ANALYTIC AND QUASI-INVARIANT MEASURES

BY

FRANK FORELLI

The University of Wisconsin, Madison, Wisconsin, U.S.A. (1)

1. Let the real line R act as a topological transformation group on the locally compact Hausdorff space S. This means that we are given a group homomorphism $t \to T_t$ from the Abelian group R into the group of homeomorphisms of the topological space S with the property that the function $(t, p) \to T_t p$ from $R \times S$ to S is continuous. The action of R on S can be used to define the convolution of a measure on S with a function on R. Let $M(S)$ be the Banach space of bounded complex Baire measures on S and let $L^1(R)$ be the group algebra of R. The convolution of λ in $M(S)$ with f in $L^1(R)$ is the measure $\lambda * f$ in $M(S)$ given by

$$(\lambda * f)\, E = \int_R \lambda(T_{-t} E)\, f(t)\, dt$$

for all Baire subsets E of S. Convolution in turn can be used to associate with a measure on S a closed subset of R called the spectrum of the measure. Let λ be in $M(S)$ and let $J(\lambda)$ be the collection of all f in $L^1(R)$ with

$$\lambda * f = 0.$$

$J(\lambda)$ is a closed ideal in $L^1(R)$. The spectrum of λ, denoted by $\mathrm{sp}(\lambda)$, is the closed subset of R where all Fourier transforms of functions in $J(\lambda)$ vanish (i.e. $\mathrm{sp}(\lambda)$ is the hull of the ideal $J(\lambda)$). λ will be called analytic if $\mathrm{sp}(\lambda)$ is contained in the nonnegative reals, and λ will be called quasi-invariant if the collection of λ null sets is carried onto itself by the action of R on S. Thus to say that λ is quasi-invariant means that

$$|\lambda|\,(T_t E) = 0$$

for all t in R whenever E is a Baire subset of S with

$$|\lambda|\, E = 0,$$

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where $|\lambda|$ is the total variation of λ. The aim of this paper is to show that *analytic measures are quasi-invariant.*

When S is a compact Abelian group and the action of R on S is given by translating with the elements of a one-parameter subgroup of S, the theorem that analytic measures are quasi-invariant is the deLeeuw–Glicksberg generalization of the F. and M. Riesz theorems [2]. Let S be the circle group, let the one-parameter subgroup be S with $t \to e^{it}$ the homomorphism of R onto S, and let

$$T_t(e^{ix}) = e^{it}e^{ix}.$$

Then $\operatorname{sp}(\lambda)$ is the set of integers where the Fourier coefficients

$$\hat{\lambda}(n) = \int_0^{2\pi} e^{-inx}\, d\lambda(e^{ix})$$

do not vanish, and consequently λ is analytic if and only if its Fourier coefficients vanish for negative indices. To say that λ is quasi-invariant means that every rotation of the circle group carries the collection of λ null sets onto itself, and this in turn means, if $\lambda \neq 0$, that λ has the same null sets as Lebesgue measure. Thus if the Fourier coefficients of λ vanish for negative indices and $\lambda \neq 0$, then λ and Lebesgue measure are mutually absolutely continuous. These are the F. and M. Riesz theorems [8], and this way of looking at them (i.e. that analytic measures are quasi-invariant) is due to deLeeuw and Glicksberg [2].

This paper has four parts in addition to the introduction. Part 2 contains a theorem about triples of one-parameter groups of linear transformations. Part 3 contains a theorem that gives a necessary and sufficient condition for a measure in $M(S)$ to be quasi-invariant. This theorem was suggested to me by work of Helson and Lowdenslager [5] and its proof uses the theorem in part 2. The theorem that analytic measures are quasi-invariant is in part 4. The proof of this theorem uses the quasi-invariance criterion given in part 3. Part 5 contains several applications of the theorem that analytic measures are quasi-invariant. These applications, with one exception, imitate applications given by deLeeuw and Glicksberg in [2].

My measure theory terminology comes from Halmos' book [3], and a convenient reference for the Fourier theory I use is Rudin's book [10].

2. This part is about one-parameter groups of linear transformations and the Fourier theory that goes with these groups. I will begin with some basic definitions and lemmas that will be used throughout the paper, and then give a theorem about triples of one-parameter groups of linear transformations. This theorem says that a certain commutation relation

and spectral condition are equivalent, and is essential to the proof of the quasi-invariance criterion given in part 3.

Let X be a Banach space and let A_t be a uniformly-bounded strongly-continuous one-parameter group of linear transformations of X.

Definition 1. The convolution of x in X with f in the group algebra $L^1(R)$ is the vector $x * f$ in X given by

$$x * f = \int_R A_t x f(t)\, dt.$$

When it is necessary to indicate the one-parameter group used to define the convolution of x with f, I will write $x *_A f$. For a given x in X, $A_t x$ is a bounded continuous function on R with values in the Banach space X, and the integral in the definition is the strong integral of $A_t x$ with the bounded complex measure $f(t)\,dt$.

X is an $L^1(R)$ module with $L^1(R)$ acting on X by convolution, and in particular

$$x * (f * g) = (x * f) * g, \tag{1}$$

where $f * g$ is the familiar convolution defined in the group algebra $L^1(R)$. (1) is true because A_t is a one-parameter group. Moreover X is a continuous $L^1(R)$ module for

$$\|x * f\| \leqslant (\operatorname*{lub}_t \|A_t\|)\, \|x\| \int_R |f(t)|\, dt.$$

Let x be in X. I will denote by $J(x)$ the collection of all f in $L^1(R)$ with

$$x * f = 0.$$

$J(x)$ is a closed ideal in $L^1(R)$.

Definition 2. The spectrum of x is the closed subset of R where all Fourier transforms of functions in $J(x)$ vanish (i.e. the spectrum of x is the hull of the ideal $J(x)$).

The spectrum of x will be denoted by $\mathrm{sp}(x)$, or by $\mathrm{sp}_A(x)$ when it is necessary to indicate the one-parameter group used to define $J(x)$. Then

$$\mathrm{sp}(x) = \bigcap\, [\hat{f} = 0]$$

where the intersection is taken over all f in $J(x)$ and

$$\hat{f}(s) = \int_R e^{-ist} f(t)\, dt.$$

I learned this very general definition of spectrum from [6, p. 992]. Definition 2 is the familiar definition of spectrum when X is the Banach algebra of all bounded complex uniformly-continuous functions on R and A_t is translation by t.

LEMMA 1. *Let f run through an approximate identity in $L^1(R)$. Then $x*f$ converges in norm to x.*

Proof.

$$x*f-x=\int_R (A_t x - x) f(t)\,dt = \int_{|t|<\varepsilon} (A_t x - x) f(t)\,dt + \int_{|t|\geq\varepsilon} (A_t x - x) f(t)\,dt$$

and hence

$$\|x*f-x\| \leqslant \operatorname*{lub}_{|t|<\varepsilon} \|A_t x - x\| + (\operatorname*{lub}_t \|A_t\| + 1)\,\|x\| \int_{|t|\geq\varepsilon} f(t)\,dt.$$

When ε is little the first term on the right is small, and when f is far along the integral on the right is small.

LEMMA 2. *Let x be in X and let f be in $L^1(R)$.*

1. $\mathrm{sp}(x*f)$ *is contained in* $\mathrm{sp}(x)$.
2. $\mathrm{sp}(x*f)$ *is contained in the support of* $\hat{f}$.
3. $x*f=0$ *whenever* $\mathrm{sp}(x)$ *is contained in the interior of the set where* $\hat{f}$ *is* 0.
4. $x*f=x$ *whenever* $\mathrm{sp}(x)$ *is contained in the interior of the set where* $\hat{f}$ *is* 1.

Proof. 1. Let s be in R but not in $\mathrm{sp}(x)$. Then there is g in $J(x)$ with $\hat{g}(s)=1$. g is in $J(x*f)$ for

$$(x*f)*g=(x*g)*f,$$

and hence s is not in $\mathrm{sp}(x*f)$.

