Proof of a conjecture of F. Nevanlinna concerning functions which have deficiency sum two

by

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In memoriam:

Frithiof Nevanlinna (1894–1977)
Rolf Nevanlinna (1895–1980)

Table of Contents

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1. Introduction

1.1. *Statement of result.* The most striking achievement of R. Nevanlinna's theory of meromorphic functions is the Deficiency Relation: if f is a non-constant meromorphic function defined in the complex plane, then its (Nevanlinna) deficiencies $\delta(a)=\delta(a,f)$ $(a\in C^*=C\cup\{\infty\})$ satisfy $0\leqslant\delta(a,f)\leqslant 1$ and

$$\sum_{C^*}\delta(a)\leqslant 2 \tag{1.1}$$

(for definitions of these terms and general information see [18], [17], [22], [23]). Those (extended) complex numbers for which $\delta(a)>0$ are called *deficient values*.

In 1929, Frithiof Nevanlinna [21] proposed the following conjecture concerning functions extremal for this relation:

CONJECTURE. *Let $f(z)$ be meromorphic in the plane and suppose the order λ of f satisfies* ($T(r)=T(r,f)$)

$$\lambda \equiv \limsup_{r\to\infty} \frac{\log T(r)}{\log r} < \infty. \tag{1.2}$$

Then if

$$\sum \delta(a) = 2 \tag{1.3}$$

we have

$$2\lambda \text{ is an integer} \geq 2; \tag{1.4}$$

if a is a deficient value, then

$$\delta(a) = \lambda^{-1} p(a) \quad \text{where } p(a) \text{ is a positive integer}; \tag{1.5}$$

$$\text{each deficient value is an asymptotic value.} \tag{1.6}$$

The central result of this article is the

THEOREM. *F. Nevanlinna's conjecture is correct.*

Significant partial results have been obtained by A. Weitsman [26] and this writer [4]. Weitsman proved that hypotheses (1.2) and (1.3) imply that the number of deficient values is $\leq 2\lambda$, and I established a weak form of (1.5): each non-zero $\delta(a)$ may be written as $\delta(a)=(\lambda^*)^{-1}p(a)$ where $\lambda^* \leq \lambda$, and both $2\lambda^*$ and $p(a)$ are integers. The theorem here implies that $\lambda^*=\lambda$.

In the special case that f is entire (or, somewhat more generally, that $\delta(\infty,f)=1$) the theorem has been known for many years [10], [11], [24]; the only change is that (1.4) becomes: λ is a positive integer.

The deficiency relation (1.1) is a special case of the more general inequality

$$\sum \{\delta(a)+\theta(a)\} \leq 2, \tag{1.7}$$

where $\theta(a)$ is a non-negative term which measures the branching "over" $w=a$ ($w=f(z)$). It is interesting that equality in (1.7) is considerably "easier" to achieve than in (1.3). For example, (1.7) holds for solutions to a large class of ordinary differential equations [30, Chapter 5], and if f is any polynomial of degree n, then

$\Sigma\{\delta(a)+\theta(a)\}=2-n^{-1}$, although all deficiencies $\delta(a)$ $(a\neq\infty)$ are zero. It is also possible to find functions of any order λ for which (1.7) holds, with no algebraic relations between λ and the numbers $\delta(a)$, $\theta(a)$.

The ideas of [4] and [5] play an important role here, but an effort has been made to make this paper reasonably self-contained.

1.2 *Outline of proof.* The most familiar functions which satisfy (1.2) and (1.3) are the exponential functions $f_p(z)=\exp(z^p)$. The behaviour of these functions is very simple. Of course, $\delta(0)=\delta(\infty)=1$. Further, the plane divides into $2p$ congruent sectors D_j $(1\leqslant j\leqslant 2p)$; in the odd D_j, f_p tends to 0, and $f_p\to\infty$ inside the even D_j. The remaining values are assumed regularly near the boundaries of the D_j.

Now let f be the function of our theorem. Our goal is to construct a *quasi-conformal modification* of $f(z^2)$, for z sufficiently large, such that

$$\omega(f(\varphi(\zeta)^2))\approx f_p(\zeta)\quad(|\zeta|>R)$$

with $p=2\lambda$, for certain quasi-conformal maps ω and φ. Quasi-conformal modifications played an essential role in [4], and in Chapter 4 we give a self-contained introduction to them. Almost all work in this paper is to show that f or a Möbius transformation of f (which may be chosen locally) shares the value-distribution properties of $f_p(z)$.

Until § 7.8, we work locally in the Pólya peak annuli of the z-plane. Recall that the Pólya peaks (of order λ) of $T(r)$ are real-intervals on which

$$T(r)\leqslant\{1+o(1)\}\,T(\varrho_n)\left(\frac{r}{\varrho_n}\right)^{\lambda}\quad(\tfrac{1}{4}(A')^{-1}\varrho_n<r<4A'\varrho_n)\tag{1.8}$$

where ϱ_n and $A_n'\to\infty$; cf. [18, p. 101]. Corresponding to (1.8) are the annuli

$$\mathfrak{A}'=\mathfrak{A}_n'=\{(A_n')^{-1}\varrho_n<|z|<A_n'\varrho_n\},\tag{1.9}$$

in which we work. Only in Lemma 7.7 of § 7.6 we are able to obtain formulae which describe the behavior of f in the subannuli $\mathfrak{A}_n$ of $\mathfrak{A}_n'$, where

$$\mathfrak{A}=\mathfrak{A}_n=\{A_n^{-1}\varrho_n<|z|<A_n\varrho_n\};\tag{1.10}$$

although $A_n\to\infty$, the ratio $A_n(A_n')^{-1}$ tends rapidly to zero. Yet to pass from this to the global result takes little more space.

A key result from Chapter 7 is needed to go from knowledge of f in the $\mathfrak{A}$'s to global information. This is Lemma 7.5 (§ 7.4) which asserts that (1.3) and (1.8) imply that 2λ is an integer greater than one. Since (cf. [7]) the set of λ for which (1.8) is possible forms an interval, it follows from Lemma 7.5 that not only (1.4) holds, but a weak form of (1.8) is satisfied for all large r: given $\varepsilon>0$, there exist $C=C(\varepsilon)$, $r_0=r_0(\varepsilon)$ such that

$$T(r) \leqslant \begin{cases} CT(\varrho)\left(\dfrac{r}{\varrho}\right)^{\lambda+\varepsilon} & r>\varrho>r_0 \\ CT(\varrho)\left(\dfrac{r}{\varrho}\right)^{\lambda-\varepsilon} & \varrho>r>r_0. \end{cases} \tag{1.11}$$

While we are not permitted to use (1.11) until after § 7.4, we note that it is weaker than (1.8) when $\varrho=\varrho_n$ and $(A_n')^{-1}\varrho_n<r<(A_n')\varrho_n$. Until Lemma 7.4 has been established, we shall mention, when using (1.8), that a similar inequality holds if (1.8) is replaced by (1.11). This means that once (1.11) is known, the arguments which have been applied to the $\mathfrak{A}_n'$ may be transferred to *any* sequence of annuli $\{A_n^{-1}t_n<|z|<A_n t_n\}$ where A_n and $t_n\to\infty$. It is this fact which leads us so quickly to the global result. In this paper, we will use $\varepsilon=\frac{1}{4}$ in (1.11).

Since the A_n' of (1.8) will be diminished in our work until they become the A_n in (1.10) and since any subsequence of the $\{\varrho_n, A_n'\}$ satisfies (1.8), we make the inessential *a priori* condition

$$A_n'<\varrho_n^{1/2}. \tag{1.12}$$

It is now possible to outline Chapters 2–6, in which f is studied only in the $\mathfrak{A}_n'$. The reader should keep the behavior of $f_p(z)=\exp(z^p)$ in mind when reading this. Choose a complex number a with $\delta(a)>0$ and consider the subsets of $\mathfrak{A}_n'$ (cf. (1.9)) for which

$$|f(z)-a|<\varepsilon_n \tag{1.13}$$

for a sequence ε_n which slowly approaches zero. The definition of deficiency

$$\delta(a)=\liminf_{r\to\infty}\frac{m(r,a)}{T(r)} \tag{1.14}$$

implies that these subsets meet each circle $\{|z|=r\}$ $((A')^{-1}\varrho_n<r<A'\varrho_n)$. In Chapter 2 we say considerably more. First (Lemma 2.1): the sets $D=D(a)$ which contribute to $\delta(a)$ in (1.14) may be thought as being simply-connected, whose f-images cover the punctured discs $\{0<|w-a|<\varepsilon\}$. Standard potential theory (Lemma 2.3) then shows that if A_n is chosen so that A_n/A'_n is sufficiently small (but still $A_n \to \infty$) then at most $2\lambda = Q(\lambda)$ of these sets $D(a)$ (as a varies) contribute to (1.14) relative to the $\mathfrak{A}_n$ of (1.10). By

$$Q(\lambda) \tag{1.15}$$

we mean any numerical expression which is bounded, with bound depending only on the order λ of f. This role for $Q(\lambda)$ will be convenient throughout this paper.

These $Q(\lambda)$ components are the basis of all further analysis. In Chapter 3, we remove a union B^* of $Q(\lambda)$ small logarithmic rectangles from $\mathfrak{A}_n$, and in the Decomposition Lemma (Lemma 3.2) show that $\mathfrak{A}_n - B^*$ divides into $Q(\lambda)$ subsets F, in each of which f has two *near-Picard* values: there exist $a=a(F)$, $b=b(F)$ among the full set of deficient values such that the counting-functions satisfy

$$n(a, F) + n(b, F) < C(A'_n)^{-2h} T(\varrho_n) \tag{1.16}$$

where h is some positive constant which depends only on f.

Inequality (1.16) is a natural re-interpretation of hypothesis (1.3). The advantage of a condition such as (1.16), is that it suggests the possibility of modifying f so that the "omitted" values a and b are always 0 and ∞.

The theory of these modifications is the subject of Chapter 4, although in Chapter 4 they are only constructed in "admissible" logarithmic rectangles. Much of this information is needed for later work, and Lemma 4.13 is needed in Chapter 7 to show that the behavior of f in any union of $Q(\lambda)$ small logarithmic rectangles (such as B^* in Lemma 3.2) is negligible. Unfortunately, it is not possible to make a single modification of f which is defined in all of $\mathfrak{A}_n$; rather, we consider $f(z^2)$ and construct $Q(\lambda)$ modifications $G_{m,n}(z)$, each valid in a specific subannular region $\mathscr{A}_{m,n}$ of $\mathfrak{A}_n$; cf. (5.1) and (5.28). These $G_{m,n}$ depend on m, but we find that

$$(G'_m/G_m)^2 - (G'_{m+1}/G_{m+1})^2 \tag{1.17}$$

is an error term in $\mathscr{A}_{m,n} \cap \mathscr{A}_{m+1,n}$. Since 0 and ∞ are near-Picard values for the G's, we may "divide them" from G and make little change in the growth of G. Thus, for each m we let $K(z) = G(z)P(z)$ (P an error term), so that $K \neq 0, \infty$ (cf. (5.31)) and then expand each $(K'_m/K_m)^2$ in a Laurent series (cf. (6.21)). The key result in this paper is Lemma 6.3, which uses (1.16) and (1.17) to show that these Laurent coefficients are essentially independent of m. Once this is known, the parallels to the classical entire function case (cf. [10, 11]) become very clear. For example, Lemma 7.5, which implies (1.4), reflects the tension between (1.8) and the usual convexity of means of analytic functions.

It is natural that these methods depend on the invariance of some differential operator as in (1.17). F. Nevanlinna was led to his conjecture by studying the Schwarzian derivative $\{f,z\}=(f''/f')'-\frac{1}{2}(f''/f')^2$, under the special assumption that f has no multiple values; then $\{f,z\}$ is a *polynomial*, and so f may be analysed by asymptotic integration [19, §§ D.3, 7.4]. While $\{f,z\}$ is invariant under all Möbius transformations, (1.17) is only invariant under the transformation $Z\to Z^{-1}$ as well as $Z\to tZ$ ($t\in\mathbf{C}$), but this is all that is needed here.

