# The analogue of Picard's theorem for quasiregular mappings in dimension three

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## 1. Introduction

The theory of quasiregular mappings has turned out to be the right extension of the geometric parts of the theory of analytic functions in the plane to real $n$-dimensional space. The study of these mappings was initiated by Rešetnjak around 1966 and his main contributions to the theory is presented in the recent book [8]. For the basic theory of quasiregular mappings we refer to [2], [3], [13]. The definition is given in Section 2.1. In 1967 Zorič [14] raised the question of the validity of a Picard's theorem on omitted values for quasiregular mappings. Such a theorem appeared in 1980 in the following form.

THEOREM 1.1. [9]. *For each $K\geq 1$ and integer $n\geq 3$ there exists an integer $q=q(n,K)$ such that every $K$-quasiregular mappings $f:\mathbf{R}^n\to\mathbf{R}^n\setminus\{u_1,\ldots,u_q\}$ is constant whenever $u_1,\ldots,u_q$ are distinct points in $\mathbf{R}^n$.*

Already from the early beginning of the theory it has been conjectured that the Picard's theorem is true in the same strong form for $n\geq 3$ as in the plane, namely that $q$ can be taken to be 2 in Theorem 1.1. The purpose of this paper is to give a solution to this question in dimension three. The result is presented in Theorem 1.2. It shows that the conjecture is false and that Theorem 1.1 is indeed qualitatively best possible for $n=3$.

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THEOREM 1.2. *For each positive integer p there exists a nonconstant quasiregular mapping $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ which omits p points in $\mathbf{R}^3$.*

The history of the problem is the following. At a rather early stage of the theory of quasiregular mappings it was proved both in [3, 4.4] and [7, Theorem 2] that a nonconstant quasiregular mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ cannot omit a set of positive $n$-capacity. The discovery of Theorem 1.1 was a result of a development of value distribution theory of quasiregular mappings some years earlier than [9]. Two proofs of Theorem 1.1 different from the original in [9] are presented in [11] and [12]. A defect relation, which is an analogue for a result from Ahlfors's theory of covering surfaces and a generalization of Theorem 1.1, was proved in [10]. For $p=1$ the construction of the map in Theorem 1.2 is easy. This was done by Zorič in [14] and his construction can be generalized for all $n$ in a straightforward manner.

Already for $p=2$ it can be shown that a map $f$ in Theorem 1.2 must satisfy hard requirements. In fact, the solution presented in this paper is geometrically one of the simplest possible in the same sense as the exponential function is the simplest in the plane to omit one point. In short terms, to construct $f$ in Theorem 1.2 is a sensitive interplay between the combinatorial properties and the dilatation of $f$. A 2-dimensional deformation theory, developed in Section 5, has an essential role in this interplay. The main reason why a map $f$ in Theorem 1.2 is at all possible to construct is that tubular neighborhoods of arcs do not separate in $\mathbf{R}^3$. To obtain Theorem 1.2 also for dimensions $n \geq 4$ in a similar way would among other things require an $(n-1)$-dimensional deformation theory. Such a theory is not available in a straightforward manner from the method presented here because we use essentially some properties of the plane.

For the sake of clarity we shall give the proof of Theorem 1.2 in detail for $p=2$. The general case is in principle almost the same and we shall indicate in the end (Section 8) what changes will be made for larger $p$. To help the reader to understand the strategy of the proof, the main features of it will be outlined below.

Let us first give a description of Zorič's construction for $p=1$. Let $f_0$ be a quasiconformal mapping of the infinite cylinder $C=\{x \in \mathbf{R}^3 \mid (x_1, x_2) \in A\}$, with the square $A=\{x \in \mathbf{R}^2 \mid 0<x_1, x_2<1\}$ as base, onto the half space $H_+=\{x \in \mathbf{R}^3 \mid x_3>0\}$ such that $f_0(C \cap H_+)=H_+ \setminus \bar{B}^3$ ($B^n$ is the unit ball in $\mathbf{R}^n$) and the edges of $C$ correspond to rays emanating from the origin. We extend $f_0$ by repeated reflections through the faces of $C$ and $\partial H_+$ and obtain a quasiregular mapping $f\colon \mathbf{R}^3 \to \mathbf{R}^3 \setminus \{0\}$. Let $D$ be a cylinder obtained from $C$ by reflection in a face and let $C_1=C \cap H_+$, $D_1=D \cap H_+$. Let $C_2$ and $D_2$ be obtained from $C_1$ and $D_1$ respectively by reflection in $\partial H_+$. Then $f$ maps $\bar{C}_1 \cup \bar{D}_1$

onto $U_1 \cup S^2$ and $\bar{C}_2 \cup \bar{D}_2$ onto $U_2 \cup S^2$ where $U_1 = \mathbf{R}^3 \setminus \bar{B}^3$ and $U_2 = B^3 \setminus \{0\}$ ($S^{n-1}$ is the unit sphere in $\mathbf{R}^n$). The preimages $W_j = f^{-1}U_j$ are halfspaces and hence very simple.

Let then $p=2$ and suppose we are given a nonconstant quasiregular mapping $f: \mathbf{R}^3 \to \mathbf{R}^3 \setminus \{u_2, u_3\}$, $u_2 = -e_3/2$, $u_3 = e_3/2$ ($e_i$ is the standard $i$th basis vector in $\mathbf{R}^n$). Write $u_1 = \infty$ and let $U_1, U_2, U_3$ be the components of $\mathbf{R}^3 \setminus (S^2 \cup B^2 \cup \{u_2, u_3\})$ such that $u_j \in \bar{U}_j$, $j=2,3$. Since $u_j$ is omitted, each component of $W_j = f^{-1}U_j$ is unbounded. In our construction of such a map $f$ the cylinders $C_j, D_j, \ldots$ in the Zorič's map will correspond to tubes in $W_j$ tending to $\infty$ and ending in $\partial W_j$. Each such tube will be mapped by $f$ onto half of $U_j$ similarly as in the Zorič's map. One of the main difficulties arises from the fact that if a tube in $W_1$ has a common end with a tube in $W_2$, then the neighboring tube in $W_1$ must have a common end with a tube in $W_3$, and this for all permutations of $W_1$, $W_2$, and $W_3$. This makes the sets $W_j$ very complicated because the third must be near every common boundary point of two of the sets. For $p=1$ there is no such problem because there is no third set $W_j$. To glue the various tubes at the ends together is essentially a combinatorial problem, the solution of which is part of Section 7.

The main idea of the proof is to first construct an approximation of $f^{-1}(S^2 \cup B^2)$, denoted by $|M_\infty|$. The complement of $|M_\infty|$ will consist of eight components $V_1, V_2, V_3(h)$, $h=0,\ldots,5$, all topologically equivalent to a 3-ball. The sets $V_1, V_2$, and $V_3 = V_3(0) \cup \ldots \cup V_3(5)$ are approximations of $W_1$, $W_2$, and $W_3$ respectively. To achieve the requirement that a common boundary point of any two of the $V_j$'s is not far from the third, we introduce an operation, called *cave refinement*, by which we can diminish the Hausdorff distance between parts of boundaries of sets like the $V_j$. To be able to decide when a cave refinement operation is needed, we have to have in mind the number of tubes ending at a given part of $f^{-1}(S^2 \cup B^2)$. For this we construct simultaneously an approximation on each $\partial V_1, \ldots, \partial V_3(5)$ of the configuration of the ends of the tubes. Such configurations are called *map complexes* and they are essentially triangulations with the property that each vertex is common to an even number of 2-simplexes. These constructions are made in Sections 2 and 3.

The definition of the mapping $f$ is started in Section 4. There it is given on certain level surfaces which lie in the sets $V_1, \ldots, V_3(5)$ and are almost obtained by similarities from $\partial V_1, \ldots, \partial V_3(5)$. Since the number of repeated cave refinement operations tends to infinity when we approach $\infty$, in order to keep the dilatation bounded we cannot take the map complexes corresponding to the tube configurations on the different level surfaces to be topologically equivalent, but we have to rearrange the order of the tubes when we pass from one level surface to the next. This has led us to develop a deformation theory in Section 5 mentioned above for discrete open maps in the plane. Deformation of such maps has been studied rather extensively by M. Morse and M. Heins (see [6]), but their results are not applicable here. By means of a deformation lemma from Section 5 together with a technique of straightening caves, we are able to extend the map to the layers between the level surfaces in Section 6. In Section 7 we glue together the obtained maps near $|M_\infty|$. Apart from the combinatorial aspects that part is somewhat similar to Section 6. Throughout the work we use mainly piecewise linear technique.

We make no effort in striving for any good bound for the dilatation of $f$ in Theorem 1.2. Let it be remarked here that for $n\geq 3$ there exists $K_n>1$ with the property that for $1\leq K\leq K_n$ every $K$-quasiregular mapping $f$ is locally homeomorphic [4, 4.6]. If in addition $f:\mathbf{R}^n\to\mathbf{R}^n$, then $f$ is quasiconformal [14], and hence does not omit any point.

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## 2. Preliminary constructions

Most of our constructions are based on special triangulations of surfaces. The main object in this section is to give a "cave" construction operation which will be used in Section 3 to construct the set $|M_\infty|$ mentioned in the introduction.

2.1. *Terminology and notation.* A continuous map $g: D\to\mathbf{R}^n$ of a domain in the Euclidean $n$-space $\mathbf{R}^n$, $n\geq 2$, is called *quasiregular* if

(1) $g$ belongs to the local Sobolev space $W^1_{n,\mathrm{loc}}(D)$, i.e. $g$ has distributional first order partial derivatives which are locally $L^n$ integrable, and

(2) there exists $K$, $1\leq K<\infty$, such that

$$|g'(x)|^n\leq KJ_g(x) \quad \text{a.e.} \tag{a}$$

Here $g'(x)$ is the formal derivative defined by means of the partial derivatives $D_i g(x)$ by $g'(x)e_i=D_i g(x)$ ($e_i$ is the standard $i$th basis vector), $|g'(x)|$ is its operator norm, and $J_g(x)$ the Jacobian determinant. These are defined a.e. by (1). Let $g$ be quasiregular. Then also

$$J_g(x)\leq K\inf_{|h|=1}|g'(x)\,h|^n \quad \text{a.e.} \tag{b}$$

holds for some $K\in[1,\infty[$. The smallest $K$ satisfying (a) and (b) is the *dilatation* $K(g)$ of $g$. If $K(g)\leq K$, $g$ is called *K-quasiregular*. Although not used in this work we make the remark that the definition of quasiregularity extends immediately to the case $g\colon M\to N$ where $M$ and $N$ are connected oriented Riemannian $n$-manifolds, see for example [5]. The term *quasimeromorphic* is reserved for the case where $M$ is a domain in $\mathbf{R}^n$ or in $\hat{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$ and $N=\hat{\mathbf{R}}^n$. $\hat{\mathbf{R}}^n$ is equipped with the spherical metric. A quasiregular homeomorphism is called a *quasiconformal mapping*.

The ball $\{y\in\mathbf{R}^n \mid |x-y|<r\}$ and the sphere $\{y\in\mathbf{R}^n \mid |x-y|=r\}$ are denoted by $B^n(x,r)$ and $S^{n-1}(x,r)$. We write $B^n(r)=B^n(0,r)$, $B^n=B^n(1)$, $S^{n-1}(r)=S^{n-1}(0,r)$, $S^{n-1}=S^{n-1}(1)$. We identify $\mathbf{R}^k$ with $\mathbf{R}^k\times\{0\}\subset\mathbf{R}^{k+m}$. The normalized $k$-dimensional Hausdorff measure in $\mathbf{R}^n$ is denoted by $H^k$. The Euclidean metric is denoted by $d$.

A *complex* means in this work a locally finite (rectilinear) simplicial homogeneous complex in $\mathbf{R}^3$. For terminology, see [1]. If $K$ is a complex, $K^i$ is the set of $i$-simplexes in $K$, $|K|$ is the space of $K$, $St(A,K)$ is the star of $A\in K$ as a subcomplex, and $K^{(k)}$ is the $k$th barycentric subdivision of $K$. If $L$ is a subcomplex of $K$, $L$ is also denoted by $K|A$ where $A=|L|$. We shall need the notion of join of a complex $K$ with a 0-simplex $\{v\}$, denoted by $vK$. If $M$ is a complex or a simplex, $\dot{M}$ means the boundary complex and $\operatorname{int} M$ the relative interior. By $\langle v_0,\dots,v_i\rangle$ we denote the (geometrically realized) $i$-simplex with vertices $v_0,\dots,v_i$. If $K$ is a 2-complex or a set of 2-simplexes, $\sigma(K)$ is the number of 2-simplexes in $K$. If $K$ is a connected complex and $v,v'\in K^0$, we let $\varrho_K(v,v')=\varrho(v,v')$ be the smallest number of 1-simplexes in a path in $|K^1|$ connecting $v$ and $v'$. The closure $\bar{A}$ and boundary $\partial A$ of a set $A\subset\mathbf{R}^3$ is taken with respect to $\mathbf{R}^3$.

2.2. *l-subdivisions and K-trees*. We fix $\nu=24\,000$ througout except Section 8. A 2-simplex $T$ in $\mathbf{R}^3$ is called *admissible* if all its angles are at least $\pi/12$. For an integer $l\geq 0$ an admissible $T$ is called an *l-triangle* if the side lengths of $T$ are between $\nu^l/4$ and $8\nu^l$.

Let $T$ be an $(l+1)$-triangle, let $s_1,s_2,s_3$ be the sides of $T$, and let $d_i$ be the length of $s_i$. We assume that for each $i=1,2,3$ we are given an integer $h_i'$, $1\leq h_i'\leq 6$. We divide $s_i$ by a set $X_i$ of points into $h_i=h_i'h_i''$ equal parts, where $h_i''$ is an integer, such that $|d_i/h_i-\nu^l|$ takes its smallest possible value. Let $a_i\in X_i$ be a point such that $|a_i-b_i|<d_i/8$, $i=1,2,3$, where $b_i$ is the midpoint of $s_i$. By elementary geometry one shows then that there exists a 2-complex $K$ with $|K|=T$ and a 1-subcomplex $L$ of $K$ such that the following conditions hold:

(1) All angles in $K$ are at least $\pi/8$ except possibly two at vertices of $T$ where it is at least $\pi/12$.

(2) $K^0\cap|\dot{T}|\subset X_1\cup X_2\cup X_3$ and $\{v\in X_i|\,|v-b_i|<d_i/6\}\subset K^0$.

(3) $|L|$ is a tree and consists of connected subcomplexes $L_i$, $i=0,\dots,3$, such that $L_0^0=\{v\in L^0|\varrho_K(v,K^0\cap|\dot{T}|)\geq 5\}$ and $|L_i|$, $i=1,2,3$, is a line segment and a minimal path in the $\varrho_K$-distance connecting $|L_0|$ and $a_i$. $d(|L_0|,|\dot{T}|)\geq 4v^l$.

(4) If $v\in L^0$, then the angles in $St(v,K)$ are in the interval $[\pi/3-20/v,\pi/3+20/v]$. If $v\in K$ and $\varrho_K(v,v')\leq 1$ for some $v'\in L_i^0$, $i=1,2,3$, the angles in $St(v,K)$ are all $\pi/3$. Hence $\sigma(St(v,K))$ is 6 if $v\in L^0\setminus\{a_1,a_2,a_3\}$ and 3 if $v\in\{a_1,a_2,a_3\}$.

(5) Let $r\in K^1$ and let $t$ be the length of $r$. Then $v^l/2\leq t\leq 6v^l$. If $v\in L^0$ and $r\in St(v,K)$, then $v^l-40v^{l-1}\leq t\leq v^l+40v^{l-1}$.

(6) Let $v,v'\in L^0$. If $\varrho_K(v,v')=1$, then $\sigma_L(v,v')\leq 2$. If $\varrho_K(v,v')=2$, then $\varrho_L(v,v')\leq 4$.

(7) If $v\in K^0$ and $\varrho_K(v,K^0\cap|\dot{T}|)\geq 4$, then $\varrho_K(v,L^0)\leq 3$.

(8) $2v^{-2l}H^2(T)\leq\sigma(K)\leq 4v^{-2l}H^2(T)$ and $\sigma(K)/2\leq\sigma(\{A\in K^2|A\cap|L|\neq\emptyset\})\leq 2\sigma(K)/3$.

If we fix one of the points $a_i$, call it $a$, to be the last point in $L^0$, we are given a natural (partial) order in $L^0$ such that each $v\in L^0\setminus\{a\}$ has a unique successor. The complex $K$ is called an *l-subdivision* of $T$ (with *partition numbers* $h_i'$) and $L$ a *K-tree*. Each 2-simplex $B$ of $K$ will be given partition numbers for its sides by the following rule. Let $t$ be a side of $B$. If $t$ is not in $|\dot{T}|$, then its partition number is 1. If $t$ is contained in $s_i$ and if the length of $t$ is $c$, then the partition number of $t$ is $ch_i/d_i$.

2.3. *Cave complexes.* Let $M$ be a finite 2-complex consisting of $(l+1)$-triangles such that the following holds:

(a) $|M|$ is homeomorphic to a disk and is contained in a plane.

(b) Each $T\in M^2$ is given partition numbers for its sides and so that the partition number for a common side $T\cap S$, $S\in M^2$, is always 1.

(c) Let $P$ be a 1-subcomplex of the dual cell complex of $M$, defined by the barycentric subdivision, such that $P^1$ is a union of pairs $\{\langle v,x\rangle,\langle x,w\rangle\}$ where $v$ and $w$ are barycenters of some $S$ and $T$ in $M^2$ with a common side $s$ and $x$ is the barycenter of $s$. We assume that $|P|$ is a tree, for every $T\in M^2$ the barycenter of $T$ is in $P^0$, and there is a barycenter $p$ of some $T\in M^2$ which belongs to only one 1-simplex of $P$. If we fix $p$ to be the last element, we are given a natural (partial) order in $P^0$. This induces an order on $M^2$.

If $S,T\in M^2$ and if $T$ is the successor of $S$, we call $S\cap T$ the last side of $S$.

For each $T\in M^2$ we choose an $l$-subdivision $K_T$ with the given partition numbers and a $K_T$-tree $L_T$ with the following condition:

(d) Let the $a_i$ and $L_i$ in Section 2.2 be for $T$ now denoted by $a_{Ti}$ and $L_{Ti}$. If

$S, T \in M^2$ and $T$ is the successor of $S$, the last $a_{Si}$ in $L_S^0$ is some $a_{Tj}$ and $a_{Tj}$ is not the last in $L_T^0$. Moreover, $L_{Si}$ and $L_{Tj}$ are then symmetric with respect to the common side of $S$ and $T$.

If $T \in M^2$ is not the last in $M^2$, we let $L'_T$ be the subcomplex of $L_T$ defined as follows. Suppose that $a_{T3}$ is the last of $L_T^0$. Let $I \subset \{1, 2\}$ be the set of indices $i$ for which $a_{Ti}$ is not the last of any $L_S^0$ such that $T$ is the successor of $S$. Then we obtain $L'_T$ from $L_T$ by removing the $L_{Ti}$'s, $i \in I$, and completing it to a complex. Then we form

$$L = \bigcup_{T < T_p} L'_T$$

where $T_p$ is the last in $M^2$.

Let $u$ be a normal of $|M|$ of length $\nu^l$. Let $a \in L$ and let $A = \langle v_0, v_1, v_2 \rangle$ be a 2-simplex in $St(a, K)$ where $K = \bigcup_T K_T$. Note that $K$ is not a complex in general but $St(a, K)$ is well defined for $a \in L$ by (2) in Section 2.2. Write $U = \{v_0, v_1, v_2\} \cap L^0$, $W = \{v_0, v_1, v_2\} \setminus L^0$, and let $A_+$ and $A_-$ be the 2-simplexes with $(U+u) \cup W$ and $(U-u) \cup W$ as the set of vertices respectively. From $K$ we form a new union of complexes, denoted by $K_L = K(L)$, as follows. All $A$'s as above are replaced by $A_+$ and $A_-$ and all other 2-simplexes in $K$ are left untouched. $K_L$ is called a *cave complex* and the bounded component of $\mathbf{R}^3 \setminus |K_L|$ the *cave* of $K_L$. All 2-simplexes in $K_L$ are $l$-triangles. The partition number for a side of a 2-simplex of $K_L$ is defined to be the partition number of the corresponding side in $K_T$ for some $T \in M^2$.