2. The proof of 2 depends on the elementary fact that given s in R and an open set containing s, then there is a function in $L^1(R)$ whose Fourier transform is 1 at s and 0 outside the open set. Let s be in the complement of the support of $\hat{f}$, and let g in $L^1(R)$ be such that $\hat{g}(s)=1$ and $\hat{f}\hat{g}=0$. Then $f*g=0$, hence g is in $J(x*f)$ for

$$(x*f)*g=x*(f*g),$$

and hence s is not in $\mathrm{sp}(x*f)$.

3. The proof of 3 is the nonelementary fact that if J is a closed ideal in the group algebra $L^1(R)$ and if the hull of J is contained in the interior of the set where $\hat{f}$ is 0, then f is in J [10, 7.2.5].

4. The proof of 4 depends on 3. Assume that $\mathrm{sp}(x)$ is contained in the interior of the set where $\hat{f}$ is 1. Let g be in $L^1(R)$ and let $h=f*g-g$. The set where $\hat{f}$ is 1 is contained in the set where $\hat{h}$ is 0 since $\hat{h}=\hat{f}\hat{g}-\hat{g}$, and therefore $\mathrm{sp}(x)$ is contained in the interior of the set where $\hat{h}$ is 0. By 3

$$(x*f-x)*g=x*h=0.$$

Thus $J(x*f-x)$ is $L^1(R)$, and hence $x*f=x$ by Lemma 1.

LEMMA 3. 1.
$$\mathrm{sp}(A_t x)=\mathrm{sp}(x)$$
$$\mathrm{sp}(cx)=\mathrm{sp}(x) \quad (c\neq 0)$$
$$\mathrm{sp}(x+y)\subset \mathrm{sp}(x)\cup \mathrm{sp}(y).$$

2. *Let E be a subset of R. The collection of all x with* sp(x) *contained in E is an A_t invariant subspace of X, and it is a closed subspace if E is closed.*

3. *The subspace consisting of all x with* sp(x) *compact is dense in X.*

Proof. 1. The first two assertions are clear. For example

$$(A_t x)*f=A_t(x*f)$$

and hence $J(A_t x)=J(x)$.

Let s be in R but not in $\mathrm{sp}(x)\cup\mathrm{sp}(y)$. Then there is f in $J(x)$ with $\hat{f}(s)=1$ and g in $J(y)$ with $\hat{g}(s)=1$. $f*g$ is in $J(x+y)$ for

$$(x+y)*(f*g)=(x*f)*g+(y*g)*f,$$

and hence s is not in $\mathrm{sp}(x+y)$ for $(f*g)\hat{}\,(s)=1$.

2. Let M be the collection of all x with $\mathrm{sp}(x)$ contained in E. That M is an A_t invariant subspace of X follows from 1.

Assume that E is closed and let x be in the norm closure of M. Let s be in the complement of E, and let f be in $L^1(R)$ with $\hat{f}(s)=1$ and E contained in the interior of the set where $\hat{f}$ is 0. Then $y*f=0$ for all y in M, and hence $x*f=0$. Thus s is not in $\mathrm{sp}(x)$ and consequently $\mathrm{sp}(x)$ is contained in E.

3. Let f with support of $\hat{f}$ compact run through an approximate identity in $L^1(R)$. Then $x*f$ converges in norm to x by Lemma 1 and $\mathrm{sp}(x*f)$ is compact by Lemma 2.

Let B_t be another uniformly-bounded strongly-continuous one-parameter group of linear transformations of X. In addition, let Y be another Banach space and let T_t be a uniformly-bounded strongly-continuous one-parameter group of linear transformations of Y. Finally, let L be a bounded linear transformation from Y into the Banach algebra $B(X)$ of all bounded linear transformations of X. For each t in R and y in Y, $B_t L(y)$ and $L(T_t y)A_t$ belong to $B(X)$, and the theorem that follows gives a necessary and sufficient condition that they be the same.

THEOREM 1. *The following commutation relation and spectral condition are equivalent.*
Commutation relation:

$$B_t L(y) = L(T_t y) A_t \quad \textit{for all } t \textit{ in } R \textit{ and } y \textit{ in } Y.$$

Spectral condition: for all x in X and y in Y, and for all a and b in R,

$$\begin{matrix} \mathrm{sp}_T(y) \subset (a, \infty) \\ \mathrm{sp}_A(x) \subset (b, \infty) \end{matrix} \quad \textit{implies } \mathrm{sp}_B(L(y)x) \subset (a+b, \infty)$$

and

$$\begin{matrix} \mathrm{sp}_T(y) \subset (-\infty, a) \\ \mathrm{sp}_A(x) \subset (-\infty, b) \end{matrix} \quad \textit{implies } \mathrm{sp}_B(L(y)x) \subset (-\infty, a+b).$$

The spectral condition is equivalent to the seemingly stronger spectral condition:

$$\mathrm{sp}_B(L(y)x) \subset \text{closure}\,(\mathrm{sp}_T(y) + \mathrm{sp}_A(x))$$

for all x in X and y in Y. The version I will use, however, is the one given in the statement of Theorem 1, and for this reason I will not say anymore about the other version. The proof of Theorem 1 will use a (known) proposition equivalent to the fact that a particular two dimensional linear subspace of $R \times R \times R$ is a spectral synthesis set, and therefore I will present this proposition before giving the proof of Theorem 1.

Let G be the linear subspace of $R \times R \times R$ consisting of all points of the form $(a, b, a+b)$.

PROPOSITION 1. *Let ϕ be a bounded continuous function on $R \times R \times R$. The spectrum of ϕ is contained in G if and only if*

$$\phi(r, s, t) = \phi(r+v, s+v, t-v) \tag{2}$$

for all r, s, t, v in R.

Proof. The fact that (2) holds when the spectrum of ϕ is contained in G is an easy corollary of (and in turn implies) the fact that G is a spectral synthesis set [10, 7.5.2]. However, one can argue more directly.

Let $J(\phi)$ be the collection of all f in the group algebra $L^1(R \times R \times R)$ with

$$\phi * f = 0,$$

where $*$ is the familiar convolution. $J(\phi)$ is a closed ideal in $L^1(R \times R \times R)$. The spectrum of ϕ, denoted by $\mathrm{sp}(\phi)$, is the closed subset of $R \times R \times R$ where all Fourier transforms of functions in $J(\phi)$ vanish.

Let T be the linear transformation from R into $R \times R \times R$ given by

$$Tv = (v, v, -v),$$

let f be in $L^1(R)$, and let g be in $L^1(R \times R \times R)$. Then

$$(\phi *_T f) * g = \phi * (g *_T f) \tag{3}$$

where
$$\phi *_T f(p) = \int_R \phi(p - Tv)\, f(v)\, dv$$

and
$$g *_T f(p) = \int_R g(p - Tv)\, f(v)\, dv.$$

Replacing $\phi(p)$ in (3) by $\exp(ip \cdot q)$ and evaluating (3) at (0, 0, 0) gives

$$(g *_T f)\hat{}\,(q) = \hat{g}(q) \hat{f}(q' + q'' - q'''), \tag{4}$$

where
$$q = (q', q'', q''').$$

Assume that $\mathrm{sp}(\phi)$ is contained in G. Let 0 be in the interior of the set where $\hat{f}$ vanishes. Then by (4) G is contained in the interior of the set where $(g *_T f)\hat{}$ vanishes, hence the hull of the ideal $J(\phi)$ is too, and thus $g *_T f$ belongs to $J(\phi)$ [10, 7.2.5]. Hence by (3)

$$(\phi *_T f) * g = 0.$$

This holds for all g in $L^1(R \times R \times R)$ and consequently

$$\phi *_T f = 0. \tag{5}$$

This holds for all f with $\hat{f}$ vanishing on an open set containing 0, and hence it holds for all f with $\hat{f}(0) = 0$ since the set whose only member is 0 is a spectral synthesis set [10, 2.6.4]. (This means that any function whose Fourier transform vanishes at 0 can be approximated in norm by functions whose Fourier transforms vanish on open sets containing 0.) Because (5) is true for all f in $L^1(R)$ with

$$\int_R f(v)\, dv = 0,$$

$\phi(p - Tv)$ is constant in v, and this is what I want.

