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Resolvent of the Laplacian on strictly pseudoconvex domains

by

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Introduction

Pseudoconvex domains in $\mathbf{C}^{n+1}$ are the natural regions in which the $\bar\partial$ equations can be generally solved. Any smooth strictly pseudoconvex domain carries a complete Kähler metric given by the mixed second derivatives of the logarithm of a plurisuperharmonic defining function

$$g = -\bar\partial\partial \log r. \tag{1}$$

In this paper we investigate in detail the kernel of the resolvent of the Laplacian of such 'Bergman' type metrics. Although we only treat the Laplacian acting on functions below, to limit the algebraic complexity of the problem, the general case of (p,q)-forms can be handled by the same method. Since an application of Hodge theory allows the $\bar\partial$ equations to be solved in terms of the inverse of such a Laplacian this leads to a detailed description of the Schwartz kernel of a particularly natural inverse of $\bar\partial$. This, and other applications to spectral and scattering theory, will be discussed in detail elsewhere. Here we only note how the Dirichlet problem for the Laplacian can be solved, with the Poisson operator arising as a limit of the resolvent.

Our basic methods are geometric and constructive. We start by abstracting from the Kähler metrics of the form (1) a class of complete metrics on the interiors of compact manifolds with boundary. As a first step the $\mathscr{C}^\infty$ structure on the domain, $\mathscr{U}$, is altered by adjoining the square root of ϱ. The new manifold with boundary so obtained

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is denoted $\mathscr{U}_{1/2}$, we call it the square root of $\mathscr{U}$. The significant behaviour of the metric at the boundary of $\mathscr{U}_{1/2}$ is captured by a conformal class of 1-forms, represented by Θ, defined at the boundary. For strictly pseudoconvex domains Θ induces the contact structure on the boundary of $\mathscr{U}$ and is determined by it.

Once the form Θ has been isolated, we proceed much in the spirit of [20], [18] and [19]. Thus we identify a Lie algebra of vector fields, $\mathscr{V}_\Theta$, which vanish at the boundary in a manner encoded by Θ (see (1.4) and (1.5)). This is an example of a 'boundary fibration structure' as described in general in [19]; in fact it is a motivating example. The Lie algebra $\mathscr{V}_\Theta$ forms the space of all smooth sections of a vector bundle, ${}^\Theta TX$, over the manifold with boundary X. Over the interior ${}^\Theta TX$ is canonically identified with the usual tangent bundle TX. By a Θ-metric on X we mean a metric on the interior of X which extends to a smooth, non-degenerate, fibre metric on ${}^\Theta TX$. The Kähler metrics (1) are Θ-metrics with $X=\mathscr{U}_{1/2}$. The Laplacian of a Θ-metric is a Θ-differential operator, in the sense that it is in the enveloping algebra, Diff_Θ, of $\mathscr{V}_\Theta$. The main construction we make is the 'microlocalization' of Diff_Θ (or $\mathscr{V}_\Theta$) to the class of Θ-pseudodifferential operators on X. The resolvent of the Laplacian is shown to lie in this class.

The main step in the definition of the Θ-pseudodifferential operators is the replacement of the product, $X^2=X\times X$, by the Θ-stretched product, which we denote X^2_Θ. This is a manifold with corners up to codimension three (X^2 has corners up to codimension two) which comes equipped with a blow-down map:

$$\beta^{(2)}_\Theta : X^2_\Theta \to X^2. \tag{2}$$

The blow-down map is a diffeomorphism from the complement of one of the boundary hypersurfaces of X^2_Θ (the front face, $\mathrm{ff}(X^2_\Theta)$) onto the complement in X^2 of the boundary of the diagonal. In fact X^2_Θ and $\beta^{(2)}_\Theta$ are constructed by parabolically blowing up the boundary of the diagonal in X^2. The choice of the submanifold blown up and the parabolic manner of this blow-up are determined, by general principles, from the Lie algebra $\mathscr{V}_\Theta$. Just as normal blow-up is the invariant version of the introduction of polar coordinates, so parabolic blow-up corresponds to the introduction of polar coordinates around the submanifold with some variables, in this case just one fixed by Θ, scaling in a quadratic rather than a linear fashion.

The relevance of the Θ-stretched product can be seen geometrically in terms of the distance function d on X^2, for a metric (1). In essence X^2_Θ is the simplest manifold, as in (2), with the property that d is simple when lifted to it in the sense that

$$0<\varrho_{\mathrm{lb}}\,\varrho_{\mathrm{rb}}[\beta^{(2)*}_\Theta \cosh d] \in \mathscr{C}^\infty(X^2_\Theta). \tag{3}$$

Here ϱ_{lb} and ϱ_{rb} are defining functions for the two boundary hypersurfaces (other than $\mathrm{ff}(X^2_\Theta)$) of X^2_Θ. Not surprisingly, given (3), the kernel of the resolvent family has a simple structure when lifted to X^2_Θ. This indeed is our basic result.

To make the conclusions more precise, and then to prove them, it is necessary to describe the spaces of Θ-pseudodifferential operators. We consider, for the moment, operators acting on metric half-densities on X (for a Θ-metric). The replacement of X^2 by X^2_Θ has the effect of physically separating the two types of singularities which occur in the resolvent kernel, namely the diagonal and boundary singularities. Let m be a real number and z, z' be complex numbers. Then a typical class of operators we consider is denoted by

$$\Psi_\Theta^{m;z,z'}(X; dg^{1/2}). \tag{4}$$

The Schwartz kernel of $A \in \Psi_\Theta^{m;z,z'}(X; dg^{1/2})$, K_A, is a half-density on X^2. To characterize the class (4) we lift the kernels to X^2_Θ, where we let $\varkappa_A$ denote that of A. Then for a fixed (though slightly singular) half-density, μ,

$$A \in \Psi_\Theta^{m;z,z'}(X; dg^{1/2}) \ \Leftrightarrow\ \varkappa_A = \varkappa'_A\mu,\ \varkappa'_A\varrho_{\mathrm{lb}}^{-z}\varrho_{\mathrm{rb}}^{-z'} \in I^m(X^2_\Theta, \Delta_\Theta). \tag{5}$$

Here $\Delta_\Theta \subset X^2_\Theta$ is the 'lifted diagonal', the closure in X^2_Θ of the interior of the diagonal. It is an embedded submanifold meeting the boundary transversally in the interior of $\mathrm{ff}(X^2_\Theta)$. This is an essential property of X^2_Θ, as it allows us to study the singularities of the kernel along the diagonal and along the boundary successively. The space $I^m(X^2_\Theta, \Delta_\Theta)$ consists of the conormal distributions, of order m, with respect to this submanifold, precisely the space which occurs in the usual theory of pseudodifferential operators. For example if $m = -\infty$ then $I^{-\infty}(X^2_\Theta, \Delta_\Theta) = \mathscr{C}^\infty(X^2_\Theta)$.

There can also be some similar, but simpler, terms in the kernel. Consider the 'smoothing operators', with power law growth, acting on metric half-densities:

$$A \in \Psi^{-\infty;z,z'}(X; dg^{1/2}) \ \Leftrightarrow\ \varrho_{\mathrm{lb}}^{-z}\varrho_{\mathrm{rb}}^{-z'}K_A = a(dg)^{1/2} \otimes \nu^{1/2}, \quad a \in \mathscr{C}^\infty(X^2), \tag{6}$$

where $\nu^{1/2}$ is a half-density such that $0 \neq dg^{1/2}\nu^{1/2} \in \mathscr{C}^\infty(X; \Omega)$. As the kernels in (6) are conormal on X^2 they are not closely related to the Θ-structure (apart from the bundle on which they act) and are residual terms.

At each $p \in X$ the fibre of the structure bundle, ${}^\Theta T_p X$, is a Lie algebra, on which any Θ-metric induces a Euclidean inner product. Two metrics (possibly on different manifolds) will be said to have the same model at respective boundary points if the fibres, at those points, of the structure bundles are linearly isomorphic simultaneously both as Lie algebras and metrically.

Set

$$\mathcal{P}=\left\{\frac{1}{4}m; m\equiv 1,3 \bmod 4,\ m<2(n+1)\right\}\cup\{-\mathbf{N}_0\}$$

and put $N=\dim X+1$.

MAIN THEOREM. *Suppose g is a $\mathscr{C}^\infty$ Θ-metric on a compact manifold with boundary, X, and that for some non-vanishing smooth function $\chi\in\mathscr{C}^\infty(X)$ the metric is modeled at each $p\in\partial X$ on the lift to $\mathbf{CB}^{n+1}_{1/2}$ of the metric of constant holomorphic sectional curvature $-\chi^{-2}(p)$ on the ball, then the modified resolvent family of the Laplacian, Δ_g, acting on metric half-densities, is a meromorphic family*

$$\begin{aligned} R(s)=[\Delta_g-\chi^2 s(n+1-s)]^{-1}&\in\Psi_\Theta^{-2;2s-N/2,2s-N/2}(X;dg^{1/2})\\ &+\Psi^{-\infty;2s-N/2,2s-N/2}(X;dg^{1/2}),\quad s\in\mathbf{C}\setminus\mathcal{P}.\end{aligned}\tag{7}$$

Here the operator $R(s)$ is defined as a bounded operator on L^2 when $\Re s>(n+1)/2$. The non-positive integers $-\mathbf{N}_0=\{0,1,2,\dots\}\subset\mathcal{P}$ are excluded because the resolvent kernel for the Bergman metric itself has poles at these points and accidental multiplicity effects lead us to exclude the non-positive quarter-integers in the general case. The meromorphy of (7) should be considered as a refined description of the spectral theory of Δ_g, really a form of scattering theory; this, and the behaviour near $-\mathbf{N}_0$, is pursued in more detail elsewhere. The poles of $R(s)$ outside $\mathcal{P}$ are all of finite rank.

For fixed dimension all the metrics (1), lifted to $\mathcal{U}_{1/2}$, have as model at every boundary point the metric of constant holomorphic sectional curvature -4 on the ball, $\mathbf{CB}^{n+1}\subset\mathbf{C}^{n+1}$, lifted to $\mathbf{CB}^{n+1}_{1/2}$. Thus we can take $\chi=1/2$ and then $R(s)$ is essentially the resolvent family, apart from the rewriting of the spectral parameter. The same is true for certain infinite volume, cuspless, qotients $\Gamma\backslash\mathbf{CB}^{n+1}$, $\Gamma\subset\mathrm{SU}(n+1,1)$ being a suitably 'sparse' discrete group. In these cases we can also show that the singularities at the true negative quarter-integers are, at worst, finite order poles, so $\mathcal{P}$ in (7) can then be replaced by $-\mathbf{N}_0$. Graham, in [9], analyzed the resolvent kernels of a family of invariant operators on the ball, including the Bergman Laplacian. He conjectured an extension of his results to general strictly pseudoconvex domains which is, essentially, born out by the Main Theorem above.

The proof of (7) is, as noted above, essentially constructive. A rather precise parametrix for $\Delta_g-\chi^2 s(n+1-s)$ is produced using the symbolic properties of the spaces of Θ-pseudodifferential operators. In the process of the construction we need slightly larger spaces than described in (5), by allowing more complicated expansions at the boundaries. These spaces are described in terms of an index family, $E=E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}$, consisting of an index set at each boundary hypersurface. An index set $E_* \subset \mathbf{C}\times\mathbf{N}_0, \mathbf{N}_0=\{0,1,\dots\}$ fixes the terms which may occur at the corresponding boundary, so if $(z,m)\in E_*$ then a term $\varrho_*^{z+k}(\log\varrho_*)^l$ may occur, where ϱ_* is a defining function for that boundary, k and l are non-negative integers and $l\leqslant m$. Using these more general classes of operators we also give a result analogous to the theorem above for the Kähler–Einstein metric of the form (1), where ϱ is the solution of the complex Monge–Ampère equation. This does not quite fall under the result as stated because the defining function in (1) may have logarithmic terms (see [8], [3] and [16]). Similar remarks apply to the Bergman metric for a general strictly pseudoconvex domain (see [7]).

The space of Θ-pseudodifferential operators (4) has four distinct symbol maps, corresponding to the singularities at the (lifted) diagonal and at the three boundary hypersurfaces. The symbol map at the diagonal is the precise analogue of the usual symbol mapping for pseudodifferential operators and allows the construction of a 'uniform interior parametrix':

(8) $$E_0(s)\in\Psi_\Theta^{-2}(X;dg^{1/2}),\quad [\Delta_g-\chi^2 s(n+1-s)]E_0(s)-\mathrm{Id}\in\Psi_\Theta^{-\infty}(X;dg^{1/2}).$$

The absence of powers, z, z', compared to (4) means that the kernels here vanish to all orders at boundary hypersurfaces other than the front face, i.e.

(9) $$\Psi_\Theta^m(X;dg^{1/2})=\bigcap_{z,z'}\Psi_\Theta^{m;z,z'}(X;dg^{1/2}).$$

This is the 'small calculus'. However the remainder term in (8) is by no means compact.

Were the kernel of the remainder to vanish at the front face of X^2_Θ it would indeed be compact. To arrange this we use the symbol mapping at the front face. This is the normal homomorphism. It is a non-commutative symbol, taking values in the convolution algebra of the solvable Lie group, G_p, at each point $p\in\partial X$, with Lie algebra the fibre ${}^\Theta T_pX$. For the case of a contact structure, as arises for the metrics (1), the group is a semidirect product with the Heisenberg group:

(10) $$G_p=\mathbf{R}^+\ltimes H^n.$$

These groups can be thought of as parabolic subgroups of $\mathrm{SU}(n+1,1)$. This symbol map leads to a global 'model problem' on each G_p, which needs to be solved exactly to get a compact error. The model problem for the metrics (1) is always the Bergman metric on the ball, $\mathbf{CB}^{n+1} \subset \mathbf{C}^{n+1}$. Using the classical theory of rank one symmetric spaces we invert the corresponding Laplacian explicitly.

To prove (7) we also need to make use of the symbol map at the left boundary (involving the indicial operator) and iteration. This makes essential use of the composition properties of the calculus. To prove such composition properties we construct a triple Θ-stretched product, X^3_Θ, equipped with projection maps to X^2_Θ, corresponding to the three projections from X^3 to X^2. We then use general results on products, push-forward and pull-back of conormal distributions on manifolds with corners from [19] to show that the kernel of the composite operator lies in the appropriate class; for the convenience of the reader these results are recalled in an appendix.

In outline our analysis of the resolvent proceeds as follows. In the first section we introduce the general notion of a Θ-structure and in the second connect it to pseudo-convex domains by describing the square root of a manifold with boundary. Next we consider the tangent (finite dimensional) Lie algebras at boundary points and, in Section 4, the classification of the model problems for Θ-metrics. The next two sections describe the general properties of parabolic blow-up of a submanifold, first without and then in the presence of boundaries. In Section 7 the parabolic blow-up is used to define, and describe, the Θ-stretched product, X^2_Θ. The motivating example (and model problem) for our constructions is the Bergman metric on the ball in $\mathbf{C}^{n+1}$. This is discussed in detail in Section 8 where it is shown that lifting to X^2_Θ simplifies the structure of the resolvent kernel, i.e. the theorem above is proved by explicit computation for this special metric on the ball. Section 9 is devoted to a commutation result for parabolic blow-ups which is then used in Section 10 to define the triple Θ-stretched product, X^3_Θ; this is of fundamental importance in the proofs of the composition properties of the Θ-pseudodifferential operators. These are introduced in Sections 11 and 12. Some mapping properties are demonstrated in Section 13 and the calculus is applied to the construction of a parametrix for the Laplacian in Section 14, which contains the formal proof of the theorem above. It also contains the extension to the Monge–Ampère and Bergman metrics. In Section 15 the Poisson operator for the Dirichlet problem is discussed. For the ball similar results were obtained by Graham [9].

There are also two appendices. In Appendix A discrete subgroups of $\mathrm{SU}(n+1, 1)$ are discussed and the applications of the results of the paper to quotients of the ball are given. Along the way generalizations of results of Patterson and Sullivan on the Hausdorff dimension of the limit set are made. Appendix B contains a summary of that part of theory of conormal distributions for a manifold with corners (taken largely from [19]) which is used in the discussion of the Θ-pseudodifferential operators.

§ 1. Θ-structure

We start by describing a general class of algebras of vector fields (boundary structures) on smooth manifolds with boundary. These Lie algebras are fixed by the projective class of a non-vanishing 1-form, Θ, at the boundary. We call them Θ-structures. As we shall show in Section 3, algebras of this type arise naturally in the study of Laplace operators of complete Kähler metrics on complex manifolds with strictly pseudoconvex boundaries. However the definition of a Θ-structure which we adopt allows considerably more general cases than arise in this way.

Let X be a $\mathscr{C}^\infty$ manifold with boundary and let Θ be a 1-form on X defined only at the boundary, i.e.

$$\Theta \in \mathscr{C}^\infty(\partial X; T^*X). \tag{1.1}$$

Let $\iota: \partial X \hookrightarrow X$ be the inclusion of the boundary. We require that Θ be non-vanishing when pulled back to the boundary

$$\iota^*\Theta \neq 0. \tag{1.2}$$

Let ϱ be a defining function for the boundary of X:

$$\varrho \in \mathscr{C}^\infty(X), \quad \varrho \geq 0, \quad \partial X \equiv \{x; \varrho(x) = 0\}, \quad d\varrho \neq 0 \text{ at } \partial X. \tag{1.3}$$

Associated to the projective class, $[\Theta]$, of Θ is a space $\mathscr{V}_\Theta$ of $\mathscr{C}^\infty$ vector fields, V, on X. This space is defined by the following two conditions at the boundary, in which $\tilde{\Theta}$ is an extension of Θ to a smooth form on X:

$$V \text{ vanishes at } \partial X, \quad \text{i.e.} \quad V \in \varrho\mathscr{C}^\infty(X, TX) \tag{1.4}$$

$$\tilde{\Theta}(V) \in \varrho^2\mathscr{C}^\infty(X). \tag{1.5}$$

Proposition 1.6. *The space, $\mathscr{V}_\Theta$, of vector fields defined by the conditions* (1.4) *and* (1.5) *is a $\mathscr{C}^\infty(X)$-module and Lie algebra independent of the choice of the defining function, of extension $\tilde{\Theta}$ and of representative of the projective class, $[\Theta]$. Conversely the algebra $\mathscr{V}_\Theta$ determines the projective class of Θ.*

Proof. That $\mathscr{V}_\Theta$ is a $\mathscr{C}^\infty(X)$-module and is independent of the choice of both the defining function and of the representative of the projective class of $\tilde{\Theta}$ follows from the $\mathscr{C}^\infty(X)$-linearity of the defining conditions. If $\tilde{\Theta}'$ is another extension of Θ then the difference $\tilde{\Theta}'-\tilde{\Theta}=\varrho\Phi$ for some $\mathscr{C}^\infty$ 1-form Φ. Since (1.4) just means $\Phi(V)\in\varrho\mathscr{C}^\infty(X)$ for every such form, (1.5) holds for $\tilde{\Theta}'$ if and only if it holds for $\tilde{\Theta}$. Next we show that $\mathscr{V}_\Theta$ is a Lie algebra.

Certainly the commutator of two vector fields vanishing at the boundary also vanishes at the boundary, thus we only have to check (1.5) for the commutator. From (1.4) it follows that $d\varrho(V)\in\varrho\mathscr{C}^\infty(X)$ for each $V\in\mathscr{V}_\Theta$. Thus

$$V[\varrho^k\mathscr{C}^\infty(X)]\subset\varrho^k\mathscr{C}^\infty(X),\quad \forall k\in\mathbf{N},\ V\in\mathscr{V}_\Theta. \tag{1.7}$$

By taking tensor products, (1.4) shows that

$$\alpha(V_1,\dots,V_k)\in\varrho^k\mathscr{C}^\infty(X)\quad \forall\alpha\in\mathscr{C}^\infty(X,\Lambda^k),\ V_1,\dots,V_k\in\mathscr{V}_\Theta. \tag{1.8}$$

The formula for the exterior derivative of a 1-form gives

$$\tilde{\Theta}([V,W])=V\tilde{\Theta}(W)-W\tilde{\Theta}(V)+d\tilde{\Theta}(V,W). \tag{1.9}$$

Applying (1.7) and (1.8) to the right side we see that (1.5) holds for the commutator, $[V,W]$ of two elements of $\mathscr{V}_\Theta$.

The null space of Θ at each boundary point is clearly determined by (1.4), (1.5) so the algebra determines the projective class of Θ as a form at the boundary. This completes the proof of the proposition.

Now that we have shown the naturality of the algebra we examine it locally. Choose a convenient frame $N, T, Y_1,\dots,Y_{d-1}$, in a neighborhood of a point $p\in\partial X$, satisfying

$$d\varrho(N)=1,\quad \tilde{\Theta}(N)=0 \tag{1.10}$$

$$d\varrho(Y_i)=0,\quad \tilde{\Theta}(Y_i)=0,\quad i=1,\dots,d-1, \tag{1.11}$$

$$d\varrho(T)=0,\quad \tilde{\Theta}(T)=1. \tag{1.12}$$

Clearly such a frame exists, since $\tilde{\Theta}$ is non-zero when pulled back to the boundary. In terms of this local basis a vector field, V in $\mathscr{V}_\Theta$ is precisely one of the form:

$$V=a\varrho N+\sum_{i=1}^{d-1}b_i\varrho Y_i+c\varrho^2T \tag{1.13}$$

where the coefficients belong to $\mathscr{C}^\infty(X)$. From (1.13) we see that $\mathscr{V}_\Theta$ itself has as a local basis

$$\varrho N,\ \varrho Y_i,\ \varrho^2 T,\quad i=1,\dots,d-1. \tag{1.14}$$

That this is a basis over $\mathscr{C}^\infty(X)$ can be re-expressed in the more geometric form:

LEMMA 1.15. *There is a $\mathscr{C}^\infty$ vector bundle ${}^\Theta TX$ over X with a C^∞ bundle map*

$$\iota_\Theta\colon {}^\Theta TX \to TX \tag{1.16}$$

which is an isomorphism over $\mathring{X}$, is the zero map over ∂X, and is such that the elements of $\mathscr{V}_\Theta$ lift under ι_Θ to the space of all smooth sections:

$$\mathscr{C}^\infty(X;{}^\Theta TX)=\iota_\Theta^*(\mathscr{V}_\Theta).$$

Proof. This is exactly the content of (1.13). More precisely we can define an equivalence relation on $\mathscr{V}_\Theta$ associated to a point p (not necessarily in the boundary) by

$$V \underset{p}{\sim} W \iff \lim\left(\frac{d\varrho(V-W)}{\varrho},\frac{\alpha_1(V-W)}{\varrho},\dots,\frac{\alpha_{d-1}(V-W)}{\varrho},\frac{\tilde{\Theta}(V-W)}{\varrho^2}\right)=0 \tag{1.17}$$

where the α_i, with $d\varrho$ and $\tilde{\Theta}$, form a dual frame and the limit is at p from the interior. If p is in the interior this is just the equivalence relation defining the fibre of the ordinary tangent bundle, but at the boundary it is stronger. This gives the map (1.16). Since the limit in (1.17), for $W=0$, just reproduces the coefficients of (1.13) the equivalence relation certainly defines a $\mathscr{C}^\infty$ vector bundle.

The existence of this vector bundle is one reason we think of $\mathscr{V}_\Theta$ as a 'boundary structure' in that it amounts to a modification of the usual tangent bundle, and hence the $\mathscr{C}^\infty$ structure, at the boundary. In our analysis the bundle ${}^\Theta TX$ is a replacement for TX. The dual bundle, ${}^\Theta T^*X$, is particularly important; it has an obvious basis, dual to (1.14):

$$\frac{d\varrho}{\varrho},\ \frac{\alpha_i}{\varrho},\ \frac{\Theta}{\varrho^2}. \tag{1.18}$$

Let ${}^\Theta\Lambda^k X$ be the kth exterior power of ${}^\Theta T^*X$. This is a vector bundle as in the usual case and moreover the exterior differential operator extends to act on its sections:

$$d\colon \mathscr{C}^\infty(X;{}^\Theta\Lambda^k X)\to \mathscr{C}^\infty(X;{}^\Theta\Lambda^{k+1})\quad \forall k. \tag{1.19}$$

Indeed this is the dual (i.e. Cartan) form of the statement that $\mathscr{V}_\Theta$ is a Lie algebra.

Let $\mathrm{Diff}^m_\Theta(X)$ denote the space of differential operators on X which can be expressed as polynomials of degree at most m in vector fields belonging to $\mathcal{V}_\Theta$ i.e. the natural filtration of the enveloping algebra of $\mathcal{V}_\Theta$. Clearly the same definition extends to local maps from sections of one smooth vector bundle, E, to another, F, defining $\mathrm{Diff}^m_\Theta(X; E, F)$. We call the elements of Diff^*_Θ 'Θ-differential operators'.

LEMMA 1.20. *Exterior differentiation,* (1.19), *gives a natural element*

$$d \in \mathrm{Diff}^1_\Theta(X; {}^\Theta\Lambda^k, {}^\Theta\Lambda^{k+1}) \quad \textit{for each } k.$$

Proof. Leibniz formula

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge d\beta \tag{1.21}$$

and induction show that it is enough to check the action of d on functions. In terms of the frame (1.10)–(1.12), and its dual, d can be written as

$$d\phi = (\varrho N\phi)\frac{d\varrho}{\varrho} + \sum_{i=1}^{d-1}(\varrho Y_i\phi)\frac{\alpha_i}{\varrho} + (\varrho^2 T\phi)\frac{\Theta}{\varrho^2}. \tag{1.22}$$

Since (1.18) gives a basis of sections of ${}^\Theta T^*X$, this proves the lemma.

§ 2. Square root of a manifold with boundary

The examples of Θ-structures most important in the sequel arise from the CR structure on the boundary of a strictly pseudoconvex domain. In this case the 1-form is the contact form on the boundary, determined by the complex structure. To reduce the boundary geometry to a Θ-structure leads us to a modification of the $\mathscr{C}^\infty$ structure of the original manifold by admitting the square root of a defining function as a smooth function.

Let $\mathcal{U}$ be a smooth $(d+1)$-manifold with boundary. Let r be a defining function for the boundary of $\mathcal{U}$. Consider the extension of the ring of smooth functions $\mathscr{C}^\infty(\mathcal{U})$ obtained by adjoining the function $r^{1/2}$. We will denote this ring by $\mathscr{C}^\infty(\mathcal{U}_{1/2})$. In local coordinates $r, y_1, \dots, y_d$ a function is in this ring if it can be expressed as a $\mathscr{C}^\infty$ function of $r^{1/2}, y_1, \dots, y_d$. This just means it is $\mathscr{C}^\infty$ in the interior and has an expansion at $\partial\mathcal{U}$ of the form:

$$f(r,x) \sim \sum_{j=0}^{\infty} r^{j/2} a_j(x)$$

where the $a_j(x)$ are smooth in the usual sense and the difference of f and the sum restricted to $j<N$ becomes increasingly smooth with N. This determines f up to a smooth function vanishing with all its derivatives at the boundary. The ring $\mathscr{C}^\infty(\mathscr{U}_{1/2})$ is clearly independent of the choice of smooth defining function and invariant under maps of $\mathscr{U}$ smooth with respect to the standard differential structure. In fact the reason for the notation is that, equipped with this ring of functions, $\mathscr{U}$ is a $\mathscr{C}^\infty$ manifold with boundary—which we denote $\mathscr{U}_{1/2}$. To simplify the notation we set $X=\mathscr{U}_{1/2}$, the 'square root' of $\mathscr{U}$. It is easy to see that X is, as an abstract manifold with boundary, globally diffeomorphic to $\mathscr{U}$. However there is no natural $\mathscr{C}^\infty$ isomorphism, since the obvious 'identity' map

$$\iota_{1/2}: X \to \mathscr{U}, \tag{2.1}$$

given by the inclusion of $\mathscr{C}^\infty(\mathscr{U})$ in $\mathscr{C}^\infty(X)$, is not smoothly invertible. The boundary ∂X is canonically isomorphic to $\partial\mathscr{U}$ and the interior $\mathring{X}$ is also canonically isomorphic to $\mathring{\mathscr{U}}$; it is only the way the boundary is attached that differs.

Let θ be a 1-form defined on $\partial\mathscr{U}$. We shall always suppose that θ is non-vanishing, although later we shall place further restrictions on it. Let $\tilde\theta \in \mathscr{C}^\infty(\mathscr{U}, \Lambda^1)$ be an extension of θ to a form on $\mathscr{U}$. Then consider the lift, $\tilde\Theta = \iota_{1/2}^*\tilde\theta$ of $\tilde\theta$ to X.

LEMMA 2.2. *The algebra $\mathscr{V}_\Theta$ of $\mathscr{C}^\infty$ vector fields on $X=\mathscr{U}_{1/2}$ determined by $\Theta=\tilde\Theta|_{\partial X}$ depends only on the conformal class of θ as a form on the boundary of $\mathscr{U}$.*

Proof. If $\tilde\Theta$ and $\tilde\Theta'$ are the pull-backs of two different extensions of θ then there is a smooth 1-form α and a smooth function β on $\mathscr{U}$ such that

$$\tilde\Theta' - \tilde\Theta = \iota_{1/2}^*(r\alpha + \beta dr) = \varrho^2\alpha' + \varrho\beta' d\varrho$$

with α' and β' smooth on X. From this it is immediate that the class of vector fields that satisfy (1.4) and (1.5) will also satisfy $\tilde\Theta'(V) \in \varrho^2\mathscr{C}^\infty(X)$.

In the extended differential structure there is a distinguished direction transverse to ∂X, namely the null space of the Jacobian of $\iota_{1/2}$. We will let ψ denote a smooth vector field in this direction. If θ is any $\mathscr{C}^\infty$ 1-form on $\partial\mathscr{U}$ then for any $\mathscr{C}^\infty$ extension $\tilde\theta$ to $\mathscr{U}$, $\Theta = \iota_{1/2}^*\tilde\theta|_{\partial X}$ is a 1-form on X at ∂X which pulls back to θ on ∂X and which vanishes on ψ. This determines $\Theta \in \mathscr{C}^\infty(\partial X; T^*X)$ uniquely.

The primary example of this construction is where $\mathscr{U}$ is a smooth domain in $\mathbf{C}^{n+1}$. Then if r is any defining function for the boundary, the form $\theta = i\partial r$ is non-zero on $\partial\mathscr{U}$ and well-defined up to a positive multiple. We consider the Θ-structure on $X=\mathscr{U}_{1/2}$. In case the domain is strictly pseudoconvex and $-r$ is plurisubharmonic the Hermitian symmetric 2-form

$$(2.3)\quad g=-\sum_{i,j}\frac{\partial^2}{\partial z_i\partial \bar z_j}\log(r)\,dz_i\otimes d\bar z_j=\sum_{i,j}\left[-\frac{\partial^2 r}{\partial z_i\partial \bar z_j}\frac{dz_i\otimes d\bar z_j}{\varrho^2}+\frac{\partial r}{\partial z_i}\frac{dz_i}{\varrho^2}\otimes\frac{\partial r}{\partial \bar z_j}\frac{d\bar z_j}{\varrho^2}\right]$$

defines a complete (Kähler) metric on the interior. This explicit form, (2.3), shows:

Proposition 2.4. *If $\mathcal{U}\subset\mathbf{C}^{n+1}$ is a $\mathcal{C}^\infty$ strictly pseudoconvex domain and $r\in\mathcal{C}^\infty(\mathcal{U})$ is a defining function with $-r$ plurisubharmonic then lifted to $X=\mathcal{U}_{1/2}$ the metric* (2.3) *is a smooth positive definite section of* $\mathrm{Sym}_2({}^\Theta T^*X)$, *where $\Theta\in\mathcal{C}^\infty(\partial X;T^*X)$ is the lift of $\theta=i\partial r$.*

We call such a smooth positive-definite section of $\mathrm{Sym}_2({}^\Theta T^*X)$ a Θ-metric.

§ 3. Tangent algebra

Now let $\mathcal{V}_\Theta$ be any Θ-structure on a manifold with boundary, X. Consider the fibre ${}^\Theta T_pX$ at a point $p\in X$. The elements of $\mathcal{V}_\Theta$ of the form (1.13) with coefficients vanishing at p form a subspace $\mathcal{I}_p$ and the quotient is just

$$(3.1)\qquad {}^\Theta T_pX=\mathcal{V}_\Theta/\mathcal{I}_p.$$

When $p\in\partial X$, $\mathcal{I}_p\subset\mathcal{V}_\Theta$ is an ideal, so ${}^\Theta T_pX$ is itself a Lie algebra. For $x\in\mathring{X}$ this is not the case.

There is a natural one-dimensional subspace which we denote

$$(3.2)\qquad K_{2,p}=\{V\in\mathcal{V}_\Theta;\,V=\varrho^2W,\ W\in\mathcal{C}^\infty(X,TX)\}/\mathcal{I}_p,\quad p\in\partial X.$$

There is also a natural hypersurface containing this line

$$(3.3)\qquad K_{1,p}=\{V\in\mathcal{V}_\Theta;\,V=\varrho W,\ W\in\mathcal{C}^\infty(X,TX),\ W\text{ tangent to }\partial X\}/\mathcal{I}_p,\quad p\in\partial X.$$

Both are ideals, related to the central series for ${}^\Theta T_pX$:

$$(3.4)\qquad \begin{aligned}&[{}^\Theta T_pX,{}^\Theta T_pX]=K_{1,p},\quad p\in\partial X\\ &[K_{1,p},K_{1,p}]\subset K_{2,p},\quad p\in\partial X.\end{aligned}$$

Lemma 3.5. *The Lie algebra $K_{1,p}$ is two-step nilpotent for each $p\in\partial X$ and is commutative only if $d\Theta=\gamma\wedge\Theta$ at p, so except in this case there is equality in* (3.4).

Proof. This is obvious from the form of the frame (1.10)–(1.12). Indeed $K_{1,p}$ is spanned by ϱY_i, $i=1, \dots, d-1$ and $\varrho^2 T$. Clearly $\varrho^2 T$ is in the center of $K_{1,p}$ and

$$[\varrho Y_i, \varrho Y_j] = d\Theta(Y_i, Y_j)\varrho^2 T \quad \forall i,j = 1, \dots, d-1. \tag{3.6}$$

Thus if $d\Theta = \gamma \wedge \Theta$ at p then the algebra is commutative. Conversely if it is commutative then $d\Theta = 0$ on the subspace spanned by the Y_i, i.e. modulo Θ itself, proving (3.6).

There are only finitely many possibilities for the Lie algebras ${}^{\Theta}T_pX$. The structure is determined by the rank of $d\Theta$, the largest integer k such that $\Theta \wedge (d\Theta)^k \neq 0$ at p. The rank in turn determines the dimension of the centre of $K_{1,p}$, which is clearly the only invariant. Since the rank is determined by a finite number of derivatives of the coefficients of a frame it is clear that the set of points in ∂X near which the structure of ${}^{\Theta}T_pX$ is locally constant is open and dense.

The annihilator of Θ in T_pX, for $p \in \partial X$, is a hyperplane which we shall denote as H_p. Suppose we choose a complementary line, which we can take to be tangent to the boundary

$$S_p \subset T_p \partial X, \quad S_p \oplus H_p = T_p X. \tag{3.7}$$

Such a decomposition fixes a homogeneity structure, an $\mathbf{R}^+$-action, by

$$M_\delta \colon T_p X \setminus 0 \to T_p X \setminus 0, \quad v \mapsto \delta^{-1} v,\ v \in H_p, \quad w \mapsto \delta^{-2} w,\ w \in S_p. \tag{3.8}$$

This transformation can be realized locally as a coordinate dilation. Taking coordinates in which $\partial_x, \partial_{y_1}, \dots, \partial_{y_{d-1}}$ span H_p and S_p is spanned by ∂_{y_d} we obtain (3.8) as the infinitesimal version of

$$M_\delta \colon (x, y_1, \dots, y_{d-1}, y_d) \mapsto (\delta x, \delta y_1, \dots, \delta y_{d-1}, \delta^2 y_d). \tag{3.9}$$

In fact such a dilation also realizes (3.1). If $V \in \mathcal{V}_\Theta$ then in local coordinates

$$(M_\delta^{-1})_* V = V_\delta \tag{3.10}$$

is a $\mathscr{C}^\infty$ vector field, defined in a neighborhood of 0 which increases as δ decreases, such that

$$\lim_{\delta \downarrow 0} V_\delta = V' \tag{3.11}$$

is a well-defined smooth vector field on T_pX:

$$\mathcal{V}_\Theta \to \mathscr{C}^\infty(T_pX; T(T_pX)). \tag{3.12}$$

The null space of (3.12) is clearly $\mathcal{I}_p$. This realizes ${}^{\Theta}T_pX$ as a Lie algebra of $\mathscr{C}^\infty$ vector fields on T_pX.