2.4. *Bending and opening of cave complexes.* We also need two modifications of the above construction. Suppose that $|M|$ is not necessarily contained in a plane. We fix a normal $u_T$ of length $\nu^l$ of each $T \in M^2$ such that $u_T$ points towards the same side of $|M|$ for all $T$. Let $S, T \in M^2$, let $T$ be the successor of $S$, and let $a = a_{Si}$ be in $L^0 \cap s$ where $s = S \cap T$. We assume that $u_S$ and $u_T$ form an angle $\varphi$, $0 \leq \varphi \leq \pi/3$. Then, when forming $K_L$ from $K$ we replace the vertex $a$ by $b$ and $c$ defined as follows. Let $C$ be the line containing the line segment $L_{Si}$ and let $X$ be the plane spanned by $s$ and $u_S + u_T + a$. Then $b$ is the point in $(C + u_S) \cap X$ and $c$ the point in $(C - u_S) \cap X$. This modification is called *bending* at $s$. If there are bendings at several sides, we assume that no self intersection occurs. The normals $u_T$ are said to be positive.

Our second modification is called *cave opening*. Let $S, T, s$ and $a$ be as above and let $T = T_0$ be the last in $M^2$. We shall first replace $T_0$ by two $(l+1)$-triangles $T_+$ and $T_-$ with partition number 1 on the sides. We assume that $u_S = u_T$. $T_+$ and $T_-$ have $s$ as a common side and they lie in planes obtained by turning the plane containing $T_0$ around $s$ by angles $\varphi_+ \in ]0, \pi/2[$ and $\varphi_- \in ]-\pi/2, 0[$ respectively. The orientation is chosen so

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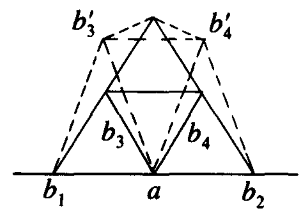


Fig. 2.1.

that $T_+$ lies in the nonnegative side of $T_0$ defined by $u_T$. In $T_+$ we choose an $l$-subdivision $K_+$ as follows. If $T_+$ is considered as a successor of $S$, then the pair $S, T_+$ satisfies the condition (d) in Section 2.3 modified to bending at $s$. Similarly we choose an $l$-subdivision $K_-$ in $T_-$. We deform $K_+$ to an isomorphic complex $K_+'$ by moving the two vertices $b_3$ and $b_4$ in $St(a, K_+)$ and not in $s$ (Figure 2.1) to $b_3'$ and $b_4'$ to a greater distance form $a$. A complex $K_-'$ is similarly defined. To form the new cave complex $K_L$ we first replace $K_{T_0}$ by $K_+'$ and $K_-'$. We replace each $A\in K^2$ with $A\not\subset T_0$ by $A_+$ and $A_-$ as before, in particular $a$ is replaced by $a+u_S$ and $a-u_S$. Each $A=\langle a, c, d\rangle$ in $St(a, K_+')^2$ is replaced by $A_+=\langle a+u_S, c, d\rangle$ and all others in $K_+'^2$ are kept untouched. The deformed $K_+'$ is so chosen that all resulting 2-simplexes are $l$-triangles. $K_-'$ is treated similarly. The 2-simplexes $T_+$ and $T_-$ are called *cave opening simplexes*, the set $Q=M^2\setminus\{T_0\}$ is called a *cave base*, $s=T_+\cap T_-$ the *last side* of $Q$, and $a$ the *cave center* in $s$. By $K$ we denote now the union

$$K=\bigcup_{S<T_0} K_S\cup K_+'\cup K_-'.$$

All 2-simplexes in $K_L$ are $l$-triangles also in these modifications. The cave opening simplexes will always be in certain positions with respect to the cave base, see Sections 2.5 and 3.

A cave complex $K_L$ is thus always obtained by replacing each 2-simplex of $K$ by one or two 2-simplexes. This gives a natural simplicial projection $\pi: |K_L|\to|K|$. If $T\in M^2\setminus\{T_0\}$, the 2-subcomplex $N=\omega(T)$ of $K_L$ with $|\omega(T)|=\pi^{-1}T$ is said to be *inherited* from $T$. The 2-subcomplex of $N$, the 2-simplexes of which are the $A_+$'s ($A_-$'s), is denoted by $N_+$ ($N_-$). The 2-subcomplex of $N$ which $\pi$ leaves invariant is denoted by $N_0$. The map $\pi$ induces from $u_T$ positive normals on the 2-simplexes of $N$.

2.5. *Cave refinements*. Let $T\in M^2\setminus\{T_0\}$ and $N$ be as in the last paragraph, and let $J=N_0\cap N_+$ $(=N_+\cap N_-)$. We shall next form opened cave complexes such that the cave bases are subsets of $N_0^2$. There exists a set $\Omega_0$ of disjoint such cave bases such that

(1) for each $Q \in \Omega_0$ $Q \subset N_0^2$ and the last side of $Q$ is in $J$,

(2) $\bigcup_{Q \in \Omega_0} Q = N_0^2$,

(3) $\sigma(Q) \leq 10\nu$ for all $Q \in \Omega_0$.

Let $Q \in \Omega_0$, let $s$ be the last side of $Q$, and let $A_+$ and $A_-$ be the 2-simplexes of $N_+$ and $N_-$ with $A_+ \cap A_- = s$. We form $(l-1)$-subdivisions of elements in $Q' = Q \cup \{A_+, A_-\}$ and an opened cave complex with $Q$ as a cave base and $A_+$ and $A_-$ as cave opening simplexes. Recall that the sides of each 2-simplex in $N$ are given partition numbers. The union of these cave complexes when $Q$ runs over $\Omega_0$ is denoted by $K(\Omega_0)$. Some 2-simplexes in $N_+ \cup N_-$, namely those which appear as cave opening simplexes, are replaced by $(l-1)$-subdivisions followed by a simplicial homeomorphism when $K(\Omega_0)$ is formed.

With $N_0$ replaced by $N_+$ and $N_-$ we form similarly corresponding sets $\Omega_+$ and $\Omega_-$. Let $Q' \in \Omega_+$ and let $s$ be the last side of $Q'$. Then $s = A \cap B$ for some $A \in N_0^2$, $B \in N_-^2$. We shall now form a union of complexes which we again call an opened cave complex with $Q'$ as a cave base and $A$ and $B$ as cave opening simplexes. The difference from the constructions in $K(\Omega_0)$ is that now we use the $(l-1)$-triangles already constructed, possibly together with some $(l-1)$-subdivisions for 2-simplexes not earlier subdivided. We require that if $s$ is also a last side of $Q \in \Omega_0$ and $a$ and $a'$ are the cave centers in $s$ corresponding to $Q$ and $Q'$, then $\varrho(a, a') \geq 6$ measured in the $(l-1)$-subdivisions. This requirement makes it possible to do the opening constructions independently. We also require that each $A \in N_0^2$ contains a last side for at most one cave base $Q' \in \Omega_+$. The union of these cave complexes is denoted by $K(\Omega_+)$. We can choose $\Omega_+$ so that the angles in the bendings for any $Q' \in \Omega_+$ do not exceed $\pi/3$. Note that $K(\Omega_0) \cap K(\Omega_+)$ is a nonempty union of 2-complexes. In a similar way we form $K(\Omega_-)$ using also $(l-1)$-triangles already constructed in $K(\Omega_0) \cup K(\Omega_+)$. We require that if $Q$ and $Q'$ are any two in $\Omega = \Omega_0 \cup \Omega_+ \cup \Omega_-$ with a common last side $s$, then $\varrho(a, a') \geq 6$ for the cave centers $a$ and $a'$ of $Q$ and $Q'$ in $s$. We write $K(\Omega) = K(\Omega_0) \cup K(\Omega_+) \cup K(\Omega_-)$ and say that $K(\Omega)$ is obtained from $N$ by *cave refinement* and that $K(\Omega)$ is inherited from $N$. We write $K(\Omega) = \omega(N) = \omega^2(T)$. All 2-simplexes of $K(\Omega)$ are admissible. In Section 3 we shall use the symbol $\Omega$ for a generic notation of a set of cave bases as presented here.

2.6. *$\delta_f$-inheriting.* Let $N$ and $K(\Omega)$ be as in Section 2.5. Fix $A_0 \in N^2$. If $a \in \operatorname{int} A_0$, $B^3(a, t) \setminus A_0$ has exactly two components for small $t$. Two such components are said to

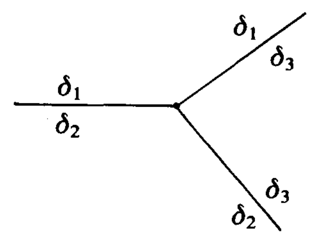


Fig. 2.2.

be equivalent if they lie on the same side of $A_0$. Let us call the equivalence classes $\delta_1$ and $\delta_2$. This equivalence relation is extended naturally over sides which are common for two 2-simplexes. If three 2-simplexes $A_1, A_2$, and $A_3$ in $N$ have a common side, we deliver the classes as in Figure 2.2 by introducing a third class $\delta_3$. We extend the definition of the equivalence classes for $K(\Omega)$ in a natural way as follows. For a 2-simplex $B$ in $K(\Omega)$ which is kept untouched when the cave refinement is performed, the classes remain the same as those for the 2-simplex in $N$ containing $B$. We use the same symbols $\delta_1, \delta_2, \delta_3$ for the classes in all cases.

Let $A \in N^2$ and let $\delta_j$ and $\delta_k$ be the classes of $A$. We define a union $\omega_j(A)$ of 2-complexes as a subset of $K(\Omega)$ as follows. The 2-simplexes of $\omega_j(A)$ are desribed by the following two conditions:

(1) If $D \in \omega(A)^2$ and if $\delta_j$ is a class of $D$, then $D \in \omega_j(A)^2$.

(2) Let $A$ be one of the cave opening simplexes for a cave complex with some base $Q \in \Omega$ and $|Q|$ has not $\delta_j$ as a class. If $D \in \omega(B)^2$ for some $B \in Q$ and if $\delta_j$ and $\delta_k$ are the classes of $D$, then $D \in \omega_j(A)$.

The condition (2) means that half of the walls of the cave corresponding to $Q$ are in $\omega_j(A)$.

2.7. *Map complexes*. Let

$$F_+ = \{x \in \mathbf{R}^2 | 0 \leq x_2 \leq x_1 \leq 1\}.$$

We divide the 2-simplex $\nu F_+$ into 2-simplexes congruent to $F_+$ by repeated reflections as shown in Figure 2.3 and call this subdivision the *first canonical subdivision* $(\nu F_+)_{(1)}$ of $\nu F_+$. Let $\lambda > 1$. A pair $(A, \varphi_A)$, also denoted by $A$, is called a *$L_\lambda$-simplex* if $A$ is contained in a plane in $\mathbf{R}^3$, $\varphi_A : A \to t_A F_+$ is a $\lambda$-bilipschitz homeomorphism for some $t_A > 0$, and if $\varphi_A^{-1}|B$ is affine for all $B \in \nu^{-1} t_A((\nu F_+)_{(1)}^{(1)})^2$. Recall that $K^{(i)}$ denotes the $i$th barycentric subdivision of a complex $K$. The map $\varphi_A$ defines vertices and sides of $A$.

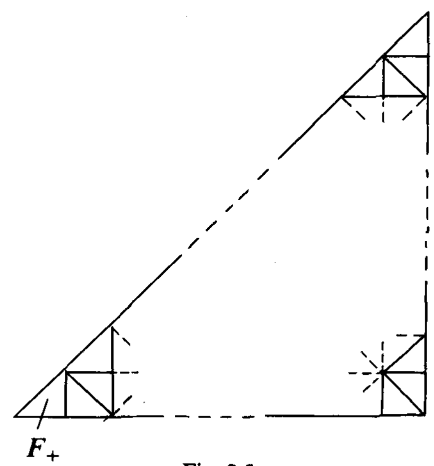


Fig. 2.3.

Let $G$ be a set of $L_\lambda$-simplexes and all their faces. $G$ is called a *map complex* (with constant $\lambda$) if the following conditions hold (we use notation similar to the simplicial case whenever applicable):

(1) $G$ is locally finite and $|G|=\cup\{A|A\in G^2\}$ is either homeomorphic to a closed disk or to $\mathbf{R}^2$.

(2) If $A, B\in G^2$, $A\cap B$ is empty or a set of faces of $A$ and $B$. Hence, because of (1), if $A\cap B$ contains a side, $A\cap B$ consists of (a) one side, (b) two sides, or (c) one side plus one vertex.

(3) $\sigma(St(v, G))$ is even for $v\in G^0\cap \operatorname{int} G$.

(4) If $A, B\in G^2$ and $A\cap B\neq\emptyset$, the map $\varphi_B\circ\varphi_A^{-1}|\varphi_A(A\cap B)$ is $x\mapsto t_Bx/t_A$.

LEMMA 2.8. *Let $G$ be a map complex. Then there exists a decomposition of $G^0$ into three classes $\alpha, \beta$, and $\gamma$ such that every $A\in G^2$ has one vertex in each class. Such a decomposition is uniquely determined if we fix the classes for the vertices of a side.*

*Proof.* We call a sequence $\Gamma=(A_1, \ldots, A_k)$ of elements of $G^2$ a chain connecting $A_1$ and $A_k$ if $A_i\cap A_{i+1}$ contains at least one side for $i=1, \ldots, k-1$. If we in $\Gamma$ replace a part $(A_i, \ldots, A_j)$, $1\leq i<j\leq k$, where $A_i, \ldots, A_j\in St(v, G)$ for some $v\in G^0$, by a chain $(B_1, \ldots, B_m)$ in $St(v, G)$ such that $B_1=A_i$, $B_m=A_j$, we say that the new chain $\Gamma'$ is obtained from $\Gamma$ by an elementary deformation. Fix $A\in G^2$ and let $B\in G^2$. A chain connecting $A$ and $B$ induces naturally an equivalence relation on the vertices of $A$ and $B$. If an elementary deformation is performed, the relation does not change because of (3). The condition (1) implies that any two chains connecting $A$ and $B$ can be obtained

from one another by a sequence of elementary deformations. Hence the equivalence relation is well defined. This proves the lemma.

2.9. Suppose we are given the decomposition $G^0=\alpha\cup\beta\cup\gamma$ as in Lemma 2.8 for a map complex $G$ and $\dot{G}$ does not contain sides with vertices $a\in\alpha$ and $b\in\beta$, called $\alpha\beta$-sides. Then $G^2$ is the disjoint union of pairs $\{T, T'\}$ such that $T\cap T'$ contains an $\alpha\beta$-side and we say that $G$ is given a *pairing with common $\alpha\beta$-sides*. For a map complex $G$ we call the elements of $G^2$ also 2-*simplexes*. If $H$ is a subcomplex of $G$ and $K$ a complex such that $|K|=|H|$, we write also $H=G|K$ (instead of $G||K|$).

We also need a slightly more general concept, namely a set $G$ as before except that we require only that $\varphi_A^{-1}|B$ be affine for $B\in t_A\nu^{-1}|((\nu F_+)_{(1)}^{(i)})^2$ for some fixed $i>1$. These are called *refined map complexes* (of subdivision order $i$).

## 3. Basic cave and map complexes

In this section we shall lay a basis for all later constructions by defining a part of a union $M_\infty$ of 2-complexes (see Section 4.1). As indicated in the introduction, the space $|M_\infty|$ will be an approximation of $f^{-1}(S^2\cup B^2)$. Along with the construction of $M_\infty$ we define certain map complexes which will guide the definition of $f$ on level surfaces defined in Section 4.

3.1. *First cave refinements*. Recall that $\nu=24\,000$. Set

$$E_0=\{x\in\mathbf{R}^2|\sqrt{3}\,|x_2|\leq x_1\leq\sqrt{3}/2\}$$

and let $M_*$ be the 2-complex with space $\mathbf{R}^2$ obtained by successive reflections in the sides of $E_0$. For $k=0,1,\dots$ we let $M_{k0}$ be the 2-complex with $M_{k0}^2=\{\nu^k E_0\}$. We let $K_{10}$ be the subcomplex of $M_*$ with $|K_{10}|=|M_{10}|$ and write $K_{k0}=\nu^{k-1}K_{10}$, $k=2,3,\dots$. Then $K_{k0}$ is a $(k-1)$-subdivision of $M_{k0}$.

We are going to define unions $M_{kq}$ of 2-complexes obtained by successive cave refinements for $1\leq q\leq k-1$, $k\geq 2$. To define $M_{21}$ we take the 1-subdivision $K_{20}$ of $M_{20}$. Let $K'_{20}$ be the complex obtained from $K_{20}$ by reflecting through the line $\{x\in\mathbf{R}^2|x_1=\sqrt{3}\,\nu^2/2\}$. We form a (nonopened) cave complex $K_L$ with $K=K_{20}\cup K'_{20}$ and $M_{20}^2$ as the cave base. We let $M_{21}$ be $\omega(M_{20})$. Recall the inheriting operation $\omega$ from Section 2.4.

We use similarity to define $M_{31}$, namely, we put $M_{31}=\nu M_{21}$. Consequently, $M_{31}=\omega(M_{30})$. It is clearly possible to choose a cave refinement of $M_{31}$ such that the obtained refinement $\omega(M_{31})$ includes $M_{21}$. We set $M_{32}=\omega(M_{31})$.

Supposing that $M_{kq}$ is obtained by some rules by cave refinements from $M_{k0}$ we shall fix the $\delta_j$-inheriting defined in Section 2.6 by letting $M_{k0}$ have classes $\delta_1$ and $\delta_2$ and by letting the elements of $\delta_1$ for $M_{k0}$ lie in $H_+=\{x\in\mathbf{R}^3|x_3>0\}$. We let $M_{kqj}$, $j=1,2,3$, be the union of 2-complexes such that $M^2_{kqj}$ consists of all $A$ in $M^2_{kq}$ having $\delta_j$ as a class.

3.2. *First map complexes.* On each $|M_{kqj}|$ we shall define a map complex $G_{kqj}$. In the definitions of $M_{kq}$ and $G_{kqj}$ we shall make use of as much similarity as possible when raising $k$ by 1.

By definition $M_{201}=M_{202}=M_{20}$. Fix $\lambda=20$. We let $G_{201}=G_{202}$ ($G_{203}$ is not defined) be a map complex with constant $\lambda$ and with a given decomposition $\alpha\cup\beta\cup\gamma$ of $G^0_{201}$ as in Lemma 2.8 such that

(i) $|G_{201}|=|M_{20}|$.

(ii) $|M^1_{20}|$ contains only $\alpha\gamma$-sides of $G_{201}$ and each point in $M^0_{20}$ is an $\alpha$-vertex,

(iii) $\sigma(G_{201})=\nu^2$.

Map complexes with the properties we need are easily constructed by starting from a sufficiently regular simplicial triangulation, taking the barycentric subdivision, and then adjusting the number of 2-simplexes by replacing some sides by pairs $A, B$ of type (b) in condition (2) in Section 2.7.

To define $G_{21j}$ we shall give the 2-simplexes of $M_{21}$ weights as follows. We can define a map $w_2$ of $M^2_{21}$ into the set of positive integers $\mathbf{N}$ such that for $B, B'\in M^2_{21}$

(a) 6 divides $w_2(B)$,

(b) $\Sigma_{B\in M^2_{21}} w_2(B)=\nu^4$,

(c) $w_2(B)/H^2(B)\leq 4$,

(d) $w_2(B)\,H^2(B')/(w_2(B')\,H^2(B))\leq 4$,

(e) $w_2(|M_{10}|)=\nu^2$.

We can construct a map complex $G_{21j}$ with an $\alpha,\beta,\gamma$ decomposition and with constant $\lambda$ on $|M_{21j}|$, $j=1,2,3$, such that the following conditions hold:

(1) $G_{21i}|M_{10}=\nu^{-1}G_{201}$, $i=1,2$.

(2) $|M^1_{21j}|\subset|G^1_{21j}|$, $M^0_{21j}\subset G^0_{21j}$.

(3) $|M^1_{21j}|$ contains only $\alpha\gamma$-sides of $G_{21j}$ and each point in $M^0_{21j}$ is an $\alpha$-vertex.

(4) $\sigma(G_{21j})=\nu^4$ and for each $A\in M^2_{21j}$

$$\sigma(G_{21j}|A) = w_2(A)+\frac{1}{2}\sum_{B\in Q(A,j)} w_2(B),$$

where

$$Q(A,j) = \{B\in M_{21}^2|A\neq B,\ \omega_j(\nu A)\cap \operatorname{int}\omega(\nu B)\neq\emptyset\}.$$

Here $\nu A$ and $\nu B$ are elements in $M_{31}^2$ and $\omega$ and $\omega_j$ are the inheriting operations for $M_{31}\to M_{32}$ and $M_{31j}\to M_{32j}$.

