The elementary theory of large e-fold ordered fields

by

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Introduction

The aim of this work is to continue Van den Dries' treatment of the theory of e-fold ordered fields along the lines of the treatment given in Jarden-Kiehne [10] to the theory of e-free Ax fields.

Van den Dries generalizes in his thesis [18] the theory of real closed fields. He considers structures $(K, P_1, \ldots, P_e)$ that consist of a field K and e orderings $P_1, \ldots, P_e$ of K. He proves in [18, p. 54]:

The theory of e-fold ordered fields OF_e has a model companion $\overline{OF}_e$, the models of which are the e-fold ordered fields $(E, P_1, \ldots, P_e)$ satisfying:

(α) P_i and P_j induce different order topologies on E for all $1 \leq i < j \leq e$.

(β) If $f \in E[T, X]$ is an irreducible polynomial and if there exists an $a_0 \in E$ such that $f(a_0, X)$ changes sign on E with respect to each of the P_i's, then there exist $a, b \in E$ such that $f(a, b) = 0$.

In particular it follows from this theorem that the absolute Galois group $G(E)$ of E is a pro-2-group generated by e involutions (cf. [18, p. 77 and p. 92]). If E is algebraic

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(1) This work was done while the author was visiting the University of California at Irvine. He was also supported in part by a BSF grant.

over $\mathbf{Q}$ and R is a real closure of $\mathbf{Q}$, this means that there exist $\sigma_1,\dots,\sigma_e\in G(\mathbf{Q})$ such that $E=R^{\sigma_1}\cap\dots\cap R^{\sigma_e}$. In general, if $\sigma_1,\dots,\sigma_e\in G(\mathbf{Q})$ we write $\mathbf{Q}_\sigma=R^{\sigma_1}\cap\dots\cap R^{\sigma_e}$ and denote by $P_{\sigma i}$ the ordering of $\mathbf{Q}_\sigma$ induced by the unique ordering of the real field R^{σ_i}. Thus we have a family $\mathscr{L}_\sigma=(\mathbf{Q}_\sigma,P_{\sigma 1},\dots,P_{\sigma e})$ of e-fold ordered fields indexed by $G(\mathbf{Q})^e$. The absolute Galois group $G(\mathbf{Q}_\sigma)$ is still generated by e involutions but it is not necessarily a pro-2-group. Indeed, Geyer proved in [5] that for almost all $\boldsymbol{\sigma}\in G(\mathbf{Q})^e$ the group $G(\mathbf{Q}_\sigma)$ is isomorphic to the free profinite product, $\hat{D}_e$, of e copies of $\mathbf{Z}/2\mathbf{Z}$. Here "almost all" is used in the sense of the Haar measure μ of the compact group $G(\mathbf{Q})^e$. The models $\mathscr{L}_\sigma$ of $\widetilde{OF}_e$ appear therefore to be 'rare' among all the models $\mathscr{L}_\sigma$. We therefore concentrate in this work on the theory T_e of sentences that are true in $\mathscr{L}_\sigma$ for almost all $\boldsymbol{\sigma}\in G(\mathbf{Q})^e$. We prove the following theorems:

THEOREM A. *Almost all $\boldsymbol{\sigma}\in G(\mathbf{Q})^e$ have the following properties:*

(γ) *If V is an absolutely irreducible variety defined over $\mathbf{Q}_\sigma$ and if $P_{\sigma 1},\dots,P_{\sigma e}$ can be extended to orderings of the function field of V, then the set $V(\mathbf{Q}_\sigma)$ of all $\mathbf{Q}_\sigma$-rational points of V is not empty. Moreover, for every $1\le i\le e$ and every simple point q_i of V which is rational over R^{σ_i}, can be P_i-approximated by points in $V(\mathbf{Q}_\sigma)$.*

(δ) *The orderings $P_{\sigma 1},\dots,P_{\sigma e}$ induce distinct topologies on $\mathbf{Q}_\sigma$.*

THEOREM B. *If $\mathscr{E}=(E,P_1,\dots,P_e)$ and $F=(F,Q_1,\dots,Q_e)$ are two e-fold ordered fields that satisfy* (γ) *and* (δ), *if $G(E)\cong G(F)\cong\hat{D}_e$ and if $\tilde{\mathbf{Q}}\cap\mathscr{E}\cong\tilde{\mathbf{Q}}\cap\mathscr{F}$ then $\mathscr{E}$ is elementarily equivalent to $\mathscr{F}$. Here $\tilde{\mathbf{Q}}\cap\mathscr{E}=(\tilde{\mathbf{Q}}\cap E,\tilde{\mathbf{Q}}\cap P_1,\dots,\tilde{\mathbf{Q}}\cap P_e)$.*

Using Theorems A and B and the above mentioned theorem of Geyer we show that every sentence θ of the theory of e-fold ordered fields is equivalent to a, so called, 'test sentence', that amounts to a statement about e-fold ordered fields which are contained in a finite normal extension L of $\mathbf{Q}$. This enables us to compute (in a recursive way) the measure of the set

$$A(\theta)=\{\boldsymbol{\sigma}\in G(\mathbf{Q})^e\mid\mathscr{L}_\sigma\models\theta\}$$

which turns to be a rational number. In particular we can find whether or not $\mu(A(\theta))=1$. Thus

THEOREM C. *T_e is a decidable theory.*

Acknowledgement. The author is indebted to Wulf-Dieter Geyer and to Dan Haran for some useful conversations on this paper.

1. Ordered fields

Let $<$ be an ordering of a field K. Then the set $P=\{x\in K|x>0\}$ of all positive elements of K determines $<$ uniquely. We can therefore view our ordered field as a couple (K,P). Then we abuse our language and speak about P as the ordering of K. The appropriate first-order predicate calculus language is denoted by $\mathcal{L}_1$.

A polynomial $f\in K[X]$ is said to *change sign* on K if there exist $a,b\in K$ such that $f(a)<0$ and $f(b)>0$. If f is in addition irreducible over K and α is a root of K, then P can be extended to an ordering of $K(\alpha)$ (cf. Ribenboim [15, p. 150]). In particular if an algebraic extension L of K has an odd degree, then P can be extended to L.

Every ordered field (K,P) has a *real closure* $(\bar{K},\bar{P})$, where $\bar{K}$ is an algebraic extension of K, $\bar{P}$ is an extension of P and no proper algebraic extension of $\bar{K}$ can be ordered. Indeed, the algebraic closure $\tilde{K}$ of K is equal to $\bar{K}(\sqrt{-1})$ and the absolute Galois group, $G(\bar{K})=\mathcal{G}(\tilde{K}/\bar{K})$, of $\bar{K}$ is a cyclic group of order 2. The field $\bar{K}$ is then said to be *real closed*. Conversely, given an involution ε in $G(K)$ (i.e. an element that satisfies $\varepsilon^2=1$ and $\varepsilon\neq1$), then its fixed field, $\tilde{K}(\varepsilon)$, in $\tilde{K}$ is a real closed field. It has a unique ordering $\bar{P}$ that consists of all non zero squares of $\bar{K}$. The restriction of this ordering to K is *the ordering of K induced by $\bar{K}$*, or, as we also say, by ε. If ε' is an additional involution in $G(K)$ and it induces the same ordering on K as ε, then ε and ε' are conjugate in $G(K)$, and $\tilde{K}(\varepsilon)$ and $\tilde{K}(\varepsilon')$ are isomorphic over K (cf. [15, p. 163]). Also there are no K-automorphisms of $\bar{K}$ besides the identity (cf. [15, p. 165]). In particular all the real closed fields which are algebraic over $\mathbf{Q}$ are isomorphic and they have no non-identity automorphisms, since $\mathbf{Q}$ has a unique ordering.

More generally, if φ: $(K_1,P_1)\to(K_2,P_2)$ is an isomorphism of ordered fields and $\bar{K}_1,\bar{K}_2$ are real closures of K_1,K_2, respectively, then φ can be uniquely extended to an isomorphism $\bar{\varphi}$: $\bar{K}_1\to\bar{K}_2$ (cf. Jacobson [6, p. 285]).

Another important extension theorem is the following:

If $(L_1,\mathbf{Q}_1)$ and $(L_2,\mathbf{Q}_2)$ are extensions of an ordered field (K,P) and if L_1,L_2 are linearly disjoint over K, then L_1L_2 has an ordering $\mathbf{Q}$ that extends both $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Moreover, if both L_1 and L_2 are algebraic over K, then the extension $\mathbf{Q}$ is unique.

A proof for the existence part of this theorem is given in Van den Dries [18, p. 75]. Both the existence and the uniqueness part in the case where L_1,L_2 are algebraic over K can be proved in the following way.(1)

First one observes that if suffices to prove the assertion for L_1 and L_2 finite over K. Second, one notes that if L is a finite extension of K and $(\bar{K},\bar{P})$ is a real closure of

(1) This argument is due to Dan Haran.

(K, P), then there is a bijection between the K-embedding of L into $\bar{K}$ and the orderings of L that extend P. Indeed, a K-embedding σ: $L \to \bar{K}$ corresponds to the ordering $P_\sigma = \sigma^{-1}(\sigma L \cap \bar{P})$ of L. Finally, one uses the fact that for every pair σ_1, σ_2 of K-embeddings of L_1, L_2, respectively, into $\bar{K}$ there exists a unique K-embedding σ: $L_1 L_2 \to \bar{K}$ that extends both σ_1 and σ_2, since L_1 and L_2 are linearly disjoint over K.

If (K, P) is an ordered field and if x is a transcendental element over K, then P can be extended to $K(x)$.