Furthermore, it follows from (3.4) and Lemma 3.5 that the full Lie algebra ${}^{\Theta}T_pX$ is always solvable and is a semi-direct product:

$$^{\Theta}T_pX = \mathbf{R}^+ \ltimes K_{1,p}$$

although there is no natural decomposition of this type.

The vector fields on T_pX in the image of (3.12) all vanish at $T_p\,\partial X$. Thus it is natural to restrict the action to X_p, the inward-pointing (closed) half-space:

$$\lambda_p\colon {}^{\Theta}T_pX \hookrightarrow \mathscr{C}^\infty(X_p; TX_p). \tag{3.13}$$

The image of λ_p consists of $\mathscr{C}^\infty$ vector fields which are homogeneous of degree zero under (3.9). At each interior point, q, of X_p the values of the image must span $T_qX_p \cong T_pX$. The local coordinates in X induce homogeneous coordinates on X_p, by using the limit (3.9):

$$\lim_{\delta\downarrow 0}\delta^{-1}M_\delta^* x,\quad \lim_{\delta\downarrow 0}\delta^{-1}M_\delta^* y_j,\ j=1,\dots,d-1,\quad \lim_{\delta\downarrow 0}\delta^{-2}M_\delta^* y_d.$$

With only slight ambiguity we shall denote these coordinates in X_p again by $x, y_1, \dots, y_d$. The range of λ_p must then be of the form

$$\lambda_p({}^{\Theta}T_pX) = \mathrm{sp}\left\{x\left(\partial_x + \sum_{j=1}^{d-1} a_j y_j \partial_{y_d}\right),\ x\left(\partial_{y_i} + \sum_{j=1}^{d-1} l_{ij} y_j \partial_{y_d}\right),\ x^2\partial_{y_d},\ i=1,\dots,d-1\right\}. \tag{3.14}$$

Conversely this always gives a Lie algebra. Since the $\mathscr{C}^\infty(X_p)$-span of (3.14) is the Lie algebra $\mathscr{V}_{\tilde{\Theta}_p}$ on X_p where

$$\tilde{\Theta}_p = dy_p - \sum_{j=1}^{d-1} a_j y_j\,dx - \sum_{i,j=1}^{d-1} l_{ij} y_i\,dy_j \tag{3.15}$$

every Lie algebra (3.14) arises in this way from some Θ-structure. Notice in fact that the class of $\tilde{\Theta}_p$ on X_p is given by

$$\tilde{\Theta}_p = \lim_{\delta\to 0}\delta^{-2}M_\delta^*\Theta. \tag{3.16}$$

The range of λ_p is fixed by the choice of S_p and of the coordinates, (3.9), consistent with the homogeneity (3.8). By making appropriate choices we can simplify its form.

The transformation on X_p:

$$y_d \mapsto y_d - x\sum_{j=1}^{d-1} a_j y_j, \tag{3.17}$$

with the other coordinates fixed alters the basis (3.14) to

$$\lambda_p({}^{\Theta}T_pX) = \mathrm{sp}\left\{x\partial_x,\ x\left(\partial_{y_i}+\sum_{j=1}^{d-1} l_{ij}y_j\partial_{y_d}\right)-x^2a_i\partial_{y_d},\ x^2\partial_{y_d},\ i=1,\dots,d-1\right\}.$$

Since (3.17) can be viewed as a homogeneous change of coordinates, (3.9), we can assume that $a_i=0$ in (3.14). Similarly the further change of y_d, only,

$$y_d \mapsto y_d - \frac{1}{2}\sum_{i,j=1}^{d-1} l_{ij}y_iy_j$$

transforms the basis to the same form (3.14) with the l_{ij} replaced by their skew parts. We can therefore assume in place of (3.14):

$$\lambda_p({}^{\Theta}T_pX) = \mathrm{sp}\left\{x\partial_x,\ x\left(\partial_{y_i}+\sum_{j=1}^{d-1} l_{ij}y_j\partial_{y_d}\right),\ x^2\partial_{y_d},\ i=1,\dots,d-1,\ l_{ij}+l_{ji}=0\right\} \tag{3.18}$$

by a suitable choice of the coordinates (3.9) for any choice of S_p.

In the analytic problem which arises the Lie algebra ${}^{\Theta}T_pX$ only appears through (3.13). Thus we need to classify the Lie algebras in this sense; fortunately this classification problem leads to the obvious notion of equivalence.

LEMMA 3.19. *Suppose $\Theta \in \mathscr{C}^\infty(\partial X; T^*X)$ and $\Theta' \in \mathscr{C}^\infty(\partial X'; T^*X')$ define Θ-structures on two manifolds with bondary and $p \in \partial X$, $p' \in \partial X'$ are such that there is a linear isomorphism*

$$A\colon {}^{\Theta}T_pX \leftrightarrow {}^{\Theta'}T_{p'}X' \quad \textit{as Lie algebras.} \tag{3.20}$$

Let $\lambda'_{p'}$ be the representation (3.13) *corresponding to some choice of transversal $S'_{p'}$ and of local coordinates* (3.9) *in X' near p', then for an appropriate choice of transversal S_p and of coordinates* (3.9) *in X near p the coordinate identification, $\mathrm{Id}_{\mathrm{coor}}$ of X_p and $X'_{p'}$ gives*

$$\mathrm{Id}_{\mathrm{coor}}\cdot\lambda_p = \lambda'_{p'}\cdot A. \tag{3.21}$$

Moreover, if the data vary smoothly with parameters then the transversal and coordinates can be chosen smoothly in the parameters so that (3.21) *holds.*

Proof. The discussion above shows that it suffices to prove the lemma under the additional hypothesis that the Lie algebra on $X'_{p'}$ takes the form (3.18) in the induced coordinates. We can then introduce coordinates on X so that the same is true on X_p, possibly with a different skew matrix l_{ij}. We need to modify the choice of transversal and coordinates on X until the image of the basis (3.18) under A^{-1} is the corresponding basis on X_p. We let $\tilde{A}$ denote the map, which we wish to reduce to the identity, induced by A, λ_p and $\lambda'_{p'}$ in terms of these bases.

Let the coordinates in X, and X_p, be denoted $x, y_1, \dots, y_d$ and those in X', and $X'_{p'}$, $x', y'_1, \dots, y'_d$. We shall assume that $K_{1,p}$ is not commutative (and leave the less interesting commutative case to the reader). Then $x^2\partial_{y_d}$ spans $K_{2,p}=[K_{1,p}, K_{1,p}]$, so must be mapped by $\tilde{A}$ to $c(x')^2\partial_{y'_d}$. Making the transformation

$$y_d \mapsto cy_d$$

ensures that $\tilde{A}$ is the identity on $K_{2,p}$.

Next consider the transformation induced by

$$A: K_{1,p}/K_{2,p} \to K_{1,p'}/K_{2,p'}. \tag{3.22}$$

In terms of the bases (3.18) this can be written

$$x\left(\partial_{y_i}+\sum_{j=1}^{d-1} l_{ij}y_j\partial_{y_d}\right) \mapsto \sum_{k=1}^{d-1} T_{ik}x'\left(\partial_{y'_i}+\sum_{j=1}^{d-1} l'_{ij}y'_j\partial_{y'_d}\right). \tag{3.23}$$

The change of coordinates

$$x \mapsto x, \quad y_i \mapsto \sum_{j=1}^{d-1} T_{ij}y_j, \; i=1,\dots,d-1, \quad y_d \mapsto y_d$$

reduces T to the identity, and from the assumption that A is a Lie algebra isomorphism and both l and l' are skew we find $l=l'$.

Thus we now have the map on $K_{1,p}$ of the form

$$\tilde{A}: x\left(\partial_{y_i}+\sum_{j=1}^{d-1} l_{ij}y_j\partial_{y_d}\right) \mapsto x'\left(\partial_{y'_i}+\sum_{j=1}^{d-1} l_{ij}y'_j\partial_{y'_d}\right)+b_i(x')^2\partial_{y'_d}. \tag{3.24}$$

The linear coordinate transformation, which is not M_δ-homogeneous,

$$y_i \mapsto y_i - b_i y_d, \quad i=1,\dots,d-1$$

makes the constants b_i in (3.24) vanish. This amounts to a change of the transversal S_p.

Thus we have arranged that the basis of $K_{1,p}$ is carried into that of $K_{1,p'}$. So now we have to consider

$$\tilde{A}: x\partial_x \mapsto x'\partial_{x'} + \sum_{i=1}^{d-1} a_i x' \left(\partial_{y'_i} + \sum_{j=1}^{d-1} l_{ij} y'_j \partial_{y'_d} \right) + \beta (x')^2 \partial_{y'_d}. \tag{3.25}$$

Here we know that the coefficient of $x'\partial_{x'}$ is 1 since commutation with $K_{1,p}$ and $K_{2,p}$ must be preserved. In fact the additional constraint

$$\sum_{i=1}^{d-1} l_{ij} a_i = 0 \tag{3.26}$$

also follows from this. The coefficient β can be made to vanish by the transformation $y_d \mapsto y_d - \frac{1}{2}\beta x^2$. Finally the transformation

$$y_i \mapsto y_i - a_i x$$

also removes the coefficients a_i. This completes the proof of the lemma since the construction is clearly smooth in parameters at each step.

In order to apply this lemma in a non-trivial way the structure of the Lie algebras ${}^{\Theta}T_pX$ will in general have to be locally constant, or at least change simply.

§ 4. Normal operator and model problem

The most general metric we consider on X in the present context is a Θ-metric, i.e. a fibre metric on ${}^{\Theta}TX$. Such a form can be written as a positive definite quadratic combination of a coframe:

$$g = g_{00}\left(\frac{d\varrho}{\varrho}\right)^2 + 2\sum_{i=1}^{d-1} g_{0i}\frac{d\varrho}{\varrho}\frac{a_i}{\varrho} + \sum_{i,j=1}^{d-1} g_{ij}\frac{a_i}{\varrho}\frac{a_j}{\varrho} + 2\sum_{i=1}^{d-1} g_{di}\frac{\Theta}{\varrho^2}\frac{a_i}{\varrho} + 2g_{0d}\frac{d\varrho}{\varrho}\frac{\Theta}{\varrho^2} + g_{dd}\frac{\Theta^2}{\varrho^4}. \tag{4.1}$$

We shall actually consider more restricted metrics below but even in this generality we note that the Laplacian is a Θ-differential operator.

PROPOSITION 4.2. *If g is a Θ-metric on a manifold with boundary then the Laplacian, acting on the Θ-form bundles, is a Θ-differential operator*

$$\Delta \in \mathrm{Diff}^2_{\Theta}(X; {}^{\Theta}\Lambda^k) \quad \forall k. \tag{4.3}$$

This is straightforward to prove but it should be noted that if the Θ-form bundles are replaced by the usual form bundles (on which the Laplacian does indeed act smoothly) then the analogue of (4.3) is not true.

Proof. Simply note that the Laplacian is

$$\Delta = d\delta + \delta d \tag{4.4}$$

where δ is the adjoint of d with respect to the Riemannian inner product and density. The inner product is non-degenerate on ${}^{\Theta}\Lambda^*$ so it is only necessary to check that taking adjoints with respect to the Riemannian density preserves Diff^1_{Θ}. This is certainly true for a smooth non-vanishing density. However the Riemannian density, dg, is of the form $\varrho^{-N}\nu$ where ν is such a density and $N=d+2$. From (1.13) it follows that this factor can be removed by conjugation since

$$\mathcal{V}_{\Theta} \ni V \mapsto \varrho^{-1} V \varrho \in \mathrm{Diff}^1_{\Theta}. \tag{4.5}$$

When $p \in \partial X$

$$0 \to \mathcal{I}_p \to \mathcal{V}_{\Theta} \to {}^{\Theta}T_p X \to 0 \tag{4.6}$$

is an exact sequence of Lie algebra homomorphisms. The projection therefore lifts to a map of the enveloping algebras:

$$\mathrm{Diff}^*_{\Theta}(X) \to \mathcal{D}({}^{\Theta}T_p). \tag{4.7}$$

The null space is $\mathcal{C}^\infty_p(X) \cdot \mathrm{Diff}^m_{\Theta}(X) \subset \mathrm{Diff}^m_{\Theta}(X)$, where $\mathcal{C}^\infty_p(X)$ is the ideal of smooth functions vanishing at p. The image of $P \in \mathrm{Diff}^m_{\Theta}(X)$ will be denoted $N_p(P)$ and called the normal operator of P at p.

The normal operators form our model problems. Thus the main step in the construction of a parametrix for the Laplacian is simply the construction of an appropriate fundamental solution for each $N_p(P)$ with smooth dependence on p. Clearly it is of the greatest importance that we understand the structure of these families of operators.

The definition of the normal operator is closely related to the definition of the indicial operator for the spaces $\mathrm{Diff}^m_b(X)$ of totally characteristic differential operators (see [20], [12]). Since

$$\mathrm{Diff}^m_{\Theta}(X) \subset \mathrm{Diff}^m_b(X)$$

this latter definition restricts to the Θ-differential operators. We can obtain it more directly in terms of a further quotient of the Lie algebra. Since $K_{1,p}$ is an ideal there is a natural exact sequence of Lie algebras

$$0 \hookrightarrow K_{1,p} \hookrightarrow {}^{\Theta}T_pX \to {}^bN_p\partial X = [{}^bT_pX/T_p\partial X] \to 0$$

and the projection from ${}^{\Theta}T_pX$ induces a map of the enveloping algebras

$$I_p\colon \mathrm{Diff}^m_{\Theta}(X) \to \mathcal{D}^m({}^bN_p\partial X).$$

The 'b-normal bundle' of the boundary, ${}^bN\partial X$, can be canonically identified with the b-tangent bundle to the fibres of the usual normal bundle, i.e. $I(P)$ is an $\mathbf{R}^+$-invariant ordinary differential operator on the fibres of the normal bundle $N\partial X$. The map, I_p, is just the quotient by the ideal $K_{1,p}\otimes\mathcal{D}^{m-1}({}^{\Theta}T_pX)$. In terms of the basis (1.14) of $\mathcal{V}_{\Theta}$ the indical operator is obtained by dropping all terms containing factors other than ϱN and freezing the coefficients at the boundary.

We have already shown that, once a transversal S_p as in (3.7) and local $\mathbf{R}^+$-action (3.9) have been chosen, ${}^{\Theta}T_pX$, and hence $\mathcal{D}({}^{\Theta}T_pX)$, acts on X_p. We think of X_p as a local model for X. Recall from (3.15) that X_p has a Θ-structure, given by $\tilde{\Theta}_p$. Any smooth section α of ${}^{\Theta}T^*X$ defines, by the same limit as in (3.11), a section of ${}^{\tilde{\Theta}_p}T^*X_p$. Thus a fibre metric g on X defines a $\tilde{\Theta}_p$-metric, $\tilde{g}_p$, on X_p. A local orthonormal frame for X, near p, defines a global orthonormal frame on each X_p. The boundary structure on X_p is given by any non-vanishing form at ∂X_p which quadratically annihilates all elements of ${}^{\Theta}T_pX$, in their action on X_p. Clearly different choices of complement, S_p, lead to isomorphic spaces.

PROPOSITION 4.8. *For any Θ-metric, g, on a manifold with boundary, X, and any choice of transversal, S_p, as in* (3.7) *near $p\in\partial X$ the normal operator of the Laplacian is, acting on X_p, the Laplacian of the induced Θ-metric on X_p:*

$$N_p(\Delta_g) = \Delta_{\tilde{g}_p}; \tag{4.9}$$

this is a left-invariant differential operator for the transitive group action, G_p, on X_p generated by the action of ${}^{\Theta}T_p$.

Proof. Following the proof of Proposition 4.2 it is clear that if g_t is a smooth 1-parameter family of Θ_t-metrics then the Laplacian varies smoothly with t. The scaling (3.9) applied to the metric gives such a family with limit $\tilde{g}_p$. Since the limiting Laplacian is, by definition, the normal operator we conclude that (4.9) holds.

In the introduction we defined the condition that Θ-metrics, g on X with structure form Θ and g' on X' with structure form Θ' should have the same model at points $p \in \partial X$ and $p' \in \partial X'$. Namely this is said to be the case if there is a linear isomorphism

$$A: {}^{\Theta}T_pX \leftrightarrow {}^{\Theta'}T_{p'}X'$$

which is both an isometry and a Lie algebra isomorphism

$$\begin{aligned} &g'_{p'}(Av) = g_p(v) \quad \forall v, v' \in {}^{\Theta}T_pX. \\ &[Av, Av'] = A[v, v'] \quad \forall v, v' \in {}^{\Theta}T_pX. \end{aligned} \tag{4.10}$$

Applying Lemma 3.19 to the Lie algebra, and using Proposition 4.8, we find

PROPOSITION 4.11. *If Θ-metrics g, g' have the same models at points p, p' then there are choices of transversals S_p, $S'_{p'}$ as in* (3.7) *and local coordinates near p in X and p' in X' such that in the homogeneous coordinates induced on X_p and $X'_{p'}$*

$$N_p(\Delta_g) = N_{p'}(\Delta_{g'}). \tag{4.12}$$

In order to prove the Main Theorem of the introduction we need to investigate the conditions under which a given Θ-metric has the same model as a multiple of the Bergman metric on the ball. We examine the metric on the ball in § 8, for the moment we express the result more intrinsically.

First we assume

$$\iota^*_{\partial X}\Theta \text{ is a contact form near } p \in \partial X \tag{4.13}$$

which is always true for strictly pseudoconvex domains. Let g be a Θ-metric. Then the ideals corresponding to the central series

$$K_{2,p} \subset K_{1,p} \subset {}^{\Theta}T_pX \tag{4.14}$$

lead to an orthogonal decomposition

$$\begin{aligned} &{}^{\Theta}T_pX = M_p \oplus J_p \oplus K_p, \\ &K_p = K_{2,p}, \quad K_{1,p} = J_p \oplus K_p. \end{aligned} \tag{4.15}$$

Let $m \in M_p$ be of unit length and consider the condition

$$\mu: J_p \to J_p \oplus K_p, \quad \mu(v) = [m, v] \quad \text{has range in } J_p \text{ and is } \frac{1}{2}\chi \cdot \mathrm{Id}. \tag{4.16}$$

Since $J_p \subset K_{1,p}$ we also have an antisymmetric bilinear form

$$J_p \times J_p \ni (v, v') \mapsto \omega(v, v') \in \mathbf{R} \quad \text{if} \quad [v, v'] = \omega(v, v')Z, \ Z \in K_p, \ g(Z) = 1 \tag{4.17}$$

Of course ω is only defined up to sign. The metric restricts to J_p so we can define the 'Hamilton' map

$$\Xi: J_p \to J_p, \quad \omega(v, \Xi v') = g(v, v') \ \forall v, v' \in J_p. \tag{4.18}$$

From standard linear algebra it follows that Ξ has eigenvalues $\pm i\lambda_i$, $i=1, \dots, \frac{1}{2}(d-1)$. We require:

(4.19) the eigenvalues of Ξ are all $\pm i\chi$.

We can easily construct examples of Lie algebras and metrics satisfying these conditions. Namely take the standard contact form

$$\Phi = du - \sum_{i=1}^{n} (y_i \, d\eta_i - \eta_i \, dy_i) \tag{4.20}$$

on

$$\tilde{X} = [0, \infty)_x \times \mathbf{R}^{2n}_{y,\eta} \times \mathbf{R}_u. \tag{4.21}$$

The homogeneous elements of $\mathcal{V}_\Phi$ are

$$\frac{1}{2} x\partial_x, \quad \frac{1}{\sqrt{2}} x(\partial_{y_j} + \eta_j \partial_u), \quad \frac{1}{\sqrt{2}} x(\partial_{\eta_j} - y_j \partial_u), \quad x^2 \partial_u. \tag{4.22}$$

Consider the metric

$$g_1 = 4\frac{dx^2}{x^2} + 2\frac{|dy|^2}{x^2} + 2\frac{|d\eta|^2}{x^2} + \frac{\Phi^2}{x^4} \tag{4.23}$$

which has (4.22) as an orthonormal basis; (4.13), (4.16) and (4.19) hold for this metric, with $\chi = 1$.

PROPOSITION 4.24. *Suppose a Θ-metric g satisfies* (4.13), (4.16) *and* (4.19), *at each boundary point of X with χ a non-vanishing smooth function on ∂X. Then for any extension of χ to a non-vanishing smooth function on X the metric $\chi^2 g$ has the same model as the metric g_1, in* (4.23), *at* 0 *and the isometric Lie algebra isomorphism between ${}^{\Theta}T_pX$ and ${}^{\Phi}T_0\tilde{X}$ can be chosen locally smooth on ∂X.*

Proof. Replacing g by $\chi^2 g$ gives the hypotheses of the proposition with $\chi\equiv 1$, so we can assume this from the start.

By Darboux's theorem we can introduce local coordinates in ∂X so that the pull-back of the form Θ is given by (4.20). If N is a normal vector field, chosen to be annihilated by Θ then extending the coordinates to be constant along it, except for the normal coordinate which should satisfy $Nx=1$, reduces Θ to the form (4.20) at ∂X. We can therefore take $\Theta=\Phi$. Thus the tangent algebras are certainly identified.

We need to choose the Lie algebra isomorphism from ${}^{\Theta}T_pX$ to ${}^{\Theta}T_0\tilde{X}$ so that the metrics are related by it. There is little choice for the Lie algebra isomorphism. Certainly it must intertwine the decompositions (4.15). In fact (4.13), (4.16) and (4.19) together allow us to choose an orthonormal basis for ${}^{\Theta}T_pX$ with the commutation relations of (4.22). This only involves a symplectic transformation of J_p and can therefore be done, at least locally, smoothly in parameters.

A simple calculation using the curvature tensor of $\tilde{g}_p$ shows that (4.13), (4.16) and (4.19) are actually necessary conditions for the conclusions of Proposition 4.24 to hold.

§ 5. Parabolic blow-up

We next treat an extension of the notion of normal blow-up, as used previously in the analysis of the $\mathcal{V}_0$ and $\mathcal{V}_b$ boundary structures, see [20] and [18]. This 'parabolic' blow-up captures the anisotropic dilation structure characteristic of Bergman type Laplacians. The construction will closely follow that given in [19] for the normal case.

First we consider a linear model. Let $n>1$ be fixed and let S be a linear subspace of the dual space $(\mathbf{R}^n)^*$. We shall choose a complementary subspace S' to the annihilator $S^\circ\subset\mathbf{R}^n$ of S and define a dilation structure on $\mathbf{R}^n\setminus\{0\}$ using the splitting $\mathbf{R}^n=S^\circ\oplus S$. For $z=x+y\in\mathbf{R}^n$ with $x\in S^\circ$ and $y\in S'$ set

$$M_\delta(z)=\delta x+\delta^2 y. \tag{5.1}$$

This is a smooth $\mathbf{R}^+$-action on $\mathbf{R}^n\setminus\{0\}$ and so induces an equivalence relation by $z\sim z'$ if $z'=M_\delta z$ for some $\delta>0$. We shall denote the quotient by $\mathbf{R}^n\setminus\{0\}/M_\delta$. We then define the S-parabolic blow-up of $\mathbf{R}^n$ along $\{0\}$ to be, as a set,

$$\mathbf{R}^n_{\{0\},S}=[\mathbf{R}^n\setminus\{0\}/M_\delta]\sqcup\mathbf{R}^n\setminus\{0\}. \tag{5.2}$$

Let $\beta_{\{0\},S}$ denote the canonical projection map $\beta_{\{0\},S}\colon\mathbf{R}^n_{\{0\},S}\to\mathbf{R}^n$ i.e. $\beta_{\{0\},S}$ is the identity on $\mathbf{R}^n\setminus\{0\}$ and maps all of $\mathbf{R}^n\setminus\{0\}/M_\delta$ to 0 and denote the front face,

$\beta^{-1}_{\{0\},S}(\mathbf{R}^n\setminus\{0\}/M_\delta)$ by $\mathrm{ff}(\mathbf{R}^n_{\{0\},S})$. Notice that '$S$-parabolic' means that the quadratic variables are those with differentials in S.

Consider the functions on $\mathbf{R}^n\setminus\{0\}$ which are $\mathscr{C}^\infty$ and homogeneous of degree one or zero under the action (5.1). Homogeneous functions of degree zero are just functions on a spherical cross-section. It is also easy to construct a function homogeneous of degree one. To do so choose a positive definite quadratic form, Q, on $\mathbf{R}^n$. If $z\in\mathbf{R}^n$ is represented as $x+y$, as above, then consider the function

$$r(z) = [Q(x)^2+Q(y)]^{1/4}. \tag{5.3}$$

Clearly $M^*_\delta r(z)=\delta r(z)$. Any other $\mathscr{C}^\infty$ function homogeneous of degree one is of the form ar where a is $\mathscr{C}^\infty$ and homogeneous of degree zero. The function r lifts to a well-defined function on $\mathbf{R}^n_{\{0\},S}$, vanishing along the boundary $\mathbf{R}^n\setminus\{0\}/M_\delta$ introduced in the process of blowing up. Similarly any $\mathscr{C}^\infty$ function homogeneous of degree zero lifts to $\mathbf{R}^n_{\{0\},S}$. We define the topology and $\mathscr{C}^\infty$ structure on $\mathbf{R}^n_{\{0\},S}$ in terms of these functions. Namely take the weakest topology with respect to which they are all continuous. To define a smooth structure we use Proposition 1.6.5 in [19]. Define the algebra of smooth functions on $\mathbf{R}^n_{\{0\},S}$ to be the algebra generated by the lifts of these smooth functions on $\mathbf{R}^n\setminus\{0\}$ homogeneous of some non-negative integral degree with respect to (5.1). It is easily verified that this algebra separates points, is local and defines coordinates for $\mathbf{R}^n_{\{0\},S}$ and hence defines a smooth structure on it as a manifold with boundary. Of course with this $\mathscr{C}^\infty$ structure M_δ extends to an action of $[0,\infty)$,

$$M\colon [0,\infty)\times\mathbf{R}^n_{\{0\},S}\to\mathbf{R}^n_{\{0\},S},$$

showing that

$$\mathbf{R}^n_{\{0\},S}\cong[0,\infty)\times[\mathbf{R}^n\setminus\{0\}/M_\delta]. \tag{5.4}$$

The definition of the parabolic blow-up depends, in principle, on the choice of complementary space S'. We will show that, as a $\mathscr{C}^\infty$ manifold with boundary, it is actually independent of the choice. Of course the definition really only uses the action (5.1). Thus any linear transformation of $\mathbf{R}^n$ which commutes with this action lifts to a diffeomorpism of $\mathbf{R}^n_{\{0\},S}$. Such a linear transformation is simply the direct sum of linear transformations of S° and S'.

LEMMA 5.5. *Let S'_1 and S'_2 be two subspaces complementary to S° and let X_1 and X_2 be the S-parabolic blow-ups of $\mathbf{R}^n$ along 0 that they define. As manifolds with* *boundary X_1 and X_2 are diffeomorphic in a manner giving a commutative diagram*

$$
\begin{tikzcd}
X_1 \arrow[r, leftrightarrow] \arrow[d, "\beta_1"'] & X_2 \arrow[d, "\beta_2"] \\
\mathbf{R}^n \arrow[r, leftrightarrow, "\mathrm{Id}"'] & \mathbf{R}^n
\end{tikzcd}
\tag{5.6}
$$

Proof. By the remark above we can assume that

$$S^\circ = \{(x_1, \dots, x_k, 0, \dots, 0)\} \quad \text{and} \quad S_1' = \{(0, \dots 0, y_1, \dots, y_{n-k})\}. \tag{5.7}$$

Then there is an $(n-k)\times k$ matrix, A, such that $S_2'=\{(Ay, y)\}$. The lemma now reduces to showing that the linear transformation $(x, y)\to(x+Ay, y)$ lifts to a diffeomorpism of $\mathbf{R}^n_{\{0\},S}$. In general this transformation does not commute with (5.1) and so does not preserve the homogeneous functions. However such a transformation can always be obtained by integration of a linear vector field of the form

$$v = \sum_{i=1}^{n-k}\sum_{j=1}^{k} a_{ij} V_{ij}, \quad V_{ij} = y_i \partial_{x_j}, \quad A = (a_{ij}). \tag{5.8}$$

Thus it suffices to show that the vector fields V_{ij} lift to be smooth on $\mathbf{R}^n_{\{0\},S}$. This in fact follows from the commutation relation with the infinitesimal generator of (5.1):

$$[R, V_{ij}] = V_{ij}, \quad R = x\partial_x + 2y\partial_y, \quad M_\delta = e^{\delta R} \tag{5.9}$$

since this shows that the lift of V_{ij} is homogeneous of degree one and smooth away from the boundary, hence is smooth (and vanishes at the boundary).

To extend this construction to submanifolds of a manifold we first consider the lifts of more general diffeomorphisms of $\mathbf{R}^n$ to $\mathbf{R}^n_{\{0\},S}$. We shall do this using an homotopy method and the lifting of vector fields. Note that all the vector fields

$$y_i\frac{\partial}{\partial y_{i'}}, \quad x_r\frac{\partial}{\partial x_{r'}}, \quad y_i\frac{\partial}{\partial x_r}, \quad x_r x_{r'}\frac{\partial}{\partial y_i}, \quad r, r' = 1 \dots, k, \quad i, i' = 1, \dots, n-k \tag{5.10}$$

lift to be $\mathscr{C}^\infty$ on $\mathbf{R}^n_{\{0\},S}$. The first set correspond to linear motions of S', the second to linear motions of S° and the third we have just considered in (5.8). As in (5.9), the last set of vector fields are also homogeneous of degree zero hence smooth up to the boundary, to which they are tangent. Any vector field which vanishes quadratically at the origin is in the span, over $\mathscr{C}^\infty(\mathbf{R}^n)$, of the vector fields in (5.10) and therefore lifts to be $\mathscr{C}^\infty$ on $\mathbf{R}^n_{\{0\},S}$. Amongst the linear vector fields only those which are tangent to S° lift—this is precisely the content of (5.10).

Now we can generalize the lifting to local diffeomorphisms.

LEMMA 5.11. *Let $f\colon U\leftrightarrow U'$, where U, U' are neighborhoods of 0 in $\mathbf{R}^n$, be such that $f(0)=0$ and f_* at 0 carries T_0S° to T_0S°, then f can be lifted to a diffeomorphism between neighborhoods of the lift of 0 in $\mathbf{R}^n_{\{0\},S}$.*

Proof. This is essentially identical to the argument given in [19]. We first observe that if we compose on the right with the inverse of f_* at 0 then we get a new map f' such that $f'(0)=0$ and f'_* is the identity at 0. Moreover, f_* is a linear map preserving S° so it lifts to the S-parabolic blow-up. Thus we can assume that $f_*=\mathrm{Id}$ at 0. Such a map can be smoothly homotoped through diffeomorphisms f_t, for $t\in[0,1]$ such that

$$
\begin{aligned}
&f_0=\mathrm{Id} \quad \text{in a neighborhood of } 0\\
&f_1=f'\\
&f_t(0)=0 \quad \forall t\in[0,1]\\
&(f_t)_*=\mathrm{Id} \quad \text{at } 0 \quad \forall t\in[0,1].
\end{aligned}
\tag{5.12}
$$

As in [19] such a map is always given by the integration of a t-dependent vector field W_t. Since both the value of f_t and its differential are fixed at 0 it follows that W_t vanishes quadratically at 0 and therefore can be lifted as a smooth vector field to $\mathbf{R}^n_{\{0\},S}$. By integrating the lifted vector field we conclude that the diffeomorphism itself lifts.

We next add parameters to the theory and define the parabolic blow-up of the zero section of a vector bundle. Let V be a vector bundle over the base Y. We can identify Y with the zero section of V. Let S be a subbundle of V^*. By introducing a positive definite metric on V we can define a subbundle of V complementary to the annihilator, S°, of S as the orthogonal complement of S°. Call this subbundle S'. Now we define the S-parabolic blow-up of V along Y, $V_{Y,S}$, as

$$
V_{Y,S}=\coprod_{y\in Y}(V_y)_{\{0\},S_y}. \tag{5.13}
$$

This is simply the parabolic blow-up in each fibre. The additional boundary introduced by this process will be called (as in the normal case) the front face:

$$
\mathrm{ff}(V_{Y,S})=\coprod_{y\in Y}[V_y\setminus\{0\}/M_\delta]. \tag{5.14}
$$

Again the $\mathscr{C}^\infty$ structure, making this a manifold with boundary, is given by the algebra generated by the functions $\mathscr{C}^\infty$ on $V\setminus Y$ which are homogeneous of degree zero or one under the dilation (5.1) on each fibre. Under a local trivialization

$$\pi^{-1}(U)\cong U\times V_y \quad \text{for} \quad y\in U \tag{5.15}$$

the $\mathscr{C}^\infty$ structure reduces to the product structure from the blow-up on each fibre.

It is an easy exercise to add parameters to Lemma 5.11 and verify that the parabolic blow-up is independent of the choice of complementary subbundle. Indeed following the discussion in Lemma 5.5 the question is reduced to the smooth lifting of a vector field which is tangent to the zero section and with linearization there tangent to $S°$. In the local trivialization, a vector field purely in the base certainly lifts so we are reduced to the previous discussion with smooth variation from fibre to fibre. Using integration to extend the discussion to diffeomorphisms we find:

LEMMA 5.16. *If $V_{Y,S}$ is the S-parabolic blow-up of the vector bundle V along its zero section Y then a vector field on V lifts to be $\mathscr{C}^\infty$ on $V_{Y,S}$ if and only if it is tangent to Y and with linearization tangent to $S°$ at Y. A $\mathscr{C}^\infty$ diffeomorphism of V lifts to be $\mathscr{C}^\infty$ if and only if it maps Y to itself and its differential at each point $y\in Y$ maps $T_yS°$ to itself.*

The next step in our discussion is to extend the theory to a closed and smoothly embedded submanifold Y of a manifold X. The appropriate vector bundles are the normal bundle of Y, NY, and a subbundle of its dual, the conormal bundle, $S\subset N^*Y$. As a set we define the S-parabolic blow-up of X along Y, $X_{Y,S}$ to be:

$$X_{Y,S}=\mathrm{ff}(NY_{Y,S})\sqcup[X\setminus Y]. \tag{5.17}$$

Thus the front face is, by definition, the front face of the S-parabolic blow-up of the normal bundle along the zero section. We have an obvious map

$$\beta_{Y,S}\colon X_{Y,S}\to X. \tag{5.18}$$

The new boundary face in $X_{Y,S}$ will again be called the front face and denoted $\mathrm{ff}(X_{Y,S})\equiv\mathrm{ff}(NY_{Y,S})$.

We will define a smooth structure by applying the normal fibration (or collar neighborhood) theorem. This asserts that if Y is closed and embedded in X then there is a diffeomorphism f from a neighborhood of the zero section, Y, in NY to a neighborhood of Y in X such that

$$
\begin{aligned}
&f\colon Y \to Y \quad \text{is the identity}\\
&f_*\colon NY \to NY \quad \text{is the identity.}
\end{aligned}
\tag{5.19}
$$

The condition on f_* is to be understood by using the canonical identification of the normal bundle of the zero section of a vector bundle with the vector bundle itself.

Using this result we can define $\mathscr{C}^\infty(X_{Y,S})$ by identifying it with $\mathscr{C}^\infty(NY_{Y,S})$ in a neighborhood of $\mathrm{ff}(NY_{Y,S})$. All we need to show is that this is independent of the choice of f. As in the normal case if we have two such maps, f_1 and f_0, then we can join them, at least in a neighborhood of the zero section, by a smooth homotopy f_t with the same properties (5.19) for each t. If we set $F_t=f_0^{-1}\cdot f_t$ then F_t is obtained by integration of a t-dependent vector field on NY. The proof is completed by showing that W_t lifts to a smooth t-dependent vector field on $NY_{Y,S}$. From the fact that both F_t and $(F_t)_*$ are fixed along Y it follows that W_t is locally a sum of vector fields tangent to Y with coefficients vanishing on it. Thus Lemma 5.16 shows that W_t can be lifted smoothly to $NY_{Y,S}$. Hence the diffeomorphism F_1 lifts to the parabolic blow-up. This shows that the $\mathscr{C}^\infty$ structure on $X_{Y,S}$ is independent of the choice of normal fibration.