This argument is due to Reiter [7]. I have given it to let the reader, if interested, keep track of what is going into my argument that analytic measures are quasi-invariant.

Conversely, assume that $\phi(p - Tv)$ is constant in v for each p in $R \times R \times R$. Let f be in $L^1(R)$ with $\hat{f}(0) = 0$. Then

$$\phi *_T f = 0$$

and hence by (3)
$$\phi * (g *_T f) = 0$$

for all g in $L^1(R \times R \times R)$. This and (4) show that $\mathrm{sp}(\phi)$ is contained in G.

Here is the proof of Theorem 1.

1. Let y be in Y, x in X, z in X^*, and let ϕ be the bounded continuous function on $R \times R \times R$ given by

$$\phi(r, s, t) = \langle B_t L(T_r y) A_s x, z \rangle,$$

where $\langle x, z \rangle$ is the value of the linear functional z at the vector x.

The commutation relation holds if and only if for all ϕ of the kind just described

$$\phi(r, s, t) = \phi(r+v, s+v, t-v) \tag{6}$$

for all r, s, t, v in R.

For assume that the commutation relation holds. Then

$$\phi(r, s, t) = \langle L(T_t T_r y) A_t A_s x, z \rangle = \langle L(T_{r+t} y) A_{s+t} x, z \rangle.$$

Conversely, assume that (6) is true for all ϕ of the kind described. Then

$$\langle B_{-t} L(T_t y) A_t x, z \rangle = \phi(t, t, -t) = \phi(0, 0, 0) = \langle L(y) x, z \rangle,$$

and since this holds for all z in X^* and x in X,

$$B_{-t} L(T_t y) A_t = L(y).$$

This of course is the commutation relation.

Now, by Proposition 1, the commutation relation holds if and only if $\mathrm{sp}(\phi)$ is contained in G for all ϕ of the kind described. I will complete the proof of Theorem 1 by showing that the spectral condition holds if and only if $\mathrm{sp}(\phi)$ is contained in G for all ϕ of the kind described.

2. *Let f, g, and h be in $L^1(R)$ and let k be the function in $L^1(R \times R \times R)$ given by*

$$k(r, s, t) = f(-r) g(-s) h(-t).$$

Then

$$\langle (L(y *_T f)(x *_A g)) *_B h, z \rangle = \phi * k(0, 0, 0). \tag{7}$$

For

$$L(y *_T f) = \int_R L(T_r y) f(r)\, dr$$

and

$$x *_A g = \int_R A_s x g(s)\, ds$$

and hence

$$L(y *_T f)(x *_A g) = \int_{R \times R} L(T_r y) A_s x f(r) g(s)\, dr\, ds.$$

Moreover,

$$(L(y *_T f)(x *_A g)) *_B h = \int_R B_t L(y *_T f)(x *_A g) h(t)\, dt$$

and hence

$$(L(y *_T f)(x *_A g)) *_B h = \int_{R\times R\times R} B_t L(T_r y) A_s x f(r) g(s) h(t)\, dr\, ds\, dt.$$

Finally

$$\langle (L(y *_T f)(x *_A g)) *_B h, z\rangle = \int_{R\times R\times R} \langle B_t L(T_r y) A_s x, z\rangle f(r) g(s) h(t)\, dr\, ds\, dt$$

and this last integral is $\phi * k$ evaluated at $(0, 0, 0)$.

3. *Assume that the spectral condition holds.* Let (a, b, c) be in $R\times R\times R$ but not in G. Then $c \neq a+b$. Suppose $a+b<c$ and choose $\varepsilon>0$ such that

$$a+b+4\varepsilon<c.$$

Let f, g, and h in $L^1(R)$ be such that

$$\begin{aligned} \hat f &= 0 \quad \text{on} \quad (-\infty, -a-\varepsilon) \\ \hat g &= 0 \quad \text{on} \quad (-\infty, -b-\varepsilon) \\ \hat h &= 0 \quad \text{on} \quad (-a-b-4\varepsilon, \infty). \end{aligned}$$

By Lemma 2

$$\mathrm{sp}_T(y *_T f) \subset (-a-2\varepsilon, \infty), \qquad \mathrm{sp}_A(x *_A g) \subset (-b-2\varepsilon, \infty),$$

hence by the spectral condition

$$\mathrm{sp}_B(L(y *_T f)(x *_A g)) \subset (-a-b-4\varepsilon, \infty),$$

and now again by Lemma 2

$$(\mathrm{L}(y *_T f)(x *_A g)) *_B h = 0.$$

This and (7) give

$$\phi * k(0, 0, 0) = 0.$$

This in turn gives

$$\phi * k = 0$$

for when a function is translated the set where its Fourier transform vanishes does not change. Now in addition let f, g, and h be such that

$$\hat f(-a) = \hat g(-b) = \hat h(-c) = 1.$$

Then $\hat k(a, b, c) = 1$, and thus (a, b, c) does not belong to $\mathrm{sp}(\phi)$.

A like argument using the second part of the spectral condition shows that (a, b, c) is not in $\mathrm{sp}(\phi)$ when $c<a+b$, and hence $\mathrm{sp}(\phi)$ is contained in G.

4. *Assume that* $\mathrm{sp}(\phi)$ *is contained in* G *for all* ϕ *of the kind described at the beginning.* Let x and y be such that

$$\mathrm{sp}_T(y) \subset (a, \infty), \qquad \mathrm{sp}_A(x) \subset (b, \infty). \tag{8}$$

I want to show that

$$\mathrm{sp}_B(L(y)x) \subset (a+b, \infty). \tag{9}$$

Suppose for the moment both $\mathrm{sp}_T(y)$ and $\mathrm{sp}_A(x)$ are compact, and hence

$$\mathrm{sp}_T(y) \subset (a+2\varepsilon, d), \qquad \mathrm{sp}_A(x) \subset (b+2\varepsilon, d),$$

where ε is positive. The argument will use the elementary fact that given a compact interval and an open set containing it, then there is a function in $L^1(R)$ whose Fourier transform is 1 on the interval and 0 outside the open set. Let f and g in $L^1(R)$ be such that

$$\hat{f} = 1 \quad \text{on} \quad (a+2\varepsilon, d), \qquad \hat{g} = 1 \quad \text{on} \quad (b+2\varepsilon, d).$$

By Lemma 2
$$y *_T f = y, \qquad x *_A g = x,$$

and thus
$$L(y)x = L(y *_T f)(x *_A g). \tag{10}$$

Let h be a third function in $L^1(R)$. Then by (10) and (7)

$$\langle (L(y)x) *_B h, z\rangle = \phi * k(0, 0, 0). \tag{11}$$

Now let f and g in addition to the condition already placed on them be such that

$$\hat{f} = 0 \quad \text{on} \quad (-\infty, a+\varepsilon), \qquad \hat{g} = 0 \quad \text{on} \quad (-\infty, b+\varepsilon)$$

and let h be such that
$$\hat{h} = 0 \quad \text{on} \quad (a+b+\varepsilon, \infty).$$

Then
$$\hat{k}(r, s, t) = 0 \quad \text{for} \quad t < r+s+\varepsilon$$

since
$$\hat{k}(r, s, t) = \hat{f}(-r)\hat{g}(-s)\hat{h}(-t),$$

and hence k belongs to $J(\phi)$ since $\mathrm{sp}(\phi)$ is contained in G and G is contained in the interior of the set where $\hat{k}$ is 0. We have $\phi * k = 0$ and in particular

$$\phi * k(0, 0, 0) = 0,$$

and now by (11)
$$\langle (L(y)x) *_B h, z\rangle = 0.$$

This holds for all z in X^*, and therefore

$$(L(y)x) *_B h = 0.$$

Thus h is in $J_B(L(y)x)$. Finally let $c \leqslant a+b$ and let h in addition to the condition already placed on it be such that

$$\hat{h}(c) = 1.$$

Thus c does not belong to $\mathrm{sp}_B(L(y)x)$, and I have shown that (9) holds.