There is a fundamental reason why this problem is more complicated for meromorphic functions than entire functions. If an entire function $f(z)$ satisfies (1.3), then $\delta(0,f')=\delta(\infty,f')=1$, so we may study $g=f'(P(z))^{-1}$ in place of f', where P is a canonical product taken over the zeros of f'. This g is a globally-defined function, and always satisfies (1.16) with $a=0$ and $b=\infty$, and the idea of [10], [11] is to study the Taylor coefficients of $\log g$.

Nothing so straightforward will work for meromorphic functions, since multiplicative terms are certain to introduce non-zero residues at the poles of f. However, when functions $G(z)$ are constructed in Chapter 5 so that they have few zeros *and poles*, these singularities may be removed very simply. But to construct the G's requires most of this paper.

1.3. *Quasi-conformal modifications.* The idea of using compositions by quasi-conformal mappings to study meromorphic functions goes back to Teichmüller and is the theme of [17, Chapter 7]. I learned of this method with Weitsman, and [8] is our first use of these ideas. A full exposition, with more history, is in [6, Chapter 2].

The method depends on having a large collection of functions ω. The next result is essentially Lemma 8 of [6] and adequate for our purpose.

LEMMA 1.1. *Let complex numbers γ and σ be given with $\sigma\neq 0$. Then for each η, $0<\eta<(50)^{-1}$ and $M'\geq 1$ we may construct a quasi-conformal homeomorphism $\omega(W)$ of the plane with*

$$\|\mu_\omega\|_\infty \equiv \left\|\frac{\omega_{\bar W}}{\omega_W}\right\|_\infty < 3\eta \tag{1.18}$$

such that

$$\omega(W)=\gamma+\sigma W \quad (|W|\geq M) \tag{1.19}$$

$$\omega(W)=W \quad (|W|\leq M') \tag{1.20}$$

so long as M is chosen with

$$\begin{aligned}\eta \log(M/M') &> C \max(|\gamma|, |\log|\sigma||+\pi)\\ \sigma \log(M/M') &> C \max(|\gamma|, |\log|\sigma||+\pi),\end{aligned} \tag{1.21}$$

where C is an absolute constant.

Finally, we may choose $\omega \in C^\infty$ with

$$|\omega_W(W)| \leq C|W|^{-1}|\omega(W)|. \tag{1.22}$$

Proof. All but the last assertion are in [4]. Let $S=|W|$, and construct smooth functions $a(S)$ and $b(S)$ with $|Sa'(S)|+|b'(S)|<\frac{1}{4}\min(\sigma, \eta)$, with the boundary conditions

$$\begin{aligned} a(S) &= 0 \quad (S \leq M'), \quad a(S) = \log \sigma \quad (S > M),\\ b(S) &= 0 \quad (S \leq M'), \quad b(S) = \gamma \quad (S > M);\end{aligned}$$

conditions (1.21) ensure that this is possible. Now consider

$$\omega(W) = e^{a(S)}W + b(S)$$

(this is a simpler formula than in [4]). Elementary manipulations, such as in [4], give (1.18)–(1.20) and (1.22). For example, we find that $|\omega(W)| \geq CSe^{a(S)}$ and $|\omega_W(W)| \leq Ce^{a(S)}$; thus (1.22) holds.

Lemma 1.1 will be used in the following manner. Let a_i and a_j be two of the deficient values, choose $\sigma=(a_i-a_j)^{-1}$, $\gamma=-\sigma$, and T_{ij} be the Möbius transformation $T_{ij}(w)=W=(w-a_j)/(w-a_i)$. Then the composition $\omega_{ij}(W)=\omega T_{ij}(w)=\omega((w-a_j)/(w-a_i))$ satisfies (1.18) as well as

$$\begin{aligned}\omega_{ij}(W) &= \frac{w-a_j}{w-a_i} \quad \left(\left|\frac{w-a_j}{w-a_i}\right| \leq M'\right),\\ \omega_{ij}(W) &= \frac{1}{w-a_i} \quad \left(\left|\frac{w-a_j}{w-a_i}\right| \geq M\right).\end{aligned} \tag{1.23}$$

1.4. *The hierarchy of perameters.* Several families of parameters will be used, which depend on the growth of the numbers $T(\varrho_n)$, and the rate at which the limits implicit in (1.3) are attained. We have already introduced A_n and A'_n in (1.9) and (1.10), and have suggested that A_n is small compared to A'_n. In general, if α_n and β_n tend to zero, we write $\alpha_n < \beta_n$ to mean that for each fixed $k>1$, $\alpha_n = o(1)\beta_n^k$ as $n \to \infty$. A chain of such relations means that each adjacent pair satisfies them. Using this notation, we shall require

$$(\log T(\varrho_n))^{-1} < \delta_n < \varepsilon_n < (A_n')^{-1} < \eta_n < \gamma_n < \sigma_n < A_n^{-1}, \tag{1.24}$$

and will freely impose further restrictions on these sequences if they are consistent with (1.24).

In our various constructions, it will sometimes be necessary to limit the rates at which any other than the left-hand sequence in (1.24) tends to zero. What (1.24) controls are the relative rates at which these can happen, but, for example, (1.24) is consistent with any of the δ's, ..., $(A_n)^{-1}$'s tending to zero arbitrarily slowly.

1.5. *Acknowledgments.* I learned of this problem in the late 1960's from my colleague A. Weitsman who in [26] obtained the first non-trivial conclusions from hypotheses (1.2) and (1.3), and I have had many useful conversations through the years with him.

The note [16] of W. H. J. Fuchs plays an important role in this work, and the germ of Lemma 7.5 is there. Estimates for means of logarithmic derivatives are basic here (cf. Chapter 6), and the techniques used go back to Fuchs's articles [13] and [14].

I have profited from several suggestions from A. Baernstein and very thorough, patient and helpful comments from W. H. J. Fuchs and W. K. Hayman. I especially thank Walter Hayman for giving so generously of his time. It was Professor Hayman who saw the need for Lemma 4.8, and both he and Professor Fuchs tightened several arguments. Judy Snider and Jane Brown have patiently produced more versions of this than any of us had expected.

Finally, the friendly encouragement shown by my family and many mathematicians has helped maintain my spirits during the mostly frustrating efforts to prove this theorem.

2. First consequences of (1.3); Significant components

2.1. *Preliminary observations.* The second fundamental theorem is a consequence of R. Nevanlinna's inequality that for any finite set $a_1, \ldots, a_q \in C^*$,

$$\sum_{i=1}^{q} N(r, a_i) \geq \{q-2-\delta(r)\}\, T(r), \tag{2.1}$$

$$N(r, a_i) \geq \{1-\delta(a_i)-\delta(r)\}\, T(r) \quad (i=1, \ldots, q) \tag{2.2}$$

where here and throughout this section $\delta(r)$ is a function for which

$$\delta(r) \to 0 \quad (r \to \infty). \tag{2.3}$$

Since (1.2) is assumed, (2.1) holds for all large r; in general an exceptional set E of finite r-measure must be excluded. Now (1.3) implies that if we choose $a_1, \ldots, a_q$ properly, we also have

$$\sum_1^q N(r, a_i) \leqslant \{q-2+\delta(r)\}\, T(r) \quad (r \to \infty), \tag{2.4}$$

where (2.3) holds again. Also, we have that $q \leqslant 2\lambda$ [26]. If $q=2$, our results are well-known ([24], [10], [11]); the methods given here work when $q=2$, but some constructions are vacuous (for example we may always take a and b in (1.16) to be the two deficient values) and thus we usually imagine that $q>2$.

By considering $T \circ f$, T a Möbius transformation, we may assume that $w=\infty$ is a normal value in the sense that

$$f \textit{ has no multiple poles}, \tag{2.5}$$

$$m(r, \infty) < \delta(r)\, T(r) \tag{2.6}$$

for some $\delta(r)$ as in (2.3). The function $\delta(r)$ tends to zero so slowly that

$$N\left(r, 0, \frac{1}{f'}\right) < \delta(r)\, T(r). \tag{2.7}$$

Throughout this article we must control error terms. Since f has at least two deficient values, there exists a positive constant $h=h(f)$ with

$$\frac{d \log T(r)}{d \log r} \geqslant 10h > 0 \quad (r > r_0), \tag{2.8}$$

and we suppose that $h<10^{-1}$ (this is proved in [27]). Inequality (2.8) implies that $\lambda \geqslant 10h$ (λ is defined in (1.2)); however Weitsman in [26] already has proved that $\lambda \geqslant 1$.

By restricting the rate at which $A'_n \to \infty$ in the Pólya peak inequality (1.8), we may assume that if $\delta(r)$ is any of the functions (2.1), (2.2), (2.4), (2.6) and (2.7), then

$$\sup_{(A'_n)^{-1}\varrho_n < r < 4A'_n \varrho_n} \{\delta(r)\, T(r)\} < \delta_n^2\, T(\varrho_n), \tag{2.9}$$

where δ_n is from (1.24). Since [22, p. 25] if g is meromorphic, we always have

$$r^{-1}\int^{r}\log^{+}M(s,g)\,ds\leqslant C(k)\,T(kg)\quad(k>1)$$

we may adjust A' slightly so that

$$\begin{gathered}\sum_{i=1}^{q}\log^{+}\frac{1}{|f(z)-a_i|}+\log^{+}|f(z)|\leqslant CT(2A'_n\varrho_n)\quad(|z|=A'_n\varrho_n),\\ \sum_{i=1}^{q}\log^{+}\frac{1}{|f(z)-a_i|}+\log^{+}|f(z)|\leqslant CT(2(A'_n)^{-1}\varrho_n)\quad(|z|=(A'_n)^{-1}\varrho_n).\end{gathered}\tag{2.10}$$

Similarly, an appeal to [13, Lemma 1] shows that we may assume that

$$\sum_{i=1}^{q}\int_{|z|=(A'_n)^{-1}\varrho_n}\left|\frac{f'(z)}{f(z)-a_i}\right||dz|<CT(2(A'_n)^{-1}\varrho_n).\tag{2.11}$$

In (2.10), (2.11) and throughout, C is a constant which depends only on the function f, at least when n is large in (1.8)–(1.10).

Since we will usually be in the Pólya peak annulus (1.8), the dependence on n will be suppressed in general, save in a few cases when we use it to avoid ambiguity.

Choose a sequence $\varepsilon=\varepsilon_n$ which approaches zero sufficiently slowly and consider the sets $\{|f(z)-a_i|<\varepsilon\}\cap\mathfrak{A}'$, $1\leqslant i\leqslant q$, where $\mathfrak{A}'$ has been introduced in (1.9). Obviously, when $(A'_n)^{-1}\varrho_n<r<A'_n\varrho_n$, $m(r,a_i)$ (asymptotically) is obtained by integrating $-\log|f(re^{i\theta})-a_i|$ over these sets, and the purpose of this and the next section is to show that only $Q(\lambda)$ components of $\{|f(z)-a_i|<\varepsilon_n\}$ relative to $\mathfrak{A}'$ need be considered (the convention concerning $Q(\lambda)$ has been described in (1.15)).

LEMMA 2.1. *Let $\{\varrho_n\}$ be the Pólya peaks of order λ of $T(r)$. Then if δ_n, ε_n and $(A'_n)^{-1}$ tend to zero sufficiently slowly, consistent with* (1.24), *we may find unions of disjoint simply-connected components $\{D\}_i$ relative to $\{|z|<A'\varrho\}$ such that if $D\in\{D\}_i$, then*

$$|f(z)-a_i|=\varepsilon_n\quad(z\in(\partial D)\cap\mathfrak{A}')\tag{2.12}$$

and

$$\sum_{i=1}^{q}n(r,a_i,\bigcup_{j=1}^{q}\{D\}_j)+n(r,\infty,\bigcup\{D\}_j)<C\delta_n T(\varrho)\quad((A')^{-1}\varrho<r<A'\varrho).\tag{2.13}$$

Moreover, all components D of $f^{-1}\{|w-a_i|<\varepsilon\}$ which meet $\{|z|=\tfrac{1}{2}A'\varrho\}$ and $\{|z|=A'\varrho\}$ belong to $\{D\}_i$.