(5) The $\alpha,\beta,\gamma$ decomposition is chosen so that the maps $\varphi_C\colon C\to t_C F_+$, $C\in G_{21j}^2$, satisfy $\varphi_C^{-1}(0)\in\alpha$, $\varphi_C^{-1}(t_C e_1)\in\beta$, $\varphi_C^{-1}(t_C(e_1+e_2))\in\gamma$.

Because of (3) $G_{21j}$ can be given a pairing with common $\alpha\beta$-sides and such a pairing is induced in each $A\in M_{21j}^2$. The set $Q(A,j)$ appearing in (4), if nonempty, corresponds to the cave base $\nu Q$ having $\nu A$ as a cave opening simplex and not having $\delta_j$ as a class when $M_{32}$ is formed from $M_{31}$ by cave refinement. Hence such $Q$ is an element of an $\Omega$ for $M_{21}$. Note, however, that no cave refinement is done for $M_{21}$.

We proceed by defining $G_{31j}$ on $|M_{31j}|$, $j=1,2,3$, by similarity, i.e. $G_{31j}=\nu G_{21j}$. Also the maps $\varphi_{\nu B}$, $B\in M_{21}^2$, are induced by similarity, i.e. $\varphi_{\nu B}(x)=\nu\varphi_B(x/\nu)$. A weight function $w_3\colon M_{32}^2\to \mathbf{N}$ is now defined so that the following conditions are satisfied where $B$ is any element in $M_{21}^2$:

(a′) 6 divides $w_3(C)$, $C\in M_{32}^2$.

(b′) $\Sigma_{C\in\omega(\nu B)^2} w_3(C)=\nu^2 w_2(B)$.

(c′) $w_3(C)/H^2(C)\leq w_2(B)/H^2(B)$, $C\in\omega(\nu B)^2$.

(d′) $w_3(C)H^2(C')/(w_3(C')H^2(C))\leq 4$ whenever $C,C'\in\omega(\nu B)^2$ or $C,C'\in M_{32}^2$, $C\cap C'\neq\emptyset$.

(e′) $w_3|M_{21}^2=w_2$.

We can construct a map complex $G_{32j}$ with a $\alpha,\beta,\gamma$ decomposition and with constant $\lambda$ on $|M_{32j}|$, $j=1,2,3$, such that the conditions (2), (3), and (5) hold with $M_{21j}$ and $G_{21j}$ replaced by $M_{32j}$ and $G_{32j}$, and such that (1) and (4) are replaced by the following conditions:

(1′) $G_{32j}|M_{21j}=G_{21j}$.

(4′) Let $B\in M_{21}^2$. Then

$$\sigma(G_{32j}|\omega(\nu B)\cap M_{32j}) = \nu^2 w_2(B)$$

and for each $A\in\omega(\nu B)\cap M_{32j}^2$

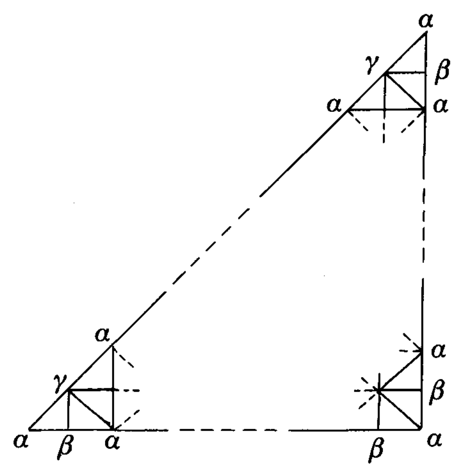


Fig. 3.1.

$$\sigma(G_{32j}|A) = w_3(A) + \frac{1}{2} \sum_{C \in Q(A,j)} w_3(C)$$

where $Q(A, j)$ is defined by means of an $\Omega$ for $\omega(\nu B)$ similarly as explained above after the condition (5).

3.3. *Canonical subdivisions.* Recall the first canonical subdivision $(\nu F_+)_{(1)}$ of $\nu F_+$ from Section 2.7. We give $(\nu F_+)^0_{(1)}$ the decomposition $\alpha \cup \beta \cup \gamma$ as in Figure 3.1. If $G$ is any map complex $G_{kqj}$ or its restriction to a map subcomplex, we form the *first canonical subdivision* $G_{(1)}$ of $G$ by giving each $A \in G^2$ the subdivision induced by $x \mapsto \varphi_A^{-1}(t_A x/\nu)$ from $(\nu F_+)_{(1)}$. Because of (4) in Section 2.7 $G_{(1)}$ is a map complex and the decomposition $\alpha \cup \beta \cup \gamma$ of $(\nu F_+)^0_{(1)}$ is induced on $G^0_{(1)}$.

The second canonical subdivision is formed as follows. In $\nu(\nu F_+)_{(1)}$ we form in the 2-simplex $\nu F_+$ the first canonical subdivision $(\nu F_+)_{(1)}$ and continue this to all 2-simplexes in $\nu(\nu F_+)_{(1)}$ in a congruent way. The obtained map complex is called the second canonical subdivision $(\nu^2 F_+)_{(2)}$ of $\nu^2 F_+$. The decomposition $\alpha \cup \beta \cup \gamma$ of $(\nu^2 F_+)^0_{(2)}$ for the part $(\nu F_+)^0_{(1)}$ is as in Figure 3.1. If $G$ is as above, the second canonical subdivision $G_{(2)}$ of $G$ is induced from $(\nu^2 F_+)_{(2)}$ via the maps $x \mapsto \varphi_A^{-1}(t_A x/\nu^2)$. Similarly we form the $k$th *canonical subdivision* $G_{(k)}$ of $G$.

The main idea in the construction of the map complexes $G_{kqj}$ is to give a rearrangement process to the pairs of 2-simplexes with common $\alpha\beta$-sides. For example, we should think $G_{32j}|\omega_j(\nu B)$ for $\nu B \in M^2_{31j}$ as a map complex obtained by a rearrangement of the pairs with common $\alpha\beta$-sides in $(G_{31j})_{(1)}|\nu B$. This is the meaning of conditions (4) and (4′) in Section 3.2.

3.4. *General induction step*. We have $\sigma(G_{32j})/\sigma(G_{31j})=\nu^2$. From (8) in Section 2.2 we obtain $\nu^2\leqslant\sigma(M_{313})\leqslant 4\nu^2/3$ and directly from the construction that $\sigma(M_{311})=\sigma(M_{312})=\nu^2$. From the construction together with (8) in Section 2.2 we get furthermore $\sigma(M_{32j})>1.55\nu^4$, $j=1,2,3$. Hence $\sigma(M_{32j})/\sigma(M_{31j})>1.55\nu^2$, $j=1,2$, and $\sigma(M_{323})/\sigma(M_{313})>1.15\nu^2$. These ratios are essentially larger than $\sigma(G_{32j})/\sigma(G_{31j})=\nu^2$, and this means in particular that if we would copy the steps in Section 3.2 to be used as an induction step, the number of 2-simplexes of $G_{kqj}$ in a 2-simplex of $M_{kqj}$ would have a tendency to decrease essentially. However, we want to keep this number above a fixed bound. This will be arranged by leaving out sometimes cave refinements when $M_{kq}$ is formed from $M_{k,q-1}$.

Suppose $M_{kq}$, $q\leqslant k-1$, is formed from $M_{k,q-1}$ by some cave refinements. The notations $\omega$ and $\omega_j$ are inheriting operations when a cave refinement is performed. We need also indexed inheriting operations $\omega_{k,q-1}\colon M^2_{k,q-1}\to\mathcal{P}(M_{kq})$ and $\omega_{k,q-1,j}\colon M^2_{k,q-1,j}\to\mathcal{P}(M_{k,q,j})$, $j=1,2,3$, defined as follows. We set $\omega_{k,q-1}(A)=A$ if $A\in M^2_{k,q-1}$ is not replaced by $\omega(A)$ when $M_{kq}$ is formed and $\omega_{k,q-1}(A)=\omega(A)$ otherwise, and similarly for $\omega_{k,q-1,j}$.

With a certain integer $m$ defined below, we shall first proceed by induction up to $k\leqslant m-1$ as follows. $M_{kq}$ and $G_{kqj}$ are defined to be $\nu M_{k-1,q}$ and $\nu G_{k-1,q,j}$ by similarity for $1\leqslant q\leqslant k-2$, $j=1,2,3$, and $M_{k,k-1}$ and $G_{k,k-1,j}$ are defined exactly as $M_{32}$ and $G_{32j}$ in Section 3.2 by shifting only indexes. This includes then the definition of the weight functions $w_k$. The integer $m$ is the first $k+2$ such that $w_k(D)<\nu$ for some $D\in M^2_{k,k-1}$. Then we set $M_{mq}=\nu M_{m-1,q}$, $G_{mqj}=\nu G_{m-1,q,j}$ for $1\leqslant q\leqslant m-2$, $j=1,2,3$. For each $D\in M^2_{m-2,m-3}$ we perform in $\omega_{m,m-3}(\nu^2D)=\omega(\nu^2D)$ a cave refinement if $w_{m-2}(D)\geqslant\nu$, otherwise not. This procedure gives $M_{m,m-1}$ as a union of complexes. Since $w_{m-2}|M^2_{m-3,m-4}=w_{m-3}$, the condition $w_{m-2}(D)\geqslant\nu$ holds for all $D\in M^2_{m-3,m-4}\subset M^2_{m-2,m-3}$. Hence we may do the construction of $M_{m,m-1}$ so that $M_{m-1,m-2}\subset M_{m,m-1}$.

In the definition of the weight function $w_m\colon M^2_{m,m-1}\to\mathbf{N}$ the conditions are as before for a $B\in M^2_{m-1,m-2}$ such that $\omega(\nu B)\subset M_{m,m-1}$. For a $B\in M^2_{m-1,m-2}$ such that $\nu B\in M^2_{m,m-1}$ we only replace (b′) in Section 3.2 by

(b″) $w_m(\nu B)=\nu^2 w_{m-1}(B)$.

We construct a map complex $G_{m,m-1,j}$ with constant $\lambda$ on $|M_{m,m-1,j}|$ by replacing

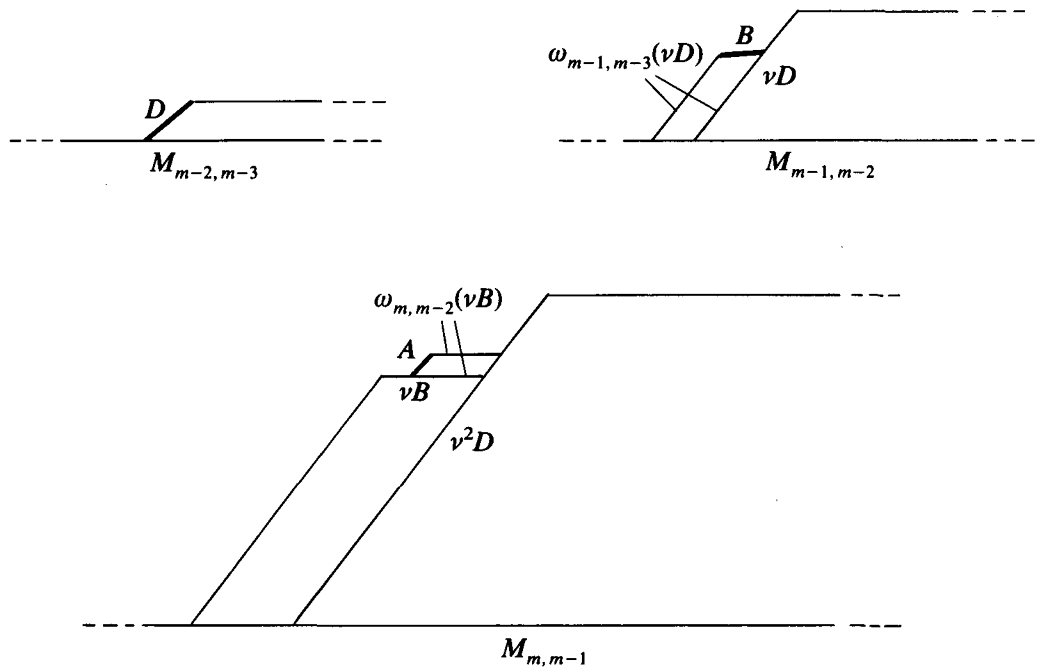


Fig. 3.2.

(4′) in Section 3.2 by the following condition (4″) and changing in the other conditions of (1′)–(5′) in Section 3.2 only indexes.

(4″) Let first $D \in M^2_{m-2,m-3}$ be such that $w_{m-2}(D) \geqslant \nu$. If $B \in \omega_{m-1,m-3}(\nu D)^2$, then

$$\sigma(G_{m,m-1,j}|\omega(\nu B) \cap M_{m,m-1,j}) = \nu^2 w_{m-1}(B)$$

and for each $A \in \omega(\nu B) \cap M^2_{m,m-1,j}$

$$\sigma(G_{m,m-1,j}|A) = w_m(A) + \frac{1}{2} \sum_{C \in Q(A,j)} w_m(C)$$

where $Q(A, j)$ is defined by means of an $\Omega$ for $\omega(\nu B)$. Secondly, if $D \in M^2_{m-2,m-3}$ is such that $w_{m-2}(D) < \nu$, then $G_{m,m-1,j}|\omega_{m,m-2}\,\omega_{m,m-3}(\nu^2 D) \cap M_{m,m-1,j} = G_{m,m-1,j}|\omega_{m,m-3}(\nu^2 D) \cap M_{m,m-2,j}$ is the first canonical subdivision $(G_{m,m-2,j}|\omega_{m,m-3}(\nu^2 D) \cap M_{m,m-2,j})_{(1)}$.

Figure 3.2 (not in scale) represents schematically the various simplexes appearing in (4″) in the first case.

In (4″) we have in the first case $w_{m-2}(D) \geqslant \nu$ that $\omega(\nu B) \subset M_{m,m-1}$ and in the second case $w_{m-2}(D)<\nu$ that $\omega(\nu^2 D)\subset M_{m,m-1}$. We use similarity and put $M_{m+1,q}=\nu M_{mq}$ and $G_{m+1,q,j}=\nu G_{m,q,j}$ for $1\leq q\leq m-1, j=1,2,3$. For each $E\in M^2_{mq}$ with $\omega(E)\subset M_{m,m-1}$, $\omega(\nu E)$ is replaced by a cave refinement if $\Sigma\{w_m(C)|C\in\omega(E)^2\}\geq\nu^3$, otherwise not. This will define $M_{m+1,m}$ if we also require, as we may, that $M_{m,m-1}\subset M_{m+1,m}$. Note that this agrees with the rule for the construction of $M_{m,m-1}$.

To construct $G_{m+1,m,j}$ we define $w_{m+1}$ as $w_m$ by raising the indexes by 1. In the definition of $G_{m+1,m,j}$ all other conditions are the same as before except for a slight change of (4″) to the following form:

(4‴) Let $E\in M^2_{mq}$ be such that $\omega(E)\subset M_{m,m-1}$. If $\Sigma\{w_m(B)|B\in\omega(E)^2\}\geq\nu^3$ and if $B\in\omega(E)^2$, then

$$\sigma(G_{m+1,m,j}|\omega(\nu B)\cap M_{m+1,m,j})=\nu^2 w_m(B)$$

and for each $A\in\omega(\nu B)\cap M^2_{m+1,m,j}$

$$\sigma(G_{m+1,m,j}|A)=w_{m+1}(A)+\frac{1}{2}\sum_{C\in Q(A,j)} w_{m+1}(C)$$

where $Q(A,j)$ is defined by means of an $\Omega$ for $\omega(\nu B)$. If $\Sigma\{w_m(B)|B\in\omega(E)^2\}<\nu^3$, then

$$G_{m+1,m,j}|\omega(\nu E)\cap M_{m+1,m,j}=(G_{m+1,m-1,j}|\omega(\nu E)\cap M_{m+1,m,j})_{(1)}.$$

The general induction step will be the same as the step from $m$ to $m+1$ just completed.

## 4. The construction of the map on some level surfaces

We shall now start the definition of the map $f$ to be constructed. The definition will be continued in Section 6 where we extend it to layers between the level surfaces given here.

4.1. *Notation*. We shall extend the notions in Section 3. Therefore we write from now on $M_{kq}(0)$ instead of $M_{kq}$, constructed in Section 3, and similarly for other notions. Let $\varepsilon_h$ be the reflection in the half plane $\{(r,\varphi,x_3)|\varphi=\pi/6+(h-1)\pi/3,\ r\geq 0,\ x_3\in\mathbf{R}^1\}$ presented in cylinder coordinates, $h=1,\dots,5$.

We let $M_{kq}(h)=\varepsilon_h M_{kq}(h-1)$, $h=1,\dots,5$, and set now

$$M_{kq}=\bigcup_{h=0}^{5} M_{kq}(h).$$

Similar extension is performed to other notions. Next we write

$$M_j(h) = \bigcup_{k=2}^{\infty} M_{k,k-1,j}(h), \quad j=1,2,3, \quad h=0,\dots,5,$$

$$M_j = \bigcup_{h=0}^{5} M_j(h), \quad j=1,2,3,$$

$$M_\infty = M_1 \cup M_2 = M_1 \cup M_2 \cup M_3.$$

Similarly we define $G_j(h)$ and $G_j$.

The set $\mathbf{R}^3 \setminus |M_\infty|$ has eight components. For $j=1,2$ the elements of the class $\delta_j$ lie in one component $V_j$ and the elements of $\delta_3$ in six components $V_3(h)$ with $\partial V_3(h) = |M_3(h)|$.

4.2. *A level surface*. We shall here give the construction of a certain surface in $V_1$ which the final map $f$ to be constructed will take onto a sphere. The constructions in $V_2$ and $V_3(h)$, $h=0,\dots,5$, are similar. The surface will be the space $|N_1|$ of a union $N_1$ of 2-complexes and it is in general approximately at the distance $\nu^{-1/2}d(A)$ from $|M_1|$ near a simplex $A \in M_1^2$. The conditions for $N_1$ are the following:

(1) There exists a simplicial homeomorphism $\psi\colon |M_1| \to |N_1|$ by which we mean that simplexes of $M_1$ are mapped onto simplexes of $N_1$ affinely. Note that $M_1$ and $N_1$ are not complexes.

(2) Let $A \in M_1^2$ be an $l$-triangle. If every $B \in M_1^2$ with $B \cap A \neq \emptyset$ is an $l$-triangle, then $|\psi(x) - x| < 3\nu^{l-1/2}$ for all $x \in A$ and the distance of $A$ and $|N_1|$ satisfies $d(A, |N_1|) > \nu^{l-1/2}$. Let $B \in M_1^2$ be an $(l+1)$-triangle and $A \cap B \neq \emptyset$. Then $|\psi(x) - x| < 9\nu^l$ for $x \in A \cap B$, $d(A, \psi A) > \nu^{l-1/2}$, and $d(B, \psi B) > 3\nu^l$.

(3) $\psi$ is 20-bilipschitz.

(4) $|N_1| \subset V_1$.

It follows from the construction rules that such $N_1$ and $\psi$ exist. $N_1$ is obtained by simple moving of the vertices of $M_1$.

4.3. *The map on level surfaces*. We shall now first define the map $f$ on $|N_1|$. Let $D \in N_1^2$. The map $\psi$ induces in a natural way a pair $\delta_1, \delta_j$ of classes for $D$. A normal of $D$ pointing towards the elements of $\delta_j$ is called an *outward normal* of $D$ (with respect to $\delta_1$). A simple closed path $\gamma$ in $D$ is *positively oriented* (with respect to $\delta_1$) if there is a sense preserving similarity map $h$ of $\mathbf{R}^3$ which takes the outward normal of $D$ to $e_3$, $hD \subset \mathbf{R}^2$, and $h \circ \gamma$ is positively oriented in $\mathbf{R}^2$. We fix a 20-bilipschitz homeomorphism $\zeta$ of $F_+$ onto $S_+^2 = \{x \in S^2 | x_3 \geq 0\}$ such that $\zeta$ induces from the positively oriented boundary

$|\dot{F}_+|$ positive orientation on its image $S^1 \subset \mathbf{R}^2$. On $|N_1|$ we have the map complex $H_1 = \psi G_1$. The vertex assignment of $G_1$ is transferred to $H_1$ by $\psi$. We call $C \in H_1^2$ *positively* (*negatively*) *oriented* (with respect to $\delta_1$) if along a positively oriented $|\dot{C}|$ the vertex classes follow in the order $\alpha, \gamma, \beta$ $(\alpha, \beta, \gamma)$. Let $\varkappa: \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection in $\mathbf{R}^2$. We define a map $g_0$ of $|N_1|$ such that for $C \in H_1^2$, $B = \psi^{-1}C$,

$$g_0|C = \zeta \circ \frac{1}{t_B} \varphi_B \circ \psi^{-1}|C \quad \text{if } C \text{ is positively oriented,}$$

$$g_0|C = \varkappa \circ \zeta \circ \frac{1}{t_B} \varphi_B \circ \psi^{-1}|C \quad \text{if } C \text{ is negatively oriented.}$$

For real numbers $\mu$ we write

$$s_\mu = e^{-1} \exp \nu^{\mu+1}.$$

Then we set

$$f|\,|N_1| = (s_0, g_0)$$

where $(s_0, g_0)$ is presented in spherical coordinates $(t, y)$, $t \geq 0$, $y \in S^2$.