Now drop the assumption that $\mathrm{sp}_T(y)$ and $\mathrm{sp}_A(x)$ are compact, but keep the assumption (8), and let $c \leqslant a+b$. In addition, let ε positive be such that $\mathrm{sp}_T(y)$ is contained in $(a+\varepsilon, \infty)$, let h in $L^1(R)$ be such that both $\hat{h}(c)=1$ and $\hat{h}$ is 0 on the open interval $(a+b+\varepsilon, \infty)$, and let f in $L^1(R)$ be such that $\hat{f}$ has compact support. By Lemma 2

$$\mathrm{sp}_T(y *_T f) \subset (a+\varepsilon, \infty), \qquad \mathrm{sp}_A(x *_A f) \subset (b, \infty),$$

and both $\mathrm{sp}_T(y *_T f)$ and $\mathrm{sp}_A(x *_A f)$ are compact, and thus by the preceding paragraph

$$\mathrm{sp}_B(L(y *_T f)(x *_A f)) \subset (a+b+\varepsilon, \infty).$$

In particular $\mathrm{sp}_B(L(y *_T f)(x *_A f))$ is contained in the interior of the set where $\hat{h}$ is 0, and so again by Lemma 2

$$(L(y *_T f)(x *_A f)) *_B h = 0. \tag{12}$$

Now let f run through an approximate identity in $L^1(R)$. Then $L(y *_T f)(x *_A f)$ converges in norm to $L(y)x$ by Lemma 1, and from this and (12) we get

$$(L(y)x) *_B h = 0.$$

Thus h belongs to $J_B(L(y)x)$, and c is not in $\mathrm{sp}_B(L(y)x)$ for $\hat{h}(c)=1$.

I have shown that the first part of the spectral condition holds, and a like argument shows that the second part holds too. This completes the proof of Theorem 1.

The application to be made of Theorem 1 in the next part will be with $X=H$ and $A_t=B_t=V_t$ where H is a Hilbert space and V_t is a strongly-continuous one-parameter group of unitary transformations of H. For this reason I will describe here some well-known things about one-parameter unitary groups.

H will always denote a Hilbert space. A one-parameter family M_t of closed subspaces of H is called decreasing if M_t is contained in M_s when $s<t$, and a decreasing one-parameter family is called a resolution of the identity if $\vee M_t$ is H and $\wedge M_t$ has as its only member the vector 0, where $\vee M_t$ is the smallest closed subspace of H containing all M_t and $\wedge M_t$ is the largest closed subspace of H contained in all M_t.

Let M_t be a one-parameter family of decreasing closed subspaces of H forming a resolution of the identity, and assume in addition that the family M_t is continuous from the left, i.e.

$$M_t = \bigwedge_{s<t} M_s = \bigcap_{s<t} M_s.$$

Because the family M_t is decreasing and continuous from the left, there is a Borel measure P on R with range contained in the lattice of orthogonal projections of the Hilbert space H and with the property

$$P_{[s,t)}H = M_s \ominus M_t, \tag{13}$$

where P_E is the value of P at the Borel set E and $M_s \ominus M_t$ is the orthogonal complement of M_t in M_s [4] [3, Ch. 2, Ch. 3]. Because of (13)

$$P_R H = \vee M_t \ominus \wedge M_t = H$$

and

$$P_{[s,\infty)} H = M_s \ominus \wedge M_t = M_s.$$

P is called the spectral measure generated by M_t. Let V_t be the Fourier transform of the spectral measure P. V_t is the strongly-continuous one-parameter group of unitary transformations of H given by

$$V_t = \int_R e^{-ist}\, dP_s.$$

LEMMA 4. *Let E be a closed subset of R and let x be in H. Then $\mathrm{sp}_V(x)$ is contained in E if and only if $P_E x = x$. In particular, $\mathrm{sp}_V(x)$ is contained in $[t, \infty)$ if and only if x belongs to M_t.*

Proof. Let f be in $L^1(R)$. Then

$$x *_V f = \int_R \hat{f}\, dPx = \int_G \hat{f}\, dPx + \int_E \hat{f}\, dPx, \tag{14}$$

where G is the complement of E.

Assume that $\mathrm{sp}_V(x)$ is contained in E, and let f be such that the support of $\hat{f}$ does not meet E. Then

$$\int_E \hat{f}\, dPx = 0$$

and (Lemma 2)

$$x *_V f = 0.$$

Hence

$$\int_G \hat{f}\, dPx = 0$$

for all such f, and consequently $P_G x = 0$. Thus

$$P_E x = P_E x + P_G x = P_R x = x.$$

Conversely, assume that $P_E x = x$. Then $P_G x = 0$ and (14) becomes

$$x *_V f = \int_E \hat{f}\, dPx.$$

Thus $x *_V f = 0$ when $\hat{f}$ is 0 on E, and hence $\mathrm{sp}_V(x)$ is contained in E.

A one-parameter unitary group in turn generates a resolution of the identity. For let V_t be a strongly-continuous one-parameter group of unitary transformations of H, and let M_t be the collection of all x in H with $\mathrm{sp}_V(x)$ contained in $[t, \infty)$. Then

LEMMA 5. *M_t is a one-parameter family of decreasing closed subspaces of H continuous from the left and forming a resolution of the identity.*

Proof. By Lemma 3, each M_t is a closed subspace of H and $\vee M_t$ is H, and clearly the one-parameter family M_t is decreasing and continuous from the left. Finally $\wedge M_t$ is trivial, for if x belongs to $\wedge M_t$, then

$$x *_V f = 0$$

whenever f has compact support (Lemma 2), and hence $x = 0$ (Lemma 1).

Stone's theorem [9, p. 383] completes the circle by telling us that V_t is the unitary group generated by the family M_t. I will not need this.

3. This part contains a criterion for quasi-invariance. The criterion will be used in the next part to show that analytic measures are quasi-invariant, and its proof will depend on the preceding theorem about triples of one-parameter groups of linear transformations.

S is a locally compact Hausdorff space with the real line R acting as a topological transformation group on S, and T_t is the one-parameter group of homeomorphisms of S that defines the action of R on S.

Definition 3. When f is a function on S, $T_t f$ is the function on S given by composing f with T_{-t}:

$$T_t f(p) = f(T_{-t} p).$$

With T_t defined on functions in this way, T_t is a uniformly-bounded strongly-continuous one-parameter group of linear transformations of $C_0(S)$, where $C_0(S)$ is the Banach algebra of all complex continuous functions on S that vanish at infinity. That T_t is a uniformly-bounded one-parameter group of linear transformations of $C_0(S)$ is clear. That this group of linear transformations is strongly-continuous follows from (and in turn implies) the assumption that the function $(t, p) \to T_t p$ from $R \times S$ to S is continuous. When f is in $C_0(S)$, $\mathrm{sp}_T(f)$ will mean the spectrum that is defined with this group of linear transformations (Definition 2).

μ will always denote a bounded positive Baire measure on S. $L^2(\mu)$ is the Hilbert space of all complex μ measurable functions f with

$$\int |f|^2 d\mu < \infty.$$

The inner product in $L^2(\mu)$ is given by

$$\langle f, g\rangle = \int f\bar{g}\, d\mu.$$

The criterion for quasi-invariance of μ is part 3 of the following theorem.

THEOREM 2. *The following are equivalent.*

1. *μ is quasi-invariant.*

2. *There is a strongly-continuous one-parameter group V_t of unitary transformations of $L^2(\mu)$ with the property (commutation relation)*

$$V_t(fg) = T_t f V_t g \tag{15}$$

for all t in R, f in $C_0(S)$, and g in $L^2(\mu)$.

3. *There is a one-parameter family M_t of decreasing closed subspaces of $L^2(\mu)$ forming a resolution of the identity and with the property (spectral condition) for all s and t in R and f in $C_0(S)$,*

$$\mathrm{sp}_T(f) \subset (s, \infty) \quad \textit{implies} \quad fM_t \subset M_{s+t}. \tag{16}$$

At least to people interested in group representations, the equivalence of 1 and 2 is very well known. Theorem 2, and in particular the equivalence of 2 and 3, was suggested to me by work of Helson and Lowdenslager [5].