Remarks (1) In (2.13) we are modifying the definition of counting-function by letting $n(r,a,J)$ be the number of solutions to $f(z)=a$ in $J\cap\{|z|<r\}$. We also define

$$N(r,a,J)=\int_0^r (n(t,a,J)-n(0,a,J))\,t^{-1}\,dt+n(0,a,J)\log r.$$

(2) Estimate (2.13) is more cumbersome than the corresponding conclusion in [5], where it is written simply as $o(T(\varrho))$. At this stage we cannot replace the right side of (2.13) by $o(T((A')^{-1}\varrho))$ since it is consistent with the Pólya peak inequalities (1.8) that the ratios $T((A')^{-1}\varrho)\,[(A')^{-\lambda}\,T(\varrho)]^{-1}$ tend to zero arbitrarily rapidly.

Proof. Choose a (small) positive number ε_0, so that the balls $\{|w-a_i|<2\varepsilon_0\}$ $(1\leq i\leq q)$ are disjoint. For a fixed $\varepsilon<\varepsilon_0$ and $1\leq i\leq q$ let $\Delta_i=\{|w-a_i|<\varepsilon\}$ and, for $r>0$, let D_i $(=D_i(r))$ be the components of $f^{-1}(\Delta_i)$ relative to $\{|z|<r\}$.

We now follow the ideas introduced by Ahlfors in his study of covering surfaces ([18, Chapter 5], [23, Chapter 13]). Thus each D_i is an *island* or *tongue* (relative to $\{|z|<r\}$), according to whether D_i is compactly contained in $\{|z|<r\}$ or not.

We start with a simple consequence of (1.3) and Ahlfors's theory. Let $S(r)$ be the area of the Riemann image on C^* of $\{|z|<r\}$ under f and $L(r)$ be the length, on C^*, of the image of $\{|z|=r\}$. In geometric language, S is the mean-covering number (over C^*) and L the length of the relative boundary. Then (cf. [18, p. 148]) there is a positive constant h which depends only on the $\{a_i\}$ and ε such that

$$\sum_{i=1}^{q}\sum_{D_i}(n(D_i)-n_1(D_i))\geq(q-2)\,S(r)-hL(r). \tag{2.14}$$

This sum is over all *islands* D_i which are compactly contained in $\{|z|<r\}$, and $n(D_i)$ is the multiplicity of D_i (i.e., the cardinality of $f^{-1}(w)$ for $w\in\Delta_i$). Also, if $\varrho(D_i)$ is the Euler characteristic of D_i, then $n_1=(n-1)+(\varrho+1)$, is non-negative, and called the *excess* of the island [18, p. 147]. In particular, each term $n(D_i)-n_1(D_i)$ which corresponds to an island of multiplicity $m>1$ satisfies

$$n(D_i)-n_1(D_i)\leq\tfrac{1}{2}n(D_i). \tag{2.15}$$

(We would like to record two observations concerning (2.14) which influence our approach. First, (2.14) is a general result, which holds for general exhaustions as long as we measure length and area with the spherical metric [23, pp. 324, 325, 341]. Also, the "error term" $L(r)$ in (2.14) may be replaced by the length of the relative boundary which lies outside the *tongues* relative to $\{|z|<r\}$.)

In the next inequality as well as in (2.18) and (2.19), Σ_s refers to summation over simple islands (i.e. islands having $n(D_i)=1$) and Σ_m over multiple islands. Then (2.14) and (2.15) yield that (cf. [18, p. 147])

$$\begin{aligned}(q-2)\,S(r)-hL(r) &\leq \sum_i \sum_s n(D_i)+\sum_i \sum_m \{n(D_i)-n_1(D_i)\} \\ &\leq \sum_i \sum_s n(r, a_i, D_i)+\tfrac{1}{2}\sum_i \sum_m n(r, a_i, D_i) \qquad (2.16) \\ &\leq \sum_i n(r, a_i)-\tfrac{1}{2}\sum_i \sum_m n(r, a_i, D_i).\end{aligned}$$

This inequality is to be integrated with respect to $r^{-1}\,dr$. Since $T(r)=\int^r S(t)\,t^{-1}\,dt+o(1)$ [18, p. 13], and [20] for each $h>0$

$$\int_1^r L(t)\,t^{-1}\,dt < T(r)^{1/2+h} \quad (r>r_0), \tag{2.17}$$

the fundamental hypothesis (1.3) (cf. (2.4)) lets us conclude that

$$\sum_i \sum_m \int^r n(t, a_i, D_i)\,t^{-1}\,dt = o(1)\,T(r) \tag{2.18}$$

from (2.16).

The integrand in (2.18) is an increasing function of r. It is then no loss of generality to retard the rate at which the $\delta_n \to 0$ in (1.24) and (2.9) so that

$$\sum_i \sum_m n(2A'\varrho, a_i, D_i) < \delta_n^2\, T(\varrho). \tag{2.19}$$

We use (2.18) and re-examine the standard proof of (2.14) (cf. [18, pp. 146–7]; our aim is to get a bit more information than is contained in (5.27) thereof). We consider the covering of $\{|z|<r\}$ over the Riemann sphere C^* for a fixed $r>0$. We first remove all *tongues* D whose image under f "lies over" the various Δ_i $(1\leq i\leq q)$. What remain are certain simply-connected regions G'. Next, remove all islands D which are over the Δ_i from the G'; what remain are certain domains $\bar{G}$ which are mapped by $w=f(z)$ to the region C', where C' is C^* with the q discs $\Delta_i=\{|w-a_i|<\varepsilon\}$, $(1\leq i\leq q)$ deleted.

We divide the G' into $q+1$ classes. G' is a G_0' if it is not compactly contained in $\{|z|<r\}$, and G' is placed in G_i' $(1\leq i\leq q)$ if it is compactly contained in $\{|z|<r\}$ with

$|f(z)-a_i|=\varepsilon$ on $\partial G'$. Those components $\bar{G}$ which are contained in a G_i' $(0\leq i\leq q)$ are denoted by $\bar{G}_i$, and we let $\mathcal{G}_0$ be the union of all $\bar{G}_0$ and $\mathcal{G}_1$ the union of the $\bar{G}_i$ $(1\leq i\leq q)$.

Recall that each $\bar{G}$ is mapped by f onto a covering of C', and [18, p. 136] C' has Euler characteristic equal to $(q-2)$. Hence [18, p. 146]

$$\max(\varrho(\bar{G}),0)\geqslant(q-2)\,S_{\bar{G}}(C')-hL_{\bar{G}}(C') \tag{2.20}$$

where, if $I(C')$ is the area of C' on C^*, we have

$$S_{\bar{G}}(C')=\{I(C')\}^{-1}\int_{\bar{G}}[|f'|(1+|f|^2)^{-1}]^2\,dx\,dy$$

($\equiv$ mean covering) and L is the length of the portion of the relative boundary which projects onto C'. Since always $\varrho(\bar{G})\geqslant-1$, we have that $\max(\varrho(\bar{G}),0)\leqslant\varrho(\bar{G})+1$, and so (2.20) implies that

$$\sum_{\mathcal{G}_0}(\varrho(\bar{G}_0)+1)\geqslant(q-2)\,S_{\mathcal{G}_0}(C')-h_2\,L_{\mathcal{G}_0}(r)\geqslant(q-2)\,S_{\mathcal{G}_0}(C')-hL(r). \tag{2.21}$$

Next, if $\bar{G}$ is a $\bar{G}_i$ with $i\geqslant1$, then $L_{\bar{G}_i}(C')=0$ and since $q>2$, (2.20) implies that $\varrho(\bar{G}_i)>0$. Each inner boundary of a $\bar{G}_i$ separates $\bar{G}_i$ from an island D, and each island inside G_i' is on an inner boundary of exactly one $\bar{G}_i$. Now $\varrho(\bar{G}_i)+1$ is precisely the number of inner boundary components of $\bar{G}_i$ [18, p. 136] so now instead of (2.21) we have that

$$\begin{aligned}\sum_{i=1}^{q}\sum_{\bar{G}_i}(\varrho(\bar{G}_i)+1)&=\text{number of islands } D\subset\mathcal{G}_1\\ &\geqslant\sum_{D\subset\mathcal{G}_1}n(D)-\sum_{\substack{D\subset\mathcal{G}_1\\ n(D)>1}}n(D) \qquad (2.22)\\ &\geqslant\sum_{D\subset\mathcal{G}_1}n(D)-\delta^2T(\varrho)\end{aligned}$$

in view of (2.19).

Since $L_{G_i'}(C')=0$ $(1\leqslant i\leqslant q)$, f assumes every value in $|w-a_i|>\varepsilon$ (including ∞) with the same multiplicity in each $\bar{G}_i$, which is $S_{(\cup\bar{G})_i}(C')$, where $S_{(\cup\bar{G})_i}(C')$, is the mean covering over C' from the $\bar{G}$ which are contained in a G_i'. In particular, we may take $D=D_j$ for any of the $(q-1)$ choices of $j\neq i$ and deduce that $n(D_j)=S_{(\cup\bar{G})_i}(C')$. Thus the last expression in (2.22) is equal to

$$\sum_{i=1}^{q}\sum_{D_i\subset\cup G_i'} n(D_i)+(q-1)\,S_{\mathscr{G}_1}(C')-\delta^2 T(\varrho).$$

We now sum (2.21) and (2.22) and recall [18, p. 133] that $S(r,C')>S(r)-hL(r)$; thus

$$\begin{aligned}\sum_{\bar{G}\subseteq\{|z|<r\}} (\varrho(\bar{G})+1) &\geq \sum_{i=1}^{q}\sum_{D_i\subset\cup G_i'} n(D_i)+(q-2)\,S_{\mathscr{G}_0}(r,C')\\ &\quad+(q-1)\,S_{\mathscr{G}_1}(r,C')-\delta_n^2\,T(\varrho)-hL(r) \qquad (2.23)\\ &=\sum_{i=1}^{q} n'(r,a_i)+(q-2)\,S(r,C')+S_{\mathscr{G}_1}(r,C')-\delta_n^2\,T(\varrho)-hL(r),\end{aligned}$$

where $n'(r,a_i)$ is the number of solutions to $f(z)=a_i$ in each G_i'.

According to [18, p. 147], the left side of (2.23) is precisely $\Sigma\,[n(D)-n_1(D)]$, where the summation is over all islands in $\{|z|<r\}$ and the non-negative term $n_1(D)$ has been introduced in (2.15).

