NONTRANSITIVE QUASI-ORBITS IN MACKEY'S ANALYSIS OF GROUP EXTENSIONS

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This paper is dedicated to George Mackey on his sixtieth birthday

0. Introduction

G. W. Mackey developed a general method for analyzing the dual of a locally compact group G (always second countable) in terms of the dual of a closed normal subgroup N and the cocycle duals of subgroups of G/N, provided that the action of G on the dual of N is sufficiently regular [9]. In this case regularity means that every ergodic quasi-invariant measure under the action of G is concentrated on an orbit, which means that the associated quasi-orbit is transitive. The theory of virtual groups was introduced by Mackey for the purpose of dealing with the less regular case [11, 12]. Section 9 of this paper gives proofs that the theorems of section 8 of [9] remain valid in the more general setting. It should be remarked that this leaves work yet to be done before a complete understanding of the general case is achieved. For instance, one of the theorems establishes a one–one correspondence between part of the dual of G and the ω-dual of a certain virtual group for a certain cocycle ω, but an example due to C. C. Moore shows that the latter can be empty [1]. This example is discussed in section 10 of this paper, and shows that representations of virtual groups need not decompose into primary representations.

The organization of the paper is as follows. The first six sections deal with the machinery of inducing representations from one group action to another. More particularly, sections 1 and 2 give preliminary material on Hilbert bundles and bundle representations of groupoids. In section 3 this is used to define induced representations, and it is proved that the definition given is an extension of the definition for subgroups. One novelty here is the proof of Proposition 3.4, which uses no special choice of Radon–Nikodym derivatives. In section 4 a lemma needed in later sections is proved, concerning intertwining operators.

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Then sections 5 and 6 deal with virtual group versions of inducing in stages and the subgroup theorem. Next, section 7 shows that cocycle representations of a groupoid can be connected with ordinary representations of another groupoid, just as for groups except more easily done. Section 8 deals with material related to section 7 of [9] as well as a lemma which allows us to deal only with invariant analytic sets even when $\hat{N}$ is not completely smooth. Then section 9 deals with the extensions of results of section 8 of [9], and section 10 has examples and applications.

We give here some notation and terminology which will be used throughout the paper: If G is a groupoid [15, section 1] then $G^{(0)}$ will denote the set of units of G and $G^{(2)}=\{(x, y)\in G\times G\colon xy$ is defined$\}$. The functions $r, d\colon G\to G^{(0)}$ are defined by $r(x)=xx^{-1}$, $d(x)=x^{-1}x$ [15, section 1]. If φ is a groupoid homomorphism, $\varphi^{(2)}=(\varphi\times\varphi)|G^{(2)}$. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The letter $\mathcal{R}$ will be used for commuting rings or sets of intertwining operators, such as $\mathcal{R}(L, M)=\{T\colon T\colon G^{(0)}\to\mathcal{L}(\mathcal{H}(L), \mathcal{H}(M))$ with $T(r(x))L(x)=M(x)T(d(x))$ for all $x\}$. If λ is a measure and f is Borel, $f_*(\lambda)(E)=\lambda(f^{-1}(E))$. Also $[\lambda]$ is the measure class of λ. The term measurable will be used for functions measurable relative to the completion of a given Borel measure. Measures are Borel and spaces are Borel spaces, i.e. with an assumed or given σ-algebra of subsets. The material on analytic and standard Borel spaces in [1, 8] is assumed, as is everything in [9, 15]. Note: Use the definition of groupoid in [15] rather than the one in [16].

1. Operations on Hilbert bundles

Let S be an analytic Borel space and let $\mathcal{H}$ be a function assigning to each $s\in S$ a Hilbert space denoted either $\mathcal{H}(s)$ or $\mathcal{H}_s$. Form $S*\mathcal{H}=\{(s, x)\colon s\in S$ and $x\in\mathcal{H}_s\}$. (If $\mathcal{H}$ happens to be constant, taking the value $\mathcal{K}$ everywhere, then $S*\mathcal{H}=S\times\mathcal{K}$.) Let π be the natural projection of $S*\mathcal{H}$ onto S. Then a section of $S*\mathcal{H}$ is a function from S to $S*\mathcal{H}$ with $\pi\circ f$ equal to the identity on S. If f is a function from S to $\bigcup\{\mathcal{H}(s)\colon s\in S\}$ such that for each $s\in S$, $f(s)\in\mathcal{H}(s)$, i.e. $f\in\Pi_{s\in S}\,\mathcal{H}(s)$, then we can define a section f^+ by $f^+(s)=(s, f(s))$, i.e. $f^+=i\times f$. Conversely if f is a section of $S*\mathcal{H}$, and f^- is defined by $(s, f^-(s))=f(s)$, then $f^-\in\Pi_{s\in S}\,\mathcal{H}(s)$. There is also an obvious Hilbert space structure on each $\{s\}\times\mathcal{H}(s)$ making $(s, x)\to x$ an isomorphism with $\mathcal{H}(s)$, and then $(f^+(s)\colon(s, x))=(f(s)\colon x)$ holds identically. Thus while sections of $S*\mathcal{H}$ and elements of $\Pi_{s\in S}\,\mathcal{H}(s)$ are not really the same, they are tied so closely that there should be no confusion if we treat them as if they were the same. We shall thus refer to elements of $\Pi_{s\in S}\,\mathcal{H}(s)$ as sections of $S*\mathcal{H}$ unless there is a real need to be careful. After the definition of induced representation, Definition 3.5, we shall return to this point again, for clarification. A Hilbert bundle over S is such a function

$\mathcal{H}$ on S together with an analytic Borel structure on $S * \mathcal{H}$ satisfying these two conditions [13]:

For $E \subseteq S$, $\pi^{-1}(E)$ is Borel iff E is Borel.

There is a sequence $f_1, f_2, \ldots$ of sections such that

(a) for each n, $(s, x) \to (f_n(s): x) = f'_n(s, x)$ is Borel,

(b) for each m and n, $s \to (f_m(s): f_n(s))$ is Borel,

(c) π and the functions f'_n separate points.

We may also refer to $S * \mathcal{H}$ as the bundle, with the Borel structure implicit. Notice that multiplying the sections f_n by non-vanishing scalar valued Borel functions does no harm, so that the functions $s \to \|f_n(s)\|$ can be taken as small as may be convenient. We say two bundles over S with functions $\mathcal{H}_1$, $\mathcal{H}_2$ are equivalent if there is a Borel isomorphism α of $S * \mathcal{H}_1$ onto $S * \mathcal{H}_2$ such that for each $s \in S$, $(s, x) \to \alpha(s, x)$ is a unitary isomorphism of $\{s\} \times \mathcal{H}_1(s)$ onto $\{s\} \times \mathcal{H}_2(s)$. Here we don't want to go from one fiber to another, though later we will. The function taking $s \in S$ to dim $(\mathcal{H}(s))$ is Borel and S is partitioned by $\{S_\infty, S_0, S_1, S_2, \ldots\}$ where $S_n = \{s \in S\colon \dim(\mathcal{H}(s)) = n\}$. If $\mathcal{K}_n$ has dimension n for $n = \infty, 0, 1, 2, \ldots$ we can define $\mathcal{H}'(s) = \mathcal{K}_n$ if $s \in S_n$ and give $S * \mathcal{H}'$ the Borel structure $\mathcal{B}'$ of $S_\infty \times \mathcal{K}_\infty \cup S_0 \times \mathcal{K}_0 \cup S_1 \times \mathcal{K}_1 \cup \ldots$, a disjoint union. This is easily shown to be a Hilbert bundle, and it is in fact isomorphic to the given bundle $(\mathcal{H}, \mathcal{B})$: since $S * \mathcal{H}$ is analytic, countably many Borel functions which separate points determine the Borel structure, so a function into $S * \mathcal{H}$ is Borel iff its compositions with π and the functions f'_n are Borel. Hence a section g is Borel iff $s \to (f_n(s): g(s))$ is Borel for $n = 1, 2, \ldots$. Therefore a sum of two Borel sections is a Borel section and a multiple of a Borel section by a scalar valued Borel function is a Borel section. The Gram–Schmidt process applied to the sections $f_1, f_2, \ldots$ in a pointwise manner yields a sequence of sections $g_1, g_2, \ldots$ for which properties (a), (b) and (c) hold, and for each s the non-zero elements of $\{g_n(s)\colon n = 1, 2, \ldots\}$ form an orthonormal basis of $\mathcal{H}(s)$. Let $h_1(s)$ be the first non-zero $g_n(s)$, $h_2(s)$ the second, etc. Then for $s \in S_n$, $h_1(s), \ldots, h_n(s)$ is an orthonormal basis of $\mathcal{H}(s)$, and $(s, (c_1, \ldots, c_n)) \to (s, c_1 h_1(s) + \ldots + c_n h_n(s))$ is an equivalence of $S_n \times \mathcal{K}_n$ with $S_n * (\mathcal{H} | S_n)$. It follows that $S * \mathcal{H}$ is standard if S is standard.

If X is another analytic space with $p\colon X \to S$ a quotient map and $s \to \lambda_s$ is a Borel function from S to finite measures on X such that λ_s is concentrated on $p^{-1}(s)$ for $s \in S$, then we can define $\mathcal{H}(s) = L^2(\lambda_s)$ and make this into a Hilbert bundle by giving $S * \mathcal{H}$ the smallest Borel structure for which π is Borel along with all the functions $(s, x) \to (f: x)_s$ for bounded Borel f, where $(f: x)_s$ is the inner product in $L^2(\lambda_s)$ [15, p. 265].

There are many ways to build new bundles out of old ones. The condition required on the projection is usually trivial to verify. If $S_1 * \mathcal{H}_1$, $S_2 * \mathcal{H}_2$, ... are bundles and the base sets S_1, S_2, ... are pairwise dispoint, then $S_1 * \mathcal{H}_1 \cup S_2 * \mathcal{H}_2 \cup \dots$ is a bundle under the disjoint union Borel structure. If $S * \mathcal{H}$ is a Hilbert bundle and T is a Borel set in S, then $\pi^{-1}(T)$ is a Hilbert bundle over T under the relative Borel structure. If T is any analytic space and g: $T \to S$ is Borel we can pull $\mathcal{H}$ back to a Hilbert bundle over T by using $\mathcal{H} \circ g$: let $\{E_1, E_2, \dots\}$ be a countable generating family of Borel sets in T which includes the inverse images of a generating family for S, let f_1, f_2, ... be the appropriate sections of $S * \mathcal{H}$, and form the sections $h_{m,n} = \varphi_{E_m}(f_n \circ g)$ of $T * (\mathcal{H} \circ g)$. Take the smallest Borel structure on $T * (\mathcal{H} \circ g)$ for which the projection is Borel and $(t, x) \to (h_{m,n}(t) \colon x)$ is always Borel.

The direct sum of Hilbert bundles is defined as follows: suppose $\mathcal{H}(s) = \mathcal{H}_1(s) \oplus \mathcal{H}_2(s) \oplus \dots$ for $s \in S$ and let $S * \mathcal{H}$ have the smallest Borel structure for which $(s, x) \to (s, x_n)$ is always Borel, where x_n denotes the component of x in $\mathcal{H}_n(s)$. The necessary sections may be gotten as follows: choose sections $f_{n1}, f_{n2}, \dots$ for $S * \mathcal{H}_n$ so that for each s and k the sequence $h_k = (f_{1k}(s), f_{2k}(s), \dots)$ is in $\mathcal{H}(s)$ (by making them small enough, say $\|f_{nk}(s)\| \leq 1/n$ everywhere). Then h_1, h_2, ... will be what we need.

If $S * \mathcal{H}$ is a bundle and E is a Borel set in S, define $(\varphi_E \mathcal{H})(s)$ to be $\mathcal{H}(s)$ for $s \in E$ and $\{0\}$ otherwise. The Borel structure is taken to be that of $\pi^{-1}(E) \cup (S \backslash E)$. This is a disjoint union of two previous constructions, and we mentioned earlier that any $\mathcal{H}$ is isomorphic to a direct sum of bundles $\varphi_{S_n} \mathcal{K}_n$ where $\mathcal{K}_n$ is constant. If $E_n = \bigcup_{n \leq k \leq \infty} S_k$, $n = 1, 2, \dots$, we also have $\mathcal{H}$ isomorphic to $\oplus \varphi_{E_n} \mathbf{C}$. Notice finally that if E_1, E_1, ... are disjoint Borel sets with union E, then $\varphi_E \mathcal{H} \cong \oplus_{n \geq 1} (\varphi_{E_n} \mathcal{H})$.

Now let $S * \mathcal{H}_1$, $S * \mathcal{H}_2$ be bundles and define $\mathcal{H}(s) = \mathcal{H}_1(s) \otimes \mathcal{H}_2(s)$ for $s \in S$. If $\mathcal{K}_1$, $\mathcal{K}_2$ are constant and E_1, E_2 are Borel, then for $\mathcal{H}_1 = \varphi_{E_1} \mathcal{K}_1$, $\mathcal{H}_2 = \varphi_{E_2} \mathcal{K}_2$, we have $\mathcal{H} = \varphi_{E_1 \cap E_2} \mathcal{K}_1 \otimes \mathcal{K}_2$. Thus in this case $S * \mathcal{H}$ has a good Borel structure. In general we can reduce to a direct sum of such tensor products by distributing products over sums, so we can give $S * \mathcal{H}$ a good Borel structure.

Let $S * \mathcal{H}$ be a Hilbert bundle and let λ be a σ-finite Borel measure on S. We will denote by $L^2(\lambda; \mathcal{H})$ the Hilbert space of sections f of $S * \mathcal{H}$ such that $\int \|f(s)\|^2 d\mu(s) < \infty$, with the natural inner product. These are the L^2-sections of $\mathcal{H}$, and $L^2(\lambda; \mathcal{H})$ is what is called the direct integral. For a constant bundle the notation is compatible.