Proof. Let $T_t\mu$ be the Baire measure given by

$$(T_t\mu)E = \mu(T_{-t}E)$$

for all Baire subsets E of S. Then

$$\int T_{-t} g\, d\mu = \int g\, d(T_t\mu) \tag{17}$$

for all Baire functions g.

1 implies 2. *Assume that μ is quasi-invariant.* This means that $T_t\mu$ is absolutely continuous with respect to μ for each t in R. Let ϕ_t be the Radon–Nikodym derivative of $T_t\mu$ with respect to μ, and define V_t by

$$V_t g = \sqrt{\phi_t}\, T_t g.$$

From (17) we get

$$\int T_{-t} g\, d\mu = \int g\phi_t\, d\mu \tag{18}$$

for all Baire functions g. This in turn gives

$$\int |V_t g|^2 d\mu = \int \phi_t T_t |g|^2 d\mu = \int |g|^2 d\mu$$

and hence V_t is a unitary transformation from $L^2(\mu)$ into $L^2(\mu)$. The group property of the T_t implies that the one-parameter family of μ measurable functions ϕ_t is a cocycle, i.e. the ϕ_t satisfy the functional equation

$$\phi_{s+t} = \phi_s T_s \phi_t \quad (\mu) \tag{19}$$

for all s and t in R. For by (18)

$$\int g\phi_{s+t} d\mu = \int T_{-s-t} g \, d\mu = \int T_{-s} g \phi_t \, d\mu = \int g \phi_s T_s \phi_t \, d\mu$$

for all Baire functions g. The functional equation (19) and the group property of the T_t in turn imply that V_t is a one-parameter group, for

$$V_{s+t} g = \sqrt{\phi_s} T_s \sqrt{\phi_t} T_{s+t} g = \sqrt{\phi_s} T_s (\sqrt{\phi_t} T_t g) = V_s V_t g.$$

Finally, the commutation relation (15) is clear from the definition of V_t.

To show that V_t is strongly-continuous is a little work, and since I will not need this to show that analytic measures are quasi-invariant, I will omit the argument.

2 implies 1. *Assume that there is a one-parameter group V_t of unitary transformations of $L^2(\mu)$ with the property* (15). (V_t need not be strongly-continuous.) Then in particular

$$T_t f = V_t(f V_{-t} 1) \quad (\mu)$$

by taking $g = V_{-t} 1$, and hence

$$\int |T_t f|^2 d\mu = \int |f|^2 |V_{-t} 1|^2 d\mu \tag{20}$$

for all f in $C_0(S)$. When E is a compact Baire set, the characteristic function of E is the pointwise limit of a decreasing sequence from $C_0(S)$, and hence by (20)

$$\mu(T_t E) = \int_E |V_{-t} 1|^2 d\mu \tag{21}$$

for all compact Baire sets E. Because Baire measures are regular, (21) is true for all Baire subsets E of S, and this shows that μ is quasi-invariant (and incidentally that $\sqrt{\phi_t} = |V_t 1|$).

2 implies 3. *Assume that there is a strongly-continuous one-parameter group V_t of unitary transformations of $L^2(\mu)$ with the property* (15). Let M_t be the collection of all g in $L^2(\mu)$ with $\mathrm{sp}_V(g)$ contained in $[t, \infty)$. By Lemma 5, M_t is a one-parameter family of decreasing closed subspaces of $L^2(\mu)$ forming a resolution of the identity. Moreover, by Theorem 1 (with

$X = L^2(\mu)$, $A_t = B_t = V_t$, $Y = C_0(S)$, and $L(f)$ multiplication by f), the spectral condition (16) holds.

3 implies 2. *Assume that there is a one-parameter family M_t of decreasing closed subspaces of $L^2(\mu)$ forming a resolution of the identity and with the property* (16). Replacing M_t by $\wedge_{s<t} M_s$ we may assume in addition that the family M_t is continuous from the left. Let P be the spectral measure generated by M_t and let V_t be the Fourier transform of P. I will again use Theorem 1 (with $X = L^2(\mu)$, $A_t = B_t = V_t$, $Y = C_0(S)$, and $L(f)$ multiplication by f) to conclude this time that the commutation relation (15) holds. The first part of the spectral condition of Theorem 1 is that for all f in $C_0(S)$, g in $L^2(\mu)$, and s and t in R,

$$\begin{matrix} \mathrm{sp}_T(f) \subset (s, \infty) \\ \mathrm{sp}_V(g) \subset (t, \infty) \end{matrix} \quad \text{implies} \quad \mathrm{sp}_V(fg) \subset (s+t, \infty),$$

and this follows from (16) and Lemma 4. The second part of the spectral condition of Theorem 1 is that for all f in $C_0(S)$, g in $L^2(\mu)$, and s and t in R,

$$\begin{matrix} \mathrm{sp}_T(f) \subset (-\infty, s) \\ \mathrm{sp}_V(g) \subset (-\infty, t) \end{matrix} \quad \text{implies} \quad \mathrm{sp}_V(fg) \subset (-\infty, s+t). \tag{22}$$

I will complete the proof of Theorem 2 by showing that (22) holds.

Let f be in $C_0(S)$. Then $\mathrm{sp}_T(\bar{f}) = -\mathrm{sp}_T(f)$. For let g be in $L^1(R)$. Then

$$\overline{f *_T g} = \bar{f} *_T \bar{g}$$

for $\overline{T_t f} = T_t \bar{f}$, and hence $J(\bar{f}) = \overline{J(f)}$. The assertion follows from this for

$$\hat{\bar{g}}(s) = \bar{\hat{g}}(-s).$$

Now let f in $C_0(S)$ and g in $L^2(\mu)$ satisfy the hypothesis of (22). By Lemma 4 there is an r smaller than t with g in the range of $P_{(-\infty, r)}$, and by Lemma 4 too the conclusion of (22) will be true if fg is in the range of $P_{(-\infty,\, s+r)}$. Let h be in M_{s+r}. By the spectral condition (16) $\bar{f}h$ is in M_r since $\mathrm{sp}_T(\bar{f})$ is contained in $(-s, \infty)$, and hence g and $\bar{f}h$ are orthogonal since $P_{(-\infty, r)}$ and $P_{[r, \infty)}$ have orthogonal ranges. Thus

$$\int fg\bar{h}\, d\mu = \langle g, \bar{f}h \rangle = 0$$

and consequently fg is orthogonal to M_{s+r}. Hence

$$P_{(-\infty, s+r)} fg = fg$$

and this is what I want. This completes the proof of Theorem 2.

4. This part contains the theorem that analytic measures are quasi-invariant. Before coming to the theorem and its proof, I want to talk at greater length, than in the introduction, about the convolution of measures and functions.

The assumption that the function $(t, p) \to T_t p$ from $R \times S$ to S is continuous implies that it is also a Baire function (i.e. the inverse image of a Baire subset of S is a Baire subset of $R \times S$). For it is the homeomorphism $(t, p) \to (t, T_t p)$ of $R \times S$ followed by the canonical projection $(t, p) \to p$ of $R \times S$ onto S, and hence is the composition of Baire functions.