Thus, if $n''(r,a_i)$ is the number of solutions to $f(z)=a_i$ which are assumed in tongues relative to $\{|z|<r\}$, we obtain from (2.23) that

$$\begin{aligned}\sum_{i=1}^{q} n(r,a_i) &= \sum_{i=1}^{q} n''(r,a_i)+\sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{D_j\subset\cup G_i'} n(D_j)\\ &\geq \sum_{i=1}^{q} n''(r,a_i)+\sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{D_j\subset\cup G_i'} \{n(D_j)-n_1(D_j)\}\\ &\geq \sum_{i=1}^{q} n''(r,a_i)+\sum_{\bar{G}} (\varrho(\bar{G})+1) \qquad (2.24)\\ &\geq \sum n''(r,a_i)+\sum n'(r,a_i)+(q-2)\,S(r,C')+S_{\mathscr{G}_1}(r,C')-\delta_n^2\,T(\varrho)-hL(r).\end{aligned}$$

This inequality is integrated from $(A')^{-1}\varrho$ to $4A'\varrho$ with respect to $d\log r$. Thus (1.24), (2.1), (2.4), (2.9) (2.17) and [18, p. 13] the equivalence of the Ahlfors–Shimizu and Nevanlinna characteristics yield that

$$\begin{aligned}
&\int_{(A')^{-1}\varrho}^{4A'\varrho} S_{\mathscr{G}_1}(r, C')\,r^{-1}\,dr+\sum_i \int_{(A')^{-1}\varrho}^{4A'\varrho} [n'(t, a_i)+n''(t, a_i)]\,t^{-1}\,dt \\
&\leqslant \left\{\sum_i N(4A'\varrho, a_i)-(q-2)\,T(4A'\varrho)\right\}-\left\{\sum_i N((A')^{-1}\varrho, a_i)-(q-2)\,T((A')^{-1}\varrho)\right\} \\
&\quad +2T(4A'\varrho)^{2/3}+3\delta^2(\log A')\,T(\varrho) \qquad (2.25)\\
&\leqslant C(\delta^2\{1+\log(A')\}\,T(\varrho)<\delta T(\varrho).
\end{aligned}$$

For $1\leqslant i\leqslant q$, let $\{D\}_i$ be the union of tongues D_i in $\{|z|<4A'\varrho\}$ which also meet $\{|z|=\frac{1}{2}A'\varrho\}$ and whose outer boundary is mapped to $\partial\Delta_i$ by f, together with all compact components of their complement relative to $\{|z|<A'\varrho\}$. These $\{D\}_i$ are thus disjoint unions of simply-connected sets relative to $\{|z|<A'\varrho\}$. Then

$$\sum_{i,j} n(2A'\varrho, a_i, \{D\}_j)\leqslant \sum n'(2A'\varrho, a_i)+n''(2A'\varrho, a_i)+S_{\mathscr{G}_1}(2A'\varrho, C'),$$

so (2.25) shows that $\Sigma_{i,j}\,n(r, a_i, \{D\}_j)$ satisfies (2.13). By Rouché's theorem,

$$n(A'\varrho, \infty, \{D\}_i)=(q-1)^{-1}\sum_{j\neq i} n(A'\varrho, a_j, \{D\}_i),$$

so a similar bound holds for the poles.

Finally, all this was done for a fixed $\varepsilon>0$, and so holds if $\varepsilon=\varepsilon_n\to 0$ sufficiently slowly, subject of course to (1.24). For example, for each fixed, ε, $0<\varepsilon<\varepsilon_0$, the mean-coverings $S_i^\circ(r)$ over any of $\{|w-a_i|=\varepsilon\}$, $\{|w-a_i|\leqslant\varepsilon\}$ satisfy

$$\int^r S_i^\circ(t)\,t^{-1}\,dt\sim T(r) \tag{2.26}$$

(Cartan's identity); then by retarding the rate at which $\varepsilon_n\to 0$ we may arrange that

$$\left|\int_0^r t^{-1}\frac{1}{2\pi}\int_0^{2\pi} n(t, a_i+\varepsilon_n e^{i\theta})\,d\theta\,dt-T(r)\right|<(A')^{-5h}\,T(\varrho) \qquad (1\leqslant i\leqslant q, r\geqslant (A_n')^{-1}\varrho, n\to\infty), \tag{2.27}$$

with a similar asymptotic equality for coverings over the solid disks $\{|w-a_i|\leqslant\varepsilon_n\}$.

2.2. *Significant components*. In principle, each set $\{D\}_i$ of Lemma 2.1 may have a large number of components. We now use some elementary potential theory and show that if A/A' is sufficiently small (compare (1.9), (1.10) and (1.24)) then only $Q(\lambda)$ components are significant in $\mathfrak{A}$, where $Q(\lambda)$ is described by (1.15). This result is similar to that of Lemmas 3 and 5 of [4], but the methods are important here, since we will be considering several different exhaustions, in this chapter, and in chapters 4 and 7.

We begin with a simple estimate on Green functions.

LEMMA 2.2. *Let D be a region in $\{|z|<R\}$, and let $g(z, z_0)$ be the Green function of D with pole at z_0, $|z_0|=r_0<R$. Then*

$$\int_{D\cap\{|z|=r\}} g(re^{i\theta}, z_0)\, d\theta \le 2\pi \min\{\log(R/r), \log(R/r_0)\}. \tag{2.28}$$

Proof. By the maximum principle,

$$g(z, z_0) \le \log\left|\frac{R^2-\bar{z}_0 z}{R(z-z_0)}\right| \equiv h_R(z, z_0).$$

Thus

$$\int g(z, z_0)\, d\theta \le \int_0^{2\pi} h_R(re^{i\theta}, z_0)\, d\theta,$$

and $\int h_R\, d\theta$ can be explicitly computed, and gives (2.28) with equality.

COROLLARY. *Under the same hypothesis, let $1\le p<\infty$. Then there exists a function $\Omega_p(s)$, $1\le s<\infty$ such that*

$$\left\{\int_{D\cap\{|z|=r\}} g(re^{i\theta}, z_0)^p\, d\theta\right\}^{1/p} \le \Omega_p(\min(R/r, R/r_0)), \tag{2.29}$$

and Ω_p is uniformly bounded in each region $1\le s\le M$.

Proof. It suffices to take $R=1$, $z_0=r_0>0$ and compute with the function $h_1(z, z_0)=\log|(1-\bar{z}_0 z)/(z-z_0)|$. The resulting integral in (2.29) is a continuous function of r and r_0 in this range, so (2.29) follows.

Remark. More refined estimates may be found, for example, in [28] and [29].

We now produce a subset of $Q(\lambda)$ tongues contained in $\bigcup_i \{D\}_i$ which will be the basis of all our constructions.

LEMMA 2.3. *Let $\mathfrak{A}'$ be the annulus* (1.9) *centered at the Pólya peak of $T(r)$. Let the unions of components $\{D\}_i$ be constructed as in Lemma* 2.1. *Then we may choose a subset of s components $\cup \mathfrak{D}_i$ of $(\cup \{D\}_i) \cap \mathfrak{A}'$ with*

$$s \leq 2\lambda + 6 = Q(\lambda), \tag{2.30}$$

such that: each component D of $\mathfrak{D}_i$ reaches $\{|z| = A'\varrho\}$ in $\mathfrak{A}'$;

$$\sum_{D \in \mathfrak{D}_i} m(r, a_i, D) > m(r, a_i) - C(A')^{-5h} T(\varrho) \quad (1 \leq i \leq q, (10A)^{-1}\varrho < r < 10A\varrho)); \tag{2.31}$$

if $D \in \mathfrak{D}_i$, then $D \cap \{(10A)^{-1}\varrho < |z| < 10A\varrho\}$ is simply-connected in $\{|z| \leq A'\varrho\}$.

Remarks. (1) The s components D of $\cup \mathfrak{D}_i$ are called the *significant components* of f (in $\mathfrak{A}'$ relative to $\mathfrak{A}$).

(2) In (2.31) and for the rest of this article, we write

$$m(r, a, J) = \frac{1}{2\pi} \int_{re^{i\theta} \in J} \log^+ \frac{1}{|f(re^{i\theta}) - a|} \, d\theta$$

where J is any open set; this complements the notation used in Remark 1 following the statement of Lemma 2.1, concerning $n(r, a, J)$.

Proof. Let Ω be any subregion of $\mathfrak{A}'$ and suppose $|f(z) - a_i| < \varepsilon$ for some $z \in \Omega$. Then z is in some component D of $f^{-1}(\Delta_i)$, as in Lemma 2.1, and according to potential theory

$$\begin{aligned} \log \frac{1}{|f(z) - a_i|} = \log \varepsilon^{-1} &+ \sum g_D^\Omega(z, z_\nu) - \sum g_D^\Omega(z, w_\mu) \\ &+ \int_{D \cap \partial\Omega} \log \frac{1}{|f(\zeta) - a_i|} \, d\omega_z(\zeta, D \cap \partial\Omega, D \cap \Omega), \end{aligned} \tag{2.32}$$

where g_D^Ω is the Green function of $D \cap \Omega$, the z_ν and w_μ are the zeros and poles of $f(z) - a_i$ in $\Omega \cap D$, and $d\omega_z$ is harmonic measure at z. If D is one of the tongues D_i selected in Lemma 2.1, enlarge $D = D_i$ in (2.32) to be the corresponding component of $\{D\}_i$; this means we may be adding some compact components of its complement, and thus D now may have poles of f, although (2.13) shows that their number is an error term.

Hypotheses (1.2) and (1.3) with the extremely elementary bound (2.28) will always show that means of Green functions are negligible: if $A < (20)^{-1}A'$, then

$$\sum_D \left\{ \sum_{z_\nu, w_\mu} \int_0^{2\pi} g_D^\Omega(re^{i\theta}, z_\nu)\, d\theta + \int_0^{2\pi} g_D^\Omega(re^{i\theta}, w_\mu)\, d\theta \right\} \leqslant C(A')^{-5h}\, T(\varrho) \qquad ((10A^{-1}\varrho < r < 10A\varrho), \tag{2.33}$$

where we set $g=0$ outside $D \cap \Omega$. For D's contained in $\cup \{D\}_i$ of Lemma 2.1 this is clear from (2.13) and (1.24), but (2.33) holds for *all* D's.

To obtain (2.33) in general, divide the z and w into classes (I) and (II). To simplify notation, let $t=z_\nu$ or w_μ. Then the pair (t, D) is in class (I) if $\{|z|=t\} \cap D \neq \emptyset$ and $D \cap \Omega$ is contained in $\{|z| \leqslant (1+(A')^{-\lambda-1-6h})\,|r|\}$; the other pairs are in class (II). Since (1.8) [or (1.11) with $\varepsilon=1$] implies that $n(r, a_i) \leqslant C(A')^{\lambda+1}\, T(\varrho)$ for $r \leqslant 2A'\varrho$, we obtain from (2.28) that

$$\sum_{\text{(I)}} \int_0^{2\pi} g_D^\Omega(re^{i\theta}, t)\, d\theta \leqslant C(A')^{\lambda+1}\, T(\varrho) \log(1+(A')^{-\lambda-1-6h}) \leqslant C(A')^{-6h}\, T(\varrho) \qquad ((10A)^{-1}\varrho < r < 10A\varrho). \tag{2.34}$$

We use (2.24) and (2.25) to show that the number of terms in class (II) is small. Indeed each pair (r, D) of class (II) contributes to $n''(r, a_i)$ in (2.24) on an r-interval of logarithmic length at least $C(A')^{-\lambda-1-6h}$. Let $Z(A'\varrho)$ be the total number of pairs (r, D) of class (II), where $|r| < A'\varrho$. Then (2.25) yields that

$$CZ(A'\varrho)\,(A')^{-\lambda-1-6h} < \sum_i \int^{2A'\varrho} n''(t, a_i)\, t^{-1}\, dt < \delta T(\varrho);$$

since $\Omega \subset \mathfrak{A}'$, (1.8), (1.24) and (2.28) now give that

$$\sum_{\text{(II)}} \int_0^{2\pi} g_D^\Omega(re^{i\theta}, t)\, d\theta \leqslant Z(A'\varrho) \log\left(\frac{A'\varrho}{r}\right) \leqslant C(A')^{3(\lambda+1)}\, \delta \log\left(\frac{A'\varrho}{r}\right) T(\varrho) \leqslant C(A')^{-5h}\, T(\varrho) \qquad ((10A)^{-1}\varrho < r < 10A\varrho),$$

and this and (2.34) establish (2.33).