If V is an absolutely irreducible variety defined over K and has a simple point with coordinates in a real closure $\bar{K}$ of (K, P), then the function field F of V over K is *formally real,* i.e. -1 is not the sum of squares of F (cf. Lang [11, p. 281]). By the same reason $\bar{K}F$ is formally real. A real closure of $\bar{K}F$ induces then an ordering of F that extends P. Conversely, if P can be extended to an ordering of F, then the set $V(\bar{K})$ of $\bar{K}$-rational points of V is Zariski-dense in V. This property of real closed fields motivates our definition of pseudo-real-closed fields in the next section.

2. Pseudo real closed fields

Let M be a field and let $<_1, \ldots, <_e$ be e orderings of M. Denote by P_i the set of $<_i$-positive elements of M. The structure $(M, P_1, \ldots, P_e)$ is said to be an *e-fold ordered field*. The corresponding first order predicate calculus language is denoted by $\mathscr{L}_e$. If K is a formally real field, then $\mathscr{L}_e(K)$ denotes the language $\mathscr{L}_e$ augmented by constant symbols for the elements of K. We also denote by $\bar{M}_i$ the real closure of M with respect to P_i.

An e-fold ordered field $(M, P_1, \ldots, P_e)$ is said to be *psuedo-real-closed* (abbreviated PRC) if it has the following properties:

(a) If V is a (non-empty) absolutely irreducible variety defined over M and if each of the P_i's can be extended to an ordering of the function field of V over M, then V has an M-rational point.

(b) The orderings $P_1, \ldots, P_e$ induce distinct order topologies on M.

Note that condition (b) implies that if for each $1 \leq i \leq e$, U_i is a non-empty P_i-open subset of M, then $U_i \cap \ldots \cap U_e \neq \emptyset$. This conclusion is known as the *approximation theorem* (cf. [14, p. 327]).

Remark. A field M is said to be PAC if every absolutely irreducible variety defined over M has an M-rational point (cf. Frey [2], or [7] where it was still called a Σ-field). The PRC fields are therefore the analogue of PAC fields for e-fold ordered fields. Also, our definition of PRC fields becomes that of McKenna [13] in the case $e=1$.

LEMMA 2.1. *Let* $(M, P_1, \ldots, P_e)$ *be a* PRC *field.*

(i) *If* V *is an absolutely irreducible variety defined over* M *and if each of the* P_i*'s can be extended to an ordering of the function field of* V *over* M*, then* $V(M)$ *is Zariski-dense in* V.

(ii) *For every absolutely irreducible polynomial* $f \in M[T_1, \ldots, T_r, X]$ *for which there exists an* $\mathbf{a}_0 \in M^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on* M *with respect to each of the* P_i*'s, and for every* $0 \neq g \in M[\mathbf{T}]$, *there exists an* $(\mathbf{a}, b) \in M^{r+1}$ *such that* $f(\mathbf{a}, b) = 0$ *and* $g(\mathbf{a}) \neq 0$.

Proof. (i) Use Rabinovitz trick (cf. [4, Lemma 3.3]). (ii) Let $t_1, \ldots, t_r$ be r algebraically independent elements over M. Extend each of the P_i's to an ordering of $M(\mathbf{t})$ such that $\mathbf{t}$ is infinitesimally close to $\mathbf{a}_0$. Hence if x is an element such that $f(\mathbf{t}, x) = 0$ we can extend P_i to $M(\mathbf{t}, x)$. By (i) there exists $(\mathbf{a}, b) \in M^{r+1}$ such that $f(\mathbf{a}, b) = 0$ and $g(\mathbf{a}) \neq 0$. Q.E.D.

LEMMA 2.2. *Let* $(M, P_1, \ldots, P_e)$ *be a* PRC *field. If* $f \in M[T_1, \ldots, T_r, X]$ *is an absolutely irreducible polynomial for which there exists an* $\mathbf{a}_0 \in M^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on* M *with respect to each of the* P_i *and if* U_i *is a* P_i*-neighbourhood of* $\mathbf{a}_0$ *for* $i = 1, \ldots, e$, *then there exists an* $(\mathbf{a}, b) \in M^{r+1}$ *such that* $\mathbf{a} \in U_1 \cap \ldots \cap U_e$ *and* $f(\mathbf{a}, b) = 0$.

Proof. In order to facilitate notation we carry out the proof only for $r = 1$. Let c be an element of M which is P_i-large for $i = 1, \ldots, e$. Then there exists a positive integer m and polynomials $0 \neq g \in M[T_1, T_2, T_3]$ and $h \in [T_1, T_2, T_3, X]$ such that

$$f\left(a_0 + \frac{1}{c + T_1^2 + T_2^2 + T_3^2}, X\right) = \frac{g(T_1, T_2, T_3)\, h(T_1, T_2, T_3, X)}{(c + T_1^2 + T_2^2 + T_3^2)^m}$$

and h is a primitive polynomial in X. By [8, Lemma 10.3] h is an absolutely irreducible polynomial. Consider now three elements t_1, t_2, t_3 which are algebraically independent over M, and extend each of the P_i's to an ordering of $M(t_1, t_2, t_3)$. Then $a_0 + (c + t_1^2 + t_2^2 + t_3^2)^{-1}$ is P_i-close to a_0 and therefore $h(t_1, t_2, t_3, X)$ changes sign on $M(t_1, t_2, t_3)$. It follows that if x is a root of $h(t_1, t_2, t_3, X)$, then P_i can be extended to an ordering of $M(t_1, t_2, t_3, X)$. This implies that there exist $m_1, m_2, m_3, b \in X$ such that $h(m_1, m_2, m_3, b) = 0$. Let $a = a_0 + (c + m_1^2 + m_2^2 + m_3^2)^{-1}$. Then a is P_i-close to a_0, for $i = 1, \ldots, e$, and $f(\mathbf{a}, b) = 0$. Q.E.D.

The converse to Lemma 2.2 is

LEMMA 2.3. *Let* $\mathcal{M} = (M, P_1, \ldots, P_e)$ *be an e-fold ordered field with the following properties:*

(i) *If* $f \in M[T_1, \ldots, T_r, X]$ *is an absolutely irreducible polynomial for which there exists an* $\mathbf{a}_0 \in M^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on* M *with respect to each of the* P_i*'s and if* U_i *is a* P_i*-neighbourhood of* $\mathbf{a}_0$ *for* $i=1, \ldots, e$, *then there exists an* $(\mathbf{a}, b) \in M^{r+1}$ *such that* $\mathbf{a} \in U_1 \cap \ldots \cap U_e$, *and* $f(\mathbf{a}, b)=0$.

(ii) $P_1, \ldots, P_e$ *induce distinct order topologies on* M.

Then $\mathcal{M}$ *is a* PRC *field.*

Proof. We prove first that M is P_i-dense in $\bar{M}_i$. Let $g(X)=X^n+c_1X^{n-1}+\ldots+c_n$ be a polynomial in $M[X]$ such that $g(s)\,g(t)<_i 0$ for two elements $s<_i t$ of M. The polynomial $f(\mathbf{T}, X)=X^n+T_1X^{n-1}+\ldots+T_n$ is absolutely irreducible and $f(\mathbf{c}, X)=g(X)$. Hence there exists $(\mathbf{a}, b) \in M^{r+1}$ such that $\mathbf{a}$ is P_1-close to $\mathbf{c}$ and $f(\mathbf{a}, b)=0$. It follows that $g(b)=f(\mathbf{c}, b)$ is P_i-close to zero. Using Van den Dries [18, p. 108], we conclude that M is indeed, P_i-dense in $\bar{M}_i$.

Let now V be an absolutely irreducible variety defined over M, let F be its function field over M and suppose that each of the P_i can be extended to an ordering of F. Using an M-birational transformation we can assume that V is defined by an absolutely irreducible polynomial $f \in M[T_1, \ldots, T_r, X]$ and $F=M(\mathbf{t}, x)$, where $t_1, \ldots, t_r$ are algebraically independent over M and $f(\mathbf{t}, x)=0$. Denote by $\bar{F}_i$ a real closure of F with respect to P_i. Then $\tilde{M} \cap \bar{F}_i$ is a real closure of M with respect to P_i, and without loss of generality we may assume that $\bar{M}_i=\tilde{M} \cap \bar{F}_i$. Then $\bar{M}_i$ is an elementary subfield of $\bar{F}_i$ (cf. [1, p. 267]). Clearly $(\partial f/\partial X)(\mathbf{t}, x) \neq 0$, hence $f(\mathbf{t}, X)$ changes sign in $\bar{F}_i$. It follows that there exists an $\mathbf{a}_i \in \bar{M}_i^r$ *such that* $f(\mathbf{a}_i, X)$ changes sign on $\bar{M}_i$. By (ii) and by the first part of the proof there exists an $\mathbf{a}_0 \in M^r$ such that $f(\mathbf{a}_0, X)$ changes sign on M with respect to each of the P_i. It follows by (i) that there exist an $(\mathbf{a}, b) \in M^{r+1}$ such that $f(\mathbf{a}, b)=0$. Moreover, $\mathbf{a}$ can be taken arbitrarily P_i-close to $\mathbf{a}_0$. In particular it can be chosen such that $(\mathbf{a}, b)$ is mapped by the birational transformation onto an M-rational point of the original variety V. Q.E.D.

It follows from Lemmas 2.2 and 2.3 that the property of an e-fold ordered field $(M, P_1, \ldots, P_e)$ to be PRC can be reformulated by conditions (i) and (ii) of Lemma 2.3. The advantage of this reformulation is that theses conditions are equivalent to an explicit countable list of sentences in the language $\mathcal{L}_e$. Indeed, condition (i) can be rewritten as a list of sentences using e.g. [10, p. 278].

We have therefore proved the following:

LEMMA 2.4. *There exists an explicit set* Π *of sentences of* $\mathcal{L}_e$ *such that an e-fold* *ordered field* $(M, P_1, \ldots, P_e)$ *is* PRC *if and only if it satisfies* Π. *In particular ultraproducts of* PRC *fields are again* PRC *fields*.