Let g be a bounded Baire function on S, let λ be in $M(S)$, and let f be in $L^1(R)$. Then $g(T_t p)$ is a Baire function on $R \times S$, and by the Fubini theorem

$$\int T_{-t} g \, d\lambda$$

is a Borel function on R,

$$\int_R T_{-t} g \, f(t) \, dt$$

is a Baire function on S, and

$$\int_R \left(\int T_{-t} g \, d\lambda \right) f(t) \, dt = \int \left(\int_R T_{-t} g \, f(t) \, dt \right) d\lambda. \tag{23}$$

In particular, if g is the characteristic function of a Baire subset E of S, then $T_{-t} g$ is the characteristic function of $T_{-t} E$, and hence $\lambda(T_{-t} E)$ is a bounded Borel function on R. The convolution of λ with f is the measure $\lambda * f$ in $M(S)$ given by

$$(\lambda * f) E = \int_R \lambda(T_{-t} E) \, f(t) \, dt$$

for all Baire subsets E of S. Then

$$\int g \, d(\lambda * f) = \int_R \left(\int T_{-t} g \, d\lambda \right) f(t) \, dt. \tag{24}$$

This is just the definition of $\lambda * f$ when g is the characteristic function of a Baire set, and now the usual argument gives (24) for all bounded Baire functions g. (23) can be written

$$\int_R \left(\int T_t g \, d\lambda \right) f(t) \, dt = \int (g * f) \, d\lambda, \tag{25}$$

where

$$g * f = \int_R T_t g \, f(t) \, dt.$$

THEOREM 3. *Analytic measures are quasi-invariant.*

Proof. The proof will be the quasi-invariance criterion given by part 3 of Theorem 2. Let λ be an analytic measure, let μ be the total variation of λ, and let M_t be the closure in $L^2(\mu)$ of the linear set of all g in $C_0(S)$ with $\mathrm{sp}_T(g)$ contained in (t, ∞). I will show that the one-parameter family M_t of decreasing closed subspaces of $L^2(\mu)$ satisfies the spectral condition (16) and that

$$\vee M_t = L^2(\mu) \tag{26}$$

$$\wedge M_t = [0]. \tag{27}$$

It turns out that (16) and (26) are true with any μ, and only to get (27) will I use the fact that μ is the total variation of an analytic measure.

The spectral condition (16) clearly follows from the spectral condition that for all f and g in $C_0(S)$ and s and t in R,

$$\begin{matrix} \mathrm{sp}_T(f) \subset (s, \infty) \\ \mathrm{sp}_T(g) \subset (t, \infty) \end{matrix} \quad \text{implies} \quad \mathrm{sp}_T(fg) \subset (s+t, \infty), \tag{28}$$

and this in turn follows from Theorem 1. For in Theorem 1 let both Banach spaces be $C_0(S)$, let the three one-parameter groups of linear transformations be T_t, and let $L(f)$ be multiplication by f. The commutation relation in this setting is just that T_t is multiplicative (T_t is an algebra homomorphism of $C_0(S)$), and hence the first part of the spectral condition of Theorem 1 holds (i.e. (28) holds).

$\vee M_t$ contains a dense subspace of $C_0(S)$, for all g in $C_0(S)$ with $\mathrm{sp}_T(g)$ compact belong to $\vee M_t$, and by Lemma 3 the set of such g is dense in $C_0(S)$. Hence (26) holds.

To get (27) I will use the following corollary of the analyticity of λ.

Let g be in $C_0(S)$ with $\mathrm{sp}_T(g)$ contained in the positive reals. Then

$$\int g \, d\lambda = 0. \tag{29}$$

To get (29) let

$$\phi(t) = \int T_t g \, d\lambda.$$

ϕ is a bounded (uniformly) continuous function on R, and moreover

$$\mathrm{sp}(\phi) \subset -\mathrm{sp}(g) \cap \mathrm{sp}(\lambda), \tag{30}$$

where $\mathrm{sp}(\phi)$ is the familiar spectrum of ϕ. For let r be a real number not in $\mathrm{sp}(\lambda)$. Then there is f in $J(\lambda)$ with $\hat{f}(r) = 1$. By (24)

$$\int_R \phi(-t) f(t) \, dt = 0. \tag{31}$$

$J(\lambda)$ is a closed ideal in the group algebra $L^1(R)$ and hence it is translation invariant. Thus (31) continues to be true when $f(t)$ is replaced by $f(s+t)$ and we get

$$\phi * f = 0.$$

This shows that r is not in $\mathrm{sp}(\phi)$, and $\mathrm{sp}(\phi)$ is contained in $\mathrm{sp}(\lambda)$. Now let r be a real number not in $-\mathrm{sp}(g)$. Then there is f in $J(g)$ with $\hat{f}(-r)=1$. By (25)

$$\int_R \phi(t)\, f(t)\, dt = 0. \tag{32}$$

Again, (32) continues to be true when $f(t)$ is replaced by $f(t-s)$ and we get

$$\phi * f^* = 0$$

where $f^*(t)=f(-t)$. This shows that r is not in $\mathrm{sp}(\phi)$, and $\mathrm{sp}(\phi)$ is contained in $-\mathrm{sp}(g)$.

But now (30) implies that $\mathrm{sp}(\phi)$ is empty, and therefore ϕ vanishes identically. In particular $\phi(0)=0$ and this is (29).

Now let h be in ΛM_t and let f be in $C_0(S)$ with $\mathrm{sp}_T(f)$ contained in (s, ∞). Then for g in $C_0(S)$ with $\mathrm{sp}_T(g)$ contained in $(-s, \infty)$

$$\int fg\phi\, d\mu = \int fg\, d\lambda = 0$$

by (28) and (29), where ϕ is the Radon–Nikodym derivative of λ with respect to μ. Since h is in M_{-s}, h can be approximated in $L^2(\mu)$ by such g, and therefore

$$\int fh\phi\, d\mu = 0.$$

This holds in particular for all f in $C_0(S)$ with $\mathrm{sp}_T(f)$ compact, and hence for all f in $C_0(S)$ (Lemma 3). Therefore $h=0$ since $|\phi|=1$ (μ). This gives (27) and the proof of Theorem 3 is complete.

The full proof of Theorem 3 uses (along with other things) two basic facts about closed ideals in Abelian group algebras:

1. A set with one member is a spectral synthesis set.
2. A closed ideal contains every function whose Fourier transform vanishes on an open set containing the hull of the ideal.

Moreover, 2 follows from 1 (in a nonelementary way), and therefore one might say that Theorem 3 is a grand corollary of 1.

5. This final part contains four applications of the theorem that analytic measures are quasi-invariant. These applications, with one exception (Theorem 6), imitate applications given by deLeeuw and Glicksberg in [2].

The first is to the way R acts on analytic measures, and is more an application of the proof of Theorem 3 than of the statement. Moreover, it is the outcome here and not the application that imitates what is in [2].

THEOREM 4. *When λ is an analytic measure, $T_t \lambda$ moves continuously in $M(S)$.*

Proof. Let μ be the total variation of λ, let ϕ_t be the Radon–Nikodym derivative of $T_t \mu$ with respect to μ, and let V_t be the strongly-continuous one-parameter group of unitary transformations of $L^2(\mu)$ given by the proof of Theorem 3. Then

$$\phi_t = |V_t 1|^2 \tag{33}$$

and

$$V_t f = T_t f V_t 1 \tag{34}$$

for all f in $L^2(\mu)$. (33) follows from (21), and (34) holds at least for all f in $C_0(S)$ by the commutation relation (15). Because of (33), the linear transformation given by the right side of (34) is an isometry of $L^2(\mu)$, and hence (34) holds for all f in $L^2(\mu)$.

Now let ϕ be the Radon–Nikodym derivative of λ with respect to μ. Then the Radon–Nikodym derivative of $T_t \lambda$ with respect to μ is $\phi_t T_t \phi$, for by (17)

$$\int g\, d(T_t \lambda) = \int T_{-t} g \phi\, d\mu = \int g \phi_t T_t \phi\, d\mu.$$

This in turn gives

$$\| T_t \lambda - T_s \lambda \| = \int | V_t \phi \overline{V_t 1} - V_s \phi \overline{V_s 1} |\, d\mu \tag{35}$$

for by (33) and (34)

$$\phi_t T_t \phi = V_t \phi \overline{V_t 1}.$$

By the Schwarz inequality the right side of (35) does not exceed

$$\left(\int d\mu\right)^{\frac{1}{2}} \text{ times } \left(\left(\int |V_t \phi - V_s \phi|^2\, d\mu\right)^{\frac{1}{2}} + \left(\int |V_t 1 - V_s 1|^2\, d\mu\right)^{\frac{1}{2}}\right).$$

This shows that $T_t \lambda$ moves continuously in $M(S)$ for V_t is strongly-continuous.