Now we choose $\Omega = \mathfrak{A}'$ in (2.32). It is necessary to estimate the boundary integrals. On $\{|z|=(A')^{-1}\varrho\}$ we use the uniform bound (2.10), (2.8), (1.24) and the obvious fact that $\omega(z, D \cap \{|z|=(A')^{-1}\varrho\}, D \cap \mathfrak{A}') \leqslant 1$. On $\{|z|=A'\varrho\}$, we have (2.10) and the classical estimate [15, p. 102]

$$\omega(z, I_{A'\varrho}, D \cap \mathfrak{A}) \leqslant 4 \exp\left\{ -\pi \int_{20A\varrho}^{A'\varrho} \frac{dt}{t\theta(t)} \right\}, \tag{2.35}$$

where $I_{A'\varrho}=D\cap\{|z|=A'\varrho\}$, and $\theta(t)$ is the angular measure of $D\cap\{|z|=t\}$. Thus we obtain using (1.8) [or (1.11) with $\varepsilon=1$] that

$$\begin{aligned}\int_{D\cap\partial\Omega}\log\frac{1}{|f-a|}\,d\omega_z &\leq CT(2A'\varrho)\,\omega(z, I_{A'\varrho}, D\cap\mathfrak{A}')+CT(2(A')^{-1}\varrho)\\ &\leq C\left[(A')^{\lambda+1}\exp\left\{-\pi\int_{20A\varrho}^{A'\varrho}\frac{dt}{t\theta(t)}\right\}+(A')^{-5h}\right]T(\varrho).\end{aligned}\tag{2.36}$$

Thus suppose D meets $\{|z|=A'\varrho\}$. Lemma 2.1 ensures that if D also meets $\{|z|\leq 10A\varrho\}$, then $D\subset\bigcup\{D\}_i$. Thus the harmonic means inequality [15, p. 108] and (1.24) show that if $|z|\leq 10A\varrho$, then

$$\omega(z, I_{A'\varrho}, D\cap\mathfrak{A}')\leq C(A'/A)^{-(\lambda+3)}\leq(A')^{-\lambda-2}\tag{2.37}$$

for all but at most $2\lambda+6=Q(\lambda)$ components D of $\bigcup\{D\}_i\cap\mathfrak{A}'$. We use (2.37) and (2.35) in (2.36), and then (2.36) and (2.33) in (2.32). Thus, if $|D(r)|$ is the angular measure of $D\cap\{|z|=r\}$, we find using (1.24) and (2.32) that in the components D for which (2.37) holds [and in particular for all D's which do not meet $\{|z|=A'\varrho\}$] we have

$$\sum_i\sum_D m(r, a_i, D)\leq\log\varepsilon^{-1}\sum_D|D(r)|+C(A')^{-1}T(\varrho)\sum_D|D(r)|+C(A')^{-5h}T(\varrho)\leq C(A')^{-5h}T(\varrho)$$
$$((10A)^{-1}\varrho<r<10A\varrho),$$

since $h<\frac{1}{10}$ in (2.8). This proves (2.31).

Each circle $\{|z|=r\}$ with large r must meet D's from at least two $\mathcal{D}_i$, since $q\geq 2$. Let D be a significant component, so in particular D is a component of $\mathfrak{A}'\cap\bigcup\{D\}_i$. We have constructed the $\{D\}_i$ in Lemma 2.1 to be simply-connected in $\{|z|<A'\varrho\}$. Suppose now that a component of D_0 of $D\cap\{(10A)^{-1}\varrho<|z|<10A\varrho\}$ were not simply-connected. Choose $j\neq i$, and let $\mathcal{D}_j^0$ be the components of $\mathcal{D}_j$ which meet $\{|z|=(10A)^{-1}\varrho\}$. Then for large r, (2.31), (2.8) and (1.24) show that

$$\begin{aligned}m((10A)^{-1}\varrho, a_j, \mathcal{D}_j^0)&\geq\tfrac{1}{2}\delta(a_j)\,T((10A)^{-1}\varrho)-C(A')^{-5h}T(\varrho)\\ &\geq C\{A^{-10h}-(A')^{-5h}]\,T(\varrho)\\ &\geq CA^{-10h}\,T(\varrho).\end{aligned}$$

However, D_0 prevents any D of $\mathcal{D}_j^0$ from reaching $\{|z|=A'\varrho\}$, so the right side of (2.35) vanishes, and this simplifies (2.36). Thus the analysis of (2.32) leads to

$$m((10A)^{-1}\varrho, a_j, \mathcal{D}_j^\circ) \leq C(A')^{-5h}T(\varrho) \quad (j \neq i).$$

By combining these last two inequalities, we would have a contradiction to the understanding of (1.24) that $(A')^{-1} < (A)^{-1}$. This contradiction completes the proof of Lemma 2.3.

COROLLARY 1. *For* $1 \leq i \leq q$, *let* $\mathcal{D}_i$ *be the union of the* $Q(\lambda)$ *significant tongues* D *of* $\mathfrak{A}'$ *described in Lemmas* 2.1 *and* 2.3, *and let*

$$S_{\mathcal{D}_i}(r) = \frac{1}{2\pi}\int_0^{2\pi} n_{\mathcal{D}_i}(r, a_i + \varepsilon\, e^{i\theta})\, d\theta, \tag{2.38}$$

where $n_{\mathcal{D}_i}(r, a_i + \varepsilon\, e^{i\theta})$ *counts those solutions to* $f(z) = a_i + \varepsilon\, e^{i\theta}$ *which arise from points of* $\partial\mathcal{D}_i$. *Then*

$$\left| \int^r S_{\mathcal{D}_i}(t)\, t^{-1}\, dt - \delta(a_i)\, T(r) \right| < C(A')^{-4h}T(\varrho) \quad ((10A)^{-1}\varrho < r < 10A\varrho). \tag{2.39}$$

Proof. Let $\mathcal{D}_i \cap \{|z| = r\} = \bigcup (\alpha_m(r), \beta_m(r))$. Since $|f(z) - a_i| = \varepsilon$ on $\partial\mathcal{D}_i$, the Cauchy–Riemann equations and the argument principle give (cf. [4, Lemma 1])

$$\begin{aligned} 2\pi r \frac{d}{dr} m(r, a_i, \mathcal{D}_i) &= \log \varepsilon^{-1} \sum_m r\{\beta_m' - \alpha_m'\} + \sum \Delta_{\alpha_m(r), \beta_m(r)}\left(\arg\frac{1}{f - a_i}\right) \\ &= \log \varepsilon^{-1} \sum r\{\beta_m' - \alpha_m'\} + S_{\mathcal{D}_i}(r) + n(r, \infty, \mathcal{D}_i) - n(r, a_i, \mathcal{D}_i) \\ &\quad - \sum \Delta_{\alpha_m((A')^{-1}\varrho), \beta_m((A')^{-1}\varrho)}\left(\arg\frac{1}{f - a_i}\right). \end{aligned} \tag{2.40}$$

Now

$$\left| \int_{A^{-1}\varrho}^r \sum (\beta_m' - \alpha_m')\, dr \right| \leq 4\pi,$$

ε tends slowly to zero as $n \to \infty$, and we also have (2.11) and (2.13). Thus we deduce (2.39) on integrating (2.40) from $(A')^{-1}\varrho$ to r and recalling (1.24), (2.8), (2.13) and (2.31).

Remark. Note that (2.40) may be applied to any collection of components D such that $|f(z) - a_i| = \varepsilon$ on each ∂D. Thus for $1 < i \leq q$, let $S_{\mathcal{R}_i}(t)$ be the *residual* covering of $\{|w - a_i| = \varepsilon\}$. This refers to the contribution to the total covering $S_i^\circ(t)$ (cf. (2.27)) which is not included in $S_{\mathscr{D}_i}(t)$ in (2.39) nor in $S_i'(t)$, the contribution from simple islands in $\{|z|<t\}$ (this is estimated from below in (2.14) and above in (2.1)). Thus (1.8) [or (1.11) with $\varepsilon=1$] and (1.24) give the

COROLLARY 2. *The residual coverings $S_{\mathscr{R}_i}$ satisfy*

$$\int^{8A\varrho} \sum_{i=1}^{q} S_{\mathscr{R}_i}(t)\, t^{-1}\, dt \leqslant C(A')^{-3h}\, T(\varrho), \tag{2.41}$$

$$\int^{8A\varrho} \sum_{i} n(t, a_i, \mathscr{R}_i)\, t^{-1}\, dt \leqslant C(A')^{-3h}\, T(\varrho). \tag{2.42}$$

Proof. The previous paragraph justifies the following lines:

$$\begin{aligned}\sum \int^{8A\varrho} S_{\mathscr{R}_i}(t)\, t^{-1}\, dt &= \sum \int^{8A\varrho} [S_i^\circ(t) - S_{\mathscr{D}_i}(t) - S_i'(t)]\, t^{-1}\, dt \\ &\leqslant CT(8A\varrho)\, [q + (A')^{-5h} + (A')^{-4h} - \textstyle\sum \delta(a_i) - (q-2) + \delta] \\ &\leqslant CA^{\lambda+1}(A')^{-4h}\, T(\varrho) < C(A')^{-3h}\, T(\varrho),\end{aligned}$$

which is (2.41).

Inequality (2.42) is a consequence of (1.24), (2.13) and (2.19). Thus, let $n_s(t, a_i)$ be the number of solutions to $f(z)=a_i$ in simple islands in $\{|z|<t\}$. Then

$$\begin{aligned}\sum \int_1^{8A\varrho} [n(t, a_i) - n_s(t, a_i)]\, t^{-1}\, dt &\leqslant \left\{ \sum_{i=1}^{q} N(8A\varrho, a_i) - (q-2)\, T(8A\varrho) + \delta T(\varrho) \right\} \\ &\leqslant 2\delta T(\varrho) < C(A')^{-3h}\, T(\varrho).\end{aligned}$$

Remark. Because of estimates such as (2.41) and (2.42), we let $k(r)$ be a generic non-negative function such that

$$\int^{8A\varrho} k(t)\, t^{-1}\, dt < C(A')^{-3h}\, T(\varrho). \tag{2.43}$$

3. The Decomposition Lemma

3.1. *A negligible grid.* Lemma 2.3 will be combined with Lemma 3.1 below to study value-distribution by means of the argument principle. If $f^{\#}$ is the spherical derivative of f, (2.17) becomes

$$\int_{\{1\leq|z|\leq r\}} f^{\#}(z)\,dr\,d\theta < T(r)^{1/2+h} \tag{3.1}$$

for large r. An immediate consequence of (3.1) and (1.24) is

LEMMA 3.1. *Let τ be a positive number. Then for n sufficiently large, the Pólya peak annulus $\mathfrak{A}'$ (cf. (1.9)) may be sliced by a grid*

$$\mathscr{G}=[\{r_\alpha\},\{\theta_\beta\}] \quad (1\leq\alpha\leq\alpha_n<\infty, 1\leq\beta\leq\beta_n<\infty) \tag{3.2}$$

with

$$\tau<|\theta_{\beta+1}-\theta_\beta|<2\tau \quad (1\leq\beta\leq\beta_n-1) \tag{3.3}$$

$$\tau<|(\theta_1+2\pi)-\theta_{\beta_n}|<2\tau, \tag{3.4}$$

$$\tau<\log\frac{r_{\alpha+1}}{r_\alpha}<2\tau, \tag{3.5}$$

$$-\tau<\log\frac{r_1}{(A')^{-1}\varrho}<0, \quad 0<\log\frac{r_{\alpha_n}}{A'\varrho}<\tau \tag{3.6}$$

such that

$$\sum_\alpha\int_0^{2\pi}|f^{\#}(r_\alpha e^{i\theta})|\,r\,d\theta+\sum_\beta\int_{r_1}^{r_{\alpha_n}}|f^{\#}(re^{i\theta_\beta})|\frac{dr}{r}\leq T(\varrho)^{2/3}. \tag{3.7}$$

In fact, Lemma 3.1 holds so long as $\tau_n>(A_n')^{-1}$ in (1.24).

These sets $\{|z|=r_\alpha\}\cap\mathfrak{A}'$, $\{\arg z=\theta_\beta\}\cap\mathfrak{A}'$ comprise what we call a *negligible logarithmic grid.* Note that if E is any subset of $\mathfrak{A}'$, then E may be surrounded by a set E_τ, such that

$$\partial E_\tau\subset\mathscr{G}, \tag{3.8}$$

and (d=non-euclidean distance with respect to the metric $r^{-1}|dz|$)

$$d(\partial E_\tau,\partial E)\leq C\tau. \tag{3.9}$$

Although the (logarithmic) separation of the sides of the grid approaches zero as $r \to \infty$, there are limitations to the rapidity. For example, if $f(z)=e^z$, no θ_β can be too near $\pm\pi/2$, since the spherical length of $\{|z|=r\} \cap \{\arg z = \pm\pi/2\}$ is proportional to r.