This completes the definition of $X_{Y,S}$ as a manifold with boundary whenever $Y\subset X$ is a closed embedded submanifold and $S\subset N^*Y$ is a subbundle. Notice that if S is the zero section then

$$X_{Y,0}=X_Y$$

is just the normal blow-up of X along Y.

For the case of a vector bundle, V, S-parabolically blown up along its zero section, the front face has two natural submanifolds. The first is just the image of S° under the $\mathbf{R}^+$-quotient. The second is the image of S'. This is also well-defined, and of course independent of the choice of S'. We denote these two submanifolds, both of which are spherical subbundles by

$$\beta^*_{Y,S}(S^\circ),\beta^*_{Y,S}(S')\subset \mathrm{ff}(V_{Y,S}). \tag{5.20}$$

In fact since the front face of $X_{Y,S}$ is just the front face of the blow-up of the normal bundle these sphere bundles are always well-defined. Notice that

$$\beta^*_{Y,S}(S^\circ)\cap\beta^*_{Y,S}(S')=\varnothing \quad \text{in} \quad \mathrm{ff}(X_{Y,S}). \tag{5.21}$$

If v is a $\mathscr{C}^\infty$ vector field on X which is tangent to Y then its linear part at Y is a bundle homomorphism

$$L_Y v\colon N^*Y \to N^*Y, \quad L_y v(df) = d_y(vf), \;\; y \in Y. \tag{5.22}$$

Then we can say that the linear part of v is tangent to S° at Y (or normal to S) if $L_Y v$ maps S to itself. Set

$$\mathcal{V}(Y,S) = \{v \in \mathscr{C}^\infty(X, TX);\ v \text{ is tangent to } Y \text{ and to } S^\circ \text{ at } Y\}. \tag{5.23}$$

LEMMA 5.24. *If Y is an embedded submanifold of X and $S \subset N^*Y$ is a subbundle of the conormal bundle to Y in X then under the S-parabolic blow-up of X along Y the space $\mathcal{V}(Y,S)$ lifts to span, over $\mathscr{C}^\infty(X_{Y,S})$ the space of all vector fields $w \in \mathscr{C}^\infty(X_{Y,S}, TX_{Y,S})$ tangent to the boundary and to $\beta^*_{Y,S}(S')$.*

Proof. This lifting property may at first seem counter-intuitive, since it states that the lifted vector fields, while tangent to the front face, are not tangent to the lift of S° there. Of course this is not so unreasonable when it is recalled that the last class of liftable vector fields in (5.10) are not tangent to S° away from zero and yet are homogeneous of degree 0 under the $\mathbf{R}^+$-action. On the other hand the vector fields in (5.10) not tangent to S' are in the third group. These are homogeneous of degree 1 and hence vanish at the front face. This shows that the lift of $\mathcal{V}(Y,S)$ is contained in the space stated. That the lifts span follows from the fact that the vector fields in (5.10) which are homogeneous of degree 0 have this property away from the origin.

COROLLARY 5.25. *If $\beta_{Y,S}\colon X_{Y,S} \to X$ is the S-parabolic blow-down map for the embedded submanifold Y then for any $p \in X_{Y,S}$ the differential*

$$(\beta_{Y,S})_*\colon {}^bT_pX_{Y,S} \to T_qX, \quad q = \beta_{Y,S}(p) \tag{5.26}$$

has range T_yY if $y \in Y$ and is otherwise surjective; its null space always contains the tangents to the fibres of the front face.

The lifting properties of functions are also of fundamental interest. Suppose that $f \in \mathscr{C}^\infty(X)$ is the defining function for an embedded hypersurface, i.e. $df \neq 0$ on $\{f=0\}$. Then there are three distinct cases where the pull-back is easy to understand. First of course

$$\begin{gathered} f \neq 0 \text{ on } Y \;\Rightarrow\; \beta^*_{Y,S}f = f' \\ \text{defines a hypersurface disjoint from } \mathrm{ff}(X_{Y,S}). \end{gathered} \tag{5.27}$$

The second case is where $f=0$ on Y but f is not itself parabolic:

(5.28)
$$f=0 \text{ on } Y,\ df|_Y \notin S\,\forall y\in Y \Rightarrow \beta^*_{Y,S}f=\varrho_{\mathrm{ff}}f'$$
where f' defines a hypersurface containing $\beta^*_{Y,S}(S')$.

Finally in the third case f is parabolic:

(5.29)
$$f=0 \text{ on } Y,\ df|_Y \in S\ \forall y\in Y \Rightarrow \beta^*_{Y,S}f=\varrho^2_{\mathrm{ff}}f'$$
where f' defines a hypersurface containing $\beta^*_{Y,S}(S^\circ)$.

In both cases (5.28) and (5.29) the differentials of f' and ϱ_{ff} are independent where $f'=0$. In these three cases then the lift of a hypersurface $H=\{f=0\}$ under $\beta_{Y,S}$ is well-defined; we denote the lift of the hypersurface by $\beta^*_{Y,S}(H)=\{f'=0\}$; it is never tangent to the front face.

§ 6. Parabolic blow-up in manifolds with corners

For our applications we will require something slightly more general than the parabolic blow-up of a manifold along a submanifold. In particular we have to deal with the case of manifolds with corners. Our conventions, together with some results, on manifolds with corners are taken from [19]. By a manifold with corners we mean a space locally homeomorphic to $\mathbf{R}^n_k$, which is just the product of k closed half-lines and $n-k$ lines, with $\mathscr{C}^\infty$ transition functions and such that any boundary hypersurface is embedded.

This last global condition is not really crucial to most developments but it greatly simplifies the notation. Moreover it is true for manifolds with boundary and is preserved under the taking of products. As shown below appropriate parabolic blow-up also leads to a new manifold with corners in this sense, so this class of spaces is large enough to handle the problems of interest here.

If we consider the model space $\mathbf{R}^n_k$ with $Y=\{0\}$ we can easily see that some additional conditions are necessary on the subspace S for the parabolic blow-up to be well-behaved. In fact the only real requirement is that the dilation structure defined by the annihilator $S^\circ\subset\mathbf{R}^n$ and a complementary subspace S' must carry $\mathbf{R}^n_k$ to itself. This is obviously the case if S° and S' are complementary in the strong sense that

(6.1)
$$\mathbf{R}^n_k=(\mathbf{R}^n_k\cap S^\circ)\times(\mathbf{R}^n_k\cap S').$$

This just means that any half-line factors which are not in S° *must* be in S'. This is both a restriction on S, and then on the choice of S'. Certainly it is a consequence of (6.1) that the S-parabolic blow-up of $\mathbf{R}^n_k$ along $\{0\}$ can be defined by

$$\mathbf{R}^n_{k\{0\},S} \subset \mathbf{R}^n_{\{0\},S} \tag{6.2}$$

and is then a submanifold with corners. In fact the existence of S' such that (6.1) holds is the usual cleanness condition on S°, or equivalently S. Thus if $x_1, \dots, x_k$ are the first k coordinates then we require

$$\sum_{i=1}^{k} a_i\, dx_i = 0 \text{ on } S^\circ \quad \Rightarrow \quad a_i\, dx_i = 0 \text{ on } S^\circ \quad \forall i = 1, \dots, k. \tag{6.3}$$

If $Y \subset X$ is a submanifold of a manifold with corners X then we say that Y is clean if it is closed and embedded in the usual sense and near each boundary point there are local coordinates $x_1, \dots, x_n$ in which X is locally $\mathbf{R}^n_k$, with the point being the origin, and Y is the intersection of $\mathbf{R}^n_k$ and a clean linear subspace in the sense of (6.3). The inward-pointing part of the conormal fibre to Y, at a boundary point, is a manifold with corners. We say that a subbundle is clean if each fibre is clean in the fibre of N^*Y in the sense of (6.3).

Any manifold with corners, X, can be embedded in a manifold without boundary, $\tilde{X}$, of the same dimension. If $Y \subset X$ is clean, closed and embedded then it can be extended to an embedded submanifold $\tilde{Y}$ of $\tilde{X}$.

THEOREM 6.4. *Let X be a manifold with corners, Y a clean (closed embedded) submanifold of X and $S \subset N^*Y$ a clean subbundle, then the closure, $X_{Y,S}$, of the pre-image of $X \setminus Y$ in $\tilde{X}_{\tilde{Y},S}$ is a manifold with corners which has a $\mathscr{C}^\infty$ structure independent of the choice of extension.*

Proof. The structure as a manifold with corners is clear from the lifting results (5.27)–(5.29).

Thus each boundary hypersurface of X lifts to a boundary hypersurface of $X_{Y,S}$:

$$\beta^*_{Y,S}(B) = \mathrm{cl}(\beta^{-1}_{Y,S}(B \setminus Y)). \tag{6.5}$$

Moreover the lift of Y is itself a boundary hypersurface, the front face. Thus, in general, the maximum codimension of a corner of $X_{Y,S}$ is one greater than of X.

For manifolds with corners X, X' there is special class of $\mathscr{C}^\infty$ maps, namely b-maps. The defining condition is just that the pull-back of a defining function for a boundary hypersurface should be a product of powers of defining functions for boundary hypersurfaces. Thus if ϱ_i, $i=1,\dots,q$ are defining functions for the q distinct boundary hypersurfaces of X and ϱ_i', $i=1,\dots,q'$ those for X' then a $\mathscr{C}^\infty$ map $F\colon X\to X'$ is a b-map if

$$F^*\varrho_i' = \alpha\prod_{j=1}^{q}\varrho_j^{e(i,j)},\quad 0\neq\alpha\in\mathscr{C}^\infty(X) \tag{6.6}$$

for some non-negative integers $e(i,j)$.

For $\beta_{Y,S}\colon X_{Y,S}\to X$ if the boundary hypersurfaces of X are labelled $B_j=\{\varrho_j=0\}$, $j=1,\dots,q$ then we can label the boundary faces of $X_{Y,S}$ as $B_0'=\mathrm{ff}(X_{Y,S})$, $B_j'=\beta^*_{X,Y}B_j$, $j=1,\dots,q$.

LEMMA 6.7. *For the S-parabolic blow-up of a clean submanifold $Y\subset X$ of a manifold with corners, $X_{Y,S}$, the blow-down map is a b-map and*

$$e(i,i') = \begin{cases} 0 & i\neq i',\ i'\neq 0 \\ 1 & i=i' \\ d(i) & i'=0 \end{cases} \tag{6.8}$$

where

$$\begin{aligned} &d(i)=0 \quad \text{if} \quad Y\cap B_i=\emptyset \\ &d(i)=1 \quad \text{if} \quad Y\subset B_i \text{ but } N^*B_i\not\subset S \\ &d(i)=2 \quad \text{if} \quad Y\subset B_i \text{ and } N^*B_i\subset S. \end{aligned} \tag{6.9}$$

We also need the following consequences of Lemma 5.24:

LEMMA 6.10. *Under the S-parabolic blow-down map for a clean submanifold, Y, of a manifold with corners, X, the space $\mathscr{V}_b(Y,S)$, of vector fields tangent to the boundary of X and in the space $\mathscr{V}(Y,S)$ defined by (5.23), lifts into $\mathscr{V}_b(X_{Y,S})$.*

COROLLARY 6.11. *If $\beta_{Y,S}\colon X_{Y,S}\to X$ is the S-parabolic blow-down map for a clean submanifold Y of a manifold with corners then for any $p\in X_{Y,S}$ the differential*

$$(\beta_{Y,S})_*\colon {}^bT_pX_{Y,S}\to {}^bT_qX,\quad q=\beta_{Y,S}(p) \tag{6.12}$$

has range ${}^bT_yY$ if $y\in Y$ and is otherwise surjective, its null space always contains the tangent space to the fibres of the front face of $X_{Y,S}$.

We have already encountered one case of parabolic blow-up. Namely if X is a manifold with boundary then

$$X_{\partial X, N^*\partial X} \cong X_{1/2}. \tag{6.13}$$

§ 7. Stretched product

Let X be a compact manifold with boundary and $\Theta \in \mathscr{C}^\infty(\partial X, T^*X)$ a 1-form on X, at ∂X, which pulls back to be non-vanishing as a 1-form on ∂X. We now proceed to define the Θ-stretched product, X^2_Θ, of X with itself.

In the product $X^2 = X \times X$ the boundary of the diagonal is an embedded submanifold sitting in the corner

$$\partial\Delta \hookrightarrow \partial X \times \partial X \hookrightarrow X \times X. \tag{7.1}$$

Moreover, $\partial\Delta$ is certainly clean in the sense of (6.3) since it is an embedded submanifold of the corner, so all differentials of boundary defining functions vanish on it.

The dual bundle to $N(\partial\Delta)$, $N^*(\partial\Delta)$, can be identified with the annihilator of $T\partial\Delta$ in $T^*_{\partial\Delta}X^2$. The 1-form Θ gives a section of $N^*\partial\Delta$, namely

$$\Theta_2 = \pi_L^*\Theta - \pi_R^*\Theta \quad \text{over } \partial\Delta. \tag{7.2}$$

By assumption on Θ this section is non-vanishing. Its span $S \subset N^*(\partial\Delta)$ is a line bundle, so its annihilator $H \subset N(\partial\Delta)$ is hyperspace bundle. We define the Θ-stretched product to be the S-parabolic blow-up of X^2 along $\partial\Delta$:

$$X^2_\Theta = X^2_{\partial\Delta, S}, \quad \beta^{(2)}_\Theta : X^2_\Theta \to X^2. \tag{7.3}$$

Of course we need to check the cleanness assumption on H to apply Theorem 6.4. This follows from the fact that Θ is non-vanishing on the boundary, so no non-trivial linear combination of boundary differentials vanishes on H, a trivial case of (6.3).

Since the inward-pointing part of the normal bundle to $\partial\Delta$ is a quarter space ($\mathbf{R}^{d+2}_2$) bundle, the front face of X^2_Θ is a quarter sphere bundle over the boundary of X:

$$\begin{array}{ccc} \mathbf{S}^{d+1}_2 & \text{———} & \mathrm{ff}(X^2_\Theta) \\ & & \Big\downarrow \beta^{(2)}_\Theta \\ & & \partial X. \end{array} \tag{7.4}$$

Recall from (5.20) that there are two natural submanifolds of the front face, we shall denote them $H_2 = \beta^{(2)*}_\Theta(H)$ and $H_2^\perp = \beta^{(2)*}_\Theta(H^\perp)$.

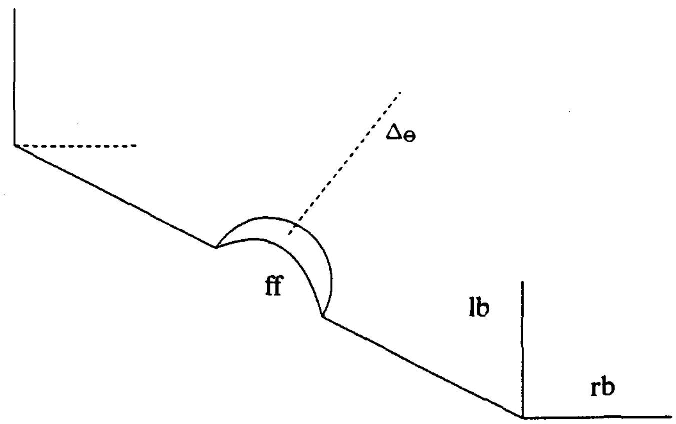

Fig. 1. The Θ-stretched product

Consider the action of the Lie algebra $\mathcal{V}_\Theta$ on X^2 on the left factor. Since all these vector fields are tangent to the left boundary of X^2 they are certainly tangent to $\partial\Delta$. By definition the pairing $\Theta(v)$ vanishes quadratically at the boundary so that the pairing with Θ_2 satisfies

$$d[\Theta_2(v)] = 0 \quad \text{at } \partial\Delta. \tag{7.5}$$

This shows that Lemma 5.24 applies and that all these vector fields lift to be $\mathcal{C}^\infty$ on X^2_Θ, where we denote the Lie algebra $\mathcal{V}_{\Theta,L}$. It therefore consists of $\mathcal{C}^\infty$ vector fields tangent to all boundaries, including the front face, and tangent to $H_2^\perp$. The submanifold $H_2^\perp$ just consists of two points in each fibre, H being a hyperplane; they lie in the corner of the front face.

The fibre, $\mathrm{ff}_p(X^2_\Theta)$, of $\mathrm{ff}(X^2_\Theta)$ over $p\in\partial X$ can be identified, although not quite naturally, as a quotient related to X_p. By definition $\mathrm{ff}_p(X^2_\Theta)$ is a quotient of the inward-pointing part of the normal space to $\partial\Delta$ at (p,p). The latter is just

$$[X_p\times X_p]/T_{(p,p)}\partial\Delta.$$

Since the tangent space to the diagonal in $\partial X\times\partial X$ can be naturally identified with $T_p\partial X$ for either factor the normal space can be identified with

$$X_p\times N_p^+X. \tag{7.6}$$

The (quotient) $\mathbf{R}^+$-action on N_pX is the standard one, so by choosing any positive element $\psi_p \in N_p X$ we obtain an identification

$$\mathrm{ff}_p(X^2_\Theta) \cong (X_p \times N_p^+)/\mathbf{R}^+. \tag{7.7}$$

Different choices of ψ_p can give different identifications (7.7). Since the action of ${}^\Theta T_p X$ on X_p consists of vector fields which are homogeneous of degree zero under this action we find

PROPOSITION 7.8. *The lift, $\mathcal{V}_{\Theta,L}$, of the structure algebra $\mathcal{V}_\Theta$ from the left factor of X to X^2_Θ restricts to* $\mathrm{ff}(X^2_\Theta)$ *to give an action of ${}^\Theta TX$ and spans, at each point of the interior of the front face, the tangent space to the fibres of* (7.4).

Let us label the three boundary hypersurfaces of X^2_Θ as

$$\mathrm{lb}(X^2_\Theta) = \beta^{(2)*}_\Theta(\partial X \times X), \quad \mathrm{rb}(X^2_\Theta) = \beta^{(2)*}_\Theta(X \times \partial X), \quad \mathrm{ff}(X^2_\Theta) = \beta^{(2)*}_\Theta(\partial \Delta). \tag{7.9}$$

Now, the diagonal $\Delta \subset X^2$ lifts to X^2_Θ as an embedded submanifold, Δ_Θ, meeting the boundary only in the interior of $\mathrm{ff}(X^2_\Theta)$ and doing so transversally. An important consequence of this is that if $P \in \mathrm{Diff}^m_\Theta(X)$ is elliptic (in the sense that $\sigma_m(P) \neq 0$ on ${}^\Theta T^*X \setminus 0$), then it lifts to X^2_Θ, from the left factor, to a $\mathcal{C}^\infty$ differential operator $\tilde{P}$ which is transversally elliptic to Δ_Θ. The lift, $\tilde{P}$, is totally characteristic and in terms of the identification (7.7), restricts to the front face to give the normal operator as a family of left-invariant differential operators on the Lie groups G_p.

More geometrically we also see that we can identify

$$N\Delta_\Theta \cong {}^\Theta TX, \quad N^*\Delta_\Theta \cong {}^\Theta T^*X. \tag{7.10}$$

This will allow the symbol of a pseudodifferential operator to be transferred to the bundle ${}^\Theta T^*X$.

We also need to consider densities on X^2_Θ. Directly from the definition of the parabolic blow-up in local coordinates it follows that a $\mathcal{C}^\infty$ density on X^2 lifts to be $\mathcal{C}^\infty$ on X^2_Θ. In fact this lifting gives

$$\beta^{(2)*}_\Theta : \mathcal{C}^\infty(X^2;\Omega) \to \varrho^N_{\mathrm{ff}} \mathcal{C}^\infty(X^2_\Theta;\Omega), \quad N = \dim X + 1. \tag{7.11}$$

This is not surjective, but

$$\nu \neq 0 \text{ on } X^2 \quad \Rightarrow \quad \beta^{(2)*}_\Theta \nu / \varrho^N_{\mathrm{ff}} \neq 0 \text{ on } X^2_\Theta. \tag{7.12}$$

Let us set $\pi^{(2)}_{\Theta,L} = \pi_L \cdot \beta^{(2)}_\Theta$ and $\pi^{(2)}_{\Theta,R} = \pi_R \cdot \beta^{(2)}_\Theta$ the two maps from X^2_Θ back to X. These are examples of b-submersions. Any b-map, as defined by (6.6) leads to a map on the b-tangent space

$$ {}^bF_*: {}^bT_pX \to {}^bT_{F(p)}X' \quad \forall p \in X. \tag{7.13} $$

Such a b-map is a b-submersion if it is surjective and ${}^bF_*$ surjective for each $p \in X$.

PROPOSITION 7.14. *Both maps $\pi^{(2)}_{\Theta,L}$ and $\pi^{(2)}_{\Theta,R}$ are b-submersions which are transversal to the lifted diagonal; if ϱ is a defining function for the boundary of X then*

$$ \begin{aligned} (\pi^{(2)}_{\Theta,L})^*\varrho &= \varrho_{\mathrm{ff}}\varrho_{\mathrm{lb}} \\ (\pi^{(2)}_{\Theta,R})^*\varrho &= \varrho_{\mathrm{ff}}\varrho_{\mathrm{rb}}. \end{aligned} \tag{7.15} $$

Proof. It suffices to consider $\pi^{(2)}_{\Theta,L}$. The transversality to the lifted diagonal follows from Proposition 7.8. That $\pi^{(2)}_{\Theta,L} = \pi_R \cdot \beta^{(2)}_\Theta$ is a b-map follows from the fact that each of the factors is a b-map. Thus we need only show the surjectivity of the b-differential. This certainly follows if we show that every element $V \in \mathcal{V}_b(X)$ can be lifted to an element of $\mathcal{V}_b(X^2_\Theta)$. Lifting $V \in \mathcal{V}_b(X)$ to the left factor of X^2 gives a vector field which is tangent to the boundary, but not necessarily to $\partial\Delta$. However, if V_L and V_R are the lifts from left and right, then $V_L + V_R$ is tangent to all boundary components and to the diagonal. Thus we can simply lift $V_L + V_R$ to X^2_Θ using Lemma 6.10. The formulae (7.15) now follow from Lemma 6.7.

§ 8. Resolvent for the ball

The Lie theoretic model problem for the Laplace operator on a strictly pseudoconvex domain is given by the Laplace operator for the metric of constant holomorphic sectional curvature on the unit ball, $\mathbf{CB}^{n+1} \subset \mathbf{C}^{n+1}$. We shall normalize the curvature to be -1. In holomorphic coordinates $(z_1, \ldots, z_{n+1})$ the appropriate metric is given by

$$ \begin{aligned} ds^2 &= \sum_{i,j=1}^{n+1} g_{ij}\, dz_i\, d\bar{z}_j \\ g_{ij} &= -4\partial_i \bar{\partial}_j[\log(1-|z|^2)]. \end{aligned} \tag{8.1} $$

The ball is an Hermitian symmetric space of rank 1, the group of automorphisms is $SU(n+1, 1)$. The Laplace operator is invariant under the automorphism group and therefore the kernel of the resolvent is simply a function of the distance between a pair of points. To determine this function it therefore suffices to study the 'radial' part of the Laplace operator. Since $SU(n+1)$ is the subgroup of the automorphism group fixing the point 0, a function of $r=|z|$ is also radial with respect to the Bergman metric. The vector field $R=\frac{1}{2}(1-r^2)D_r$ is easily seen to have unit length and the volume form of the Bergman metric is given by

$$dV=\frac{4^{n+1}dV_e}{(1-r^2)^{n+2}}$$

where dV_e is the Euclidean volume form on $\mathbf{C}^n$.

Thus the Dirichlet form for a radial function f is

$$D(f)=\int_0^1 |Rf|^2 r^{2n+1}\frac{4^{n+1}\,dr}{(1-r^2)^{n+2}}. \tag{8.2}$$

If we make the substitution $\tau=r^2$, setting $g(\tau)=f(r)$, the Dirichlet form and norm become:

$$\begin{aligned} D(g)&=\frac{1}{2}\int_0^1\left|\frac{Dg}{D\tau}\right|^2\frac{4^{n+1}\tau^{n+1}\,d\tau}{(1-\tau)^n}\\ \|g\|^2&=\frac{1}{2}\int_0^1|g|^2\frac{4^{n+1}\tau^n\,d\tau}{(1-\tau)^{n+2}}. \end{aligned} \tag{8.3}$$

Integrating by parts we obtain the radial eigenvalue equation for the Bergman Laplacian:

$$g_{\tau\tau}+\left[\frac{n+1}{\tau}+\frac{n}{1-\tau}\right]g_\tau+\frac{s(n+1-s)}{\tau(1-\tau)^2}g=0. \tag{8.4}$$

We have introduced the analytic energy parameter $\lambda=s(n+1-s)$; the 'physical' resolvent set corresponds to $\Re(s)>(n+1)/2$ and the spectrum to the line $\Re(s)=(n+1)/2$. The radial equation is a classical Riemann $\mathscr{P}$-equation:

$$\mathscr{P}\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & s & ;\tau \\ -n & 0 & n+1-s & \end{Bmatrix}. \tag{8.5}$$

The solution with a pole at $\tau=0$ and the correct behavior at $\tau=1$ is given by

$$r(\tau;s)=c_n\frac{\Gamma(s)^2}{\Gamma(2s-n)}(\tau-1)^s\,{}_2F_1(s,s,2s-n;1-\tau). \tag{8.6}$$

The constant, c_n, is determined so as to make r a fundamental solution. Using the Gauss relations for hypergeometric functions found in [1; p. 560] one can show that, in a neighborhood of $\tau=0$, r can be written as:

$$r(\tau; s) = \tau^{-n} G_1(\tau; s) + \log(\tau)\, G_2(\tau; s) \tag{8.7}$$

where G_1 and G_2 are analytic in a neighborhood of $\tau=0$.

The kernel $r(\tau; s)$ has simple poles at

$$s \in -\mathbf{N}_0 = \{0, -1, -2, \ldots\}.$$

An easy computation shows that

$$P_k = \lim_{s \to -k} (s+k)\, r(\tau; s) = \sum_{m=0}^{k} a_{m,k}(1-\tau)^{m-k}.$$

As a residue of the resolvent P_k acts on $\mathscr{C}_c^\infty(\mathbf{CB}^{n+1})$ as a projection onto a space of solutions to the equation $[\Delta + k(n+1+k)]\,u=0$; if $k>0$ these spaces are infinite dimensional. These solutions are of the form $u \in (1-\tau)^{-k}\mathscr{C}^\infty(\mathbf{CB}^{n+1})$. The fact that the rank of the projection is infinite means that, for the general Θ-metrics these points require special treatment; this will be done elsewhere.

To solve the indicial problem in the general case for s a half-integer less than $\frac{1}{2}(n+1)$ we will need the detailed behaviour of r at such values. The ratio

$${}_2F_1(s, s, 2s-n; 1-\tau)/\Gamma(2s-n)$$

is analytic at these points. A simple calculation shows that

$$r\left(\tau; \frac{1}{2}k\right) = \Gamma^2\left(\frac{1}{2}k\right) \sum_{m=n+1-k}^{\infty} a_{k,m}(1-\tau)^{m+k/2} \tag{8.8}$$

for k an odd integer less than $n+1$, or an even integer between 0 and $n+1$.

Since each complex line through the origin is a totally geodesic submanifold and the restriction of the metric to such a line is the standard hyperbolic metric on the disk, one easily shows that the variable τ satisfies:

$$1-\tau = \left[\cosh \frac{1}{2} d(z, 0)\right]^{-2} \tag{8.9}$$

where $d(z, w)$ is the distance between z and w measured in the Bergman metric. To complete the story we need to find a formula for the distance between two points in the

Bergman metric. It is more convenient to work in the biholomorphically equivalent domain defined by the hyperquadric. If z and w are two points in $\mathbf{C}^n$ then we define:

$$Q(z, w) = -\frac{i}{2}(z_1 - \bar{w}_1) - \frac{1}{2}(z_2 \bar{w}_2 + \ldots + z_{n+1} \bar{w}_{n+1}). \tag{8.10}$$

The region equivalent to the ball is $\mathbf{Q}^+ = \{z; Q(z, z) > 0\}$. The fundamental point pair invariant on $\mathbf{Q}^+$ is:

$$(z, w) = \frac{Q(z, w)}{[Q(z, z)\, Q(w, w)]^{1/2}}. \tag{8.11}$$

The surface $\{z_2 = \ldots = z_{n+1} = 0\}$ is totally geodesic and the metric restricted to this set is given by $ds^2 = 4dw_1\, d\bar{w}_1/(w_1 - \bar{w}_1)^2$. Using this and the homogeneity of the domain one easily shows that:

$$|(z, w)| = \cosh \frac{1}{2} d(z, w). \tag{8.12}$$

To summarize:

THEOREM 8.13. *For the Bergman Laplacian on the ball the resolvent kernel is given by*

$$R(z, w; s) = r(|z, w)|^{-2}; s) \otimes dV_{\mathrm{Berg}} \tag{8.14}$$

where r is given by (8.6) *and the eigenvalue, λ, is related to s by*

$$\lambda = s(n+1-s). \tag{8.15}$$

We shall generalize this rather explicit formula for the resolvent kernel to more general metrics below. We express this general result in terms of the regularity of the kernel when pulled-back to the Θ-stretched product $[\mathbf{CB}^{n+1}_{1/2}]^2_\Theta$. Naturally we wish to see how the kernel for the ball lifts in this sense. Let $R_\Theta = \beta^{(2)*}_\Theta R$.

PROPOSITION 8.16. *The resolvent kernel, lifted to $[\mathbf{CB}^{n+1}_{1/2}]^2_\Theta$, satisfies*

$$R_\Theta(s) = \varrho_{\mathrm{lb}}^{2s} \varrho_{\mathrm{rb}}^{2s} R'_\Theta(s) \tag{8.17}$$

where R'_Θ is $\mathscr{C}^\infty$ away from the lifted diagonal and is a classical conormal distribution of order -2 along Δ_Θ (see also section 11).

Proof. Obviously the way to understand the lift of the kernel to the Θ-stretched product is to first understand the lift of the point-pair invariant (z, w). We shall show that

$$|\beta_{\Theta}^{(2)*}(z, w)|^{-1} = \varrho_{\mathrm{lb}}\varrho_{\mathrm{rb}} \tag{8.18}$$

is simply the product of defining functions for the let and right boundaries of the Θ-stretched product.

It is convenient to introduce 'polar' coordinates for the stretched product. First we introduce convenient coordinates for $\mathbf{Q}^{+}_{1/2}$. Since $Q(z, z)$ is a defining function for the boundary we can take $\varrho = Q(z, z)^{1/2}$, $u = \Re(z_1)$ and $(z_2, \ldots, z_{n+1})$ as coordinates. We shall denote coordinates on the left factor with primes, $(\varrho', u', z_2, \ldots, z'_{n+1})$. Setting $t = u - u' + \Im\sum_{j=2}^{n+1}\bar{z}_j(z_j - z'_j)$, an easy calculation shows that

$$dt|_{\partial\Delta} = \Theta|_{\partial\Delta}$$

and thus we can define 'polar' coordinates for the parabolic blowup by:

$$R^4 = \left(\varrho^2 + \varrho'^2 + \frac{1}{2}\sum_{i=2}^{n}|z_i - z'_i|^2\right)^2 + t^2$$

$$\varrho_{\mathrm{lb}} = \frac{\varrho}{R}, \quad \varrho_{\mathrm{rb}} = \frac{\varrho'}{R}, \quad T = \frac{t}{R^2}, \quad Z_i = \frac{z_i - z'_i}{\sqrt{2}R}.$$

Near any point on the front face an appropriately chosen subset of $(\varrho_{\mathrm{lb}}, \varrho_{\mathrm{rb}}, T, Z)$ along with (R, u, z) will define a coordinate system for the Θ-stretched product. Using (8.10) and (8.11) we see that in these coordinates:

$$\beta_{\Theta}^{(2)*}(z, w) = (\varrho_{\mathrm{lb}}^2 + \varrho_{\mathrm{rb}}^2 + |Z|^2 - iT)/(2\varrho_{\mathrm{lb}}\varrho_{\mathrm{rb}}).$$

It follows easily from the definition of the polar coordinates that the numerator in this formula is of modulus 1. Thus, changing the defining functions by a factor of $\sqrt{2}$, we arrive at (8.18). Notice that although we have proved (8.18) for the upper half-space model it must hold globally for the stretched-product of the ball with itself, because of the invariance of (z, w). Note also that $|(z, w)|$ is, when lifted to $[\mathbf{CB}^{n+1}_{1/2}]^2_{\Theta}$, homogeneous of degree zero under the $\mathbf{R}^+$-action corresponding to the blow-up. Thus the behaviour of R_{Θ} is determined by its behaviour away from the front face.

To prove (8.17) we simply need to examine the behaviour of r in (8.6) and use (8.14). As a function of τ, r is certainly smooth away from $\tau=0$ and $\tau=1$, which are the singular points of (8.5). From (8.6), (8.14) and (8.18) we see that (8.17) holds away from the lifted diagonal. The singularity at the diagonal is, away from the boundary, necessarily a conormal distribution of order -2. From the homogeneity this extends down to the front face. This proves the proposition.

If we let $z_j=\eta_j+iy_j$ then the metric of constant holomorphic sectional curvature -1 is given by

$$ds^2=4\left(\frac{d\varrho}{\varrho}\right)^2+2\sum_{j=1}^{n}\left[\left(\frac{d\eta_j}{\varrho}\right)^2+\left(\frac{dy_j}{\varrho}\right)^2\right]+\frac{\Phi^2}{\varrho^4}; \tag{8.19}$$

where $\Phi=du-\sum_{j=1}^{n}(y_j\,d\eta_j-\eta_j\,dy_j)$. A unit frame is given by

$$\frac{1}{2}\varrho\partial_\varrho,\quad \frac{1}{\sqrt{2}}\varrho(\partial_{\eta_j}+y_j\partial_u),\quad \frac{1}{\sqrt{2}}\varrho(\partial_{y_j}-\eta_j\partial_u),\quad \varrho^2\partial_u.$$

In the bulk of the paper we will consider operators acting on half densities, so for the sake of consistency we will restate this theorem in that context. The natural way to extend the Laplacian to half-densities in the interior is to use the Riemannian half-density to trivialize the bundle, so setting

$$P(s)\psi=\mu P(\psi\mu^{-1}),\quad \psi\in\mathscr{C}^\infty(\mathring{X};\Omega^{1/2}),\quad \mu=dg^{1/2}. \tag{8.20}$$

Of course dg is a non-vanishing section of the Θ-density bundle $\Omega_\Theta=\varrho^{-N}\Omega$, $N=\dim X+1$. Thus, (8.20) gives an action of P on $\mathscr{C}^\infty(X;\Omega_\Theta^{1/2})$.

The resolvent of P acting on the Θ-half-densities is also obtained by conjugation, so can be expressed in terms of the kernel R by

$$R''=R\,dg_L^{1/2}\,dg_R^{1/2}. \tag{8.21}$$

If we use the discussion of the lifts of densities in § 7 we see that R'' lifts to a half-density on $[\mathbf{CB}_{1/2}^{n+1}]_\Theta^2$ which is of the form

$$\beta_\Theta^{(2)*}(R''(s))=R_\Theta(s)\varrho^{-N/2}\nu,\quad 0\neq\nu\in\mathscr{C}^\infty([\mathbf{CB}_{1/2}^{n+1}]_\Theta^2;\Omega^{1/2}). \tag{8.22}$$

We absorb the extra singular factor at the front face of the stretched product into the density bundle, so consider the bundle $\Omega_\Theta^{1/2}=\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$. Thus R'' is just a non-vanishing $\mathscr{C}^\infty$ section of this bundle multiplied by R_Θ in (8.17). Although the kernel in this form is attractively symmetric we have chosen (perhaps unwisely!) to express the results on the Θ-pseudodifferential calculus in terms of operators acting on ordinary half-densities. This amounts to conjugating the operator by a power of a boundary defining function, since by definition

$$\Omega_\Theta^{1/2} = \varrho^{-N/2}\Omega^{1/2}. \tag{8.23}$$

Having chosen such a defining function let us set

$$\tilde{P}(s)\,u = \varrho^{N/2}P(s)\,\varrho^{-N/2}u, \quad u \in \mathscr{C}^\infty(\mathbf{CB}_{1/2}^{n+1};\Omega^{1/2}), \tag{8.24}$$

giving the action on ordinary half-densities. Similarly set

$$\tilde{R}(s) = \varrho_L^{N/2}\,R''(s)\,\varrho_R^{-N/2} = R[\varrho_L^{N/2}\,dg_L^{1/2}]\,[\varrho_R^{-N/2}\,dg_R^{1/2}]. \tag{8.25}$$

Of course since ϱ_L/ϱ_R lifts to $[\mathbf{CB}_{1/2}^{2n}]_\Theta^2$ to be of the form $\varrho_{\mathrm{lb}}/\varrho_{\mathrm{rb}}$ we see that the lift of the kernel $\tilde{s}$ can be written in the form

$$\beta_\Theta^{(2)*}\tilde{R}(s) = \varrho_{\mathrm{lb}}^{2s}\,\varrho_{\mathrm{rb}}^{2s-N}\,R'_\Theta(s)\,\nu, \quad 0 \neq \nu \in \mathscr{C}^\infty([\mathbf{CB}_{1/2}^{n+1}]_\Theta^2;\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}), \tag{8.26}$$

with R' as described in Proposition 8.16.