Now to form direct images of Hilbert bundles, suppose p: $S \to T$ is Borel, where S and T are analytic, and suppose $t \to \lambda_t$ is a Borel function to finite measures on S with λ_t concentrated on $p^{-1}(t)$. Then if $S * \mathcal{H}$ is a Hilbert bundle we can define $p_*(\mathcal{H})(t) = L^2(\lambda_t; \mathcal{H})$ for $t \in T$. The notation $p_*(\mathcal{H})(t)$ suppresses the measures λ_t but should cause no confusion in our use of this construction. If $\mathcal{H}$ is a constant $\mathcal{K}$, $p_*(\mathcal{H})(t) \cong L^2(\lambda_t) \otimes \mathcal{K}$, and we know this has a good Borel structure. In general, let $\mathcal{B}$ be the smallest Borel structure on $T * p_*(\mathcal{H})$

for which the projection is Borel and $(t, x)\to\psi_f(t, x)=(x: f)_t$ (the inner product in $L^2(\lambda_t;\ \mathcal{H})$) is Borel whenever f is a bounded Borel section of $S * \mathcal{H}$. Notice that the function f is always in $L^2(\lambda_t;\ \mathcal{H})$ because bounded functions are in L^2 for finite measures. To show that this renders $T * p_*(\mathcal{H})$ a Hilbert bundle, we begin with the case that $\mathcal{H}=\varphi_E\mathbf{C}$. Then we have the bundle of Hilbert spaces $L^2(\varphi_E\lambda_t)$ over T. This is okay because it is gotten from a Borel family of measures. Now suppose $\mathcal{H}=\oplus_{n\geqslant 1}\mathcal{H}_n$. Then $p_*(\mathcal{H})(t)\cong\oplus_{n\geqslant 1}p_*(\mathcal{H}_n)(t)$ for each t. The natural projection from $T * p_*(\mathcal{H})$ to $T * p_*(\mathcal{H}_n)$ is Borel, since if f_n is a bounded section of $S * \mathcal{H}_n$ then $(0, 0, ..., f_n, 0, ...)$ is a bounded section of $S * \mathcal{H}$, so that the projection followed by one of the determining functions on $T * p_*(\mathcal{H}_n)$ is a determining function on $T * p_*(\mathcal{H})$. On the other hand, if f is a bounded section of $S * \mathcal{H}$, then $f=(f_1, f_2, ...)$ where for each n f_n is a bounded section of $S * \mathcal{H}_n$. Now for $t\in T$, if $x=(x_1, x_2, ...)\in p_*(\mathcal{H})(t)$. $(f: x)_t=\Sigma_{n\geqslant 1}(f_n: x_n)_t$. It follows that $\mathcal{B}$ is the smallest σ-algebra for which each projection $T * p_*(\mathcal{H})\to T * p_*(\mathcal{H}_n)$ is Borel. Hence if $\mathcal{H}=\oplus_{n\geqslant 1}\varphi_{E_n}\mathbf{C}$, we see that $T * p_*(\mathcal{H})$ is a Hilbert bundle. Since isomorphisms preserve what is needed, we see that $T * p_*(\mathcal{H})$ is always a Hilbert bundle.

To conclude this section, we prove a fact about double applications of the direct image process which will be useful in proving the theorem on inducing in stages. Since the fibers in a direct image bundle are direct integrals, the result can be seen as a slight addition to the theorem on refinements of direct integral decompositions [6 Theorem 2.11].

THEOREM 1.1. *Suppose S, T, U are analytic Borel spaces and $p: S\to T$, $q: T\to U$ are Borel surjections, and set $r=q\circ p$. Let $t\to\mu_t$ be a Borel function from T to the finite measures on S such that $\mu_t(S\backslash p^{-1}(t))=0$ for $t\in T$ and let $u\to\nu_u$ be a similar function for U, T and q. For $u\in U$ define $\lambda_u=\int\mu_t d\nu_u(t)$. Then $u\to\lambda_u$ is Borel from U to the finite measures on S and $\lambda_u(S\backslash r^{-1}(u))=0$ for $u\in U$. Let $S * \mathcal{H}$ be a Hilbert bundle over S and form $p_*(\mathcal{H})$ using the measures μ_t and then $q_*(p_*(\mathcal{H}))$ using the measures ν_u. Form $r_*(\mathcal{H})$ using the measures λ_u. Then $U * r_*(\mathcal{H})$ and $U * q_*(p_*(\mathcal{H}))$ are equivalent.*

Proof. The statement about the measures λ_u is proved in a straightforward manner. Recall that if $\mathcal{A}$ is a countable algebra of sets generating the Borel σ-algebra of subsets of S, then the smallest set of functions containing $\{\varphi_A: A\in\mathcal{A}\}$ which closed under addition and multiplication by numbers $a+bi$ where a and b are rational is countable and is L^2-dense relative to every finite Borel measure on S. Every Hilbert bundle over S is equivalent to a countable direct sum of bundles of the form $S * (\varphi_E\mathbf{C})$, so this density property generalizes easily: If $S * \mathcal{H}$ is a Hilbert bundle there is one countable family F of bounded sections of $S * \mathcal{H}$ which gives a dense set in $L^2(\lambda';\ \mathcal{H})$ for every finite Borel measure λ' on S.

LEMMA. *If π is the projection of $U * r_*(\mathcal{H})$ onto U and $\psi_f(u, x) = (x: f)_u$ for $(u, x) \in U * r_*(\mathcal{H})$ and f a bounded section of $S * \mathcal{H}$, then $\{\pi\} \cup \{\psi_f: f \in F\}$ determines the Borel structure on $U * r_*(\mathcal{H})$.*

Proof of Lemma. Let (u_1, x_1), (u_2, x_2) be distinct. If $u_1 \neq u_2$, there is no difficulty, so suppose $u_1 = u_2 = u$ but $x_1 \neq x_2$. Since ψ_f is linear in its second variable, we may suppose $x_2 = 0$ and $x_1 = x \neq 0$. Now λ_u is a finite measure on S, so F gives a total set in $r_*(\mathcal{H})(u)$ and hence there is an $f \in F$ such that $\psi_f(u, x) \neq 0$. The result now follows from a general fact about analytic Borel spaces.

Now define a mapping V from sections of $S * \mathcal{H}$ to sections of $T * p_*(\mathcal{H})$ as follows: for $t \in T$ and f a section let $(Vf)(t)$ be the class of f in $L^2(\mu_t; \mathcal{H}) = p_*(\mathcal{H})(t)$ if f is square integrable and let $(Vf)(t) = 0$ otherwise. If f is bounded then f is always square integrable, so Vf has no artificial zeros. For any section f, $\int\int \|f(s)\|^2 d\mu_t(s) d\nu_u(t) = \int \|f(s)\|^2 d\lambda_u(s)$ so if f is square integrable relative to λ_u it is also square integrable for almost all μ_t and Vf has at most a null set of artificial zeros. Also, V defines an isometry V_u of $r_*(\mathcal{H})(u)$ into $q_*(p_*(\mathcal{H}))(u)$. To prove that V_u is unitary, suppose g is orthogonal to its range. Let F be the set of bounded sections determined above, and let $f \in F$. Then let $(\ :\ )_t$ denote the inner product of sections of $S * \mathcal{H}$ relative to μ_t and notice that the orthogonality condition is just $\int (f: g(t))_t d\nu_u(t) = 0$ (this makes sense because g is a Borel section of $T * p_*(\mathcal{H})$). Choose h to be a Borel function on T with values in the unit circle, such that $h(t)(f: g(t))_t \geqslant 0$ for all t. If $f_1(s) = h(p(s)) f(s)$ for $s \in S$, then f_1 is a bounded section of $S * \mathcal{H}$ and $\int (f_1: g(t))_t d\nu_u(t) = 0$. The integrand is non-negative, so $(f_1: g(t))_t = 0$ for almost all t and hence $(f: g(t))_t = 0$ for almost all t. Since F is total in each $p_*(\mathcal{H})(t)$, we see that g vanishes a.e.

Now define $\alpha(u, x) = (u, V_u x)$ for $(u, x) \in U * r_x(\mathcal{H})$. Then α is one–one and onto, it is unitary on each fiber and its domain and range are analytic, so to prove α is a bundle isomorphism we only need to prove α is a Borel function. First, define ψ'_f on $U * q_*(p_*(\mathcal{H}))$ for bounded sections f of $T * p_*(\mathcal{H})$ and notice that for $f \in F$, $\psi'_{Vf} \circ \alpha = \psi_f$ and hence is a Borel function. Now if we choose F to be closed under multiplication by characteristic functions of sets of the form $p^{-1}(A)$ for A in some countable generating family of Borel sets in T, as well as rational complex linear combinations, it is not difficult to see that the set $\{Vf: f \in F\}$ is dense in every $L^2(\mu'; p_*(\mathcal{H}))$ for finite measures μ'. In that case the projection and the functions ψ'_{Vf} determine the Borel structure on $U * q_*(p_*(\mathcal{H}))$, which is all we needed.

2. Representations and bundle representations

Let G be an analytic Borel groupoid, let σ be a 2-cocycle on G and let L be a σ-representation of G on a Hilbert space $\mathcal{K}$. Then G has a 'bundle σ-representation' on $G^{(0)} \times \mathcal{K}$, where $G^{(0)}$ is the set of units of G, defined as follows:

$$g \cdot (d(g), x) = (r(g), L_g x).$$

If $g_1 g_2$ is defined then

$$d(g_1 g_2) = d(g_2),\ r(g_1 g_2) = r(g_1)$$

and

$$(g_1 g_2) \cdot (d(g_2), x) = (r(g_1), L_{g_1 g_2} x) = (r(g_1), \sigma(g_1, g_2) L_{g_1} L_{g_2} x) = g_1(g_2 \cdot (d(g_2), \sigma(g_1, g_2) x)).$$

Conversely, suppose we are given a 'bundle cocycle representation', i.e. for each $g \in G$, a mapping of $\{d(g)\} \times \mathcal{K}$ to $\{r(g)\} \times \mathcal{K}$ which gives a unitary operator L_g on $\mathcal{K}$ and $(g_1 g_2)(d(g_2), x) = g_1(g_2(d(g_2), \tau x)$ for some τ. Then if L is a Borel function, i.e. the mapping of $G \times'(G^{(0)} \times \mathcal{K}) \to G^{(0)} \times \mathcal{K}$ is Borel, L must be a cocycle representation. In fact, L will be a σ-representation if we started with a 'bundle σ-representation' of G.

If $G^{(0)} * \mathcal{H}$ is a bundle over $G^{(0)}$ for which the spaces $\mathcal{H}(u)$ all have the same dimension, then $G^{(0)} * \mathcal{H}$ is equivalent to $G^{(0)} \times \mathcal{K}$ for some Hilbert space $\mathcal{K}$, and we get a correspondence between bundle σ-representations of G on $G^{(0)} * \mathcal{H}$ and σ-representations of G on $\mathcal{K}$. If α is an equivalence of $G^{(0)} * \mathcal{H}_1$ with $G^{(0)} * \mathcal{H}_2$ carrying one bundle σ-representation to another and α_1, α_2 are equivalences of $G^{(0)} * \mathcal{H}_1$ with $G^{(0)} \times \mathcal{K}_1$ and $G^{(0)} * \mathcal{H}_2$ with $G^{(0)} \times \mathcal{K}_2$ respectively, then $\alpha_2 \circ \alpha \circ \alpha_1^{-1}$ induces a similarity between the corresponding σ-representations on $\mathcal{K}_1$ and $\mathcal{K}_2$. Thus we really have a one–one correspondence between classes of 'bundle σ-representations', and classes of σ-representations.

For measurable groupoids, we often must restrict to an inessential contraction to get a strict representation, but then we simply consider bundle representations for the contraction. If $(G, [\mu])$ is an ergodic groupoid, and G has a bundle representation (or bundle cocycle representation) on $G^{(0)} * \mathcal{H}$, it follows that dim $(\mathcal{H}(u))$ is constant on some saturated Borel conull set of units. Thus for the ergodic case we can achieve the situation discussed above by passing to an i.c.

3. Inducing from one group action to another

It is possible to study groupoid homomorphisms and their 'kernels' and characterize those homomorphisms one would want to regard as injections or inclusions, and a paper giving the details is intended for publication elsewhere [17]. Here we simply start with an equivalent formulation of the notion of subobject and develop the notion of inducing from one subobject of a group to a larger one. This definition is the one suggested by Mackey in [14]. We show in this section that the definition extends the definition for subgroups and is independent of the measures in the given measure class.

A virtual subgroup of a group G is a virtual group given by an ergodic action of G on an analytic Borel measure space (S, μ) [12, 17]. Strictly speaking, we want to work with an equivalence class of G-spaces, where S_1 and S_2 are equivalent if they contain invariant conull analytic sets S_1', S_2' such that S_1' and S_2' are strictly isomorphic as G-spaces, under an isomorphism which preserves measure classes. We shall make such changes in S only if they simplify matters.

If S and T are G-spaces and f: $S \to T$ is a Borel function, then f is equivariant iff $f \times i$: $S \times G \to T \times G$ is a homomorphism. If λ and μ are quasi-invariant measures on S and T respectively, and m is finite and equivalent to Haar measure on G, we will say that an equivariant Borel f: $S \to T$ represents $S \times G$ as a subobject of $T \times G$ or that $f \times i$ is an imbedding of $S \times G$ into $T \times G$, provided that $f_*(\lambda) \sim \mu$. Given an imbedding, we may as well suppose $f_*(\lambda) = \mu$, since only the measure class of μ is really important. We also may suppose $f(S) = T$, by replacing T by $f(S)$. This is acceptable since $f(S)$ is analytic.

We shall do inducing even for the non-ergodic case for two reasons: It goes exactly the same way, and in the end we see that 'inducing in stages' includes 'inducing a direct integral' as a special case.

We need to use Proposition 2.6, page 72 of [1]. We give here a slightly different proof. Recall that if λ is a finite measure on a G-space and $x \in G$ then $(\lambda x)(A) = \lambda(Ax^{-1})$ for Borel sets A.

LEMMA 3.1. *Let S and T be analytic G spaces, let p: $S \to T$ be Borel, equivariant and onto. Let μ be a quasi-invariant finite measure on S and define $\nu = p_*(\mu)$. Then there is a decomposition $\mu = \int \mu_t d\nu(t)$ relative to p such that $\mu_t x \sim \mu_{tx}$ for t in T and x in G.*

Proof. Let $\mu = \int \mu_t d\nu(t)$ be any decomposition and let $K = \{(t, x) \in T \times G: \mu_t x \sim \mu_{tx}\}$. Then K is Borel, because $(t, x) \to (\mu_t, x) \to \mu_t x$ and $(t, x) \to tx \to \mu_{tx}$ are Borel and $\sim$ is determined by a Borel set of pairs of measures (Lemma 1.1 of [16]). Also K is $\nu \times m$-conull in $T \times G$ by Proposition 2.5 on page 72 of [1], where m is a finite measure in the class of Haar measure. Let T_0 be the set of $t \in T$ for which the t-section of K is m-conull in G. Then T_0 is Borel and conull, by the Fubini Theorem. Now if $t \in T_0$ and $y \in G$, then $\mu_{tyx} \sim \mu_t \cdot (yx)$ for almost all x. For the same t and almost all $y \in G$ we have $\mu_t y \sim \mu_{ty}$, so for almost all y $(ty, x) \in$

K for almost all x. Thus $t \in T_0$ implies $ty \in T_0$ for almost all $y \in G$. By the proof of Lemma 6.3 of [15], the saturation T_1 of T_0 is a Borel set.

Define $\lambda_t = \int_G \mu_{tx} x^{-1} dm(x)$ for $t \in T_1$ and $\lambda_t = 0$ for $t \notin T_1$; and let $\lambda = \int \lambda_t d\nu(t)$. Now $t \in T_0$ implies $\lambda_t \sim \mu_t$, so $\lambda \sim \mu$. Let $\varrho = d\mu/d\lambda$ be a Radon–Nikodym derivative which is positive and finite everywhere and redefine $\mu_t = \varrho\lambda_t$ for $t \in T$. Then $\mu_t \sim \lambda_t$ for $t \in T$ so it will suffice to show that $\lambda_t x \sim \lambda_{tx}$ for $(t, x) \in T \times G$. This is clear if $t \notin T_1$. Now let $t \in T_0$ and $y \in G$. The discussion above showed that $\mu_{tyz} z^{-1} \sim \mu_t y$ for almost all z, so $\lambda_{ty} \sim \mu_t y$. Hence $\lambda_{tyx} \sim \mu_t(yx) \sim \lambda_{ty} x$. Now $T_0 G = T_1$ so this proves the result for $t \in T_1$.