Another conclusion that can be drawn from Lemma 2.2 and from the proof of Lemma 2.3 is

COROLLARY 2.5. *Let* $(M, P_1, \ldots, P_e)$ *be a* PRC *field. Then* M *is* P_i*-dense in* $\tilde{M}_i$ *for* $i=1, \ldots, e$.

Using the original definition for PRC fields we deduce a uniqueness theorem for the orderings. However the advantage of this definition can be recognized only in the following section.

LEMMA 2.6. *Let* $(M, P_1, \ldots, P_e)$ *be a* PRC *field. Then* $P_i \cap \ldots \cap P_e = M^2 + M^2$ *and* $P_1, \ldots, P_e$ *are the only orderings of* M.

Proof. If $c \in P_1 \cap \ldots \cap P_e$, then $(0, \sqrt{c})$ is a simple point of the absolutely irreducible equation $X^2 + Y^2 - c = 0$, which is contained in each of the fields $\tilde{M}_i$, for $i=1, \ldots, e$. It follows that each of the orderings P_i can be extended to the function field of the equation. Hence there exist $a, b \in M$ such that $a^2 + b^2 = c$.

This conclusion implies that if Q is an ordering of M, then Q contains $P_1 \cap \ldots \cap P_e$. The assumption that $P_1, \ldots, P_e$ induce distinct topologies on M implies, by a theorem of Van den Dries [18, p. 90], that Q coincides with one of the P_i's. Q.E.D.

3. The elementary equivalence theorem for PRC fields

Let $(M, P_1, \ldots, P_e)$ be an e-fold ordered field and let D be an integral domain that contains M. Then D is said to be *real and absolutely entire* (*with respect to the* P_i's) if each of the P_i's can be extended to an ordering of D and if M is algebraically closed in the quotient field of D.

Clearly, $(M, P_1, \ldots, P_e)$ is PRC if and only for every real and absolutely entire integral domain D which is finitely generated over M, there exists an M-homomorphism $\varphi: D \to M$. We use this observation as a motivation for the following definition.

An e-fold ordered field $(F, P_1, \ldots, P_e)$ is said to be *hyper real closed* (abbreviated HRC) if for every real and absolutely entire integral domain D which is countably generated over F there exists an F-homomorphism $\varphi: D \to F$.

Obviously, every HRC field is a PRC field. Conversely, every $\aleph_1$-saturated PRC field is HRC. In particular, by Lemma 2.4, every non-principal ultra-product on a countable index set of PRC fields is an HRC field (cf. [10, Lemma 1.2]).

Our main algebraic tool in analysing PRC fields is the following analogue to [10, Lemma 2.1].

LEMMA 3.1. *Let $(E, P_1, \ldots, P_e)$ be a countable e-fold ordered field and let $(F, Q_1, \ldots, Q_e)$ be an* HRC *field. Let ε_i and ζ_i be involutions in $G(E)$ and $G(F)$ that induce P_i and Q_i on E and F respectively. Suppose that L is a common, subfield of E and F. Suppose further that there exists a homomorphism φ: $G(F) \to G(E)$ such that:*

(a) $\mathrm{Res}_{\tilde{L}}\, \varphi(\sigma) = \mathrm{Res}_{\tilde{L}}\, \sigma$ *for every $\sigma \in G(F)$.*

(b) $\varphi(\zeta_i) = \varepsilon_i$ *for $i = 1, \ldots, e$.*

Then there exists an $\tilde{L}$-embedding Φ: $\tilde{E} \to \tilde{F}$ such that:

(i) $\Phi(\varphi(\sigma)x) = \sigma\Phi(x)$ *for every $x \in \tilde{E}$ and every $\sigma \in G(F)$.*

(ii) $\Phi(\tilde{E}(\varepsilon_i)) \subseteq \tilde{F}(\zeta_i)$ *for $i = 1, \ldots, e$.*

(iii) *The restriction of Φ to E gives an embedding of $(E, P_1, \ldots, P_e)$ into $(F, Q_1, \ldots, Q_e)$.*

Proof. Without loss of generality we may assume that $\tilde{E}$ is free from $\tilde{F}$ over $\tilde{L}$. Then $\tilde{E}$ is also linearly disjoint from $\tilde{F}$ over $\tilde{L}$ and hence every $\sigma \in G(F)$ can be uniquely extended to an element $\bar{\sigma} \in \mathcal{G}(\tilde{E}\tilde{F}/EF)$ such that

$$\bar{\sigma}x = \begin{cases} \sigma x & \text{if } x \in \tilde{F} \\ \varphi(\sigma)x & \text{if } x \in \tilde{E}. \end{cases}$$

The map $\sigma \mapsto \bar{\sigma}$ is an embedding of $G(F)$ into $\mathcal{G}(\tilde{E}\tilde{F}/EF)$ whose inverse is the restriction map. Denote by D the fixed field of the image of $G(F)$. Then Res: $\mathcal{G}(\tilde{E}\tilde{F}/D) \to G(F)$ is an isomorphism and hence $D \cap \tilde{F} = F$ and $D\tilde{F} = \tilde{E}\tilde{F}$.

For every $1 \leq i \leq e$ let $\bar{E}_i = \tilde{E}(\varepsilon_i)$ and $\bar{F}_i = \tilde{F}(\zeta_i)$. Then $\bar{E}_i$ and $\bar{F}_i$ are real closed and by (a) and (b) $\bar{L}_i = \tilde{L} \cap E_i = \tilde{L}(\varepsilon_i) = \tilde{L}(\zeta_i) = \tilde{L} \cap \bar{F}_i$ is also real closed. It follows that there exists an ordering $P_i Q_i$ of $\bar{E}_i \bar{F}_i$ that extends the orderings P_i and Q_i of $\bar{E}_i$ and $\bar{F}_i$, respectively. In addition $\tilde{E} = \bar{E}_i(\sqrt{-1})$ and $\tilde{F} = \bar{F}_i(\sqrt{-1})$, hence $\tilde{E}\tilde{F} = \bar{E}_i \bar{F}_i(\sqrt{-1})$. It follows that $\mathcal{G}(\tilde{E}\tilde{F}/\bar{E}_i \bar{F}_i)$ is a group of order 2, generated by $\bar{\zeta}_i$. In particular it follows that $D \subseteq \bar{E}_i \bar{F}_i$. The restriction of $P_i Q_i$ to D is therefore an ordering that extends Q_i.

Now $\tilde{E}\tilde{F}$ is an algebraic extension of D. Hence $\tilde{E} \subseteq \tilde{E}\tilde{F} = D[\tilde{F}] = \tilde{F}[D]$. Every element $x \in \tilde{E}$ can be therefore written in the form

$$x = \Sigma y_j d_j, \quad \text{where } y_j \in \tilde{F}, d_j \in D. \tag{3.1}$$

The set D_0 of all the d_j appearing in the expressions (3.1) (one expression for every $x \in \tilde{E}$) is countable, since $\tilde{E}$ is, and $\tilde{E} \subseteq \tilde{F}[D_0]$. The ring $F[D_0]$ is a real and absolutely entire domain which is countably generated over F. Hence there exists an F-homomorphism ψ: $F[D_0] \to F$. Also, $F[D_0]$ is linearly disjoint from $\tilde{F}$ over F. Hence ψ can be extended to an $\tilde{F}$-homomorphism $\tilde{\psi}$: $\tilde{F}[D_0] \to \tilde{F}$. It satisfies

$$\tilde{\psi}(\tilde{\sigma}x) = \sigma\tilde{\psi}(x), \quad \text{for every } \sigma \in G(F) \tag{3.2}$$

and for every x that belongs to $\tilde{F}$ or to D_0. It follows that (3.2) is true for every $x \in \tilde{F}[D_0]$ and in particular for every $x \in \tilde{E}$.

Let $\Phi = \mathrm{Res}_{\tilde{E}}\tilde{\psi}$. Then Φ is an $\tilde{L}$-embedding of $\tilde{E}$ into $\tilde{F}$ that satisfies (i), hence it satisfies (ii) and (iii) too. Q.E.D.

THEOREM 3.2. *Let $(E, P_1, \dots, P_e)$ and $(F, Q_1, \dots, Q_e)$ be two PRC fields. Let ε_i and ζ_i be involutions in $G(E)$ and $G(F)$ that induce P_i and Q_i on E and F respectively. Let L be a common subfield of E and F respectively and suppose that there exists an isomorphism φ: $G(F) \to G(E)$ such that:*

(a) $\mathrm{Res}_{\tilde{L}}\,\varphi(\sigma) = \mathrm{Res}_{\tilde{L}}\,\sigma$ *for every* $\sigma \in G(F)$.

(b) $\varphi(\zeta_i) = \varepsilon_i$ *for* $i = 1, \dots, e$.

Then $(E, P_1, \dots, P_e)$ is elementarily equivalent over L to $(F, Q_1, \dots, Q_e)$.

Proof. In every sentence of $\mathscr{L}_e(L)$ there appear only finitely many elements of L. We can therefore suppose that L is a countable field. Secondly, by going over to non-principal ultrapowers with respect to a countable index set we may assume that $(E, P_1, \dots, P_e)$ and $(F, Q_1, \dots, Q_e)$ are HRC fields. (Here we use the remarks preceeding Lemma 3.1.)