We have not actually shown here that $R(s)$ is the resolvent kernel of the Laplacian, although we have certainly checked that it is a right inverse on $\dot{\mathscr{C}}^\infty(\mathbf{CB}_{1/2}^{n+1})$. To show that it is the resolvent it suffices to show that it is bounded on $L^2(\mathbf{CB}_{1/2}^{n+1};dg)$. This is a consequence of the boundedness properties of Θ-pseudodifferential operators discussed in § 13 and the interpretation of the form (8.26) for the kernel, which in the notation of § 12 is just

$$\tilde{R}(s) \in \Psi_\Theta^{-2;2s,2s-N}(\mathbf{CB}_{1/2}^{n+1};\Omega^{1/2}). \tag{8.27}$$

As noted above the asymmetry between the indices at the left boundary and the right boundary is a consequence of our forcing the operator to act on regular half-densities, on which it is not formally self-adjoint for s real.

§ 9. Commutation of blow-ups

When the technique of parabolic blow-up is used below in the construction of the triple product of the space X with itself the iterative blow-up of submanifolds becomes necessary. We now give a sufficient condition to allow such iterative blow-up and then discuss a commutativity result permitting the order of the blow-ups to be changed.

Suppose that $Y_1 \subset Y_2 \subset X$ are two embedded submanifolds where, for the moment, we assume that $\partial X = \emptyset$. Let $S_1 \subset N^*Y_1$ and $S_2 \subset N^*Y_2$ be the subbundles of the respective conormal bundles with respect to which the blow-ups are to be parabolic. The extra condition we impose is

$$S_1 \cap N^*_{Y_1} = S_2 \cap N^*_{Y_1} Y_2. \tag{9.1}$$

Notice that, since $Y_1 \subset Y_2$, $N^*_{Y_1} Y_2 \subset N^* Y_1$ in a natural way.

PROPOSITION 9.2. *Suppose $Y_1 \subset Y_2 \subset X$ are embedded submanifolds and $S_i \subset N^* Y_i$, $i=1,2$ satisfy* (9.1). *Let $\beta_1 : X_{Y_1,S_1} \to X$ be the blow-down map for X, S_1-parabolically blown-up along Y_1, then the lift of Y_2,*

$$Y_{2,1} = \beta_1^*(Y_2) = \mathrm{cl}[\beta_1^{-1}(Y_2 \setminus Y_1)] \subset X_{Y_1,H_1}, \tag{9.3}$$

is an embedded submanifold meeting the boundary cleanly with the closure, $S_{2,1}$, of $S_2 \restriction_{(Y_2 \setminus Y_1)}$ being a clean subbundle of $N^* \beta_1^*(Y_2)$.

Proof. Clearly the regularity statements are local near points of Y_1. Thus we may linearize the geometry and take local coordinates (x, y, z) in which

$$\begin{aligned} Y_1 &= \{x=0, y=0\}, \quad S_1 = \mathrm{sp}\{dx', dy'\}, \\ Y_2 &= \{x=0\}, \quad S_2 = \mathrm{sp}\{dx'\} \end{aligned} \tag{9.4}$$

where $x=(x', x'') \in \mathbf{R}^k$ and $y=(y', y'') \in \mathbf{R}^p$ is a further splitting of these coordinates. The form of S_1 and S_2 is possible precisely because of (9.1).

In these local coordinates the result is obvious since Y_2 and S_2 are homogeneous with respect to the dilation used to define the S_1-parabolic blow-up along Y_1 if the transversal surface is taken to be $\{x''=y''=0\}$.

The coordinate form (9.4) also shows that a result analogous to Proposition 9.2 is true for the S_2-parabolic blow-up along Y_2. Thus the lift

$$Y_{1,2} = \beta_2^{-1}(Y_1) \tag{9.5}$$

is a $\mathscr{C}^\infty$ submanifold meeting the boundary cleanly and $\beta_2^* : N^*_{\tau'} Y_1 \to N^*_\tau Y_{1,2}$, for each $\tau' = \beta_2(\tau), \tau \in Y_{1,2}$. Then we set

$$S_{1,2} = \beta_2^* S_1; \tag{9.6}$$

it is a clean subbundle of $N^* Y_{1,2}$. An important tool in the manipulation of the blow-ups is the following commutativity result.

PROPOSITION 9.7. *Under the hypotheses of Proposition* 9.2 *there is a diffeomor-*

phism between the two iterated blow-ups, giving a commutative diagram:

$$\begin{tikzcd}
(X_{Y_1,S_1})_{Y_{2,1},S_{2,1}} \arrow[r, leftrightarrow, "\gamma"] \arrow[d, "\beta_{2,1}"'] & (X_{Y_2,S_2})_{Y_{1,2},S_{1,2}} \arrow[d, "\beta_{1,2}"] \\
X_{Y_1,S_1} \arrow[d, "\beta_1"'] & X_{Y_2,S_2} \arrow[d, "\beta_2"] \\
X \arrow[r, leftrightarrow, "\mathrm{Id}"'] & X
\end{tikzcd}$$

Proof. Clearly the result is local near the lifts of points in Y_1. Thus we can use the local coordinate form (9.4). We proceed to compute the successively blown-up spaces. First

$$X_{Y_2,S_2}=[0,\infty)\times\{(X',X'')\in\mathbf{R}^k;\ |X'|^2+|X''|^4=1\}\times\mathbf{R}^p\times\mathbf{R}^n,\quad p+k+n=\dim X. \tag{9.9}$$

The blow-down map is

$$\beta_{Y_2,S_2}\colon (r,(X',X''),y,z)\mapsto(r^2X',rX'',y,z). \tag{9.10}$$

The lifts are also easily computed

$$Y_{1,2}=\{r=0,y=0\},\quad S_{1,2}=\operatorname{sp}\{dy'\}. \tag{9.11}$$

Thus the second blow-up gives

$$\begin{gathered}(X_{Y_2,S_2})_{Y_{1,2},S_{1,2}}=[0,\infty)\times\{(X',X'');|X'|^2+|X''|^4=1\}\\ \times\{(R,Y',Y'')\subset[0,\infty)\times\mathbf{R}^p;R^4+|Y'|^2+|Y''|^4=1\}\times\mathbf{R}^n\\ \beta_{1,2}\cdot\beta_2\colon(\varrho,(X',X''),(R,Y',Y''))\mapsto(\varrho^2R^2X',\varrho RX'',\varrho^2Y',\varrho Y'',z)\in X.\end{gathered} \tag{9.12}$$

Similarly starting up the left side of (9.8) we find

$$\begin{gathered}X_{Y_1,S_1}=[0,\infty)\times\{(\xi',\xi'',\eta',\eta'')\in\mathbf{R}^k\times\mathbf{R}^p;|\xi'|^2+|\xi''|^4+|\eta'|^2+|\eta''|^4=1\}\times\mathbf{R}^n\\ \beta_{Y_1,S_1}(s,(\xi,\eta),z)\mapsto(s^2\xi',s\xi'',s^2\eta',s\eta'',z)\in X.\end{gathered} \tag{9.13}$$

Thus we see that there is indeed a $\mathscr{C}^\infty$ map, which covers the identity on X,

$$\gamma' : (X_{Y_2,S_2})_{Y_{1,2},S_{1,2}} \ni (\varrho, (X', X''), (R, Y', Y'')) \mapsto (\varrho, R^2\xi', R\xi'', Y', Y'', z) \in X_{Y_1,S_1}, \tag{9.14}$$

which is just the S'-parabolic blow-up of the surface $\{\xi=0\}$, where $S'=\mathrm{sp}\{d\xi'\}$. This is just the definition of $(X_{Y_1,S_1})_{Y_{2,1},S_{2,1}}$ and the resulting map γ is therefore an isomorphism as claimed.

(9.15) *Remark.* This result extends immediately to the case of manifolds with corners, provided all the submanifolds and bundles are clean. We can also analyze the differential of the composite blow-down map.

COROLLARY 9.16. *Let $\beta=\beta_{2,1}\cdot\beta_1$ be the iterated blow-down map from the space $X_{2,1}=(X_{Y_1,S_1})_{Y_{2,1},S_{2,1}}$ under the conditions of Proposition 9.7 then for all $p\in X_{2,1}$*

$$\beta_* : {}^bT_pX_{2,1} \to {}^bT_qX, \quad q=\beta(p) \tag{9.17}$$

has range ${}^bT_qY_1, {}^bT_qY_2$ or ${}^bT_qX$ as $q\in Y_1, q\in Y_2\setminus Y_1$ or $q\in X\setminus Y_2$ and has null space containing the tangents to the fibres of both boundary hypersurfaces introduced in the blow-up.

§ 10. Triple product

The most intricate construction we need to carry out is of the Θ-triple product of X, which we denote X^3_Θ. This plays a fundamental rôle in our (geometric) proof of composition formulae for the Θ-pseudodifferential operators defined above. In these proofs we need certain basic properties.

First, X^3_Θ should be obtained from X^3 by a sequence of (parabolic) blow-ups of embedded submanifolds. The overall blow-down map will be denoted

$$\beta^{(3)}_\Theta : X^3_\Theta \to X^3. \tag{10.1}$$

The blow-up should be symmetric in the sense that the permuation group on X^3, generated by the exchange of factors, lifts to a group of diffeomorphisms on X^3_Θ.

Even more important than the symmetry is the requirement that the three projections

$$\pi_F, \pi_C, \pi_S : X^3 \to X^3 \tag{10.2}$$

off the left, middle and right factors of X when lifted to X^3_Θ should factor through smooth maps

$$\pi_{O,\Theta} : X^3_\Theta \to X^2_\Theta, \quad O=F,C,S. \tag{10.3}$$

Here the three suffixes 'F', 'C' and 'S' refer to the three operators involved in a composition formula $A\cdot B=C, B$ being the 'first' operator, corresponding to π_F, A being the 'second' operator, corresponding to π_S and C being the composite operator, associated to π_C. These maps lead to commutative diagrammes

$$\begin{array}{ccc} X^3_\Theta & \xrightarrow{\pi_{O,\Theta}} & X^2_\Theta \\ {\scriptstyle\beta^{(3)}_\Theta}\big\downarrow & & \big\downarrow{\scriptstyle\beta^{(2)}_\Theta} \\ X^3 & \xrightarrow{\pi_O} & X^2 \end{array} \qquad O=F,C,S.$$

The existence of the smooth maps (10.3) is important because they allow the kernels of the two operators to be lifted to X^3_Θ and the product to be projected to X^2_Θ as the kernel of the composite operator. An important property of the construction is that the three maps $\pi_{O,\Theta}$ are b-submersions in the sense of (B5.2).

Now to the actual construction. We shall construct X^3_Θ in two stages, both involving parabolic blow-ups. In X^3 we need to consider the boundaries of the three partial diagonals, since these are blown up in the process of defining X^2_Θ, so must be blown up in the construction of X^3_Θ. These three embedded submanifolds can be written

$$\gamma_O=\pi_O^{-1}[\partial\Delta],\quad \partial\Delta=\{(x,x);x\in\partial X\},\quad O=F,C,S. \tag{10.5}$$

We cannot simply blow up these submanifolds and keep the symmetry of X^3 because they meet at the boundary of the triple diagonal:

$$\begin{gathered}\partial\Delta_T=\{(p,p,p);p\in\partial X\}\subset[\partial X]^3\subset X^3\\ \partial\Delta_T=\gamma_F\cap\gamma_C=\gamma_C\cap\gamma_S=\gamma_S\cap\gamma_F=\gamma_F\cap\gamma_C\cap\gamma_S.\end{gathered} \tag{10.6}$$

This embedded submanifold lies in the codimension three boundary of X^3. At $\partial\Delta_T$ there are three forms of the type (7.2):

$$\begin{aligned}\Theta_F&=\pi_M^*\Theta-\pi_R^*\Theta\\ \Theta_C&=\pi_R^*\Theta-\pi_L^*\Theta\\ \Theta_S&=\pi_L^*\Theta-\pi_M^*\Theta.\end{aligned} \tag{10.7}$$

Any pair of these are linearly independent, but of course

$$\Theta_F+\Theta_C+\Theta_S=0. \tag{10.8}$$

The span is therefore a subbundle of rank two

$$G = \mathrm{sp}\{\Theta_F, \Theta_S, \Theta_C\} \subset N^*\partial\Delta_T. \tag{10.9}$$

Clearly G is independent of the span of the boundary conormals at $\partial\Delta_T$.

In the first step of the construction we G-parabolically blow up X^3 along the boundary of the triple diagonal and set

$$Z = (X^3)_{\partial\Delta_T, G}, \quad \beta_1 : Z \to X^3, \quad \mathrm{ff}(Z) = \beta_1^{-1}(\partial\Delta_T). \tag{10.10}$$

The three boundaries of the partial diagonals in X^3 each contain $\partial\Delta_T$ as an embedded submanifold and Proposition 9.2 applies to each of them. Thus they lift to Z as smooth manifolds

$$\gamma_{1,O} = \beta_1^*(\gamma_O) = \mathrm{cl}[\beta_1^{-1}(\gamma_O \setminus \partial\Delta_T)] \subset Z \quad \text{for} \quad O = F, C, S. \tag{10.11}$$

The images of the γ_O's in the parabolic normal bundle to $\partial\Delta_T$ are disjoint so the $\gamma_{1,O}$'s are three disjoint embedded submanifolds in Z. Moreover the three line bundles, $S_O \subset N^*\gamma_O$, spanned by Θ_O, for $O = F, C, S$ also, separately, satisfy (9.1) with respect to G. Thus Proposition 9.2 applies to show that these bundles lift to $\tilde{S}_O \subset N^*\gamma_{1,O}$. The disjointness of the three submanifolds means that we can set

$$\gamma_{1,*} = \gamma_{1,F} \cup \gamma_{1,C} \cup \gamma_{1,S}, \quad \tilde{S} = S_O \text{ over } \gamma_{1,O}, \quad O = F, C, S \tag{10.12}$$

and thereby blow up the $\gamma_{1,O}$ jointly, but independently. We set

$$X^3_\Theta = Z_{\gamma_{1,*}, \tilde{S}}. \tag{10.13}$$

This is the triple Θ-product.

We shall label the boundary hypersurfaces of X^3_Θ, all of which are embedded, as

$$\partial X^3_\Theta = \mathrm{ff} \cup \mathrm{fs} \cup \mathrm{cs} \cup \mathrm{ss} \cup \mathrm{lb} \cup \mathrm{mb} \cup \mathrm{rb}. \tag{10.14}$$

Here $\mathrm{ff}(X^3_\Theta)$ is the lift of the front face of Z, i.e. the face introduced by the blow-up of $\partial\Delta_T$. The 'sides' fs, cs and ss are the faces introduced in (10.13) by the blow-up along $\gamma_{1,O}$ for $O = F, C, S$. Finally the remaining boundary hypersurfaces lb, mb and rb are the lifts of $\partial X \times X^2, X \times \partial X \times X$ and $X^2 \times \partial X$ from X^3. To prevent confusion with the defining functions for the similarly named boundary faces of X^2_Θ we will denote by

$\varrho_{\mathrm{ff},3}, \varrho_{\mathrm{lb},3}$ and $\varrho_{\mathrm{rb},3}$ defining functions for the front face, right boundary and left boundary of X^3_Θ; the others are simply denoted $\varrho_{\mathrm{fs}}, \varrho_{\mathrm{ss}}, \varrho_{\mathrm{cs}}$ and ϱ_{mb}.

The full partial diagonals $\Delta_O \subset X^3$, also lift as clean submanifolds to X^3_Θ where they are denoted

$$\Delta_{O,\Theta} \subset X^3_\Theta. \tag{10.15}$$

PROPOSITION 10.16. *For $O=F, C, S$ there is a commutative diagram* (10.4) *where $\pi_{O,\Theta}: X^3_\Theta \to X^2_\Theta$ is a b-submersion which is transversal to the lifted partial diagonals $\Delta_{O',\Theta}$ for $O' \neq O$ and which embeds the triple lifted diagonal*

$$\Delta_{T,\Theta} = \Delta_{F,\Theta} \cap \Delta_{C,\Theta} \cap \Delta_{S,\Theta} \tag{10.17}$$

as the diagonal of X^2_Θ.

Proof. The existence of $\pi_{F,\Theta}$ is clear from Proposition 9.7, since this shows that X^3_Θ can alternatively be constructed by two blow-ups from $X \times X^2_\Theta$. Namely, first parabolically blow up the lift of $\partial\Delta_T$, then blow up the lifts of the boundaries of the two partial diagonals, i.e. the lifts of γ_C and γ_S. Thus $\pi_{F,\Theta}$ is the product of three b-maps, two blow-down maps and a projection hence is itself a b-map.

Next consider the surjectivity of the b-differential, i.e. (B5.2). From Corollary 9.16 we can calculate the range of the b-differential at each point of X^3_Θ; it is just the b-tangent space to the smallest submanifold through that point which is blown up in the construction. However these submanifolds are both graphs over X^2_Θ, so $\pi_{F,\Theta}$ always has surjective b-differential.

Similarly the transversality to the lifted diagonals corresponding to the 'other' two projections follows from Corollary 9.16, since where they meet the new boundary hypersurfaces they are transversal to the tangents to the fibres; elsewhere the transversality is obvious. That $\pi_{F,\Theta}$ restricts to Δ_T to give an embedding as the diagonal of X^2_Θ can be checked by a simple computation. The assertions for the other projections follow from the symmetry of the construction of X^3_Θ.

We note, for later reference, precisely how the defining functions of X^2_Θ lift under these blown-up projections:

$$\begin{gathered}
\pi^*_{F,\Theta}\varrho_{\mathrm{lb}} = \varrho_{\mathrm{mb}}\varrho_{\mathrm{ss}}, \quad \pi^*_{S,\Theta}\varrho_{\mathrm{lb}} = \varrho_{\mathrm{lb},3}\varrho_{\mathrm{cs}}, \quad \pi^*_{C,\Theta}\varrho_{\mathrm{lb}} = \varrho_{\mathrm{lb},3}\varrho_{\mathrm{ss}}, \\
\pi^*_{F,\Theta}\varrho_{\mathrm{rb}} = \varrho_{\mathrm{rb},3}\varrho_{\mathrm{cs}}, \quad \pi^*_{S,\Theta}\varrho_{\mathrm{rb}} = \varrho_{\mathrm{mb}}\varrho_{\mathrm{fs}}, \quad \pi^*_{C,\Theta}\varrho_{\mathrm{rb}} = \varrho_{\mathrm{rb},3}\varrho_{\mathrm{fs}}, \\
\pi^*_{F,\Theta}\varrho_{\mathrm{ff}} = \varrho_{\mathrm{ff},3}\varrho_{\mathrm{fs}}, \quad \pi^*_{S,\Theta}\varrho_{\mathrm{ff}} = \varrho_{\mathrm{ff},3}\varrho_{\mathrm{ss}}, \quad \pi^*_{C,\Theta}\varrho_{\mathrm{ff}} = \varrho_{\mathrm{ff},3}\varrho_{\mathrm{cs}},
\end{gathered} \tag{10.18}$$

§ 11. Small calculus

We now discuss the 'small' calculus of Θ-pseudodifferential operators. This symbol-filtered algebra extends the enveloping algebra $\mathrm{Diff}^*_\Theta(X)$ of $\mathscr{V}_\Theta$. The operators are fixed in terms of their Schwartz' kernels, which have conormal singularities at the diagonal of the Θ-stretched product and are otherwise smooth, as sections of the appropriate bundle, and vanish to infinite order near the left and right boundary components lifted from X^2. Such a calculus cannot contain good parametrices for its elliptic elements, the inversion of which involves boundary conditions in a generalized sense. Thus one must augment the 'small' calculus to include operators with kernels that are singular on various of the other boundary components, this is discussed in Section 12.

For the usual reasons of symmetry we shall work with half-densities. Consider continuous linear operators on sections of the half-density bundle of the rather general type:

$$A\colon \dot{\mathscr{C}}^\infty(X;\Omega^{1/2})\to \mathscr{C}^{-\infty}(X;\Omega^{1/2}). \tag{11.1}$$

The Schwartz kernel theorem shows that these operators are in $1-1$ correspondence with the (extendible) distributional half-densities on the product

$$A\leftrightarrow K_A\in \mathscr{C}^{-\infty}(X^2;\Omega^{1/2}). \tag{11.2}$$

The kernels of the operators we consider are much simpler in form on the Θ-stretched product. Since X^2_Θ is obtained from X^2 by the blow-up of a submanifold of the boundary there is a natural isomorphism

$$\beta^{(2)*}_\Theta\colon \mathscr{C}^{-\infty}(X^2)\leftrightarrow \mathscr{C}^{-\infty}(X^2_\Theta) \tag{11.3}$$

so we can just as well look at the kernels there. For half-densities (11.3) becomes

$$\mathscr{C}^{-\infty}(X^2;\Omega^{1/2})\to \mathscr{C}^{-\infty}(X^2_\Theta;\beta^{(2)*}_\Theta\Omega^{1/2}). \tag{11.4}$$

Now, from (7.11) it follows that the lift of densities gives an isomorphism

$$\beta^{(2)*}_\Theta\Omega^{1/2}(X^2)\equiv(\varrho_{\mathrm{ff}})^{N/2}\Omega^{1/2}(X^2_\Theta),\quad N=\dim X+1. \tag{11.5}$$

Rather than use this to rewrite the right side of (11.4) we insert further powers of ϱ_{ff} and consider the kernel, on X^2_Θ, of an operator (11.1) to be

$$\begin{gathered}\varkappa_A\in \mathscr{C}^{-\infty}(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2})\\ K_A\leftrightarrow\varkappa_A\quad\text{under}\quad \mathscr{C}^{-\infty}(X^2;\Omega^{1/2})\leftrightarrow \mathscr{C}^{-\infty}(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}),\quad N=\dim X+1.\end{gathered} \tag{11.6}$$

That is, if the lift of K_A to X^2_Θ is written $\tilde{K}\cdot\mu$, with $\mu\in(\varrho_{\rm ff})^{N/2}\mathscr{C}^\infty(X^2_\Theta;\Omega^{1/2})$ using (11.4) and (11.5) then

$$\varkappa_A=[\varrho_{\rm ff}^N\tilde{K}]\cdot\mu',\quad \mu'=\varrho_{\rm ff}^{-N}\cdot\mu\in\mathscr{C}^\infty(X^2_\Theta;\varrho_{\rm ff}^{-N/2}\,\Omega^{1/2}). \tag{11.7}$$

The reason for the normalization (11.6), (11.7) is easily seen by considering the identity operator. The kernel, in terms of local coordinates x, y, x', y' in X^2 is just

$$K_{\rm Id}=\delta(x-x')\,\delta(y-y')\,\gamma\gamma', \tag{11.8}$$

where γ and γ' are non-vanishing $\mathscr{C}^\infty$ half-densisites on the two factors of X. Since this has support equal to the diagonal, its lift to X^2_Θ will have support equal to Δ_Θ. We can choose local coordinates (x, z, t, z') in X^2, near Δ_Θ, so that the local $\mathbf{R}^+$-action used to define X^2_Θ is just

$$(x,z,t,x',z')\mapsto(\delta x,\delta z,\delta^2 t,\delta x',z') \tag{11.9}$$

and

$$\Delta=\{x=x',\ z=0,\ t=0\},\quad \partial\Delta=\{x=x'=0,\ z=0,\ t=0\}. \tag{11.10}$$

Suitable local coordinates in X^2_Θ, near the lifted diagonal, are then given by

$$x',\quad s=\frac{x-x'}{x'},\quad Z=\frac{z}{x'},\quad T=\frac{t}{(x')^2}\quad\text{and}\quad z'. \tag{11.11}$$

When $K_{\rm Id}$ is lifted to X^2_Θ it becomes

$$\delta(s-1)\,\delta(Z)\,\delta(T)\,(x')^{-N}\beta^{(2)}_\Theta(\gamma\gamma'). \tag{11.12}$$

Using (7.12) and the definition (11.6) we conclude that in these local coordinates

$$\varkappa_{\rm Id}=\delta(X-1)\,\delta(Z)\,\delta(T)\,\nu_2,\quad 0\neq\nu_2\in\mathscr{C}^\infty(\beta^{(2)}_\Theta;\varrho_{\rm ff}^{-N/2}\,\Omega^{1/2}) \tag{11.13}$$

as is only reasonable!

As noted in Section 7 the diagonal lifts into X^2_Θ to be the smooth submanifold Δ_Θ which meets the boundary only in the interior of the front face. The intersection is transversal and therefore obviously clean. Using the conventions of § B1 and the ordering of the boundary hypersurfaces of X^2_Θ:

$$\partial X^2_\Theta=\mathrm{lb}\cup\mathrm{rb}\cup\mathrm{ff} \tag{11.14}$$

let $E=E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}$ be the index family

$$E=\varnothing,\ \varnothing,\ \{(0,0)\}. \tag{11.15}$$

Following the discussion in § B6 we then define

$$\Psi^m_\Theta(X;\Omega^{1/2})=\mathscr{A}^E_{\mathrm{phg}}I^m(X^2_\Theta,\Delta_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}). \tag{11.16}$$

This space of kernels is to be interpreted as a space of operators on half-densities using (11.6), (11.7) and (11.2). By the choice of index family (11.15) these kernels are, as sections of the bundle $\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$, smooth up to all boundary faces and they vanish to infinite order at both lb and rb.

Before proceeding to the discussion of the properties of these operators let us examine the Schwartz kernels of the differential operators in $\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$. Of course if $P\in\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ then we can write its action in the form

$$P\phi=P\cdot\mathrm{Id}\,\phi. \tag{11.17}$$

For any operator A as in (11.1) the kernel of $P\cdot A$ is just $P_L K_A$ where P_L is the differential operator on X^2 given by the action of P in the left factor. Since P acts on half-densities it is given by a sum of products, with $\mathscr{C}^\infty$ coefficients, of up to m factors, each the Lie action of an element of $\mathscr{V}_\Theta$ or the identity. From Proposition 7.8 we know that this action lifts to the X^2_Θ. Since the vector fields are all tangent to the boundary the Lie action extends to the scaled bundle $\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$. Thus we see that the kernels of the elements of $\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ are just given by the repeated application of the elements of $\mathscr{V}_{\Theta,L}$ to the kernel of the identity. Since this Lie algebra is transversal to the lifted diagonal we conclude that:

LEMMA 11.18. *The taking of kernels gives a linear isomorphism between* $\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ *and the space of all conormal sections of order at most m of* $\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$ *associated with, and supported by,* $\Delta_\Theta\subset X^2_\Theta$. *In particular*

$$\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})\subset\Psi^m_\Theta(X;\Omega^{1/2})\quad\forall m\in\mathrm{N}. \tag{11.19}$$

This simple result justifies describing the small calculus $\Psi^*_\Theta(X)$ as the 'microlocalization' of $\mathscr{V}_\Theta$, or $\mathrm{Diff}^*_\Theta(X)$, in the sense that the relationship is the same as that of the pseudodifferential operators to the differential operators, or the $\mathscr{C}^\infty$ vector fields, on a compact manifold without boundary.

The symbol mapping for the small calculus arises directly from the symbol calculus for general conormal distributions. Thus, from (11.16) and (B6.10) we get

$$\sigma_m: I^m(X^2_\Theta, \Delta_\Theta; \varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}) \to S^{\{m\}}(N^*\Delta_\Theta)\otimes\Omega_{\mathrm{fibre}}(N^*\Delta_\Theta)\otimes\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2} \tag{11.20}$$

with the tensor product being over $\mathscr{C}^\infty(\Delta_\Theta)$ using the translation action on densities on the fibres. To accord with the usual convention for pseudodifferential operators we have not compactified the conomal bundle and reinterpreted the symbols as conormal functions as in § B6; symbols are more convenient to deal with here, as there are no other singularities at the boundaries of Δ_Θ. We can also remove the density factors altogether:

LEMMA 11.21. *There are natural bundle isomorphisms covering $X \leftrightarrow \Delta_\Theta$ which give identifications*

$$\Omega_{\mathrm{fibre}}(N^*\Delta_\Theta) \leftrightarrow \Omega_{\mathrm{fibre}}({}^\Theta T^*X) \tag{11.22}$$

and

$$\begin{gathered} \varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2} \leftrightarrow \Omega_{\mathrm{fibre}}({}^\Theta TX) \\ \Omega_{\mathrm{fibre}}({}^\Theta T_pX) \cong \Omega_{\mathrm{fibre}}(\mathrm{ff}_p), \quad p\in \mathrm{ff}(X^2_\Theta). \end{gathered} \tag{11.23}$$

Proof. To get (11.22) one only need use (7.10), so consider (11.23). Over any submanifold the density bundle of the whole manifold splits into the tensor product of the density bundle of the submanifold and the fibre densities on the normal bundle. Thus

$$\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2} \cong \varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}(X)\otimes\Omega^{1/2}_{\mathrm{fibre}}(N\Delta_\Theta) \tag{11.24}$$

where ϱ is now a defining function for the boundary of X. Using (7.10) again, on the second factor, and the identifications

$$\varrho^{-N}\Omega(X) \cong {}^\Theta\Omega(X) \cong \Omega_{\mathrm{fibre}}({}^\Theta TX), \quad N = \dim X + 1, \tag{11.25}$$

we get (11.23).

Of course the immediate consequence of this lemma is that the density factor in (11.20) is canonically trivial. Thus we find the more satisfactory symbol map

$${}^\Theta\sigma_m: \Psi^m_\Theta(X;\Omega^{1/2}) \to S^{\{m\}}({}^\Theta T^*X) \quad \forall m\in\mathbf{R}. \tag{11.26}$$

Carrying through the computation directly we see that

$$ {}^{\Theta}\sigma_0(\mathrm{Id}) = 1. \tag{11.27} $$

The discussion above of the kernels of the Θ-differential operators shows that (11.26) is consistent with the usual symbol map, i.e. if $P \in \mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ then ${}^{\Theta}\sigma_m(P)$ is just the polynomial of degree m on ${}^{\Theta}T^*X$ obtained from its principal part in the usual sense.

The exactness of the symbol map follows from (B6.12). Hence if $\phi \in \mathscr{C}^\infty(X)$ is a non-vanishing function we conclude that

$$ \Psi^m_\Theta(X;\Omega^{1/2}) \ni A \mapsto \phi^{-1}A\phi \in \Psi^m_\Theta(X;\Omega^{1/2}) \tag{11.28} $$

is an isomorphism which induces the identity on the symbol map. Thus if we define the space $\Psi^m_\Theta(X)$ of Θ-pseudodifferential operators acting on functions by using a non-vanishing $\mathscr{C}^\infty$ half-density ν to write

$$ \Psi^m_\Theta(X) \ni A \leftrightarrow \nu A \nu^{-1} \in \Psi^m_\Theta(X;\Omega^{1/2}) \tag{11.29} $$

we conclude that the symbol map is well-defined, independent of the choice of ν.

We summarize these properties in the following theorem, in which only the composition properties remain to be proved.

THEOREM 11.30. *For a compact manifold with boundary, X and a projective class of non-vanishing 1-forms at ∂X, $[\Theta]$, the Θ-pseudodifferential operators form a symbol-filtered ring of operators*

$$ A\colon \dot{\mathscr{C}}^\infty(X) \to \dot{\mathscr{C}}^\infty(X). \tag{11.31} $$

For any $m \in \mathbf{R}$ the symbol map defines a short exact sequence:

$$ 0 \to \Psi^{m-1}_\Theta(X) \hookrightarrow \Psi^m_\Theta(X) \xrightarrow{{}^{\Theta}\sigma_m} S^{\{m\}}({}^{\Theta}T^*X) \to 0 \tag{11.32} $$

and composition gives

$$ \begin{gathered} \Psi^m_\Theta(X)\cdot\Psi^{m'}_\Theta(X) \subset \Psi^{m+m'}_\Theta(X) \\ {}^{\Theta}\sigma_{m+m'}(A\cdot B) = {}^{\Theta}\sigma_m(A)\cdot{}^{\Theta}\sigma_{m'}(B) \quad \mathrm{mod}\ S^{m+m'-1}({}^{\Theta}T^*X). \end{gathered} \tag{11.33} $$

Proof. First we need to check the mapping property (11.31); this also shows that the composition of Θ-pseudodifferential operators is well-defined.

If $\phi \in \dot{\mathscr{C}}^\infty(X;\Omega^{1/2})$ and $A \in \Psi^m_\Theta(X;\Omega^{1/2})$ then $A\phi \in \mathscr{C}^{-\infty}(X;\Omega^{1/2})$ by the easy direction of the Schwartz kernel theorem. We can make this explicit in terms of the push-forward of distributional densities by choosing a non-vanishing half-density $\nu \in \mathscr{C}^\infty(X;\Omega^{1/2})$. Then as a density on X

$$(A\phi)\,\nu = (\pi_L)_*[\pi_L^*\nu\cdot K_A\cdot\pi_R^*\phi]$$

in terms of the push-foward from X^2. Combining the two half-density and lifting to X^2_Θ we can write this as

$$(A\phi)\,\nu = (\pi_L\beta^{(2)}_\Theta)_*[\varkappa_A\cdot(\beta^{(2)}_\Theta)^*(\nu\cdot\phi)]. \tag{11.34}$$

Now, by (7.11) we know that

$$(\beta^{(2)}_\Theta)^*(\nu\cdot\phi)\in(\varrho_{\mathrm{ff}})^{N/2}\mathscr{C}^\infty(X^2_\Theta;\Omega^{1/2}). \tag{11.35}$$

Recalling the density factor in the definition of the kernels in (11.16), and the fact that ϕ vanishes to infinite order at the boundary, we conclude that the distribution on the right in (11.34) is an element of $\mathscr{A}^{\varnothing}_{\mathrm{phg}}I^m(X^2_\Theta,\Delta_\Theta;\Omega)$, all index sets empty. It follows from Proposition B7.11 and Proposition 7.14 that the push-forward under $\beta^{(2)}_\Theta$ can be carried out, giving

$$(A\phi)\,\nu\in(\pi_L)_*[\mathscr{A}^{\varnothing}_{\mathrm{phg}}I^m(X^2,\Delta;\Omega)]\subset\dot{\mathscr{C}}(X;\Omega) \tag{11.36}$$

since the projection is transversal to the diagonal. This proves (11.31).

Our proof of (11.33) is in a similar spirit, using the geometric properties on the Θ-stretched triple product discussed in Section 10. Figure 2 may be of help in this discussion. Thus we first find a suitable representation of the kernel of the composite as a push-forward. Suppose that $A\in\Psi^m_\Theta(X;\Omega^{1/2})$ and $B\in\Psi^{m'}_\Theta(X;\Omega^{1/2})$ and let $C=A\cdot B$. If $\mu\in\varrho_{\mathrm{ff}}^{N/2}\,\mathscr{C}^\infty(X^2_\Theta;\Omega^{1/2})$ then

$$\varkappa_C\cdot\mu=(\pi_C)_*[(\pi_S)^*\varkappa_A\cdot(\pi_F)^*\varkappa_B\cdot(\pi_C)^*\mu] \tag{11.37}$$

is given as the push-forward of a distributional density on X^3_Θ. The product in (11.37) is justified by Proposition B7.20, which together with Proposition 10.16 also shows that

$$\begin{gathered}(\pi_C)_*[\mathscr{A}^{E}_{\mathrm{phg}}I^{m-n/4}(X^3_\Theta,\Delta_S)\cdot\mathscr{A}^{E'}_{\mathrm{phg}}I^{m'-n/4}(X^3_\Theta,\Delta_F)\cdot\gamma]\subset\mathscr{A}^{E''}_{\mathrm{phg}}I^{m+m'}(X^2_\Theta,\Delta_\Theta)\gamma'\\ E=(\varnothing,\varnothing,E_{\mathrm{ff}}),\quad E'=(\varnothing,\varnothing,E'_{\mathrm{ff}}),\quad E''=(\varnothing,\varnothing,E'_{\mathrm{ff}}+E''_{\mathrm{ff}}).\end{gathered} \tag{11.38}$$

The orders in (11.38) reflect the effect of the pull-back, $n=\dim X$ being the codimension.