By the construction of $M_1$ and $N_1$, the surfaces $\nu^{2i}|N_1|$, $i = 0, 1, 2, \ldots$, are disjoint. We shall next define $f$ on $\nu^{2i}|N_1|$, $i = 1, 2, \ldots$. For this we use similarity as follows. We take the $2i$th canonical subdivision $(\nu^{2i}G_1)_{(2i)}$ of $\nu^{2i}G_1$. In this subdivision each $\nu^{2i}A \in \nu^{2i}G_1^2$ is divided into $\nu^{4i}$ 2-simplexes. Let $A \in G_1^2$ and let $\xi_A(x) = \nu^{2i} t_A^{-1} \varphi_A(\psi^{-1}(x/\nu^{2i}))$ for $x \in \nu^{2i}\psi A$. Let $C = \xi_A^{-1} F_+$. We define the restriction on $C$ of a map $g_{2i}$ by

$$g_{2i}|C = \zeta \circ \xi_A|C \quad \text{if } C \text{ is positively oriented,}$$

$$g_{2i}|C = \varkappa \circ \zeta \circ \xi_A|C \quad \text{if } C \text{ is negatively oriented.}$$

Repeated reflections in the sides of $(\nu^{2i}F_+)_{(2i)}$ gives a map $w: \nu^{2i}F_+ \to F_+$. If $B \in (\nu^{2i}\psi A)^2_{(2i)}$, we set

$$g_{2i}|B = \zeta \circ w \circ \xi_A|B \quad \text{if } B \text{ is positively oriented,}$$

$$g_{2i}|B = \varkappa \circ \zeta \circ w \circ \xi_A|B \quad \text{if } B \text{ is negatively oriented.}$$

In other words, $g_{2i}|C$ is extended to $\nu^{2i}\psi A$ by these natural reflections. We can glue all such maps together to get a map

$$g_{2i}: \nu^{2i}|N_1| \to S^2. \tag{4.4}$$

Then we set

$$f|\nu^{2i}|N_1| = (s_{2i}, g_{2i}), \quad i=0,1,\dots. \tag{4.5}$$

The map $g_{2i}$ factorizes now as

$$g_{2i} = v_{2i} \circ \bar{g}_{2i} \tag{4.6}$$

where $\bar{g}_{2i}$ is defined by similarity from $g_0$, i.e.

$$\bar{g}_{2i}(x) = g_0(x/\nu^{2i}), \quad i=1,2,\dots, \tag{4.7}$$

and $v_{2i}\colon S^2 \to S^2$ is a discrete open map of degree $\nu^{4i}$. We call this factorization the *canonical factorization* of $g_{2i}$. The map $v_{2i}$ depends only on $\zeta$ and $i$.

We extend the definition of $g_{2i}$ and $\bar{g}_{2i}$ to the surfaces $|N_2|$ and $|N_3(h)|$, $h=0,\dots,5$, corresponding to $|N_1|$ by the same rules as above when $\delta_1$ is changed to $\delta_2$ and $\delta_3$. To define $f$ on these surfaces we need homeomorphisms $\varkappa_2\colon \bar{U}_1 \to \bar{U}_2 \setminus \{u_2\}$, $\varkappa_3\colon \bar{U}_1 \to \bar{U}_3 \setminus \{u_3\}$ where $U_1, U_2$, and $U_3$ are the components of $\mathbf{R}^3 \setminus (S^2 \cup B^2 \cup \{u_2, u_3\})$ such that $u_j \in \bar{U}_j$, $j=2,3$, and $u_2=-e_3/2$, $u_3=e_3/2$. Let $w_2$ be the Möbius transformation of $\bar{\mathbf{R}}^3$ which keeps $S^1$ fixed and takes $S^2_+$ onto $\bar{B}^2$. We define $\varkappa_2$ and $\varkappa_3$ such that $\varkappa_2|U_1\colon U_1 \to U_2$ and $\varkappa_3|U_1\colon U_1 \to U_3$ are quasiconformal and on the boundary $\varkappa_2|S^2_+ = \varkappa|S^2_+$, $\varkappa_2|S^2_- = w_2 \circ \varkappa|S^2_-$, $\varkappa_3|S^2_+ = w_2|S^2_+$, $\varkappa_3|S^2_- = \varkappa|S^2_-$. Then set

$$f|\nu^{2i}|N_2| = \varkappa_2 \circ (s_{2i}, g_{2i}), \quad i=0,1,\dots, \tag{4.8}$$

$$f|\nu^{2i}|N_3(h)| = \varkappa_3 \circ (s_{2i}, g_{2i}), \quad i=0,1,\dots, \quad h=0,\dots,5. \tag{4.9}$$

## 5. Deformation of 2-dimensional maps

In this section we shall present a method of deforming discrete open maps of plane domains into $S^2$. Our maps to be deformed are given by means of map complexes in the same way as the maps in Section 4.3. In fact, the deformations are used in Sections 6 and 7 to define $f$ between the level surfaces given in Section 4. The extensive studies of deformations by M. Morse and M. Heins for example in [6] have not turned out to be useful here. In our method it is important to have the map at hand at every stage of the deformation.

5.1. *Elementary deformations*. Let $G$ be a finite map complex with some constant $\lambda>1$ such that $|G|\subset\mathbf{R}^2$. Recall that by definition $|G|$ is then homeomorphic to a closed disk. We assume that $G^0$ is given a decomposition $\alpha \cup \beta \cup \gamma$ and that $\dot{G}$ does not contain

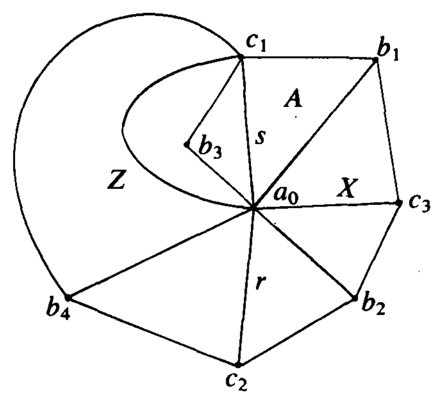


Fig. 5.1.

any $\alpha\beta$-sides. Hence we can give $G$ a pairing with common $\alpha\beta$-sides. We fix $-e_3$ to be the outward normal for $|G|$. This defines then positively oriented 2-simplexes of $G$. Let $\zeta: F_+ \to S^2_+$ be as in Section 4.3. We define $g: |G| \to S^2$ (cf. the definition of $g_0$ in Section 4.3) by

$$g|C = \zeta \circ \frac{1}{t_C} \varphi_C \quad \text{if } C \in G^2 \text{ is positively oriented,}$$

$$g|C = \varkappa \circ \zeta \circ \frac{1}{t_C} \varphi_C \quad \text{if } C \in G^2 \text{ is negatively oriented,}$$

and say that $g$ is represented by the map complex $G$. Its restriction to $\operatorname{int} G$ is a discrete open mapping into $S^2$. We shall in the following perform deformations of $g$ by one parameter families of maps $g_t$ so that $g_t$ depends piecewise smoothly on $t$ and each $g_t|\operatorname{int} G$ is discrete open. In each complete deformation the final map is induced by some refined map complex (see Section 2.9) in the same way as $g$.

We shall first define a deformation called *exchange of sides*. We start with a topological description. Let $r$ and $s$ be two $\alpha\gamma$-sides of $G$ with a common $\alpha$-vertex $a_0$ and different $\gamma$-vertices $c_1$ and $c_2$. In Figure 5.1 we have $St(a_0, G)$ of an example of this. In what follows we shall in general use the letters $a, b, c$ for points in the classes $\alpha, \beta, \gamma$ respectively, possibly with subscripts. Image points will be indicated by primes. We assume that $St(r, G) \cup St(s, G)$ does not meet $|\dot{G}|$. According to our convention, the simplex $A$ is positively oriented and $g$ maps it onto the upper hemisphere $S^2_+$. Let $X$ and $Z$ be the two middle sides between $r$ and $s$ with vertex $a_0$ (see Figure 5.1). We define a family $g_t$, $t \in [0, 1]$, of maps as follows. During $0 \leq t \leq 1/2$ we move $g$ in $St(a_0, G)$ by a $C^1$

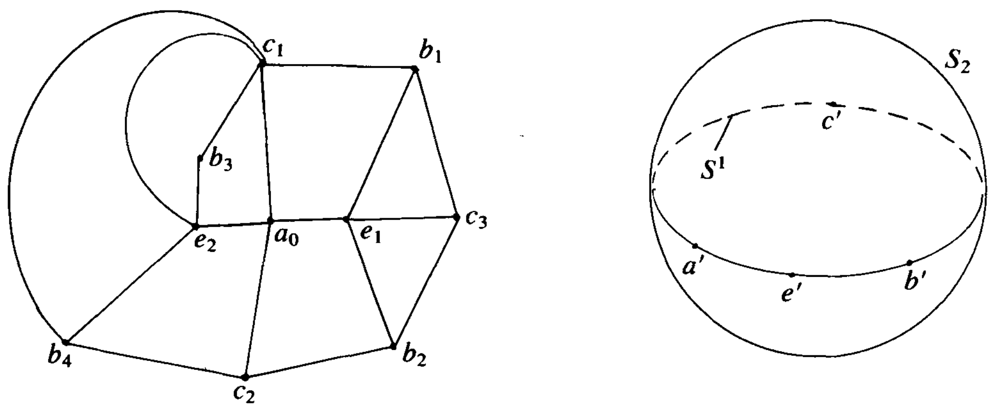


Fig. 5.2.

move so that the point $a_0$ becomes a branch point with local index 2 (Figure 5.2). The sides in Figure 5.2 form the preimage of $S^1$ under $g_{1/2}$. The points $e_1$ and $e_2$ are mapped onto a point $e'$ between $a'$ and $b'$. If there are three or more sides emanating from $a_0$ on the same "side" of $r \cup s$ as $X$ (as in Figure 5.2), then $e_1$ becomes a branch point and this branch point moves from $a_0$ to $e_1$ along $X$ during $0 \leq t \leq 1/2$ and the image of this branch point moves from $a'$ to $e'$ along $S^1$, and similarly for $e_2$. If the local index of $g$ is already 2 at $a_0$, $g_t = g$ for $t \in [0, 1/2]$. During $1/2 \leq t \leq 1$ we move $c_1$ and $c_2$ to $a_0$ and at the same time we move the image of $a_0$ to $c'$ and the image of $e_1$ and $e_2$ from $e'$ to $a'$. We rename points for obvious reasons: $a_0$ will be $c_0$ and $e_1$ and $e_2$ will be $a_1$ and $a_2$ (Figure 5.3). We get topologically a new set $G_1$ of 2-simplexes and their faces. We see that $G_1$ is obtained from $G$ by replacing the pair $r, s$ of sides by the sides $c_0 a_1, c_0 a_2$.

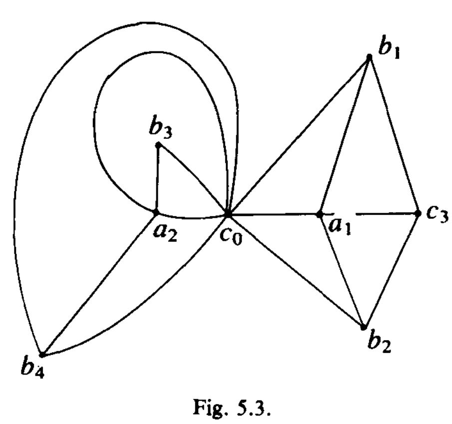


Fig. 5.3.

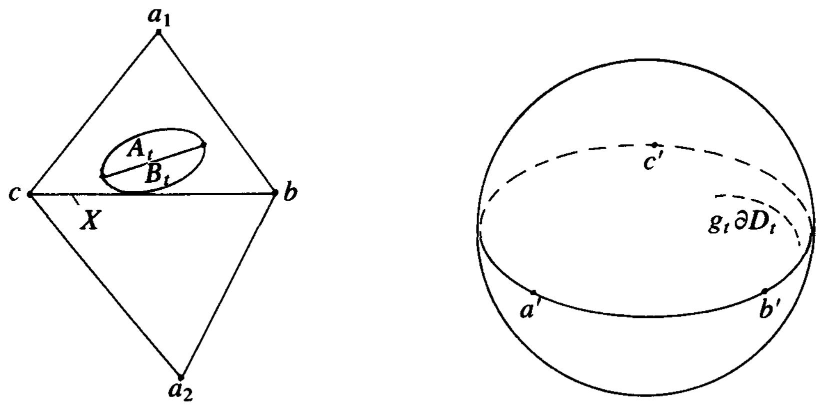


Fig. 5.4.

We can do the reformation $g_t$, $t\in[0,1]$, in addition so that the following metrical conditions are fullfilled:

(1) The map $g^x\colon[0,1]\to S^2$, $g^x(t)=g_t(x)$, is $L$-Lipschitz for all $x\in|G|$, $L$ depending only on $\lambda$.

(2) for $x\in C\in G^2$ and $t\in[0,1]$

$$\frac{1}{t_C Q(\lambda)}\leq l(x,g_t)\leq L(x,g_t)\leq\frac{Q(\lambda)}{t_C}$$

where the constant $Q(\lambda)>0$ depends only on $\lambda$ and where

$$l(x,g_t)=\liminf_{h\to0}\frac{|g_t(x+h)-g_t(x)|}{|h|},$$

$$L(x,g_t)=\limsup_{h\to0}\frac{|g_t(x+h)-g_t(x)|}{|h|}.$$

(3) The sides of $G_1$ are contained in $|H^1|$ where $H$ is the $i$th barycentric subdivision $(G)_{(1)}^{(i)}$ of the first canonical subdivision $(G)_{(1)}$ of $G$ where $i$ depends only on $\lambda$. Barycentric subdivisions are defined by means of the maps $\varphi_A$, $A\in G^2$.

(4) $G_1$ can be made to be a refined map complex of subdivision order $i$ and constant $\lambda_1$, depending only on $\lambda$, and $g_1$ is represented by $G_1$.

To achieve these conditions only elementary methods are needed and the details are omitted. This completes the definition of exchange of sides of the pair $r,s$.

Next we shall define a deformation called moving of a simple cover. A *simple cover* at $X$ in the map complex $G$ is the restriction of $g$ to $A\cup B$ where $A,B$ is a pair with

$X=A\cap B$ being two sides, i.e. we have case (2) (b) in Section 2.7. Let $A, B$ be such a pair so that $St(A, G)\cup St(B, G)$ does not meet $|\dot{G}|$ and let $Y$ be an $\alpha\gamma$-side or a $\beta\gamma$-side of $G\setminus(St(A, G)\cup St(B, G)\cup\dot{G})$. Set $D=A\cup B$. We shall define a family $g_t$, $t\in[0,1]$, of maps so that $g_1$ is represented by a refined map complex $G_1$ obtained from $G$ by "collapsing" the pair $A, B$ to one side $X$ and "expanding" the side $Y$ to a pair $A_1, B_1$ of type (2) (b) in Section 2.7 with $Y=A_1\cap B_1$. The simple cover $g|D$ has the properties that $g|(D\setminus\partial D)$ covers once $S^2\setminus g\partial D$, $g\partial D$ is an arc in $S^1$, and the endpoints of $X$ are branch points of $g$. During $0\le t\le 1$ we move the simple cover $g|D$ to a simple cover $g_1|D_1$ where $D_1=A_1\cup B_1$. This means that we at the same time move the arc $g\partial D$. For $0\le t\le 1$ let the moving simple cover be $g_t|D_t$ and $D_t=A_t\cup B_t$ (Figure 5.4). Let $\hat{G}$ be the set of 2-simplexes and their faces obtained from $G$ by collapsing the pair $A, B$ to one side $X$. If $E$ is the union of elements in $\hat{G}^2$ which meet $D_t$ for some $t\in]0,1[$, $g_t$ will coincide with $g$ outside $\operatorname{int} E$ for $t\in[0,1]$. The map $g_t|(D_t\setminus\partial D_t)$ covers once $S^2\setminus g_t\partial D_t$ for $t\in[0,1]$, and $g_1\partial D_1$ is $gY\subset S^1$. The set $E$ is called the *joining set* for the move.

The metrical conditions (1)–(4) are the same except that in (1) we replace $L$ by $mL$ where $m$ is the number of 2-simplexes in the set $E$ above. The obtained family $g_t$, $t\in[0,1]$, is called a *move of a simple cover at the arc X to the side Y*, or also a *move of the pair A, B to Y*.

Let now $A, B$ be a pair in $G$ of type (2) (a) in Section 2.7, i.e. $X=A\cap B$ is one side. Then one endpoint $a_0$ of $X$ is an $\alpha$-point. Let the $\gamma$-points of $A$ and $B$ be $c_1$ and $c_2$ and let $r$ and $s$ be the sides $a_0c_1$ and $a_0c_2$ of $A$ and $B$ respectively. Let us assume that $St(r, G)\cup St(s, G)$ does not meet $|\dot{G}|$. We perform a deformation $g_t$, $t\in[0,1]$, of exchange of sides to the pair $r, s$. Then the pair $A, B$ is deformed to a pair $A_1, B_1$ of type (b) in the new refined map complex $G_1$ with $X=A\cap B=A_1\cap B_1$ and $g_1|A_1\cup B_1$ is a simple cover. If we move this simple cover to a side $Y$ of $G$ as above by a deformation $g_t$, $t\in[1,2]$, we also here say that we *move the pair A, B to the side Y* by the deformation $g_t$, $t\in[0,2]$. The various deformations presented above are called elementary. We can apply these deformations also to a refined map complex instead of $G$.

As a main result of this section we shall prove a deformation lemma suitable for our purposes. I want to point out, however, that the method presented here is quite general and could be used to produce many other results as well. In the proof of the deformation lemma we need the following result for $PL$ homeomorphisms. The idea of the proof of Lemma 5.2 was given to me by Pekka Tukia.

LEMMA 5.2. *Let $Q>1$, $\iota$ a positive integer, $\Delta$ an equilateral triangle in $\mathbf{R}^2$ with side length 1 and origin as center, let K and M be 2-complexes with space $|K|=|M|=\Delta$ such* *that* $\iota^2=\sigma(K^2)=\sigma(M^2)$ *and each* $A\in K^2\cup M^2$ can be mapped affinely onto $\iota^{-1}\Delta$ *with bilipschitz constant* $Q$. *Let* $K|\dot{\Delta}=M|\dot{\Delta}$ *and let* $h\colon\Delta\to\Delta$ *be a simplicial homeomorphism with respect to* $K$ *and* $M$ *such that* $h||\dot{\Delta}|$ *is the identity. Then there is a PL isotopy* $h_t$, $t\in[0,1]$, *of PL homeomorphisms* $h_t\colon\Delta\to\Delta$ *such that*

(1) $h_0$ *is the identity,* $h_1=h$,

(2) $h_t$ *is* $L_0$*-bilipschitz,* $L_0$ *depending only on* $\iota$ *and* $Q$,

(3) *the map* $h^x\colon[0,1]\to\Delta$, $h^x(t)=h_t(x)$, *is* $L_0$*-Lipschitz for all* $x\in\Delta$,

(4) $h_t||\dot{\Delta}|$ *is the identity.*

*Proof.* Let $W$ be the family of all maps $h$ as in the lemma for constants $Q$ and $\iota$. Let $W$ be the family of all triples $(h, K, M)$ as in the lemma. To simplify notation, we write $h$ for such a triple. There is a finite subset $W'$ of $W$ depending on $\iota$ and $Q$ with the following property: If $h\in W$, there exists $u\in W'$ and an isotopy $u_t$, $t\in[1/2,1]$, between $u_{1/2}=u$ and $u_1=h$ which is obtained by moving vertices along line segments and such that the bilipschitz constant of $u_t$ is bounded by $L$, depending only on $Q$, and $u^x\colon[1/2,1]\to\Delta$, $u^x(t)=u_t(x)$, is 2-Lipschitz. Let $W'=\{h^1,\dots,h^m\}$. For each $h^i$ we can (see [1, Lemma IV.24]) choose a $PL$ isotopy $h^i_t\colon\Delta\to\Delta$, $t\in[0,1/2]$, such that

(a) $h^i_0$ is the identity, $h^i_{1/2}=h^i$,

(b) $h^i_t|\dot{\Delta}$ is the identity for all $t\in[0,1/2]$.