By a repeated use of the Skolem-Löwenheim theorem together with Lemma 3.1 we can construct by the go and forth method two elementary submodels $(E', P_1', \dots, P_e')$ and $(F', Q_1', \dots, Q_e')$ of $(E, P_1, \dots, P_e)$ and $(F, Q_1, \dots, Q_e)$ respectively, which are isomorphic over L, (cf. the proofs of Lemma 2.2, Lemma 3.1 and Theorem 3.2 of [10]). It follows that

$$(E, P_1, \dots, P_e) \equiv_L (F, Q_1, \dots, Q_e). \qquad \text{Q.E.D.}$$

COROLLARY 3.3. *Let $(E, P_1, \dots, P_e) \subseteq (F, Q_1, \dots, Q_e)$ be two* PRC *fields. If* Res: $G(F) \to G(E)$ *is an isomorphism, then $(E, P_1, \dots, P_e)$ is an elementary submodel of $(F, Q_1, \dots, Q_e)$.*

4. Geyer-fields

Denote by D_e the free product in the category of groups of e copies of $\mathbf{Z}/2\mathbf{Z}$. Consider its completion $\hat{D}_e = \varprojlim D_e/N$, where N runs over all the normal subgroups of finite index. $\hat{D}_e$ has a system of e generators $\varepsilon_1, \ldots, \varepsilon_e$ satisfying $\varepsilon_1^2 = \ldots = \varepsilon_e^2 = 1$. If $x_1, \ldots, x_e$ are involutions in a profinite group G, then the map $\varepsilon_i \mapsto x_i$, $i=1, \ldots, e$, can be extended to a homomorphism of $\hat{D}_e$ into G. Indeed, every system of e involutions that generates $\hat{D}_e$ has this property. Thus $\hat{D}_e$ is the free product in the category of profinite groups of e copies of $\mathbf{Z}/2\mathbf{Z}$. It follows that a finite group H is a homomorphic image of $\hat{D}_e$ if and only if it is generated by e involutions. Conversely:

LEMMA 4.1. *If a profinite group G satisfies "a finite group H is a homomorphic image of G if and only if it is generated by e involutions", then G is isomorphic to $\hat{D}_e$.*

Proof. See e.g. Schuppar [17, Satz 2.1]. Q.E.D.

LEMMA 4.2. *If $\varepsilon_1, \ldots, \varepsilon_e$ are involutions generating $\hat{D}_e$ then no two of them are conjugate.*

Proof. Indeed the map of $\varepsilon_1, \ldots, \varepsilon_e$ onto a basis of $(\mathbf{Z}/2\mathbf{Z})^e$ can be extended to an epimorphism of $\hat{D}_e$ onto $(\mathbf{Z}/2\mathbf{Z})^e$. Q.E.D.

PRC fields $(M, P_1, \ldots, P_e)$ for which $G(M) \cong \hat{D}_e$ are called *Geyer-fields* of *corank e*. The motivation for this name comes from a theorem of Wulf-Dieter Geyer, where 'a lot' of e-fold ordered fields are proved to have $\hat{D}_e$ as their absolute Galois group. For Geyer fields we have the following stronger form of Lemma 2.6.

LEMMA 4.3. *Let $(E, P_1, \ldots, P_e)$ be a Geyer-field and let $\varepsilon_1, \ldots, \varepsilon_e$ be involutions that generate $G(E)$. Then $P_1 \cap \ldots \cap P_e = E^{\times 2}$ and the orderings of E induced by $\varepsilon_1, \ldots, \varepsilon_e$ coincide (possibly after a permutation) with $P_1, \ldots, P_e$.*

Proof. The orderings $Q_1, \ldots, Q_e$ of E induced by $\varepsilon_1, \ldots, \varepsilon_e$ are distinct. Indeed, if $Q_i = Q_j$, then ε_i is conjugate to ε_j in $G(E)$, hence $i=j$, by Lemma 4.2.

If $x \in P_1 \cap \ldots \cap P_e$, then x is a square in $\tilde{E}_i = \tilde{E}(\varepsilon_i)$, for $i=1, \ldots, e$. Hence $\sqrt{x} \in \tilde{E}_1 \cap \ldots \cap \tilde{E}_e = E$, i.e. $x \in E^2$.

It follows from Van den Dries [18, p. 90], that $P_1, \ldots, P_e$ are the only orderings of E. Hence $\{P_1, \ldots, P_e\} = \{Q_1, \ldots, Q_e\}$. Q.E.D.

Using the characterizations of $\hat{D}_e$ given by Lemma 4.1 and imitating the arguments in section 5 of [10], one can write down a sequence of sentences in $\mathscr{L}_e$ such that a field

E satisfies this sequence if and only if $G(E) \cong \hat{D}_e$. If we combine this sequence with the sequence Π of Lemma 2.4 we obtain an elementary characterization of Geyer fields.

LEMMA 4.4. *There exist an explicit set Π_e of sentences of $\mathcal{L}_e$ such that an e-fold ordered field $(E, P_1, \ldots, P_e)$ is a Geyer-field of corank e if and only if it satisfies Π_e.*

5. The elementary equivalence theorem for Geyer-fields

Let $\mathcal{E}=(E, P_1, \ldots, P_e)$ be an e-fold ordered field and let K be a subfield of E. Then $\tilde{K} \cap \mathcal{E}=(\tilde{K} \cap E, \tilde{K} \cap P_1, \ldots, \tilde{K} \cap P_e)$ is a substructure of $\mathcal{E}$.

LEMMA 5.1. *Let $\mathcal{E}=(E, Q_1, \ldots, Q_e)$ and $\mathcal{E}'=(E', Q_1', \ldots, Q_e')$ be two e-fold ordered fields and let $\mathcal{K}=(K, P_1, \ldots, P_e)$ be a common substructure. If $\mathcal{E} \equiv_K \mathcal{E}'$, then $\tilde{K} \cap \mathcal{E} \cong_K \tilde{K} \cap \mathcal{E}'$.*

Proof. The proof applies a general principle in the theory of models which was communicated to the author by Saharon Shelah.

Let $a_1, \ldots, a_n$ be elements of $\tilde{K} \cap E$ and denote by $\varphi_0(X_1, \ldots, X_n)$ the formula $[f_1(X_1)=0 \wedge \ldots \wedge f_n(X_n)=0]$, where $0 \neq f_i \in K[X_i]$ is the monic irreducible polynomial such that $f_i(a_i)=0$. For every quantifierfree formula $\varphi(X_1, \ldots, X_n)$ in $\mathcal{L}_e(K)$ such that $\mathcal{E} \models \varphi(a_1, \ldots, a_n)$ denote by $A'(\varphi)$ the set of all n-tuples $(a_1', \ldots, a_n') \in E'^n$ such that $\mathcal{E}' \models \varphi_0(a_1', \ldots, a_n') \wedge \varphi(a_1', \ldots, a_n')$. Clearly $A'(\varphi)$ is a finite non-empty set. Also, if $(a_1', \ldots, a_n') \in A'(\varphi)$, then $a_1', \ldots, a_n' \in \tilde{K} \cap E'$ and $\tilde{K} \cap E' \models \varphi_0(\mathbf{a}') \wedge \varphi(\mathbf{a}')$. The intersection $B'(a_1, \ldots, a_n)$ of all the sets $A'(\varphi)$ is a non-empty finite set, by e.g. the compactness theorem. Also, if $(a_1', \ldots, a_n') \in B'(a_1, \ldots, a_n)$, then $(a_1', \ldots, a_{n-1}') \in B'(a_1, \ldots, a_{n-1})$. Using compactness arguments once more one concludes the existence of a map $f\colon \tilde{K} \cap E \to \tilde{K} \cap E'$ such that if $a_1, \ldots, a_n \in \tilde{K} \cap E$ and if $\varphi(X_1, \ldots, X_n)$ is a quantifier-free formula of $\mathcal{L}_e(K)$ such that $\tilde{K} \cap \mathcal{E} \models \varphi(a_1, \ldots, a_n)$, then $\tilde{K} \cap \mathcal{E}' \models \varphi(f(a_1), \ldots, f(a_n))$. It is now not difficult to see that f is the desired isomorphism. Q.E.D.

In order to go in the other direction we need an analogue to a lemma of Gaschütz that plays an essential role in [10]. We start with some definitions.

Let $(E, P_1, \ldots, P_e)$ be an e-fold ordered field and let F be a normal extension of E which is not formally real. Then $(\beta_i, P(\beta_i))$ is said to be an *ordering-pair* for F/E if β_i is an involution of $\mathcal{G}(F/E)$ and $P(\beta_i)$ is an ordering of $F(\beta_i)$. If in addition, $P(\beta_i)$ extends P_i, then $(\beta_i, P(\beta_i))$ is said to be an *ordering pair* for $F/(E, P_i)$.

An e-tuple $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))=((\beta_1, P(\beta_1)), \ldots, (\beta_e, P(\beta_e)))$ is said to be *an e-tuple of ordering pairs for $F/(E, P_1, \ldots, P_e)$* if $(\beta_i, P(\beta_i))$ is an ordering pair for $F/(E, P_i)$, for $i=1, \ldots, e$.

If N is an additional normal extension of E that contains F, then an e-tuple $(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$ of ordering pairs for $N/(E, P_1, \ldots, P_e)$ is said to be an *extension* of $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ if $\mathrm{Res}_F \gamma_i = \beta_i$ and if $F \cap P(\gamma_i) = P(\beta_i)$ for $i = 1, \ldots, e$.