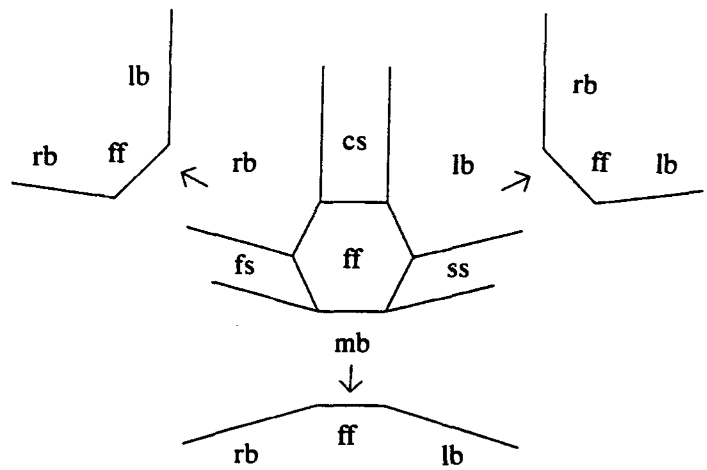

Fig. 2. Triple blow-up and projections

Since the symbolic property in (11.33) also follows from Proposition B7.20 this completes the proof of the composition theorem.

We also remark that the basic mapping properties of these operators are easily proved using the same techniques. Thus from Proposition B7.11 it also follows that

$$A: \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X) \quad \forall A \in \Psi^m_\Theta(X). \tag{11.39}$$

If $\phi \in \mathscr{C}^\infty(\partial X)$ is extended to $\tilde{\phi} \in \mathscr{C}^\infty(X)$ then

$$A(\tilde{\phi})|_{\partial X} = A_\partial \phi \in \mathscr{C}^\infty(\partial X) \tag{11.40}$$

is independent of the choice of extension. In fact A_∂ is just a multiplication operator.

In addition to the symbol associated to the singularity at the diagonal a Θ-pseudodifferential operator also has a well-defined 'normal operator;' this just arises from the symbol mapping at the front face, as in (B6.11). Since the index set at the front face is just $\{(0,0)\}$, we identify this with the restriction of its kernel to the front face of X^3_Θ.

$$\Psi^m_\Theta(X; \Omega^{1/2}) \ni A \mapsto N(A) = (\varkappa_A)|_{\mathrm{ff}} \in I^{m+1/4}(\mathrm{ff}(X^3_\Theta), \partial\Delta_\Theta; \Omega_{\mathrm{fibre}}(\mathrm{ff})). \tag{11.41}$$

Here we have identified density factors at the front face using (11.23). The action of this kernel as an operator on the tangent Lie group at each boundary point is readily described. Since we shall not use this action we simply note that in the special case of a differential operator it can easily be identified with the normal operator as defined in § 4 and even more importantly

$$N(P\cdot A)=N(P)\,N(A)\quad \forall P\in \mathrm{Diff}^m_\Theta(X),\quad A\in\Psi^{m'}_\Theta(X). \tag{11.42}$$

In the general case the normal operator of the composite is given by convolution in terms of the group structure on each fibre of the front face:

$$N_p(A\cdot B)=N_p(A)*_p N_p(B).$$

From the definition of the normal operator it is clear that $N(A)=0$ if and only if the kernel of A is of the form

$$\varrho_{\mathrm{ff}}\varkappa \quad\text{with}\quad \varkappa\in\Psi^*_\Theta(X).$$

In addition to the symbolic filtration we therefore have a second filtration by order of vanishing at the front face, we shall denote those elements of $\Psi^*_\Theta(X)$ which vanish to order k at the front face by $\varrho^k_{\mathrm{ff}}\Psi^*_\Theta(X)$. By reviewing the proof of (11.33) it is clear that

$$\varrho^k_{\mathrm{ff}}\Psi^m_\Theta(X)\cdot\varrho^{k'}_{\mathrm{ff}}\Psi^{m'}_\Theta(X)\subset\varrho^{k+k'}_{\mathrm{ff}}\Psi^{m+m'}_\Theta(X). \tag{11.43}$$

The residual space of the small calculus, $\Psi^{-\infty}_\Theta(X)$, consists of kernels which are smooth on the stretched product and vanish to infinite order at the top and bottom faces. As shown in § 13 these kernels define bounded operators on $L^2(X)$ which are moreover infinitely smoothing in the interior, however they do not give any additional decay near the boundary and hence do not in general define compact operators. Since

$$A\in\varrho_{\mathrm{ff}}\Psi^m_\Theta(X)\Leftrightarrow A=\varrho B,\ B\in\Psi^m_\Theta(X) \tag{11.44}$$

it is the case that $\varrho_{\mathrm{ff}}\Psi^{-\infty}_\Theta(X)$ consists of compact operators. Using the symbol map, (11.32), and (11.33) in an iterative manner it is possible to construct an approximate inverse for an elliptic element of $P\in\Psi^m_\Theta(X)$, i.e. $Q\in\Psi^{-m}_\Theta(X)$ such that

$$P\cdot Q-\mathrm{Id},\ Q\cdot P-\mathrm{Id}\in\Psi^{-\infty}_\Theta(X). \tag{11.45}$$

The non-compactness of these operators means that such an approximate inverse is not really a parametrix. In order to construct parametrices it is necessary to remove at least the first term in the Taylor series of the kernel of the Ψ_Θ-residual operator. This amounts to inverting the normal operator and as we shall shortly see this leads inevitably to kernels which have singularities along the top and bottom faces. In light of this we proceed to a detailed discussion of the full calculus.

§ 12. Full calculus

In the previous section we considered a class of operators with Schwartz kernels concentrated near the diagonal in X^2_Θ. From our formulae (8.6) and (8.14) for the resolvent of the Bergman Laplacian on the ball it is apparent that it does not belong to this 'small calculus' since its kernel does not vanish to infinite order at the left and right boundary faces. Rather, the kernel has a simple power law behaviour at these faces with the leading order of singularity determined by the null space of the indicial operator of the Laplacian. In this section we extend the small calculus to include boundary terms of this type. It is in this context that the full power of the blow-up method becomes evident. Its effect is to physically separate the diagonal and the various other boundary components on the product space. Once the interaction between them has been removed these two types of singularities can be treated successively.

Recall that the small calculus of Θ-pseudodifferential operators was defined by fixing the space of kernels as a subspace of the extendible distributions on X^2_Θ, see (11.16). The most obvious class of kernels which contains the resolvent of the Bergman Laplacian on the ball is obtained by adjoining the smooth sections of the bundle $\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}$ with a power law behaviour at the left and right boundary faces. In terms of index sets as defined in § B1 we set, for any $a, b \in \mathbf{C}$,

$$\Psi_\Theta^{-\infty;a,b}(X;\Omega^{1/2}) = \mathscr{A}_{\mathrm{phg}}^{\{(a,0)\},\{(b,0)\},\{(0,0)\}}(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}). \tag{12.1}$$

We generalize (12.1) by choosing an index family $E = E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}$, corresponding to the ordering of the boundary hypersurfaces of X^2_Θ in (11.14) and then set

$$\begin{aligned}\Psi_\Theta^{m;E}(X;\Omega^{1/2}) &= \Psi_\Theta^{m;E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}}(X;\Omega^{1/2})\\ &= \mathscr{A}^E_{\mathrm{phg}} I^m(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2})\\ &= \mathscr{A}_{\mathrm{phg}}^{\varnothing,\varnothing,E_{\mathrm{ff}}} I^m(X^2_\Theta,\Delta_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}) + \mathscr{A}^E_{\mathrm{phg}}(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}),\end{aligned} \tag{12.2}$$

where we use the fact that Δ_Θ only meets the boundary in the front face. If $m = -\infty$ then the first term is contained in the second and

$$\Psi_\Theta^{-\infty;E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}} = \mathscr{A}^E_{\mathrm{phg}}(X^2_\Theta;\varrho_{\mathrm{ff}}^{-N/2}\Omega^{1/2}). \tag{12.3}$$

These operators, without singularities on the lifted diagonal, form the first and main class of "boundary terms" in the calculus.

If $E_{\mathrm{lb}}, E_{\mathrm{rb}}$ or E_{ff} consists of a single element $(z, 0)$ we replace it by z in the notation, so for example

$$\Psi_\Theta^{m;\{(z,0)\},\{(z',0)\},E_{\mathrm{ff}}}(X;\Omega^{1/2}) = \Psi_\Theta^{m;z,z',E_{\mathrm{ff}}}(X;\Omega^{1/2}) \tag{12.4}$$

for any index set E_{ff} for the front face. Moreover, if $E_{\mathrm{ff}}=\{(0,0)\}$ then we drop it altogether and set

$$\Psi_\Theta^{m;E_{\mathrm{lb}},E_{\mathrm{rb}}}(X;\Omega^{1/2}) = \Psi_\Theta^{m;E_{\mathrm{lb}},E_{\mathrm{rb}},0}(X;\Omega^{1/2}) = \Psi_\Theta^{m;E_{\mathrm{lb}},E_{\mathrm{rb}},\{(0,0)\}}(X;\Omega^{1/2}). \tag{12.5}$$

On the other hand we drop the remaining two index sets, for the left and right boundaries, only if both are empty. This gives a notation quite consistent with (12.1) and also with the notation for the small calculus, for instance:

$$\begin{aligned} \Psi_\Theta^m(X;\Omega^{1/2}) &= \Psi_\Theta^{m;\varnothing,\varnothing}(X;\Omega^{1/2}) \\ \varrho_{\mathrm{ff}}^k\,\Psi_\Theta^m(X;\Omega^{1/2}) &= \Psi_\Theta^{m;\varnothing,\varnothing,k}(X;\Omega^{1/2}). \end{aligned} \tag{12.6}$$

We also need to admit certain 'residual' terms, which are compact operators, into the calculus. These are not special to the Θ-structure and are in fact best described on X^2 itself. Thus for index sets E_{lb} and E_{rb} giving an index family, E, for X^2 set

$$\Psi^{-\infty;E}(X;\Omega^{1/2}) = \Psi^{-\infty;E_{\mathrm{lb}},E_{\mathrm{rb}}}(X;\Omega^{1/2}) = \mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{lb}},E_{\mathrm{rb}}}(X^2;\Omega^{1/2}). \tag{12.7}$$

We shall even use this notation when E is an index family for X^2_Θ, then E_{ff} should simply be ignored. Thus a general 'boundary term' is an element of

$$\Psi_\Theta^{-\infty;E}(X;\Omega^{1/2})+\Psi^{-\infty;E}(X;\Omega^{1/2}) = \Psi_\Theta^{-\infty;E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}}(X;\Omega^{1/2})+\Psi^{-\infty;E_{\mathrm{lb}},E_{\mathrm{rb}}}(X;\Omega^{1/2}). \tag{12.8}$$

Of course, since all these kernels are extendible distributional half-densities on X^2 or on X^2_Θ, and hence on X^2, the corresponding operators are well defined as maps (11.1). The following basic mapping property allows us to check when the composite of two such operators is defined. We use the following notation for the 'extended union' of two index sets

$$\begin{aligned} E_1 \bar{\cup} E_2 = \{(z,m)\in \mathbf{C}\times\mathbf{N}_0;\ &(z,m)\in E_1 \text{ or } (z,m)\in E_2 \text{ or } m=m_1+m_2+1 \\ &\text{if } (z-k_1,m_1)\in E_1,\ (z-k_2,m_2)\in E_2 \text{ for some } k_1,k_2\in\mathbf{N}_0\}. \end{aligned} \tag{12.9}$$

PROPOSITION 12.10. *For any index family $E=E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}$, for X^2_Θ, and index set E_I, for X,*

$$A \in \Psi_\Theta^{m;E}(X;\Omega^{1/2}) \quad \Rightarrow \quad A\colon \mathcal{A}_{\mathrm{phg}}^{E_I}(X;\Omega^{1/2}) \to \mathcal{A}_{\mathrm{phg}}^{E_O}(X;\Omega^{1/2}),$$
$$\text{provided } E_I + E_{\mathrm{rb}} > -1 \text{ and } E_O = E_{\mathrm{lb}} \,\overline{\cup}\, (E_{\mathrm{ff}} + E_I). \tag{12.11}$$

Proof. We start by considering the action on $\dot{\mathcal{C}}^\infty(X;\Omega^{1/2})$. If ν is a $\mathcal{C}^\infty$ half-density on X, and $\phi \in \dot{\mathcal{C}}^\infty(X;\Omega^{1/2})$

$$A\phi \cdot \mu = (\pi_L \cdot \beta_\Theta^{(2)})_*[(\pi_L \cdot \beta_\Theta^{(2)})^* \mu \cdot \varkappa_A \cdot (\pi_R \cdot \beta_\Theta^{(2)})^* \phi] \in \mathcal{C}^{-\infty}(X;\Omega). \tag{12.12}$$

Since $(\pi_R \cdot \beta_\Theta^{(2)})^* \phi$ vanishes to infinite order at $\mathrm{rb}(X_\Theta^2)$ and $\mathrm{ff}(X_\Theta^2)$ the density on X_Θ^2 pushed forward in (12.12) is an element, v, of

$$\mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{lb}},\varnothing,\varnothing}(X_\Theta^2;\Omega). \tag{12.13}$$

Using Proposition B5.6 and Proposition 7.14 we see that this pushes forward into the appropriate space of conormal distributions, $\mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{lb}}}(X;\Omega)$. This is the trivial case of (12.11), when $E_I = \varnothing$:

$$A \in \Psi_\Theta^{m,E}(X;\Omega^{1/2}) \quad \Rightarrow \quad A\colon \dot{\mathcal{C}}^\infty(X;\Omega^{1/2}) \to \mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{lb}}}(X;\Omega^{1/2}). \tag{12.14}$$

Applying the obvious adjoint identity

$$[\Psi_\Theta^{m;E}(X;\Omega^{1/2})]^* = \Psi_\Theta^{m;E'}(X;\Omega^{1/2}), \quad E' = (E_2, E_1, E_3) \text{ if } E = (E_1, E_2, E_3) \tag{12.15}$$

we conclude by duality from (12.14) that

$$A \in \Psi_\Theta^{E}(X;\Omega^{1/2}) \quad \Rightarrow \quad A\colon [\mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{rb}}}(X;\Omega^{1/2})]' \to \mathcal{C}^{-\infty}(X;\Omega^{1/2}). \tag{12.16}$$

From the multiplicative properties of conormal distributions, Lemma B3.5, we know that

$$\mathcal{A}_{\mathrm{phg}}^{E_I}(X;\Omega^{1/2}) \subset [\mathcal{A}_{\mathrm{phg}}^{E_{\mathrm{rb}}}(X;\Omega^{1/2})]' \quad \Leftrightarrow \quad E_I + E_{\mathrm{rb}} > -1 \tag{12.17}$$

so we conclude that the operator in (12.11) is well-defined, with range in $\mathcal{C}^{-\infty}(X;\Omega^{1/2})$.

To check the regularity of the range in (12.11) we first apply Proposition B4.3 and Lemma B3.5 and so conclude that in (12.12),

$$A\phi \cdot \mu \in (\pi_L \cdot \beta_\Theta^{(2)})_* \mathcal{A}_{\mathrm{phg}}^{F}(X_\Theta^2; {}^b\Omega).$$
$$F = (E_{\mathrm{lb}} + 1, E_{\mathrm{rb}} + E_I + 1, E_{\mathrm{ff}} + E_I + 1). \tag{12.18}$$

Applying Proposition B5.6 we find that

$$A\phi\cdot\in\mathcal{A}^{G}_{\mathrm{phg}}(X;{}^b\Omega),$$

where $G=(\pi_L\cdot\beta^{(2)}_\Theta)_b(F)$ and we use the assumption $E_l+E_{\mathrm{rb}}>-1$. Using (B5.4) and (7.15) to compute we find that

$$G=(E_{\mathrm{lb}}+1)\,\overline{\cup}\,(E_{\mathrm{ff}}+E_l+1). \tag{12.19}$$

Changing the density bundle from ${}^b\Omega$ back to Ω gives (12.11) and proves the proposition.

We are mostly interested in the spaces with kernels 'smooth up to the front face'. By definition

$$\Psi^{m;E,F}_\Theta(X;\Omega^{1/2})=\Psi^m_\Theta(X;\Omega^{1/2})+\Psi^{-\infty;E,F}_\Theta(X;\Omega^{1/2}). \tag{12.20}$$

Moreover, for any index sets E and F

$$\Psi^m_\Theta(X;\Omega^{1/2})\cap\Psi^{-\infty;E,F}_\Theta(X;\Omega^{1/2})=\Psi^{-\infty}_\Theta(X;\Omega^{1/2}). \tag{12.21}$$

This allows us to extend the symbol map from the small calculus to

$$\begin{gathered}{}^\Theta\sigma_m\colon\Psi^{m;E,F}_\Theta(X;\Omega^{1/2})\to S^{\{m\}}({}^\Theta T^*X)\\ {}^\Theta\sigma_m(B)={}^\Theta\sigma_m(B')\quad\text{if}\quad B=B'+B'',\ B'\in\Psi^m_\Theta(X;\Omega^{1/2}),\ B''\in\Psi^{-\infty;E,F}_\Theta(X;\Omega^{1/2}).\end{gathered} \tag{12.22}$$

This clearly results in a short exact sequence as for the small calculus in (11.32):

$$0\to\Psi^{m-1;E,F}_\Theta(X)\hookrightarrow\Psi^{m;E,F}_\Theta(X)\xrightarrow{{}^\Theta\sigma_m}S^{\{m\}}({}^\Theta T^*X)\to 0. \tag{12.23}$$

We can also extend the definition of the normal operator to these spaces. Namely by interpreting

$$N(A)=[\varkappa_A]|_{\mathrm{ff}}\in\mathcal{A}^{(E,F)}_{\mathrm{phg}}(\mathrm{ff}(X^2_\Theta);\Omega^{1/2}_{\mathrm{fibre}}\otimes\Omega^{1/2}_{\mathrm{fibre}}) \tag{12.24}$$

using the identification of densities in Lemma 11.21. In fact this normal operator is just the symbol of the kernel at the front face in the sense of (B6.11). We certainly have an exact sequence

$$0\to\Psi^{m;E,F,1}_\Theta(X;\Omega^{1/2})\hookrightarrow\Psi^{m;E,F}_\Theta(X;\Omega^{1/2})\xrightarrow{N}\mathcal{A}^{m,E,F}_{\mathrm{phg}}(\mathrm{ff}(X^2_\Theta);\Omega^{1/2}_{\mathrm{fibre}}\otimes\Omega^{1/2}_{\mathrm{fibre}})\to 0. \tag{12.25}$$

Note that the smoothness of the normal operator, $m=-\infty, E=F=\emptyset$ does not imply that it is negligible. Moreover there are also symbols at the left and right boundary faces.

$$\sigma_{\mathrm{lb}}(A) \in \mathscr{A}_{\mathrm{phg}}^{\{E\},F,G}({}^{\Theta}\pi_L^* N(\partial X); \varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}), \quad A \in \Psi_{\Theta}^{m;E,F,G}(X;\Omega^{1/2}). \tag{12.26}$$

Here we have used the identification of the normal bundle, in X^2_{Θ}, to $\mathrm{lb}(X^2_{\Theta})$ with the lift of the normal bundle to the boundary of X, from the left factor:

$${}^{\Theta}\pi_L(\mathrm{lb}(X^2_{\Theta})) = \partial X, \quad {}^{\Theta}\pi_L^*: N(\mathrm{lb}(X^2_{\Theta})) \cong {}^{\Theta}\pi_L^* N\partial X. \tag{12.27}$$

Thus we have an exact sequence

$$0 \hookrightarrow \Psi_{\Theta}^{m;E+1,F,G}(X;\Omega^{1/2}) \to \Psi_{\Theta}^{m;E,F,G}(X;\Omega^{1/2}) \xrightarrow{\sigma_{\mathrm{lb}}} \mathscr{A}_{\mathrm{phg}}^{\{E\},F,G}({}^{\Theta}\pi_L^* N(\partial X); \varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}) \to 0 \tag{12.28}$$

and similarly for the right boundary.

These four symbol maps are not completely independent. The common range is subject to the following conditions:

$$\sigma_m(N(A)) = \sigma_m(A)|_{\mathrm{ff}(X^2_{\Theta})} \tag{12.29}$$

$$\sigma_{\mathrm{lb}}(N(A)) = \sigma_{\mathrm{lb}}(A)|_{\mathrm{ff}(X^2_{\Theta})} \tag{12.30}$$

$$\sigma_{\mathrm{rb}}(N(A)) = \sigma_{\mathrm{rb}}(A)|_{\mathrm{ff}(X^2_{\Theta})} \tag{12.31}$$

although these conditions do not appear explicitly in our step-by-step construction.

To start the composition results we consider first the 'trivial' case when one of the operators is differential. See Figure 2.

PROPOSITION 12.32. *Let* $P \in \mathrm{Diff}^m_{\Theta}(X;\Omega^{1/2})$ *and* $A \in \Psi_{\Theta}^{m';E_{\mathrm{lb}},E_{\mathrm{rb}}}(X;\Omega^{1/2})$ *then*

$$P \cdot A \in \Psi_{\Theta}^{m+m';E_{\mathrm{lb}},E_{\mathrm{rb}}}(X;\Omega^{1/2})$$

satisfies

$$N(P\cdot A) = N(P)\cdot N(A) \tag{12.33}$$

$$\sigma_{m+m'}(P\cdot A) = \sigma_m(P)\cdot \sigma_{m'}(A) \tag{12.34}$$

$$\sigma_{\mathrm{lb}}(P\cdot A) = I(P)\cdot \sigma_{\mathrm{lb}}(A). \tag{12.35}$$

Proof. Since Diff^m_{Θ} is the part of the enveloping algebra of $\mathscr{V}_{\Theta}$ of order at most m it suffices to consider the special case when $P=\mathscr{L}_V$ is given by the Lie action of $V \in \mathscr{V}_{\Theta}$ on half-densities. Thus the kernel of $P\cdot A$ is just given by the lift of $\mathcal{L}_V$ to X^2_Θ acting on the kernel of A:

$$\varkappa(\mathcal{L}_V\cdot A)=\widetilde{\mathcal{L}_V}\varkappa(A). \tag{12.36}$$

This is just the Lie action of the lift, $\tilde{V}$ of V to X^2_Θ, from the left factor, on the weighted density bundle $\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$. Since $\tilde{V}$ is tangent to all boundaries it is clear that

$$\mathcal{L}_V\cdot A\in\Psi_\Theta^{m+1;E}(X;\Omega^{1/2}). \tag{12.37}$$

The formula (12.33) for the normal operators is a direct consequence of the tangency of $\tilde{V}$ to the front face of X^2_Θ and the definition of the normal operator of an element of $\mathrm{Diff}^1_\Theta(X;\Omega^{1/2})$. Similarly (12.34) follows from (12.36) and the definition of the symbol and (12.35) follows from (12.36), the definition of the boundary symbol and the definition of the indicial operator of a differential operator.

From this general composition result we deduce two corollaries which will be used in the construction of parametrices.

COROLLARY 12.38. *Suppose $P\in\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ is elliptic then for any index family $E=E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}$ and any $R\in\Psi_\Theta^{r;E}(X;\Omega^{1/2})$ there exists $F\in\Psi_\Theta^{r-m;E}(X;\Omega^{1/2})$ such that*

$$P\cdot F-R\in\Psi_\Theta^{-\infty;E}(X;\Omega^{1/2}); \tag{12.39}$$

moreover the construction of F can be carried out smoothly in parameters in any compact manifold.

Proof. This just uses standard iteration with the symbol map associated to the diagonal singularities.

COROLLARY 12.40. *Suppose $P\in\mathrm{Diff}^m_\Theta(X;\Omega^{1/2})$ and the indicial operator $I(P)$, on the fibres of $N\partial X$, is elliptic with constant characteristic roots (independent of the base point) and $E=(E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}})$ are index sets such that there is no point $(z,m)\in E_{\mathrm{lb}}$ with $z+k$ a characteristic root of $I(P)$ root for $k\in\mathbb{N}_0$, then for any $R\in\Psi_\Theta^{r;E}(X;\Omega^{1/2})$ there exists $F\in\Psi_\Theta^{-\infty,E}(X;\Omega^{1/2})$ with*

$$P\cdot F-R\in\Psi_\Theta^{r;E'}(X;\Omega^{1/2}),\quad E'=\varnothing,E_{\mathrm{rb}},E_{\mathrm{ff}}. \tag{12.41}$$

Proof. This is only a matter of solving the indicial operator iteratively.

These composition formulae allow us to construct the leading term of the parametrix rather directly in Section 14 below. To remove the error terms we need to compose more general elements of the calculus.

THEOREM 12.42. *If* $E=E_{\rm lb}, E_{\rm rb}, E_{\rm ff}$ *and* $E'=E'_{\rm lb}, E'_{\rm rb}, E'_{\rm ff}$ *are index families for* X^2_Θ *such that*

$$E_{\rm rb}+E'_{\rm lb}>-1 \tag{12.43}$$

then for any $A\in\Psi^{m;E}_\Theta(X;\Omega^{1/2})$ *and* $B\in\Psi^{m';E'}_\Theta(X;\Omega^{1/2})$ *the composite operator* $A\cdot B$ *is well-defined and*

$$\begin{gathered} A\cdot B\in\Psi^{m+m';E''}_\Theta(X;\Omega^{1/2}) \quad where \quad E''=E''_{\rm lb}, E''_{\rm rb}, E''_{\rm ff}, \\ E''_{\rm lb}=E_{\rm lb}\,\overline{\cup}\,[E_{\rm ff}+E'_{\rm lb}],\quad E''_{\rm rb}=[E'_{\rm ff}+E_{\rm rb}]\,\overline{\cup}\,E'_{\rm rb},\quad E''_{\rm ff}=[E'_{\rm ff}+E_{\rm ff}]\,\overline{\cup}\,[E'_{\rm rb}+E_{\rm lb}+N]; \end{gathered} \tag{12.44}$$

the symbol formula at the diagonal (12.34) *continues to hold.*

We reduce the proof to the following push-forward lemma.

LEMMA 12.45. *Let* $G=G_{\rm ff}, G_{\rm fs}, G_{\rm ss}, G_{\rm cs}, G_{\rm lb}, G_{\rm rb}, G_{\rm mb}$, *be an index family for* X^3_Θ *with*

$$G_{\rm mb}>-1 \tag{12.46}$$

then push-forward from the triple Θ*-stretched product gives a continuous linear map*

$$\begin{gathered} ({}^\Theta\pi_C)_*\colon \mathscr{A}^G_{\rm phg}I^m(X^3_\Theta,\Delta_L;\Omega)\to\mathscr{A}^E_{\rm phg}(X^2_\Theta;\Omega), \\ E_{\rm lb}=G_{\rm lb}\,\overline{\cup}\,G_{\rm ss},\quad E_{rb}=G_{\rm rb}\,\overline{\cup}\,G_{\rm fs},\quad E_{\rm ff}=G_{\rm ff}\,\overline{\cup}\,G_{\rm cs}. \end{gathered} \tag{12.47}$$

Proof. Given the change of density bundle this is just Proposition B7.20 applied to the map ${}^\Theta\pi_C$, using (10.18).

Proof of Theorem 12.42. The formula (11.37) for the kernel of the composite still holds. Thus, if μ is a non-vanishing smooth section of $\varrho_{\rm ff}^{N/2}\,\Omega^{1/2}$ over X^2_Θ we see that the kernel of the composite operator is determined by the push-forward formula:

$$\varkappa_C\cdot\mu=(\pi_{C,\Theta})_*[(\pi^*_{S,\Theta}(\varkappa_A)\,\pi^*_{F,\Theta}(\varkappa_B)\,\pi^*_{C,\Theta}\mu] \tag{12.48}$$

where the bracketed expression is a distributional density. Let $\mu'\in\mathscr{C}^\infty(X^2_\Theta;\varrho_{\rm ff}^{-N/2}\,\Omega^{1/2})$ be a non-vanishing section. Applying Proposition B7.6 to the two kernels we see that

$$\begin{aligned} &\pi_{S,\Theta}^*(\varkappa_A/\mu') \in \mathscr{A}_{\text{phg}}^{F} I^{m-d/4}(X_\Theta^3, \Delta_S) \\ &\pi_{F,\Theta}^*(\varkappa_B/\mu') \in \mathscr{A}_{\text{phg}}^{F'} I^{m'-d/4}(X_\Theta^3, \Delta_F) \end{aligned} \tag{12.49}$$

where the index families on X_Θ^3 are:

$$\begin{aligned} &F_{\text{ff}} = E_{\text{ff}},\ F_{\text{fs}} = E_{\text{rb}},\ F_{\text{ss}} = E_{\text{ff}},\ F_{\text{cs}} = E_{\text{lb}},\ F_{\text{lb}} = E_{\text{lb}},\ F_{\text{rb}} = \{(0,0)\},\ F_{\text{mb}} = E_{\text{rb}} \\ &F'_{\text{ff}} = E'_{\text{ff}},\ F'_{\text{fs}} = E'_{\text{ff}},\ F'_{\text{ss}} = E'_{\text{lb}},\ F'_{\text{cs}} = E'_{\text{rb}},\ F'_{\text{lb}} = \{(0,0)\},\ F'_{\text{rb}} = E'_{\text{rb}},\ F'_{\text{mb}} = E'_{\text{lb}}. \end{aligned} \tag{12.50}$$

Taking the densities together we find that

$$\pi_{S,\Theta}^*\mu'\pi_{F,\Theta}\mu'\pi_{C,\Theta}^*\mu \in \varrho_{\text{cs}}^N \mathscr{C}^\infty(X_\Theta^3;\Omega). \tag{12.51}$$

Applying Proposition B7.20 now gives the theorem as stated.

It is also straightforward to generalize the formulae (12.33)–(12.35) for the normal operator, symbol and boundary symbol of the composite, since we do not need them below they will nevertheless be omitted here.

In view of the form of the terms which actually arise in the iterative procedure below, we note some special cases of the general composition formula, Theorem 12.42.

COROLLARY 12.52. *For any index sets $E_{\text{rb}}, E'_{\text{rb}}$ and any integers k, l,*

$$\Psi_\Theta^{-\infty;\varnothing,E_{\text{rb}},k}(X;\Omega^{1/2})\cdot\Psi_\Theta^{-\infty;\varnothing,E'_{\text{rb}},l}(X;\Omega^{1/2}) \subset \Psi_\Theta^{-\infty;\varnothing,[E_{\text{rb}}+l]\bar{\cup}E'_{\text{rb}},k+l}(X;\Omega^{1/2}). \tag{12.53}$$

COROLLARY 12.54. *For any index sets $E_{\text{lb}}, E_{\text{rb}}$ and E'_{rb} and any $k \in \mathbf{N}$*

$$\Psi_\Theta^{m;E_{\text{lb}},E_{\text{rb}}}(X;\Omega^{1/2})\cdot\Psi_\Theta^{-\infty;\varnothing,E'_{\text{rb}},k}(X;\Omega^{1/2}) \subset \Psi_\Theta^{-\infty;E_{\text{lb}},[E_{\text{rb}}+k]\bar{\cup}E'_{\text{rb}},k\bar{\cup}[E_{\text{lb}}+E'_{\text{rb}}+N]}(X;\Omega^{1/2}). \tag{12.55}$$

We also need the corresponding composition properties for the spaces $\Psi^{-\infty;E}(X;\Omega^{1/2})$:

THEOREM 12.56. *For index sets $E_{\text{lb}}, E_{\text{rb}}$ and E_{ff} forming an index family for X_Θ^2 and index sets E'_{lb} and E'_{rb} forming an index family for X^2 satisfying* (12.43)

$$\Psi_\Theta^{m;E}(X;\Omega^{1/2})\cdot\Psi^{-\infty;E'}(X;\Omega^{1/2}) \subset \Psi^{-\infty;E''}(X;\Omega^{1/2}) \tag{12.57}$$

where $E''_{\text{lb}} = E_{\text{lb}}\bar{\cup}[E_{\text{ff}}+E'_{\text{lb}}]$ and $E''_{\text{rb}} = E'_{\text{rb}}$. Similarly if E' is an index family for X^2 and (12.43) *holds then*

$$\Psi^{-\infty;E'}(X;\Omega^{1/2})\cdot\Psi^{-\infty;E}(X;\Omega^{1/2}) \subset \Psi^{-\infty;E'_{\text{lb}},E_{\text{rb}}} \tag{12.58}$$

and if F is an index set for X with $F+E_{\mathrm{rb}}>-1$ then

$$\Psi^{-\infty;E}(X;\Omega^{1/2})\mathcal{A}^{F}_{\mathrm{phg}}(X;\Omega^{1/2})\subset\mathcal{A}^{E'_{\mathrm{lb}}}_{\mathrm{phg}}(X;\Omega^{1/2}). \tag{12.59}$$

Proof. These are simpler versions of Theorem 12.42 and Proposition 12.10. They can be proved in the same way. For example in (12.57) one should lift to the partial Θ-stretched product $X^2_\Theta\times X$; all three projections are b-submersions:

$$\begin{array}{ccccc} X^2_\Theta & \longleftarrow & X^2_\Theta\times X & \longrightarrow & X^2 \\ & & \big\downarrow & & \\ & & X^2. & & \end{array} \tag{12.60}$$

For (12.58) and (12.59) one only needs lift to the standard triple product, X^3, and double product, X^2, respectively.

§ 13. Mapping properties

The basic mapping property we need to establish is the boundedness, on L^2, of the operators which are of order at most zero and which have sufficiently rapid decay at the right and left boundaries.

PROPOSITION 13.1. *If $m\leqslant 0$ and the exponent sets satisfy $E_{\mathrm{lb}}>-1/2$, $E_{\mathrm{rb}}>-1/2$ and $E_{\mathrm{ff}}\geqslant 0$ then each $A\in\Psi^{m;E}_\Theta(X;\Omega^{1/2})$ defines a bounded operator on $L^2(X;\Omega^{1/2})$; the operators in $\Psi^{-\infty;E}(X;\Omega^{1/2})$ are compact on $L^2(X;\Omega^{1/2})$.*

Proof. We first use Hörmander's elegant symbolic argument to reduce to the case $m=-\infty$. In fact if A has order $m\leqslant 0$ then by choosing a large positive constant and exploiting the symbolic formula (12.34) an approximate square root can be constructed:

$$C\,\mathrm{Id}-A^*\cdot A=B^*\cdot B+R,\quad R\in\Psi^{-\infty;E'}_\Theta(X;\Omega^{1/2}) \tag{13.2}$$

for a new index family which however still satisfies the hypotheses of the proposition.

So now assume $m=-\infty$. Let ν be a non-vanishing smooth half-density on X. Then we need to show that for some constant C_A

$$\left|\int_X f\nu A(g\nu)\right|\leqslant C_A\|f\nu\|_{L^2(X)}\|g\nu\|_{L^2(X)}\quad\forall f,g\in\dot{\mathcal{C}}^\infty(X). \tag{13.3}$$

As usual we lift the integral on the left to X^2_Θ. There it can be written

$$\int_{X^2_\Theta} \varkappa_A (\pi_{L,\Theta})^*(g\nu)\,(\pi_{R,\Theta})^*(f\nu). \tag{13.4}$$

The lift of ν from the two factors combine with the singular density factor, $\varrho_{\mathrm{ff}}^{-N/2}\,\Omega^{1/2}$, in $\varkappa_A$ to give a smooth density on X^2_Θ, so (13.4) can be written

$$\int_{X^2_\Theta} a(\pi_{L,\Theta})^*(g)\,(\pi_{R,\Theta})^*(f)\,\mu, \quad a\in \mathcal{A}^E_{\mathrm{phg}}(X^2_\Theta), \quad \mu\in \mathscr{C}^\infty(X^2_\Theta;\Omega). \tag{13.5}$$

Applying the Cauchy–Schwarz inequality gives the bound

(13.6)

$$\left|\int_X f\nu A(g\nu)\right| \leqslant \left[\int_{X^2_\Theta} |a|\,|(\pi_{L,\Theta})^*(g)|^2\mu(\varrho_{\mathrm{lb}}/\varrho_{\mathrm{rb}})^{1/2}\right]^{1/2}\left[\int_{X^2_\Theta} |a|\,|(\pi_{R,\Theta})^*(f)|^2\mu(\varrho_{\mathrm{rb}}/\varrho_{\mathrm{lb}})^{1/2}\right]^{1/2}.$$

Thus it suffices to estimate these integrals.