The bilipschitz constant $L^i_t$ of $h^i_t$ depends continuously on $t$, hence $L_1=\max\{L^i_t|1\leq i\leq m,\ 0\leq t\leq 1/2\}<\infty$. If $h^{ix}(t)=h^i_t(x)$ and $L^i_x$ is the Lipschitz constant of $h^{ix}$, also $L_2=\max\{L^i_x|1\leq i\leq m,\ x\in\Delta\}<\infty$. For $u=h^i$ we define the isotopy $h_t$, $t\in[0,1]$, by

$$h_t=h^i_t,\quad 0\leq t\leq 1/2,$$

$$h_t=u_t,\quad 1/2\leq t\leq 1.$$

We can make $L_1$ and $L_2$ depend only on $\iota$ and $Q$. This proves the lemma.

DEFORMATION LEMMA 5.3. *Let* $\Delta$ *be the* 2*-simplex as in Lemma* 5.2 *and let* $G$ *and* $G'$ *be map complexes with constant* $\lambda$ *such that the following conditions hold:*

(1) $2\Delta\subset|G|=|G'|\subset\mathbf{R}^2$.

(2) $G^0$ *and* $G'^0$ *are given decompositions into classes* $\alpha,\beta,\gamma$ *and pairings with common* $\alpha\beta$*-sides.*

(3) $|\dot{\Delta}|\subset|G^1|$, *and* $G$ *and* $G'$ *coincide together with the* $\alpha,\beta,\gamma$ *decompositions and maps* $\varphi_A$ *in* $|G|\setminus\Delta$.

(4) $|\dot{\Delta}|$ *contains only* $\alpha\gamma$-*sides of* $G$, *and hence of* $G'$.

(5) $\sigma(G|\Delta)=\sigma(G'|\Delta)=\tau$ *and* $0<\delta_1\leq d(A)\leq\delta_2<1/50$ *for all* $A\in G^2\cup G'^2$.

*Let* $g:|G|\to S^2$ *and* $g':|G'|\to S^2$ *be the maps as in Section* 5.1 *with respect to* $G$ *and* $G'$ *respectively. Then* $g=g_0$ *can be deformed as a family* $g_t$, $t\in[0,\mu]$, *to the map* $g_\mu=g'$ *such that the following conditions hold:*

(i) *Each* $g_t$ *is a mapping of* $|G|$ *onto* $S^2$ *such that it is discrete open in* int $G$, $g_t(|G|\setminus 2\Delta)=g|(|G|\setminus 2\Delta)$ *and the inequalities*

$$0<C_1(\lambda,\tau,\delta_1,\delta_2)\leq l(x,g_t)\leq L(x,g_t)\leq C_2(\lambda,\tau,\delta_1,\delta_2)$$

*hold for all* $x\in|G|$.

(ii) $\mu\leq P\tau$ *where* $P$ *is an absolute constant.*

(iii) *Let* $g^x(t)=g_t(x)$. *Then for all* $x\in|G|$, $g^x:[0,\mu]\to S^2$ *is an L-Lipschitz map with* $L\leq C(\lambda,\tau,\delta_1,\delta_2)$.

*Proof.* The strategy is to deform both $g$ and $g'$ so that $g|\Delta$ and $g'|\Delta$ are replaced by a bunch of simple covers.

We start by deforming $g$. Because of (4), $G|\Delta$ has a pairing with common $\alpha\beta$-sides. First we show that there exists at least one pair of type (2) (a) or (2) (b) in Section 2.7 in $G|\Delta$. Suppose all pairs in $G|\Delta$ are of type (c). Let $A_1, B_1$ be a pair in $G|\Delta$. Then $\mathbf{R}^2\setminus\text{int}(A_1\cup B_1)$ has a bounded component $C_1\subset\Delta$. Also $C_1$ contains a pair $A_2, B_2$ and $\mathbf{R}^2\setminus\text{int}(A_2\cup B_2)$ a bounded component $C_2\subset C_1$, $C_2\neq C_1$. By repeating this we end up with a contradiction because $G|\Delta$ is finite.

By (5) we can fix a side $Y_1$ of $G$ in $(3/2)\,\Delta\setminus(4/3)\,\Delta$ which is an $\alpha\gamma$-side or an $\beta\gamma$-side. We can take a pair $A_1, B_1$ of type (a) or (b) in $G|\Delta$ and move it to the side $Y_1$. During this elementary deformation $g$ is fixed outside the set $|St(A_1,G)|\cup|St(B_1,G)|\cup E$ where $E$ is the joining set. By (5) it is possible to do this deformation so that $g|(|G|\setminus 2\Delta)$ remains fixed. Let the obtained refined map complex be $G_1$ and let $g_1$ be the map represented by $G_1$. When the pair $A_1, B_1$ is collapsed, the part $G|\Delta$ has changed to $G_1|V_1$ where int $V_1$ may have several components. Let $U$ be a component of int $V_1$. Since $\partial V_1$ is connected, $U$ is homeomorphic to a disc. Also $\partial U$ contains only $\alpha\gamma$-sides and we can repeat the above by replacing $\Delta$ by $\bar{U}$ and $G$ by $G_1$. The simple cover $g|A_1\cup B_1$ was moved to a simple cover $g_1|D_1$ at $Y_1$. Let $Y_2$ be a side in $G_1|\partial D_1$. There is a pair $A_2, B_2$ of type (a) or (b) in $G_1|\bar{U}$. We move the pair $A_2, B_2$ to the side $Y_2$. We can repeat until all pairs originating from $G|\Delta$ have been moved similarly. We end up with a refined map complex $G_i$ and a map $g_i$: $|G|\to S^2$ represented by $G_i$ where the original $G|\Delta$ has been transformed to a 1-subcomplex $\Lambda$ which is a tree plus a bunch


Fig. 5.5.

$g_i|D$ of simple covers, see Figure 5.5. The deformation from $g$ to $g_i$ induces a bijective map $\psi\colon |\dot{\Delta}| \to \tilde{\Lambda}$ onto the set $\tilde{\Lambda}$ of boundary elements of $\mathbf{R}^2 \setminus |\Lambda|$. If $\pi\colon \tilde{\Lambda} \to |\Lambda|$ is the natural projection, we have

$$g||\dot{\Delta}| = g_i \circ \pi \circ \psi.$$

Note that $\Lambda$ consists of $\alpha\gamma$-sides only. Similarly we deform $g'$ to a map $g'_j$ and get a refined map complex $G'_j$, a 1-subcomplex $\Lambda'$, maps $\psi'\colon |\dot{\Delta}| \to \tilde{\Lambda}'$ and $\pi'\colon \tilde{\Lambda}' \to |\Lambda'|$, and a bunch $g'_j|D'$ of simple covers. We do the construction so that $D=D'$ and $g_i|D=g'_j|D$. This is possible because of (3) and (5).

Next we fix a $\gamma$-vertex $c \in G^0 \cap |\dot{\Delta}| = G'^0 \cap |\dot{\Delta}|$. It corresponds to vertices $c_0 \in \Lambda$ and $c'_0 \in \Lambda'$ by the maps $\pi \circ \psi$ and $\pi' \circ \psi'$. We shall continue deformations so that $\Lambda$ and $\Lambda'$ become simple arcs where the points corresponding $c_0$ and $c'_0$ are endpoints. First we define a deformation called moving of an endside. Let $r$ be a side of $\Lambda$ which is an end of $\Lambda$, i.e. the other endpoint of $r$, say $c_1$, is a vertex of one side of $\Lambda$ only. Let us assume that $c_1$ is a $\gamma$-vertex. Let $s \neq r$ be a side of $\Lambda$ so that $r$ and $s$ are induced from neighboring sides in $G||\dot{\Delta}|$ by the map $\pi \circ \psi$. We deform $g_i$ by moving two pairs from $D$ to the sides $r$ and $s$ (Figure 5.6). Then the arcs $r$ and $s$ are divided by $\beta$-points $b_1$ and $b_2$. Let $r_1$ and $s_1$ be the new sides in the arcs $r$ and $s$ with the common vertex $a_0$. Next we perform the exchange of sides first to the pair $r_1, s_1$ and obtain Figure 5.7 and then to

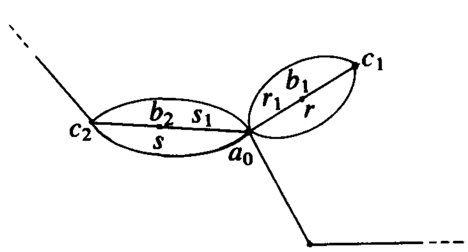


Fig. 5.6.

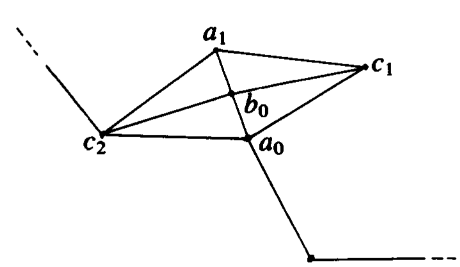


Fig. 5.7.

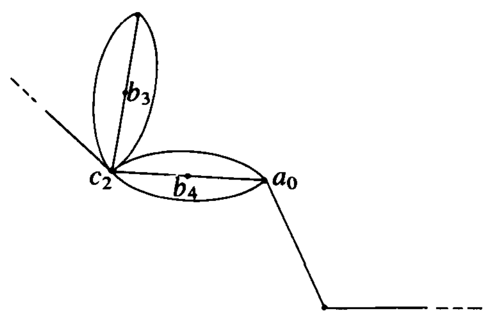


Fig. 5.8.

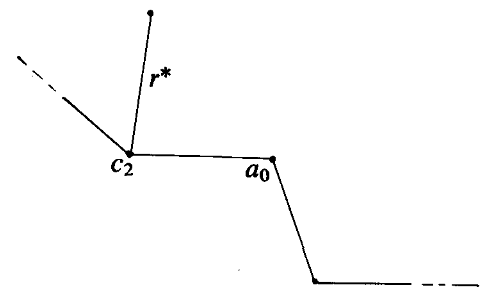


Fig. 5.9.

the sides $b_0c_1$ and $b_0c_2$ in Figure 5.7 and obtain Figure 5.8. There we see two simple covers which we finally move back to $D$ to the original places. As a result, $r$ is moved a distance of one side along $\tilde{\Lambda}, \Lambda$ is changed to a 1-subcomplex $\Lambda_1$ of a new refined map complex, and the new side $r^*$ is again an end of $\Lambda_1$ (Figure 5.9). We call this deformation the *moving of the endside $r$ to $c_2$*.

By using repeatedly the deformation of moving an endside we may now deform $g_i$ as follows. Take a subcomplex $\Lambda_1$ of $\Lambda$ such that $|\Lambda_1|$ is a maximal simple arc having $c_0$ as one endpoint. Let $d$ be the other endpoint of $\Lambda_1$. Since $\Lambda_1$ is maximal, only one side of $\Lambda$ contains $d$. We may assume $\Lambda_1 \neq \Lambda$. There exists an endside $r$ not belonging to $\Lambda_1$. We move $r$ by repeatedly using the moving of an endside to $d$, call the new side $r_1$, and add $r_1$ to $\Lambda_1$ to form a new subcomplex $\Lambda_2$ of a 1-complex which replaces $\Lambda$. Repeating this we obtain a deformed $g_k$ of $g_i$ such that $\Lambda$ is deformed to a 1-subcomplex $\Lambda_0$ of the refined map complex $G_k$ corresponding to $g_k$ such that $|\Lambda_0|$ is a simple arc with $c_0$ as one endpoint. Similarly we deform $g'_j$ to a map $g'_l$ such that $\Lambda'$ is deformed to a 1-subcomplex $\Lambda'_0$ of a map complex $G'_l$ such that $|\Lambda'_0|$ is a simple arc with $c'_0$ as one endpoint. The maps $g_k$ and $g'_l$ are topologically equivalent.

We still have to deform $g_k$ to $g'_l$. For this we will apply Lemma 5.2. It follows from (5) and the metrical conditions for the elementary deformations that the deformations $g_t$, $t \in [0, k]$, and $g'_t$, $t \in [0, l]$, satisfy conditions of the form (i), (ii), and (iii). The refined map complexes $G_k$ and $G'_l$ are of some subdivision orders $m$ and $n$ respectively, have a constant $\lambda_0$, and $m, n$, and $\lambda_0$ depend only on $\lambda, \tau, \delta_1$, and $\delta_2$. Let $q = \max(m, n)$. Set $K = (G'_{k(1)})^{(q)}|2\Delta$, $M = (G'_{l(1)})^{(q)}|2\Delta$, and let $h: 2\Delta \to 2\Delta$ be the simplicial homeomorphism with respect to $K$ and $M$ defined by $g'_l \circ h|2\Delta = g_k|2\Delta$. Then, if $\Delta$ is replaced by $2\Delta$ in Lemma 5.2, $h$ satisfies the conditions of Lemma 5.2 with constants $\iota$ and $Q$ depending only on $\lambda, \tau, \delta_1$, and $\delta_2$. The isotopy $h_t$ given by Lemma 5.2 defines a deformation $g_t$, $t \in [k, k+1]$, by $g_t|2\Delta = g_k \circ h_{t-k}^{-1}$, $g_t|(|G| \setminus 2\Delta) = g$. By taking the deformation $g_t$, $t\in[0,k+1]$, and after this the deformation $g'_t$, $t\in[0,l]$, backwards we get the required deformation. The lemma is proved.

5.4. *Remark.* It follows from the proof of Lemma 5.3 that the deformation lemma is valid also for refined map complexes $G$ and $G'$ in a form where the constants in the statements (i)–(iii) depend also on the subdivision orders of $G$ and $G'$.

## 6. The construction of the map between level surfaces

In this section we shall extend the definition of $f$ to the layers between the level surfaces defined in Section 4. An essential role is played by the deformation lemma from Section 5. These extensions must still be glued together near $|M_\infty|$, which will be accomplished in the next section.

6.1. *Straightening of layers.* We shall first define bilipschitz maps which straighten pieces between level surfaces. Let us consider the layer $\Lambda$ between $|N_1|$ and $|\nu^2N_1|$ which is the closure of the domain bounded by $|N_1|\cup|\nu^2N_1|$.

As a preliminary step we shall remove the caves of "finest order" in $|N_1|$. Let $C, D$ be a pair of cave opening simplexes with $\delta_1$ as one class such that $\omega_1(C)$, $\omega_1(D)\subset M_1$ and $C,D\in\omega_1(C')\cup\omega_1(D')$ for some $C',D'\in\nu^2M_1$. Let $P_C$ be the 2-complex $\psi(\omega_1(C)|(|\omega_1(C)|\setminus\operatorname{int}\omega(C)))$ and $P_D=\psi(\omega_1(D)|(|\omega_1(D)|\setminus\operatorname{int}\omega(D)))$. Then $P=P_C\cup P_D$ forms the "walls" of the cave in $|N_1|$ corresponding to the pair $C,D$. The 1-complex $\dot{P}$ has four vertices $a,b,c,d$ so that $c\in P_C\setminus P_D$, $d\in P_D\setminus P_C$, $a,b\in P_C\cap P_D$. Let $E$ be the third 2-simplex in $M_{kq}$ which has a common side with $C$ and $D$ if $C,D\in M_{kq}$.

Let $T$ be the plane containing $E$ and let $e\in T\setminus E$ be a point such that the 2-simplex $\langle\psi^{-1}(a),\psi^{-1}(b),e\rangle\subset T$ is an equilateral triangle. Then the 3-simplexes $X=\langle a,b,c,e\rangle$ and $Y=\langle a,b,d,e\rangle$ are in $\Lambda$. The polyhedron $R$ bounded by $|P|\cup\langle a,b,c\rangle\cup\langle a,b,d\rangle$ is the "cave part" and will be pushed into $X\cup Y$ as follows. Set $R^*=R\cup X\cup Y$.

We need some elementary maps. Let $L$ be the 1-complex which defines the cave in $\omega_1(C)\cup\omega_1(D)$ and let $K$ be the union of complexes such that $K_L$ is the corresponding cave complex. Let $v$ be an end vertex of $L$ which is not the last vertex of $L$, and let $s$ be the side in $L$ which has $v$ as a vertex. Set $L'=L\setminus\{s,v\}$. We shall here push $R$ by a $PL$ homeomorphism into a polyhedron $R'$ corresponding to $L'$. Let $w$ be the other endpoint of $s$. Let us assume that the $\varrho_K$-distance of $v$ to $L^0\setminus\{v,w\}$ is at least 2. Let $r$ be the side in $St(v,K)$ opposite $s$, let $V$ and $V'$ ($W$ and $W'$) be the 2-simplexes in $St(v,K)$ with $r$ ($s$) as a side, and let $\{Z,Z'\}=St(v,K)^2\setminus\{V,V',W,W'\}$. We assume that $\operatorname{int}(V\cup W\cup Z)$ lies in one component of $|St(v,K)|\setminus(r\cup s)$. When $K$ is formed, each $U$ of these

2-simplexes is replaced by two 2-simplexes $U_-$ and $U_+$. We may assume that each $U_+$ is in $\omega_1(C)$.

We first push the part $R_0$ of $R$ spanned by $I_0=\psi(V_-\cup V_+\cup V'_-\cup V'_+)$ into the part $R_1$ of $R$ defined correspondingly by $Z, Z', W$, and $W'$ by a $PL$ homeomorphism $h_0: R^*\to\overline{R^*\setminus R_0}$ which is simplicial on the boundary. More precisely, let $\{v_-\}=\psi(V_-\cap W_-)$, $\{v_+\}=\psi(V_+\cap W_+)$, let $v_0$ be the barycenter of $\langle v_-, v_+\rangle$, and let $P_0$ be the subcomplex of $P$ with $|P_0|=I_0$. The map $h_0$ will take $I_0$ simplicially with respect to $P_0$ onto $|v_0\dot{P}_0|$ and it is the identity on $(\partial R^*\setminus I_0)\cup(R^*\setminus(R_0\cup R_1))$. We can in addition choose $h_0$ to be locally $\theta_0$-bilipschitz with $\theta_0$ an absolute constant.

Next we push the part $R_2$ of $h_0R^*$ spanned by $\psi(Z_-\cup Z_+)$ into the part $R_3$ spanned by $\psi(W_-\cup W_+)$ similarly by a $PL$ homeomorphism $h_1: \overline{R^*\setminus R_0}\to\overline{R^*\setminus(R_0\cup R_2)}$ such that $h_1$ takes $I_1=h_0(\psi(V_-\cup V_+))\cup\psi(Z_-\cup Z_+)$ simplicially onto $|v_1\dot{P}_1|$ where $v_1$ is the barycenter of the triangle $R_2\cap R_3$. Here $h_0(\psi(V_-\cup V_+))$ is given the subdivision induced by $h_0$ from $P_0$, and $P_1$ is defined as a complex with $|P_1|=I_1$ similarly as earlier. Let us define corresponding sets with primes. Then we push $R'_2$ into $R'_3$ similarly by a $PL$ homeomorphism $h'_1: \overline{R^*\setminus(R_0\cup R_2)}\to\overline{R^*\setminus(R_0\cup R_2\cup R'_2)}$.

Let $\{z\}=\psi(W_-\cap W_+)$, $\{z'\}=\psi(W'_-\cap W'_+)$. One endpoint of $\psi(W_+\cap W'_+)$ is $v_+$. Let the other be $w_+$ and define $w_-$ similarly. Let $w_0$ be the barycenter of $\langle w_-, w_+\rangle$. Let $R_4$ and $R_5$ be the polyhedra spanned by $\{z, z', v_0, v_+, w_+\}$ and $\{z, z', v_0, w_+, w_0\}$ respectively. We push $R_4$ into $R_5$ similarly as above by a $PL$ homeomorphism $h_2^+$: $h'_1h_1h_0R^*\to h_2^+h'_1h_1h_0R^*$ so that $h_2^+(v_+)$ is the barycenter of $\langle v_0, w_+\rangle$. Similarly we define a map $h_2^-: h_2^+h'_1h_1h_0R^*\to h_2^-h_2^+h'_1h_1h_0R^*$. Set $h=h_2^-h_2^+h'_1h_1h_0$. The map $h$ is now the required $PL$ homeomorphism which pushes $R$ into $R'$. If the $\varrho$-distance of $v$ to $L^0\setminus\{v, w\}$ is 1, the map $h$ is defined with obvious modifications. The complex $L'$ is again a tree.