LEMMA 5.2 (D. Haran). *Let (E, P) be an ordered field and let F be a finite normal extension of E which is not formally real. Then the number of ordering-pairs $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ for $F/(E, P)$ is equal to $\frac{1}{2}[F:E]$.*

Proof. Let $\bar{E}$ be a real closure of (E, P) and denote $L = F \cap \bar{E}$, $Q = L \cap \bar{E}^{\times 2}$. Then $L = F(\varepsilon)$, where ε is an involution of $\mathcal{G}(F/E)$, since F is not formally real, and Q is an ordering of L that extends P. The set Σ of all E-isomorphisms of L into $\bar{E}$ has $\frac{1}{2}[F:E]$ elements. To every $\sigma \in \Sigma$ there corresponds an ordering-pair for $F/(E, P)$, namely $(\varepsilon^\sigma, Q^\sigma)$. If $\sigma, \tau \in \Sigma$ and $(\varepsilon^\sigma, Q^\sigma) = (\varepsilon^\tau, Q^\tau)$, then $\sigma\tau^{-1}$ is an automorphism of L over E that leaves Q invariant. It can be extended to an automorphism of $\bar{E}/E$, and hence $\sigma = \tau$. If $(\boldsymbol{\delta}, \mathbf{P}(\boldsymbol{\delta}))$ is an ordering-pair for $F/(E, P)$ and $\overline{F(\delta)}$ is a real closure of $(F(\delta), P(\delta))$, then $\overline{F(\delta)}$ is also a real closure of (E, P). Hence there exists a $\bar{\sigma} \in G(E)$ such that $\overline{F(\delta)} = \bar{E}^{\bar{\sigma}}$. Then $(\delta, P(\delta)) = (\varepsilon^\sigma, Q^\delta)$, where $\sigma = \mathrm{Res}_L \bar{\sigma}$. It follows that the number of ordering-pairs for $F/(E, P)$ is equal to $|\Sigma| = \frac{1}{2}[F:E]$. Q.E.D.

LEMMA 5.3. *Let $(E, P_1, \ldots, P_e)$ be an e-fold ordered field, let F be a normal algebraic extension of E which is not formally real and let $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ be an e-tuple of ordering pairs for $F/(E, P_1, \ldots, P_e)$ such that $\beta_1, \ldots, \beta_e$ generate $\mathcal{G}(F/E)$.*

Let N be a normal extension of E that contains F. Suppose that there exists an e-tuple of ordering pairs $(\boldsymbol{\zeta}, \mathbf{P}(\boldsymbol{\zeta}))$ for $N/(E, P_1, \ldots, P_e)$ such that $\zeta_1, \ldots, \zeta_e$ generate $\mathcal{G}(N/E)$.

Then there exists an e-tuple $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ of ordering pairs for N/E that extends $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ such that $\varepsilon_1, \ldots, \varepsilon_e$ generate $G(N/E)$.

Proof. Using compactness arguments twice one is reduced to the case where both F and N have finite degrees over E.

Denote by J the set of all fields K between E and N such that $F \cap K = E$ *and* $P_1, \ldots, P_e$ can be extended to orderings of K.

Let $K \in J$. Denote by $J(K, \mathbf{P})$ the set of all e-tuples $\mathbf{Q} = (Q_1, \ldots, Q_e)$ of orderings of K that extend $\mathbf{P}$.

For every $\mathbf{Q}$ in $J(K, \mathbf{P})$ there exists a unique e-tuple $(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$ of ordering pairs for $L/(K, Q_1, \ldots, Q_e)$ that extends $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ (see Section 1) where $L = FK$. Denote by

$H(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$ the set of all e-tuples $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ of ordering pairs for N/K that extend $(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$. By Lemma 5.2 we have

$$|H(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))| = (\tfrac{1}{2}[N: L(\gamma_1)])^e = [N: L]^e \tag{5.1}$$

Note that $\gamma_1, \ldots, \gamma_e$ generate $\mathscr{G}(L/K)$. Hence, it makes sense to consider the set $I(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma})))$ of all $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ in $H(K,\mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$ that satisfy $K=N(\varepsilon_1, \ldots, \varepsilon_e)$.

Claim. If $(\boldsymbol{\gamma}', \mathbf{P}(\boldsymbol{\gamma}'))$ is an additional e-tuple of ordering pairs in $L/(K,\mathbf{Q})$ such that $\gamma_1', \ldots, \gamma_e'$ generate $\mathscr{G}(L/K)$, then $I(K, \mathbf{Q},(\boldsymbol{\gamma}', \mathbf{P}(\boldsymbol{\gamma}')))$ has the same number of elements as $I(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma})))$. In other words, $f(K, \mathbf{Q})=|I(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma})))|$ does not depend on $(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$.

We prove our claim by induction on $[N: K]$. Suppose the claim has already been proved for every K in J that properly contains E. We prove that also $|I(E, \mathbf{P},(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta})))|$ does not depend on $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$. Indeed, let $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ be in $H(E, \mathbf{P},(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta})))$. Then $K=N(\varepsilon_1, \ldots, \varepsilon_e)$ belongs to J. Let $Q_i=K \cap P(\varepsilon_i)$ and $\gamma_i=\mathrm{Res}_{FK}\, \varepsilon_i$ and $P(\gamma_i)=FK(\gamma_i) \cap P(\varepsilon_i)$ for $i=1, \ldots, e$. Then $Q_1, \ldots, Q_e$ are orderings of K, $(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma}))$ is the unique e-tuple for $FK/(K, Q_1, \ldots, Q_e)$ that extends $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ and $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ belongs to $I(K,\mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma})))$. Obviously, these objects are uniquely determined by $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$. It follows that

$$H(E, P,(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))) = \bigcup_{K \in J} \bigcup_{\mathbf{Q} \in J(K, \mathbf{P})} I(K, \mathbf{Q},(\boldsymbol{\gamma}, \mathbf{P}(\boldsymbol{\gamma})))$$

Obviously $J(E, \mathbf{P})$ contains only one element, namely $\mathbf{P}$. Hence, by an induction hypothesis and by (1)

$$[N: F]^e = |I(E, \mathbf{P},(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))| + \sum_{\substack{K \in J \\ K \neq E}} \sum_{Q \in J(K, \mathbf{P})} f(K, \mathbf{Q})$$

and our claim is proved.

Return now to the given e-tuple $(\boldsymbol{\zeta}, \mathbf{P}(\boldsymbol{\zeta}))$ and denote its restriction to F by $(\boldsymbol{\zeta}', \mathbf{P}(\boldsymbol{\zeta}'))$. Then $\zeta_1', \ldots, \zeta_e'$ generate $\mathscr{G}(F/E)$ and $(\boldsymbol{\zeta}, \mathbf{P}(\boldsymbol{\zeta}))$ belongs to $I(E, \mathbf{P},(\boldsymbol{\zeta}', \mathbf{P}(\boldsymbol{\zeta}')))$. It follows that $I(E, \mathbf{P},(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta})))$ is not empty either. Every e-tuple $(\boldsymbol{\varepsilon}, \mathbf{P}(\boldsymbol{\varepsilon}))$ in it extends $(\boldsymbol{\beta}, \mathbf{P}(\boldsymbol{\beta}))$ and $\varepsilon_1, \ldots, \varepsilon_e$ generate $\mathscr{G}(N/E)$. Q.E.D.

Remark. The proof Lemma 5.3 is modeled after an unpublished version of the proof of Gaschütz' lemma, due to Peter Roquette.

THEOREM 5.4. *Let $\mathscr{E}=(E, P_1, \ldots, P_e)$ and $\mathscr{F}=(F, Q_1, \ldots, Q_e)$ be two Geyer fields and let L be a common subfield of E and F. If $\tilde{L} \cap \mathscr{E} \cong_L \tilde{L} \cap \mathscr{F}$, then* $\mathscr{E} \equiv_L \mathscr{F}$.

Proof. Without loss of generality we may assume that $\tilde{L}\cap\mathscr{E}=\tilde{L}\cap\mathscr{F}=(M,S_1,\dots,S_e)$. Let $\varepsilon_1,\dots,\varepsilon_e$ be involutions that generate $G(E)$. By Lemma 4.3 they induce $P_1,\dots,P_e$, respectively, on E. Let $\gamma_i=\mathrm{Res}_{\tilde{L}}\,\varepsilon_i$, for $i=1,\dots,e$. Then $\gamma_1,\dots,\gamma_e$ are involutions that generate $G(M)$ and induce $S_1,\dots,S_e$ on M, respectively. The fields $M(\gamma_i)$ and F are linearly disjoint over M, hence there exists an e-tuple of ordering pairs $(\boldsymbol{\delta},\mathbf{P}(\boldsymbol{\delta}))$ for $\tilde{M}F/(F,Q_1,\dots,Q_e)$ that extends $(\boldsymbol{\gamma},\mathbf{P}(\boldsymbol{\gamma}))$. Again, by Lemma 4.3, there exist involutions $\zeta_1',\dots,\zeta_e'$ that generate $G(F)$ and induce $Q_1,\dots,Q_e$ on F, respectively. By Lemma 5.3 there exists an e-tuple of ordering pairs $(\boldsymbol{\zeta},\mathbf{P}(\boldsymbol{\zeta}))$ for $\tilde{F}/F$ that extends $(\boldsymbol{\delta},\mathbf{P}(\boldsymbol{\delta}))$ such that $\zeta_1,\dots,\zeta_e$ generate $G(F)$. The map $\zeta_i\mapsto\varepsilon_i$ for $i=1,\dots,e$ can be extended to an isomorphism φ: $G(F)\to G(E)$ such that $\mathrm{Res}_{\tilde{L}}\,\varphi(\sigma)=\mathrm{Res}_{\tilde{L}}\,\sigma$, since both $G(E)$ and $G(F)$ are isomorphic to $\hat{D}_e$. It follows, by Theorem 3.2, that $\mathscr{E}\equiv_L\mathscr{F}$. Q.E.D.

COROLLARY 5.5. *If $(E,P_1,\dots,P_e)\subseteq(F,Q_1,\dots,Q_e)$ are two Geyer-fields such that E is algebraically closed in F, then $(E,P_1,\dots,P_e)$ is an elementary substructure of $(F,Q_1,\dots,Q_e)$.*

6. The existence of Geyer-fields

We start by proving some new technical lemmas on the normalized Haar measure μ of the absolute Galois group of a field K.