Consider the lift, ω_R, to X^2_Θ, from the right, of a non-vanishing volume form, ω, on X. From the (7.15) it follows that a smooth form $\alpha = d\varrho_{\mathrm{rb}} \dots \alpha_n$ can be chosen on X^2_Θ so that

$$\omega_R \wedge \alpha = \varrho_{\mathrm{rb}}\mu. \tag{13.7}$$

By assumption on the index family the kernel in (13.6) satisfies

$$|a|(\varrho_{\mathrm{rb}}/\varrho_{\mathrm{lb}})^{1/2} \leqslant C\varrho^\varepsilon_{\mathrm{rb}}, \quad \varepsilon>0. \tag{13.8}$$

Applying Fubini's theorem to the second integral in (13.6), using (13.7) and (13.8), gives

$$\int_{X^2_\Theta} |a|\,|(\pi_{R,\Theta})^*(f)|^2(\varrho_{\mathrm{rb}}/\varrho_{\mathrm{lb}})^{1/2}\mu \leqslant C\int_X |f|^2\omega.$$

The bound for the other integral follows in the same way, proving the first part of the proposition. The second part follows directly from the Cauchy–Schwarz inequality on X^2.

From this basic result we can easily investigate more general boundedness and compactness results. For each positive integer consider the non-homogeneous Sobolev spaces defined by

$$\mathcal{H}^k_\Theta(X;\Omega^{1/2}) = \{u \in L^2(X;\Omega^{1/2});\ \mathcal{V}^j_\Theta u \subset L^2(X;\Omega^{1/2})\ \forall j \leq k\}. \tag{13.9}$$

Clearly this space has a natural Hilbert space topology (although no natural inner product) with respect to which $\dot{\mathcal{C}}^\infty(X;\Omega^{1/2})$ is a dense subset. The dual of $\mathcal{H}^k_\Theta(X;\Omega^{1/2})$, which we denote $\mathcal{H}^{-k}_\Theta(X;\Omega^{1/2})$ is characterized by the condition

$$u \in \mathcal{H}^{-k}_\Theta(S;\Omega^{1/2}) \quad \Leftrightarrow \quad u = \sum_i P_i u_i,\ P_i \in \mathrm{Diff}^k_\Theta(X;\Omega^{1/2}),\ u_i \in L^2(X;\Omega^{1/2}). \tag{13.10}$$

More generally we can replace the ring of Θ-differential operators by the small calculus and define, for any real number, m,

$$\begin{gathered}\mathcal{H}^m_\Theta(X;\Omega^{1/2}) = \{u \in L^2(X;\Omega^{1/2});\ \Psi^m_\Theta u \subset L^2(X;\Omega^{1/2})\},\quad m \geq 0\\ \mathcal{H}^m_\Theta(X;\Omega^{1/2}) = \{u \in \mathcal{C}^{-\infty}(X;\Omega^{1/2});\ u = \sum_i P_i u_i,\ P_i \in \Psi^{-m}_\Theta u,\ u_i \in L^2(X;\Omega^{1/2})\},\quad m \leq 0.\end{gathered} \tag{13.11}$$

We can also add weighting by the real powers of a defining function to the boundary and set

$$\varrho^s\mathcal{H}_\Theta(X;\Omega^{1/2}) = \{u \in \mathcal{C}^{-\infty}(X;\Omega^{1/2});\ \varrho^{-s}u \in \mathcal{H}_\Theta(X;\Omega^{1/2})\}\quad \forall s, m \in \mathbf{R}. \tag{13.12}$$

Using the composition and conjugation properties the calculus, together with Proposition 13.1 the boundedness of the elements of the full calculus on these spaces is easily described:

PROPOSITION 13.13. *If r, m, s, s' are real number such that the index sets satisfy*

$$E_{\mathrm{lb}} > s' - \frac{1}{2},\quad E_{\mathrm{rb}} + s > -\frac{1}{2},\quad s + E_{\mathrm{ff}} \geq s', \tag{13.14}$$

then each element of $\Psi^{m;E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}}_\Theta(X;\Omega^{1/2})$ defines a continuous linear operator

$$\varrho^s\mathcal{H}^r_\Theta(X;\Omega^{1/2}) \to \varrho^{s'}\mathcal{H}^{r-m}_\Theta(X;\Omega^{1/2}).$$

Proof. This just follows by applying Proposition 13.1 to the operator

$$\varrho^{-s'}A\varrho^s \in \Psi^{m;E_{\mathrm{lb}}-s',E_{\mathrm{rb}}+s,E_{\mathrm{ff}}+s-s'}_\Theta(X;\Omega^{1/2}). \tag{13.15}$$

The inclusions between the weighted Θ-Sobolev spaces are obvious enough,

$$\varrho^s \mathscr{H}^m_\Theta(X;\Omega^{1/2}) \hookrightarrow \varrho^{s'} \mathscr{H}^{m'}_\Theta(X;\Omega^{1/2}) \quad \Leftrightarrow \quad s \geqslant s', \quad m \geqslant m' \tag{13.16}$$

and

$$\text{(13.16) is compact iff } s > s' \text{ and } m > m'. \tag{13.17}$$

From this we easily deduce that:

Lemma 13.18. *$A \in \Psi^{m;E_{\mathrm{lb}},E_{\mathrm{rb}},E_{\mathrm{ff}}}_\Theta(X;\Omega^{1/2})$ is compact as an operator from $\varrho^s \mathscr{H}^r_\Theta(X;\Omega^{1/2})$ to $\varrho^{s'} \mathscr{H}^{r'}_\Theta(X;\Omega^{1/2})$ if (13.14) holds and*

$$E_{\mathrm{ff}} > s'-s, \quad m < r'-r. \tag{13.19}$$

It is also relatively easy to define, and show boundedness properties for, the Θ-Hölder spaces.

§ 14. The resolvent

Now we turn to the proof of the main result, that the resolvent for an appropriate class of Θ-metrics is in the space of Θ-pseudodifferential operators. Thus let g be a Θ-metric on a compact manifold with boundary, X, where Θ induces a contact structure on the boundary. We shall assume in addition that it satisfies the normalization conditions described in § 4, thus as well as (4.13), (4.16) and (4.19) hold for some $\chi \in \mathscr{C}^\infty(\partial X)$. From Proposition 4.24 we know that this implies that the normal operator of the Laplacian is always reducible to the Bergman Laplacian on the ball, locally smoothly in the parameters in ∂X.

To conform with our consistent emphasis on operators acting on half-densities we shall replace the usual action of the Laplacian on functions by such an action. If dg is the Riemannian density of g the half-density

$$\mu = \varrho^{N/2} dg^{1/2}, \quad N = \dim X + 1, \quad \dim X = 2(n+1) \tag{14.1}$$

trivializes the half-density bundle. Thus we consider the Laplacian to act on half-densities by setting

$$Qu = (\Delta\phi)\mu \quad \text{if} \quad u = \phi\mu \in \mathscr{C}^\infty(X;\Omega^{1/2}), \quad \phi \in \mathscr{C}^\infty(X). \tag{14.2}$$

We are interested in the modified resolvent family of Q. By definition this is a bounded family of operators on $\varrho^{N/2}L^2(X;\Omega^{1/2})$:

$$R(s) = [Q - \chi^2 s(n+1-s)]^{-1}, \quad \Re(s) > n+1. \tag{14.3}$$

The boundedness for s in this range follows from the positivity of Q.

Our main result is an analytic continuation theorem for the family $R(s)$; for a general Θ-metric satisfying the appropriate normalization conditions, this family may fail to be analytic on the set:

$$\mathcal{P}=\left\{\frac{1}{4}m;\ m\equiv 1,3 \bmod 4,\ m<2(n+1)\right\}\cup\{-\mathbf{N}_0\}. \tag{14.4}$$

However, if there is a defining function, ϱ for ∂X such that the Taylor series for the metric at ∂X only involves even powers of ϱ then the singular set reduces to $-\mathbf{N}_0$. We will call such a metric D-smooth. If the manifold, X is the square root of another manifold, $\mathcal{U}$ as in § 2 and the metric is pulled up from $\mathcal{U}$ then it will obviously be D-smooth on X.

THEOREM 14.5. *Let g be a Θ-metric on a compact manifold with boundary X where Θ induces a contact structure on the boundary and the normalization conditions* (4.16) *and* (4.19) *hold, then the modified resolvent family* (14.3) *extends to be meromorphic as a family of operators,*

$$\mathbf{C}\setminus\mathcal{P}\ni s\mapsto R(s)\in\Psi_{\Theta}^{-2;2s,2s-N}(X;\Omega^{1/2})+\Psi^{-\infty;2s,2s-N}(X;\Omega^{1/2}) \tag{14.6}$$

with poles (*at points other than $\mathcal{P}$ given by* (14.4)) *of finite rank. If g is D-smooth then $R(s)$ is meromorphic on $\mathbf{C}\setminus[-\mathbf{N}_0]$. There are only real poles in $\mathbf{R}(s)>\frac{1}{2}(n+1)$, i.e. in the interval $(\frac{1}{2}(n+1),n+1)$, and these correspond to $\varrho^{N/2}L^2$ eigenspaces of finite dimension.*

The Main Theorem of the introduction is an immediate consequence of this result.

Proof. We proceed by constructing a parametrix for $Q-\chi^2 s(n+1-s)$ which is a good approximation, in terms of regularity both in the interior and at the boundary, to $R(s)$.

By assumption the normal operator of Q is a family of differential operators depending smoothly on parameters in ∂X and locally smoothly reducible to the Laplacian on the upper half-space. Thus, using the results of § 8, and the surjectivity of the normal mapping (12.25) we can construct

$$\begin{gathered} E_0(s)\in\Psi_{\Theta}^{-2;2s,2s-N}(X;\Omega^{1/2})\quad\text{such that}\\ [Q-\chi^2 s(n+1-s)]\cdot E_0(s)=\mathrm{Id}-R_1(s),\quad R_1(s)\in\Psi_{\Theta}^{0;2s,2s-N,1}(X;\Omega^{1/2}). \end{gathered} \tag{14.7}$$

The indicial roots of $I(Q-\chi^2 s(n+1-s))$ are $2s, 2(n+1-s)$. Hence from (12.35) we conclude that in fact

$$R_1(s) \in \Psi_\Theta^{0;2s+1,2s-N,1}(X;\Omega^{1/2}). \tag{14.8}$$

Thus Corollary 12.40 applies and we can remove part of the error term $R_1(s)$ by taking

$$\begin{gathered} E_1(s) \in \Psi_\Theta^{-\infty;2s+1,\emptyset,1}(X;\Omega^{1/2}) \quad \text{such that} \\ [Q-\chi^2 s(n+1-s)]\cdot E_1(s)-R_1(s) \in \Psi_\Theta^{0;\emptyset,2s-N,1} \end{gathered} \tag{14.9}$$

Using this we find that

$$\begin{gathered} E_{(1)}(s) = E_0(s)+E_1(s) \in \Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2}) \quad \text{satisfies} \\ [Q-\chi^2 s(n+1-s)]\cdot E_{(1)} = \mathrm{Id}-R_2(s), \quad R_2(s) \in \Psi_\Theta^{0;\emptyset,2s-N,1}(X;\Omega^{1/2}). \end{gathered} \tag{14.10}$$

In order to obtain the optimal meromorphy results one needs to carefully consider the solution of the indicial equation in (14.9). Along the left boundary $R_1(s)$ will have an asymptotic expansion:

$$R_1(s) \simeq \sum_{l=1}^{\infty} \varrho_{\mathrm{lb}}^{2s+l} a_l. \tag{14.11}$$

The solution to the equation in (14.9) is of the form:

$$E_1(s) \sim \sum_{l=1}^{\infty} \frac{\varrho_{\mathrm{lb}}^{2s+l}\alpha_l}{\frac{1}{2}l(2s+\frac{1}{2}l-n-1)}. \tag{14.12}$$

The coefficients, α_l are functions smooth in a neighborhood of the left boundary. A moments consideration shows that α_l depends only on the a_i for $i \leqslant l$ and their derivatives. From the formula it is apparent that poles may occur whenever $s=\frac{1}{2}(n+1)-\frac{1}{4}l$. There are essentially two cases in which the poles can be removed:

(14.13) s is a quarter-integer less than $\dfrac{1}{2}(n+1)$ but not a half integer

(14.14) s is a half integer less than $\dfrac{1}{2}(n+1)$ or an integer between 0 and $\dfrac{1}{2}(n+1)$.

The situation described in (14.13) only arises if the operator Q is not D-smooth. If it is D-smooth the expansion of $N(E_0(s))$ along $\mathrm{ff} \cap \mathrm{lb}$ involves only even powers of ϱ_{lb}, and it can be extended to retain this property. Then the series (14.11) will only involve terms with l even. In this case the poles at the quarter-integers will not arise. To treat (14.14) we use formula (8.8) to show that the expansion of $N(R_1(s))$ near to $\mathrm{lb}\cap\mathrm{rb}$ will be of the form:

$$N\left(R_1\left(\frac{1}{2}k\right)\right)\sim\sum_{m=2(n+1-k)+1}^{\infty}a_m\left(\frac{1}{2}k\right)(\varrho_{\mathrm{lb}})^{m+k}. \tag{14.15}$$

As $E_0(s)$ is defined by extending $N(E_0(s))$ one can easily arrange to have:

$$R_1\left(\frac{1}{2}k\right)\sim\sum_{m=2(n+1-k)+1}^{\infty}a_m\left(\frac{1}{2}k\right)(\varrho_{\mathrm{lb}})^{m+k}, \tag{14.16}$$

as well. From (14.16) and the way in which α_l depends on the $[a_i]$, it follows easily that $\alpha_l(\frac{1}{2}k)=0$ for $l=0,\dots,2(n+1-k)$. Since the $\alpha_l(s)$ are analytic near $\frac{1}{2}k$ it follows from (14.12) that $E_1(s)$ extends analytically across these values of s. Thus we see that the poles of $E_{(1)}(s)$ lie in $\mathscr{P}$ in the general case and in $-\mathbf{N}_0$ in case Q is D-smooth. Next we use the ellipticity of Q and apply Corollary 12.38 to find

$$\begin{gathered}E_2(s)\in\Psi_\Theta^{-2}(X;\Omega^{1/2})\quad\text{such that}\\ [Q-\chi^2s(n+1-s)]\cdot E_2(s)-R_2(s)\in\Psi_\Theta^{-\infty;\varnothing,2s-N,1}(X;\Omega^{1/2}).\end{gathered} \tag{14.17}$$

This means that

$$\begin{gathered}E_{(2)}=E_0(s)+E_1(s)+E_2(s)\in\Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2})\quad\text{satisfies}\\ [Q-\chi^2s(n+1-s)]\cdot E_{(2)}(s))=\mathrm{Id}-R_3(s),\quad R_3(s)\in\Psi_\Theta^{-\infty;\varnothing,2s-N,1}(X;\Omega^{1/2}).\end{gathered} \tag{14.18}$$

To proceed further we use Corollary 12.52 to examine the Neumann series for the inverse of $\mathrm{Id}-R_3(s)$. By (12.53) we have

$$R_3^k(s)\in\Psi_\Theta^{-\infty;\varnothing,\mathscr{I}_k,k}(X;\Omega^{1/2})\quad\forall k\in\mathbf{N}, \tag{14.19}$$

where $\mathscr{I}_k$ is the index set:

$$\mathscr{I}_k=\{(2s-N,0),\,(2s-N+1,1),\dots,(2s-N+k,k-1)\}. \tag{14.20}$$

We shall also set

$$\mathscr{I}=\bigcup_{k>0}\mathscr{I}_k. \tag{14.21}$$

We can asymptotically sum the series and so construct

$$E_3(s) \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{I},1}(X;\Omega^{1/2}) \quad \text{such that} \tag{14.22}$$
$$\mathrm{Id}-(\mathrm{Id}-R_3(s))\cdot(\mathrm{Id}-E_3(s)) \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{I},\emptyset}(X;\Omega^{1/2}).$$

Then we find that

$$E_{(3)} = E_{(2)}\cdot(\mathrm{Id}-E_3(s)) \in \Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2})+\Psi_\Theta^{-\infty;2s,\mathcal{I},\mathcal{I}'}(X;\Omega^{1/2}) \quad \text{satisfies} \tag{14.23}$$
$$[Q-\chi^2 s(n+1-s)]\cdot E_{(3)} = \mathrm{Id}-R_4(s), \quad R_4(s) \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{I},\emptyset}(X;\Omega^{1/2})$$

where

$$\mathcal{I}' = \{(4s,0),\ (4s+1,1),\ \dots,\ (4s+k,k-1)\}. \tag{14.24}$$

The remainder term here is 'extremely compact'. In particular for any $a \in \mathbf{R}$ such that $a+2s-N>-1/2$,

$$R_4(s)\colon \varrho^a L^2(X;\Omega^{1/2}) \to \dot{\mathscr{C}}^\infty(X;\Omega^{1/2}) \hookrightarrow \varrho^a L^2(X;\Omega^{1/2}) \tag{14.25}$$

defines a compact holomorpohic family in $2s-N>-a-1/2$. This allows us to apply analytic Fredholm theory. If the choice of operators above is made with inordinate bad luck it might happen that $\mathrm{Id}-R_4(s)$ is not invertible in any open set in s. So we first make sure that this is so. The operator $\mathrm{Id}-R_4(N)$, has finite dimensional null space and is of index zero. From the mapping property (14.25) it follows that the null space is in $\dot{\mathscr{C}}^\infty(X;\Omega^{1/2})$. Thus by subtracting from $R_4(N)$ a finite rank operator with Schwartz kernel of the form

$$\sum_i \phi_i \otimes \psi_i, \quad \phi_i, \psi_i \in \dot{\mathscr{C}}^\infty(X;\Omega^{1/2}) \tag{14.26}$$

we can ensure that $\mathrm{Id}-R_4(N)$ is invertible on $\varrho^a L^2(X;\Omega^{1/2})$. To do this we need to solve the equations

$$[Q+\chi^2 N(n+2)]\, u_i = \phi_i. \tag{14.27}$$

Now, since $R(N)$ is bounded on $L^2(X;\Omega^{1/2})$ this has a (unique) solution in $L^2(X;\Omega^{1/2})$. Since Q is real and self-adjoint our construction above of a right parametrix, $E_{(3)}$, can equally well be applied to obtain a left parametrix $E_{(3)}^t$. By the mapping properties this can indeed be applied to $L^2(X;\Omega^{1/2})$ and shows that $u_i=\varrho^N v_i$, $v_i \in \mathscr{C}^\infty(X;\Omega^{1/2})$. Thus we only need modify $E_{(3)}$ to

$$E_{(4)} = E_{(3)} + \sum_i \varrho^{2s} v_i \otimes \psi_i \in \Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2}) + \Psi_\Theta^{-\infty;2s,\mathcal{I},\mathcal{I}'}(X;\Omega^{1/2}) \tag{14.28}$$

to ensure that

$$\begin{gathered} \mathrm{Id} - R_5(s) = [Q - \chi^2 s(n+1-s)] \cdot E_{(4)} \quad \text{with} \\ R_5(s) \in \Psi_\Theta^{-\infty;\varnothing,\mathcal{I},\varnothing}(X;\Omega^{1/2}) \quad \text{and} \quad (\mathrm{Id} - R_5(N)) \text{ invertible.} \end{gathered} \tag{14.29}$$

Now consider the meromorphic family

$$(\mathrm{Id} - R_5(s))^{-1} = \mathrm{Id} - S(s), \quad S(s) \in \Psi_\Theta^{-\infty;\varnothing,\mathcal{I},\varnothing}(X;\Omega^{1/2}). \tag{14.30}$$

This is, by analytic Fredholm theory, meromorphic away from $-\mathbf{N}_0$ or $\mathcal{P}$ depending on the smoothness of Q at ∂X. This proves that

$$R(s) = E_{(4)} \cdot (\mathrm{Id} - S(s)) \in \Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2}) + \Psi_\Theta^{-\infty;2s,\mathcal{I},\mathcal{I}'}(X;\Omega^{1/2}). \tag{14.31}$$

Since Q is a self-adjoint operator on $L^2(X,dg)$ it follows that, if we use the volume form of the metric to trivialize the density bundles then the kernel of $R(s)$ must be symmetric. The index sets on the left and right boundaries would then be the same. Thus, the logarithmic terms are absent from the right boundary:

$$R(s) = P_1(s) + P_2(s), \quad P_1(s) \in \Psi_\Theta^{-2;2s,2s-N}(X;\Omega^{1/2}), \quad P_2(s) \in \Psi^{-\infty;2s,2s-N,\mathcal{I}'_\infty}(X;\Omega^{1/2}). \tag{14.32}$$

The only point remaining in the proof of (14.6) is the analysis of $P_2(s)$. We need to show that it actually lies in $\Psi^{-\infty;2s,2s-N}(X;\Omega^{1/2})$. Since the index set $\mathcal{I}'$ involves the powers $4s+j$ it follows that at the front face there can be no cancellation between P_1 and P_2. We conclude that

$$[Q - \chi^2 s(n+1-s)] P_2(s) \in \Psi_\Theta^{-\infty;2s+1,2s-N,\varnothing}(X;\Omega^{1/2}) \subset \Psi^{-\infty;2s,2s-N}(X;\Omega^{1/2}). \tag{14.33}$$

Thus we can apply Corollary 12.40 to find a new term

$$P_2(s)' \in \Psi_\Theta^{-2;2s+1,\varnothing,\varnothing}(X;\Omega^{1/2}) \subset \Psi^{-\infty;2s,2s-N}$$

such that

$$[Q - \chi^2 s(n+1-s)](P_2 - P_2') \in \Psi^{-\infty;\varnothing,2s-N}(X;\Omega^{1/2}). \tag{14.34}$$

Now we see that:

(14.35) $[Q-\chi^2 s(n+1-s)](P_1(s)+P_2(s)') = \mathrm{Id}-R_6(s)$ where $R_6(s) \in \Psi^{-\infty;\varnothing,2s-N}(X;\Omega^{1/2})$.

We can therefore apply the resolvent to R_6 on the left to obtain $P_3(s)=R(s)\cdot R_6(s)$. Using the composition formula (12.57) it follows that $P_3(s) \in \Psi^{-\infty;2s,2s-N}(X^2;\Omega^{1/2})$. Thus

$$[Q-\chi^2 s(n+1-s)](P_1(s)+P_2(s)'+P_3(s)) = \mathrm{Id}, \tag{14.36}$$

and for $\Re(s)$ large this kernel maps $L^2(X;\Omega^{1/2},dg)$ to itself. By the uniqueness of the resolvent kernel in the resolvent set it follows that $R(s)=P_1(s)+P_2(s)'+P_3(s)$ and therefore $R(s)$ has the smoothness claimed in (14.6).

To complete the proof of the theorem we need to consider the poles in the interval $(n+1/2, n+1)$. These poles are introduced in the inversion of $\mathrm{Id}-R_5(s)$ and are therefore of finite rank. If such a pole occurs at s_0 then we can find a function ψ such that $(\mathrm{Id}-R_5(s_0))\psi=0$ evidently $\psi \in \dot{\mathscr{C}}^\infty(X;\Omega^{1/2})$ hence $u=E_{(4)}(s_0)\psi \in \mathscr{A}^{2s}_{\mathrm{phg}}(X;\Omega^{1/2})$ is either zero or an $L^2(X,dg)$ eigenfunction of Q with eigenvalue $s(n+1-s)$. The value $s=n+1$ is excluded as a simple integration by parts argument shows that u would have to be a constant and therefore, since it is in L^2, zero. This completes the proof of the theorem.

If the defining function r in (1) is taken to be either the solution of the complex Monge–Ampère equation (see [8], [3] and [16]) or the kernel of the Bergman projector restricted to the diagonal ([7]) then it is not necessarily smooth, but rather of the form

$$r = r_0 + a\log r_0, \quad a \in \mathscr{C}^\infty(\mathscr{U}),$$

where r_0 is a smooth defining function. The resulting metric can be handled by the arguments above with only modifications to handle the logarithmic terms. Consider the index set

$$M(s) = \{(2s,0),\ (2s+2n+2,1),\ \dots,\ (2s+k(2n+2),k), \dots\}.$$

Then Theorem 14.5 continues to hold provided Theorem 14.5 is replaced by

$$\mathbf{C}\setminus\mathscr{P} \ni s \mapsto R(s) \in \Psi_\Theta^{-2;M(s),M(s)-N}(X;\Omega^{1/2}) + \Psi^{-\infty;M(s),M(s)-N}(X;\Omega^{1/2}). \tag{14.37}$$

§ 15. Dirichlet problem

If u is a solution to

$$(\Delta-\chi^2 s(n+1-s))\,u = 0 \tag{15.1}$$

belonging to $\mathscr{A}_{\text{phg}}(X)$, then the roots of the indicial equation correspond to the leading terms in the asymptotic expansion of u:

$$u \sim \sum_{j=0}^{\infty} \left[a_j \varrho^{2s+j} + b_j \varrho^{2(n+1-s)+j}\right]. \tag{15.2}$$

For s a quarter integer equal to or larger than $\frac{1}{2}(n+1)$, logarithmic terms can also arise, we will ignore them for the time being. For the remainder of this section we'll assume that $\Re s > \frac{1}{2}(n+1)$. In this case (15.1) has well defined Dirichlet and Neumann problems:

$$(\Delta - \chi^2 s(n+1-s))\, u = 0, \quad b_0 = f \quad \text{Dirichlet,} \tag{15.3}$$

$$(\Delta - \chi^2 s(n+1-s))\, u = 0, \quad a_0 = f \quad \text{Neumann,} \tag{15.4}$$

the Dirichlet problem is always well-defined and the Neumann problem is well defined provided $4s - 2(n+1) \notin \mathbb{N}$. In this section we will solve the Dirichlet problem by constructing the 'Poisson kernel' as a limit of the resolvent kernel constructed in § 14. To define this kernel we begin with the resolvent kernel for the action of $\Delta - \chi^2 s(n+1-s)$ on singular half densities,

$$R_\Theta(s) \in \Psi_\Theta^{-2;2s-N/2,2s-N/2}(X, \varrho_{\text{ff}}^{-N/2}\,\Omega^{1/2}) + \Psi^{-\infty;2s-N/2,2s-N/2}(X, \varrho_{\text{ff}}^{-N/2}\,\Omega^{1/2}). \tag{15.5}$$

If dg is the metric density then the kernel defined by

$$R'_\Theta(s) = R_\Theta/(dg_l\, dg_r)^{1/2} \tag{15.6}$$

belongs to $\Psi_\Theta^{-2;2s,2s}(X) + \Psi^{-\infty;2s,2s}(X)$ and satisfies:

$$(\Delta_O - \chi^2 s(n+1-s))\, R'_\Theta = \delta_{\Delta_\Theta}. \tag{15.7}$$

Here O is either R or L. Let ϱ denote a defining function for ∂X, then we define the s-Poisson kernel, $E(s)$ by:

$$E(p,s) = \lim_{q \to p} \pi_{L,\Theta}^*(\varrho^{-2s})\, R'_\Theta(q,s) \tag{15.8}$$

here p is a point in the interior of the left boundary $\mathring{\text{lb}}$; the limit is computed in X_Θ^2. To begin our analysis of E we identify the left boundary, which is the closure of its interior, with a parabolic blow-up of $\partial X \times X$. At $\partial\Delta \subset \partial X^2$ in $\partial X \times X$ the form $\iota_{\partial X}^*\Theta - \Theta$ is normal to $\partial\Delta$, i.e. spans a line bundle $\tilde{S} \subset N^*(\partial\Delta)$.

PROPOSITION 15.9. *The left boundary of the Θ-stretched product, $\mathrm{lb}(X^2_\Theta)$, is canonically isomorphic to the $\tilde{S}$-parabolic blow-up of $\partial\Delta$ in $\partial X\times X$; we will denote this by $(\partial X\times X)_\Theta$.*

Proof. For clarity let us denote by $N^+(\partial\Delta, \partial X\times X)$ the inward-pointing part of the normal bundle of $\partial\Delta$ in $\partial X\times X$ and let $N^+(\partial\Delta, X^2)$ denote the corresponding normal space in X^2. Clearly $\tilde{S}$ satisfies the cleanness conditions in §6, thus the $\tilde{S}$-parabolic blow-up of $\partial\Delta$ in lb is well defined. In fact in order to define the $\tilde{S}$-parabolic blow-up of $\partial\Delta$ in X^2 we need to choose a complement to the annihilator H' of $\tilde{S}$. In the construction of X^2_Θ in §7 the complement $H^\perp$ to the annihilator H of the line bundle G spanned by (7.2) was chosen to lie in $T(\partial X\times\partial X)$. Thus we can use the same space $H^\perp$ as complement here. From this it is immediate that the S-dilation structure defined on $N^+(\partial\Delta, X^2)$ restricts to define a $\tilde{S}$-dilation structure on $N^+(\partial\Delta, \partial X\times X)$. As the ring of $\mathscr{C}^\infty$ functions on the parabolic blow-up is defined by homogeneity properties it is immediately clear that the canonical inclusion of $N^+(\partial\Delta, \partial X\times X)$ into $N^+(\partial\Delta, X^2)$ lifts to a $\mathscr{C}^\infty$ map:

$$\iota': N^+(\partial\Delta, \partial X\times X)_{\partial\Delta, \tilde{S}} \hookrightarrow N^+(\partial\Delta, X^2)_{\partial\Delta, S}. \tag{15.10}$$

From this it follows that the inclusion of $\partial X\times X$ into X^2 lifts to a smooth inclusion of $(\partial X\times X)_\Theta$ into X^2_Θ. Since the image is closed it is clearly just $\mathrm{lb}(X^2_\Theta)$.

This identification shows that the maps

$$\begin{aligned} &\beta'_\Theta: (\partial X\times X)_\Theta \to \partial X\times X \\ &\pi'_{R,\Theta}: (\partial X\times X)_\Theta \to X \quad \text{and} \\ &\pi'_{L,\Theta}: (\partial X\times X)_\Theta \to \partial X \end{aligned} \tag{15.11}$$

are all b-submersions.

The Θ-blown-up boundary, $(\partial X\times X)_\Theta$ has two boundary components, the right boundary, rb$'$ and the front face ff$'$. From Theorem 14.5 and (15.8) it follows that $E(s)$ is a classical conormal distribution on $(\partial X\times X)_\Theta$ of the form:

$$(\varrho'_{\mathrm{rb}}/\varrho'_{\mathrm{ff}})^{2s}E_1(s)+(\varrho'_{\mathrm{rb}}\varrho'_{\mathrm{ff}})^{2s}E_2(s). \tag{15.12}$$

Here $E_1(s)$ is in $\mathscr{C}^\infty([\partial X\times X]_\Theta)$ and E_2 is the pullback via β'_Θ of a function in $\mathscr{C}^\infty(\partial X\times X)$.

To define a map from $\mathscr{C}^\infty(\partial X)$ to $\mathscr{C}^{-\infty}(X)$, we choose a smooth volume form, ν, on

∂X and set:

$$P_s f = (\pi'_{R,\Theta})_*(\pi'_{L,\Theta}{}^*(f\nu)\, E(s)). \tag{15.13}$$

It follows from the fact that the projections onto the factors are b-submersions that:

Proposition 15.14. *If $f \in \mathscr{C}^\infty(\partial X)$ then $P_s f$ belongs to $\mathscr{A}^E_{\text{phg}}(X)$ where*

$$E = \{(2s, 0)\} \,\overline{\cup}\, \{(2(n+1-s), 0)\}.$$

Thus we see that if s is not a quarter integer larger than $\frac{1}{2}(n+1)$ then $P_s f$ has an expansion as in (15.2), in any case the 'Dirichlet' data is obtained by computing:

$$\lim_{p \to p'} \varrho^{2(s-n-1)} P_s f(p) \tag{15.15}$$

where p' is a point on ∂X.

15.16 **Theorem.** *If $\Re s > \frac{1}{2}(n+1)$ then the limit in* (15.15) *equals*

$$\frac{w(p')^s f(p')}{2s-n-1},$$

where $w \in \mathscr{C}^\infty(\partial X)$ is positive.

Proof. Choose a smooth density μ on X, then

$$(P_s f)\,\mu = \pi'_{R,\Theta *}[(\pi'_{L,\Theta})^* f \beta'_\Theta{}^*(\nu\mu)\, E(s)].$$

The pull-back of the density $\nu\mu$ is of the form $(\varrho'_{\text{ff}})^{2(n+1)}\omega$, where ω is a smooth, nonvanishing density on $(\partial X \times X)_\Theta$. Thus we are pushing forward an element of $\mathscr{A}^{E'}_{\text{phg}}((\partial X \times X)_\Theta, \Omega)$ where $E' = \{2s, 2(n+1)-2s\} + \{2s, 2(n+1)+2s\}$, provided s is not a quarter integer. We will leave the modifications necessary for this case to the reader. Now $\varrho^{2(s-n-1)} P_s f \mu$ is the pushforward of an element of $\mathscr{A}^{E''}_{\text{phg}}((\partial X \times X)_\Theta)$ where

$$E'' = \{4s - 2(n+1), 0\} + \{4s - 2(n+1), 4s\}. \tag{15.17}$$

If p is a point in $\mathring{X}$ then $(\pi'_{L,\Theta})^{-1}(p)$ is simply a copy of ∂X however as p tends to p' in the boundary the fiber of the projection converges to a union of $p \times \partial X \setminus \{p, p\}$ with the fiber of the front face of $(\partial X \times X)_\Theta$ lying over p. We will denote this fiber by F_p. From (15.17) and the fact that $\Re s > \frac{1}{2}(n+1)$ it follows that as p tends to the boundary only the part of the integral along $F_{p'}$ contributes to (15.15). Along this fiber f reduces to the constant $f(p')$. Moreover, this fiber is simply the intersection of the fiber of the front face of X^2_Θ lying over p' with lb. It is therefore easy to see from (15.8) and (12.28) that the integrand in (15.15) is simply

$$\varrho_{\mathrm{rb}}^{2s-2(n+1)}\varrho_{\mathrm{lb}}^{-2s}N_{p'}(R'_\Theta)|_{\mathrm{lb}\cap\mathrm{ff}}\,\omega'. \tag{15.18}$$

We can choose coordinates so that on the fiber of the front face this is simply the kernel for the ball with the Bergman metric. Hence it follows that there exists a smooth, positive function w defined on ∂X such that the limit in (15.15) is

$$f(p')\,w(p')^s\lim_{|Y|\to 1}(1-|Y|^2)^{s-n-1}\int_{\mathbb{S}^{2n+1}}E_0(Y,\eta;s)\,d\sigma_\eta \tag{15.19}$$

where

$$E_0(Y,\eta,s)=c_n\frac{\Gamma(s)^2(1-|Y|^2)^s}{\Gamma(2s-n)|1-\eta\cdot\bar{Y}|^{2s}}$$

is the s-Poisson kernel for the ball.

We are reduced to computing the limit in (15.19). Let $u(Y)$ denote the integral in (15.19). This function is a radially symmetric solution to the eigenvalue equation which moreover satisfies $u(0)=c_n(\Gamma(s)^2/\Gamma(2s-n))$ and $D_r u(0)=0$. Using the analysis presented in § 8, one easily concludes that:

$$u(Y)=c'_n\frac{\Gamma(2)^2}{\Gamma(2s-n)}(1-|Y|^2)^{n+1-s}{}_2F_1(n+1-s,n+1-s;n+1;|Y|^2). \tag{15.20}$$

Because $\Re s>\frac{1}{2}(n+1)$ the hypergeometric function is analytic in the closed unit disk and its limit as $|Y|\to 1$ is

$$\frac{\Gamma(n+1)\,\Gamma(2s-n-1)}{\Gamma(s)^2}.$$

From this and (15.20) it follows that the limit in (15.19) is $c''_n/(2s-n-1)$. This completes the proof of the theorem.

