\qquad\leqslant C\left[|t-s|^{-(n-1)/2}(|t-t_0|+|s-t_0|)^{-1/2}\log\frac{|t-t_0|^2+|s-t_0|^2}{|t-t_0|\cdot|s-t_0|}\right]^{2/q-1}.\end{aligned}$$

By the Minkowski inequality for integrals we would have

$$\begin{aligned}&\|S_{t_0,\pm}^*S_{t_0,\pm}(g)(\,\cdot\,,t)\|_{L_x^{q'}}\\&\quad\leqslant\int_{\mathbf{R}}\left\|\int_{\mathbf{R}^n}g(y,s)L_{4,t_0,\pm}^2(x,y,t,s)\,dy\right\|_{L_x^{q'}}ds\\&\quad\leqslant C\int_{\mathbf{R}}\|g(\,\cdot\,,s)\|_{L_y^q}\left[|t-s|^{-(n-1)/2}(|t-t_0|+|s-t_0|)^{-1/2}\log\frac{|t-t_0|^2+|s-t_0|^2}{|t-t_0|\cdot|s-t_0|}\right]^{2/q-1}ds.\end{aligned}$$

We apply Lemma 6.4 below with $\delta=1/n$ to conclude that the kernel

$$\left[|t-s|^{-(n-1)/2}(|t-t_0|+|s-t_0|)^{-1/2}\log\frac{|t-t_0|^2+|s-t_0|^2}{|t-t_0|\cdot|s-t_0|}\right]^{2/q-1}$$

defines a bounded operator from L^p_s to $L^{p'}_t$. Thus

$$\|S^*_{t_0,\pm}S_{t_0,\pm}(g)\|_{L^{p'}_t L^{q'}_x} \leqslant C\|g\|_{L^p_s L^q_y}$$

as desired. It remains to prove (6.4) and (6.5). The L^2-bound (6.4) was already proved in the remark at the end of §5. For (6.5) we need to control the absolute value of the kernels $L^2_{4,t_0,\pm}$. These kernels split as products of n integrals; as in Proposition 6.2 the integrals in $z_2, ..., z_n$ are each bounded by $C|t-s|^{-1/2}$. Thus it remains to prove that

$$\begin{aligned}\left|\int_{\mathbf{R}} e^{-(\varepsilon')^2 z_1^2} k_4^2(z_1, y_1, t_0, s)\chi_\pm(t_0-s) e^{-(\varepsilon')^2 z_1^2} \bar{k}_4^2(z_1, x_1, t_0, t)\chi_\pm(t_0-t)\, dz\right| \\ \leqslant C(|t-t_0|+|s-t_0|)^{-1/2}\log\frac{|t-t_0|^2+|s-t_0|^2}{|t-t_0|\cdot|s-t_0|}\end{aligned} \tag{6.6}$$

uniformly in all the variables. We substitute the formula (4.42) and integrate the variable z_1 first. As before let $A=2|s-t_0|^{1/2}$ and $B=2|t-t_0|^{1/2}$. It suffices to prove that

$$\begin{aligned}\int_{\mathbf{R}} \frac{1}{A}|J_4(t_0, s, (z_1-\alpha_1)/A, y_1/A)|\frac{1}{B}|J_4(t_0, t, (z_1-\alpha_2)/B, x_1/B)|\, dz_1 \\ \leqslant C(A^2+B^2)^{-1/2}\log\frac{A^2+B^2}{AB}.\end{aligned}$$

This follows easily from (4.44). □

In the proof of Proposition 6.3 we used the following lemma:

LEMMA 6.4. *For any $t, s>0$, $\delta\in(0,1]$ and $p\in[1,2]$ let*

$$L_\delta(t,s) = |t-s|^{(-2+2\delta)/p'}(t+s)^{-2\delta/p'}\left[\log\frac{t^2+s^2}{ts}\right]^{4\delta/p'}.$$

For any continuous compactly supported function $f:(0,\infty)\to\mathbf{C}$ let

$$S_\delta f(t) = \int_0^\infty f(s)L_\delta(t,s)\, ds.$$

Then

$$\|S_\delta f\|_{L^{p'}((0,\infty))} \leqslant C_\delta\|f\|_{L^p((0,\infty))}. \tag{6.7}$$

Proof. By analytic interpolation, using the family of kernels

$$L_\delta^z(t,s) = |t-s|^{(-2+2\delta)z}(t+s)^{-2\delta z}\left[\log\frac{t^2+s^2}{ts}\right]^{4\delta z}$$

defined for $\operatorname{Re} z\in[0,\frac{1}{2}]$, we see that it suffices to prove the lemma for $p=p'=2$. In this case, (6.7) is equivalent to a Hardy inequality. Let $(Y,d\mu)=((0,\infty),dt/t)$, and for any $t,s>0$ let $\tilde f(s)=s^{1/2}f(s)$ and $\widetilde F(t)=t^{1/2}S_\delta f(t)$. Then

$$\widetilde F(t)=t^{1/2}\int_0^\infty |t-s|^{-1+\delta}(t+s)^{-\delta}\left[\log\frac{t^2+s^2}{ts}\right]^{2\delta} f(s)\,ds=\int_Y \tilde L_\delta(t,s)\tilde f(s)\,d\mu(s), \tag{6.8}$$

where $\tilde L_\delta(t,s)=t^{1/2}s^{1/2}|t-s|^{-(1-\delta)}(t+s)^{-\delta}[\log(t^2+s^2)/ts]^{2\delta}$. The inequality (6.7) with $p=p'=2$ is equivalent to

$$\|\widetilde F\|_{L^2(Y,d\mu)}\leqslant C_\delta\|\tilde f\|_{L^2(Y,d\mu)}.$$

This follows from (6.8) and the observation that $\|\tilde L_\delta(\,\cdot\,,s)\|_{L^1(Y,d\mu(t))}\leqslant C_\delta$ uniformly in s, and $\|\tilde L_\delta(t,\cdot\,)\|_{L^1(Y,d\mu(s))}\leqslant C_\delta$ uniformly in t, provided that $\delta\in(0,1]$. □

This completes the proof of the $L^p_sL^q_y\to L^\infty_tL^2_x$ boundedness of the operators T_j. We now turn to the question of $L^1_sL^2_y\to L^{p'}_tL^{q'}_x$ boundedness.