Lemma 3.1 will be used with the particular choice

$$\tau = \sigma^5, \tag{3.10}$$

and for this choice of τ, we study our function f relative to a special network of (logarithmic) rectangles. A *logarithmic rectangle* centered at z_0 of (logarithmic) side-length 2σ is defined by

$$B = B_\sigma(z_0) = \left\{ z;\ \left|\log\left|\frac{z_0}{z_0'}\right|\right| < \sigma',\ \left|\arg\left(\frac{z}{z_0'}\right)\right| < \sigma''\right\} \tag{3.11}$$

where $z_0 \in \mathfrak{A}$,

$$|\sigma' - \sigma| + |\sigma'' - \sigma| < 3\sigma^5 \tag{3.12}$$

and z_0' is chosen so that (for some branch of logarithm)

$$\left|\log\left(\frac{z_0}{z_0'}\right)\right| < 2\sigma^5 \tag{3.13}$$

and, with $\mathscr{G}$ the grid of Lemma 3.1,

$$\partial B_\sigma \subset \mathscr{G}. \tag{3.14}$$

If B is a logarithmic rectangle in the sense (3.11)–(3.14) and $k>0$, then kB is a rectangle similarly defined, except that (3.12) is modified to

$$|\sigma' - k\sigma| + |\sigma'' - k\sigma| < 3\sigma^5. \tag{3.15}$$

The slight inaccuracy of thinking of z_0 as the center and of 2σ (resp. $2k\sigma$) as the side-length of B (resp. kB) when only (3.11)–(3.15) are true is compensated by (3.14).

3.2. *The Decomposition Lemma* (Lemma 3.2). We now use the $s=Q(\lambda)$ significant components $D \in \bigcup \mathscr{D}_i$ (of Lemma 2.3) to partition almost all of the annulus $\mathfrak{A}'$. The purpose of this is to show that the $\mathscr{D}_i$'s divide a large family of regions Ω into subregions, in each of which f has two near-Picard values in the sense (1.16). Lemma 2.1 shows that all a_i and ∞ are near-Picard in the $\mathscr{D}_i$, but the analysis in $\mathfrak{A}' - \bigcup \mathscr{D}_i$ is more subtle and depends on properties of plane sets.

DECOMPOSITION LEMMA 3.2. *There exists a union of $Q(\lambda)$ rectangles B_σ so that if*

$$B^* = \bigcup B_\sigma, \tag{3.16}$$

then the subset $\mathfrak{A}^*$ of $\mathfrak{A}$ *defined by*

$$\mathfrak{A}^* = \mathfrak{A} - B^* \tag{3.17}$$

has the following property. Let Ω be any region in $\mathfrak{A}^*$, whose boundary relative to $\mathfrak{A}^* \cup \partial B^*$ consists of $Q(\lambda)$ closed Jordan curves with

$$\partial\Omega \subset \mathcal{G}. \tag{3.18}$$

Then $\mathcal{F} \equiv \Omega - \bigcup \mathcal{D}_i$ may be partitioned into sets $\mathcal{F}(i,j)$ $(=\mathcal{F}(i,j)(\Omega))$ $(i \neq j, 1 \leq i,j \leq q)$ and $\mathcal{F}^{\#} = \mathcal{F}^{\#}(\Omega)$ so that the relative boundary of each $F \in \mathcal{F}$ is contained in $\mathcal{G}$ and $\bigcup \partial\mathcal{D}_i$, *and*

$$n(a_i, \mathcal{F}(i,j)) + n(a_j, \mathcal{F}(i,j)) < C(A')^{-2h}\, T(\varrho). \tag{3.19}$$

In $\mathcal{F}^{\#}$, all values a_i are near-Picard in the sense that

$$\sum_{i=1}^{q} n(a_i, \mathcal{F}^{\#}) < C(A')^{-2h}\, T(\varrho). \tag{3.20}$$

Remarks. (1) Inequalities (3.19) and (3.20) complement the bounds (2.13) for value distribution in $\bigcup \mathcal{D}_i$.

(2) The description of the sets $\mathcal{F}(i,j)$ and $\mathcal{F}^{\#}$ is quite explicit. Thus a component F of $\mathcal{F}$ is assigned to $\mathcal{F}(i,j)$ if $\Omega \cap \partial F$ includes portions of $\partial\mathcal{D}_i$, $\partial\mathcal{D}_j$ $(i \neq j)$ and subsets of $\mathcal{G}$ while $\partial F \cap \partial\mathcal{D}_k = \emptyset$ $(k \neq i,j)$. All other components F are assigned to $\mathcal{F}^{\#}$. More precisely, we place F in $\mathcal{F}(\emptyset)$ or $\mathcal{F}(i_1, \ldots, i_p)$ if $(\Omega \cap \partial F) - \mathcal{G}$ meets a subset of $\partial\mathcal{D}_{i_1}, \ldots, \partial\mathcal{D}_{i_p}$.

The Decomposition Lemma is motivated by many examples. For instance, if $f(z) = e^z$ and $D_1 = \{|f(z)| < \varepsilon\}$, $D_2 = \{|f(z)| > \varepsilon^{-1}\}$, then D_1 and D_2 are half-planes bounded by vertical lines. In the region F between D_1 and D_2, f assumes all values w with $|\log|w|| < \varepsilon$ but, independent of ε, 0 and ∞ are Picard values in F. This suggests that the Picard values a_i correspond to the indices i_k such that $\partial F \subset (\bigcup_k \partial\mathcal{D}_{i_k})$, and "usually" there are two such i_k, at least in $\mathbf{R}^2$. This is not the case in $\mathbf{R}^n$ $(n \geq 3)$, and this difference is crucial in the construction of quasi-regular mappings in $\mathbf{R}^n$ which have a large (albeit finite) Picard set [25].

We now begin the proof of the Decomposition Lemma. Choose $\sigma > 0$ and let D be a fixed component of $\cup \mathscr{D}_i$. For each $\zeta \in \partial D \cap \mathfrak{A}'$, consider a rectangle $B_\sigma(\zeta)$ as described in (3.11)–(3.15).

Definition. A component $D' \in \cup \mathscr{D}_i$ $(D' \neq D)$ is called *adjacent* to D (at ζ, through B_σ) if D may be joined to D' by a continuum contained in B_σ which (other than endpoints) is disjoint from $\cup \mathscr{D}_i$.

The next lemma is proved by an elementary connectedness argument.

LEMMA 3.3. *Let D' be adjacent to D at ζ_1 and ζ_2, and let $\Gamma(\zeta_1, \zeta_1)$ be that portion of $\partial D \cap \{|z| < A'\varrho\}$ with ζ_1 and ζ_2 as endpoints. Let $\zeta \in \Gamma(\zeta_1, \zeta_2)$ with*

$$B_\sigma(\zeta) \cap [B_\sigma(\zeta_1) \cup B_\sigma(\zeta_2)] = \emptyset.$$

Then no D_j $(D_j \neq D')$ can be adjacent to D at ζ.

Proof. For $j=1,2$ let z_j' be a point of D' which may be joined to ζ_j by an arc γ_j in $B(\zeta_j) - (\cup \mathscr{D}_i)$. Since D' belongs to exactly one component of $\{|z| < A'\varrho\} - D$, it follows that z_1' and z_2' are endpoints of a unique arc Γ' of $\partial D' \cap \{|z| < A'\varrho\}$.

Now $\Gamma(\zeta_1, \zeta_2)$, Γ', γ_1 and γ_2 form the boundary of a Jordan region J whose boundary (other than the arcs Γ, Γ') is disjoint from $\cup \mathscr{D}_i$. Let ζ and $B(\zeta)$ be as in the hypotheses the Lemma. If D_j $(\neq D')$ were adjacent at ζ through $B(\zeta)$, then there would be a continuum in $B(\zeta)$ which connects ζ to D_j without otherwise encountering D or D'. Thus D_j would meet the interior of J. Since we saw in Lemma 2.3 that D_j must also meet $\{|z| = A'\varrho\}$, it follows that D_j must meet ∂J, and this is a contradiction.

3.3. *Removing rectangles.* Choose $D \in \cup \mathscr{D}_i$, and for each $D' \neq D$, $D' \in \cup \mathscr{D}_i$, we construct boxes $B_\sigma(\zeta')$, $B_\sigma(\zeta'')$ with $\zeta', \zeta'' \in \mathfrak{A} \cap \partial D$ subject to two conditions:

$$D' \text{ is adjacent to } D \text{ at } \zeta', \zeta'' \text{ through } B(\zeta'), B(\zeta''); \tag{3.21}$$

further, if $\Gamma(\zeta', \zeta'')$ is the arc of $\partial D \cap \{|z| < A'\varrho\}$ which contains ζ' and ζ'' as endpoints (cf. statement of Lemma 3.3) then

$$\begin{gathered} \text{if } D' \text{ is adjacent to } D \text{ at some } \zeta \in \partial D \cap \mathfrak{A}', \\ \text{then either } \zeta \in \Gamma(\zeta', \zeta'') \text{ or } B_\sigma(\zeta) \cap [B_\sigma(\zeta') \cup B_\sigma(\zeta'')] \neq \emptyset. \end{gathered} \tag{3.22}$$

This may be achieved in the following manner. We consider all components D' which are adjacent through (at least) two distinct rectangles $B(\zeta_1)$, $B(\zeta_2)$ with ζ_1 and ζ_2 in

$\cup\,\partial D$. Now each D of $\cup\,\mathfrak{D}_i$ is a component of $\mathfrak{A}'$ which meets $\{|z|=A'\varrho\}$, and thus ζ_1 and ζ_2 are always in the same component of $\partial D \cap \{|z|<A'\varrho\}$.

Under the obvious ordering of points on $\partial D \cap \{|z|<A'\varrho\}$, we may assume that $\zeta_1<\zeta_2$. By choosing ζ' close to $\inf \zeta_1$ and ζ'' close to $\sup \zeta_2$, we obtain (3.21) and (3.22). To each ordered pair D, D' thus corresponds at most $2(Q(\lambda)-1)$ such rectangles, and thus $Q(\lambda)$ as D and D' vary. We replace each of these B_σ by the similar $B_{3\sigma}$, and let B_1^* be the union of these. Finally, B^* is the union of B_1^* with compact components of $\mathfrak{A}-B_1^*$ and those noncompact components of $\mathfrak{A}-B_1^*$ whose closure meets only one component of $\partial\mathfrak{A}$. Then B^* is also a union of $Q(\lambda)$ rectangles whose (noneuclidean) area is $O(\sigma^2)$. The set $\mathfrak{A}^*$, defined by (3.17), is connected.

3.4. *The Decomposition Lemma in an ideal case.*

LEMMA 3.4. *Let F be a union of regions in $\{\tfrac{1}{2}A^{-1}\varrho<|z|<2A\varrho\}-[B^*\cup(\cup\,\mathfrak{D}_i)]$, each component of which is bounded by portions of $\partial\mathfrak{D}_i \cup \partial\mathfrak{D}_j$ (where $i\neq j$) and a subset Γ of $\mathcal{G}$. We assume that Γ consists of Jordan curves or Jordan arcs, each of whose endpoints lies on $\partial\mathfrak{D}_i\cup\partial\mathfrak{D}_j$. Then as $A'\to\infty$, we have independent of F that*

$$\sum_1^q n(r,a_k,F) \geq (q-2)\,S(r,F)+n(r,a_i,F)+n(r,a_j,F)-C(A')^{-3h}\,T(\varrho)-C\varepsilon^{-1}\,L(r)-k(r) \qquad (\tfrac{1}{2}A^{-1}\varrho<r<2A\varrho) \tag{3.23}$$

where $L(r)$ is the length on C^* of the image of $\{|z|=r\}$, and $k(r)$ satisfies (2.43).