Remark: In applications of this lemma it will often be necessary to replace T by T_1, for instance if we want all the measures μ_t to be probability measures. It should be remarked that if T is standard then T_1 is also standard, so nothing essential ordinarily is lost by this change.

LEMMA 3.2. *Let (X, λ) and (Y, μ) be analytic Borel spaces with finite measures and let $T: X \to Y$ be a Borel isomorphism such that $T_*(\lambda) \sim \mu$. Let P and Q be the canonical projection valued measures on $L^2(\lambda)$ and $L^2(\mu)$ respectvely. Then there is a Borel function ϱ from X to $\mathbf{C}$ such that the operator U taking g to $\varrho g \circ T$ is unitary from $L^2(\mu)$ to $L^2(\lambda)$. Of all such operators, there is only one with $\varrho \geq 0$, namely the one with $\varrho^2 = (d\mu/dT_*(\lambda)) \circ T$. Let $\mathcal{E}$ be a countable algebra generating the Borel sets in Y. Then U is the only unitary operator from $L^2(\mu)$ to $L^2(\lambda)$ such that*

(a) *for E in $\mathcal{E}$, $P_{T^{-1}(E)} = UQ_E U^{-1}$*

(b) *for E in $\mathcal{E}$, $(U1, \varphi_{T^{-1}(E)}) \geq 0$.*

Proof. It is well known, and easy to verify, that if $\varrho^2(x) = [d\mu/dT_*(\lambda)](T(x))$ for $x \in X$ then the formula defines a unitary operator U from $L^2(\mu)$ onto $L^2(\lambda)$. The inverse is determined in the same way by T^{-1}. Suppose $U_1 g = \varrho_1 g \circ T$ and U_1 is unitary. Then $U_1 U^{-1}$ is a multiplication operator on $L^2(\lambda)$, namely $U_1 U^{-1} f = \varrho_1 f/\varrho$. (We may assume ϱ never vanishes; then $U^{-1} f = (f/\varrho) \circ T^{-1}$). The only positive function whose multiplication operator is unitary is the function identically 1, up to null sets.

It is easy to see that U satisfies conditions (a) and (b), so to complete the proof we suppose U_1 is unitary and satisfies those conditions. Let $\varrho_1 = U_1 1$. Then for each $E \in \mathcal{E}$, $U_1 \varphi_E$ vanishes off $T^{-1}(E)$ and so must be a multiple f_E of $\varphi_{T^{-1}(E)} = \varphi_E \circ T$. The same holds for $Y \setminus E$, and since U_1 is linear, we have $f_E \varphi_E \circ T + f_{Y \setminus E} \varphi_{Y \setminus E} \circ T = \varrho_1$. Then f_E agrees with ϱ_1 a.e. on $T^{-1}(E)$, so $U_1 \varphi_E = \varrho_1 \varphi_E \circ T$. Since U_1 is linear and continuous, $U_1 g = \varrho_1 g \circ T$ for g in $L^2(\mu)$. Condition (b) implies that $\varrho_1 \geq 0$ a.e.

Let S, T be analytic Borel G-spaces and suppose $p: S \to T$ is an equivariant Borel surjection. Let $t \to \mu_t$ be a Borel function from T to the finite measures on S, such that μ_t is concentrated on $p^{-1}(t)$ for each $t \in T$. Suppose that $\mu_t x \sim \mu_{tx}$ for $t \in T$ and $x \in G$. Let $\mathcal{H}(t) = L^2(\mu_t)$ for $t \in T$ and form the Hilbert bundle $T * \mathcal{H}$. Then $\mathcal{H}(t)$ and $\mathcal{H}(tx)$ are isomorphic for $t \in T$ and $x \in G$, so the Borel set where dim $\mathcal{H}(t)$ takes a given value is invariant, and we may as well suppose that all the $\mathcal{H}(t)$ are isomorphic, since we can deal with the various subsets one at a time. Then we have an equivalence of $T * \mathcal{H}$ with $T \times \mathcal{K}$. Denote the corresponding unitary operators from $\mathcal{H}(t)$ onto $\mathcal{K}$ by V_t.

For $t \in T$ and $E \in \mathrm{Bor}\,(S)$ define $P^t(E)$ on $\mathcal{H}(t)$ by $(P^t(E)f)(s) = \varphi_E(s) f(s)$. The dependence on t occurs because equivalence of functions f depends on the measure μ_t.

LEMMA 3.3. *For each Borel set $E \subseteq S$, $t \to P^t(E)$ is a Borel function in the weak (or strong) operator sense.*

Proof. The meaning of this statement is as follows: the space $\bigcup \{\{t\} \times \mathcal{L}\mathcal{H}(t)): t \in T\}$ is given the smallest Borel structure for which the projection onto T is Borel together with all the functions $\psi_{f,g}$ for bounded f, g, where $\psi_{f,g}(t, A) = (A[f]_t, [g]_t)([f]_t$ is the equivalence class of f in $L^2(\mu_t))$. The equivalence of $T * \mathcal{H}$ with $T \times \mathcal{K}$ carries this space isomorphically to $T \times \mathcal{L}(\mathcal{K})$. The strong operator version uses $\theta_f(t, A) = A[f]_t$ and the weak and strong Borel structures are the same, just as they are on $\mathcal{L}(\mathcal{K})$. Now $P^t(E)[f_t] = [\varphi_E f]_t$, and since $\varphi_E f$ is also a bounded Borel function, the truth of the Lemma is clear.

Now if we define $Q^t(E) = V_t P^t(E) V_t^{-1}$ for each Borel set $E \subseteq S$, we get a Borel function from T to $\mathcal{L}(\mathcal{K})$. We want to know that $(t, x) \to Q^{tx}(Ex)$ is also Borel. To show that, we look at another bundle equivalent to $T * \mathcal{H}$. Project $S \times G$ onto T by taking (s, x) to $p_1(s, x) = p(s)$. Let $\lambda_t = \mu_t \times \varepsilon_e$, where ε_e is the unit point mass at e and define $\mathcal{H}'(t) = L^2(\lambda_t)$. Then $T * \mathcal{H}'$ is clearly equivalent to $T * \mathcal{H}$; in fact $s \to (s, e)$ induces the equivalence. Define $\tau(s, x) = (sx, x^{-1})$ for $(s, x) \in S \times G$ and notice that τ is a Borel automorphism of period 2. Let $P_1^t(E)$ be the projection in $L^2(\lambda_t)$ corresponding to a Borel set $E \subseteq S \times G$, as above. Since $\varphi_{E \times G}(s, x) = \varphi_E(s)$ and $\varphi_{\tau(E \times G)}(s, x) = \varphi_{Ex}(s)$ for E a Borel set in S, we see that $P_1^t(E \times G)$ corresponds to $P^t(E)$ and $P_1^{tx}(\tau(E \times G))$ corresponds to $P^{tx}(Ex)$ under the equivalence of the two bundles. Now $(t, x) \to tx$ is Borel, so we see that $(t, x) \to P^{tx}(Ex)$ is Borel, by applying the above lemma to P_1.

PROPOSITION 3.4. *Let S and T be analytic Borel G-spaces for a locally compact group G, and suppose $p: S \to T$ is equivariant, Borel and onto. Let $t \to \mu_t$ be a Borel function from T to the finite measures on S, such that μ_t is always concentrated on $p^{-1}(t)$, and suppose that $\mu_t x \sim \mu_{tx}$ for $t \in T$ and $x \in G$. Let $\mathcal{H}(t) = L^2(\mu_t)$ for $t \in T$ and form the Hilbert bundle $T * \mathcal{H}$. Define $W(t, x)$: $\mathcal{H}(tx) \to \mathcal{H}(t)$ by $(W(t, x)g)(s) = [(d\mu_{tx}/d(\mu_t x))(sx)]^{1/2} g(sx)$. Then W is a bundle representation of $T \times G$ on $T * \mathcal{H}$.*

Proof. It is convenient to work both with $T * \mathcal{H}$ and with $T \times \mathcal{K}$, where all the $\mathcal{H}(t)$ are isomorphic to $\mathcal{K}$, and we will in fact construct W as a representation on $\mathcal{K}$. Recall the unitary operators V_t which establish the equivalence. The only thing that remains to be proved is that W is a Borel function. Let $\mathcal{E}$ be a countable algebra which generates the Borel sets in S. Then the intersections with $p^{-1}(t)$ generate the Borel sets in $p^{-1}(t)$ for $t \in T$. For each $E \in \mathcal{E}$ let $\mathcal{W}_E = \{(t, x, U) \in T \times G \times \mathcal{U}(\mathcal{K}) \colon UQ^{tx}(Ex)U^{-1} = Q^t(E)$ and $(UV_{tx}1, V_t\varphi_E) \geqslant 0\}$. The functions involved in defining $\mathcal{W}_E$ are Borel on $T \times G \times \mathcal{U}(\mathcal{K})$, so $\mathcal{W}_E$ is Borel. Now let $\mathcal{W} = \bigcap \{\mathcal{W}_E \colon E \in \mathcal{E}\}$. By Lemma 3.2, we see that $\mathcal{W}$ projects one–one onto $T \times G$ and that for each $(t, x) \in T \times G$ the corresponding unitary operator is the $W(t, x)$ defined in the statement of the theorem. Thus $\mathcal{W}$ is the graph of W. Since $T \times G$ is analytic and $\mathcal{W}$ is Borel, W is Borel, as desired.

Remark. Notice that there is no need to choose Radon–Nikodym derivatives in a smooth way. Since the operator does not depend on the choice, the Borel character of the function can be made global rather than simply on a conull set, as would be the case if we were forced to prove the Borel character by making a smooth choice of derivatives as in [5, 8, 15].

The representation W is the simplest induced representation and it is an ingredient in the general inducing process. The same construction and formulas apply equally well to give a representation on the bundle of Hilbert spaces $L^2(\mu_t; \mathcal{K})$ whenever $\mathcal{K}$ is a Hilbert space. This is simply a multiple of the W constructed above.

Two more facts need to be mentioned before the actual definition of inducing in general. If $S * \mathcal{H}$ is a Hilbert bundle and μ is a finite measure on S we can define $L^\infty(\mu; \mathcal{L}(\mathcal{H}))$ to be the algebra of functions A on s such that: for each s $A(s)$ is in $\mathcal{L}(\mathcal{H}(s))$, if f is a Borel section of $S * \mathcal{H}$ so is $s \to A(s)\, f(s)$, and $s \to \|A(s)\|$ is bounded. For such an A we define $A^\sim[f]$ be the class of $s \to A(s) f(s)$ when $[f] \in L^2(\mu; \mathcal{H})$, i.e. the direct integral operator; e.g. see [7]. Now consider the case of constant $\mathcal{H}$, say $S * \mathcal{H} = S \times \mathcal{K}$, and if $A \in L^\infty(\mu_{tx}; \mathcal{L}(\mathcal{K}))$ define $A^{(t,x)}(s) = A(sx)$ for $s \in p^{-1}(t)$. Let $\mathcal{H}(t) = L^2(\mu_t; \mathcal{K})$ for $t \in T$. For each (t, x) choose a Radon–Nikodym derivative as used in Proposition 3.4 and let $\varrho(s, t, x) = ((d\mu_{tx}/d(\mu_t x))(sx))^{\frac{1}{2}}$. Then we calculate, for $f \in \mathcal{H}(tx)$, $A \in L^\infty(\mu_{tx}; \mathcal{L}(\mathcal{K}))$, that for μ_t-almost all s,

$$(W(t, x) A^\sim W(t, x)^{-1} f)(s) = \varrho(s; t, x)(A^\sim W(t, x)^{-1} f)(sx) = \varrho(s; t, x) A(sx)(W(t, x)^{-1} f)(sx)$$
$$= \varrho(s; t, x) A(sx) \varrho(sx; tx, x^{-1}) f(s) = A^{(t,x)}(s) f(s) = (A^{(t,x)\sim} f)(s)$$

Continuing as in [9, 15] we let σ be a strict 2-cocycle on $T \times G$ and define $\sigma(s, x; sx, y) = \sigma(p(s), x; p(s)x, y)$ for $s \in S$, $x, y \in G$, so we think of σ as a cocycle on $S \times G$ as well as $T \times G$. This corresponds to the restriction of σ in the case of subgroups, as $S \times G$ is regarded as a subobject of $T \times G$. Now if R is a strict σ-representation of $S \times G$, let $R'(t, x)$ be the function on $p^{-1}(t)$ whose value at a point s is $R(s, x)$. Then $R'(t, x) \in L^\infty(\mu_t; \mathcal{L}(\mathcal{K}))$ and we can form $R'(t, x)^\sim$, which will be a unitary operator on $L^2(\mu_t; \mathcal{K}) = \mathcal{H}(t)$. Now for $s \in p^{-1}(t)$, and $x, y \in G$, we have $sx \in p^{-1}(tx)$ and $R(sx, y) = R'(tx, y)^{(t,x)}(s)$. Thus $R'(t, xy) = \sigma(t, x; tx, y) \times R'(t, x)\, R'(tx, y)^{(t,x)}$, where equality means actual equality as functions from $p^{-1}(t)$ to $\mathcal{U}(\mathcal{K})$. If R were not strict, but became strict on the contraction to a set S_0, then the functions would agree a.e. on $p^{-1}(t)$ if S_0 were conull for both μ_t and μ_{tx}.

Definition 3.5. The bundle σ-representation U of $T \times G$ induced by R, denoted Ind $(T \times G, S \times G; R)$ is defined by

$$U(t, x) = R'(t, x)^\sim W(t, x).$$

The corresponding σ-representation will be denoted by ind $(T \times G, S \times G; R)$.

That we actually have a σ-representation follows from the calculations made above. We do not give an imprimitivity theorem for this generality here. We only need that characterization for representations of groups.

Suppose R is a strict σ-representation of $S \times G$ on $\mathcal{K}$ and let L be the corresponding bundle σ-representation: $L(s, x)(sx, v) = (s, R(s, x)v)$. Then the formula for the induced bundle representation in terms of L and sections of $S \times \mathcal{K}$ is the same as the formula in terms of R and $\mathcal{K}$ valued functions on S. Since we have systematically blurred the distinction between the two classes of functions, it seems good to take note of this equivalence. What it means is this: let ϱ be a function on $S \times T \times G$ whose square gives the relevant Radon–Nikodym derivatives as in Proposition 3.4. If we let $\mathcal{H}(t)$ be the Borel $\mathcal{K}$-valued functions on S square integrable relative to μ_t and let $\mathcal{H}^+(t)$ be the Borel sections of $S \times \mathcal{K}$ square integrable relative to μ_t, then $\mathcal{H}(t)$ is isomorphic to $\mathcal{H}^+(t)$ under $f \to f^+$ where $f^+(s) = (s, f(s))$ for $s \in S$. Also, $L(s,x)f^+(sx) = (s, R(s,x)f(sx))$, so $(\mathrm{Ind}\,(R)(t,x)f)^+(s) = \varrho(s;t,x)L(s,x)f^+(sx)$.