PROPOSITION 6.5. *The operators T_1, T_2 and T_4^1 are bounded from $L^1_sL^2_y$ to $L^{p'}_tL^{q'}_x$.*

Proof. As in Proposition 6.2, by using Lemma 6.1 it suffices to prove the uniform bound

$$\left|\int_{\mathbf{R}^n} e^{-(\varepsilon')^2|z|^2}K(x,z,t,s_0)\,e^{-(\varepsilon')^2|z|^2}\overline{K}(y,z,s,s_0)\,dz\right|\leqslant C|t-s|^{-n/2} \tag{6.9}$$

for the kernels $K=K_1$, $K=K_2$ and $K=K_4^1$, under the assumption that $(t-s_0)(s-s_0)>0$. The integrals in (6.9) split as products of n integrals. By the same argument as in Proposition 6.2 the integrals in $z_2,\dots,z_n$ in (6.9) are bounded by $C|t-s|^{-1/2}$ as desired. It remains to prove a similar bound for the integral in z_1. To summarize, it suffices to prove that

$$\left|\int_{\mathbf{R}} e^{-(\varepsilon')^2z_1^2}k(x_1,z_1,t,s_0)\,e^{-(\varepsilon')^2z_1^2}\bar k(y_1,z_1,s,s_0)\,dz_1\right|\leqslant C|t-s|^{-1/2} \tag{6.10}$$

for $k=k_1$, $k=k_2$ and $k=k_4^1$, where $t,s,s_0\in[-1,1]$ and $(t-s_0)(s-s_0)>0$. Assume that $t-s_0>0$ and $s-s_0>0$ (the case $t-s_0<0$ and $s-s_0<0$ is similar). Notice that the bound (6.10) is trivial for $j=2$, since $\mu_2(\,\cdot\,,\cdot\,,t,s_0)\equiv0$ if $t-s_0>0$. Also, for the kernel $k=k_4^1$ the estimate (6.10) can be obtained as in Proposition 6.2. It remains to consider the case $k=k_1$.

Recall that $\gamma\geqslant\beta+1$. Fix $h=h_{x_1,y_1}:\mathbf{R}\to[0,1]$, a smooth function with the following properties:

$$\begin{aligned} h(z_1)&=1 &&\text{if } \min\{|z_1-x_1|,|z_1-y_1|\}\leqslant 10\gamma,\\ h(z_1)&=0 &&\text{if } \min\{|z_1-x_1|,|z_1-y_1|\}\geqslant 20\gamma,\\ |\partial^l_{z_1}h(z_1)|&\leqslant C\gamma^{-l} &&\text{for any } z_1\in\mathbf{R} \text{ and } l=0,1,2.\end{aligned}$$

We use this function to break up the integral in the left-hand side of (6.10) into two parts. For the term that contains the function $1-h$, i.e. when z_1 is far from x_1 and y_1, we integrate by parts in (4.27) and use the fact that $|(x_1-z_1)-2(t-s_0)\xi_1|\geqslant\frac{1}{10}|x_1-z_1|$ if $|\xi_1|\leqslant 2\gamma$ and $t,s_0\in[-1,1]$. The result is

$$|k_1(x_1,z_1,t,s_0)|\leqslant C|x_1-z_1|^{-1}$$

and

$$|\bar{k}_1(y_1,z_1,s,s_0)|\leqslant C|y_1-z_1|^{-1}$$

if $\min\{|z_1-x_1|,|z_1-y_1|\}\geqslant 10\gamma$. Thus

$$|k_1(x_1,z_1,t,s_0)\,\bar{k}_1(y_1,z_1,s,s_0)|\leqslant C(|x_1-z_1|^{-2}+|y_1-z_1|^{-2})$$

if $\min\{|z_1-x_1|,|z_1-y_1|\}\geqslant 10\gamma$. It follows that

$$\left|\int_{\mathbf{R}} e^{-(\varepsilon')^2z_1^2}k_1(x_1,z_1,t,s_0)\,e^{-(\varepsilon')^2z_1^2}\bar{k}_1(y_1,z_1,s,s_0)(1-h(z_1))\,dz_1\right|\leqslant C\gamma^{-1}\leqslant C|t-s|^{-1/2} \tag{6.11}$$

if $|t-s|\leqslant 2$, as desired.

To estimate the term that contains the function h, assume first that $|x_1-y_1|\leqslant 100\gamma$. Let

$$\tilde{k}_1(x_1,z_1,t,s_0)=\int_{\mathbf{R}} e^{i(x_1-z_1)\xi_1}e^{-i(t-s_0)\xi_1^2}e^{-\varepsilon^2\xi_1^2}\mu_1(x_1,\xi_1,t,s_0)\,d\xi_1 \tag{6.12}$$

and

$$\tilde{k}_1(y_1,z_1,s,s_0)=\int_{\mathbf{R}} e^{i(y_1-z_1)\eta_1}e^{-i(s-s_0)\eta_1^2}e^{-\varepsilon^2\eta_1^2}\mu_1(y_1,\eta_1,s,s_0)\,d\eta_1. \tag{6.13}$$

Since $|x_1-z_1|+|y_1-z_1|\leqslant C\gamma$, we can use (3.8) and the formula (3.16) to see that

$$|\mu_1(z_1,\xi_1,t,s_0)-\mu_1(x_1,\xi_1,t,s_0)|+|\mu_1(z_1,\eta_1,s,s_0)-\mu_1(y_1,\eta_1,s,s_0)|\leqslant C\frac{\gamma^3}{\lambda}.$$

By integrating we have

$$|k_1(x_1,z_1,t,s_0)-\tilde{k}_1(x_1,z_1,t,s_0)|+|k_1(y_1,z_1,s,s_0)-\tilde{k}_1(y_1,z_1,s,s_0)|\leqslant C\frac{\gamma^4}{\lambda} \tag{6.14}$$

if $|x_1-z_1|+|y_1-z_1|\leqslant C\gamma$. Also

$$|k_1(x_1,z_1,t,s_0)|+|\tilde{k}_1(x_1,z_1,t,s_0)|+|k_1(y_1,z_1,s,s_0)|+|\tilde{k}_1(y_1,z_1,s,s_0)|\leqslant C\gamma. \tag{6.15}$$

By (6.14) and (6.15) and the fact that $|t-s|\leqslant 2$,

$$\begin{aligned}
&\left|\int_{\mathbf{R}} h(z_1)\, e^{-(\varepsilon')^2 z_1^2} k_1(x_1,z_1,t,s_0)\, e^{-(\varepsilon')^2 z_1^2}\, \bar{k}_1(y_1,z_1,s,s_0)\, dz_1\right| \\
&\qquad \leqslant \left|\int_{\mathbf{R}} h(z_1)\, e^{-(\varepsilon')^2 z_1^2}\, \tilde{k}_1(x_1,z_1,t,s_0)\, e^{-(\varepsilon')^2 z_1^2}\, \bar{\tilde{k}}_1(y_1,z_1,s,s_0)\, dz_1\right| + C\frac{\gamma^6}{\lambda} \qquad (6.16)\\
&\qquad \leqslant \left|\int_{\mathbf{R}} h(z_1)\, e^{-(\varepsilon')^2 z_1^2}\, \tilde{k}_1(x_1,z_1,t,s_0)\, e^{-(\varepsilon')^2 z_1^2}\, \bar{\tilde{k}}_1(y_1,z_1,s,s_0)\, dz_1\right| + C|t-s|^{-1/2},
\end{aligned}$$