LEMMA 6.1. *Let K be a field, let $K_1,K_2,\dots,K_n$ be finite separable extensions of K, linearly disjoint over K. Denote by K' the composition of $K_1,\dots,K_n$ and let L be a finite Galois extension of K that contains K'. Let $\sigma_1,\dots,\sigma_n\in\mathscr{G}(L/K)$.* Then

$$\frac{1}{[L:K]}\#\{\sigma\in\mathscr{G}(L/K)|\mathrm{Res}_{K_i}\sigma_i=\mathrm{Res}_{K_i}\sigma_i\quad \textit{for } i=1,\dots,n\}=\frac{1}{[K':K]}$$

Proof. Our assumption implies that $K'\cong_K K_1\otimes_K K_2\otimes_K\dots\otimes_K K_n$. Hence there exists a unique K-isomorphism σ' of K' into L such that $\mathrm{Res}_{K_i}\sigma'=\mathrm{Res}_{K_i}\sigma_i$ for $i=1,\dots,n$. This isomorphism can be extended exactly in $[L:K']$ ways to a K-automorphism of L. Our formula follows. Q.E.D.

COROLLARY 6.2. *Let K be a field, let $K_1,\dots,K_n$ be finite separable extensions of K, linearly disjoint over K and let $\sigma_1,\dots,\sigma_n\in G(K)$.* For $i=1,\dots,e$ *denote*

$$S_i=\{\sigma\in G(K)|\mathrm{Res}_{K_i}\sigma=\mathrm{Res}_{K_i}\sigma_i\}$$

Then

$$\mu(S_i) = \frac{1}{[K_i : K]}$$

and $S_1, \ldots, S_n$ *are* μ*-independent.*

LEMMA 6.3. *Let* K *be a field and let* $K_1, K_2, K_3, \ldots,$ *be a linearly disjoint sequence of finite separable extensions of* K. *Let also*

$\{(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{ie}) | i = 1, 2, 3, \ldots\}$ *be a sequence of e-tuples of elements of* $G(K)$. *For every i denote*

$$S_i = \{(\sigma_1, \ldots, \sigma_e) \in G(K)^e | \mathrm{Res}_{K_i} \sigma_j = \mathrm{Res}_{K_i} \sigma_{ij} \quad for\ j = 1, \ldots, e\}.$$

Then

$$\mu\left(\bigcup_{i=1}^{\infty} S_i\right) = 1 \quad \text{if} \sum_{i=1}^{\infty} \frac{1}{[K_i : K]^e} = \infty.$$

Proof. Corollary 6.2 implies that the sequence $S_1, S_2, S_3, \ldots$ is μ-independent and that $\mu(S_i) = [K_i : K]^{-e}$. Our conclusion follows now in the usual way (cf. [7, Lemma 1.10] or [9, Lemma 1.1]). Q.E.D.

Suppose now that K is a countable Hilbertian field equipped with e-orderings $P_1, \ldots, P_e$. Let $\bar{K}_1, \ldots, \bar{K}_e$ be some fixed real closures of K that induce the orderings $P_1, \ldots, P_e$, respectively. For every $\sigma_1, \ldots, \sigma_e \in G(K)$ let $K_\sigma = \bar{K}_1^{\sigma_1} \cap \ldots \cap \bar{K}_e^{\sigma_e}$ and denote by $P_{\sigma 1}, \ldots, P_{\sigma e}$ the orderings of K_σ induced by $\bar{K}_1^\sigma, \ldots, \bar{K}_e^\sigma$, respectively. Then $\mathcal{K}_\sigma = (K_\sigma, P_{\sigma 1}, \ldots, P_{\sigma e})$ is an e-fold ordered field that extends $\mathcal{K} = (K, P_1, \ldots, P_e)$.

LEMMA 6.4 *Let* $\mathcal{L} = (L, Q_1, \ldots, Q_e)$ *be a finite extension of* $\mathcal{K}$. *Let* $f \in L[T_1, \ldots, T_r, X]$ *be an absolutely irreducible polynomial and let* $0 \neq g \in L[T_1, \ldots, T_r]$. *Suppose that there exists an* $\mathbf{a}_0 \in L^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on L with respect to each of the* Q_i*'s. Let* U_i *be a* Q_i*-neighbourhood of* $\mathbf{a}_0$ *in* L^r. *Then for almost all* $(\sigma_1, \ldots, \sigma_e) \in G(K)^e$ *for which* $Q_1, \ldots, Q_e$ *are induced by* $\bar{K}_1^{\sigma_1}, \ldots, \bar{K}_e^{\sigma_e}$, *respectively, there exists an* $(\mathbf{a}, b) \in K^{r+1}$ *such that* $\mathbf{a} \in U_1 \cap \ldots \cap U_e$, $f(\mathbf{a}, b) = 0$ *and* $g(\mathbf{a}) \neq 0$.

Proof. Let $\bar{L}_i$ be a real closure of L that induces Q_i. Then there exists a $\tau_i \in G(K)$ such that $\bar{L}_i = \bar{K}_i^{\tau_i}$. If $\sigma_i \in G(K)$ is an additional element such that $\bar{K}_i^{\sigma_i}$ induces Q_i on L, then there exists a $\lambda \in G(L)$ such that $\bar{K}_i^{\sigma_i} = \bar{K}_i^{\tau_i \lambda}$. Hence $\sigma_i = \tau_i \lambda$, since $\bar{K}_i$ has no K-

automorphisms besides the identity (cf. Ribenboim [15, p. 163]). Conversely $\bar{K}_i^{\tau_i\lambda}$ induces the ordering Q_i on L for every $\lambda \in G(L)$. It follows that $(\tau_1 G(L), \dots, \tau_e G(L))$ is the set of all e-tuples $(\sigma_1, \dots, \sigma_e)$ in $G(K)^e$ for which Q_i is induced by $\bar{K}_i^{\sigma_i}$ for $i=1, \dots, e$.

Without loss of generality we may assume that $f(\mathbf{a}, X)$ changes sign with respect to Q_i for every $\mathbf{a} \in U_i$ for $i=1, \dots, e$. By Lemma 8.4 of Geyer [5] and since L is Hilbertian the set $H \cap U_1 \cap \dots \cap U_e$ is not empty for every Hilbertian set H in L^r. Using the fact that f is absolutely irreducible one can find a sequence $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots,$ of elements L^r and a sequence $b_1, b_2, b_3, \dots$ of elements of $\tilde{L}$ such that

(a) $\mathbf{a}_j \in U_1 \cap \dots \cap U_e$ and $f(\mathbf{a}_j, X)$ is an irreducible polynomial over L of degree $n=\deg_X f$ and changes sign on L with respect to Q_i for every i and j;

(b) $f(\mathbf{a}_j, b_j)=0$ and $g(\mathbf{a}_j)\neq 0$ for every j;

(c) denoting $L_j=L(b_j)$, we have that $L_1, L_2, L_3, \dots$ is a linearly disjoint sequence of extensions of L of degree n (cf. the proof of Lemma 2.2 of [7]).

Condition (a) implies that each of the Q_i can be extended to an ordering Q_{ij} of L_j. Let $\bar{L}_{ij}$ be a real closure of L_j that induces Q_{ij}. Then $\bar{L}_{ij}$ induces Q_i on L. It follows that there exists $\varrho_{ij} \in G(L)$ such that $\bar{L}_{ij}=\bar{L}_i^{\varrho_{ij}}$. Denote by Λ the set of all $(\lambda_1, \dots, \lambda_e) \in G(L)^e$ for which there exists a j such that λ_i coincides with ϱ_{ij}^{-1} on L_j for $i=1, \dots, e$. By Lemma 6.3, Λ differs from $G(L)^e$ only by a zero set. In addition, the map $\sigma \mapsto \sigma^{-1}$ is measure preserving, hence

$$\Sigma = \{(\sigma_1, \dots, \sigma_e) \in G(K)^e | \exists (\lambda_1, \dots, \lambda_e) \in \Lambda \colon \sigma_i = \tau_i \lambda_i^{-1} \quad \text{for } i=1, \dots, e\}$$

differs from the set $(\tau_1 G(L), \dots, \tau_e G(L)$ only by a zero set.

Let $(\sigma_1, \dots, \sigma_e) \in \Sigma$. Then there exists $(\lambda_1, \dots, \lambda_e) \in \Lambda$ such that $\sigma_i = \tau_i \lambda_i^{-1}$ for $i=1, \dots, e$. By construction, there exists a j such that λ_i coincides with ϱ_{ij}^{-1} on L_j, hence there exists a $\nu_i \in G(L_j)$ such that $\lambda_i = \nu_i^{-1} \varrho_{ij}^{-1}$ for $i=1, \dots, e$. Hence $\bar{K}_i^{\sigma_i} = \bar{K}_i^{\tau_i \varrho_{ij} \nu_i} = \bar{L}_{ij}^{\nu_i} \supseteq L_j$. Hence $(\mathbf{a}_j, b_j) \in K_\sigma^{r+1}$. Q.E.D.

COROLLARY 6.5. *Almost all* $(\boldsymbol{\sigma}) \in G(K)^e$ *have the following property: If* $f \in K_\sigma[T_1, \dots, T_r, X]$ *is an absolutely irreducible polynomial for which there exists an* $\mathbf{a}_0 \in K_\sigma^r$ *such that* $f(\mathbf{a}_0, X)$ *changes sign on* K_σ *with respect to each of the orderings* $P_{\sigma i}$, *if* U_i *is a* $P_{\sigma i}$*-neighbourhood of* $\mathbf{a}_0$ *for* $i=1, \dots, e$, *and if* $0 \neq g \in K_\sigma[T_1, \dots, T_r]$, *then there exists an* $(\mathbf{a}, b) \in K_\sigma^{r+1}$ *such that* $\mathbf{a} \in U_1 \cap \dots \cap U_e$, $f(\mathbf{a}, b)=0$ *and* $g(\mathbf{a})\neq 0$.