To see that ind (R) depends only on the measure classes and not the measure, first notice that it was defined without using any measures on T, so it does not depend on any measure on T. Whenever a measure class on T is relevant we must assume that it is the image of the class on S, but otherwise it has no effect. We then ask what happens if we pass to a function $t \to \lambda_t$ from T to finite measures on S for which $\lambda_t \sim \mu_t$ for all t. Then using Lemma 3.2 as in the proof of Proposition 3.4 we see that there is a Borel family V_t of unitary operators from $L^2(\mu_t; \mathcal{K})$ onto $L^2(\lambda_t; \mathcal{K})$. In this case, for each t we have $V_t f = \varrho_t f$ where ϱ_t is a non-negative Borel function. By the uniqueness of such operators as proved in Lemma 3.2, the operators $W(t, x)$: $L^2(\mu_{tx}; \mathcal{K}) \to L^2(\mu_t; \mathcal{K})$ and $W'(t, x)$: $L^2(\lambda_{tx}; \mathcal{K}) \to L^2(\lambda_t; \mathcal{K})$ match up under the $V_t's$. Also $R'(t, x)^\sim$ is defined by the same formula in either case, so the two versions of inducing give equivalent representations.

Let us show that this definition of inducing produces equivalent results in case S and T are transitive G-spaces, in which case they correspond to subgroups of G, say H and K, Let us be more precise. This definition gives a representation of $T \times G$. If t_0 is the point of T stabilized by K, then $k \to (t_0, k) = \varphi(k)$ is a homomorphism of K into $T \times G$ which is one component of a similarity. If the induced respresentation as defined above is composed with φ, we should get a representation equivalent to the usual induced representation of K.

We know that representations of H and those of $S \times G$ correspond, so we can start with a representation of H, say L. Let ψ: $S \times G \to H$ be the homomorphism which is part of a similarity, of the form $\psi(s, x) = \gamma(s)x\gamma(sx)^{-1}$ where γ: $S \to G$ is a cross section, thinking of S as the right coset space for H. We induce using $R = \beta L \circ \psi$, where β makes R a σ-representation [15, p. 314]. The usual definition for inducing from H to K involves K/H rather than G/H. To help keep this straight, let γ_1 be a cross section of K over K/H and let γ_2 be a cross section of G over $G/K = T$. Then we can define γ by $\gamma(s) = \gamma_1(s\gamma_2(p(s))^{-1})\gamma_2(p(s))$. This makes sense since $K/H \subseteq G/H$ and γ_1 is defined on that subset, while $s \to s\gamma_2(p(s))^{-1}$ maps S into K/H. Also γ is in fact a section.

Now all the Hilbert spaces in the bundle over T used to define the induced representation U are equivalent to $\mathcal{H}_{t_0} = L^2(K/H;\ \mathcal{K})$, and we can let the induced representation act on $\mathcal{H}_{t_0}$. Also translation by $\gamma_2(t)^{-1}$ combined with a Radon–Nikodym derivative can be used as the unitary equivalence of $\mathcal{H}_t$ onto $\mathcal{H}_{t_0}$ if $t \in T$. We may as well assume the induced representation is strict, and then for $k \in K$ and $f \in \mathcal{H}_{t_0}$, $(U(t_0, k)f)(s) = \varrho(s, k)\beta(s, k)L(\gamma(s) k\gamma(sk)^{-1})f(sk)$ for almost all s in $p^{-1}(t_0) = K/H$. For $s \in p^{-1}(t_0)$, $\gamma_2(p(s)) = e$ so $\gamma(s) = \gamma_1(s)$. Then the formula clearly agrees with the usual formula for inducing [15]. At points other than t_0, we simply get equivalent representations. Thus we get agreement with the previous definition when it applies.

4. Intertwining operators

Suppose (S, μ) is an analytic G-space with finite quasi-invariant measure and let $\mathcal{K}_1$, $\mathcal{K}_2$ be Hilbert spaces. Set $\mathcal{H}_i = L^2(\mu;\ \mathcal{K}_i)$ $i = 1, 2$. Then for $T \in L^\infty(\mu;\ \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2))$ we can define $T^\sim$: $\mathcal{H}_1 \to \mathcal{H}_2$ by $(T^\sim f)(s) = T(s)f(s)$. If P_i is the canonical projection valued measure on $\mathcal{H}_i$ $(i = 1, 2)$, then $T \to T^\sim$ is a Banach space isomorphism of $L^\infty(\mu;\ \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2))$ onto $\mathcal{R}(P_1, P_2)$, by direct integral theory. It is a $*$-algebra isomorphism if $\mathcal{K}_1 = \mathcal{K}_2$. The following lemma is a convenient variation on Theorem 10.8 of [15].

LEMMA 4.1. *Let σ be a 2-cocycle on G and let L_1, L_2 be σ-representations of $S \times G$ on $\mathcal{K}_1$, $\mathcal{K}_2$ respectively. Let $U_1 = \mathrm{ind}\,(G;\ L_1)$ and $U_2 = \mathrm{ind}\,(G;\ L_2)$. Then $T \to T^\sim$ is a Banach space isomorphism of $\mathcal{R}(L_1, L_2)$ onto $\mathcal{R}(P_1, P_2) \cap \mathcal{R}(U_1, U_2)$, taking equivalences to equivalences.*

Proof. If $T \in \mathcal{R}(L_1, L_2)$, by definition we have $T(s)L_1(s, x) = L_2(s, x)\,T(sx)$ for almost all (s, x). The set of pairs for which equality holds is closed under multiplication and hence contains an inessential contraction, say to the conull Borel set S_0. Let $F_0 = (S \times G)\,|\,S_0 = \{(s, x)\colon s \text{ and } sx \in S_0\}$. Then for any x in G, (s, x) is in F_0 whenever $s \in S_0 \cap S_0 x^{-1}$ which is almost all s. Let $\varrho\colon S \times G \to (0, \infty)$ be Borel for fixed x and give the factor needed to make unitary operators out of translations. Then if $x \in G$ and $f \in \mathcal{H}_1$ we have, for almost all s,

$$(T^{\sim} U_1(x) f)(s) = T(s)(U_1(x) f)(s)$$
$$= \varrho(s, x)\,T(s)\,L_1(s, x) f(sx) = \varrho(s, x) L_2(s, x)\,T(sx) f(sx) = (U_2(x)\,T^{\sim} f)(s).$$

Conversely, if $T^{\sim} U_1(x) = U_2(x)\,T^{\sim}$ for all x in G, then for each x, $T(s)L_1(s, x) = L_2(s, x)\,T(sx)$ for almost all s, so $T \in \mathcal{R}(L_1, L_2)$. Now $T^{\sim}$ is unitary iff $T(s)$ is unitary for almost all s, so all that was claimed is true.

5. Inducing in stages

Here we prove that induced representations can be formed in several steps or in one, with equivalent results. This will be useful in section 10, and also has a corollary on inducing direct integrals. This generalizes Theorem 4.1 of [5].

THEOREM 5.1. *Let S, T, U be analytic Borel G-spaces for a locally compact group G, with quasi invariant measures λ, μ, ν respectively and suppose $p\colon S \to T$ and $q\colon T \to U$ are equivariant, Borel, and onto, with $p_*(\lambda) \sim \mu$ and $q_*(\mu) \sim \nu$. Then $r = q \circ p$ is equivariant from S onto U and $r_*(\lambda) \sim \nu$. If σ is a cocycle on $U \times G$ and R is a σ-representation of $S \times G$, then* $\operatorname{ind}(U \times G, S \times G;\, R) \cong \operatorname{ind}(U \times G, T \times G;\, \operatorname{ind}(T \times G;\, S \times G;\, R))$.

Proof. We may as well suppose $p_*(\lambda) = \mu$ and $q_*(\mu) = \nu$. Let $\lambda = \int \lambda(p, t)\,d\mu(t)$ and $\mu = \int \mu(q, u)\,d\nu(u)$ be decompositions of λ relative to p and μ relative to q, respectively, such $\lambda(p, t)x \sim \lambda(p, tx)$ for $(t, x) \in T \times G$ and $\mu(q, u)x \sim \mu(q, ux)$ for $(u, x) \in U \times G$. Also suppose that all the measures are probabilities (discard invariant null set if necessary). Then define $\lambda(r, u) = \int \lambda(p, t)\,d(\mu(q, u))(t)$ for $u \in U$. This gives a decomposition of λ relative to r. The quasi-invariance of the $\lambda(p, t)$'s and the $\mu(q, u)$'s combines to quarantee that $\lambda(r, u)x \sim \lambda(r, ux)$ always holds, by applying Lemma 1.2 of [16].

Let $M_1 = \operatorname{Ind}(U \times G, S \times G;\, R)$, let $M = \operatorname{Ind}(T \times G, S \times G;\, R)$ and let $M_2 = \operatorname{Ind}(U \times G, T \times G;\, M)$. Then M_1 acts in $U * r_*(\mathcal{K})$ and M_2 acts in $U * q_*(p_*(\mathcal{K}))$. According to Theorem 1.1, these bundles are equivalent. The equivalence arises from the mapping V taking a bounded section f of $S * \mathcal{H} = S \times \mathcal{K}$ to the bounded section Vf of $T * p_*(\mathcal{H})$, where $(Vf)(t)$ is the class of f in $p_*(\mathcal{H})(t) = L^2(\lambda(p, t);\, \mathcal{K})$.

Now if f is a bounded section of $S*\mathcal{H}$, $(M_1(u, x)f)(s)$ is a positive multiple of $R(s, x)f(sx)$, since it is $R(s, x)(W_1(u, x)f)(s)$. Let $\varrho_1(s; u, x)$ be the positive multiplier. Likewise for some $\varrho(s; t, x)$ we have $(M(t, x)f)(s)=\varrho(s; t, x)R(s, x)f(sx)$ and for some $\varrho_2(t; u, x)$ we have $(M_2(u, x)g)(t)=\varrho_2(t; u, x)M(t, x)g(tx)$. If $g=Vf$ for a bounded section f of $S*\mathcal{H}$, $((M_2(u, x)Vf)(t))(s)=\varrho_2(t; u, x)\varrho(s; t, x)R(s, x)f(sx)$, while $((VM_1(u, x)f)(t))(s)=\varrho_1(s; u, x)R(s, x)f(sx)$. Since $V^{-1}M_2(u, x)V$ and $M_1(u, x)$ are both unitary from $r_*(\mathcal{H})(ux)$ to $r_*(\mathcal{H})(u)$, each obtained by composing with translation by x, and multiplying by $R(s, x)$ and then by a positive function, the positive functions must agree a.e. relative to $\lambda(r, u)$. and hence $M_1(u, x)=V^{-1}M_2(u, x)V$.

COROLLARY 5.2. *Let (S, λ) and (T, μ) be analytic G-spaces with quasi-invariant finite measures and suppose p: $S\to T$ is Borel onto and equivariant with $p_*(\lambda)\sim\mu$. Let σ be a 2-cocycle on $T\times G$. Suppose U is an analytic space and L is a Borel function from $U\times S\times G$ to $\mathcal{U}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$, such that $L(u; \cdot, \cdot)$ is a σ-representation of $S\times G$ for each $u\in U$. Let ν be a finite measure on U and define $(L_1(s, x)f)(u)=L(u, s, x)f(u)$ for $f\in L^2(\nu; \mathcal{K})$. Let M be defined on $U\times T\times G$ by $M(u; t, x)=\operatorname{ind}(T\times G, S\times G; L(u; \cdot, \cdot))$ and define $M_1=\operatorname{ind}(T\times G, S\times G; L_1)$. If $(M_1'(t, x)f)(u)=M(u; t, x)f(u)$ for $f\in L^2(\nu; \mathcal{H})$ where $\mathcal{H}(M(u; \cdot, \cdot))$ for $u\in U$, then M_1 and M_1' are equivalent.*

Proof. We let G act trivially on U. Then $U\times S$ and $U\times T$ are G-spaces. The projection of $U\times S$ onto S is equivariant, L is a representation of $U\times S\times G$, $L_1=\operatorname{ind}(S\times G, U\times S\times G; L)$, and M_1 is the representation of $T\times G$ induced in stages $U\times S\to S$ and $S\to T$. But M_1' is induced via $U\times S\to U\times T$ and then $U\times T\to T$.

It should be mentioned that the transitivity of inducing seems not to imply that the representation of G induced by the regular representation of a virtual subgroup is the regular representation. Of course there may be a question about what the regular representation of a virtual subgroup $S\times G$ should be. One natural possibility is to take the bundle over S which is $S\times L^2(G)$. This is the same as the bundle induced over S by the decomposition of the measure on $S\times G$ relative to the 'range' mapping, since $r(s, x)=(s, e)$. Now $S\times G$ has a left action on $S\times L^2(G)$, and that bundle representation is a candidate for the regular representation. If S has only one element, it is the regular representation of G. However, if we induce this representation the result is a multiple of the regular representation, $I\otimes L$, acting on $L^2(S)\otimes L^2(G)$. Now the measure on $S\times G$ can also be decomposed relative to the mapping $(s, x)\to(s, sx)$ of $S\times G$ onto $E\subseteq S\times S$, and a bundle representation of $S\times G$ can be given on a bundle over E. However if G acts freely then the corresponding representation is trivial because the measures in the decomposition are point masses and the only translation-generated unitary between such L^2 spaces is the trivial one taking 1 to 1. Thus the induced representation is just the representation on $L^2(S)$ given by the action of G on S. If S is the circle and Z acts on S by an irrational rotation, the result is not the regular representation of Z. It is not clear what other spaces might be tried. Perhaps one should always deal with an infinite multiple of the regular representation. Then the first attempt behaves well.

6. The subgroup theorem

Although we will not be using the information in the rest of the paper, we want to include a discussion of the subgroup theorem for virtual subgroups because it is a useful part of the apparatus for dealing with induced representations. If H and K are closed subgroups of G, L is a representation of H, and $U = \mathrm{ind}\,(G, H; L)\,|\,K$, the theorem gives a decomposition of U over the H: K double coset space, assuming that the image of the Haar measure class is standard [5, Theorem 12.1]. The integrands are induced from subgroups of K conjugate to subgroups of H. We shall see that the assumption that the image of the Haar measure class be standard is not necessary if virtual subgroups are allowed into the process [11, page 62], though it may happen that no decomposition of the representation occurs. On the other hand the theorem does not always even have meaning in its usual form when applied to virtual subgroups, because it can happen that no subobject of H can be imbedded in K. We can think of several special cases: two subgroups not necessarily regularly related, inducing from a virtual subgroup and restricting to a subgroup, inducing from a subgroup and restricting to a virtual subgroup, and two virtual subgroups arising from ergodic actions with finite invariant measures. The first two can be handled together, but some remarks on the first case are in order to begin with.