provided that $|x_1-y_1|\leqslant 100\gamma$ and

$$\gamma^6 \leqslant \lambda. \tag{6.17}$$

It remains to estimate the first integral in the right-hand side of (6.16). For this we substitute the formulas (6.12) and (6.13), and integrate the variable z_1 first, as in Proposition 6.2. Let $\widetilde{H}=\widetilde{H}_{x_1,y_1}$ denote the Fourier transform of the function $z_1\mapsto h(z_1)\, e^{-2(\varepsilon')^2 z_1^2}$. The properties of the cutoff function h guarantee that

$$\|\widetilde{H}\|_{L^1(\mathbf{R})} \leqslant C. \tag{6.18}$$

We have

$$\begin{aligned}
&\int_{\mathbf{R}} h(z_1)\, e^{-(\varepsilon')^2 z_1^2}\, \tilde{k}_1(x_1,z_1,t,s_0)\, e^{-(\varepsilon')^2 z_1^2}\, \bar{\tilde{k}}_1(y_1,z_1,s,s_0)\, dz_1 \\
&\qquad = \int_{\mathbf{R}} \widetilde{H}(\theta)\, e^{ix_1\theta - i(t-s_0)\theta^2} \int_{\mathbf{R}} e^{-i(t-s)\eta_1^2} e^{i[(x_1-y_1)-2(t-s_0)\theta]\eta_1} \qquad (6.19)\\
&\qquad\qquad \times e^{-\varepsilon^2\eta_1^2}\, \bar{\mu}_1(y_1,\eta_1,s,s_0)\, e^{-\varepsilon^2(\eta_1+\theta)^2} \mu_1(x_1,\eta_1+\theta,s,s_0)\, d\eta_1\, d\theta.
\end{aligned}$$

The multiplier μ_1 belongs to the symbol class $\mathrm{HM}^0_{\eta_1}$. This was checked in Proposition 6.2. By Lemma 4.2, the η_1-integral in (6.19) is bounded by $C|t-s|^{-1/2}$. By (6.16) and (6.18) we have

$$\left|\int_{\mathbf{R}} h(z_1)\, e^{-(\varepsilon')^2 z_1^2} k_1(x_1,z_1,t,s_0)\, e^{-(\varepsilon')^2 z_1^2}\, \bar{k}_1(y_1,z_1,s,s_0)\, dz_1\right| \leqslant C|t-s|^{-1/2} \tag{6.20}$$

if $|x_1-y_1|\leqslant 100\gamma$, as desired.

If $|x_1-y_1|\geqslant 100\gamma$, we break up the integral in (6.20) into two parts, depending on whether z_1 is close to x_1, or z_1 is close to y_1. Assume that we are looking to estimate the integral over $|z_1-x_1|\leqslant 20\gamma$. We argue as before: the only difference is that we replace the kernels $k_1(x_1,z_1,t,s_0)$ and $k_1(y_1,z_1,s,s_0)$ with the kernels

$$\tilde{k}_1(x_1,y_1,z_1,t,s_0) = \int_{\mathbf{R}} e^{i(x_1-z_1)\xi_1} e^{-i(t-s_0)\xi_1^2} e^{-\varepsilon^2\xi_1^2} \mu_1(x_1,\xi_1,t,s_0)\, d\xi_1$$

and

$$\tilde{k}_1(x_1, y_1, z_1, s, s_0) = \int_{\mathbf{R}} e^{i(y_1-z_1)\eta_1} e^{-i(s-s_0)\eta_1^2} e^{-\varepsilon^2\eta_1^2} \mu_1(x_1, \eta_1, s, s_0)\, d\eta_1.$$

The only difference compared to (6.12) and (6.13) is that we replace the multipliers $\mu_1(z_1, \cdot, \cdot, \cdot)$ with $\mu_1(x_1, \cdot, \cdot, \cdot)$ in both integrals. Since $|z_1-x_1|\leqslant 20\gamma$, all the previous estimates apply, so the integral in the left-hand side of (6.20) over the set $|z_1-x_1|\leqslant 20\gamma$ is bounded by $|t-s|^{-1/2}$, as desired. The integral over the set $|z_1-y_1|\leqslant 20\gamma$ is similar, the only difference being that we replace the multipliers $\mu_1(z_1, \cdot, \cdot, \cdot)$ with $\mu_1(y_1, \cdot, \cdot, \cdot)$ in both integrals. Together with (6.11) this completes the proof of (6.10) for the kernel k_1. $\square$

PROPOSITION 6.6. *The operators T_3 and T_4^2 are bounded from $L_s^1 L_y^2$ to $L_t^{p'} L_x^{q'}$.*

Proof. By the same argument as in Proposition 6.3 it suffices to prove that

$$\begin{aligned} \left| \int_{\mathbf{R}} e^{-(\varepsilon')^2 z_1^2} k(x_1, z_1, t, s_0)\, e^{-(\varepsilon')^2 z_1^2} \bar{k}(y_1, z_1, s, s_0)\, dz_1 \right| \\ \leqslant C(|t-s_0|+|s-s_0|)^{-1/2} \log \frac{|t-s_0|^2+|s-s_0|^2}{|t-s_0|\cdot|s-s_0|} \end{aligned} \tag{6.21}$$

for $k=k_3$ and $k=k_4^2$, provided that $(t-s_0)(s-s_0)>0$. For the kernel k_4^2 this follows as in Proposition 6.3—see the proof of (6.6). For the kernel k_3 we prove a bound similar to (4.42) and (4.44): by examining (5.3) and integrating by parts we see easily that

$$|\xi_1^l \partial_{\xi_1}^l \mu_3(y_1, \xi_1, t, s)| \leqslant C(1+|t-s|\xi_1^2)^{-2} \tag{6.22}$$

for $l=0, 1, 2$. We use this in (4.27) and integrate by parts (because of the decay in (6.22) as $|t-s|\xi_1^2 \to \infty$, the factor $e^{-i(t-s)\xi_1^2}$ may be absorbed in $\mu_3(y_1, \xi_1, t, s)$). It follows easily that

$$|k_3(x_1, y_1, t, s)| \leqslant \frac{C}{|t-s|^{1/2}+|x_1-y_1|}. \tag{6.23}$$

This estimate can be used to prove (6.21) for the kernel k_3, as in Proposition 6.3. $\square$

7. Boundedness of the operators T_j, III

It remains to prove the following result:

PROPOSITION 7.1. *The operators T_j, $j=1, 2, 3, 4$, are bounded from $L_s^p L_y^q$ to $L_t^{p'} L_x^{q'}$.*