Proof. Here one uses the countability of K together with the observation that if f is as above, then there exists a finite extension L of K over which everything happens. Compare also the proof of Theorem 2.5 of [7]. Q.E.D.

LEMMA 6.6. *The orderings $P_{\sigma 1}, \ldots, P_{\sigma e}$ induce distinct topologies on K_σ for almost all $\boldsymbol{\sigma} \in G(K)^e$.*

Proof. It suffices to prove that for every $1 \leq i < j \leq e$, for every $d \in K$ such that $0 <_i d$, $0 <_j d$ and for almost all $\boldsymbol{\sigma} \in G(K)^e$

(1) there exists a $z \in K_\sigma$ such that $-d <_i z <_i d$ and $1-d <_j z <_j 1+d$. Also, without loss, let $i=1$ and $j=2$.

Consider the polynomial $g(Z)=Z^2-Z$ and consider the absolutely irreducible polynomial $f(X, Y, Z)=Z^2-XZ+Y$. As in the proof of Lemma 6.4 we construct a sequence of triples (a_j, b_j, c_j) for $j=1,2,3,\ldots$, such that

(a) $(a_j, b_j) \in K \times K$ are P_k-close to $(1,0)$ for $k=1,\ldots,e$,

(b) $g_j(Z)=Z^2-a_jZ+b_j$ is irreducible having c_j as a root,

(c) the sequence $L_j=K(c_j)$, $j=1,2,3,\ldots$ of quadratic extensions of K is linearly disjoint.

The polynomial $g_j(Z)$ us P_k-close to $g(Z)=Z(Z-1)$. Hence, in every real closure R of K that induces P_k there exists two roots z_1 and z_2 of $g_j(Z)$ satisfying $-d <_k z_1 <_k d$ and $1-d <_k z_2 <_k 1+d$. In particular $L_j \subset K_\sigma$ for every $\boldsymbol{\sigma} \in G(K)^e$. Taking $R=\bar{K}_1$, the map $c_j \to z_i$ induces an ordering P_{1j} of L_j such that $-d <_{1j} c_j <_{1j} d$. Similarly the map $c_j \to z_2$ induces an ordering P_{2j} of L_j such that $1-d <_{2j} c_j <_{2j} 1+d$. Let $\bar{L}_{1j}$ and $\bar{L}_{2j}$ be real closures of L_j that induce the orderings P_{1j} and P_{2j}, respectively. Then there exist $\varrho_{1j}, \varrho_{2j} \in G(K)$ such that $\bar{L}_{kj} = \bar{K}_k^{\varrho_{kj}}$ for $k=1,2$.

By Lemma 6.3 and by (c), for almost all the e-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_e) \in G(K)^e$ there exists a j such that σ_k coincides with ϱ_{kj} on L_j, for $k=1,2$. Consider such an e-tuple. Then there exist $\lambda_1, \lambda_2 \in G(L_j)$ such that $\sigma_k = \varrho_{kj}\lambda_k$ for $k=1,2$. Hence $\bar{K}_k^{\sigma_k} = \bar{L}_{kj}^{\lambda_k}$ and therefore $\bar{K}_k^{\sigma_k}$ induces the orderings P_{kj} on L_j. Recalling that $c_j \in K_\sigma$ we obtain that $\boldsymbol{\sigma}$ satisfies (1). Q.E.D.

To Lemmas 2.3, 6.5 and 6.6 we add now the following remarkable result of Wulf-Dieter Geyer:

$$G(K_\sigma) \cong \hat{D}_e \quad \text{for almost all } \boldsymbol{\sigma} \in G(K)^e$$

(cf. [5, Theorem 4.3]) and conclude:

THEOREM 6.7. *Let K be a countable Hilbertian field with e orderings $P_1, \ldots, P_e$. Then $\mathcal{K}_\sigma$ is a Geyer field for almost all $\boldsymbol{\sigma} \in G(K)^e$.*

7. The theory of almost all $\mathcal{K}_\sigma$

We continue to consider an e-fold ordered field $(K, P_1, \dots, P_e)$, where K is a countable Hilbertian field, and retain the convention made in Section 6.

Recall that an ultrafilter $\mathcal{D}$ of $G(K)^e$ is said to be *regular*, if $\mathcal{D}$ contains all subsets of $G(K)^e$ of measure 1 (cf. [10, p. 287]).

LEMMA 7.1. *For every $\boldsymbol{\tau} \in G(K)^e$ there exists a regular ultrafilter D of $G(K)^e$ such that $\mathcal{K}_\tau \cong_K \tilde{K} \cap \Pi\mathcal{K}_\sigma/\mathcal{D}$.*

Proof. Let L be a finite Galois extension of K. For every $\lambda \in \mathcal{G}(L/K)$ we denote

$$S(L, \lambda) = \{\boldsymbol{\sigma} \in G(K)^e | \mathrm{Res}_L\, \sigma_i = (\mathrm{Res}_L\, \tau_i)\lambda \quad \text{for } i = 1, \dots, e\}$$

and let $S(L)$ to be the union of the $S(L, \lambda)$'s. Clearly $S(L, \lambda)$ and hence $S(L)$ are non-empty open subsets of $G(K)^e$. If L' is a finite Galois extension of K that contains L, then $S(L') \subseteq S(L)$. It follows that the intersection of finitely many sets of the form $S(L)$ is a non-empty open set. By [10, Lemma 6.1] there exists a regular ultrafilter $\mathcal{D}$ of $G(K)^e$ that contains all the sets $S(L)$.

Let $F = \Pi K_\sigma/\mathcal{D}$, $Q_i = \Pi P_{\sigma i}/\mathcal{D}$ and $\bar{F}_i = \tilde{F} \cap \Pi \bar{K}_{\sigma i}/\mathcal{D}$. Then $\mathcal{F} = (F, Q_1, \dots, Q_e) = \Pi\mathcal{K}_\sigma/\mathcal{D}$ and $\bar{F}_i$ is a real closure of (F, Q_i).

Consider a finite Galois extension L of K. Then $S(L) \in \mathcal{D}$. Hence there exists a $\lambda \in \mathcal{G}(L/K)$ such that $S(L, \lambda) \in \mathcal{D}$. If $\boldsymbol{\sigma} \in S(L, \lambda)$, then $L \cap \bar{K}_{\sigma i} = (L \cap \bar{K}_{\tau i})^\lambda$, hence $L \cap \bar{F}_i = (L \cap \bar{K}_{\tau i})^\lambda$ for $i = 1, \dots, e$.

Let now L run over all the finite Galois extensions of K. Using compactness arguments one obtains an element $\lambda \in G(K)$ such that $\tilde{K} \cap \bar{F}_i = \bar{K}_{\tau i}^\lambda$ for $i = 1, \dots, e$. Hence $\tilde{K} \cap \mathcal{F} \cong_K \mathcal{K}_\tau$. Q.E.D.

THEOREM 7.2. *Let $\mathcal{K} = (K, P_1, \dots, P_e)$ be an e-fold ordered field such that K is countable and Hilbertian. Then a sentence θ of $\mathcal{L}_e(K)$ is true in all Geyer fields of co-rank e that extend $\mathcal{K}$ if and only if θ is true in $\mathcal{K}_\sigma$ for almost all $\boldsymbol{\sigma} \in G(K)^e$.*

Proof. One direction follows from Theorem 6.7.

Suppose in the other direction that θ is true in $\mathcal{K}_\sigma$ for almost all $\boldsymbol{\sigma} \in G(K)^e$ and let $\mathcal{E} = (E, Q_1, \dots, Q_e)$ be a Geyer field that extends $\mathcal{K}$. Then $\tilde{K} \cap \mathcal{E} = \mathcal{K}_\tau$ for some $\boldsymbol{\tau} \in G(K)^e$. By Lemma 7.1, there exists a regular ultrafilter $\mathcal{D}$ of $G(K)^e$ such that $\tilde{K} \cap \Pi\mathcal{K}_\sigma/\mathcal{D} \cong_K \mathcal{K}_\tau$. It follows from Theorem 5.4 that $\Pi\mathcal{K}_\sigma/\mathcal{D} \equiv_K \mathcal{E}$. The sentence θ is true in $\Pi\mathcal{K}_\sigma/\mathcal{D}$, since $\mathcal{D}$ is regular, hence it is true in $\mathcal{E}$. Q.E.D.

8. The probability of a sentence to be true

Let $\mathcal{K}=(K, P_1, \ldots, P_e)$ be again an e-fold ordered field with K a countable Hilbertian field. For every sentence θ of $\mathcal{L}_e(K)$ denote $A(\theta)=\{\boldsymbol{\sigma}\in G(K)^e | \mathcal{K}_\sigma \models \theta\}$. The measure of $A(\theta)$, which we shall prove to exist, can be viewed as the probability of θ to be true among the Geyer fields.

Denote by $\mathcal{A}_0$ the family of all sentences of the form $A(\lambda_0)$, where λ_0 is a sentence of the form

(1) $\exists X_1 \ldots \exists X_n \varphi(X_1, \ldots, X_n)$, where $\varphi(X_1, \ldots, X_n)$ is a quantifier-free formula of $\mathcal{L}_e(K)$ for which there exist finitely many n-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_m$ in $\tilde{K}^n$ such that if $\mathcal{L}=(L, Q_1, \ldots, Q_n)$ is an extension of $\mathcal{K}$ and if $\mathbf{a}$ is an n-tuple in L^n such that $\mathcal{L} \models \varphi(\mathbf{a})$, then $\mathbf{a}$ is equal to one of the a_i's.