Let S and T be the right coset spaces of H and K respectively. Then $S \times T$ is also a G-space and the function taking (Ha, Kb) to $Hab^{-1}K$ induces a Borel isomorphism of the orbit space $(S \times T)/G$ onto the double coset space. Of course the double coset space is naturally taken to be the orbit space of S under the action of K, i.e. S/K. If $t_0 \in T$ is the identity coset of K, $S \times \{t_0\}$ meets each G-orbit in $S \times T$, and the orbit $(s, t_0)G$ intersects $S \times \{t_0\}$ in $(sK) \times \{t_0\}$. This is another way to see that $(S \times T)/G$ and S/K are the same. The space $(S \times T)/G$ is what we must consider in general, but for purposes of inducing from an ordinary or virtual subgtoup and restricting to a subgroup, the space S/K is very convenient. Notice that the following theorem generalizes Theorem 12.1 of [5] by allowing subgroups which are not regularly related, through the use of virtual subgroups of K. We let m be a finite measure in the class of Haar measure on G.

THEOREM 6.1. *Let (S, λ) be an analytic G-space with finite quasi-invariant measure, and let L be a σ-representation of $(S \times G, [\lambda \times m])$ for some 2-cocycle σ on G. Set $M_1 =$*

ind $(G, S \times G; L)$. *Suppose K is a closed subgroup of G and let $M = M_1 | K$. Then there is an analytic Borel space Z and a projection q of S onto Z such that $q^{-1}(z)$ is invariant under K for each z in Z and if $\lambda = \int \lambda_z dq_*(\lambda)(z)$ is the decomposition of λ relative to q, then almost every λ_z is ergodic under K. Furthermore, $L | q^{-1}(z) \times K$ is a σ-representation of $(q^{-1}(z) \times K, [\lambda_z \times m'])$ for almost every z ($m' \sim$ Haar measure on K), which we denote by L^z. Then $M = \int^{\oplus}$ ind $(K, (q^{-1}(z) \times K, [\lambda_z \times m']); L^z) dq_*(\lambda)(z)$.*

Proof. The existence of Z and q can be gotten from Theorem 4.3 of [4]. If L is strict on $(S \times G) | S_0$ for some conull Borel set, then S_0 is conull for almost every λ_z, say for z in the conull Borel set Z_0, and then L is a σ-representation on $(q^{-1}(z) \times K, [\lambda_z \times m'])$ for $z \in Z_0$. Let $\mathcal{K}$ be the Hilbert space of L, and set $M^z = \text{ind}\,(K; L^z)$. Then $\mathcal{H}(M^z) = L^2(\lambda_z; \mathcal{K})$ and $\mathcal{H}(M) = L^2(\lambda; \mathcal{K}) \cong \int^{\oplus} L^2(\lambda_z, \mathcal{K}) dq_*(\lambda)(z)$. Now $M(t, k)$ and the $M^z(t, k)$ are all given by composing with translation by k and multiplying by a scalar to get unitarity and then multiplying by $L(\cdot, k)$. The positive multiplier needed for M will suffice for almost every M^z. Hence $M = \int^{\oplus} M^z dq_*(\lambda)(z)$.

Remarks. (a) This gives M as a direct integral of representations induced from virtual subgroups of K. Examples such as compact K show that these need not be virtual subgroups of $(S \times G, [\lambda \times m])$ if (S, λ) is a properly ergodic G-space. Thus part of the effect of the original theorem is lost, but it does not seem to be an essential part.

(b) If S is a coset space and S/K is analytic (or even standard for the quotient measure class) then almost every λ_z is carried on an orbit: a cross-section of S over a conull standard set allows one to prove that such is the case, by a standard argument. In that case $q^{-1}(z) \times K$ is similar to the stabilizer in K of any point of $q^{-1}(z)$. If the orbit is the orbit of Ha, one such subgroup will be $K \cap a^{-1}Ha$. This returns us to the same conclusion as in Theorem 12.1 of [5].

For the remainder of this section we let (S, λ) and (T, μ) be analytic G-spaces with ergodic probability measures and let m be a finite element of the Haar measure class on G. Let π: $S \times G \to G$ and ω: $T \times G \to G$ be the projections, i.e. inclusion homomorphisms, thinking of $(S \times G, [\mu \times m])$ and $(T \times G, [\mu \times m])$ as subobjects of G. Let $U = S \times T$, $\nu = \lambda \times \mu$ and let p: $U \to S$, q: $U \to T$ be the projections. Then $\varphi = p \times i$ and $\psi = q \times i$ are 'inclusion homomorphisms' of $(U \times G, [\nu \times m])$ into $(S \times G, [\lambda \times m])$ and $(T \times G, [\mu \times m])$, though ν need not be ergodic. Suppose σ is a 2-cocycle on G and let L be a σ-representation of $S \times G$ on the Hilbert space $\mathcal{K}$. Let $M_1 = \text{ind}\,(G, S \times G; L)$ and $M = M_1 | T \times G = M_1 \circ \omega$. Then $\mathcal{H}(M) = \mathcal{H}(M_1) = L^2(\lambda; \mathcal{K})$. Our goal is to find a decomposition of $S \times T$ which generalizes the decomposition into orbits corresponding to double cosets, and which also gives rise to a decomposition of M. The most natural decomposition of $S \times T$ is the one into ergodic parts.

First, let $L' = L\circ\varphi$ and consider $M' = \operatorname{Ind}(T\times G, U\times G; L')$, the induced bundle σ-representation. Now $\nu = \int \lambda\times\varepsilon_t d\mu(t)$ is a decomposition of ν relative to q, where ε_t represents the unit point mass at t for $t\in T$. Also, $(\lambda\times\varepsilon_t)x = \lambda x\times\varepsilon_{tx}$ always and $s\to(s, t)$ induces an isomorphism of $L^2(\lambda\times\varepsilon_t; \mathcal{H})$ onto $L^2(\lambda; \mathcal{H})$. It follows that the non-bundle version of M' can be taken to be $\operatorname{ind}(T\times G, U\times G; L')$ as acting on $L^2(\lambda; \mathcal{H})$. If $\varrho(s, x) = ((d\lambda/d\lambda x)(sx))^{1/2}$ then for $f\in L^2(\lambda; \mathcal{H})$, $(M'(t, x)f)(s) = \varrho(s, x)L'(s, t, x)f(sx) = \varrho(s, x)L(s, x)f(sx)$ for almost all s. This is the same formula as for M, so $M'\cong M$. Thus we have M a representation induced from the common subobject $U\times G$ of $S\times G$ and $T\times G$, which is close to the spirit of the original subgroup theorem.

It is not difficult to show that if A and $U\setminus A$ are invariant Borel sets of positive ν measure then M is a direct sum of representations induced from $A\times G$ and $(U\setminus A)\times G$. The difficulty is in passing to the continuous version of this. For example, let $S = \mathbf{Z} = G$, T = the unit circle and let G act on T by an irrational rotation. Then each orbit in $S\times T$ meets $\{0\}\times T$ exactly once, so $S\times T$ has T as its orbit space, and this is its ergodic decomposition. Now the fiber measures are carried on orbits and hence are discrete. Thus the projection of a fiber measure onto T is not equivalent to the measure on T, and we cannot induce from (fiber) $\times G$ to $T\times G$. This example illustrates the need for the hypothesis in the following theorem.

THEOREM 6.2. *Suppose $r\colon U\to Z$, $\zeta = r_*(\nu)$, $\nu = \int \nu_t d\zeta(z)$ is the ergodic decomposition of ν under the action of G, and suppose that $q_*(\nu_z)\sim\mu$ for ζ-almost all z. Then for z in a conull Borel set $Z_0\subseteq Z$, $L^z = (L\circ\varphi)\,|\,r^{-1}(z)\times G$ is σ-representation. If we set $M^z = \operatorname{ind}(T\times G, r^{-1}(z)\times G; L^z)$ for $z\in Z_0$, then $M = \int^{\oplus} M^z d\zeta(z)$.*

Remarks. This gives M as a direct integral of representations obtained by inducing from virtual subgroups of $T\times G$. These would be similar to virtual subgroups of S if the measures $p_*(\nu_z)$ were not carried by negligible sets in S (ones whose saturation is λ-null). However, interchanging S and T in the example preceding the theorem shows this need not be the case. Just the same, this seems to be a reasonable generalization of the subgroup theorem, so we shall say the subgroup theorem holds for $S\times G$ and $T\times G$ if the hypotheses of Theorem 6.2 are satisfied. Notice that if μ is not ergodic it could never hold; even the discrete summand result for an invariant set $A\subseteq U$ would fail. Also, notice that this proof works for any decomposition of ν into quasi invariant measures.

Proof of theorem. Form $Z\times T$ and let p_1, p_2 be the coordinate projections onto Z and T respectively. Let $\nu' = (r, q)_*(\nu)$. Then $p_{1*}(\nu') = \zeta$ and $p_{2*}(\nu') = \mu$. Let $\nu = \int \nu_{z,t} d\nu'(z, t)$ be a decomposition of ν relative to (r, q), and suppose $\nu_{z,t}x\sim\nu_{z,tx}$ for all z, t, x. Now $(r, q)_*(\nu_z) =$

$\varepsilon_z \times q_*(\nu_z) \sim \varepsilon_z \times \mu$ for almost all z, so $(r, q)_*(\nu) \sim \zeta \times \mu$. By replacing ν by an equivalent measure we may arrange $(r, q)_*(\nu) = \zeta \times \mu$. Let $M_2 = \operatorname{Ind}(Z \times T \times G, U \times G; L \circ \varphi)$. Then $(M_2(z, t, x)f)(u) = \varrho(u; z, t, x)L(p(u), x)f(ux)$ for $f \in L^2(\nu_{z,tx}; \mathcal{K})$, where ϱ is a positive function. Thus $M_2(z, \cdot, \cdot) \cong \operatorname{Ind}(T \times G, U \times G; L^z) \cong M^z$. Also, it is clear that $\operatorname{ind}(T \times G, Z \times T \times G; M_2) = \int^{\oplus} M_2(z, \cdot, \cdot)d\zeta(z)$ since the action of G on Z is trivial. By the theorem on inducing in stages the proof is complete.

COROLLARY 6.3. *If T is transitive then the subgroup theorem holds for (S, λ) and (T, μ).*

Proof. The measures $q_*(\nu_z)$ are quasi-invariant and T has only one class of quasi-invariant measures so $q_*(\nu_z) \sim \mu$.

COROLLARY 6.4. *Suppose (S, λ) and (T, μ) are analytic G-spaces with invariant ergodic probability measures. Then the subgroup theorem holds for $(S \times G, [\lambda \times m])$ and $(T \times G, [\mu \times m])$.*

Proof. Return to $\nu = \lambda \times \mu$, $\nu = \int \nu_z d\zeta(z)$ as before. Then for E Borel in Z, $\int_E q_*(\nu_z)d\zeta(z)$ is a measure whose value at a Borel set $A \subseteq T$ is $\nu(r^{-1}(E) \cap q^{-1}(A))$. This is an invariant measure of total measure $\zeta(E)$ and is $\ll \mu$ so it is $\zeta(E)\mu$. Thus the function $z \to q_*(\nu_z)(A)$ has the value $\mu(A)$ almost everywhere on Z. Letting A vary over a countable generating algebra we see that $q_*(\nu_z) = \mu$ for almost all z.

7. Connecting cocycle representations with ordinary ones

The method used to reduce some questions about cocycle representations of groups to questions about ordinary representations [9, section 2] also is helpful in treating cocycle representations of groupoids. Let $(G, [\mu])$ be a measurable groupoid and let U be its set of units, with $\tilde{\mu}$ the measure induced on U. Suppose σ is a strict cocycle with values in the circle, T, and set $G^\sigma = G \times T$. Let $G^{\sigma(2)} = \{((x, s), (y, t)) \colon (x, y) \in G^{(2)}$ and $(s, t) \in T^2\}$ and for a pair in $G^{\sigma(2)}$ define the product to be $(xy, \sigma(x, y)^{-1}st)$. The cocycle condition enables the associative law to hold, the units in G^σ are the elements of $U \times \{1\}$, $(x, s)^{-1} = (x^{-1}, \sigma(x^{-1}, x)\, s^{-1})$ and the set $d^{-1}(u, 1)$ of elements of G^σ with unit $(u, 1)$ is $d^{-1}(u) \times T$. Let ν be Haar measure on T and let $\mu = \int \mu_u d\tilde{\mu}(u)$ be a decomposition of μ relative to d. Then $\mu \times \nu = \int (\mu_u \times \nu)d\tilde{\mu}(u)$ can be identified with the decomposition of $\mu \times \nu$ relative to d, by noticing that $\mu_u \times \nu$ is concentrated on $d^{-1}(u, 1) = d^{-1}(u) \times T$. By working with rectangles it is not hard to see that for $(x, t) \in G^\sigma$, $(\mu_{r(x)} \times \nu) \cdot (x, t) = (\mu_{r(x)} \cdot x) \times \nu$. If G_0 is an inessential contraction (i.c.) of G such that $\mu_{r(x)} \cdot x \sim \mu_{d(x)}$ for $x \in G_0$, then $G_0^\sigma = G_0 \times T$ is an i.c. of G^σ such that $(\mu_{r(x)} \times \nu) \cdot (x, t) \sim \mu_{d(x)} \times \nu$ for $(x, t) \in G_0^\sigma$. Hence $\mu \times \nu$ is right quasi-invariant. Suppose μ is symmetric, i.e. $\mu(\{x^{-1} \colon x \in A\})$ is always $\mu(A)$, which we may do if $[\mu]$ is symmetric. Then since ν is both symmetric and invariant under either right or left translations we can look at rectangles to see that $\mu \times \nu$ is also symmetric. Hence $(G^\sigma, [\mu \times \nu])$ is a measurable groupoid. Now G and G^σ induce the same equivalence relation on U, so if $(G, [\mu])$ is ergodic so is $(G^\sigma, [\mu \times \nu])$. For groupoids we do not require a locally compact topology, so the construction of G^σ is easier than for groups.

The relationship of σ-representations of G with some of the ordinary representations of G^σ goes as for groups [9, section 2]. If R is a σ-representation of G define R^0 on G^σ by $R^0(x, t) = tR(x)$. Then R^0 is an ordinary representation of G^σ on $\mathcal{K}$, $R^0(u, t) = tI$ for units u in G and elements $t \in T$ and $R \to R^0$ is one–one from the set of σ-representations of G on $\mathcal{K}$ onto the set of ordinary representations S of G^σ on $\mathcal{K}$ such that $S(u, t) = tI$ for units u in G and elements $t \in T$. This map preserves equivalence and multiplicity theory.

To see that there always are σ-representations, we imitate the σ-regular representation for groups. For each y in G choose a positive Borel function ϱ_y such that $(W_1(y)f)(x) = \varrho_y(x)\, f(xy)$ defines a unitary operator from $L^2(\mu_{d(y)})$ to $L^2(\mu_{r(y)})$. As in the proof of Proposition 3.4 we see that W_1 is a bundle representation of G on $G^{(0)} * \mathcal{H}$ where $\mathcal{H}(u) = L^2(\mu_u)$ for $u \in G^{(0)}$. Now define $W_2(y)$ by $(W_2(y)f)(x) = \sigma(x, y)^{-1}(W_1(y)f)(x)$. This is still a Borel function and $W_2(yz) = \sigma(y, z)\, W_2(y)\, W_2(z)$ for all (y, z) in $G^{(2)}$, by a straightforward calculation. Next choose a Hilbert space $\mathcal{K}$ of the dimension of all the $L^2(\mu_u)$'s and unitary operators $V(u)\colon L^2(\mu_u) \to \mathcal{K}$ so that V is a Borel function making our bundle isomorphic to $U \times \mathcal{K}$. Define $W(y) = V(r(y))\, W_2(y)\, V(d(y))^{-1}$. W is a σ-representation W of G (see section 5 and Lemma 10.9 of [15]).