In dimensions $n\geqslant 3$ we need an interpolation lemma of Keel and Tao [7] (see pp. 964–967 for the proof):

LEMMA 7.2. (Keel and Tao [7]) *Assume that* $n\geqslant 3$ *and*

$$U(f)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} f(y,s)K(x,y,t,s)\,ds\,dy$$

is an operator with a locally integrable kernel K. *Let*

$$U_l(f)(x,t)=\int_{\mathbf{R}^n}\int_{|t-s|\in[2^l,2^{l+1}]} f(y,s)K(x,y,t,s)\,ds\,dy.$$

Let

$$\beta(a,b)=\frac{n}{2}-1-\frac{n}{2}\left(\frac{1}{a'}+\frac{1}{b'}\right)$$

and assume that for any $f\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R})$ *the estimate*

$$\|U_l(f)\|_{L_t^2L_x^{b'}}\leqslant C2^{-l\beta(a,b)}\|f\|_{L_s^2L_y^a} \tag{7.1}$$

holds for the exponents

(i) $a=b=1$;

(ii) $2n/(n+2)\leqslant a\leqslant 2$ *and* $b=2$;

(iii) $2n/(n+2)\leqslant b\leqslant 2$ *and* $a=2$.

Then

$$\|U(f)\|_{L_t^2L_x^{2n/(n-2)}}\leqslant C\|f\|_{L_s^2L_y^{2n/(n+2)}}.$$

Proof of Proposition 7.1. We claim first that an operator of the form

$$T(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu(y_1,\xi_1,t,s)\,d\xi\,ds\,dy$$

is bounded from $L_t^pL_x^q$ to $L_t^{p'}L_x^{q'}$ if $p\in[1,2)$, and the operator

$$S_{t,s}(h)(x)=\int_{\mathbf{R}^n} h(y)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu(y_1,\xi_1,t,s)\,d\xi\,dy$$

satisfies the bounds

$$\|S_{t,s}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)}\leqslant C \tag{7.2}$$

and

$$\|S_{t,s}\|_{L^1(\mathbf{R}^n)\to L^\infty(\mathbf{R}^n)}\leqslant C|t-s|^{-n/2} \tag{7.3}$$

uniformly in t and s. Assuming (7.2) and (7.3) we would have by interpolation

$$\|S_{t,s}\|_{L^q(\mathbf{R}^n)\to L^{q'}(\mathbf{R}^n)}\leqslant C|t-s|^{-n(1/q-1/2)}.$$

By the Minkowski inequality for integrals we would have

$$\|T(g)(\cdot,t)\|_{L_x^{q'}} \leqslant \int_{\mathbf{R}} \left\| \int_{\mathbf{R}^n} g(y,s) \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} e^{-i(t-s)|\xi|^2} e^{-\varepsilon^2|\xi|^2} \mu(y_1,\xi_1,t,s)\, d\xi\, dy \right\|_{L_x^{q'}} ds$$
$$\leqslant \int_{\mathbf{R}} \|g(\cdot,s)\|_{L_y^q} |t-s|^{-n(1/q-1/2)}\, ds.$$

Since $1/p-1/p'=1-n(1/q-1/2)$ and $p<p'$, by fractional integration it would follow that

$$\|T(g)\|_{L_t^{p'}L_x^{q'}} \leqslant C_p \|g\|_{L_s^p L_y^q},$$

as desired.

It is easy to check the estimates (7.2) and (7.3) for our multipliers μ_j. Notice that the L^2-bounds (7.2) were proved in Proposition 5.1. For the $L^1\to L^\infty$ bounds, it suffices to prove that for $j=1,2,3,4$,

$$\left| \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} e^{-i(t-s)|\xi|^2} e^{-\varepsilon^2|\xi|^2} \mu_j(y_1,\xi_1,t,s)\, d\xi \right| \leqslant C|t-s|^{-n/2} \tag{7.4}$$

uniformly in t, s, x and y. For this we use Lemma 4.1 for $j=1,2,3$ (the Hörmander–Mikhlin bounds for μ_j were verified in Proposition 6.2), and the formula (4.42) for $j=4$.

This completes the proof if $(p,q)\in\mathcal{A}$ and $p<2$. It remains to prove the endpoint estimate $(p,q)=(2,2n/(n+2))$ in dimensions $n\geqslant3$. For this we use Lemma 7.2; we have to verify the estimate (7.1) for our operators $T_{j,l}$, $j=1,2,3,4$. Notice that we can assume $l\leqslant0$; in addition we can assume that f is supported in a time interval of length 2^{l+1}, say $\mathbf{R}^n\times[s_0-2^l,s_0+2^l]$. Then $T_{j,l}(f)$ is supported in $\mathbf{R}^n\times[s_0-3\cdot2^l,s_0+3\cdot2^l]$. For the bound in the case $a=b=1$ we have

$$\|T_{j,l}(f)\|_{L_t^2L_x^\infty} \leqslant C2^{l/2}\|T_{j,l}(f)\|_{L_t^\infty L_x^\infty} \leqslant C2^{l/2} \sup_{|t-s|\in[2^l,2^{l+1}]} |K_j(x,y,t,s)|\cdot\|f\|_{L_s^1L_y^1}$$
$$\leqslant C2^{l/2}2^{-ln/2}2^{l/2}\|f\|_{L_s^2L_y^1} = C2^{-l\beta(1,1)}\|f\|_{L_s^2L_y^1},$$

as desired (we used the bound (7.4)). In the case $a\in[2n/(n+2),2]$ and $b=2$, let $p(a)\in[1,2]$ be the exponent with the property that $(p(a),a)\in\mathcal{A}$. We use Propositions 6.2 and 6.3 to get

$$\|T_{j,l}(f)\|_{L_t^2L_x^2} \leqslant C2^{l/2}\sup_{t_0}\|T_{j,l}(f)(t_0,\cdot)\|_{L_x^2} \leqslant C2^{l/2}\|f\|_{L_s^{p(a)}L_y^a}$$
$$\leqslant C2^{l/2}2^{l(1/p(a)-1/2)}\|f\|_{L_s^2L_y^a} = C2^{-l\beta(a,2)}\|f\|_{L_s^2L_y^a},$$

as desired. The estimate in the case $a=2$ and $b\in[2n/(n+2),2]$ is similar, by using Propositions 6.5 and 6.6 instead of Propositions 6.2 and 6.3. This completes the proof of the proposition. □

8. Boundedness of the operators R_j

In this section we prove that the operators R_j, $j=1,2,3$, are bounded from $L_s^\infty L_y^2$ to X' with small norm. Recall that the operators R_j are of the form

$$R_j(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}s_j(y_1,\xi_1,t,s)\,d\xi\,ds\,dy. \quad (8.1)$$

The multipliers s_j are defined in (3.17), (3.19) and (3.23). The following proposition gives the main estimate in this section:

PROPOSITION 8.1. *If* $(p,q)\in\mathcal{A}$ *is as in* §5 *then*

$$\|R_jg\|_{L_t^{p'}L_x^{q'}}\leqslant C\frac{\gamma^5}{\lambda}\|g\|_{L_s^\infty L_y^2}$$

for $j=1,2,3$.