Note that if $L=(L, Q_1, \ldots, Q_n)$ and $\mathcal{L}'=(L', Q_1', \ldots, Q_n')$ are two extensions of $\mathcal{K}$, if $\mathbf{a}\in L^n$ and if $L\subseteq L'$, then $\mathcal{L} \models \varphi(\mathbf{a})$ if and only if $\mathcal{L}' \models \varphi(\mathbf{a})$. Also, we have seen in the proof of Lemma 5.1 that if $\mathcal{E}$ and $\mathcal{F}$ are two extensions of $\mathcal{K}$ and if they satisfy the same sentences (1), then $\tilde{K}\cap\mathcal{E}\cong_K\tilde{K}\cap\mathcal{F}$.

Denote by $\mathcal{A}$ the Boolean algebra of $G(K)^e$ generated by $\mathcal{A}_0$ and all the zero sets. If θ is an arbitrary sentence of $\mathcal{L}_e(K)$, then $A(\theta)$ belongs to $\mathcal{A}$. Otherwise there exist two regular ultrafilters $\mathcal{D}_1$ and $\mathcal{D}_2$ of $G(K)^e$ such that $\mathcal{D}_1\cap\mathcal{A}=\mathcal{D}_2\cap\mathcal{A}$ and $A(\theta)\in\mathcal{D}_1$ and $A(\theta)\notin\mathcal{D}_2$ (cf. the proof of Theorem 7.4 in [10]). Hence $\tilde{K}\cap\Pi\mathcal{K}_\sigma/\mathcal{D}_1\cong_K\tilde{K}\cap\Pi\mathcal{K}_\sigma/\mathcal{D}_2$. It follows by Theorem 5.4 that $\Pi\mathcal{K}_\sigma/\mathcal{D}_1\equiv_K\Pi\mathcal{K}_\sigma/\mathcal{D}_2$, a contradiction. It follows that there exists a sentence λ which is a Boolean combination of sentences of the form (1) such that $A(\theta)$ differs from $A(\lambda)$ only by a zero set.

By construction there exists a finite Galois extension L of K and there exist finitely many extensions $\mathcal{K}_i=(K_i, P_{i1}, \ldots, P_{ie})$, $i=1, \ldots, s$ of $\mathcal{K}$ such that $K_i\subseteq L$ and such that λ is true in $\mathcal{K}_\sigma$ if and only if $L\cap\mathcal{K}_\sigma=\mathcal{K}_i$, for some $1\leq i\leq s$. It follows that $A(\lambda)$ is an open-closed subset of $G(K)^e$ and that $\mu(A(\lambda))$ is a rational number (see also the proof of Theorem 7.5 of [10]).

We have therefore proved

THEOREM 8.1. *If θ is a sentence of $\mathcal{L}_e(K)$, then $A(\theta)$ is a measurable set and $\mu(A(\theta))$ is a rational number.*

9. A decision procedure

We come now to the case where the basic field is the field $\mathbf{Q}$ of rational numbers and all the e orderings are equal to the one $\mathbf{Q}$ has. We show that in this case there is a recursive procedure to compute the rational numbers $\mu(A(\theta))$. In particular it is possible to determine when $\mu(A(\theta))=1$, i.e. when θ is true in all Geyer-fields. In order to do it we have first to improve Lemma 5.1. We start with the following notion.

A *test sentence* in $\mathcal{L}_e(\mathbf{Q})$ is a sentence of the form

$$(\exists ! X)\left[f(X)=0 \wedge \bigwedge_{i=1}^{e} a_i <_i X <_i a_i + r\right], \tag{9.1}$$

where $f \in \mathbf{Q}[X]$, $a_1, \dots, a_e, r \in \mathbf{Q}$, $r>0$ and '$\exists ! X$' means there exists a unique X. A *one variable sentence* in $\mathcal{L}_e(\mathbf{Q})$ is a Boolean combination of test sentences.

LEMMA 9.1. *Let $\mathcal{L}=(L, P_1, \dots, P_e)$ and $\mathcal{L}'=(L'_1, P'_1, \dots, P'_e)$ be two e-fold ordered fields such that L and L'* are algebraic over $\mathbf{Q}$. *If $\mathcal{L}$ and $\mathcal{L}'$ satisfy the same test sentences, then they are isomorphic.*

Proof. Let $x \in L$ and let $f \in \mathbf{Q}[X]$ be an irreductible polynomial such that $f(x)=0$. Then there exist $a_1, \dots, a_e, r \in \mathbf{Q}$ such that

$$a_i <_i x <_i a_i + r \quad \text{for } i=1, \dots, e \tag{9.2}$$

and x is the unique root of f that satisfies these inequalities. By assumption there exists a unique $x' \in L'$ such that $f(x')=0$ and $a_i <_i x' <_i a_i + r$ for $i=1, \dots, e$. This x' then does not depend on $a_1, \dots, a_e, r$. Indeed, let also $b_1, \dots, b_e, s$ be elements of $\mathbf{Q}$ such that x is the unique root of f in L that satisfies $b_i < x < b_i + s$ for $i=1, \dots, e$ and let x'' be the unique element of L' that satisfies $f(x'')=0$ and $b_i < x'' < b_i + s$ for $i=1, \dots, e$. Define $c_i = \max\{a_i, b_i\}$, $i=1, \dots, e$, and $t=\min\{r, s\}$. Then x is the unique root of f in L that satisfies $c_i <_i x <_i c_i + t$ for $i=1, \dots, e$. Both x' and x'' satisfy the same inequalities. Hence $x'=x''$.

The map φ: $x \mapsto x'$ from L into L' is therefore well defined. It satisfies

$$\bigwedge_{i=1}^{e} a_i <_i x <_i a_i + r \Rightarrow \bigwedge_{i=1}^{e} a_i <_i \varphi(x) <_i a_i + r$$

for every $a_1, \dots, a_e, r \in \mathbf{Q}$.

Clearly φ maps every element of $\mathbf{Q}$ on itself. We show that φ is additive. Let $x, y, z \in L$ such that $x+y=z$. Let $a_i, b_i, c_i, r \in \mathbf{Q}$ such that

$$a_i <_i x <_i a_i + r \text{ and } b_i <_i y <_i b_i + r \text{ and } c_i <_i z <_i c_i + r, \text{ for } i=1, \dots, e.$$

Then
$a_i <_i \varphi(x) <_i a_i + r$ and $b_i <_i \varphi(y) <_i b_i + r$ and $c_i <_i \varphi(z) <_i c_i + r$ for $i=1, \dots, e$. It follows that $c_1 - 2r <_1 \varphi(x) + \varphi(y) <_1 c_1 + 3r$, hence $-3r <_1 \varphi(z) - (\varphi(x) + \varphi(y)) <_1 3r$. This inequality

is true for every $r>0$, hence $\varphi(z)=\varphi(x)+\varphi(y)$. Similarly one shows that φ is multiplicative, preserves the orderings and surjective. It follows that φ is an isomorphism of $\mathcal{L}$ onto $\mathcal{L}'$. Q.E.D.

THEOREM 9.2. *The function that assigns to every sentence θ of $\mathcal{L}_e(\mathbf{Q})$ the rational number $\mu(A'(\theta))$ is recursive. In particular the theory of Geyer-fields of corank e is (recursively) decidable.*

Proof. As in Section 8, one concludes that for every sentence θ there exists a one variable sentence λ such that $A(\theta)=A(\lambda)$. Using Gödel completeness theorem and Lemma 4.4 one can find λ, once θ is given (cf. [10, the proof of Theorem 8.2]). We may therefore assume that $\mathbf{Q}$ is a one variable sentence.

In order to simplify notation assume that θ is a sentence of the form (9.1) and $f(X)$ is an irreducible polynomial. For every $1\leq i\leq e$ we check, e.g. by Sturm's sequences if $f(X)$ has a root in R in the interval (a_i, a_i+r). If there exists an i such that there is none, $\mu(A(\theta))=0$. Suppose therefore that the answer is positive. By dividing (a_i, b_i) in $\mathbf{Q}$ to subintervals and changing θ appropriately we can finally assume that each interval (a_1, a_i+r) in R contains exactly one root x_i of f.

Construct the splitting field L of $f(X)$ over $\mathbf{Q}$. Let x be a root of $f(X)$, let $K=\mathbf{Q}(x)$ and let $\varepsilon\in\mathcal{G}(L/K)$ such that $\varepsilon^2=1$ and $\varepsilon x=x$. Find an irreducible polynomial $g\in\mathbf{Q}[x]$, a root y of which generates $L(\varepsilon)$. Find a polynomial $h\in\mathbf{Q}[X]$ such that $x=h(y)$ and check whether the following sentence is ture in $\mathbf{R}$.

$$(\exists Y)[g(Y)=0\wedge f(h(Y))=0\wedge a_i<h(Y)<b_i].$$

If this sentence is true and $y'\in\mathbf{R}$ satisfies the formula in the brackets, then the map $y\mapsto y'$ induces an ordering on $L(\varepsilon)$ and $h(y')=x_i$. In this ordering we have $a_i<x<b_i$. We do it for every $1\leq i\leq e$ and for every ε as above. In this way we find all the e-tuples $(\varepsilon_1,\ldots,\varepsilon_e)$ of involutions in $\mathcal{G}(L/K)$ such that $L(\varepsilon_i)$ can be ordered and (9.1) is true in $L(\varepsilon_1)\cap\ldots\cap L(\varepsilon_e)$. If we take one such e-tuples then all the others will have the form $(\varepsilon_1^{\sigma_1},\ldots,\varepsilon_e^{\sigma_e})$, where $\sigma_1,\ldots,\sigma_e\in\mathcal{G}(L/K)$. We can therefore count the number of e-tuples $(\sigma_1,\ldots,\sigma_e)\in\mathcal{G}(L/K)^e$ for which is true in $L(\varepsilon^{\sigma_1})\cap\ldots\cap L(\varepsilon^{\sigma_e})$. The measure $\mu(A(\theta))$ is then this number divided by $[L:Q]^e$. Q.E.D.

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Received November 17, 1980