Remark. For each pair (i,j), the set F so described may consist of many components.

Proof. Estimate (3.23) follows from the argument principle applied to the function $(f(z)-a_k)^{-1}$ (where $k\neq i,j$) in the region $F(r)=F\cap\{|z|\leq r\}$.

Choose $a_k\neq a_i$, a_j and consider

$$\Delta_{\partial F(r)} \arg \frac{1}{f-a_k},$$

with $\tfrac{1}{2}A^{-1}<r<2A\varrho$. The key estimate is that

$$\left|\Delta_{\partial F(r)} \arg \frac{1}{f-a_k}\right| < C[(A')^{-3h}\,T(\varrho)+\varepsilon^{-1}L(r)+k(r)] \quad (\tfrac{1}{2}A^{-1}\varrho<r<2A\varrho). \tag{3.24}$$

On the level-sets $\{|f-a_i|=\varepsilon\}$, $\{|f-a_j|=\varepsilon\}$, we find, using the notation suggested by (2.26), that the change in $\arg(w-a_i)$ and $\arg(w-a_j)$ is at most $S_i^\circ(4A\varrho)$ or $S_j^\circ(4A\varrho)$. Thus since $a_k \neq a_i$, a_j, this means that on these level-sets $|\arg(f-a_k)|$ changes by at most

$$C\varepsilon[S_i(4A\varrho)+S_j(4A\varrho)] \leq C\varepsilon T(8A\varrho) \leq C\varepsilon A^{\lambda+1} T(\varrho) = o(1)(A')^{-3h} T(\varrho) \qquad (3.25)$$

since we have (1.24).

It is next necessary to obtain a bound similar to (3.25) for the change of $|\arg(f-a_k)|$ on $\partial_1(r)$ (where we set $\partial_1(r)=\partial F(r)-(\partial\mathscr{D}_i \cup \partial\mathscr{D}_j)$). Let $\{\bigcup \alpha_k\}$ $(=\bigcup \alpha_k(r))$, $\{\bigcup \alpha_k^*\}$ $(=\bigcup \alpha_k^*(r))$ be those subarcs of $\partial_1(r)$ on which, respectively, $|f-a_k|<\varepsilon$ or $|f-a_k|>\varepsilon^{-1}$, and let $\{\beta\}$ $(=\beta(r))$ be the complement of the α's relative to $\partial_1(r)$. The length of the image of $\partial_1(r)$ is at most $L(r)+CT(\varrho)^{2/3}$, since $\partial_1(r) \subset \mathscr{G} \cup \{|z|=r\}$ and (3.7) holds. In particular, this gives that

$$\sum_\beta |\Delta_\beta \arg(f-a_k)| < C\varepsilon^{-1}(L(r)+T(\varrho)^{2/3}) \quad (r>\tfrac{1}{2}A^{-1}\varrho). \qquad (3.26)$$

We next show why

$$\sum_{\alpha_k} |\Delta_{\alpha_k} \arg(f-a_k)| + \sum_{\alpha_k^*} |\Delta_{\alpha_k^*} \arg(f-a_k)| \leq C(A')^{-3h} T(\varrho)+k(r) \quad (r>\tfrac{1}{2}A^{-1}\varrho), \qquad (3.27)$$

by considering the first sum of (3.27) and leaving the analogous estimate of the α_k^* to the reader (this latter discussion is a bit simpler, since by hypothesis $\delta(\infty)=0$). Since $F \subset \mathfrak{A}' - \bigcup \mathscr{D}_i$ and the components of ∂F are closed Jordan curves, or Jordan arcs with endpoints in $\partial\mathscr{D}_i \cup \partial\mathscr{D}_j$, it follows that the α_k are cross-cuts I of preimages D of $\{|w-a_k|<\varepsilon\}$ which are not among the significant components of $\mathscr{D}_k$. (It is to ensure that the α_k be cross-cuts that we require that Γ consist of Jordan curves or Jordan arcs, whose endpoints are in $\bigcup \partial\mathscr{D}_i$.)

There are two possibilities. First, let $I^*=I^*_{\alpha_k}$ be those arcs I such that $|f(z)-a_k|>\frac{1}{2}\varepsilon$ for all $z \in I$. Then exactly as in (3.26) we have

$$\sum_{I^*} |\Delta \arg(f-a_k)| \leq C\varepsilon^{-1}(L(r)+T(\varrho)^{2/3}) \quad (r>\tfrac{1}{2}A^{-1}\varrho). \qquad (3.28)$$

Next, if n^* is the number of arcs I in $\bigcup \alpha_k$ which are not in I^*, then the image of each I has spherical length $|f(I)|$ at least $C\varepsilon$, so (2.17) and (3.7) give that

$$\varepsilon n^* \leq C \sum |f(I)| \leq C(L(r)+T(\varrho)^{2/3}).$$

Now if I is also contained in a simple island D_s over $\{|w-a_k|<\varepsilon\}$, the argument principle shows that $|\Delta_I \arg(w-a_k)| \leq 2\pi$, so the total contribution from these I's satisfies

$$\sum_{I \subset \cup D_s} \Delta_I |\arg(f-a_k)| \leq C\varepsilon^{-1}(L(r)+T(\varrho)^{2/3}). \tag{3.29}$$

Another idea is needed to estimate the contribution to (3.24) which comes from the I's of $\{a_k\}-I^*$ which are not cross-cuts of D_s's. Instead, these I's are cross-cuts of residual coverings $\mathcal{R}_k$ which were introduced at the end of Chapter 2 and satisfy (2.41) and (2.42). Thus (2.41), (2.42) and the argument principle show that

$$\sum_{I \in \mathcal{R}_k} \Delta_I |\arg(f-a_k)| \leq C(A')^{-3h}\, T(\varrho)+k(r), \tag{3.30}$$

where $k(r)$ satisfies (2.43), and (3.27) follows from (3.28), (3.29) and (3.30). Finally, (3.24) follows from (3.25), (3.26), (3.27) and (1.24). Since there are $(q-2)$ choices of $a_k \neq a_i$, a_j, (3.24) yields that

$$\sum_{k \neq i,j} n(r, a_k, F) \geq (q-2)\, n(r, \infty, F) - C(A')^{-3h}\, T(\varrho) - C\varepsilon^{-1} L(r) - k(r). \tag{3.31}$$

This is (3.23) once we derive the general result: if $\partial F - \mathcal{G} \subset \{\partial \mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_k}\}$, then

$$\begin{gathered} |n(r,a,F)-S(r,F)| \leq C(A')^{-3h}\, T(\varrho) + C\varepsilon^{-1} L(r) + k(r) \\ \left(\min_i (|a-a_i|) > 2\varepsilon^{1/2}, \tfrac{1}{2}A^{-1}\varrho < r < 2A\varrho\right); \end{gathered} \tag{3.32}$$

in particular, since here $\partial F - \mathcal{G} \subset \partial \mathcal{D}_i \cup \partial \mathcal{D}_j$, (3.32) holds for $a=\infty$, and we use (3.32) (with $a=\infty$) in (3.31) to get (3.23).

We now prove (3.32). To compute $S(r,F)$, let $C^{\#}$ be the sphere with the discs $|w-a_i|<2\varepsilon_n^{1/2}$ deleted ($1 \leq i \leq q$). It is not hard to see that we may replace a_k by a ($a \in C^{\#}$) in the analysis of (3.24) and deduce

$$\begin{gathered} \left|\Delta_{\partial F(r)} \arg\left(\frac{1}{f-a}\right)\right| \equiv |n(r,\infty,F)-n(r,a,F)| \leq (A')^{-3h}\, T(\varrho) + C\varepsilon^{-1} L(r) + k(r) \\ (a \in C^{\#}, \tfrac{1}{2}A^{-1}\varrho < r < 2A\varrho); \end{gathered} \tag{3.33}$$

the only modification is that here ε is to be replaced by $\varepsilon^{1/2}$, but (1.24) implies that this makes no difference in the final bound. Next, if $a \notin C^{\#}$, the first fundamental theorem implies that $n(r, a, F) \leqslant CT(A'\varrho) \leqslant C(A')^{\lambda+1}T(\varrho)$. The spherical area of such a's is $O(\varepsilon)$, so the contribution to $S(r, F)$ from the complement of $C^{\#}$ is at most $\varepsilon(A')^{\lambda+1}T(\varrho)$. This, (3.33) and (1.24) give (3.32).

COROLLARY 1. *Let F be as in Lemma 3.4, except that $\partial F \subset [\partial \mathcal{D}_i \cup \mathcal{G}]$. Then*

$$\sum_1^q n(r, a_k, F) \geqslant (q-1)S(r, F) + n(r, a_i, F) - C(A')^{-3h}T(\varrho) - C\varepsilon^{-1}L(r) - k(r) \qquad (\tfrac{1}{2}A^{-1}\varrho < r < 2A\varrho). \tag{3.34}$$

Proof. All that need be observed is, since there are $(q-1)$ choices of $a_k \neq a_i$, the sum (3.31) becomes

$$\sum_{k \neq i} n(r, a_k, F) \geqslant (q-1)S(r, F) - C(A')^{-3h}T(\varrho) - C\varepsilon^{-1}L(r) - k(r)$$

and (3.34) follows as did (3.22).

COROLLARY 2. *Let F be as in Lemma 3.4, except that we suppose $\partial F \subset \mathcal{G}$. Then*

$$\sum_1^q n(r, a_k, F) \geqslant qS(r, F) - C(A')^{-3h}T(\varrho) - C\varepsilon^{-1}L(r) - k(r) \quad (\tfrac{1}{2}A^{-1}\varrho < r < 2A\varrho). \tag{3.35}$$

3.5. *Regions F which meet several $\mathcal{D}_i$.* We have observed in Remark 2, § 3.2, that the set $\mathfrak{A}^* - \bigcup \mathcal{D}_i$ is a union of components F, where each F is put in a class $\mathcal{F}(i), \ldots, \mathcal{F}(i_1, \ldots, i_r)$, or $\mathcal{F}(\emptyset)$. Classes $\mathcal{F}(i, j)$, $\mathcal{F}(i)$ and $\mathcal{F}(\emptyset)$ have already been analysed in Lemma 3.4 and its two corollaries. Our next result uses properties (3.21) and (3.22) of our construction to reduce the analysis of classes $\mathcal{F}(i_1, \ldots, i_r)$ (with $r \geqslant 3$) to these simpler situations.

LEMMA 3.5. *Let $\mathcal{F}^* = \mathcal{F}(i_1, i_2, \ldots, i_r)$ for a given r-tuple with $r \geqslant 3$. Then $\mathcal{F}^*$ may be divided into at most $r+1$ classes of type $\mathcal{F}(i)$, $\mathcal{F}(\emptyset)$, by means of curves and arcs from the logarithmic grid $\mathcal{G}$. These curves may be chosen to be Jordan arcs, each of whose endpoints is in $\bigcup \partial \mathcal{D}_i$.*

Proof. Choose $F \in \mathcal{F}(i_1, \ldots, i_r)$ with $r \geqslant 3$, and suppose $\partial D \cap \partial F \neq \emptyset$, with $D \in \mathcal{D}_i$. According to (3.8)–(3.10) and (3.14), each component of $\partial D \cap \partial F$ may be enclosed in a

Jordan domain J whose boundary in $\mathfrak{A}'$ is contained in $\mathscr{G}$ and such that (d=noneuclidean distance)

$$d(\partial J, \partial D\cap\partial F)\leqslant C\tau = C\sigma^5. \tag{3.36}$$

Let $\mathfrak{A}^*$ be the set described in (3.17), and suppose that the closure of $F\cap J$ relative to $\mathfrak{A}^*$ meets a significant component D'' for some $D''\neq D$, with $D''\in\bigcup\mathscr{D}_j$. According to (3.36), D'' and D may be joined within some $B_\sigma(\zeta)$, with $\zeta\in\partial D\cap F$. Thus some $D'\neq D$, $D'\in\bigcup\mathscr{D}_j$, is adjacent to D (in the sense of § 3.2) through some $B_\sigma(\zeta_0)$, with $\zeta_0\in\partial D\cap F$. Properties (3.21) and (3.22) guarantee that $B_\sigma(\zeta')$ and $B_\sigma(\zeta'')$ satisfy (3.21) and (3.22), and so are in the set B^* of (3.16). Thus, the portion of F inside J is either in $\mathscr{F}(i,j)$ (if $j\neq i$) or in $\mathscr{F}(i)$ (if $j=i$).