Proof. Notice that it suffices to prove the stronger bound

$$\|R_jg\|_{L_t^{p'}L_x^{q'}}\leqslant C\frac{\gamma^5}{\lambda}\|g\|_{L_s^1 L_y^2}.$$

For $j=1,2,3$ let

$$m_j(x_1,y_1,t,s)=\int_{\mathbf{R}} e^{i(x_1-y_1)\xi_1}e^{-i(t-s)\xi_1^2}e^{-\varepsilon^2\xi_1^2}s_j(y_1,\xi_1,t,s)\,d\xi_1.$$

As in Propositions 5.1 and 6.6 it suffices to prove that for $j=1,2,3$,

$$\|s_j(\,\cdot\,,\xi_1,t,s)\|_{\mathrm{BV}_{y_1}}\leqslant C\frac{\gamma^5}{\lambda} \quad (8.2)$$

for the $L_s^1L_y^2\to L_t^\infty L_x^2$ bound, and

$$\left|\int_{\mathbf{R}} e^{-(\varepsilon')^2z_1^2}m_j(x_1,z_1,t,s_0)e^{-(\varepsilon')^2z_1^2}\overline{m}_j(y_1,z_1,s,s_0)\,dz_1\right| \\ \leqslant C\left(\frac{\gamma^5}{\lambda}\right)^2(|t-s_0|+|s-s_0|)^{-1/2}\log\frac{|t-s_0|^2+|s-s_0|^2}{|t-s_0|\cdot|s-s_0|} \quad (8.3)$$

for any $t,s,s_0\in[-1,1]$.

Assume first that $j=1$ or $j=2$. The bound (8.2) follows easily from (3.14) and the formulas (3.17) and (3.19). Also, by (3.14) we have

$$|m_{1,2}(x_1,z_1,t,s_0)|\leqslant C\frac{\gamma^4}{\lambda},$$

and, by integrating by parts, we have

$$|m_{1,2}(x_1, z_1, t, s_0)| \leqslant C|x_1 - z_1|^{-1} \frac{\gamma^5}{\lambda}.$$

Thus

$$\left| \int_{\mathbf{R}} e^{-(\varepsilon')^2 z_1^2} m_{1,2}(x_1, z_1, t, s_0) e^{-(\varepsilon')^2 z_1^2} \overline{m}_{1,2}(y_1, z_1, s, s_0)\, dz_1 \right| \leqslant C\left(\frac{\gamma^5}{\lambda}\right)^2,$$

which is better than (8.3). The proposition follows for $j=1,2$.

Assume now that $j=3$. We examine the formula (3.23). Recall that in this formula $|\xi|_1 \geqslant \gamma$ and $|\tau| \geqslant \frac{1}{10}\xi_1^2$. The symbol in the second line of (3.23) can be written in the form

$$-a'_{\beta,\lambda}(y_1)\tau^{-1} + Q_3(y_1, \xi_1, \tau),$$

where

$$\lambda^{l_1} |\xi_1|^{l_2} |\tau|^{l_3} |\partial_{y_1}^{l_1} \partial_{\xi_1}^{l_2} \partial_{\tau}^{l_3} Q_3(y_1, \xi_1, \tau)| \leqslant C_{l_1, l_2, l_3} \frac{\gamma}{\lambda} \frac{\xi_1^2}{\tau^2}$$

for any nonnegative integers l_1, l_2 and l_3 (using (3.8)). In addition, $Q_3(y_1, \cdot, \cdot)$ is supported in the set $y_1 \in [\lambda, 2\lambda]$. It follows easily that $\|s_3(\cdot, \xi_1, t, s)\|_{\mathrm{BV}_{y_1}} \leqslant C\gamma/\lambda$, which is better than (8.2). Also, by integrating by parts as in (6.22) we have

$$|\xi_1^l \partial_{\xi_1}^l s_3(y_1, \xi_1, t, s)| \leqslant C \frac{\gamma}{\lambda} \frac{1}{(1+|t-s|\xi_1^2)^2}.$$

Thus, as in Proposition 6.6,

$$|m_3(x_1, y_1, t, s)| \leqslant \frac{C\gamma/\lambda}{|t-s|^{1/2} + |x_1 - y_1|},$$

which suffices to prove (8.3) in the case $j=3$. This completes the proof of the proposition. □

We can now establish the precise condition on λ and β. The two relevant estimates are (6.17) and Proposition 8.1. Thus we need to assume that

$$\lambda \geqslant \Lambda(\beta) = C(1+\beta)^6 \tag{8.4}$$

for some large constant C.

9. Applications

In this section we prove Theorems 2.3, 2.4 and 2.5. To simplify the notation, we write X for $X([0,1])$, X' for $X'([0,1])$, and Y for $Y([0,1])$. Recall that $\overline{C}$ is the constant in Theorem 2.1. For Theorem 2.3 we simply apply Theorem 2.1:

$$\begin{aligned}\|e^{\beta\varphi_\lambda(x_1)}u(x,t)\|_{X'} &\leqslant \overline{C}[\|e^{\beta\varphi_\lambda(x_1)}Hu(x,t)\|_X+\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,0)\|_{L^2(\mathbf{R}^n)}\\ &\qquad+\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,1)\|_{L^2(\mathbf{R}^n)}]\\ &\leqslant \overline{C}\|V\|_Y\|e^{\beta\varphi_\lambda(x_1)}u(x,t)\|_{X'}\\ &\qquad+\overline{C}[\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,0)\|_{L^2(\mathbf{R}^n)}+\|e^{\beta\varphi_\lambda(x_1)}u(\,\cdot\,,1)\|_{L^2(\mathbf{R}^n)}].\end{aligned}$$

If $\|V\|_Y\leqslant 1/2\overline{C}$, the first term of the right-hand side of the inequality above can be absorbed into the left-hand side (this term is finite since $u\in X'$ by Theorem 2.1 and $e^{\beta\varphi_\lambda(x_1)}$ is bounded). Theorem 2.3 follows by letting $\lambda=\infty$.