We can apply maps like $h$ repeatedly to get a $PL$ homeomorphism $\varphi: R^*\to X\cup Y$ which has the following properties:

(1) $\varphi|(\partial R^*\setminus|P|)$ is the identity and $\varphi$ is simplicial on $P$.

(2) $\varphi$ is locally $\exp(\theta_1\nu^3)$-bilipschitz where $\theta_1$ is an absolute constant.

We assume now that all caves of the described type have been pushed in. We extend the obtained maps $\varphi$ to the rest of $\Lambda$ by identity and call the extended map $\varphi_0$. The set $\varphi_0N_1$ is again a union of complexes.

Next we define the union $M'_1$ of complexes which we obtain from $\nu^2M_1$ by applying the operation $\omega_1$ once. More precisely, $M'_1=\cup\{\omega_1(A)|A\in\nu^2M_1^2\}$. Then $|M'_1|\cap|\nu^2N_1|=\emptyset$. Let $\Lambda'$ be the layer bounded by $|M'_1|\cup|\nu^2N_1|$. At this point it is elementary to perform a $PL$ homeomorphism $\varphi_1\colon \varphi_0\Lambda\to\Lambda'$ with the following properties:

(a) $\varphi_1$ is 10-bilipschitz.

(b) $\varphi_1||\nu^2N_1|$ is the identity.

(c) $\varphi_1$ maps $\varphi_0|N_1|$ onto $|M_1'|$ simplicially in the sense that $\varphi_1$ is affine in each simplex of $\varphi_0 N_1$.

(d) Let $A\in\nu^2M_1^2$ and $B\in\omega_1(A)^2$. If $B\in M^2_{k,k-2,1}$, $\varphi_1\varphi_0\psi|\omega_{k,k-2,1}(B)|=B$. If $B\in M^2_{k,k-1,1}$, $\varphi_1\varphi_0\psi B=B$.

We proceed now somewhat similary with $\Lambda'$ as we did with $\Lambda$. First we push in all caves of finest order in $M_1'$ by the same method as before and obtain a $PL$ homeomorphism $\varphi_0'$ of $\Lambda'$ which is locally $\exp(\theta_1\nu^3)$-bilipschitz, the identity on $|\nu^2N_1|$, and simplicial on $M_1'$. After this we perform a $PL$ homeomorphism $\varphi_1'$ of $\varphi_0'\Lambda'$ onto the layer $\Lambda''$ bounded by $|\nu^2M_1|\cup|\nu^2N_1|$ such that (c) and (d) are replaced by the following conditions:

(c′) $\varphi_1'$ maps $\varphi_0'|M_1'|$ onto $|\nu^2M_1|$ simplicially.

(d′) For each $A\in\nu^2M_1$, $\varphi_1'\varphi_0'|\omega_1(A)|=A$.

Finally we shall map pieces of $\Lambda''$ onto products as follows. First we construct a family of $PL$ homeomorphisms $\psi_A$, $A\in\nu^2M_1^2$, with the following properties:

(i) If $A$ is an $l$-triangle, $\psi_A$ is a map of $A^*\times[0,\nu^l]$ into $\Lambda''$, where $A^*=uA\subset\mathbf{R}^2$ and $u$ is a motion in $\mathbf{R}^3$. Moreover, $\psi_A$ is $u^{-1}$ on $A^*$ and maps $A^*\times\{\nu^l\}$ affinely onto $\nu^2\psi(\nu^{-2}A)$. We set here $I_A=[0,\nu^l]$, $\tau_A=\nu^l$.

(ii) The sets $X_A=\operatorname{Im}\psi_A$ form a decomposition of $\Lambda''$ such that for $A\neq B\in\nu^2M_1^2$, $X_A\cap X_B$ is either empty or $\psi_A(r\times I_A)=\psi_B(s\times I_B)$ where $r$ and $s$ are sides of $A^*$ and $B^*$ respectively. Let in the latter case $r=\langle a,b\rangle$, $s=\langle c,d\rangle$, and $\psi_A(a)=\psi_B(c)$. If $\tau\in I_A$, $\tau'=\tau_B\tau/\tau_A$, $x=\mu a+(1-\mu)b$, $x'=\mu c+(1-\mu)d$, and $0\leq\mu\leq 1$, then $\psi_A(x,\tau)=\psi_B(x',\tau')$.

(iii) Every $\psi_A$ is 40-bilipschitz.

The construction of the maps $\psi_A$ is elementary. For each $A\in\nu^2M_1^2$ we write $W_A=|St(A,\nu^2M_1)|$ where $St(A,\nu^2M_1)$ is the union of subcomplexes of $\nu^2M_1$ whose 2-simplexes meet $A$. By the construction of $M_1$ we can extend $\psi_A|A^*\times\{0\}$ to a $PL$ homeomorphism $\psi_A^0\colon V_A\to W_A$, where $V_A\subset\mathbf{R}^2\times\{0\}=\mathbf{R}^2$, such that $(\psi_A^0)^{-1}$ is affine in each simplex in $St(A,\nu^2M_1)$ and such that $\psi_A^0$ is $\theta'$-bilipschitz where $\theta'$ is an absolute constant. We let $\eta_A\colon V_A\times I_A\to\Lambda''$ be the extension of $\psi_A$ defined by

$$\eta_A(x,\tau)=\psi_B(\psi_B^{-1}\psi_A^0(x),\tau_B\tau/\tau_A)\quad\text{if } x\in(\psi_A^0)^{-1}B,\ B\in St(A,\nu^2M_1)^2,\ \tau\in I_A.$$

The map $\eta_A$ is a $PL$ homeomorphism onto $\eta_A(V_A\times I_A)$ and it is $\theta''$-bilipschitz where $\theta''$ is an absolute constant.

6.2. *The extension of the map to layers between level surfaces.* We continue to consider the layer $\Lambda$ between $|N_1|$ and $|\nu^2 N_1|$. The map $\xi=\varphi_1'\varphi_0'\varphi_1\varphi_0\colon \Lambda\to\Lambda''$ induces the map complex $\xi H_1$ on $\xi|N_1|=|\nu^2 M_1|$. On $|\nu^2 N_1|$ $\xi$ is the identity and there we have the map complex $\nu^2 H_1$. Recall the maps $g_{2i}$, $i=0,1,\dots$ from (4.4). Our task is to deform $\varrho_0'=g_0\circ\xi^{-1}||\nu^2 M_1|\colon |\nu^2 M_1|\to S^2$ to $\varrho_1'=g_2\colon |\nu^2 N_1|\to S^2$ when we move from $|\nu^2 M_1|$ to $|\nu^2 N_1|$. These maps are represented by $\Gamma_0'=\xi H_1$ and the second canonical subdivision $\Gamma_1'=(\nu^2 H_1)_{(2)}$ of $\nu^2 H_1$ respectively.

To apply the Deformation lemma 5.3 we perform a preliminary deformation which adjusts the map complexes so that they correspond on the sides of $\nu^2 M_1$ and $\nu^2 N_1$. According to the construction $\Gamma_0'$ and $\Gamma_1'$ contain only $\alpha\gamma$-sides on the sides of $\nu^2 M_1$ and $\nu^2 N_1$ respectively and the vertices are $\alpha$-vertices. For $0\le t\le 1$ we let $S_t$ be the surface $\{\psi_A(x,\tau)|x\in A^*,\ \tau=t\tau_A,\ A\in\nu^2 M_1^2\}$ and call $S_t$ the $t$-level.

Let $A\in\nu^2 M_1^2$ be an $l$-triangle such that all $B\in\nu^2 M_1^2$ with a common side with $A$ are $l$-triangles or $(l+1)$-triangles. Let $r$ be a side of $A^*$. Suppose there are $p$ pairs of sides less in $\Gamma_0'|\eta_A r$ than in $\Gamma_1'|\eta_A(r\times\{\tau_A\})$. We apply repeatedly the elementary deformation of exchange of sides (see Section 5.1) to pairs of sides in $\Gamma_1'|\eta_A(r\times\{\tau_A\})$. More precisely, let $\pi\colon \mathbf{R}^3\to\mathbf{R}^2$ be the orthogonal projection, $\sigma_t\colon V_A\to V_A\times\{t\tau_A\}$ the map with $\pi\circ\sigma_t$ the identity, $0\le t\le 1$, let $\Gamma_i$ be the map complex $\pi\eta_A^{-1}\Gamma_i'$, and $\varrho_i\colon V_A\to S^2$ the map $\varrho_i'\circ\eta_A\circ\sigma_i$ which is represented by $\Gamma_i$, $i=0,1$. We perform a deformation $\varrho_t$, $t\in[3/4,1]$ of $\varrho_1$ starting from $t=1$ which consists of $p$ repeated exchanges of sides, where in each a pair of side in $r$ is removed, and in addition such a $PL$ homeomorphism which moves the vertices of the obtained refined map complex along $r$ so that we end up with a refined map complex $\Gamma_{3/4}$ which coincides with $\Gamma_0$ on $r$. Such a $PL$ homeomorphism clearly exists. If $p$ is negative, we perform a similar deformation during $t\in[0,1/4]$ and change the role of $\varrho_0$ and $\varrho_1$. The part of exchanges of sides can clearly be done for each side of $\nu^2 M_1$ corresponding to a side like $r$ simultaneously during $t\in[0,1/4]$ and $t\in[3/4,1]$. On the level surface $S_t$ we obtain for these $t$ a map $\varrho_t'\colon S_t\to S^2$ defined by $\varrho_t'|Y_A^t=\varrho_t\circ\pi\circ\eta_A^{-1}|Y_A^t$ where $Y_A^t=\eta_A(V_A\times I_A)\cap S_t$.

According to the construction of $M_1$ there is a decomposition $\mathscr{A}_1\cup\dots\cup\mathscr{A}_i$ of $\nu^2 M_1^2$ into disjoint subsets $\mathscr{A}_j$ with the property that $\operatorname{int} St(A,\nu^2 M_1)\cap\operatorname{int} St(B,\nu^2 M_1)=\emptyset$ for all $A,B\in\mathscr{A}_j$ with $A\neq B$, and $i$ depends only on $\nu$. We divide $[1/4,3/4]$ into intervals $I_j=[t_{j-1},t_j]$ $j=2,\dots,i+1$, of equal length corresponding to the sets $\mathscr{A}_j$.

Let $A\in\mathscr{A}_1$ and let the notation be as above. By performing a preliminary homeomorphism of $V_A$, which is simplicial with respect to $\eta_A^{-1}(\nu^2 M_1|W_A)$, if necessary, we may assume that $A^*$ is the $\Delta$ as in the Deformation lemma 5.3 and $2\Delta\subset V_A$. We now want to apply Lemma 5.3 together with Remark 5.4 to the map $g=\varrho_{t_1}=\varrho_{1/4}$ and to

$g'\colon V_A \to S^2$ defined by

$$g'|A^* = \varrho_{3/4}|A^*,$$
$$g'|(V_A \setminus A^*) = \varrho_{t_1}|(V_A \setminus A^*).$$

All other assumptions in Lemma 5.3 are satisfied except possibly $d_2 < 1/50$ in (5), but this we achieve by choosing the various preliminary maps properly. We obtain a deformation $\varrho_t, t \in [t_1, t_2]$, with $\varrho_{t_2} = g'$. On the $t$-levels we have then the maps $\varrho'_t\colon Y^t_A \to S^2$, $\varrho'_t = \varrho_t \circ \pi \circ \eta_A^{-1}|Y^t_A$, $t \in [t_1, t_2]$. By the definition of the sets $\mathcal{A}_j$ we can do such a deformation for all $A \in \mathcal{A}_1$ simultaneously and we obtain maps $\varrho'_t\colon S_t \to S^2$, $t \in [t_1, t_2]$, if we in addition put $\varrho'_t|X_B \cap S_t = \varrho_{t_1} \circ \pi \circ \psi_B^{-1}|X_B \cap S_t$ for $B \in \nu^2 M_1^2$ with $\operatorname{int} B \cap \bigcup\{W_A | A \in \mathcal{A}_1\} = \emptyset$. By continuing similarly for the subsets $\mathcal{A}_2, \ldots, \mathcal{A}_i$ we get all together a family $\varrho'_t\colon S_t \to S^2$, $t \in [0, 1]$, of maps which deforms $\varrho'_0$ to $\varrho'_1$.

We are now in a position to define $f$ in the layer $\Lambda$ between $|N_1|$ and $|\nu^2 N_1|$. Recall the formula (4.5) for the definition of $f$ on original level sets:

$$f|\nu^{2i}|N_1| = (s_{2i}, g_{2i}), \quad i = 0, 1, \ldots.$$

Here $s_\mu = e^{-1} \exp \nu^{\mu+1}$ for all $\mu \in \mathbf{R}^1$. The definition in $\Lambda = \Lambda_0$ is given by (the map $\xi$ is given in the beginning of Section 6.2)

$$f|(\xi^{-1} S_t) = (s_{2t}, g_{2t}), \quad t \in [0, 1],$$

where $g_{2t} = \varrho'_t \circ \xi|(\xi^{-1} S_t)$. Write $\Sigma_{2t} = \xi^{-1} S_t$, $t \in [0, 1]$.

To define $f$ in any other layer $\Lambda_i$ between $|\nu^{2i} N_1|$ and $|\nu^{2i+2} N_1|$, $i = 1, 2, \ldots$, we use similarity as follows. In the deformation we replace $g_0$ and $g_2$ by the maps $\tilde{g}_{2i}$ and $v_2 \circ \tilde{g}_{2i+2}$ where we recall the formulae (4.6) and (4.7): $g_{2i} = v_{2i} \circ \tilde{g}_{2i}$, $\tilde{g}_{2i}(x) = g_0(x/\nu^{2i})$. From the deformation we obtain maps $\tilde{g}_{2t}\colon \Sigma_{2t} \to S^2$, $t \in [i, i+1]$, where $\Sigma_{2t}$ is again a level surface between $|\nu^{2i} N_1|$ and $|\nu^{2i+2} N_1|$, and $\tilde{g}_{2i} = \tilde{g}_{2i}$, $\tilde{g}_{2i+2} = v_2 \circ \tilde{g}_{2i+2}$. Set $g_{2t} = v_{2i} \circ \tilde{g}_{2t}\colon\ \Sigma_{2t} \to S^2$, $t \in [i, i+1]$. Observe that $v_{2i} \circ v_2 = v_{2i+2}$, so that this agrees with the earlier definitions of $g_{2i}$ and $g_{2i+2}$. In $\Lambda_i$ we define $f$ then by

$$f|\Sigma_{2t} = (s_{2t}, g_{2t}), \quad t \in [i, i+1]. \tag{6.3}$$

We have now defined $f$ in

$$W'_1 = \bigcup_{i=0}^{\infty} \Lambda_i. \tag{6.4}$$

The construction has been carried out so that $f|\text{int } W_1'$ is $K$-quasiregular, $K$ depending only on $\lambda$ and $\nu$.

The set $W_1'$ is contained in $V_1$ (see Section 4.2). The construction of $f$ in a corresponding subset $W_2'$ of $V_2$ and in a subset $W_3'(h)$ of $V_3(h)$, $h=0,\dots,5$, is similar. We use formulae corresponding to (4.8) and (4.9).

## 7. Final glueing

In this section we shall complete the construction for the case $p=2$ by performing a glueing near the set $|M_\infty|$ of the maps $f|W_1'$, $f|W_2'$, $f|W_3'(h)$, $h=0,\dots,5$, constructed in Section 6. The constructions resemble the cave construction, but we work this time directly with map complexes rather than complexes.

7.1. *New map complexes.* On $|M_1|$ we have the map complex $G_1$. Transferring it to $|N_1|$ by the map $\psi: |M_1| \to |N_1|$ we obtained the map complex $H_1$. Corresponding to $M_2$ and $M_3(h)$, $h=0,\dots,5$, we have map complexes $G_2$, $G_3(h)$, $H_2$, $H_3(h)$, $h=0,\dots,5$. To simplify notation in the conditions (1)–(4) below we write for a moment $M_3(h)=M^*_{3+h}$, $h=0,\dots,5$, $M_j=M_j^*$, $j=1,2$. We define new map complexes $\mathscr{G}_j$ with constant $\lambda=20$ on $M_j^*$, $j=1,\dots,8$, which satisfy the following conditions:

(1) $\mathscr{G}_i$ and $\mathscr{G}_j$ coincide together with the maps $\varphi_A$ on common parts of $|M_i^*|$ and $|M_j^*|$.

(2) $|M^1_\infty| \subset |\mathscr{G}_1^1 \cup \mathscr{G}_2^1|$, $M^0_\infty \subset \mathscr{G}_1^0 \cup \mathscr{G}_2^0$.

(3) $\mathscr{G}_j^0$ has a decomposition into classes $\alpha, \beta, \gamma$, and these coincide on common parts for different $j$. $|M^1_\infty|$ contains only $\alpha\gamma$-sides and $M^0_\infty$ consists only of $\alpha$-vertices.

(4) If $C \in M^2_{k,k-1} \cap M_j^*$ then $\sigma(\mathscr{G}_j|C)=w_k(C)/2$, where $w_k$ was defined in Sections 3.2 and 3.4.

We have $\mathscr{G}_1 \cup \dots \cup \mathscr{G}_8 = \mathscr{G}_1 \cup \mathscr{G}_2$. Set $\mathscr{G}_\infty = \mathscr{G}_1 \cup \mathscr{G}_2$. We shall now construct a new kind of cave refinement on the basis of $\mathscr{G}_\infty$. Let $A \in M^2_{kq}$ be such that $\omega(A) \subset M_{k,k-1}$. Write $\mathscr{G}_A = \mathscr{G}_\infty|\omega(A)$. The definition of $w_k: M^2_{k,k-1} \to \mathbf{N}$ includes an assignment of an $\Omega$ for $\omega(A)$. Choose one point $a_B$ in the interior of each $B \in \mathscr{G}^2_\infty$. For $Q \in \Omega$ and for $T \in Q$ we choose a union $L_T$ of simple arcs $s(B, C)$ in $T$ each of which joins two points $a_B, a_C$ through exactly one common side of some $B$ and $C$. Such points form the set $L^0_T$ of vertices of $L_T$. We require the following:

(a) $L_T$ is a tree and an endvertex $p_T=a_B \in L^0_T$ exists such that a side of $B_T=B \in \mathscr{G}^2_A$ is contained in the last side of $T$. We choose $p_T$ to be the last vertex in $L^0_T$. This gives $L^0_T$ a natural order.

(b) $L_T^0=\{a_B|B\in\mathscr{G}_\infty^2|T\}$.

(c) Let
$$J=\bigcap_{j=1}^{3} M_{k,k-1,j}|\omega(A).$$

If $T$ is the last in $Q$ and $S$ is the last in $Q'\in\Omega$, $Q'\neq Q$, then $B_T\cap|J|$ is different from $B_S\cap|J|$.

(d) Let $T$ be in $Q$ which is not the last, and let $S$ be the successor of $T$. We can join $L_T$ to $L_S$ by an additional arc $s_T$ of the type $s(B,D)$ where $a_B=p_T$ and $a_D$ is an endvertex in $L_S^0$. The union of the arcs $s_T$, $T\in Q$ not the last, and all $L_T$ is denoted by $L$.

The set $L$ is again a tree and the order in each $L_T^0$ induces an order in $L^0=\bigcup_T L_T^0$. This gives an order in $\mathscr{G}_Q^2$ where $\mathscr{G}_Q=\mathscr{G}_A|Q$. If $B,C\in\mathscr{G}_Q^2$ and $C$ is the successor of $B$, then the side in $B\cap C$ which meets $s(B,C)$ is the *last side* of $B$.

We shall next define positive and negative elements of $\mathscr{G}_A^2$. We define outward normals in $\omega(A)$ and positively oriented simple closed paths in any element in $\omega(A)^2$ with respect to $\delta_j$ as in Section 4.3. We say that $B\in\mathscr{G}_A^2$ is *positive* if for some $j$ the order $\alpha,\gamma,\beta$ is positively oriented in $|\dot{B}|$ with respect to $\delta_j$ and the classes of $B$ are $\delta_j$ and $\delta_k$ where $k=j+1 \pmod 3$. Otherwise $B$ is *negative*. Let $B$ and $C$ be two elements in $\mathscr{G}_A^2$ with at least one common side. It follows from the definition that $B$ and $C$ have different (same) signs if $B$ and $C$ have same (different) classes.