The cases of interest to us in this paper concern groupoids $(S \times G, [\mu \times \nu])$ where (S, μ) is an ergodic analytic G-space. In that case it may be done more simply as follows. Let $\mathcal{K} = L^2(G)$ with left Haar measure on G and define $(W(s, x)f)(y) = \sigma((s, y)^{-1}, (s, x))^{-1} f(x^{-1}y)$ for $(s, x) \in S \times G$ and $y \in G$. Notice that $(s, y)^{-1}(s, x)$ is defined, so the formula is meaningful. We have the left regular representation of G followed by a multiplication operator (depending on s), so it is a unitary operator. Clearly, W is a Borel function, and again a straightforward calculation shows W is a σ-representation, using the cocycle property of σ.

8. Measures on $\hat{N}^\sigma$ and the action of G

Let G be a (second countable) locally compact group and let σ be a 2-cocycle on G. If N is a normal subgroup of G, we also refer to σ-representations of N, when we actually mean $\sigma \mid N \times N$ representations. Choosing one concrete Hilbert space of each dimension $\leqslant \aleph_0$, we form the space $N^{c,\sigma}$ of concrete σ-representations of N, with its usual Borel structure [9, section 3]. We let $N^{i,\sigma}$ denote the subspace of irreducible σ-representations, and $\hat{N}^\sigma(N^{r,\sigma})$ denotes the quotient space of $N^{i,\sigma}(N^{c,\sigma})$ modulo unitary equivalence, with p the canonical projection.

For L in $N^{c,\sigma}$ and x in G, define the σ-representation L^x by $L^x(y)=\beta(x, y)L(xyx^{-1})$, where $\beta(x, y)=\sigma(x^{-1}, x)\sigma(xy, x^{-1})^{-1}\sigma(x, y)^{-1}$. This defines a Borel action of G on $N^{c,\sigma}$ which preserves $N^{i,\sigma}$ and also preserves equivalence, so there is an induced Borel action on $N^{r,\sigma}$ and on $\hat{N}^\sigma$ which we denote the same way [9; sections 4 and 7], and p is then equivariant. If $L=M|N$ where M is a σ-representation of G, then for each x in G the operator M_x is an equivalence of L with L^x. The equivalence class of L is then invariant under G as a point of $N^{r,\sigma}$. In the same way, the action of N on $N^{r,\sigma}$ and $\hat{N}^\sigma$ is trivial, so these are in fact also G/N spaces.

The method for analyzing $\hat{G}^\sigma$ in terms of $\hat{N}^\sigma$ and the action of G on $\hat{N}^\sigma$ as developed by Mackey in [9] depended on $\hat{N}^\sigma$ being both smooth and of type I. It is known that these conditions are equivalent [2, 3], and we still need that condition eventually. Mackey's results also depended on another assumption, which his theory of virtual groups was intended to remove. It comes about as follows: If U is a primary σ-representation of G then $U|N$ is a multiple of a representation of the form $\int L d\mu(L)$, and the measure μ is quasi-invariant and ergodic for the action of G on $\hat{N}^\sigma$ [9, Theorem 7.6]. The results of section 8 of Mackey's paper deal with the case that μ is carried by a single orbit of the action of G (the transitive quasi-orbit case), and makes use of the closed subgroup consisting of the elements of G which fix a particular point in that orbit. Our purpose is to show how these results extend if nontransitive quasi-orbits are allowed.

Because no extra effort is required and in fact the proof of Theorem 9.2 is simplified, we will not use the full strength of the type I assumption at first. Instead we shall work with type I measure classes as we now define them, noting that all measure classes on a smooth dual are type I.

Definition 8.1. If σ is a cocycle on a locally compact group K, a measure μ (or the measure class $[\mu]$) on K^σ will be called type I if it is standard and $\int L d\mu(L)$ is type I.

Recall that $[\mu]$ is standard if there is a standard set $B=B[\mu]$ whose complement is of measure zero. In that case $p|p^{-1}(B)$ has a measurable cross section γ by the von Neumann selection lemma, and there is a conull Borel set $B_0 \subseteq B$ on which γ is Borel. Then $\int_B L d\mu(L)$ is defined to be the equivalence class of $\int_{B_0} \gamma(L) d\mu(L)$, which depends only on $[\mu]$. If this is type I it is automatically multiplicity free, and the direct integral is its central decomposition [7, chapter 2].

Here is a general fact which will help us:

LEMMA 8.2. *Let S be a Borel space and suppose a locally compact group G has a Borel action, a, on S (a: $S \times G \to S$ is Borel; we write sx for $a(s, x)$). Let μ be a finite quasi-invariant Borel measure on S and suppose there is a conull Borel subset S_0 which is analytic as a Borel space. Then there is a conull invariant subset S_1 which is analytic as a Borel space.*

Proof. First of all, notice that the set S_0 may be assumed to be standard since an analytic space is metrically standard. Next, let ν be a probability measure in the class of Haar measure on G. Then $a^{-1}(S_0)$ is Borel in $S \times G$ so the measure $\nu(a^{-1}(S_0)_s)$ of the s-section of $a^{-1}(S_0)$ is a Borel function of s. Thus the set, S_0^*, of s in S_0 for which $\nu(a^{-1}(S_0)_s) = 1$, is a Borel set. For each x in G, the x-section of $a^{-1}(S_0)$ is $S_0 x^{-1}$ which is μ-conull since μ is quasi-invariant. Thus $a^{-1}(S_0)$ is conull, and hence S_0^* is conull. For each s in S, a_s: $x \to sx$ is a Borel map of G into S and $S_0^* = \{s \in S_0 : a_s^{-1}(S_0) \text{ is } \nu\text{-conull}\}$. If $y \in G$ then $a_{sy}(x) \in S_0$ iff $syx \in S_0$ iff $a_s(yx) \in S_0$, so $a_{sy}^{-1}(S_0) = y^{-1} a_s^{-1}(S_0)$. Hence if $s \in S_0^*$ and $sy \in S_0$, then $sy \in S_0^*$. It follows that if we apply the same procedure to S_0^* we find that $(S_0^*)^* = S_0^*$. By replacing S_0 by S_0^*, we may suppose $S_0^* = S_0$.

Now let $S_1 = a(S_0 \times G)$. Since S_1 is the Borel image of a standard space, it will be analytic if it is countably separated. Let α be a Borel isomorphism of S_0 with a Borel subset in $[0, 1]$ and extend α to have the value 0 on $S_1 \backslash S_0$. Then α is Borel from S_1 to $[0, 1]$. Then as in the proof of Lemma 2 of [10] or Lemma 3.2 of [15], we can define ψ: $S_1 \to L^2_{\text{loc}}(G)$ by letting $\psi(s)$ be the equivalence class of the function whose value at x is $\alpha(sx)$. ψ is Borel, and if $s_1 \neq s_2$ in S_1, then $\alpha(s_1 x) \neq \alpha(s_2 x)$ whenever $s_1 x$ and $s_2 x$ are in S_0 which happens for ν-almost all x. Thus ψ is one–one and since $L^2_{\text{loc}}(G)$ is standard, we see that S_1 is countably separated, as desired.

COROLLARY 3.3. *If an invariant measure class C on $\hat{N}^\sigma$ is carried by a standard Borel subset (C is a standard measure class), then C is carried by an analytic invariant set in $\hat{N}^\sigma$.*

Now let us reformulate two theorems from [9] to suit our purposes. The first is Mackey's Theorem 7.4.

THEOREM 8.4. *Let N be a closed normal subgroup of the locally compact group G and let σ be a 2-cocycle on G. Let $[\mu]$ be a type* I *measure class in $\hat{N}^\sigma$. Then $\int L d\mu(L)$ is an invariant σ-representation of N iff $[\mu]$ is invariant under the canonical action of G on $\hat{N}^\sigma$. If $[\mu]$ is invariant, $\int L d\mu(L)$ is ergodic iff $[\mu]$ is ergodic.*

Proof. Let B be a standard set in $\hat{N}^\sigma$ which supports μ. Then Bx supports the transformed measure $\mu \cdot x$, and if γ is a cross-section over B then $L \to \gamma(L^{x^{-1}})^x = \gamma_x(L)$ is a cross-section over Bx. Now for $L \in B$, $\gamma(L)^x = \gamma_x(L^x)$, so $(\int_B \gamma(L) d\mu(L))^x$ is equivalent to

$\int_{Bx} \gamma_x(L)\, d(\mu x)(L)$, i.e. $(\int L d\mu(L))^x = \int L d(\mu\cdot x)(L)$. Hence $[\mu\cdot x]=[\mu]^x$ is also type I, and so is $[\mu+\mu\cdot x]$. Except for this obvious point, Mackey's proof also proves our formulation.

THEOREM 8.5. (Theorem 7.6 of [9]) *Let M be a σ-representation of the locally compact group G and let N be a closed normal subgroup of G. Suppose the restriction of M to N, $M|N$, is equivalent to $(\int_B \gamma(L)\, d\mu(L))\otimes I_0$, where I_0 is the identity on a Hilbert space $\mathcal{K}_0$, μ is a type* **I** *measure carried on the standard set B, and γ is a cross-section over B. Let P be the canonical projection valued measure on $\int \mathcal{H}(\gamma(L))\otimes\mathcal{K}_0 d\mu(L)$. Then P is a system of imprimitivity for M based on $\hat{N}^\sigma$ and is ergodic if M is primary.*

Proof. We may assume $B\subseteq X$ where X is analytic and invariant, and regard P as based on X. Using the fact that $[\mu]$ is type I, Mackey's proof works.

9. The σ-representations associated with quasiorbits

The purpose of this section is to extend the results of section 8 of [9] to nontransitive quasi-orbits. Throughout this section we fix a locally compact group G, a 2-cocycle σ on G, a probability measure ν on G equivalent to Haar measure and a closed normal subgroup N of G. If a representation M satisfies the hypotheses of Theorem 8.5 and μ is ergodic, we say M is associated with the quasi-orbit $[\mu]$. Before getting into the theorems and their proofs, notice that Theorem 7.6 of [9] shows that if $\hat{N}^\sigma$ is type I then every primary σ-representation of G is associated with some quasi-orbit. Also when $\hat{N}^\sigma$ is type I, every measure class on $\hat{N}^\sigma$ is type I.

THEOREM 9.1. (cf. Theorem 8.1 of [9]) *Let $[\mu]$ be a type* I *quasi-orbit in $\hat{N}^\sigma$. Then $[\mu]$ is carried by an analytic invariant set $X\subseteq\hat{N}^\sigma$ over which there is a measurable cross-section γ and the function taking M to* ind $(G; M)$ *maps those σ-representations M of $(X\times G, [\mu\times\nu])$ for which $M(L,\cdot)|N$ is almost always (equivalent to) a multiple of $\gamma(L)$ to σ-representations of G associated with the quasi-orbit $[\mu]$. This mapping induces a bijection of equivalence classes and preserves multiplicity, i.e. $\mathcal{R}(M, M)\cong\mathcal{R}$* (ind (G, M), ind (G, M)).

Proof. The existence of X is given by Corollary 3.3. Let us show that the map is onto at the equivalence class level.

Suppose U is a σ-representation of G whose restriction to N has the quasi-orbit $[\mu]$. If $V=U|N$, this means V is equivalent to a multiple of $\int^{\oplus}\gamma(L)\, d\mu(L)$ [9, Theorem 7.6]. Now $\gamma(L)$ and $\gamma(L)^x$ always have the same Hilbert space, and the latter is that of $\gamma(L^x)$. Thus $L\to\mathcal{H}(\gamma(L))$ is a measurable function from $\hat{N}^\sigma$ to a discrete set, which is constant on G orbits, so it is constant on some G-invariant Borel set $E\subseteq X$ which is conull relative to

M. Thus, we may as well suppose that $\mathcal{H}(\gamma(L))$ is constant on X. Then taking $\mathcal{K}=\mathcal{H}(\gamma(L))$ and $\mathcal{K}_0$ to be a space whose dimension is the same as the multiplicity of $\int^{\oplus}\gamma(L)d\mu(L)$ in V, we may replace U by a unitarily equivalent representation so that $\mathcal{H}(U)=\mathcal{H}(V)$ $[=L^2(\mu;\ \mathcal{K})\otimes\mathcal{K}_0]=L^2(\mu;\ \mathcal{K}\otimes\mathcal{K}_0)$. This can be done in such a way that if $x\in N$ and $f\in\mathcal{H}(V)$, then for μ-almost all $L\in X$,

$$(V(x)f)(L)=(\gamma(L)(x)\otimes I_0)f(L),$$

where I_0 is the identity operator on $\mathcal{K}_0$. Then the canonical projection valued measure on $L^2(X,\mu;\ \mathcal{K}\otimes\mathcal{K}_0)$, P, is a system of imprimitivity for U. It follows from the imprimitivity theorem for virtual subgroups that there is a σ-representation of H on $\mathcal{K}\otimes\mathcal{K}_0$, say M, such that $U=\operatorname{ind}(G, M)$, i.e. such that $(U(x))f(L)=\varrho(L,x)^{1/2}M(L,x)f(L^*)$ for μ-almost all L (section 1). Now if $x\in N$, then $L\in\hat{N}^\sigma$ implies $L^x=L$, so $(V(x)f)(L)=\varrho(L,x)^{1/2}M(L,x)f(L)$. Since N acts trivially on $\hat{N}^\sigma$, we may assume that $\varrho(L,x)=1$ for $x\in N$. Hence if $x\in N$ we have for almost all L, $M(L,x)=\gamma(L)(x)\otimes I_0$. Now $M(L,\ \cdot)$ and $\gamma(L)(\cdot)\otimes I_0$ are σ-representations of N, and hence determined by their values on any countable dense set, so there is one conull Borel set $B\subseteq X$ such that for $(L,x)\in B\times N$, $M(L,x)=\gamma(L)(x)\otimes I_0$. This shows that every U associated with the quasiorbit $[\mu]$ is induced by a representation of the type indicated.