For Theorem 2.4 we use a variant of the Carleman argument. Let $u=u_1-u_2$; we have

$$Hu=\widetilde{W}u, \tag{9.1}$$

where

$$\widetilde{W}(x)=\begin{cases} V(x)+\dfrac{F(u_1(x))-F(u_2(x))}{u_1(x)-u_2(x)} & \text{if } u_1(x)\neq u_2(x),\\ V(x) & \text{if } u_1(x)=u_2(x).\end{cases}$$

Since $\widetilde{W}\in Y$, we have $\widetilde{W}u\in X$, and thus the identity (9.1) holds in X and $u\in Z([0,1])$. By (2.9) and (2.10),

$$\|\widetilde{W}\chi_{bw_0+D(w_0)}(x)\|_Y\leqslant \bar{c}.$$

By rotation we may assume without loss of generality that $w_0=(1,0,...,0)$. Let $E=\|u(\,\cdot\,,0)\|_{L^2}+\|u(\,\cdot\,,1)\|_{L^2}$. By Theorem 2.1 with $\lambda\geqslant\max\{b+1,\Lambda(\beta)\}$, the identity (9.1) and the support property of the functions $u(\,\cdot\,,0)$ and $u(\,\cdot\,,1)$, we have

$$\begin{aligned}\|e^{\beta\varphi_\lambda(x_1)}\chi_{\{x:x_1>b\}}u\|_{X'} &\leqslant \|e^{\beta\varphi_\lambda(x_1)}u\|_{X'}\\ &\leqslant \overline{C}\|e^{\beta\varphi_\lambda(x_1)}Hu\|_X+\overline{C}e^{\beta b}E\\ &\leqslant \overline{C}\|e^{\beta\varphi_\lambda(x_1)}(\chi_{\{x:x_1>b\}}\widetilde{W})(\chi_{\{x:x_1>b\}}u)\|_X\\ &\qquad+\overline{C}\|e^{\beta\varphi_\lambda(x_1)}\chi_{\{x:x_1\leqslant b\}}Hu\|_X+\overline{C}e^{\beta b}E\\ &\leqslant \tfrac{1}{2}\|e^{\beta\varphi_\lambda(x_1)}\chi_{\{x:x_1>b\}}u\|_{X'}+\overline{C}e^{\beta b}\|Hu\|_X+\overline{C}e^{\beta b}E.\end{aligned} \tag{9.2}$$

We can absorb the first term of the right-hand side (which is clearly finite) into the left-hand side. The theorem follows by letting $\beta,\lambda\to\infty$.

To prove Theorem 2.5 we define the functions u and $\widetilde{W}$ as before. We have to show that if $u\in Z([0,1])$ vanishes in a half-space and $Hu=\widetilde{W}u$, then $u\equiv 0$. This is similar to the type of unique continuation theorems proved by Kenig, Ruiz and Sogge [12] for wave equations. As in [12, p. 331], we should remark that the classical examples of smooth solutions of the equation $Hu=0$ which vanish in a half-space (as in [3]) are not decaying, and are thus not in $C([0,1]:L^2(\mathbf{R}^n))$. Our argument is somewhat similar to the proofs of Theorem 1.3 and Corollary 6.1 in the work of Isakov [6]. The difference is that we use L^p Carleman inequalities to cover rough potentials (in the space Y), as opposed to only bounded potentials as in [6].

The identity (9.1) holds and, by Theorem 2.4, $u\equiv 0$ in $[bw_0+D(w_0)]\times[0,1]$. We will prove now that $u\equiv 0$ in $\mathbf{R}^n\times[0,1]$. By (2.11), there is $\varepsilon_0\in(0,1]$ with the property that

$$\|W\chi_{\{x:x\cdot w_0\in[b',b'+\varepsilon_0]\}}\|_Y \leqslant \frac{1}{4\overline{C}_1} \quad \text{for any } b'\in(-\infty,b-\varepsilon_0], \tag{9.3}$$

where $\overline{C}_1$ is the constant C in Lemma 9.1 below. This is the only assumption we need on W to carry out the proof. By rotation we may assume that $w_0=(1,0,...,0)$. Let $b_j=b-j\varepsilon_0$. By induction it suffices to prove that $u\equiv 0$ in $\{x:x_1>b_{j+1}\}\times[0,1]$ assuming that $u\equiv 0$ in $\{x:x_1>b_j\}\times[0,1]$ and $j\geqslant 0$. The Carleman inequality in Theorem 2.1 (or Corollary 2.2) does not apply directly, mainly because we do not have good control over the boundary terms $u(\,\cdot\,,0)$ and $u(\,\cdot\,,1)$. To avoid these boundary terms we make a change of variables (as in [6]). For any $\delta\in(0,\frac{1}{10}]$ fix a smooth function $\omega=\omega_{\delta,j}:[0,1]\to[b_{j+1},b_j]$ with the property that $\omega(t)=b_{j+1}$ if $t\in[2\delta,1-2\delta]$, $\omega(t)=b_j$ if $t\in[0,\delta]\cup[1-\delta,1]$, and $\delta|\omega'(t)|+\delta^2|\omega''(t)|\leqslant C$. We will show that

$$u\equiv 0 \quad \text{in } \{(x,t):x_1>\omega(t),\ t\in[0,1]\}. \tag{9.4}$$

Since δ is arbitrary and $u\in C([0,1]:L^2(\mathbf{R}^n))$ this would suffice to complete the proof of the induction step. Let

$$v(y_1,y',s)=e^{-i\omega'(s)y_1/2}u(y_1+\omega(s),y',s).$$

An elementary calculation using (9.1) shows that

$$Hv(y_1,y',s)=v(y_1,y',s)\big[\widetilde{W}(y_1+\omega(s),y',s)+\tfrac{1}{2}\omega''(s)y_1-\tfrac{1}{4}\omega'(s)^2\big]. \tag{9.5}$$

The role of the exponential $e^{-i\omega'(s)y_1/2}$ is to cancel the $\partial_{y_1}u$-term in the commutator. By the support properties of the function u, we know that $v\equiv 0$ in the sets

$$\{(y_1,y',s):y_1>\varepsilon_0\} \quad \text{and} \quad \{(y_1,y',s):y_1>0 \text{ and } s\in[0,\delta]\cup[1-\delta,1]\}.$$

It remains to prove that $v\equiv 0$ in the set $\{(y_1,y',s):y_1>0\}$. The equation (9.5) may be written in the form

$$Hv=(W_0+M)\cdot v, \tag{9.6}$$

where

$$W_0(y_1, y', s) = \widetilde{W}(y_1+\omega(s), y', s)$$

and

$$M(y_1, y', s) = \tfrac{1}{2}\omega''(s)y_1 - \tfrac{1}{4}\omega'(s)^2.$$

Theorem 2.1 cannot be applied in this case because the potential M does not belong to the space Y. We use instead the following Carleman inequality:

LEMMA 9.1. *Assume that* $u \in C([0,1]:L^2(\mathbf{R}^n))$, $u\equiv 0$ *in the set* $\{(x_1, x', t): x_1>1\}$, *and* $(1+|x_1|)^{-N} Hu \in X(\mathbf{R}^n\times(0,1))$ *for some* $N\geqslant 0$. *Then, for some constant* $c_0>0$,

$$\begin{aligned}\|e^{\phi_\beta(x_1)}\chi_{[0,1]}(x_1)u(x,t)\|_{X'([0,1])} + \beta^{c_0}\|e^{\phi_\beta(x_1)}\chi_{[0,1]}(x_1)u(x,t)\|_{L^1_tL^2_x(\mathbf{R}^n\times[0,1])} \\ \leqslant C[\|e^{\phi_\beta(x_1)}Hu(x,t)\|_{X([0,1])} + \|e^{\phi_\beta(x_1)}u(\,\cdot\,,0)\|_{L^2} + \|e^{\phi_\beta(x_1)}u(\,\cdot\,,1)\|_{L^2}]\end{aligned} \tag{9.7}$$

for any $\beta\in[2,\infty)$. *The function* $\phi_\beta:\mathbf{R}\to\mathbf{R}$ *is given by* $\phi_\beta(x_1)=\beta x_1$ *if* $x_1\in[-1,\infty)$, *and* $\phi_\beta(x_1)=x_1-\beta+1$ *if* $x_1\in(-\infty,-1]$.

The same argument as in the proof of Theorem 2.4 (see (9.2)), using (9.3) and the identity (9.6), shows that Lemma 9.1 suffices to prove that $v\equiv 0$ in the set $\{(y_1, y', s): y_1>0\}$, which gives (9.4).

Proof. By the same argument as in §3 we may assume that $u\in C_0^\infty(\mathbf{R}^n\times\mathbf{R})$ is supported in $\mathbf{R}^n\times[0,1]$. The bound for the first term in the left-hand side of (9.7) follows from the stronger inequality

$$\|e^{\beta x_1}u\|_{X'([0,1])} \leqslant C\|e^{\beta x_1}Hu\|_{X([0,1])}$$

for any $\beta\geqslant 2$ and any $u\in C_0^\infty(\mathbf{R}^n\times\mathbf{R})$, which is a consequence of Theorem 2.1 with $\lambda=\infty$.

To control the second term in the left-hand side of (9.7) it suffices to prove that for some $c_0>0$,

$$\beta^{c_0}\|e^{\phi_\beta(x_1)}\chi_{[0,1]}(x_1)u(x,t)\|_{L^1_tL^2_x(\mathbf{R}^n\times[0,1])} \leqslant C\|e^{\phi_\beta(x_1)}Hu(x,t)\|_{X([0,1])} \tag{9.8}$$

for any $\beta\geqslant 2$ and any $u\in C_0^\infty(\mathbf{R}^n\times\mathbf{R})$ supported in $\mathbf{R}^n\times[0,1]$. Let $f=Hu$. Let P^1_ε denote the operator defined by the Fourier multiplier $(\xi_1,\xi',\tau)\mapsto e^{-\varepsilon^2|\xi'|^2}e^{-\varepsilon^2(\tau+|\xi'|^2)^2}$. It suffices to prove that

$$\|\chi_{[0,1]}(t)e^{\phi_\beta(x_1)}\chi_{[0,1]}(x_1)P^1_\varepsilon(u)\|_{L^1_tL^2_x(\mathbf{R}^n\times[0,1])} \leqslant C\beta^{-c_0}\|e^{\phi_\beta(x_1)}f(x,t)\|_{X([0,1])}. \tag{9.9}$$

As in §3, let $\tilde{u}(x_1,\xi',\tau)$ and $\tilde{f}(x_1,\xi',\tau)$ denote the partial Fourier transforms of the functions u and f in the variables x' and t. The equation $(i\partial_t+\Delta_x)u=f$ becomes

$$[\partial^2_{x_1}-(\tau+|\xi'|^2)]\tilde{u}(x_1,\xi',\tau) = \tilde{f}(x_1,\xi',\tau).$$

By integration by parts we have for any $x_1 \in [-1,0]$,

$$\tilde{u}(x_1,\xi',\tau)=\int_{\mathbf{R}} \tilde{f}(y_1,\xi',\tau)G(x_1,y_1,\tau+|\xi'|^2)\,dy_1,$$

where

$$G(x_1,y_1,\mu)=\begin{cases} -\chi_+(y_1-x_1)\dfrac{\sin\big[(x_1-y_1)\sqrt{-\mu}\,\big]}{\sqrt{-\mu}} & \text{if } \mu\leqslant 0,\\[2mm] -\chi_+(y_1-x_1)\dfrac{\sinh\big[(x_1-y_1)\sqrt{\mu}\,\big]}{\sqrt{\mu}} & \text{if } 0\leqslant\mu\leqslant\beta^2,\\[2mm] -\dfrac{e^{-|x_1-y_1|\sqrt{\mu}}}{2\sqrt{\mu}} & \text{if } \beta^2<\mu. \end{cases} \tag{9.10}$$

By taking the inverse Fourier transform,

$$P_\varepsilon^1 u(x_1,x',t)=C\int_{\mathbf{R}^n}\int_{\mathbf{R}} f(y,s)\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} e^{i(x'-y')\cdot\xi'}e^{i(t-s)\tau}e^{-\varepsilon^2|\xi'|^2} \\ \times e^{-\varepsilon^2(\tau+|\xi'|^2)^2}G(x_1,y_1,\tau+|\xi'|^2)\,d\tau\,d\xi'\,ds\,dy.$$

Thus, to prove (9.9), it suffices to prove that the operator

$$\widetilde{T}(g)(x,t)=\int_{\mathbf{R}^n}\int_{\mathbf{R}} g(y,s)\widetilde{K}(x,y,t,s)\,ds\,dy$$

has the property that

$$\|\widetilde{T}(g)\|_{L_t^1L_x^2}\leqslant C\beta^{-c_0}\|g\|_X \tag{9.11}$$

for any bounded compactly supported function g, where

$$\begin{aligned}\widetilde{K}(x,y,t,s)&=\chi_{[0,1]}(t)\chi_{[0,1]}(s)\chi_{[0,1]}(x_1)\chi_{(-\infty,1]}(y_1)e^{\phi_\beta(x_1)-\phi_\beta(y_1)}\\ &\quad\times\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} e^{i(x'-y')\cdot\xi'}e^{i(t-s)\tau}e^{-\varepsilon^2|\xi'|^2}e^{-\varepsilon^2(\tau+|\xi'|^2)^2}G(x_1,y_1,\tau+|\xi'|^2)\,d\tau\,d\xi'\\ &=\chi_{[0,1]}(t)\chi_{[0,1]}(s)\chi_{[0,1]}(x_1)\chi_{(-\infty,1]}(y_1)e^{\phi_\beta(x_1)-\phi_\beta(y_1)}\\ &\quad\times\int_{\mathbf{R}^{n-1}} e^{i(x'-y')\cdot\xi'}e^{-i(t-s)|\xi'|^2}e^{-\varepsilon^2|\xi'|^2}\int_{\mathbf{R}} e^{i(t-s)\mu}e^{-\varepsilon^2\mu^2}G(x_1,y_1,\mu)\,d\mu\,d\xi'.\end{aligned}$$