7.2. *New caves*. In our new cave constructions we shall now replace each positive 2-simplex in $\mathscr{G}_A$ by two and each negative by four sheets. The purpose of this is to get the right order of entering when we switch from one set $W_j=f^{-1}U_j$, $j=1,2,3$, to another. Recall that the sets $U_j$ are the components of $\mathbf{R}^3\setminus(X_0\cup\{u_2,u_3\})$ where $X_0=S^2\cup B^2$ and $u_2=-e_3/2$, $u_3=e_3/2$. Note that $f$ is not completely defined yet. The sheets will be attached to each other with certain identifications on the boundaries and they will all be mapped into $X_0$ by the final map $f$.

Let $Q\in\Omega$. For each negative $B\in\mathscr{G}_Q^2$ we choose one side $r_B$, called the *connection side for B*, as follows. If $B$ is not the last in $\mathscr{G}_Q^2$, we let $r_B$ be the last side of $B$. If $B$ is the last in $\mathscr{G}_Q^2$, choose one $C\in\mathscr{G}_Q^2$ such that $B$ is the successor of $C$ and let $r_B$ be the last side of $C$. By the construction such a $C$ exists and it is positive. Suppose the classes of $|Q|$ are $\delta_1$ and $\delta_2$, $|Q|\subset\mathbf{R}^2$, and that the elements of $\delta_1$ are in $H_+=\{x\in\mathbf{R}^3|x_3>0\}$. Note that $Q$ belongs then to $\Omega_0$, see Section 2.5. For $C\in\mathscr{G}_Q^2$ positive we let $\Psi_C^1,\Psi_C^2\colon C\to\mathbf{R}^3$ be maps which are $PL$ homeomorphisms onto their images and which are parametric

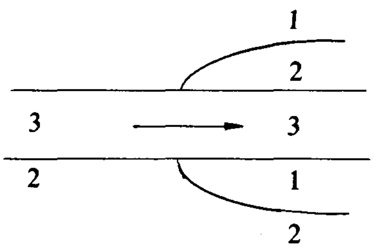


Fig. 7.1.

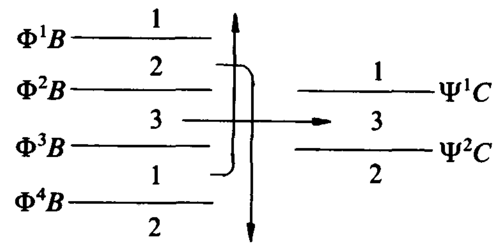


Fig. 7.2.

representations of the *sheets* for $C$ to be defined below. Let $\Phi_B^1, \ldots, \Phi_B^4\colon B \to \mathbf{R}^3$ be corresponding maps for a negative $B \in \mathscr{G}_Q^2$. The upper indexing shows the order of the sheets when we pass in the negative direction of the $x_3$-axis. We set $N = \bigcup\{B \in \mathscr{G}_Q^2 | B \text{ negative}\}$, $P = \bigcup\{C \in \mathscr{G}_Q^2 | C \text{ positive}\}$, and define $\Phi^m\colon N \to \mathbf{R}^3$ and $\Psi^n\colon P \to \mathbf{R}^3$ by means of the maps $\Phi_B^m$ and $\Psi_C^n$ respectively in the obvious way. We require that all open sheets $\Phi^m \operatorname{int} B$ and $\Psi^n \operatorname{int} C$ are mutually disjoint and do not touch the sets $W_1'$, $W_2'$, $W_3'(h)$, $h=0, \ldots, 5$ (see end of Section 6.2). Let $s$ be a side in $\mathscr{G}_Q$ which is not the last side $r$ of the last in $\mathscr{G}_{Q'}^2$ for some $Q' \in \Omega$, $Q' \neq Q$. We call sides like $r$ *excluded for* $Q$ in this context. To define the identifications of the maps $\Phi^m$ and $\Psi^n$ on $s$ we separate the following cases:

(1) $s$ is not a last side of any element in $\mathscr{G}_Q^2$.

(2) $s$ is the last side of $D \in \mathscr{G}_Q^2$, $D$ is not the last of $\mathscr{G}_Q^2$, and $s$ is not a connection side.

(3) $s$ is a connection side.

(4) $s$ is the last side of the last in $\mathscr{G}_Q^2$.

The maps $\Phi^m$ and $\Psi^n$ are defined in these cases as follows. Here $n$ runs over $1, 2$ and $m$ over $1, 2, 3, 4$.

Case (1): We let $\Phi^m$ and $\Psi^n$ be the identity on $s$.

Case (2): Let $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$ be the ortogonal projection. We set the following conditions in this case:

(a) The endpoints of $s$ are fixpoints for all $\Phi^m$ and $\Psi^n$.

(b) On $s$, $\pi \circ \Phi^m$ and $\pi \circ \Psi^n$ are the identity.

(c) On $s$, $\Phi^1 = \Phi^2 = \Psi^1$, $\Phi^3 = \Phi^4 = \Psi^2$.

(d) On $\operatorname{int} s$ the third coordinates satisfy $\Phi_3^2 > 0 > \Phi_3^3$.

The purpose of the identifications in case (2) is to get a cave connection in $W_3$ between the sheets $\Phi^2B, \Phi^3B, \Psi^1C, \Psi^2C$ where $B$ and $C$ have $s$ as a side. In Figure 7.1

this connection is shown by an arrow. The arrow shows here, and in other figures below, the way towards $\infty$ in $W_j$ (here $j=3$). The sets $W_j$ are marked simply by $j$ in the figures. As $f$ is not fully defined yet, we use only the sets $W_j$ in the figures to help the reader to follow the arguments. Formally we give each sheet a pair of classes $\delta_j$ such that the elements of the class lie in the set marked by $j$.

Case (3): Let $s$ be a connection side for $B$. We shall do the identifications so that cave connections appear as in Figure 7.2. (The arrow for $W_3$ is in the opposite direction if $B$ is the last of $\mathscr{G}^2_Q$.) For this let $s$ be divided into three consecutive closed arcs $s_1, s_2$ and $s_3$ of equal length. The conditions (a) and (b) remain the same, we only replace $s$ by $s_i$, $i=1,2,3$, in (a), but (c) and (d) are replaced by the following:

(c′) On $s_i$ we have

$$\Phi^1=\Psi^1=\Psi^2,\quad \Phi^2=\Phi^3=\Phi^4 \quad \text{if } i=1,$$

$$\Phi^1=\Phi^2=\Psi^1,\quad \Phi^3=\Phi^4=\Psi^2 \quad \text{if } i=2,$$

$$\Phi^1=\Phi^2=\Phi^3,\quad \Phi^4=\Psi^1=\Psi^2 \quad \text{if } i=3,$$

(d′) On $\operatorname{int} s_i$ we have

$$\Phi^1_3>0>\Phi^2_3 \quad \text{if } i=1,$$

$$\Phi^2_3>0>\Phi^3_3 \quad \text{if } i=2,$$

$$\Phi^3_3>0>\Phi^4_3 \quad \text{if } i=3.$$

The identifications on $s_1(s_3)$ will give a cave connection of $W_2$ $(W_1)$ to the layer between $\Phi^1B$ and $\Phi^2B$ ($\Phi^3B$ and $\Phi^4B$). The identifications on $s_2$ are similar to the case (2).

Case (4): Let first the last in $\mathscr{G}^2_Q$, say $C$, be positive. There are two other 2-simplexes in $\mathscr{G}^2_A$ with $s$ as a side, say $D$ and $E$. Suppose $D$ is the one with non-negative $x_3$-coordinates. According to the definition, $D$ and $E$ are also positive. The conditions for the maps $\Psi^n_C, \Psi^n_D, \Psi^n_E$ on $s$ are the following:

(a′) The endpoints of $s$ are fixpoints for $\Psi^n_C, \Psi^n_D, \Psi^n_E$.

(b′) On $s$, $\pi\circ\Psi^n_C$ is the identity.

(c″) On $s$, $\Psi^1_C=\Psi^1_D=\Psi^2_D$, $\Psi^2_C=\Psi^1_E=\Psi^2_E$.

(d″) On $\operatorname{int} s$, $(\Psi^1_C)_3>0>(\Psi^2_C)_3$.

Let then $C$ be negative. The 2-simplexes $D$ and $E$ are then also negative. In (a′) and

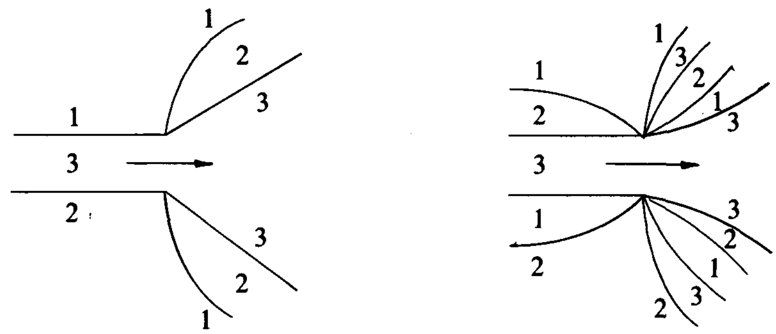


Fig. 7.3.

(b′) we replace $\Psi$ by $\Phi$ and $n$ by $m$. The conditions (c″) and (d″) are replaced by the following:

(c‴) On $s$, $\Phi^1_C=\Phi^2_C=\Phi^1_D=\Phi^2_D=\Phi^3_D=\Phi^4_D$, $\Phi^3_C=\Phi^4_C=\Phi^1_E=\Phi^2_E=\Phi^3_E=\Phi^4_E$.

(d‴) On int $s$, $(\Phi^2_C)_3>0>(\Phi^3_C)_3$.

These indentifications are illustrated in Figure 7.3. The conditions for the maps $\Phi^m$ and $\Psi^n$ on $s$ in each case is now completed. If $Q\notin\Omega_0$, the geometry is slightly different, but we achieve the same topological connections by modifying the conditions (b), (b′), (d), (d′), (d″), and (d‴). When we let $Q$ run over $\Omega$, we obtain the identifications for all possible $\Phi^m$ and $\Psi^n$ maps in *all* sides $s$. This is because in case (4) above we gave also conditions for maps $\Phi^m_D$, $\Phi^m_E$, $\Psi^n_D$, $\Psi^n_E$ on $s$ which is an excluded side for some $Q', Q''\in\Omega$ with $D\in Q', E\in Q''$.

We have above given a topological description of the sheet maps $\Phi^m_B$ and $\Psi^n_C$ for $B$ and $C$ in $\mathscr{G}^2_A$. On the boundary $|\dot G_A|$ these sheet maps are the identity. Hence we can simply put together all sheet maps for the elements in $\mathscr{G}^2_\infty$. The upper indexes in the maps $\Phi^m_B$ and $\Psi^n_C$ are not important, all that matters is what classes $\delta_j$ are attached to each sheet. Let the set of all sheets for elements in $\mathscr{G}^2_\infty$ be denoted by $\mathscr{H}^2$. According to the construction of $\mathscr{H}^2$, the set $\mathbf{R}^3\setminus|\mathscr{H}^2|$ contains eight components $W_1, W_2, W_3(0), \ldots, W_3(5)$ such that $W_1'\subset W_1$ etc. (see (6.4)). A notion called cave part is defined in Section 7.6.

7.3. *Definition of the map on the sheets.* We define the notion of a positively oriented sheet with respect to $\delta_j$ topologically as was done for elements in $H^2_1$ in Section 4.3. Let $D$ be a sheet with classes $\delta_j$ and $\delta_i$. Let $D$ be positively oriented with respect to $\delta_j$. Then $D$ is negatively oriented with respect to $\delta_i$. Recall the quasiconformal maps $\varkappa_2$ and $\varkappa_3$ from the formulae (4.8) and (4.9). Let $\varkappa_1$ be the identity in $\bar{U}_1$. Let the sheet map of $D$ be $\theta: B \to D$ which means that $\theta$ is some $\Phi_B^m$ or $\Psi_B^n$. Then we set

$$f|D = \varkappa_j \circ \zeta \circ \frac{1}{t_B} \varphi_B \circ \theta^{-1}. \tag{7.4}$$

According to the definitions of the maps $\varkappa_j$, we may rephrase (7.4) by

$$f|D = \varkappa_i \circ \varkappa \circ \zeta \circ \frac{1}{t_B} \varphi_B \circ \theta^{-1}. \tag{7.5}$$

Here $\varkappa$ is the reflection in $\mathbf{R}^2$ and the map $\zeta$ was defined in Section 4.3. We give more conditions for the sheet maps in Section 7.6 to guarantee that we are able to construct a quasiregular map.

7.6. *Final extension.* We shall now use more or less the principles in Section 6 to extend $f$ to the remaining parts $W_1 \setminus W_1'$ etc. However, one of the differences is that $f|\partial W_1$ etc. is not represented topologically by a map complex, but rather by something which could be called a generalized map complex. We shall confine ourselves to the part $W_1 \setminus W_1'$. The treatment in $W_2 \setminus W_2'$ etc. is similar.

We let $\mathscr{H}_1^2$ be the set of sheets which form the boundary $\partial W_1$, i.e. the sheets which have $\delta_1$ as one class. Let $Q$ be in some $\Omega$ for $\omega(A) \subset M_{k,k-1}$ where $A$ is in some $M_{kq}^2$. Let the classes of $Q$ be $\delta_2$ and $\delta_3$. Let the set of sheets for $B$ when $B$ runs over $\mathscr{G}_Q^2$ be $\Gamma_Q$. We call $\Gamma_Q \cap \mathscr{H}_1^2$ a *cave part* of $\mathscr{H}_1^2$. We also say that $|\Gamma_Q \cap \mathscr{H}_1^2|$ is a cave part of $|\mathscr{H}_1^2|$. Let then $Q$ have $\delta_1$ as one class and let $B \in \mathscr{G}_Q^2$ be negative. If the sheets $\Phi^1 B, \ldots, \Phi^4 B$ lie like in Figure 7.2, we call $\Gamma = \{\Phi^3 B, \Phi^4 B\} \subset \mathscr{H}_1^2$ (resp. $|\Gamma|$) a *hollow part* of $\mathscr{H}_1^2$ (resp. $|\mathscr{H}_1^2|$).

Let $\Gamma = \Gamma_Q \cap \mathscr{H}_1^2$ be a cave part of $\mathscr{H}_1^2$. Let $s = \langle a, b \rangle$ be the last side of the last in $\mathscr{G}_Q^2$. We may assume that the sheet maps take $s$ onto $\langle a, c \rangle \cup \langle c, b \rangle$ and $\langle a, d \rangle \cup \langle d, b \rangle$ for some points $c$ and $d$, and that $Z = \operatorname{int}(\{a, b, c\rangle \cup \langle a, b, d\rangle) \subset W_1$. We call the component $R$ of $W_1 \setminus Z$, with $|\Gamma|$ contained in $\partial R$, the *cave part* of $W_1$ corresponding to $\Gamma$. We choose a point $e \in W_1 \setminus \bar{R}$ similarly as in the beginning of Section 6.1 and write $X = \langle a, b, c, e \rangle$, $Y = \langle a, b, d, e \rangle$. There is an obvious decomposition of $\Gamma$ into the sets $\Gamma_C$ and $\Gamma_D$ where $C$ and $D$ in $M_1^2$ are cave opening simplexes for $Q$ (cf. Section 6.1). We may assume $c \in |\Gamma_C|$. A sort of $\delta_1$-inheriting is also defined from elements of $M_1^2$ into $\mathscr{H}_1^2$. To define it for $C$ we let $\mathscr{H}_1^2(C)$ be the set of sheets $\theta B$ in $\mathscr{H}_1^2$ for which $B \in \mathscr{G}_1|C$. We say that $\mathscr{H}_1^2(C, \delta_1) = \mathscr{H}_1^2(C) \cup \Gamma_C$ is $\delta_1$-*inherited* from $C$. If

$E \in M_1^2$ and if $E$ is not a cave opening simplex for any $Q$ with classes $\delta_2$ and $\delta_3$ we set $\mathscr{H}_1^2(E, \delta_1) = \mathscr{H}_1^2(E)$.

We shall next push $R$ into $W_1 \setminus \bar{R}$. Now $R$ is not locally connected on the boundary, but the boundary elements of $R$ have an obvious meaning. We perform a preliminary $PL$ homeomorphism $h$ of the domain $W_1$ which is the identity on $W_1 \setminus R$ and which shrinks $R$ slightly onto a domain $R'$ which is locally connected on the boundary. The boundary elements of $R$ are transferred to boundary points of $R'$ bijectively. Let $W^* = (W_1 \setminus R) \cup R'$. We can then modify the method in 6.1 to obtain a $PL$ homeomorphism $\Psi^*: \bar{W}^* \to \overline{W_1 \setminus R}$ such that $\Psi^*$ is the identity on $W_1 \setminus (R \cup X \cup Y)$. It is possible to choose the sheet maps $\theta$ in the beginning, the shrinking map $h$, and the map $\Psi^*$ such that the composed maps $\Psi^* \circ h \circ \theta$ (here $h$ is extended to the boundary elements), with $\theta: B \to B'$ running over the sheet maps giving the sheets $B'$ in $\Gamma$, are simplicial with respect to $B_{(1)}^{(1)}$. Recall that $B_{(1)}$ is the first canonical subdivision of $B$, see Section 3.3. Furthermore, we can do the construction so that

(1) the sheets in $\Gamma_C$ $(\Gamma_D)$ are taken into $\langle a, b, c \rangle$ $(\langle a, b, d \rangle)$ by the boundary element correspondence of $\Psi^* \circ h$,

(2) $\Psi^* \circ h$ is locally $\exp(\theta_1 \nu^3)$-bilipschitz where $\theta_1$ is an absolute constant.

Similarly we can define and push in hollow parts of $W_1$. We can perform all these pushings simultaneously and we get a map $\Psi_0$ of $W_1$ onto a domain $X_1$. If needed, we extend $\Psi_0$ to the boundary elements and use the same notation. The domain $X_1$ is locally connected at the boundary. The map $\Psi_0$ is the identity in $W_1'$ (see (6.4)). If the sheet maps are originally chosen in a suitable simple way, $\bar{X}_1$ can be mapped by a $PL$ homeomorphism $\Psi_1$ onto $\bar{V}_1$ (see Section 4.1) in a straightforward manner such that

(a) $\Psi_1$ is 10-bilipschitz,

(b) $\Psi_1 | W_1'$ is the identity,

(c) $|\mathscr{H}_1^2(C, \delta_1)|$ is taken onto $C$ by the boundary element correspondence of $\Psi_1 \circ \Psi_0$ for every $C \in M_1^2$.

(d) If $\theta: B \to B'$ is a sheet map with $B' \in \mathscr{H}_1^2$, then $\Psi_1 \circ \Psi_0 \circ \theta$ is simplicial with respect to $B_{(1)}^{(1)}$.

The map $f \,|\, |\mathscr{H}_1^2|$, given in (7.4) and considered as defined on the boundary elements of $W_1$, splits as $f \,|\, |\mathscr{H}_1^2| = F \circ \Psi_1 \circ \Psi_0$, where $F: |M_1| \to S^2$ is a discrete open map. Since $f$ maps each sheet homeomorphically onto $S_+^2$ or $S_-^2$, we get a topological description of $F$ if we study $F^{-1}S^1$. In particular, we shall describe how $F^{-1}S^1$ can be formed from $|\mathscr{G}_1^1|$. Each hollow part corresponds to a configuration of $F^{-1}S^1$ shown in Figure 7.4

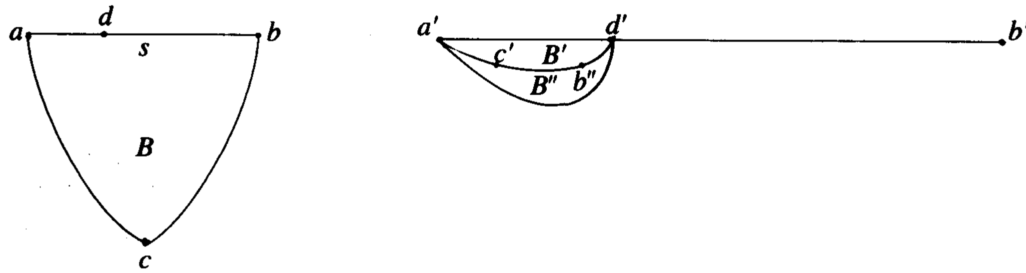


Fig. 7.4.

where we have assumed that the corresponding connection side $ab$ in $\mathcal{G}^1$, call it $s$, is an $\alpha\beta$-side. Here and in other figures below we indicate points in the classes $\alpha, \beta, \gamma$ by letters $a, b, c$, possibly with subscripts, when they are vertices of $\mathcal{G}_1$. Corresponding points in $F^{-1}S^1$ are denoted by adding one or more primes. We use the latter principle also for elements in $\mathcal{G}_1^2$. We see that the hollow part corresponds for $F$ to a simple cover with branching at $a'$ and $d'$, and we obtain this part of $F^{-1}S^1$ by replacing the subarc $ad$ in $ab$ by the boundaries of the pair $B', B''$. We may also do the construction such that $a'=a$, $b'=b$, $d'=d$, and the side $s$ is contained in $F^{-1}S^1$ as shown in Figure 7.4.