When λ is in $M(S)$ and E is a Baire set, the trace of λ on E, denoted by λ_E, is the measure in $M(S)$ given by

$$\lambda_E F = \lambda(EF)$$

for all Baire sets F.

Let λ be in $M(S)$, let σ be a positive measure on the Baire subsets of S, let E'' be a Baire set with the property that E'' is a σ null set and has greatest $|\lambda|$ measure among all σ null Baire sets, and finally let E' be a Baire set with the property that E' is contained in $S\backslash E''$ and has greatest $|\lambda|$ measure among all Baire sets contained in $S\backslash E''$. Then $E' \cup E''$ carries λ. Let λ' be the trace of λ on E' and λ'' the trace of λ on E''. Then λ' is absolutely continuous with respect to σ, λ'' and σ are singular, and

$$\lambda = \lambda' + \lambda''. \tag{36}$$

The (unique) decomposition (36) is called the Lebesgue decomposition of λ with respect to σ.

The next application of Theorem 3 is to the Lebesgue decomposition of an analytic measure with respect to a quasi-invariant measure. This will use Lemma 6 and Lemma 8.

Definition 4. Let λ be in $M(S)$ and let E be a Baire set. E is called λ invariant under the action of R if $E \Delta T_t E$ is a λ null set for all t in R.

LEMMA 6. *Let λ in $M(S)$ be quasi-invariant, let σ be a positive quasi-invariant measure on the Baire subsets of S, and let E' and E'' be the disjoint Baire sets used to define the Lebesgue decomposition* (36) *of λ with respect to σ. Then E' and E'' are λ invariant under the action of R.*

Proof. $T_t E''$ is a σ null set since σ is quasi-invariant, and hence $T_t E''\backslash E''$ is a λ null set (for all t in R) because of the extremal property that E'' has. Now

$$E''\backslash T_t E'' = T_t(T_{-t} E''\backslash E'')$$

and hence $E''\backslash T_t E''$ is also a λ null set since λ is quasi-invariant. Thus $E'' \Delta T_t E''$ is a λ null set.

$E' \cup E'' T_t E''$ carries λ, for $E' \cup E''$ carries λ and $E'' \Delta T_t E''$ is a λ null set, and hence $T_t E'\backslash E'$ is a λ null set (for all t in R) since $T_t E'$ does not meet $T_t E''$. The roles of E' and $T_t E'$ in this argument can be interchanged to give that $E'\backslash T_t E'$ is a λ null set, for $T_t E' \cup T_t E''$ also carries λ (since λ is quasi-invariant). Thus $E' \Delta T_t E'$ is a λ null set.

LEMMA 7. *Let λ in $M(S)$ be quasi-invariant and let the Baire set E be λ invariant under the action of R. Then the trace of λ on E is also quasi-invariant.*

Proof. Let the Baire set F be a λ_E null set. Now

$$FT_{-t}E = F(T_{-t}E\backslash E) \cup FET_{-t}E$$

and

$$ET_tF = T_t(FT_{-t}E).$$

Hence $FT_{-t}E$ is a λ null set since $T_{-t}E\backslash E$ and FE are, and thus T_tF is a λ_E null set since λ is quasi-invariant.

A corollary to Lemma 6 and Lemma 7 is that the components in the Lebesgue decomposition of a quasi-invariant measure with respect to a positive quasi-invariant measure are also quasi-invariant.

LEMMA 8. *Let λ be in $M(S)$ and let the Baire set E be λ invariant under the action of R. Then*

$$(\lambda * f)_E = \lambda_E * f$$

for all f in $L^1(R)$, and thus $\mathrm{sp}(\lambda_E)$ *is contained in* $\mathrm{sp}(\lambda)$.

Proof.
$$(\lambda * f)_E F = \int_R \lambda(T_{-t}ET_{-t}F)\, f(t)\, dt$$

for all Baire sets F. Now

$$\lambda(T_{-t}ET_{-t}F) = \lambda(ET_{-t}ET_{-t}F) = \lambda(ET_{-t}F)$$

since $E\Delta T_{-t}E$ is a λ null set, and hence

$$(\lambda * f)_E F = \int_R \lambda(ET_{-t}F)\, f(t)\, dt = (\lambda_E * f)\, F.$$

THEOREM 5. *Let λ be an analytic measure, let σ be a positive quasi-invariant measure on the Baire subsets of S, and let*

$$\lambda = \lambda' + \lambda''$$

be the Lebesgue decomposition (36) *of λ with respect to σ. Then both* $\mathrm{sp}(\lambda')$ *and* $\mathrm{sp}(\lambda'')$ *are contained in* $\mathrm{sp}(\lambda)$. *In particular, λ' and λ'' are analytic.*

Proof. Theorem 3, Lemma 6, and Lemma 8.

When S is the circle group and R acts on S in the standard way ($T_t(e^{ix}) = e^{it}e^{ix}$), the collection of all analytic measures coincides with the annihilator in $M(S)$ of the algebra A_0 where A_0 is the collection of all functions in $C(S)$ whose Fourier coefficients vanish for nonpositive indices. A_0 is an ideal, and in fact a maximal ideal, in the familiar algebra A of all functions in $C(S)$ whose Fourier coefficients vanish for negative indices. There is a like characterization, generalizing this, of the collection of all analytic measures when there is no assumption about S or the way R acts on S.

Let A be the collection of all f in $C_0(S)$ with $\mathrm{sp}_T(f)$ contained in the nonnegative reals, and let A_0 be the collection of all f in $C_0(S)$ with $\mathrm{sp}_T(f)$ contained in the positive reals. By Lemma 3 and Theorem 1 (again with $X = Y = C_0(S)$, $A_t = B_t = T_t$, and $L(f)$ multiplication by f), A is an algebra and A_0 is an ideal in A. Though A is a closed subalgebra of $C_0(S)$ (Lemma 3), A_0 in general will not be closed, and the closure of A_0, though a proper closed ideal in A when S is compact, will not in general be a maximal ideal.

PROPOSITION 2. *The collection of analytic measures coincides with the annihilator in $M(S)$ of the algebra A_0.*

Proof. Let λ be in $M(S)$. I showed in the proof of Theorem 3 that λ annihilates A_0 when λ is analytic. Assume now that

$$\int g\,d\lambda = 0.$$

for all g in A_0.

Let g be in $C_0(S)$ and let f be in $L^1(R)$. Then by (24) and (25)

$$\int g\,d(\lambda * f) = \int (g * f^*)\,d\lambda, \tag{37}$$

where $f^*(t) = f(-t)$. Now let $s < 0$ and choose f with $\hat{f}(s) = 1$ and the support of $\hat{f}$ contained in the negative reals. Then $g * f^*$ belongs to A_0 by Lemma 2, and therefore

$$\int g\,d(\lambda * f) = 0$$

by (37). This is true for all g in $C_0(S)$, and hence

$$\lambda * f = 0.$$

Thus s is not in $\mathrm{sp}(\lambda)$.

Proposition 2 implies that the linear space of analytic measures is weak star closed in $M(S)$, and consequently the unit ball of this linear space is weak star compact, and therefore (by the Krein–Milman theorem) this unit ball is the weak star closure of the convex hull of its extreme points. Theorem 6 gives a little information about these extreme points.

Definition 5. Let λ be in $M(S)$. λ is called ergodic if every Baire set that is λ invariant under the action of R is either a λ null set or carries λ.

Being ergodic may not mean anything. For example, when S is the circle group and R acts on S in the standard way, every measure in $M(S)$ is ergodic.

Definition 6. Let λ in $M(S)$ be quasi-invariant. λ is called minimal if λ is either absolutely continuous or singular with respect to every positive quasi-invariant measure on the Baire subsets of S.

LEMMA 9. *Let λ in $M(S)$ be quasi-invariant. Then λ is minimal if and only if λ is ergodic.*

Proof. Assume that λ is ergodic. Let σ be a positive quasi-invariant measure on the Baire subsets of S, and let E' and E'' be the disjoint Baire sets used to define the Lebesgue decomposition (36) of λ with respect to σ. E'' is λ invariant under the action of R (Lemma 6), and therefore E'' is a λ null set or E'' carries λ. With the first alternative λ is absolutely continuous with respect to σ, and with the second λ and σ are singular. Thus λ is minimal.