Conversely, suppose M is a σ-representation of H and m is a cardinal number $\leqslant\aleph_0$ such that for almost all L, $M(L,\ \cdot)|N$ is equivalent to $m\gamma(L)$. Taking $\mathcal{K}_0$ of dimension m, we may suppose $\mathcal{H}(M)=\mathcal{K}\otimes\mathcal{K}_0$, with $\mathcal{K}$ as above. Then our assumption is that for almost every $L\in X$ there is a unitary V on $\mathcal{K}\otimes\mathcal{K}_0$ such that for all $x\in N$, $VM(L,x)V^{-1}=\gamma(L)(x)\otimes I_0$. Let $\mathcal{U}=\mathcal{U}(\mathcal{K}\otimes\mathcal{K}_0)$ and notice that for $x\in N$, $\{(N,L)\in\mathcal{U}\times X_0\colon VM(L,x)V^{-1}=\gamma(L)(x)\otimes I_0\}$ is a Borel set, if X_0 is conull and Borel and $\gamma|X_0$ is Borel. If D is countable and dense in N, $\{(V,L)\in\mathcal{U}\times X_0\colon x\in D \text{ implies } VM(L,x)V^{-1}=\gamma(L)(x)\otimes I_0\}=\{(V,L)\in\mathcal{U}\times X_0\colon x\in N \text{ implies } VM(L,x)V^{-1}=\gamma(L)(x)\otimes I_0\}$, so the latter set is Borel. It projects onto a conull set in X_0, by assumption, so there is a Borel function V on X such that for almost all L we have $V(L)M(L,x)V(L)^{-1}=\gamma(L)(x)\otimes I_0$ for all $x\in N$. Now define M_1 by $M_1(L,x)=V(L)M(L,x)V(L^x)^{-1}$ for $(L,x)\in X\times G=H$. Then $M_1\cong M$, so $U_1=\operatorname{ind}(G,M_1)\cong\operatorname{ind}(G,M)=U$. Also $M_1(L,x)=\gamma(L)(x)\otimes I_0$ if $x\in N$, so if $x\in N$ and $f\in L^2(E,\mu;\ \mathcal{K}\otimes\mathcal{K}_0)$, $(U_1(x))f(L)=M_1(L,x)f(L^x)=M_1(L,x)f(L)=(\gamma(L)(x)\otimes I_0)f(L)$. Thus U_1 is associated with the quasiorbit $[\mu]$ and hence so is U.

From Lemma 4.1 we know that if M_1, M_2 are σ-representations of $X\times G$, $U_1=\operatorname{ind}(G,M_1)$, $U_2=\operatorname{ind}(G,M_2)$, and P_1, P_2 are the associated systems of imprimitivity, then $\mathcal{R}(M_1,M_2)$ is isomorphic to $\mathcal{R}(P_1,P_2)\cap\mathcal{R}(U_1,U_2)$. If $T\in\mathcal{R}(U_1,U_2)$, then $T\in\mathcal{R}(U_1|N,\ U_2|N)$. Let $T=WH$ be the polar decompoisition of T: W is a partial isometry and $H=(T^*T)^{1/2}$.

Then $H \in \mathcal{R}(U_1, U_1)$ and W is an equivalence of $V_1 = U^{Q_1}$ with $V_2 = U_1^{Q_2}$ where Q_1 projects onto $\mathcal{N}(T)^\perp$ and Q_2 projects onto $\overline{\mathcal{R}(T)}$. If $H \in \mathcal{R}(P_1, P_1)$ and $W \in \mathcal{R}(P_1, P_2)$ the proof is complete. We have $H \in \mathcal{R}(U_1 | N, U_1 | N)$ and the range of P_1 is contained in the center of this ring so $H \in \mathcal{R}(P_1, P_1)$. We know that $V_1 | N$, $V_2 | N$ are equivalent subrepresentations of type I representations based on μ. (In fact each "is" a multiple of $\int L d\mu(L)$.) Now for Borel $E \subseteq X$, $(U_i | N)^{P_i(E)}$ is the largest subrepresentation of $U_i | N$ disjoint from $\int_{X-E}^{\oplus} \gamma(L) d\mu(L)$ and $P_i(E)$ commutes with Q_i so $(U_i | N)^{Q_i P_i(E)}$ is the largest subrepresentation of $V_i | N$ disjoint from $\int_{X-E}^{\oplus} \gamma(L) d\mu(L)$. It follows that W carries $Q_1 P_1(E) \mathcal{K}_1$ to $Q_2 P_2(E) \mathcal{K}_2$. Since W is an isometry of $Q_1 \mathcal{K}_1$ onto $Q_2 \mathcal{K}_2$, and $(1 - Q_2) W = W(1 - Q_1) = 0$, this implies that $W \in \mathcal{R}(P_1, P_2)$.

THEOREM 9.2. (cf. Theorem 8.2 of [9]) *Let μ and X be as in Theorem 9.1, let γ be a measurable section over X, and set $H = (X \times G, [\mu \times \nu])$. Then there is a cocycle τ on H and a τ-representation M of H such that for μ-almost all L we have $M(L, \cdot) | N = \gamma(L)$. τ may be chosen of the form $\sigma / \omega \circ F^{(2)}$ where ω is a cocycle on $H/N = (X \times (G/N), [\mu \times \nu'])$ and F is the quotient homomorphism of H onto H/N. (Here f: $G \to G/N$ is the quotient homomorphism, $\nu' = f_*(\nu)$ and $F(L, x) = (L, f(x))$.) In that case, the cohomology class of ω is determined by σ and $[\mu]$.*

Proof. First suppose $\sigma = 1$ and consider the question of existence. Following a line of reasoning pointed out privately by L. W. Baggett in the transitive case, we write $M(L, x)$ as a product of $\gamma(L)(n(x))$ ($n(x)$ is the N-component of x) and an operator determined by L and $f(x)$, i.e. by $F(L, x)$. Explicitly, let c: $G/N \to G$ be a cross section and for $x \in G$ let $n(x) = xc(f(x))^{-1}$ so that $x = n(x) c(f(x))$. We find a unitary operator valued function A on H/N such that for $(L, y) \in H/N$ and $x \in N$, $A(L, y) \gamma(L^y)(u) A(L, y)^{-1} = \gamma(L)^{c(y)}(u)$, proceeding as follows (see the proof of Theorem 8.2 of [9]):

Let $\mathcal{E} = \{(L, y, V) \in X \times G/N \times \mathcal{U}$: $u \in N$ implies $V \gamma(L^y)(u) V^{-1} = \gamma(L)^{c(y)}(u)\}$, where $\mathcal{U} = \mathcal{U}(\mathcal{K}) = \mathcal{U}(\mathcal{H}(\gamma(L)))$. Let p denote the projection of $X \times G/N \times \mathcal{U}$ onto $X \times G/N$, and let B be a conull Borel set in X such that $\gamma | B$ is Borel. Let $(H/N)_0 = \{(L, y) \in H/N$: L and $L^y \in B\}$, which is an inessential contraction of H/N. Then the two functions involved in defining $\mathcal{E}$ are Borel on $p^{-1}((H/N)_0)$ for each $u \in N$. As functions of u each is determined by its values on a countable dense set in N. Hence $\mathcal{E} \cap p^{-1}((H/N)_0)$ is a Borel set.

Now for $L \in X$ and $y \in G/N$, the equivalence class of $\gamma(L)^{c(y)}$ is L^y (a similar formula holds for $y \in G$ also). Hence $\gamma(L^y)$ and $\gamma(L)^{c(y)}$ are equivalent. Thus $p(\mathcal{E}) = H/N$ and $p(\mathcal{E} \cap p^{-1}((H/N)_0)) = (H/N)_0$. If S is a Borel set in $\mathcal{U}$ meeting each coset of the scalars exactly once then p maps the Borel set $\mathcal{E} \cap p^{-1}(H/N)_0) \cap (X \times G/N \times S)$ one–one onto $(H/N)_0$. Define A to be the inverse of the latter function on $(H/N)_0$, and define $A(L, y) = I$ if $(L, y) \notin$

$(H/N)_0$. Then A is a Borel function on H/N, and for $(L, y) \in (H/N)_0$, and $u \in N$, we have $A(L, y)\gamma(L^y)(u)A(L, y)^{-1} = \gamma(L)^{c(y)}(u)$.

Now set $H_0 = F^{-1}((H/N)_0)$, define M on H_0 by $M(L, x) = \gamma(L)(n(x))A(L, f(x))$, and let $M(L, x) = I$ if $(L, x) \notin H_0$. Then for $(L, x) \in H_0$ and $u \in N$, using the fact that $L^x = L^{f(x)}$ for $L \in \hat{N}$, we have

$$\begin{aligned} M(L, x)\gamma(L^x)(u) &= \gamma(L)(n(x))A(L, f(x))\gamma(L^{f(x)})(u) \\ &= \gamma(L)(n(x))\gamma(L)^{c\circ f(x)}(u)A(L, f(x)) \\ &= \gamma(L)(n(x))\gamma(L)(c\circ f(x)uc\circ f(x)^{-1})A(L, f(x)) \\ &= \gamma(L)^{n(x)}(c\circ f(x)uc\circ f(x)^{-1})\gamma(L)(n(x))A(L, f(x)) \\ &= \gamma(L)^x(u)\gamma(L)(n(x))A(L, f(x)) = \gamma(L)^x(u)M(L, x). \end{aligned}$$

Now if (L, x) and $(L^x, y) \in H_0$, $M(L, x)M(L^x, y)$ and $M(L, xy)$ both intertwine $\gamma(L^{xy})$ and $\gamma(L)^{xy}$. Since these are irreducible, there is a scalar $\tau(L, x; L^x, y)$ of modulus 1 such that $\tau(L, x; L^x, y)M(L, x)M(L^x, y) = M(L, xy)$. Thus M is a strict τ-representation of H_0.

To see that τ really depends only on L, $f(x)$ and $f(y)$, compute as follows, with $\tau_1 = \tau(L, x; L^x, y)$, using the fact that $\gamma(L)$ is a representation:

$$\begin{aligned} &\tau_1\gamma(L)(n(x))\gamma(L)^{c\circ f(x)}(n(y))A(L, f(x))A(L^x, f(y)) \\ &\quad = \tau_1\gamma(L)(n(x))A(L, f(x))\gamma(L^x)(n(y))A(L^x, f(y)) \\ &\quad = \tau_1 M(L, x)M(L^x, y) = M(L, xy) \\ &\quad = M(L, n(x)c\circ f(x)n(y)c\circ f(x)^{-1}c\circ f(x)c\circ f(y)c\circ f(xy)^{-1}c\circ f(xy)) \\ &\quad = \gamma(L)(n(x))\gamma(L)^{c\circ f(x)}(n(y))\gamma(L)(c\circ f(x)c\circ f(y)c\circ f(xy)^{-1})A(L, f(xy)). \end{aligned}$$

Solving this equation for τ_1 gives the desired result.

Still considering the question of existence, let σ be any 2-cocycle now. Then form G^σ and notice that N^σ is normal in G^σ. If L is in the set $N^{c,\sigma}$ of concrete σ-representations of N and we define $L^0(x, s) = sL(x)$ for $(x, s) \in N \times T = N^\sigma$, then L^0 is an ordinary representation of N^σ. If $L_1, L_2 \in N^{c,\sigma}$, then $\mathcal{R}(L_1^0, L_2^0) = \mathcal{R}(L_1, L_2)$, so equivalence and multiplicity are preserved. Now if $(x, s), (y, t) \in G^\sigma$,

$$\begin{aligned} (x, s)(y, t)(x, s)^{-1} &= (x, s)(y, t)(x^{-1}, \sigma(x, x^{-1})s^{-1}) \\ &= (xy, \sigma(x,y)^{-1}st)(x^{-1}, \sigma(x, x^{-1})s^{-1}) = (xyx^{-1}, \sigma(x, y)^{-1}\sigma(xy, x^{-1})^{-1}\sigma(x, x^{-1})t), \end{aligned}$$

so $L^{0,(x,s)} = L^{x,0}$. In particular $L^{0,(x,s)} = L^{0,(x,1)}$, which must hold, because T is contained in the center of G^σ. In other words, G acts on both $N^{c,\sigma}$ and $(N^\sigma)^c$ and the action of G^σ on

$(N^\sigma)^c$ is via the homomorphism of G^σ onto G. Also, the map $L \to L^0$ is equivariant. The action of G preserves irreducibility, so we also have an equivariant restriction of this map, $N^{i,\sigma} \to (N^\sigma)^i$ and hence an equivariant imbedding $\hat{N}^\sigma \to (N^\sigma)\hat{}$. Let f_1: $G^\sigma \to G^\sigma/N^\sigma$ be the quotient homomorphism, and $F_1(L, (x, t)) = (L, f_1(x, t))$. Also, let $\gamma_1(L^0) = \gamma(L)^0$ for $L \in X$, so γ_1 is a cross-section over X^0.

The assumption on $[\mu]$ carries over to the image of the measure class on $(N^\sigma)\hat{}$, so there is a cocycle, ω_1, on $X^0 \times G^\sigma/N^\sigma$ and an $(\omega_1 \circ F_1^{(2)})^{-1}$-representation of $X^0 \times G^\sigma$, M_1, such that $M_1(L^0, \cdot)|N^\sigma = \gamma_1(L^0)$ for almost all L^0 in X^0. Then for almost all $L \in X$ and $t \in T$, $M_1(L^0, (e, t)) = tI$. Define $M(L, x)$ for $(L, x) \in H$ by $M(L, x) = M_1(L^0, (x, 1))$. To see that M is a $\sigma/\omega \circ F^{(2)}$-representation of H, where $\omega(L, f(x); L^x, f(y)) = \omega_1(L^0, f_1(x, 1); L^{0,(x,1)}, f_1(y, 1))$, note first that M is a Borel function because $L \to L^0$ and $x \to (x, 1)$ are. Next, we have M_1 and $\omega_1 \circ F_1^{(2)}$ strict on $\{(L, (x, s))$: L and $L^{(x,s)}$ are in $B_1\}$, where B_1 is a conull Borel set in X^0. Let $B = \{L \in X$: $L^0 \in B_1\}$. If L, L^x and L^{xy} are in B, then

$$M(L, xy) = M_1(L^0, (xy, 1)) = M_1(L^0, (x, 1)(y, \sigma(x, y))$$
$$= \omega_1(L^0, f_1(x, 1); L^{0,(x,1)}, f_1(y, \sigma(xy)))^{-1} M_1(L^0, (x, 1)) M_1(L^{0,(x,1)}, (y, \sigma(x, y))$$

Now $N^\sigma \supseteq T$, so $f_1(y, t) = f_1(y, 1)$ for $t \in T$, so this equals

$$\sigma(x, y)\omega(L, f(x); L^x, f(y))^{-1} M(L, x) M(L^x y),$$

as desired. Furthermore, if $L, L^x \in B$ and $u \in N$,

$$M(L, x)\gamma(L^x)(u) = M_1(L^0, (x, 1))\gamma_1(L^{0,(x,1)})(u, 1)$$
$$= \gamma_1(L^0)^{(x,1)}(u, 1) M_1(L^0, (x, 1)) = \gamma(L)^x(u) M(L, x),$$

and if $u \in N$, $M(L, u) = M_1(L^0, (u, 1)) = \gamma_1(L^0)(u, 1) = \gamma(L)^0(u, 1) = \gamma(L)(u)$.