If we define $u=P_s((2s-n-1)f/w^s)$ then u is a solution to (15.3). Note in particular that if $s=n+1$ then u is a harmonic function, so the $(n+1)$-Poisson kernel is the Poisson kernel in the usual sense. In contradistinction to the general case of pinched negative curvature treated in [2], the Poisson kernel is smooth on a compactification of X rather than Hölder continuous. For the case of a metric on a strictly pseudoconvex domain the Poisson kernel is not smooth in the standard $\mathscr{C}^\infty$ structure but only in the square root structure. In the standard structure the kernel will be Hölder 1/2; as the curvature is asymptotically 1/4 pinched this is precisely what is predicted by the result in [2].

Appendix A: Discrete subgroups of SU(n+1, 1)

In this appendix we present the basic facts about the geometry of the unit ball, $\mathbf{CB}^{n+1} \subset \mathbf{C}^{n+1}$, equipped with the Bergman metric. This is a model for the Hermitian symmetric space with constant biholomorphic sectional curvature. We then discuss discrete subgroups with infinite volume quotients and a generalization of the results of Patterson and Sullivan on the Hausdorff dimension of the limit set of a convex co-compact group. Similar results were obtained independently, a bit earlier, by Kevin Corlette. He has studied the case of a general rank 1 symmetric space, see [4].

§ A1. Complex hyperbolic geometry

We first introduce the elements of the geometry of the Hermitian symmetric space $\mathrm{SU}(n+1,1)/\mathrm{SU}(n+1)$. This is the complex hyperbolic space; the canonical bi-invariant metric has constant holomorphic sectional curvature. As with the real hyperbolic space there is a profusion of different models, each with its own advantages and disadvantages.

The first model arises from the fact that the group $\mathrm{SU}(n+1,1)$ acts as the orthogonal group of the hermitian form:

$$Q(\zeta) = \zeta_0 \zeta_{\bar{0}} + \ldots + \zeta_n \zeta_{\bar{n}} - \zeta_{n+1} \zeta_{\overline{n+1}}.$$

On the set $\{Q(\zeta) = -1\}$ the quadratic form restricts to give a positive semi-definite inner product. The null direction can be eliminated by dividing out by the $\mathbf{S}^1$-action $\zeta \mapsto \exp(i\theta)\zeta$. A somewhat more natural description of this model is to take the quotient $\{Q(\zeta)<0\}/\mathbf{C}^*$. The quotient of the light cone, $\{Q(\zeta)=0\}/\mathbf{C}^*$ defines the geometric boundary. The main advantage of this model is that the group action is linear. In the interior of the light cone we can define a potential function by $\phi = Q(\zeta)/|\zeta_{n+1}|^2$. The Kähler metric defined by this potential agrees with that defined by the restriction of Q to the tangent space modulo the $\mathbf{S}^1$-action. The potential is well-defined on the quotient by $\mathbf{C}^*$ and defines a Kähler metric there as well.

We can obtain a model for the complex hyperbolic space in the unit ball in $\mathbf{C}^{n+1}$ by setting:

$$w = \zeta_0/\zeta_{n+1}, \quad z_1 = \zeta_1/\zeta_{n+1}, \quad \ldots, \quad z_n = \zeta_n/\zeta_{n+1}. \tag{A1.1}$$

One easily sees that the image of the interior of the light cone is the set

$$\mathbf{CB}^{n+1} = \{|w|^2 + |z_1|^2 + \ldots + |z_n|^2 < 1\}$$

and moreover the defining function ϕ becomes the usual defining function of the ball which we will also denote by ϕ. If (w, z) lies in the ball then $\zeta(w, z)=(w, z, 1)$ lies in the interior of the light cone. We can use this identification to deduce the action of $SU(n+1, 1)$ on the ball:

$$A \cdot (w, z) = ((A\zeta(w, z))_0, \ldots, (A\zeta(w, z))_n)/(A\zeta(w, z))_{n+1}$$

In Section 8 we introduced the fundamental point pair invariant:

$$(\zeta, \xi) = Q(\zeta, \xi)/(Q(\zeta)\, Q(\xi))^{1/2} \tag{A1.2}$$

and we showed that if ζ and ξ are thought of a points in the quotient then

$$|(\zeta, \xi)| = \cosh(d(\zeta, \xi)/2).$$

If X and Y are points in the unit ball then the fundamental invariant is given by

$$(X, Y) = \frac{1-(X, Y)}{(1-|X|^2)^{1/2}(1-|Y|^2)^{1/2}}, \tag{A1.3}$$

where

$$\langle X, Y\rangle = X_1\, Y_{\bar{1}} + \ldots + X_{n+1}\, Y_{\overline{n+1}}.$$

From Fefferman's transformation formula for solutions of the Monge–Ampère equation we can deduce the Jacobian determinant of an element of $\mathrm{Aut}(\mathbf{CH}^{n+1})$. If ψ is a solution of the Monge–Ampère equation on a domain $\mathcal{U}$ then the volume form $dV = dV_{\text{euclid}}/(\psi)^{n+2}$. If F is a biholomorphic map from $\mathcal{U}'$ to $\mathcal{U}$ then we can pull back the defining function, ψ, by

$$\psi'(p) = \psi(F(p))|F'(p)|^{2/n+2}$$

to obtain a solution of the Monge–Ampère equation on $\mathcal{U}'$, see [8]. The volume form is given by the same formula with ψ' replacing ψ. If we apply this in the linear model to the defining function, ϕ, which is a solution of the Monge–Ampère equation we obtain a new defining function, ϕ_A, given by

$$\phi_A(\zeta) = \phi(\zeta)\, (|\zeta_{n+1}| \,/\, |(A\zeta)_{n+1}|)^2$$

from which we can deduce that the complex Jacobian determinant is

$$|J_A(w, z)| = 1 \,/\, |A\zeta(w, z)|^{2n+2}, \tag{A1.4}$$

generalizing a well known formula in real hyperbolic geometry.

The final model is a generalization of the upper half space model of the real hyperbolic space. The boundary of this model is the hyperquadratic; the model is defined by:

$$\mathcal{Q}^+ = \{(x,y); \Im(x) - |y|^2/2 > 0\}.$$

A transformation that carries the ball into $\mathcal{Q}^+$ is given by

$$x = i(1+w)/(1-w), \quad y_i = 2^{1/2} z_i/(1-w), \quad i = 1, \ldots, n.$$

The fundamental invariant in this model is given in Section 8.

§ A2. Classification of group elements

As in the case of real hyperbolic space the elements of the automorphism group can be divided into three disjoint sets. Each set is determined by the geometry of the fixed point set. The classification is most easily deduced in the linear model.

A general fact that follows easily from the defining equations of the group is that the spectrum of A^{-1} is the same as the spectrum of A^*, from which it follows that if μ is an eigenvalue of A then so is $1/\bar{\mu}$. We will set

$$S^- = \{Q(\zeta) < 0\}, \quad S^+ = \{Q(\zeta) > 0\}, \quad L = \{Q(\zeta) = 0\}. \tag{A2.1}$$

If A has an eigenvector, v, in S^- then the fact that $Q(A\zeta) = Q(\zeta)$ implies that the eigenvalue must be of modulus 1. Let

$$v^\perp = \{w; Q(w,v) = 0\}. \tag{A2.2}$$

Since v lies in S^- its orthocomplement lies in S^+. As A is orthogonal it also preserves $v^\perp$. The quadratic form restricted to $v^\perp$ is positive definite and therefore the spectrum of A on this subspace must lie on the unit circle. From these considerations it follows easily that an element with a fixed point in the interior of the light cone has all of its spectrum on the unit circle and is conjugate to an element of the form

$$\tilde{A} = \begin{bmatrix} e^{i\theta_0} & & & \\ & \ddots & & \\ & & e^{i\theta_n} & \\ & & & e^{i\psi} \end{bmatrix} \tag{A2.3}$$

where $\theta_0 + \ldots + \theta_n + \psi \equiv 0$ modulo 2π. We will call such elements elliptic.

If A does not have an eigenvector in S^- then it must have an eigenvector on the light cone. An easy calculation shows that this is the only way in which A can have an eigenvalue which is not on the unit circle. If A has an eigenvalue μ of modulus greater than one then by the above remark it also has an eigenvalue, $1/\bar{\mu}$, of modulus less than one. Let v_1 and v_2 be the corresponding eigenvectors. Both lie in the light cone and one easily sees that the orthogonal complement of $\mathrm{sp}\{v_1, v_2\}$ lies in S^+. As before this set is invariant under A and the spectrum of A on this set lies on the unit circle. One can show that an element of this type is conjugate to one of the form:

$$\text{(A2.4)}\qquad \begin{bmatrix} e^{i\theta_1} & & & & \\ & \ddots & & & \\ & & e^{i\theta_n} & & \\ & & & e^{it}\cosh(x) & e^{it}\sinh(x) \\ & & & e^{it}\sinh(x) & e^{it}\cosh(x) \end{bmatrix}$$

here $\theta_1+\ldots+\theta_n+2t\equiv 0$ modulo 2π. We will call such an element hyperbolic.

The final case is that A has an eigenvector, v, on the light cone with eigenvalue, μ, of modulus 1. In this case $v^\perp=v+H$ where H lies in S^+. If we let $T=(A-\mu)\,v^\perp$, then T is contained in $V^\perp$ and moreover has dimension at most n as v belongs to $v^\perp$. One can easily show that T is invariant under A and lies in S^+. Therefore A restricted to T has spectrum on the unit circle. In the generic case $\dim(T)=n$ and $T^\perp$ is two dimensional. The eigenvector v belongs to $T^\perp$ and either A is diagonalizable on this subspace in which case A is elliptic or A is not. In the latter case we will say that A is parabolic; it will be conjugate to an element of the form:

$$\text{(A2.5)}\qquad \begin{bmatrix} e^{i\theta_1} & & & & \\ & \ddots & & & \\ & & e^{i\theta_n} & & \\ & & & e^{i\psi} & 1 \\ & & & 0 & e^{i\psi} \end{bmatrix}$$

here $\theta_1+\ldots+\theta_n+2\psi\equiv 0$ modulo 2π.

The nongeneric cases are again either elliptic or parabolic depending on whether A is diagonalizable or not. As we will be primarily interested in groups composed entirely of hyperbolic elements we will not pursue this question further.

From the normal forms it is easy to deduce the geometric actions of the different types of elements on the complex hyperbolic space. Note in particular that a hyperbolic element will fix a two dimensional subspace which will project to a totally geodesic complex 'line' in the complex hyperbolic space. On this line the metric restricts to the standard real hyperbolic metric and the element leaves a geodesic lying within the line invariant. We call this the axis of the element. The endpoints of the axis on the geometric boundary are the fixed points of the element. The distance which the element translates a point on its axis is called its translation length.

For the remainder of this appendix we will work in the nit ball model. We use $X\cdot\bar{Y}$ to denote the standard Hermitian product on $\mathbf{C}^{n+1}$ and $|X|^2=X\cdot\bar{X}$.

§ A3. Boundary geometry

As remarked above the quotient of the light cone by the $\mathbf{C}^*$-action identifies the geometric boundary with the unit sphere in $\mathbf{C}^{n+1}$. The metric in the interior of the unit ball blows up as the boundary is approached, at two different rates. To make this precise we consider the restriction of the metric to the sets

$$S_r=\{\phi=-r\}.$$

On vectors that lie in $T^{1,0}S_r$ the metric blows up like $1/\phi$, whereas on vectors transverse to this subspace the metric blows up like $1/\phi^2$. From this it is evident that in order to find a curve of shortest distance between two points on S_r one should try to make the tangent vector lie in the subspace $T^{1,0}S_r$. From this we can deduce that if we rescale the metric on S_r so as to keep the diameter constant then as we approach the boundary the geometry of S_r will tend to the Carnot geometry of the $(2n+1)$-sphere. This is a metric geometry defined by considering only paths whose tangent vectors lie in the contact $2n$-plane field. This distribution is defined as the annihilator of the contact form $(\bar{\partial}\phi-\partial\phi)/2i$. Such a curve is called Legendrian. We define the distance between two points as the infimum of the lengths of Legendrian curves joining them. We will measure the lengths of these curves in the standard spherical metric. For the details of this geometry we refer the reader to [15].

One can show that the distance between two points X and Y on $\mathbf{S}^{2n+1}$ is given by:

$$d_c(X,Y)=|1-\langle X,Y\rangle|^{1/2}. \tag{A3.1}$$

This metric is manifestly $\mathrm{SU}(n+1)$-invariant, the full group of automorphisms act as conformal maps in the sense of Mostow, see [15].

Using this metric one can define Hausdorff measures on the sphere and the corresponding Hausdorff dimensions. The sphere itself has dimension $2n+2$ in this sense. This arises because a Legendrian curve has dimension 1 whereas a curve transverse to the contact field has dimension 2. In the sections that follow we will consider a generalization of the results of Patterson and Sullivan relating the Hausdorff dimension, in the sense above, of the limit set of a discrete subgroup of $SU(n+1,1)$ to the exponent of convergence of a Poincaré series and also to the lowest eigenvalue of the Laplace operator.

To that end it is useful to have a normal form for hyperbolic elements that is adapted to the Carnot geometry of the boundary. A moment's thought indicates that what we want are representatives of the double cosets:

$$SU(n+1)\backslash SU(n+1,1)/SU(n+1).$$

There are essentially two invariants. The first is the translation length and other is the inner product of the fixed points, which is in this case a complex number within the unit disk. We normalize so that one fixed point is $(1,0,\ldots,0)$ and the other lies in the set $\{(w,z_1,0,\ldots,0)\}$. We further normalize to remove the rotational part about the axis. The most general element of this type is of the form:

$$\begin{bmatrix} \cosh(x)-s-it & (2s)^{1/2}e^{-x/2} & 0 & \ldots & 0 & -\sinh(x)+s+it \\ -(2s)^{1/2}e^{x/2} & 1 & 0 & \ldots & 0 & (2s)^{1/2}e^{x/2} \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ -\sinh(x)-s-it & (2s)^{1/2}e^{-x/2} & 0 & \ldots & 0 & \cosh(x)+s+it \end{bmatrix} \tag{A3.2}$$

where $s\geqslant 0$ and x,t are real numbers.

If $x>0$ above then $\xi_-=(1,0,\ldots,0)$ is the repelling fixed point and

$$\begin{aligned} \xi_+ = (2s-(\sinh(x)+s-it)(1-e^{-x}),& \\ 2\sinh(x)(2s)^{1/2}e^{-x/2},0,\ldots,0,(2s+(\sinh(x)-s+it)(1-e^{-x}))& \end{aligned} \tag{A3.3}$$

is the attracting fixed point.

§ A4. Discrete groups

There is a rich theory of discrete subgroups of $SU(n+1,1)$ though it is much less developed than the corresponding theory in the real case. One important difference is that the complex hyperbolic space has no totally geodesic submanifolds of real codimension one. This complicates the construction of a fundamental domain for the action of a discrete group. The construction of the Dirichlet region still works to give a fundamental domain with real analytic boundaries.

For purely hyperbolic groups one can also use a modification of the isometric hemisphere construction. For an element A we define the 'halfspace'

$$H_A = \{p; |J_A(p)| < 1\}. \tag{A4.1}$$

From the formula, (A1.4), for $|J_A|$ it is apparent that the set

$$\{p; |J_A(p)| = 1\}$$

is an embedded disk which separates the ball into two components. The set defined as the intersection of the H_A over all elements, A, of a discrete group defines another fundamental domain with real analytic boundaries.

If Γ is a discrete subgroup of $\mathrm{SU}(n+1, 1)$ then one defines the limit set, Λ_Γ, as the set of cluster points of the orbit of a point in the interior which is not the fixed point of any elliptic element. Since the group is discrete this set necessarily lies on the boundary. As in the real case the limit set is independent of the choice of the interior point.

Since we are primarily interested in compact manifolds with a strictly pseudoconvex boundary we restrict ourselves to groups whose quotient is a manifold of this type. By considering the Dirichlet region one easily sees that such a group is composed entirely of hyperbolic elements. In order for the quotient to be of this simple type one can show that there is a finite sided fundamental region whose closure is disjoint from the limit set. This is the so called convex co-compact case.

One can easily construct examples by using the Schottky construction: take a finite collection of hyperbolic elements which identify pairs of imbedded codimension 1 disks. If the pairs of disks are pairwise disjoint with boundaries lying in the unit sphere and their union is the boundary of a connected region in $\mathbf{CH}^{n+1}$ then the argument used in the real case suffices to show that the group generated by these elements will be discrete and convex co-compact. The underlying abstract groups are free groups.

A distinctive feature of Schottky groups is the fact that the boundary of the quotient is a connected manifold. This is of course a substantial restriction in the real case. However it follows from a theorem of Kohn and Rossi, see [14] that this is always the case for a strictly pseudoconvex manifold. One easy consequence of this is that Λ_Γ cannot separate the sphere. Hence in some sense all convex co-compact subgroups of $\mathrm{SU}(n+1, 1)$ of infinite volume are of 'Schottky type.' It seems an interesting question to determine if the theorems of Phillips, Sarnak and Doyle on nontrivial upper bounds on the dimension of the limit set of a Schottky group extend to a result for all convex co-compact subgroup of $\mathrm{SU}(n+1, 1)$. See [21], [6].

§ A5. The exponent of convergence and the Patterson measure

We follow the presentation in Sullivan's paper, [22], stressing only the differences which arise as a consequence of the change from real to complex hyperbolic geometry.

Following Patterson and Sullivan we define the Poincaré series for a discrete group Γ:

$$g_s(X, Y) = \sum_{\gamma \in \Gamma} e^{-sd(X, \gamma Y)}. \tag{A5.1}$$

As in the real case, it follows from the discreteness of Γ that this series converges for $s > n+1$. Following Sullivan's argument one easily shows that if the series converges for one pair (X_0, Y_0) then it converges for any pair. We denote the infimum of the s for which the series converges by δ; this is called the convergence exponent for the group.

Next consider the family of measures:

$$\mu_s(X) = \frac{1}{g_s(Y, Y)} \sum_{\gamma \in \Gamma} e^{-sd(X, \gamma Y)} \delta_{\gamma Y}(p). \tag{A5.2}$$

It is easy to show that the total measure is bounded above and below independently of s. We assume that the series, g_s diverges at the convergence exponent; as in [22] this hypothesis can be removed but will be seen to hold in the convex co-compact case. Let

$$\mu_\delta(x) = \lim_{s_i \to \delta} \mu_{s_i}(x), \tag{A5.3}$$

denote a weak limit of these measures. In light of our assumption that the Poincaré series diverges at the convergence exponent this defines a measure concentrated on the limit set of Γ.

If X' is a different point then $\mu_\delta(X)$ and $\mu_\delta(X')$ are easily seen to be mutually absolutely continuous. From our formula for the fundamental invariant we see that:

$$\frac{e^{d(X, \gamma Y)}}{e^{d(X', \gamma Y)}} \sim \left| \frac{(X, \gamma Y)}{(X', \gamma Y)} \right|^2 = \frac{|1 - X \cdot \overline{\gamma Y}|^2 (1 - |X'|^2)}{(1 - |X|^2)\, |1 - X' \cdot \overline{\gamma Y}|^2}. \tag{A5.4}$$

If we allow γY to approach ξ on the boundary of $\mathbf{CB}^{n+1}$ then we see that

$$\text{(A5.5)}\qquad \frac{e^{d(X,\gamma Y)}}{e^{d(X',\gamma Y)}}\to\frac{|1-X\cdot\bar{\xi}|^2(1-|X'|^2)}{|1-X'\cdot\bar{\xi}|^2(1-|X|^2)}=e^{(X,X')_\xi}.$$

Therefore we obtain the formula:

$$\text{(A5.6)}\qquad \frac{d\mu_\delta(X')}{d\mu_\delta(X)|_\xi}=e^{\delta(X,X')_\xi}.$$

Following [22] we wish to obtain a comparison between the measure μ_δ and the Carnot–Hausdorff measure of dimension 2δ. To that end we prove the following lemmas; to simplify notation we will let $\mu=\mu_\delta(0)$.

LEMMA A5.7. *Let γ be a hyperbolic element, let*

$$r_\gamma=e^{-d(0,\gamma^{-1}0)/2},$$

and set

$$\xi_\gamma=\frac{\gamma^{-1}0}{|\gamma^{-1}0|}.$$

There is constant C_ω such the image under γ of the Carnot ball centered at ξ, of radius $C_\omega r_\gamma$, will cover the solid angle ω. The constant C_ω does not depend on γ.

Proof. Since a solid angle on the sphere can be identified with a solid angle in the tangent space to $\mathbf{CH}^{n+1}$ at zero by following geodesic rays originating from zero, it suffices to show that the image of the endpoints of these geodesic rays under a transformation that carries 0 to $\gamma^{-1}0$ and preserves the line that joins these two points lies in a ball of radius $C_\omega r_\gamma$ as in the statement of the lemma. Without loss of generality we can assume that $\xi=(1,0,\ldots,0)$; if we let $d=d(0,\gamma^{-1}0)$ then a transformation carrying 0 to $\gamma^{-1}0$ and preserving the line between them is given by

$$\text{(A5.8)}\qquad A:(w,z)\to\frac{(\operatorname{ch}(d/2)\,w+\operatorname{sh}(d/2),z)}{(\operatorname{sh}(d/2)+\operatorname{ch}(d/2))}$$

A solid angle centered on $(1,0,\ldots,0)$ is defined by

$$\Re(X\cdot(1,0,\ldots,0))=\operatorname{Re}(w)<\tau;\quad -1<\tau<1.$$

With $X=(w,z)$ it follows from (A3.1) and (A5.8) that the square of the Carnot distance of AX from $(1,0,\ldots,0)$ is:

$$\text{(A5.9)}\quad \left|1-\frac{\cosh(d/2)\,w+\sinh(d/2)}{\sinh(d/2)\,w+\cosh(d/2)}\right| = \frac{e^{-d/2}|1-w|}{|w\sinh(d/2)+\cosh(d/2)|} < \frac{e^{-d/2}}{\sinh(d/2)}|1+\tau|.$$

This completes the proof of the lemma.

LEMMA A5.10. *For ξ in a ball as in Lemma* A5.7

$$d\mu(\gamma\xi) = J(\xi)\,d\mu(\xi)$$

where $J(\xi)$ satisfies:

$$\text{(A5.11)}\qquad c_1 r_\gamma^{2\delta} < J(\xi) < c_2 r_\gamma^{2\delta}.$$

here the constants depend only on the solid angle in Lemma A5.7.

Proof. Let A_γ denote the element of $\mathrm{SU}(n+1,1)$ which defines γ. Using the formula for the fundamental invariant we find that:

$$\text{(A5.12)}\qquad \frac{e^{d(0,\gamma' Y)}}{e^{d(0,\gamma\gamma' Y)}} \sim \frac{(1-|\gamma\gamma' Y|^2)}{(1-|\gamma' Y|^2)}.$$

By using the formula for the fundamental invariant in the linear model one easily sees that as $\gamma' Y$ tends to ξ this tends to

$$\text{(A5.13)}\qquad \frac{1}{|A_\gamma(\xi,1)_{n+1}|^2}.$$

It now follows easily that

$$\text{(A5.14)}\qquad J(\xi) = (1/|A_\gamma(\xi,1)_{n+1}|)^{2\delta}.$$

Hence all that is required is an estimate for $|(A_\gamma(\xi,1))_{n+1}|^2$ for ξ in the Carnot ball produced in Lemma A5.7. To that end we will employ the normal form (A3.2). A simple calculation shows that:

$$\text{(A5.15)}\qquad \cosh\left(\frac{1}{2}d(0,\gamma^{-1}0)\right) = |\cosh(x)+s-it|.$$

Moreover

$$\text{(A5.16)}\qquad \frac{\gamma^{-1}0}{|\gamma^{-1}0|} = \xi_\gamma = \frac{(\sinh(x)+s-it,\,-(2s)^{1/2}e^{-x/2},0,\ldots,0)}{(\cosh(x)+s-it)(1-1/|\cosh(x)+s-it|^2)^{1/2}}$$

It is now a simple exercise to show that if ξ satisfies

$$|1-\xi\cdot\overline{\xi_\gamma}|<Cr_\gamma^2$$

then

$$c_1 e^{-d/2}<1/|A_\gamma(\xi,1)_{n+1}|<c_2 e^{-d/2} \tag{A5.17}$$

as desired.

Using these lemmas and Sullivan's observation that any sufficiently large solid angle must contain a fixed proportion of the total measure of μ we obtain Sullivan's result in this case, that for these special balls, $B_{Cr_\gamma}(\xi_\gamma)$:

$$c_1(r_\gamma)^{2\delta}<\mu(B_{Cr_\gamma}(\xi_\gamma))<c_2(r_\gamma)^{2\delta}. \tag{A5.18}$$

where as usual c_1 and c_2 are independent of γ.

Sullivan has shown that one can compare the Patterson measure to the 2δ Hausdorff measure on the radial limit set. A point, ξ, is in the radial limit set provided that for l a geodesic ending at ξ there are infinitely many points in the set $\{\gamma 0;\gamma\in\Gamma\}$ within a fixed distance of l. Evidently if this is true for some geodesic ending at ξ then it is true for any such geodesic. In the case that Γ is convex co-compact the radial limit set coincides with the whole limit set. Using the following geometric lemma we can generalize Sullivan's results to the complex case:

LEMMA A5.19. *Let ξ be a point on the unit sphere and X a point in $\mathbf{CB}^{n+1}$ such that the complex hyperbolic distance from X to the line through* 0, *ending at ξ, is less than L, then*

$$\left|1-\frac{X}{|X|}\cdot\bar{\xi}\right|<|X|^{-2}Ce^{-d(0,X)}. \tag{A5.20}$$

Here the constant depends only on L.

Proof. Let Y be the point on the line closest to X. By the triangle inequality $d(0,Y)\geq d(0,X)-L$. Moreover we have

$$|1-X\cdot\bar{Y}|^2<C(1-|X|^2)(1-|Y|^2). \tag{A5.21}$$

Using the relation between the fundamental invariant and the distance and the estimate for $d(0,Y)$ we see that the right hand side of (A5.21) is smaller than $C'e^{-2d(0,X)}$. On the other hand:

$$\left|1-\frac{X}{|X|}\cdot\bar{\xi}\right| = \left|1-\frac{X}{|X|}\cdot\frac{\bar{Y}}{|Y|}\right| = \frac{||X||Y|-X\cdot\bar{Y}|}{|X||Y|}$$
$$\leqslant\frac{((1-|X||Y|)+|1-X\cdot\bar{Y}|)}{|X||Y|}$$
$$\leqslant\frac{2|1-X\cdot\bar{Y}|}{|X||Y|}.$$

This completes the proof of the lemma.

Using this lemma and Sullivan's argument we can show that for balls of sufficiently small radius centered on points lying in the radial limit set, $B_r(\xi)$ we have:

$$c_1 r^{2\delta}<\mu(B_r(\xi))<c_2 r^{2\delta} \tag{A5.22}$$

for some positive constants independent of r and ξ. This gives:

THEOREM A5.23. *For a convex co-compact group,* $\Gamma\subset \mathrm{SU}(n+1,1)$, *the Poincaré series diverges at the critical exponent and the limit set has finite and positive* 2δ *Carnot–Hausdorff measure.*

If we define the counting function:

$$N(R)=\#\{\gamma: d(0,\gamma 0)<R\}$$

then we have the estimate:

$$ce^{\delta R}<N(R)<Ce^{\delta R}. \tag{A5.24}$$

§ A6. The Poisson kernel

On the complex hyperbolic space there is a family of fundamental eigenfunctions for the Laplace operator with their singular support a single point on the boundary. If we define the kernel:

$$P(X,\xi,s)=\left[\frac{1-|X|^2}{|1-X\cdot\bar{\xi}|^2}\right]^{is+(n+1)/2} \tag{A6.1}$$

then P satisfies:

$$\Delta P-(s^2+(n+1/2)^2)P=0. \tag{A6.2}$$

This assertion is most easily proved in the hyperquadratic model where one easily sees that $(\Im x)^{is+(n+1)/2}$ is an eigenfunction. The kernel is then obtained by composing this function with automorphisms.

If we specialize the formula (A5.6) for the Radon–Nikodym derivative, $d\mu(X')/d\mu(X)$ to $X=0$ we obtain:

$$d\mu(X')|_{\xi} = P(X', \xi, \sigma)\, d\mu \tag{A6.3}$$

here $\sigma = i((n+1)/2-\delta)$.

From the construction of $\mu(X)$ it is apparent that, as a function of X the total mass of $\mu(X)$ is invariant under the action of Γ; let $\psi(X)$ denote the total mass of the measure $\mu(X)$. From formulae (A6.1) and (A6.3) it is obvious that $\psi(X)$ is an eigenfunction of the Laplace operator with eigenvalue $\lambda=-\delta(\delta-n-1)$. In the convex co-compact case it follows from the fact that there is a fundamental domain whose closure is disjoint from the limit set that

$$|\psi(X)| < Ce^{-\delta d(0,X)} \tag{A6.4}$$

on $\Gamma\diagdown\mathbf{CH}^{n+1}$. Thus we see that if $\delta>(n+1)/2$ then $\psi(X)$ will be an L^2 eigenfunction on the quotient manifold. Since $\psi(X)$ is positive it follows by a standard argument that λ is the lower bound of the spectrum of the Laplacian on the quotient. Hence we obtain a generalization of the Theorem of Patterson and Sullivan:

THEOREM A6.5. *If Γ is a convex co-compact subgroup of* $\mathrm{SU}(n+1, 1)$ *such that the convergence exponent, δ, is larger than $(n+1)/2$ then the Laplace operator on the quotient has an L^2 eigenvalue below the continuous spectrum and the lower bound of the spectrum, λ_0, is related to the convergence exponent by:*

$$\lambda_0 = -\delta(\delta-n-1). \tag{A6.6}$$

§ A7. The resolvent kernel for a convex co-compact group

The results of Sections 9, A5 and 14 can now be combined to yield a representation of the resolvent as a convergent sum for $\Re(s)>\delta$. If the spectral parameter is given by $\lambda=s(n+1-s)$ it follows from (8.6), (8.9) and (8.12) that the resolvent kernel on $\mathbf{CB}^{n+1}$ is given by:

$$R(x, y; s) = (\cosh((d(x, y)/2))^{-2s} r(\cosh(d(x, y)/2)^{-2}; s) \tag{A7.1}$$

here $r(\tau; s)$ is a smooth function for $\tau \in [0, \infty)$. It follows easily from (A7.1) and (A5.24) that the series:

$$R_\Gamma(x,y;s)=\sum_{\gamma\in\Gamma}R(x,\gamma y;s) \tag{A7.2}$$

converges locally uniformly for $\Re s>\delta$. For s in this range and $\phi\in\dot{\mathscr{C}}^\infty(\mathbf{CB}^{n+1})$

$$u=R_\Gamma(s)\cdot\phi=\int R_\Gamma(x,y;s)\,\phi(y)\,dV_{\mathrm{Berg}}$$

is a solution to $[\Delta-s(n+1-s)]u=\phi$ and moreover $u\in L^2(\mathbf{CB}^{n+1},dV_{\mathrm{Berg}})$. From the uniqueness of such a solution it follows that $R_\Gamma(s)$ is the resolvent kernel in this region.

From Theorem 14.5 we conclude that the resolvent has a meromorphic extension to the whole plane, the essential spectrum of Δ is the interval $[((n+1)/2)^2,\infty)$ it is continuous spectrum of uniform infinite multiplicity and any point spectrum is of finite multiplicity.

From the series representation (A7.2) it is apparent that for $\Re s$ large enough

$$R_\Gamma(x,y;s)=(1-|x|^2)^s(1-|y|^2)^s\left[\frac{G_0(x,y;s)}{|1-\langle x,y\rangle|^{2s}}+G_1(x,y;s)\right] \tag{A7.3}$$

where G_0 and G_1 are $\mathscr{C}^\infty$ functions near $x=y$. From this we see that R_Γ actually has a nontrivial second component in the decomposition (14.6).

In a subsequent publication we will show that there is no point spectrum embedded in the continuous spectrum (cf. [17]) and hence all such specrum lies in the interval $[\delta^2,((n+1)/2)^2)$, provided $\delta>(n+1)/2$. Otherwise there is no point spectrum. We will also construct an eigenfunction expansion from the resolvent kernel.

Appendix B. Polyhomogeneous conormal distributions

In the definition of Θ-pseudodifferential operators, and even more in the proofs of their composition and mapping properties, we make use of the theory of polyhomogeneous conormal distributions on a manifold with corners. In this appendix the definition of these distributions is recalled and the symbolic properties and behaviour under various operations are described. For a more complete treatment, and proofs, of the results described here the reader is referred to [19].

For simplicity we shall assume throughout that the manifolds with corner satisfy an additional global constraint. Namely we require that:

(B.1) Each boundary hypersurface is embedded.

This is not a severe constraint and in any case all the manifolds in the body of the paper satisfy it. The useful consequence of (B.1) is that each boundary hypersurface B has a defining function ϱ_B such that there is a neighbourhood $\mathcal{U}$ of B which is diffeomorphic to a product

$$\mathcal{U} \cong [0,1) \times B \tag{B.2}$$

where the normal variable, in the first factor, is just ϱ_B.

§ B1. Multiple expansions

The spaces of polyhomogeneous conormal functions on a manifold with corners, X, satisfying (B.1), are fixed by an index set for each boundary hypersurface. By an index set we mean a discrete subset $E \subset \mathbf{C} \times \mathbf{N}_0$, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ such that

$$(z_l, m_l) \in E, \ |(z_l, m_l)| \to \infty \quad \Rightarrow \quad \Re(z_l) \to \infty. \tag{B1.1}$$

Let us number the boundary hypersurfaces B_j, $j=1, \ldots, q$ and let E_j be an index set for each $1 \leqslant j \leqslant q$ with ϱ_j corresponding defining functions. The collection of index sets will be written $E = \{E_1, \ldots, E_q\}$. Now to define the space $\mathcal{A}^E_{\mathrm{phg}}(X)$ of polyhomogeneous conormal distributions associated to these index sets we proceed by induction over the maximum codimension of a boundary face, i.e. the maximum number of boundary hypersurfaces which have a common intersection.

The trivial case is a compact manifold without boundary; there are no index sets and we adopt the superfluous notation

$$\mathcal{A}_{\mathrm{phg}}(X) = \mathcal{C}^\infty(X) \quad \text{if} \quad \partial X = \varnothing. \tag{B1.2}$$

Proceeding inductively we suppose that $\mathcal{A}^E_{\mathrm{phg}}(Y)$ is defined as a topological vector space whenever Y has boundary codimension at most $k-1$. In particular this means the associated spaces

$$\mathcal{C}^\infty_c([0,1); \mathcal{A}^E_{\mathrm{phg}}(Y)) \quad \text{and} \quad \dot{\mathcal{C}}^j_c([0,1); \mathcal{A}^E_{\mathrm{phg}}(Y)), \ j \in \mathbf{N}_0 \cap \{\infty\} \tag{B1.3}$$

are well-defined topological vector spaces. The second space consists of the j-times continuously differentiable functions on $[0,1)$ with values in $\mathcal{A}^E_{\mathrm{phg}}(Y)$ and which vanish to order j at 0 and in some neighbourhood of 1. Now consider one of the boundary hypersurfaces B_j. Let $E(j)$ be the collection of index sets for the boundary hypersurfaces of X which meet B_j (other than B_j itself). Thus $E(j)$ is a collection of index sets for the hypersurfaces boundaries of B_j. Consider a function on $[0,1) \times B_j$ which has an asymptotic expansion

$$u \sim \sum_{(z,m)\in E_j} \sum_{k\le m} \varrho_j^z (\log \varrho_j)^k f_{z,k}, \quad f_{z,k} \in \mathscr{C}_c^\infty([0,1); \mathscr{A}_{\mathrm{phg}}^{E(j)}(B_j)). \tag{B1.4}$$

Of course we need to specify the precise meaning of this asymptotic expansion. For each $r\in\mathbf{R}$ set $E_j(r)=\{(z,m)\in E_j; \Re(r)<r\}$. This is a finite set, because of (B1.1). Then (B1.4) means that for each $p\in\mathbf{N}$ there exists $R=R(p)\in\mathbf{R}$ such that

$$u - \sum_{(z,m)\in E_j} \sum_{k\le m} \varrho_j^z (\log \varrho_j)^k f_{z,k} \in \dot{\mathscr{C}}_c^p([0,1); \mathscr{A}_{\mathrm{phg}}^{E(j)}(B_j)) \quad \forall r > R(p). \tag{B1.5}$$

Conversely, the asymptotic completeness of these spaces is the result (really a form of E. Borel's lemma) that for any sequence, $f_{z,k}$, parametrized by E_j,

$$f_{*,*}: E_j \to \mathscr{C}_c^\infty([0,1); \mathscr{A}_{\mathrm{phg}}^{E(j)}(B_j))$$

there exists u satisfying (B1.4), i.e. (B1.5).