Let

$$\tilde{k}(x_1,y_1,t,s)=\chi_{[0,1]}(x_1)\chi_{(-\infty,1]}(y_1)e^{\phi_\beta(x_1)-\phi_\beta(y_1)}\int_{\mathbf{R}} e^{i(t-s)\mu}e^{-\varepsilon^2\mu^2}G(x_1,y_1,\mu)\,d\mu.$$

For $l\leqslant 0$, let

$$\widetilde{K}_l(x,y,t,s)=\chi_{[2^l,2^{l+1}]}(|t-s|)\widetilde{K}(x,y,t,s),$$
$$\tilde{k}_l(x_1,y_1,t,s)=\chi_{[2^l,2^{l+1}]}(|t-s|)\tilde{k}(x_1,y_1,t,s)$$

and $\widetilde{T}_l$ be the operator defined by the kernel $\widetilde{K}_l$. By (9.10),

$$|\tilde{k}_l(x_1,y_1,t,s)| \leqslant C2^{-l/2}[e^{-\beta|x_1-y_1|/10}+(1+\beta 2^{l/2})^{-1}\chi_{[0,2]}(|x_1-y_1|)]. \tag{9.12}$$

To see this, we substitute the formula (9.10) and break up the integral into three parts. To control the integral over $\mu\leqslant 0$ we make the change of variable $\mu=-\eta^2$ and use Lemma 4.2. To control the integral over $\mu\in[0,\beta^2]$ we make the change of variable $\mu=\eta^2$, use Lemma 4.2 for the integral over $\eta\in[0,\frac{1}{2}\beta]$, use again Lemma 4.2 for the integral over $\eta\in[\frac{1}{2}\beta,\beta]$ if $\beta 2^{l/2}\leqslant 1$, and integrate by parts if $\beta 2^{l/2}\geqslant 1$. To control the integral over $\mu\geqslant\beta^2$ we make the change of variable $\mu=\eta^2$, use Lemma 4.2 if $\beta 2^{l/2}\leqslant 1$, and integrate by parts if $\beta 2^{l/2}\geqslant 1$. The estimate (9.12) is the only estimate we need for the kernel $\tilde{k}_l$.

As in §5 we fix an acceptable pair (p,q), with $p\leqslant\frac{4}{3}$ if $n=1$, $p\leqslant p_0$ if $n=2$, and $p\leqslant 2$ if $n\geqslant 3$. For (9.11) it suffices to prove that

$$\sum_{l\leqslant 0}\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^1_tL^2_x}\leqslant C\beta^{-c_0}$$

for some $c_0>0$. Since

$$\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^1_tL^2_x}\leqslant C\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^p_tL^2_x}\leqslant C2^{l/p}\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^\infty_tL^2_x},$$

it suffices to prove that

$$\sum_{l\leqslant 0}2^{l/p}\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^\infty_tL^2_x}\leqslant C\beta^{-c_0}. \tag{9.13}$$

To estimate $\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^\infty_tL^2_x}$ we argue as in Proposition 6.3. For any $t_0\in[0,1]$ let $\widetilde{T}_{l,t_0}$ denote the operator defined by the kernel $e^{-(\varepsilon')^2|z|^2}\widetilde{K}_l(z,y,t_0,s)$. As in Proposition 6.3,

$$\|\widetilde{T}_l\|_{L^p_sL^q_y\to L^\infty_tL^2_x}\leqslant\sup_{t_0,\varepsilon'}\|\widetilde{T}_{l,t_0}\|_{L^p_sL^q_y\to L^2_z}\leqslant\sup_{t_0,\varepsilon'}\|\widetilde{T}^*_{l,t_0}\widetilde{T}_{l,t_0}\|^{1/2}_{L^p_sL^q_y\to L^{p'}_tL^{q'}_x}. \tag{9.14}$$

The kernel of the operator $\widetilde{T}^*_{l,t_0}\widetilde{T}_{l,t_0}$ is

$$\tilde{L}_{l,t_0}(x,y,t,s)=\int_{\mathbf{R}^n}e^{-(\varepsilon')^2|z|^2}\widetilde{K}_l(z,y,t_0,s)\,e^{-(\varepsilon')^2|z|^2}\overline{\widetilde{K}}_l(z,x,t_0,t)\,dz.$$

Let $\widetilde{U}_{l,t_0,t,s}$ denote the operator defined by the kernel $\tilde{L}_{l,t_0}(\cdot,\cdot,t,s)$. By (9.12), the L^1-norm in both the variables x_1 and y_1 of the kernel $\tilde{k}_l(\cdot,\cdot,t,s)$ is bounded by $C2^{-l/2}(1+\beta 2^{l/2})^{-1}$. Thus

$$\|\widetilde{U}_{l,t_0,t,s}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)}\leqslant C2^{-l}(1+\beta^2 2^l)^{-1}.$$

The kernel $\tilde{L}_{l,t_0}(x,y,t,s)$ splits as a product of n integrals. As in Propositions 6.2 and 6.3, the integrals over the variables $z_2, ..., z_n$ are each bounded by $|t-s|^{-1/2}$. For the integral over the variable z_1 we use (9.12). The result is

$$\|\widetilde{U}_{l,t_0,t,s}\|_{L^1(\mathbf{R}^n)\to L^\infty(\mathbf{R}^n)} \leqslant C|t-s|^{-(n-1)/2}\chi_{[0,2^{l+2}]}(|t-s|)2^{-l}(1+\beta^2 2^l)^{-1}(1+\beta 2^l).$$

By interpolation and the Minkowski inequality for integrals, as in Proposition 6.3,

$$\|\widetilde{T}^*_{l,t_0}\widetilde{T}_{l,t_0}\|_{L^p_s L^q_y \to L^{p'}_t L^{q'}_x} \leqslant C2^{-l}(1+\beta^2 2^l)^{-1}(1+\beta 2^l)^{2/q-1}2^{l/2(2/q-1)}.$$

By (9.14) and the fact that $1/p+1/2q-\frac{3}{4}\geqslant 1/2n$ if $(p,q)\in\mathcal{A}$,

$$\sum_{l\leqslant 0} 2^{l/p}\|\widetilde{T}_l\|_{L^p_s L^q_y \to L^\infty_t L^2_x} \leqslant \sum_{l\leqslant 0} C2^{l(1/p+1/2q-3/4)}\frac{1+\beta^{1/2}2^{l/2}}{1+\beta 2^{l/2}} \leqslant C\beta^{-1/2n}.$$

The main estimate (9.13) follows with $c_0=1/2n$. □

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