Now suppose ω' is a cocycle on H/N and M' is a $\sigma/\omega' \circ F^{(2)}$-representation of H such that $M'(L, \cdot)/N = \gamma(L)$ for $L \in C$, where C is μ-conull. We want to show that ω' and ω are cohomologous. Using the form of the cocycle for M' and the restriction property we see that $M'(L, x)\gamma(L^x)(u) M'(L, x)^{-1} = \gamma(L)^x(u)$ for L, $L^x \in C$ and $u \in N$. Hence there is a Borel function α: $H \to T$ such that $M'(L, x) = \alpha(L, x)\, M(L, x)$ for $L, L^x \in B \cap C$. Then $L, L^x \in B \cap C$ and $u \in N$ implies $M'(L, ux) = \alpha(L, ux) M(L, ux)$. Hence

$$\sigma(u, x)\omega'(L, f(u); L, f(x))^{-1}\gamma(L)(u) M'(L, x)$$
$$= \alpha(L, ux)\sigma(u, x)\omega(L, f(u); L, f(x))^{-1}\gamma(L)(u) M(L, x),$$

so $\alpha(L, ux) = \alpha(L, x)$, and α "determines" a Borel function α_1 on H/N. Now

$$\omega(L, f(x); L^x, f(y)) = \omega'(L, f(x); L^x, f(y))\, \alpha(L, xy)\, \alpha(L, x)^{-1} \alpha(L^x, y)^{-1}$$

so $\omega = \omega' d\alpha_1$ on an inessential contraction.

THEOREM 9.3. (Cf. Theorem 8.3 [9]) *Let μ, X, F, ω, M be as in Theorem 9.2. The mapping $S \to M \otimes (S \circ F)$ takes ω-representations of H/N to σ-representations R of H such that $R(L, \cdot)\,|\,N$ is a multiple of $\gamma(L)$ for μ-almost all L. It preserves equivalence and multiplicity, and is one–one onto at the equivalence class level.*

Proof. If S is an ω-representation of H/N and $R = M \otimes (S \circ F)$ then R is a σ-representation of H. Take $\mathcal{K} = \mathcal{H}(M)$ $\mathcal{K}_0 = \mathcal{H}(S)$, so $\mathcal{H}(R) = \mathcal{K} \otimes \mathcal{K}_0$. The units of H/N can be identified with the analytic space $X \subseteq \hat{N}^\sigma$, as can those of H, and the measure classes are both $[\mu]$. If S_1 and S_2 are ω-representation of H/N and $A \in \mathcal{R}(S_1, S_2)$, define $A'(L) = I \otimes A(L)$ for $L \in X$. Then taking $R_1 = M \otimes (S_1 \circ F)$, $R_2 = M \otimes (S_2 \circ F)$, we have

$$\begin{aligned} A'(L)\, R_1(L, x) &= (I \otimes A(L))(M(L, x) \otimes S_1(L, f(x)) \\ &= (M(L, x) \otimes S_2(L, f(x))(I \otimes A(L^{f(x)})) \end{aligned}$$

for (L, x) in some i.c. of H, and since $L^x = L^{f(x)}$, $A' \in \mathcal{R}(R_1, R_2)$ follows. Now if $B \in \mathcal{R}(R_1, R_2)$, $B(L)$ commutes with $\gamma(L)(u) \otimes I_0$ for all $u \in N$, for almost all L. Since $\gamma(L)$ is irreducible, we can write $B(L) = I \otimes A(L)$, by absorbing a scalar into $A(L)$. Thus $B = A'$, i.e. $A \to A'$ is an isomorphism of $\mathcal{R}(S_1, S_2)$ onto $\mathcal{R}(R_1, R_2)$.

It remains only to show that the mapping is onto at the equivalence class level, so suppose R is a σ-representation of H such that $R(L, \cdot)\,|\,N$ is almost always equivalent to a multiple of $\gamma(L)$. Let H_0 be an i.c. of H on which R is strict. Then for $(L, x) \in H_0$ and $u \in N$, $R(L, x)\, R(L^x, u)\, R(L, x)^{-1}$ is a scalar multiple of $R(L, xux^{-1})$. It follows that the multiple for $\gamma(L)$ is the same as for $\gamma(L^x)$, and by ergodicity that the multiple is independent of L. Hence we may choose a Hilbert space $\mathcal{K}_0$ whose dimension is that multiple. Then the von Neumann selection lemma gives the existence of a Borel function V from X to unitary operators from $\mathcal{H}(R)$ to $\mathcal{K} \otimes \mathcal{K}_0$ such that for almost all L in X,

$$V(L)\, R(L, \cdot)\, V(L)^{-1}\,|\,N = \gamma(L)(\cdot) \otimes I_0.$$

If R_1 is defined by $R_1(L, x) = V(L)\, R(L, x)\, V(L^x)^{-1}$, then $R_1(L, \cdot)\,|\,N = \gamma(L)(\cdot) \otimes I_0$ for almost all L. Thus by passing to an equivalent representation we may suppose R has that property.

Now M is a τ-representation and R is a σ-representation. Taking $H_0 = H\,|\,C_0$ to be an i.e. on which they are both strict, we compute for $(L, x) \in H_0$ and $u \in N$, since then (L, xu) and $(L, xux^{-1}) \in H_0$, that

$$R(L, xux^{-1}) = \sigma(x, u)\sigma(xu, x^{-1})\sigma(x, x^{-1})\, R(1, x)\, R(L^x, u)\, R(L, x)^{-1}$$

and

$$\begin{aligned} R(L, xux^{-1}) &= M(L, xux^{-1}) \otimes I_0 \\ &= \tau(L, x; L^x, u)\tau(L, xu; L^x, x^{-1})\tau(L, x; L^x, x^{-1})^{-1} M(L, x)\, M(L^x, u)\, M(L, x)^{-1} \otimes I_0. \end{aligned}$$

The expressions in σ and τ are the same because of the form of τ, so we see that $R(L, x)^{-1} M(L, x) \otimes I_0$ is in the commuting ring of $\gamma(L^x)(\cdot) \otimes I_0$. Hence there is a unique unitary $W(L, x)$ on $\mathcal{H}_0$ such that $R(L, x) = M(L, x) \otimes W(L, x)$. Clearly W is a Borel function on H_0 and we can extend it by the constant identity operator off H_0. Now if (L, x) and $(L^x, y) \in H_0$, using the fact that R is a σ-representation and that M is a τ-representation,

$$\begin{aligned} M(L, xy) \otimes W(L, xy) &= \sigma(x, y)(M(L, x)\, M(L^x, y) \otimes W(L, x)\, W(L^x, y)) \\ &= M(L, xy) \otimes (\sigma(x, y)\tau(L, x; L^x, y)^{-1} W(L, x)\, W(L^x, y)) \end{aligned}$$

Thus W is a σ/τ-representation, but $\sigma/\tau = \omega \circ F^{(2)}$, and $W(L, u) = I_0$ for $u \in N$ and $L \in X_0$, so it follows that W is of the form $S \circ F$, for some ω-representation S on H/N. Hence $R = M \otimes (S \circ F)$, as desired.

Remark. Theorem 8.3 of [9] establishes a one-one correspondence of the same kind, but in that case one knows that there do exist primary ω-representations of H/N. In the present case, there are ω-representations, as we saw in section 2, but there may be no primary ω-representations, which is related to the breakdown of direct integral theory for virtual groups.

10. Applications and examples

Consider a locally compact group G with a closed normal abelian subgroup N. Then $\hat{N}$ may be identified with the character group of N, i.e. with a subset of the concrete dual, so the cross-sections γ used in section 4 can be replaced by the identity function. Thus the function A on $X \times G/N$ in Theorem 9.2, which must satisfy $A(\chi, x)\gamma(\chi^x)(u) = \gamma(\chi)^{c(x)}(u)$ $A(\chi, x)$, can take the value I everywhere. Then $M(\chi, x) = \chi(n(x))$ for $(\chi, x) \in X \times G$, where $n(x) = xc \circ f(x)^{-1}$ as in section 4. Since N is abelian we have $\chi^x = \chi^{f(x)} = \chi^{c \circ f(x)}$ for $x \in G$. Hence we compute the cocycle for M as follows

$$
\begin{aligned}
&(M(\chi, x)M(\chi^x, y))^{-1}M(\chi, xy) \\
&\qquad = (\chi(n(x))\chi^x(n(y)))^{-1}\chi(n(xy)) \\
&\qquad = \chi((n(x)c\circ f(x)n(y)c\circ f(x)^{-1})^{-1}n(xy)) = \chi(c\circ f(x)c\circ f(y)c\circ f(xy)^{-1}).
\end{aligned}
$$

Thus we have an explicit formula for a τ-representation of $\hat{N}\times G$, where τ is lifted from $\hat{N}\times(G/N)$, i.e. $\tau(\chi, x; \chi^x, y)$ really depends only on χ and the cosets of x and y. We can define $\omega(\chi, x; \chi^x, y)=\chi(c(x)c(y)c(xy)^{-1})^{-1}$ for $\chi\in\hat{N}$, $x, y\in G/N$. Let F be the quotient homomorphism of $\hat{N}\times G$ onto $\hat{N}\times(G/N)$ as in section 4. Then $1/\tau=\omega\circ F^{(2)}$. The formula $(S(\chi, x)\psi)(y)=\chi(c(x)c(y)c(xy)^{-1})\psi(x^{-1}y)$ defines an ω-representation S of $\hat{N}\times(G/N)$ on $L^2(G/N)$ (using left Haar measure). Both M and S are Borel functions and are algebraically cocycle representations, so if $[\mu]$ is any quasiorbit in $\hat{N}$ they give strict cocycle representations of the corresponding virtual groups.

Let $[\mu]$ be a quasi-orbit and let ϱ: $\hat{N}\times G\to(0, \infty)$ be the Radon-Nikodym derivative needed to form the induced representation $U=\operatorname{ind}(G, M\otimes(S\circ F))$, acting on $\mathcal{H}(U)=L^2(\hat{N}, \mu; L^2(G/N))$ (Section 1). Since N acts trivially on $\hat{N}$, we may assume $\varrho(\chi, x)=1$ for $x\in N$. Then for $x\in N$, $\psi\in\mathcal{H}(U)$, $\chi\in\hat{N}$ and $y\in G/N$, we have

$$((U(x)\psi)(\chi))(y)=\varrho(\chi, x)^{1/2}M(\chi, x)S(\chi, f(x))(\psi(\chi^x)(f(x)^{-1}y)) = \chi(x)\psi(\chi)(y).$$

Thus $U|N$ is a multiple of $\int\chi d\mu(\chi)$. At this point, one might hope that the central decomposition of U would yield primary representations of G whose restriction to N are multiples of $\int\chi d\mu(\chi)$. The following example due to Calvin Moore shows that this need not happen [1, Chapter II, Section 4].

Let $\mathbf{Z}$ act on $\mathbf{R}^2$ by means of the powers of the matrix

$$\alpha=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix},$$

and form $G=\mathbf{R}^2 \circledS \mathbf{Z}$. Since α is unimodular, α preserves $\mathbf{Z}^2$ as a subgroup of $\mathbf{R}^2$ and hence it may be taken as a normal subgroup N of G. The action of $\mathbf{Z}$ induced on the torns $T^2=\hat{\mathbf{Z}}^2$ is ergodic for Haar measure (and the action of G factors through $\mathbf{Z}$), but the action on $\mathbf{R}^2$ has a Borel cross-section of the orbits. Now if V is a primary representation of G, $V|\mathbf{R}^2$ decomposes as a multiple of the direct sum of all the characters in an orbit. Further restriction to $N=\mathbf{Z}^2$ simply gives the same multiple of the direct sum of the characters obtained by restricting those characters of $\mathbf{R}^2$ to $\mathbf{Z}^2$. This is again a multiple of the sum of the characters in an orbit, while Haar measure gives measure zero to that set, so $V|N$ is not a multiple of $\int\chi d\mu(\chi)$ if μ is Haar measure on T^2.

The other example in section 4 of chapter 2 of [1] also gives negative results because of an intermediate normal subgroup which forces primary representations of G to be associated only with transitive quasi-orbits even though non-transitive quasiorbits are present.

THEOREM 10.1. *If G is a semidirect product of an abelian normal subgroup N and a subgroup K, then every quasi-orbit in $\hat{N}$ is associated with at least one primary representation of G.*

Proof. Let $[\mu]$ be a quasi-orbit and define $M(\chi, x) = \chi(n(x))$ as before. If we let c be the natural isomorphism of G/N with K, then $c(xy) = c(x)c(y)$ for x and y in G/N, so the cocycle ω is trivial. Thus M is an ordinary representation, and is irreducible because it is one-dimensional. Hence $U = \text{ind}\,(G, M)$ is irreducible and it is associated with $[\mu]$.

Remark. The formula for U is

$$(U(x)\psi)(\chi) = \chi(n(x))\varrho(\chi, f(x))^{1/2}\psi(\chi^{f(x)}),$$

so $\mathcal{R}(U\,|\,N, U\,|\,N)$ is clearly the multiplication operators and ergodicity of the action of K makes the functions constant.

In many cases there are irreducible representations of $\hat{N} \times K$ of all dimensions obtainable by mapping $\hat{N} \times K$ to various groups by homomorphisms φ with dense range. If (S, μ) is an ergodic G space and H is locally compact, we say φ: $S \times G \to H$ has dense range if the action of G on $S \times H$ defined by $(s, x)z = (sz, x\varphi(s, z))$ is ergodic. If φ is any homomorphism and L is any representation of H, then $L \circ \varphi$ is a representation of $S \times G$. If $A \in \mathcal{R}(L, L)$ then $A'(s) = A$ defines a function from S to $\mathcal{B}(\mathcal{H}(L))$ and clearly $A' \in \mathcal{R}(L\circ\varphi, L\circ\varphi)$. Now suppose $T \in \mathcal{R}(L\circ\varphi, L\circ\varphi)$ and define a function $g(s, x) = L(x)\,T(s)L(x)^{-1}$ on $S \times H$. From $L\circ\varphi(y)\,T(sy) = T(s)L\circ\varphi(y)$ it follows that g is constant on orbits in $S \times H$. Since g is clearly Borel and the operators form a standard space, we see that g is essentially constant if φ has dense range. Let C_1 be conull and suppose $g(s, \cdot)$ is constant a.e. on H for $s \in C_1$. Since $g(s, \cdot)$ is weakly continuous, it is constant if constant a.e. Now choose y_1 such that $g(\cdot, y_1)$ is constant on some conull set C_2 in S. Then g is constant on $C \times H$, where $C = C_1 \cap C_2$. Thus T is constant a.e. and on C the value is in $\mathcal{R}(L, L)$, so $T = A'$ for some A. Now if we assume L is irreducible, so is $L\circ\varphi$.

Since many virtual groups of the form $\hat{N} \times K$ have homomorphisms with dense range into any (separable) compact group H [18], we can get many irreducible representations S of $\hat{N} \times K$ and form ind $(G, M \otimes (S \circ F))$ to get more representations associated with the given quasi-orbit.

Work is under way, but much remains to be done to gain a thorough understanding of what goes wrong in the other cases, namely what the intermediate normal subgroup and the non-trivial cocycle really involve.

Remark added in proof. For the decomposition of an action into ergodic parts, see also the paper by Dang-Ngoc Nghiem, Decomposition et classification des systemes dynamiques, Bull. Soc. Math. France, 103 (1975), 149–175.

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