Assume now that λ is minimal, and let the Baire set E be λ invariant under the action of R. Since λ is quasi-invariant, the trace of λ on E is also quasi-invariant (Lemma 7), and therefore λ is absolutely continuous with respect to λ_E or λ and λ_E are singular. With the first alternative E carries λ, and with the second E is a λ null set. Thus λ is ergodic.

THEOREM 6. *The extreme points of the unit ball in the linear space of analytic measures are minimal.*

Proof. Let λ be an extreme point in ball $A_0^\perp$. Then λ is ergodic. For let the Baire set E be λ invariant under the action of R. Then λ_E is analytic (Lemma 8). If neither λ_E nor $\lambda - \lambda_E$ is 0, then

$$\lambda = \|\lambda_E\| \frac{\lambda_E}{\|\lambda_E\|} + \|\lambda - \lambda_E\| \frac{\lambda - \lambda_E}{\|\lambda - \lambda_E\|},$$

and this contradicts the assumption that λ is an extreme point in ball $A_0^\perp$ for

$$\|\lambda_E\| + \|\lambda - \lambda_E\| = \|\lambda\| = 1.$$

Hence $\lambda_E = 0$ or $\lambda = \lambda_E$. With the first alternative E is a λ null set, and with the second E carries λ. Thus λ is ergodic.

λ is now minimal by Theorem 3 and Lemma 9.

Let $\mathrm{cl}A_0$ be the uniform closure of the algebra A_0. A closed Baire set E is called an interpolation set for $\mathrm{cl}A_0$ if every function in $C_0(E)$ is the restriction to E of a function in $\mathrm{cl}A_0$. The last application of Theorem 3 is to interpolation sets for the algebra $\mathrm{cl}A_0$.

LEMMA 10. *Let E be a Baire set with the property that for each p in S the Borel set*

$$[t \in R \mid T_t p \in E]$$

has linear measure zero. (Roughly speaking, E meets each orbit in a set of linear measure zero.) Then E is a null set for every quasi-invariant measure in $M(S)$.

Proof. Let g be the characteristic function of E, let f in $L^1(R)$ be nonnegative and not the zero vector, and let μ in $M(S)$ be positive and quasi-invariant. Then

$$\int_R \left(\int T_{-t}\, g\, d\mu \right) f(t)\, dt = 0$$

by (23) and the assumption about E. Hence

$$\mu(T_{-t}E) = \int T_{-t}\, g\, d\mu = 0$$

for almost all t in the set where f is positive, and thus E is a μ null set since μ is quasi-invariant.

LEMMA 11. *Let p be in S and let f be in A. Then the closed set*

$$[t \in R \,|\, f(T_t p) = 0]$$

either has linear measure zero or is all of R.

Proof. Let g be the bounded continuous function on R given by $g(t) = f(T_t p)$, and let u be in $L^1(R)$ with the support of $\hat{u}$ contained in the nonnegative reals. Then the support of $(gu)\hat{}$ is also contained in the nonnegative reals. For let s be in R and let

$$v(t) = e^{ist} u(-t).$$

Then
$$(gu)\hat{}(s) = \int_R e^{-ist} f(T_t p)\, u(t)\, dt = (f * v)(p).$$

Now assume $s < 0$. Then $f * v = 0$ by Lemma 2 for sp(f) is contained in the nonnegative reals and $\hat{v}$ is 0 on (s, ∞), and hence $(gu)\hat{}(s) = 0$.

Since the Fourier transform of gu vanishes on the negative reals, either the closed set where g vanishes has linear measure zero or g vanishes identically (for $g(t)u(t)dt$ is an analytic measure with R acting on itself by translating with the members of R, and by Theorem 3 it has the same null sets as Lebesgue measure if g and u are not identically zero).

THEOREM 7. *Let E be a closed Baire subset of S. The following are equivalent.*

1. *E is a null set for all analytic measures.*
2. *E is an interpolation set for* clA_0.
3. *For each p in E the closed set $[t \in R \,|\, T_t\, p \in E]$ has linear measure zero.*
4. *For each p in S the closed set $[t \in R \,|\, T_t\, p \in E]$ has linear measure zero.*

Proof.

3 implies 4. Because T_t is a one-parameter group of homeomorphisms, every non-empty set in 4 is a translate of a set in 3.

4 implies 1. Theorem 3 and Lemma 10.

1 implies 2. Proposition 2 and Bishop's general Rudin–Carleson theorem [1]. (Though the spaces in [1] are compact, the arguments in [1] apply to locally compact spaces.)

2 implies 3. Let p be in E and let ϕ be the function from R to S given by $\phi(t) = T_t p$. Then ϕ is continuous and the set in 3 is $\phi^{-1}E$. Moreover,

$$E = E' \cup E''$$

and

$$\phi^{-1}E = \phi^{-1}E' \cup \phi^{-1}E'',$$

where E' has p as its only member and E'' is E with p taken out. Now t in R belongs to $\phi^{-1}E'$ if and only if T_t does not move p, and hence $\phi^{-1}E'$ is a closed subgroup of R.

Suppose $\phi^{-1}E'$ has positive linear measure. Then, being a closed subgroup of R, it must be all of R, and hence

$$T_t p = p \tag{38}$$

for all t in R. This implies that

$$g(p) = 0 \tag{39}$$

for all g in $\mathrm{cl}A_0$. For let g be in $C_0(S)$ and let f be in $L^1(R)$ with the support of $\hat{f}$ disjoint from $\mathrm{sp}(g)$. Then

$$g \ast f = 0$$

by Lemma 2, and, on the other hand,

$$g \ast f(p) = g(p)\hat{f}(0)$$

by (38). Hence (39) holds for g in A_0, and therefore it holds for g in $\mathrm{cl}A_0$. ((38) implies too that a mass concentrated at p is an analytic measure.)

(39) contradicts the assumption that E is an interpolation set for $\mathrm{cl}A_0$, and hence $\phi^{-1}E'$ has linear measure zero.

Let K be a compact set contained in $\phi^{-1}E''$. Then ϕK is compact and contained in E'', and now because E is an interpolation set for $\mathrm{cl}A_0$ there is an f in $\mathrm{cl}A_0$ with $f(p) = 1$ and $f = 0$ on ϕK. By Lemma 11, K has linear measure zero.

Since every compact set contained in $\phi^{-1}E''$ has linear measure zero, $\phi^{-1}E''$ too has linear measure zero.

Clearly, in the proof of Theorem 3 it is not necessary to assume that the measure λ is analytic, but only that $\mathrm{sp}(\lambda)$ is bounded below or above, and for this reason there are the following more general versions of theorems 3 through 6.

THEOREM 3'. *When λ is in $M(S)$ with* $\mathrm{sp}(\lambda)$ *bounded below or above, λ is quasi-invariant.*

THEOREM 4'. *When λ is in $M(S)$ with* $\mathrm{sp}(\lambda)$ *bounded below or above, $T_t \lambda$ moves continuously in $M(S)$.*

THEOREM 5′. *Let λ be in $M(S)$ with* $\mathrm{sp}(\lambda)$ *bounded below or above, let σ be a positive quasi-invariant measure on the Baire subsets of S, and let*

$$\lambda = \lambda' + \lambda''$$

be the Lebesgue decomposition (36) *of λ with respect to σ. Then both* $\mathrm{sp}(\lambda')$ *and* $\mathrm{sp}(\lambda'')$ *are contained in* $\mathrm{sp}(\lambda)$.

PROPOSITION 2′. *Let E be a closed subset of R and let G be the complement of E. The closed linear set of all measures in $M(S)$ with spectra contained in E coincides with the annihilator in $M(S)$ of the linear set of all functions in $C_0(S)$ with spectra contained in $-G$.*

THEOREM 6′. *Let E be a closed subset of R bounded below or above. The extreme points of the unit ball in the linear space of all measures in $M(S)$ with spectra contained in E are minimal.*

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