Definition B1.6. *By induction over the maximum codimension of a boundary face we define, for any manifold with corners satisfying* (B.1) *and any collection of index sets E,*

$$\mathscr{A}_{\mathrm{phg}}^E = \left\{ u = \tilde{u} + \sum_{j=1}^q u_j;\ \tilde{u} \in \dot{\mathscr{C}}^\infty(X) \text{ and } u_j \text{ satisfies (B1.4) for each } j \right\}. \tag{B1.7}$$

The topology on $\mathscr{A}_{\mathrm{phg}}^E(X)$ is given by the seminorms on the coefficients $f_{z,k}$ in (B1.4) *and on the remainders in* (B1.5).

It is straightforward to check that $\mathscr{A}_{\mathrm{phg}}^E(X)$ is defined independently of the choice of product structure near the boundary hypersurfaces. It is always a $\mathscr{C}^\infty(X)$-module. This implies in particular that we can define such polyhomogeneous sections of any $\mathscr{C}^\infty$ vector bundle over X by using the tensor product definition

$$\mathscr{A}_{\mathrm{phg}}^E(X;G) = \mathscr{A}_{\mathrm{phg}}^E \otimes_{\mathscr{C}^\infty(X)} \mathscr{C}^\infty(X;G). \tag{B1.8}$$

This just means that the coefficients are polyhomogeneous in the sense above in any trivialization of the bundle in a suitable local coordinate patch.

Note also that if E' and E are index sets and $E_j'\subset E_j$ for all j then

$$\mathscr{A}_{\mathrm{phg}}^{E'}(X) \subset \mathscr{A}_{\mathrm{phg}}^E(X). \tag{B1.9}$$

In fact there may be redundancy in the index sets. If $(z,m)\in E_j$ then dropping any element $(z+l,m')\in E_j$ where $l,m'\in \mathbb{N}_0$ and $m'\leq m$ does not affect the space.

Certain 'trivial' cases are worth noting. If one of the index sets should be empty $E_j=\emptyset$ then the space is, locally near that face, just

$$u\in \mathcal{A}^{E}_{\mathrm{phg}}(X),\ \mathrm{supp}(u)\subset \Omega_j \iff u\in \dot{\mathscr{C}}^{\infty}_c([0,1);\mathcal{A}^{E(j)}_{\mathrm{phg}}(B_j)) \tag{B1.10}$$

where Ω_j is a neighbourhood as in (B.2) for B_j. Another similar case is where $E_j=\{(0,0)\}$. Since B_j is embedded it is possible to double X across B_j to a manifold with corners 2_jX (the double of X across the jth boundary.) Then if $E'(j)$ is the family of index sets obtained by omitting E_j

$$\mathcal{A}^{E}_{\mathrm{phg}}(X)=\mathcal{A}^{E'(j)}_{\mathrm{phg}}(2_jX)|_X. \tag{B1.11}$$

As a particular case of this,

$$E_j=\{(0,0)\}\ \forall j \Rightarrow \mathcal{A}^{E}_{\mathrm{phg}}(X)=\mathscr{C}^{\infty}(X). \tag{B1.12}$$

§B2. Symbol mappings

One of the important properties of conormal distributions, and in particular the polyhomogeneous conormal functions just defined is the existence of a symbol mapping, which is an effective replacement for the Taylor series expansion of a smooth function. Since we are assuming each boundary hypersurface to be embedded we can interpret the map (B.2) more abstractly as a local identification of a neighbourhood of the boundary hypersurface in the manifold with a neighbourhood of the zero section of its normal bundle.

Recall that the normal bundle NB is a line bundle over B with fibre

$$N_pB=T_pX/T_pB\quad \forall p\in B. \tag{B2.1}$$

The subset consisting of the image of the inward-pointing part of the tangent space is a half-line bundle N^+B over B; it is a manifold with corners, since B may have corners up to one less codimension than X. As a vector budle the zero section is naturally identified with B and the inward-pointing normal bundle to the zero section in N^+B is naturally identified with N^+B. Thus we can place the following restrictions on a local diffeomorphism, χ, from a neighbourhood, Ω, of B in X to a neighbourhood Ω' of the zero section in N^+B:

$$\begin{gathered}\chi : X \supset \Omega \leftrightarrow \Omega' \subset N^+B \\ \chi \text{ is the identity on } B \\ \chi_* \text{ is the identity on } N^+B.\end{gathered} \tag{B2.2}$$

The collar neighbourhood theorem asserts the existence of such a diffeomorphism; it follows in any case from (B.2) and it is by no means unique.

If E is a family of index sets for the boundary hypersurfaces of X and B is a boundary hypersurface set

$$\begin{gathered}E+1_B = \{E_1, \ldots, E_{j-1}, E_j+1, E_{j+1}, \ldots, E_q\}, \\ E_j+1 = \{(z+1, m); (z, m) \in E_j\} \quad \text{if} \quad B = B_j\end{gathered} \tag{B2.3}$$

and similarly with 1 replaced by any integer n. The space of polyhomogeneous conormal distributions associated with this shifted index set can also be written

$$\mathscr{A}^{E+1_B}_{\text{phg}}(X) = \{\phi \in \mathscr{C}^\infty(X);\ \phi = 0 \text{ on } B\} \cdot \mathscr{A}^{E}_{\text{phg}}(X). \tag{B2.4}$$

In particular we can consider the quotient which we write as

$$\mathscr{A}^{\{E\}_B}_{\text{phg}}(X) = \mathscr{A}^{E}_{\text{phg}}(X) / \mathscr{A}^{E+1_B}_{\text{phg}}(X). \tag{B2.5}$$

The boundary hypersurfaces of N^+B are uniquely associated with boundary hypersurfaces of X and so in particular the same space is defined there. Now a map (B2.2), being a local diffeomorphism, identifies the spaces with supports in the appropriate sets

$$\chi : \mathscr{A}^{E}_{\text{phg}}(X) \cap \mathscr{C}^{-\infty}_c(\Omega) \leftrightarrow \mathscr{A}^{E}_{\text{phg}}(N^+B) \cap \mathscr{C}^{-\infty}_c(\Omega'). \tag{B2.6}$$

More significantly the quotient is well-defined, independent of the choice of χ satisfying (B2.2); this is called the symbol mapping for the face B:

$$\sigma_B : \mathscr{A}^{E}_{\text{phg}}(X) \to \mathscr{A}^{\{E\}_B}_{\text{phg}}(N^+B). \tag{B2.7}$$

It is such that

$$0 \to \mathscr{A}^{E+1_B}_{\text{phg}}(X) \to \mathscr{A}^{E}_{\text{phg}}(X) \to \mathscr{A}^{\{E\}}_{\text{phg}}(N^+B) \to 0 \tag{B2.8}$$

is always exact.

§ B3. Elementary operations

There are three 'elementary' operations we consider on polyhomogeneous conormal distributions: addition, multiplication and the action of differential operators. Using the definition it is easy to analyze each of these.

For addition the corresponding operation on index sets, $E, F \subset (\mathbf{C}\times\mathbf{N}_0)^q$ associated to the boundary hypersurfaces of a manifold with corners, is set-theoretic union of the components, we write

$$E \tilde{\cup} F = \{E_1 \cup F_1, \dots, E_q \cup F_q\}. \tag{B3.1}$$

LEMMA B3.2. *Addition gives a separately continuous bilinear form*

$$\mathcal{A}^E_{\text{phg}}(X) + \mathcal{A}^F_{\text{phg}}(X) \subset \mathcal{A}^{E\tilde{\cup}F}_{\text{phg}}(X). \tag{B3.3}$$

The operation on index sets corresponding to multiplication is addition, in the following sense. If E and F are collections of index sets put

$$\begin{gathered} E_j + F_j = \{(z, m);\ z = z' + z'',\ (z', m') \in E_j,\ (z'', m'') \in F_j,\ m = m' + m''\} \\ E \tilde{+} F = \{E_1 + F_1, \dots, E_q + F_q\}. \end{gathered} \tag{B3.4}$$

LEMMA B3.5. *For any compact manifold, satisfying* (B.1), *multiplication of* $\mathscr{C}^\infty$ *functions in the interior extends to give a separately continuous bilinear map*

$$\mathcal{A}^E_{\text{phg}}(X) \cdot \mathcal{A}^F_{\text{phg}}(X) \subset \mathcal{A}^{E\tilde{+}F}_{\text{phg}}(X) \tag{B3.6}$$

and for any boundary hypersurface B

$$\sigma_B(u \cdot v) = \sigma_B(u) \cdot \sigma_B(v). \tag{B3.7}$$

As far as the action of differential operators is concerned we shall limit our attention to totally characteristic operators, elements of the enveloping algebra, $\text{Diff}_b(X)$, of $\mathcal{V}_b(X)$, the space of $\mathscr{C}^\infty$ vector fields on X tangent to all boundary components. If $P \in \text{Diff}^m_b(X)$ and B is a boundary hypersurface then the indicial operator at B, $I_B(P)$, is well-defined as an element of $\text{Diff}^m_b(N^+B)$.

LEMMA B3.8. *If X is a manifold with corners satisfying* (B.1) *then for any index set E and any* $P \in \text{Diff}^m_b(X)$

$$P\colon \mathcal{A}^E_{\text{phg}}(X) \to \mathcal{A}^E_{\text{phg}}(X) \tag{B3.9}$$

and for any boundary hypersurface, B,

$$\sigma_B(Pu) = I_B(P)\cdot\sigma_B(u) \quad \forall u \in \mathcal{A}^E_{\text{phg}}(X). \tag{B3.3}$$

§ B4. Pull-back

After these elementary operations we consider the pull-back of conormal functions under a $\mathcal{C}^\infty$ map between manifolds with corners. Naturally we cannot expect to have a simple result for an arbitrary smooth map, since it need not be really related to the boundaries. We therefore consider the notion of a b-map. This is really a local notion, but for simplicity we stick to the case of manifolds, X and Y, satisfying (B.1).

Let $\varrho_{i,Y}$, $i=1,\dots,q'$, be defining functions for the boundary hypersurfaces of Y and similarly $\varrho_{i,X}$, $i=1,\dots,q$, defining functions for X. The condition we impose on a $\mathcal{C}^\infty$ map $F\colon X\to Y$ in order that it be a b-map is that

$$F^*\varrho_{i,Y} = \alpha_i \prod_{1\le j\le q} \varrho_{j,X}^{e(i,j)}, \quad 0 \neq \alpha_i \in \mathcal{C}^\infty(X), \quad i = 1,\dots,q'. \tag{B4.1}$$

Of course the powers $e(i,j)$ are then necessarily non-negative integers. Clearly the composite of two b-maps is again a b-map, so it is easy to see that manifolds with corners satisfying (B.1) and b-mappings form a category, i.e. this is a reasonable notion of 'smooth map'!

With a b-map we shall associate a mapping of index sets. Namely if E' is a family of index sets for the boundary hypersurfaces of Y let

$$\begin{gathered} F^{\mathrm{b}}(E') = \{E_1,\dots,E_q\}, \\ E_j = \left\{(Z,M);\ Z_j = \sum_{1\le i\le q'} e(i,j)\, z_i,\ M_j = \sum_{e(i,j)\neq 0} m_i,\ (z_i,m_i)\in E'_i\right\}. \end{gathered} \tag{B4.2}$$

PROPOSITION B4.3. *Let $F\colon X\to Y$ be a b-map between manifolds with corners satisfying* (B.1) *then for any family of index sets, E', for Y the pull-back of $\mathcal{C}^\infty$ functions in the interior extends to a continuous linear map*

$$F^*\colon \mathcal{A}^{E'}_{\text{phg}}(Y) \to \mathcal{A}^{F^{\mathrm{b}}(E')}_{\text{phg}}(X). \tag{B4.4}$$

Notice that if for some j, $e(i,j)=0$ for all i, then the component of $F^{\mathrm{b}}(E')$ corresponding to B_j is $E_j=\{(0,0)\}$ i.e. the pull-back is "$\mathcal{C}^\infty$ up to" that boundary hypersurface.

§ B5. Push-forward

Recall that the tangent vector fields $\mathcal{V}_b(X)$ form all the smooth sections of the compressed tangent bundle ${}^bTX$, with dual ${}^bT^*X$. A direct consequence of the definition of a b-map is that associated to it there is a pull-back map on this bundle:

$$ {}^bF_p^*\colon {}^bT^*_{F(p)}Y \to {}^bT_p^*X \quad \forall p\in X. \tag{B5.1} $$

In order to have a useful result for push-forward we need to add a *b-submersion conditon,* namely that

$$ {}^bF_p^* \text{ is injective} \quad \forall p\in X. \tag{B5.2} $$

As another consequence of the b-mapping condition (B5.1) we can divide the boundary hypersurface of X into two sets, corresponding to those indices j for which $e(i,j)>0$ for some i and those with $e(i,j)\equiv 0$. We can write the corresponding decomposition of the index sets as

$$ E=E'\oplus E''. \tag{B5.3} $$

Now we want to define the push-forward $G=F_b(E')$ of the index set; as indicated this only depends on E'. Thus $G=\{G_1,\dots,G_{q'}\}$ is to be an index set for Y. We set

$$ G_i=\Big\{(Z,M);\exists\,(z_j,m_j)\in E_j' \text{ for those } j \text{ with } e(i,j)\neq 0 \text{ and } E_j\neq\varnothing \text{ such that} \\ e(i,j)\,Z=z_j \text{ for one } j \text{ and } M+1=\sum_j (m_j+1)\Big\} \tag{B5.4} $$

Notice here the big difference between how we have defined the push-forward and the pull-back of index sets. In (B5.4) the multiplicity corresponds to the sum of the multiplicities, plus one less than the number of hypersurfaces in X which 'contribute' to a given hypersurfaces in Y.

To simplify the form of the push-forward we consider b-densities, i.e. (conormal) sections of the density bundle ${}^b\Omega$ associated to ${}^bTX$. This is just

$$ {}^b\Omega\cong\Big[\prod_{1\le i\le q}\varrho_i\Big]^{-1}\Omega, \tag{B5.5} $$

where ϱ_i, $i=1,\dots,q$ are defining functions for the boundary hypersurfaces.

PROPOSITION B5.6. *Let $F: X \to Y$ be a b-submersion, in the sense that it is a b-map satisfying* (B5.2), *then for any index sets E for X with a decomposition* (B5.3) *such that*

$$\Re(z) > 0 \quad \forall (z, m) \in E_j'', \tag{B5.7}$$

push-forward gives a continuous map

$$F_*: \mathcal{A}^E_{\mathrm{phg}}(X; {}^b\Omega) \to \mathcal{A}^{F_b(E)}_{\mathrm{phg}}(Y; {}^b\Omega). \tag{B5.8}$$

§ B6. Clean submanifolds

We need to further extend these results to allow for certain conormal singularities in the interior of the manifold. A subset $Y \subset X$, with $Y \cap \mathring{X} \neq \emptyset$, is a *clean* submanifold of a manifold with corners, X, if near each point $p \in Y$ there are $l = \mathrm{codim}(Y)$ smooth functions $z_1, \dots, z_l$ locally defining Y,

$$Y \cap \Omega = \{q \in \Omega;\ z_i(q) = 0,\ i = 1, \dots, l\} \tag{B6.1}$$

for some neighbourhood Ω of p, and such that

$$dz_1, \dots, dz_l, dx_1, \dots, dx_k \quad \text{are independent at } p. \tag{B6.2}$$

Here $x_1, \dots, x_k$ are local defining functions for the k boundary hypersurfaces through p. This condition just means that locally X can be decomposed into a product

$$X \cap \Omega = X' \times X'' \quad \text{such} \qquad \text{that} \quad Y = X' \times \{p''\},\ p'' \in X'' \tag{B6.3}$$

in a way which trivializes Y. Here we can assume that X'' has no boundary. One particular case is if the submanifold Y is embedded in the interior of X; not surprisingly we need more general cases than this.

The theory (although not the name) of conormal distributions associated to an embedded submanifold of a manifold without boundary was formalized by Hörmander in [10]. The local definition is reduced by (B6.3) to the case of a point in Euclidean space. There

$$\begin{gathered} u \in I^m(\mathbf{R}^n, \{0\}),\quad \mathrm{supp}(u) \subset\subset \mathbf{R}^n \quad \Leftrightarrow \\ u \in \mathscr{C}_c^{-\infty}(\mathbf{R}^n),\quad \hat{u}(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\xi} u(x)\, dx \in S^{m-n/4}(\mathbf{R}^n). \end{gathered} \tag{B6.4}$$

The space of symbols, $S^m(\mathbf{R}^n)$, can be directly related to the conormal functions we have been discussing. Namely let SP: $\mathbf{R}^n \hookrightarrow \mathbf{P}^n$ be the stereographic compactification of $\mathbf{R}^n$ to a ball. Then SP gives an isomorphism

$$\text{(B6.5)} \qquad \mathrm{SP}^*: \mathscr{A}^M_{\mathrm{phg}}(\mathbf{P}^n) \leftrightarrow S^{-M}(\mathbf{R}^n) \quad \forall M \in \mathbf{R}$$

where the single exponent M stands for the index set $\{(M,0)\}$. Under the local product decomposition the space of conormal distributions associated to Y is just

$$\text{(B6.6)} \qquad I^m(X,Y) \cap \mathscr{C}_c^{-\infty}(\Omega) = \mathscr{C}^\infty(X'; I^{m+(\dim X')/4}(X'', \{p''\})) \cap \mathscr{C}_c^{-\infty}(\Omega).$$

The strange looking order normalization is useful in the theory of pseudodifferential operators (including the ones in this paper). These spaces are independent of the local decomposition (B6.3) of Y. The local coordinate symbol leads to a coordinate independent symbol. If we let $\mathbf{P}N^*Y$ be the stereographic compactification (fibre by fibre) of the conormal bundle to Y and $\Omega_{b,\mathrm{fibre}}$ be the b-density bundle on the fibres of $\mathbf{P}N^*Y$ then the symbol map can be written

$$\text{(B6.7)} \qquad \sigma_m: I^m(X,Y) \to \mathscr{A}^{\{\mathrm{SP}(m)\}}_{\mathrm{phg}}(\mathbf{P}N^*Y; \Omega_{b,\mathrm{fibre}}).$$

Here

$$\text{(B6.8)} \qquad \mathrm{SP}(m) = \left(-m + \frac{1}{4}\dim X + \frac{1}{2}\operatorname{codim} Y, 0\right)$$

is the index set corresponding to one leading power and, as before, the brackets in the exponent mean the quotient by the space of functions one order lower in growth (i.e. 'vanishing' to one order higher). The symbol mapping gives an exact sequence

$$\text{(B6.9)} \qquad 0 \to I^{m-1}(X,Y) \hookrightarrow I^m(X,Y) \xrightarrow{\sigma_m} \mathscr{A}^{\{\mathrm{SP}(m)\}}_{\mathrm{phg}}(\mathbf{P}N^*Y; \Omega_{b,\mathrm{fibre}}) \to 0.$$

To generalize this to the case of a clean submanifold of a manifold with corners we proceed just as in Definition B1.6, only working locally and starting with $I^m(X,Y)$ in place of $\mathscr{C}^\infty(X)$ in the case of a manifold without boundary. We denote the general space, corresponding to order m at Y and to the index sets $E_1, \dots, E_q$ at the boundary hypersurfaces by $\mathscr{A}^E_{\mathrm{phg}} I^m(X,Y)$. In terms of an asymptotic expansion (B1.4) we just allow the coefficients to be elements of $\mathscr{C}_c^\infty([0,1); \mathscr{A}^{E(j)}_{\mathrm{phg}} I^{m+1/4}(B_j, Y_j))$ where $Y_j = B_j \cap Y$ and the decomposition (B.2) is of X'.

For this extension of the notion of a polyhomogeneous conormal distribution (the elements need no longer be functions) we need to reconsider the results described above, starting with the symbol mappings. There are three types of symbol mappings.

First is the direct extension of (B6.9):

$$\sigma_m\colon \mathscr{A}^{E}_{\mathrm{phg}} I^m(X, Y) \to \mathscr{A}^{\{\mathrm{SP}(m), E(Y)\}_Y}_{\mathrm{phg}}(\mathbf{P}N^*Y; \Omega_{b,\mathrm{fibre}}). \tag{B6.10}$$

Here $E(Y)$ denotes the index family for the boundary of Y, i.e. the index sets from the boundary hypersurface components of X which meet Y. Together with $\mathrm{SP}(m)$, given by (B6.8), this gives an index family for $\mathbf{P}N^*Y$, which is a ball-bundle over Y. The suffix Y on the index set denotes, as usual, the quotient by the space with index set increased by 1 at, in this case, the sphere bundle over Y forming a boundary hypersurface of $\mathbf{P}N^*Y$.

The second type of symbol map corresponds to the symbol at boundary hypersurfaces of X which meet Y:

$$\sigma_B\colon \mathscr{A}^{E}_{\mathrm{phg}} I^m(X, Y) \to \mathscr{A}^{\{E\}_B}_{\mathrm{phg}} I^m(N^+B, N^+_{B\cap Y}). \tag{B6.11}$$

The third type is at a boundary face which does not meet Y. This is just (B2.7), (B2.8) again.

In all cases the symbol maps lead to exact sequences:

$$0 \to \mathscr{A}^{E}_{\mathrm{phg}} I^{m-1}(X, Y) \hookrightarrow \mathscr{A}^{E}_{\mathrm{phg}} I^m(X, Y) \xrightarrow{\sigma_m} \mathscr{A}^{\{\mathrm{SP}(m), E(Y)\}_Y}_{\mathrm{phg}}(\mathbf{P}N^*Y; \Omega_{b,\mathrm{fibre}}) \to 0 \tag{B6.12}$$

$$0 \to \mathscr{A}^{E+1_B}_{\mathrm{phg}} I^m(X, Y) \hookrightarrow \mathscr{A}^{E}_{\mathrm{phg}} I^m(X, Y) \xrightarrow{\sigma_B} \mathscr{A}^{\{E\}_B}_{\mathrm{phg}} I^m(N^+B, N^+_{B\cap Y}B) \to 0, \quad B\cap Y \neq \varnothing \tag{B6.13}$$

$$0 \to \mathscr{A}^{E+1_B}_{\mathrm{phg}} I^m(X, Y) \hookrightarrow \mathscr{A}^{E}_{\mathrm{phg}} I^m(X, Y) \xrightarrow{\sigma_B} \mathscr{A}^{\{E\}_B}_{\mathrm{phg}}(N^+B) \to 0, \quad B\cap Y \neq \varnothing. \tag{B6.14}$$

As a consequence of the assumption that $Y \subset X$ is a clean submanifold its conormal bundle, spanned by the dz_l in (B6.2) can be consistently considered as a subbundle of the compressed cotangent bundle, so giving a commutative diagram:

$$\begin{tikzcd} & T^*_YX \arrow[d] \\ N^*Y \arrow[ur] \arrow[dr] & \\ & {}^bT^*_YX. \end{tikzcd} \tag{B6.15}$$

Thus if $P \in \mathrm{Diff}^m_b(X)$ its symbol, $\sigma_m(P)$, restricts from ${}^bT^*_YX$ to N^*Y. Of course this restriction is the same as if P is regarded as an element of $\mathrm{Diff}^m(X)$.

PROPOSITION B6.16. *If Y is a clean submanifold meeting the interior of a manifold with corners then for any $P\in \mathrm{Diff}_b^m(X)$, any $M\in \mathbf{R}$ and any index family, E, for the boundary hypersurfaces of X*

$$\begin{aligned} &P\colon \mathcal{A}^E_{\mathrm{phg}} I^M(X,Y)\to \mathcal{A}^E_{\mathrm{phg}} I^{M+m}(X,Y),\\ &\sigma_{M+m}(Pu)=\sigma_m(P)|_{N^*Y}\cdot\sigma_M(u),\\ &\sigma_B(Pu)=I_B(P)\,\sigma_B(u)\end{aligned} \tag{B6.17}$$

for each boundary hypersurface B.

§ B7. Operations on $\mathcal{A}^E_{\mathrm{phg}} I^m(X,Y)$

There are also some less elementary results we need concerning the pull-back and push-forward operations.

The pull-back result is a straightforward extension of Proposition B4.3. Let $Y'\subset X'$ be a clean submanifold. It follows that the compressed tangent bundle of Y' injects into that of X'

$$ {}^bT_yY' \hookrightarrow {}^bT_yX' \quad \forall y\in Y', \tag{B7.1}$$

which is to say that the inclusion map is a b-map. A b-map $F\colon X\to X'$ is said to be transversal to Y' if

$$ {}^bF_*({}^bT_xX)+{}^bT_{F(x)}Y'={}^bT_{F(x)}X' \quad \forall x\in F^{-1}(Y'). \tag{B7.2}$$

If F is transversal to Y' in this sense it follows that $Y=F^{-1}(Y')$ is a clean submanifold of X. Another consequence of (B7.2) is that the pull-back map defines a linear isomorphism

$$ {}^bF^*\colon N^*_{f(x)}Y'\leftrightarrow N^*_xY \quad \forall x\in Y. \tag{B7.3}$$

This means that the fibre-b-density bundle over $\mathbf{P}N^*Y'$, lifts under ${}^bF^*$ to the fibre-b-density bundle over $\mathbf{P}N^*Y$. From (B7.3) we therefore have a b-map, which is an isomorphism on the fibres,

$$ \tilde F\colon \mathbf{P}N^*Y\to \mathbf{P}N^*Y'. \tag{B7.4}$$

Thus Proposition B4.3 applies to define a pull-back map, which we shall denote

$$ F^{\#}\colon \mathcal{A}^{E'}_{\mathrm{phg}}(\mathbf{P}N^*Y';\Omega_{\mathrm{fibre}})\to \mathcal{A}^{E}_{\mathrm{phg}}(\mathbf{P}N^*Y;\Omega_{\mathrm{fibre}}). \tag{B7.5}$$

PROPOSITION B7.6. *If $F: X \to X'$ is a b-map of compact manifolds with corners, satisfying* (B.1), *and $Y' \subset X'$ is a clean submanifold to which F is transversal then for any index family E' for the boundary hypersurfaces of X' and any $m \in \mathbf{R}$ there is a naturally defined continuous pull-back map*

$$F^*: \mathcal{A}^{E'}_{\mathrm{phg}} I^m(X', Y') \to \mathcal{A}^{E}_{\mathrm{phg}} I^M(X, Y) \tag{B7.7}$$

where, in terms of (B4.2),

$$E = F^{\mathrm{b}}(E), \quad M = m + \frac{1}{4}(\dim X - \dim X') \tag{B7.8}$$

and the symbol at Y is given by

$$\sigma_M(F^* u) = F^{\#} \sigma_m(u) \quad \forall u \in \mathcal{A}^{E'}_{\mathrm{phg}} I^m(X', Y'). \tag{B7.9}$$

There is a small difficulty here, which we simply note, with the naturality of the pull-back operation. Namely smooth functions are not dense in the space $\mathcal{A}^{E'}_{\mathrm{phg}} I^m(X', Y')$ so continuity alone is not enough. This is easily overcome by enlarging the conormal space to include non-polyhomogeneous distributions and thereby ensuring the density of $\mathcal{C}^\infty$, or by using duality in terms of the push-forward of smooth functions.

Turning to the push-forward operation we shall give two results, although the first is actually contained in the second. The first is the obvious extension of Proposition B5.6. Namely we consider a clean submanifold $Y \subset X$ and a b-submersion $F: X \to X'$. The condition relating the two is that F should be transversal to Y, meaning

$$\mathrm{null}({}^bF_*) + {}^bT_y Y = {}^bT_y X \quad \forall y \in Y, \quad {}^bF_*: {}^bT_y X \to {}^bT_{F(y)} X'. \tag{B7.10}$$

PROPOSITION B7.11. *Let $F: X \to X'$ be a b-submersion transversal to a clean submanifold $Y \subset X$, then for any index family E for X with a decomposition* (B5.5) *such that* (B5.7) *holds, push-forward gives a continuous linear map*

$$F_*: \mathcal{A}^{E}_{\mathrm{phg}} I^m(X, Y; {}^b\Omega) \to \mathcal{A}^{F_b(E)}_{\mathrm{phg}}(X'; {}^b\Omega). \tag{B7.12}$$

Thus under such a push-forward all the interior conormal singularities are integrated out. We also need a more complex result which is essentially the standard composition formula for pseudodifferential operators. Here we have two clean submanifolds $Y_1, Y_2 \subset X$ each satisfying the hypotheses of Proposition B7.11. The distribution we consider is the product of two conormal distributions. As in the case without boundary we need to assume that $Y_1 \pitchfork Y_2 = Y$ for this product to be defined, i.e.

(B7.13) $$^bT_yY_1 + {}^bT_yY_2 = {}^bT_yX \quad \forall y \in Y = Y_1 \cap Y_2.$$

Then Y is necessarily a clean submanifold too. We do not want to assume that F is transversal to Y, rather we suppose that

(B7.14) $F\colon Y \to X'$ embeds Y as a clean submanifold, Y'.

As a consequence of this assumption the conormal bundle of Y' embeds as a subbundle of that of Y:

(B7.15) $$F^*\colon N^*Y' \hookrightarrow N^*Y.$$

Now, by the transversality of the intersection of Y_1 and Y_2,

(B7.16) $$N^*Y = N^*_Y Y_1 \oplus N^*_Y Y_2.$$

Moreover from the assumed transversality of F to both Y_1 and Y_2 the image in (B7.15) can meet neither of the components in (B7.16). This means that there is a well-defined product:

(B7.17) $$\mathscr{A}^{\mathrm{SP}(m_1),E_1(Y_2)}_{\mathrm{phg}}(\mathbf{P}N^*Y_1)\cdot\mathscr{A}^{\mathrm{SP}(m_2),E_2(Y_2)}_{\mathrm{phg}}(\mathbf{P}N^*Y_2) \to \mathscr{A}^{\mathrm{SP}(m_1+m_2),E_1\bar{+}E_2(Y)}_{\mathrm{phg}}(\mathbf{P}N^*Y).$$

This is just a consequence of Lemma B3.5 and Proposition B4.3. Also as a consequence of (B7.14) and the transversality of the intersection of Y_1 and Y_2 there is a natural identification of b-density bundles:

(B7.18) $$[{}^b\Omega(X)\otimes\Omega_{b,\mathrm{fibre}}(\mathbf{P}N^*Y_1)\otimes\Omega_{b,\mathrm{fibre}}(\mathbf{P}N^*Y_2)]|_Y \cong {}^b\Omega(X')|_Y\Omega_{b,\mathrm{fibre}}(Y).$$

Combining these maps gives us a bilinear map, by restriction to N^*Y':

(B7.19) $$\begin{aligned} F_\#\colon \mathscr{A}^{\{\mathrm{SP}(m_1),E_1(Y_1)\}_{Y_1}}_{\mathrm{phg}}(\mathbf{P}N^*Y_1;{}^b\Omega(X)\otimes\Omega_{b,\mathrm{fibre}})\cdot\mathscr{A}^{\{\mathrm{SP}(m_2),E_2(Y_2)\}_{Y_2}}_{\mathrm{phg}}(\mathbf{P}N^*Y_2;\Omega_{b,\mathrm{fibre}}) \\ \to \mathscr{A}^{\{\mathrm{SP}(m_1+m_2),E_1\bar{+}E_2\}_Y}_{\mathrm{phg}}(\mathbf{P}N^*Y;{}^b\Omega(X')\otimes\Omega_{b,\mathrm{fibre}}). \end{aligned}$$

With these preliminaries out of the way we can now state the result:

PROPOSITION B7.20. *Let $F\colon X \to X'$ be a b-submersion between manifolds with corners satisfying* (B.1) *and suppose that $Y_1, Y_2 \subset X$ are clean submanifolds meeting* *transversally in $Y=Y_1 \pitchfork Y_2$ such that F is transversal to Y_1 and Y_2 and embeds Y as a submanifold Y' of X', then for any index families E_1 and E_2 for the boundary hypersurfaces of X and any $m_1, m_2 \in \mathbf{R}$ multiplication on X followed by push-forward to X' gives a separately continuous bilinear form*

$$
\text{(B7.21)} \qquad \begin{gathered} u\times v \mapsto F_*(u\cdot v), \\ \mathcal{A}^{E_1}_{\mathrm{phg}} I^{m_1}(X, Y_1)\times \mathcal{A}^{E_2}_{\mathrm{phg}} I^{m_2}(X, Y_2; {}^b\Omega) \to \mathcal{A}^{F_b(E_1 \bar{+} E_2)}_{\mathrm{phg}} I^{m_1+m_2}(X', Y'; {}^b\Omega) \end{gathered}
$$

such that the symbol map at Y' satisfies

$$
\text{(B7.22)} \qquad \sigma_{m_1+m_2}[F_*(u\cdot v)] = F_\#[\sigma_{m_1}(u)\cdot \sigma_{m_2}(v)].
$$

References

[1] ABRAMOWITZ, M. & STEGUN, I. A., *Handbook of Mathematical Functions*. Dover Publ. Inc., New York, 1965.

[2] ANDERSON, M. T. & SCHOEN, R., Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math.*, 121 (1985), 429–461.

[3] CHENG, S.-Y. & YAU, S.-T., On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33 (1980), 507–544.

[4] CORLETTE, K., Hausdorff dimensions of limit sets, part I. Preprint.

[5] DONELLY, H. & FEFFERMAN, C. L., L^2-cohomology and index theorem for the Bergman metric. *Ann. of Math.*, 118 (1983), 593–619.

[6] DOYLE, P., On the base note of a Schottky group. *Acta Math.*, 160 (1988), 249–284.

[7] FEFFERMAN, C. L., The Bergman kernel and biholomorphic equivalence of pseudoconvex domains. *Invent. Math.*, 26 (1974), 1–65.

[8] — Monge–Ampère equations, the Bergman kernel, and the geometry of pseudoconvex domains. *Ann. of Math.*, 103 (1976), 395–416.

[9] GRAHAM, C. R., The Dirichlet problem for the Bergman Laplacian, I and II. *Comm. Partial Differential Equations*, 8 (1983), 433–476, 563–641.

[10] HÖRMANDER, L., Fourier integral operators, I. *Acta Math.*, 127 (1971), 79–183.

[11] — *An Introduction to Complex Analysis in Several Variables*. D. van Nostrand, Princeton, N.J., 1965.

[12] — *The Analysis of Linear Partial Differential Operators*, III. Springer Verlag, Heidelberg, 1985.

[13] KOHN, J. J., Harmonic integrals on strongly pseudoconvex manifolds, I, II. *Ann. of Math.*, 78 (1963), 112–148; 79 (1964), 450–472.

[14] KOHN, J. J. & ROSSI, H., On the extension of holomorphic functions from the boundary of a complex manifold. *Ann. of Math.*, 81 (1965), 451–472.

[15] KORÁNYI, A. & REIMANN, H., Quasiconformal mappings on the Heisenberg group. *Invent. Math.*, 80 (1985), 309–338.

[16] LEE, J. & MELROSE, R., Boundary behavior of the complex Monge–Ampère equation. *Acta Math.*, 148 (1982), 159–192.

[17] MAZZEO, R. R., Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds. To appear.

[18] MAZZEO, R. R. & MELROSE, R. B., Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75 (1987), 260–310.

[19] MELROSE, R. B., *Differential Analysis on Manifolds with Corners*. To be published.

[20] — Transformation of boundary value problems. *Acta Math.*, 147 (1981), 149–236.

[21] PHILLIPS, R. S. & SARNAK, P., The Laplacian for domains in hyperbolic space and limit sets of Kleinian groups. *Acta Math.*, 155 (1985), 173–241.

[22] SULLIVAN, D., The density at infinity of a discrete group of hyperbolic motions. *Publ. Math.*, 50 (1979), 419–450.

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