To study to what a cave part corresponds for $F$ we let $Q$ be an element of an $\Omega$ such that $|Q|$ has classes $\delta_2$ and $\delta_3$. Let $B$ be the last in $\mathcal{G}_Q^2$ and the side $ab$ the last side $r$ of $B$ (Figure 7.5). The side $r$ is also a side of $\mathcal{G}_1$ and it is always a line segment. To see the principle here it is enough to consider the case in Figure 7.5 where a part of $\mathcal{G}_Q$ is presented. The arrows have the same meaning as in Figures 7.1 and 7.3, and show also the increasing order in $\mathcal{G}_Q^2$. The 2-simplex $B$ corresponds to a pair $B', B''$. The set $B' \cup B''$ has one hole because there is one arrow entering $B$. The 2-simplex $C$ corresponds to a pair $C', C''$ such that $C' \cup C''$ has two holes because there are two arrows entering $C$. The outer boundary curve of $C' \cup C''$ coincides with the boundary of the hole of $B' \cup B''$. This way we can go through all of $\mathcal{G}_Q^2$ because the order is based on a tree. The part of $F^{-1}S^1$ corresponding to the cave part is obtained by replacing the side $r$ by the 1-dimensional configuration inside the outer boundary curve of $B' \cup B''$. Points like $d', e', d_1', e_1'$ are branch points for $F$.

We may perform the map $\Psi_1 \circ \Psi_0$ so that apart from the sides like $s$ and $r$ above all other sides in $\mathcal{G}^1$ remain untouched in $F^{-1}S^1$. The map $F$ can be deformed in an obvious way to obtain a map represented by a refined map complex as follows. For a hollow part we move the simple cover $F|B' \cup B''$ (Figure 7.4) such that the branch point $d'$ moves to $b'$. Then also $b''$ moves to $b'$ (cf. Section 5.1). For a cave part we deform

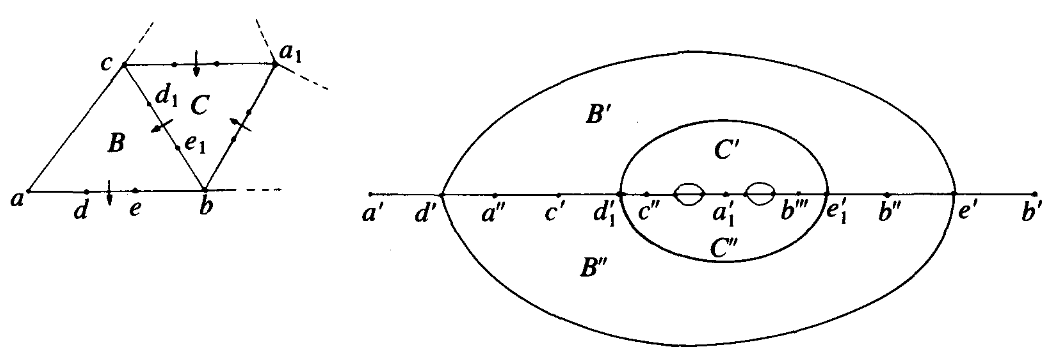


Fig. 7.5.

similarly. In Figure 7.5 we move first the branch point $d'$ to $a'$ and the branch point $e'$ to $b'$. Then $a''$ moves to $a'$ and $b''$ moves to $b'$. In the next step we move $d_1'$ to $c'$ and $e_1'$ to $b'$ and then also $c''$ moves to $c'$ and $b'''$ to $b'$. We continue similarly. In Figure 7.6 we see the part of $F^{-1}S^1$ shown in Figure 7.5 after the deformation. The line segment $r$ is required to divide it in an obvious manner. All hollow and cave parts for $F$ can be deformed simultaneously. As in Section 6.2 we define $t$-levels $S_t, t\in[-1,0]$, in the layer $\Lambda''_{-1}$ between $|M_1|$ and $|N_1|$, in fact we can transfer the $t$-levels $S_t, t\in[0,1]$, between $|\nu^2 M_1|$ and $|\nu^2 N_1|$ by the similarity $x\mapsto\nu^{-2}x$. The local product representation of the layer $\Lambda''_{-1}$ (see Section 6.2) makes it possible to see each $B\in M_1^2$ on $S_t$ as a subset $B_t$ for $t\in[-1,0]$. As in Section 6.2 we transfer the deformations to $t$-levels. For this, let $F_{-1}=F$. We do the deformations described above during $t\in[-1,-1/2]$ and obtain a family of maps $F_t\colon S_t\to S^2$, $t\in[-1,-1/2]$. Now $F_{-1/2}$ is represented by a refined map complex, call it $\Gamma'_{-1/2}$.

Our final task is to deform $F_{-1/2}$ on the $t$-levels, $t\in[-1/2,0]$, to the map $g_0\colon|N_1|\to S^2$. The map $g_0$ is represented by the map complex $H_1$. We have made the construction so that the number $\sigma(H_1|\psi B)$ of 2-simplexes in $H_1|\psi B$ equals $\sigma(\Gamma'_{-1/2}|B_{-1/2})$ for every $B\in M_1^2$. In order to do the deformation from $F_{-1/2}$ to $g_0$ similarly as in Section 6.2, we should in addition have only $\alpha\gamma$-sides on the boundary and $\alpha$-points at vertices for each $B_{-1/2}$. The latter condition is satisfied, but the first is not on those parts which correspond to cave parts. This difficulty is circumvented by the observation that cave opening simplexes for a $Q$ with classes $\delta_2$ and $\delta_3$ are grouped into pairs $B, B'$ of elements in $M_1^2$ such that $E=B\cup B'$ is homeomorphic to a disk. For such $E\,\Gamma'_{-1/2}|\dot{E}_{-1/2}$ has only $\alpha\gamma$-sides. We replace each cave opening simplex $B\in M_1^2$ by such $E\supset B$ and get from $M_1^2$ a new set $\tilde{M}_1^2$ with $|\tilde{M}_1^2|=|M_1|$. We decompose $\tilde{M}_1^2$ similarly as $\nu^2 M_1^2$ was decomposed into sets $\mathscr{A}_1, \ldots, \mathscr{A}_i$ in Section 6.2, and transfer all to the

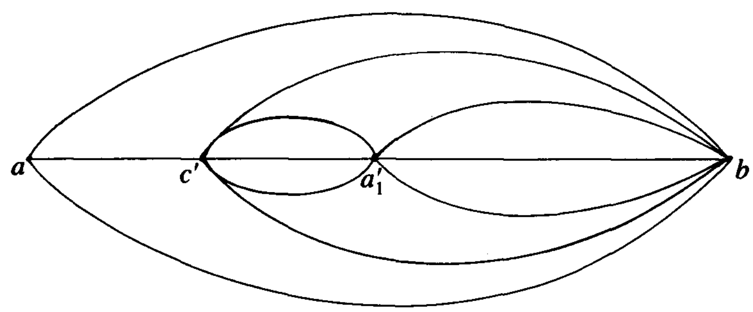


Fig. 7.6.

$-1/2$-level. Also the condition $\delta_2<1/50$ in (5) of Lemma 5.3 causes here a little problem because it may not be automatically satisfied for our application of Lemma 5.3. This can be arranged simply by taking it into account in the construction of $\mathscr{G}_\infty$. After these remarks we can adjust the sides and apply the Deformation lemma 5.3 as in Section 6.2 to get a family of maps $F_t\colon S_t\to S^2$, $t\in[-1/2,0]$, with $F_0=g_0$.

It remains to define $f$ in $W_1\setminus W_1'$. We write $\Psi_2=\Psi_1\circ\Psi_0$ and set

$$f|\Psi_2^{-1}S_t=(s_t, F_t\circ\Psi_2|\Psi_2^{-1}S_t),\quad t\in[-1,0] \tag{7.7}$$

where the coordinate presentation is as in (6.3). Recall $s_t=e^{-1}\exp\nu^{t+1}$. The construction has been carried out so that $f|W_1\setminus W_1'$ is $K$-quasiregular, $K$ depending only on $\lambda$ and $\nu$.

Combining (7.7) with the definition (6.3) in $W_1'$ we have completed the definition of $f$ in $\bar{W}_1$. The definitions of $f$ in the domains $W_2$, $W_3(h)$, $h=0,\ldots,5$, are similar. Since all these match on common boundaries, we have defined a $K$-quasiregular map $f$ of $\mathbf{R}^3$. By construction $f$ omits $u_2,u_3\in\mathbf{R}^3$ and the proof of Theorem 1.2 for $p=2$ is hereby completed.

## 8. The case $p>2$

In this final section we shall sketch the proof of Theorem 1.2 for $p>2$ by indicating the main changes to be made to our construction for $p=2$. The idea will be to replace the caves both in Section 2 and Section 7 by caves with $p-1$ passages (the original are said to have one passage). On the other hand, the parts involving deformation of maps remain almost unchanged.

8.1. *Cave complexes with $p-1$ passages.* The notions $l$-triangle, $l$-subdivision, and $K$-tree from Section 2.2 are all defined similarly by changing only constants, for

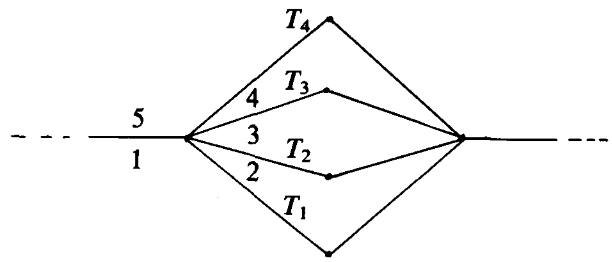


Fig. 8.1.

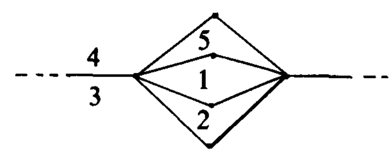


Fig. 8.2.

example $v$ will be chosen larger for larger $p$. In the cave complex $K_L$ in Section 2.3 we replaced those $A \in K^2$ with $A \cap L \neq \emptyset$ by two 2-simplexes $A_-$ and $A_+$ and in the construction we used two normals $-u$ and $u$. To get a *cave complex with* $p-1$ *passages* we use vectors $r_1 u, \ldots, r_p u$, where $-1 \leq r_1 < \ldots < r_p \leq 1$, instead of $-u$ and $u$. The new cave complex $K_L$ has $p-1$ bounded components.

It is clear how bending will be performed in the general case. The opening of a cave complex with $p-1$ passages can be described as follows. Let $P$ be a cave complex with $p-1$ passages based on an $(l+1)$-subdivision. Let $T_1, \ldots, T_p \in P^2$ be the $(l+1)$-triangles obtained from a 2-simplex $T$ by the replacement operation and let $T_1 \cap \ldots \cap T_p$ be a side $s$ which is also a side of $S \in P^2$, $S \neq T_i$, $i=1, \ldots, p$. We write $T_{p+1}=S$. Let classes $\delta_j$, $j=1, \ldots, p+1$, be given in a similar way as before for 2-simplexes of $P$ such that the $\delta_j$'s appear in a cyclic order $(\mathrm{mod}\,(p+1))$ around a side like $s$. In Figure 8.1 we see the side profile for $p=4$ where $T_5$ has classes $\delta_5$ and $\delta_1$ (the classes $\delta_j$ is denoted by $j$ in Figure 8.1). In general $T_{p+1}$ may have any pair $\delta_j, \delta_{j+1}$ $(\mathrm{mod}\,(p+1))$ of classes.

Suppose $T_{p+1}$ is the last in a cave base $Q$ for an opened cave complex $I=K_L$ with $p-1$ passages. Opened means then that the passages have connections to the passages of $P$ in the same order, i.e. the side profile for the passages of $I$ are like in Figure 8.1. The construction to obtain these connections is similar to the one in Section 2.4. The 2-simplexes $T_1, \ldots, T_p$ are called cave opening simplexes for $Q$. If some $T_i$, $i=1, \ldots, p$, is the last of a cave base $Q'$ for an opened cave complex $I'$, the connections of the passages of $P$ to those of $I'$ are similar. For example, if $i=3$ in Figure 8.1, then the classes $\delta_j$ in the passages of $I'$ appear as in Figure 8.2. These connections are obtained by a suitable modification of the construction in Section 2.4. The connections corresponding to various 2-simplexes $T_1, \ldots, T_{p+1}$ are here also performed at a sufficiently large $\varrho$-distance from each other to avoid interaction.

The cave refinement operation is generalized in an obvious manner. The same is true for the inheriting operation $\omega$. The $\delta_j$-inheriting operation $\omega_j$ is defined by the same

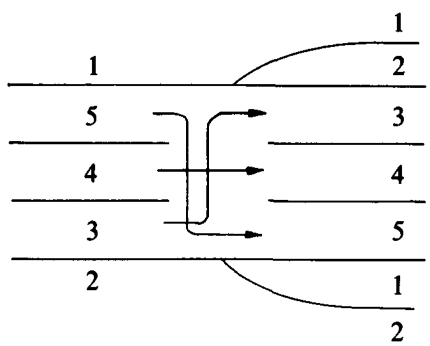


Fig. 8.3.

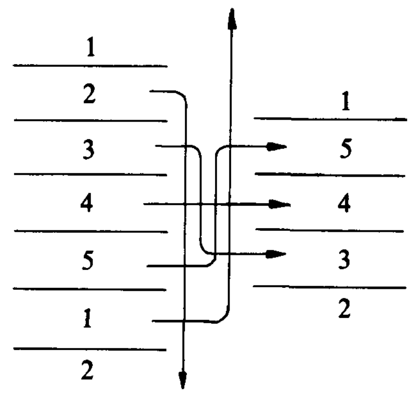


Fig. 8.4.

rules as before ((1) and (2) in Section 2.6). Note that now we may have plenty of $Q$'s for one $A$ in (2) in Section 2.6 whereas earlier there were at most one.

8.2. *Definition on level surfaces.* Section 3 can be carried out for the general case with only obvious changes. As a result we obtain (by the notation in Section 4.1) $M_{kq}(0)$, $M_{kqj}(0)$ and $G_{kqj}(0)$, $j=1,\dots,p+1$. As in Section 3, we let $M_{k0}$ have classes $\delta_1$ and $\delta_2$. The set $\mathbf{R}^3\setminus|M_\infty|$, where $M_\infty=M_1\cup\dots\cup M_{p+1}$ (see Section 4.1), has now $2+6(p-1)$ components $V_1, V_2, V_j(h)$, $j=3,\dots,p+1$, $h=0,\dots,5$. The level surfaces in Section 4.2 are defined as before with obvious changes. To define $f$ on these surfaces we only need a discussion of the quasiconformal maps $\varkappa_j$ since the maps $g_{2i}, v_{2i}, \bar{g}_{2i}$ are defined formally as before in (4.4), (4.6), and (4.7). Let $w_p$ be the Möbius transformation of $\bar{\mathbf{R}}^3$ which keeps $S^1$ fixed, maps $S^2_+$ into $\bar{B}^3$, and for which $S^2_+$ and $w_p S^2_+$ form a dihedral angle $\pi/p$. Let $U'_{p+2-j}$ be the bounded domain bounded by $w_p^{j-1}S^2_+$ and $w_p^j S^2_+$, $j=1,\dots,p$. Let $u_{p+1}$ be the midpoint of the part of the $x_3$-axis which lies in $U'_{p+1}$. We define a homeomorphism $\varkappa_{p+1}\colon \bar{U}_1\to\bar{U}'_{p+1}\setminus\{u_{p+1}\}$ such that it maps $U_1=\mathbf{R}^3\setminus\bar{B}^3$ onto $U_{p+1}=U'_{p+1}\setminus\{u_{p+1}\}$ quasiconformally and $\varkappa_{p+1}|S^2_+=w_p|S^2_+$, $\varkappa_{p+1}|S^2_-=\varkappa|S^2_-$. Then set $\varkappa_{p+2-j}=w_p^{j-1}\circ\varkappa_{p+1}$, $j=2,\dots,p$. As before, $\varkappa_1\colon \bar{U}_1\to\bar{U}_1$ is the identity. We write $u_j=w_p^{p+1-j}(u_{p+1})$, $U_j=U'_j\setminus\{u_j\}$, $j=2,\dots,p$. The only change after this in the definition of $f$ on the level surfaces is that we replace the index 3 by $j$ and let $j$ run from 3 to $p+1$ in (4.9).

8.3. *Final glueing.* The extension of the definition of $f$ to the layers between the level surfaces is done in the general case in the same manner as in Section 6. However,

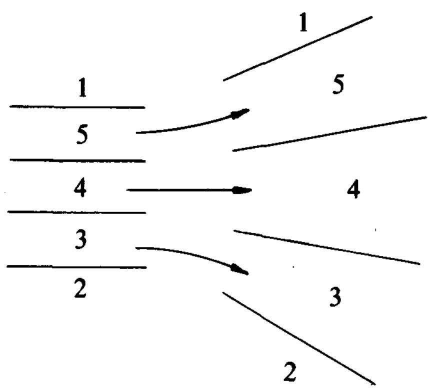


Fig. 8.5.

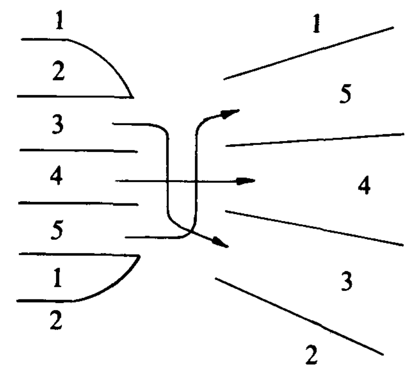


Fig. 8.6.

comments must be made to the final glueing presented for $p=2$ in Section 7. We define $\mathscr{G}_j$ on $M_j^*$ for $j=1,\ldots,2+6(p-1)$, similarly as in Section 7.1 and set $\mathscr{G}_\infty=\mathscr{G}_1\cup\ldots\cup\mathscr{G}_{2+6(p-1)}$. Order in a $\mathscr{G}_Q$, $Q\in\Omega$, and positive and negative elements in $\mathscr{G}_\infty^2$ are defined as in Section 7.1 ((mod 3) is replaced by (mod $(p+1)$)). Similarly we adopt the notion of a connection side for a negative $B\in\mathscr{G}_Q^2$ from Section 7.2.

Each positive element in $\mathscr{G}_\infty^2$ will be replaced by $p$ sheets and each negative by $p+2$ sheets. By a technique similar to that in Section 7.2 we make identifications on the boundaries of the sheets. The principle is that connections between the layers between the sheets is done so that the $\delta_j$-classes appear in cyclic order and the direction of this order is opposite for neighboring elements in $\mathscr{G}_j^2$ for each $j$. In the case (1) in Section 7.2 the identifications are as before. In the case (2) the connections are shown in Figure 8.3 for $p=4$, which illustrates the general case adequately too. At a connection side (case (3)) these connections are shown in Figure 8.4 for $p=4$. We see that now we have to switch the order of the layers in the classes $\delta_3$ and $\delta_5$. In the case (4) we form connections between the passages in a cave of $M_\infty$ and the passages in a cave part formed by the sheets so that the same classes $\delta_j$ will be connected. If in (4) the last in $\mathscr{G}_Q^2$ is positive, the connections are made as in the opening of a cave complex with $p-1$ passages. This is illustrated in Figure 8.5 for $p=4$. If the last $\mathscr{G}_Q^2$ is negative, we have again to switch arround the order of passages as shown in Figure 8.6 to get the right cyclic orders. This completes the description of the set $\mathscr{H}^2$ of sheets. The set $\mathbf{R}^3\setminus|\mathscr{H}^2|$ has now $2+6(p-1)$ components $W_1, W_2, W_j(h)$, $j=3,\ldots,p+1$, $h=0,\ldots,5$. The rest of Section 7 can be carried out like before and this ends the proof of Theorem 1.2 for the general case.

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