This argument may be applied to each D which meets ∂F. Thus we obtain $Q(\lambda)$ unions of these Jordan domains J which may be divided into r classes $\mathscr{J}_k$ which correspond to $i_1,\ldots,i_r$. For example, $\mathscr{J}_k$ consists of those J for which the closure of $F\cap J$ relative to $\mathfrak{A}^*$ meets only a component $D\in\mathscr{D}_k$. The complementary set $F-J$ is, relative to $\mathfrak{A}^*$, a union of regions whose closure does not meet $\partial(\bigcup\mathscr{D}_i)$ and whose boundary is contained in $\mathscr{G}$; this set is an $\mathscr{F}(\emptyset)$. This proves Lemma 3.5.

3.6. *Completion of proof of the Decomposition Lemma.* (Recall the statement of this Lemma as Lemma 3.2 in § 3.2).

Let Ω be as in the statement of Lemma 3.2. According to (3.5) and (3.10), we may choose a sequence r_i, $1\leqslant i\leqslant M=M(n,\Omega)$ with $\frac{1}{4}A^{-1}\varrho\leqslant r_1<r_2<\ldots<r_M\leqslant 10A\varrho$, such that each circle $\{|z|=r_i\}\subset\mathscr{G}$, and that each region

$$\{r_i<|z|<r_{i+1}\}-\Omega \quad \textit{is simply-connected.} \tag{3.37}$$

Note from (1.24), (3.5), (3.6), (3.10) and (3.14) that we always have $M<A^2\tau^{-1}<\sigma^{-12}$. Since $\partial\Omega\subset\mathscr{G}\cup(\bigcup\partial D_i)$, (3.37) may readily be arranged. We also let $r_0=(A')^{-1}\varrho$, and let B_p be the annulus $\{r_{p-1}<|z|<r_p\}$.

As we observed in the discussion of (2.14), property (3.37) allows Ahlfors's estimate (2.14) to be applied in each $B_p-\Omega$. Thus (2.14) and (3.7) give that

$$\begin{aligned}\sum n(r,a_i,B_p-\Omega) &\geqslant (q-2)\,S(r,B_p-\Omega)-hL(r)-L(\partial(B_p-\Omega))\\ &\geqslant (q-2)\,S(r,B_p-\Omega)-hL(r)-CT(\varrho)^{2/3}.\end{aligned} \tag{3.38}$$

Consider next the value distribution in $\Omega_p \equiv B_p \cap \Omega$. Each $\Omega_p - \{\bigcup \mathcal{D}_i\}$ becomes a union of $\mathcal{F}(\emptyset)$'s, $\mathcal{F}(i)$'s, and $\mathcal{F}(i,j)$'s, perhaps, as in Lemma 3.5, by introducing additional cross-cuts from $\mathcal{G}$. Using (3.23), (3.34) and (3.35), we find that

$$\begin{aligned}\sum n(r,a_i,\Omega_p) &\geq \sum n(r,a_i,\mathcal{F}^\#)+\sum n\Big(r,a_i,\bigcup_{j\neq k}\mathcal{F}(j,k)\Big)\\ &\geq qS(r,\mathcal{F}(\emptyset))+\sum n(r,a_i,\mathcal{F}(i))\\ &\quad+(q-1)\,S(r,\bigcup\mathcal{F}(i))+\tfrac{1}{2}\Big\{\sum_i n\Big(r,a_i,\bigcup_{j\neq i}\mathcal{F}(i,j)\Big)\\ &\quad+\sum_j n\Big(r,a_j,\bigcup_{j\neq i}\mathcal{F}(i,j)\Big)\Big\}+(q-2)\,S\Big(r,\bigcup_{i\neq j}\mathcal{F}(i,j)\Big)\\ &\quad-C(A')^{-3h}\,T(\varrho)-C\varepsilon^{-1}L(r)-k(r).\end{aligned} \tag{3.39}$$

For $\{|z|<r_0\}$ and in the set B^* of (3.16) we use (2.14) and (3.7):

$$\sum n(r,a_i)\geq(q-2)\,S(r)-hL(r)\quad(r\leq r_0=(A')^{-1}\varrho), \tag{3.40}$$

$$\sum n(r,a_i,B^*)\geq(q-2)\,S(r,B^*)-hL(r)-CT(\varrho)^{2/3}\quad((A')^{-1}\varrho<r<A'\varrho). \tag{3.41}$$

We combine (3.38)–(3.41) and integrate with respect to $\log r$ from $R_1\equiv(A')^{-1}\varrho$ to r, where $4A\varrho<r<8A\varrho$. An appeal to (2.17) and (2.43) gives that

$$\begin{aligned}\sum N(r,a_i)-\sum N(R_1,a_i)&\geq(q-2)\int_{R_1}^{r}S(t)\,t^{-1}\,dt\\ &\quad+2\int_{R_1}^{r}S(t,\mathcal{F}(\emptyset))\,t^{-1}\,dt+\int_{R_1}^{r}S(t,\bigcup\mathcal{F}(i))\,t^{-1}\,dt\\ &\quad+\sum\int_{R_1}^{r}n(t,a_i,\mathcal{F}(i))\,t^{-1}\,dt+\tfrac{1}{2}\sum_{i\neq j}\int_{R_1}^{r}[n(t,a_i,\mathcal{F}(i,j))\\ &\quad+n(t,a_j,\mathcal{F}(i,j))]\,t^{-1}\,dt-C\sigma^{-12}[(A')^{-5h/2}\,T(\varrho)+T(\varrho)^{2/3}]\log A'\end{aligned}$$

since there are at most σ^{-12} annuli B_p. All terms on the right (other than the last) are increasing functions of r. Thus (1.24), (2.1), (2.4), (2.8) and (2.9) imply that

$$S(A\varrho, \mathcal{F}(\emptyset))+S(A\varrho, \cup\, \mathcal{F}(i))+\sum n(A\varrho, a_i, \mathcal{F}(i))+\sum_i n\Big(A\varrho, a_i, \bigcup_{i\neq j} \mathcal{F}(i,j)\Big)$$
$$\leq \delta T(\varrho)+C\sigma^{-12}[(A')^{-5h/2}\, T(\varrho)+T(\varrho)^{2/3}]\log A' < (A')^{-2h}\, T(\varrho). \tag{3.42}$$

Inequality (3.19) is an immediate consequence of this. In order to obtain (3.20), we see from (3.42) that we need only show that

$$\sum_i n(A\varrho, a_i, \mathcal{F}(\emptyset))+\sum_{j\neq i} n(A\varrho, a_j, \mathcal{F}(i)) < C(A')^{-2h}\, T(\varrho),$$

but this follows from (3.42) and (3.32). This proves Lemma 3.2.

COROLLARY. *In addition to* (3.19) *and* (3.20), *we have*

$$S(A\varrho, \mathcal{F}^{\#}) < C(A')^{-2h}\, T(\varrho). \tag{3.43}$$

Further, if $S_{\mathcal{D}_i}(F)$ *is the mean-covering over* $|w-a_i|=\varepsilon$ *which comes from* $\partial\mathcal{D}_i\cap\Omega$, *then*

$$|S_{\mathcal{D}_i}(F)-S_{\mathcal{D}_j}(F)|\leq C(A')^{-2h}\, T(\varrho) \quad (F\in\mathcal{F}(i,j)) \tag{3.44}$$

$$|n(\infty,\Omega)-\tfrac{1}{2}\sum_i S_{\mathcal{D}_i}(\Omega)|\leq C(A')^{-2h}\, T(\varrho). \tag{3.45}$$

Proof. Conclusion (3.43) is immediate from (3.42) and Lemma 3.5. According to (2.13), it is only necessary to compute in the various F's, but if F is not an $\mathcal{F}(i,j)$, then (3.20) (relative to Ω) applies. Thus (3.44) and (3.45) need only be checked on the $\mathcal{F}(i,j)$. Let $F\in\mathcal{F}(i,j)$. Then $\Delta_{\partial F}(\arg(f-a_k))$ is small unless $k=i$ or j and in that case the significant effect is from $S_{\mathcal{D}_i}(F)$ and $S_{\mathcal{D}_j}(F)$. Thus if $F\in\mathcal{F}(i,j)$, we have by (3.32) (where $L(r)\equiv 0$) and (3.19) that

$$\begin{aligned}|S_{\mathcal{D}_i}(F)-S_{\mathcal{D}_j}(F)| &\leq |\Delta_{\partial F}(\arg(f-a_i))|+|\Delta_{\partial F}(\arg(f-a_j))|+C(A')^{-3h}\, T(\varrho)+k(r)\\ &\leq |n(\infty,F)-n(a_i,F)|+|n(\infty,F)-n(a_j,F)|+n(a_i,F)\\ &\quad +n(a_j,F)+C(A')^{-3h}\, T(\varrho)+k(r)\\ &\leq C(A')^{-2h}\, T(\varrho),\end{aligned}$$

and (3.44) follows on integrating from $4A\varrho$ to $8A\varrho$.

In order to obtain (3.45), we observe that estimates (3.25) and (3.26) apply when $k=i$ or j, and, from (3.24), that the only significant contribution to $\Delta_{\partial F}\arg(f-a_i)$ comes from $\partial F \cap \partial \mathscr{D}_i$, where $F \in \mathscr{F}(i,j)$. Thus (3.19), (3.20), (3.24) and (3.26) yield after integrating that

$$|n(\infty,\Omega)-\tfrac{1}{2}\sum S_{\mathscr{D}_i}(\Omega)| \leq \tfrac{1}{2}\left\{\sum_{i\neq j}\sum_{F\in\mathscr{F}(i,j)} |(n(\infty,F)-n(a_i,F))-S_{\mathscr{D}_i}(\Omega)| \right.$$
$$\left. +\sum_{i\neq j}\sum_{\mathscr{F}(i,j)} |(n(\infty,F)-n(a_j,F))-S_{\mathscr{D}_j}(\Omega)|\right\}$$
$$+C(A')^{-2h}\,T(\varrho) \leq C(A')^{-2h}\,T(\varrho),$$

which proves (3.45).

4. Applications of local quasi-conformal modifications

4.1. *Introduction.* In this chapter we make our first use of quasi-conformal modifications. In the z-plane, rectangles $B_\sigma(z)$ (as in (3.11)–(3.14)) were the natural domains; now we will also use circular regions $\Delta(\zeta_0,h)$, $\Delta(\zeta_0,h_1,h_2)$ and $C(\zeta_0,h)$ with

$$\begin{aligned}&\Delta(\zeta_0,h)=\{|\zeta-\zeta_0|\leq h|\zeta_0|\},\ \Delta(\zeta_0,h_1,h_2)=\Delta(\zeta_0,h_2)-\Delta(\zeta_0,h_1),\\ &C(\zeta_0,h)=\{|\zeta-\zeta_0|=h|\zeta_0|\}\end{aligned} \tag{4.1}$$

to simplify many later formulas.

Let $\mathfrak{A}^*$ be as in (3.17) and

$$\mathfrak{A}^*_\sigma=\{z\in\mathfrak{A}^*,\ d(z,B^*)\geq 2\sigma\} \tag{4.2}$$

where B^* is from (3.16) and d is non-euclidean distance with metric $|z|^{-1}|dz|$. We will study f in rectangles $B_\sigma(z_0)$ with $z_0\in\mathfrak{A}^*_\sigma$.

Definition. Let the $\mathscr{D}_i$ $(1\leq i\leq q)$ be the $Q(\lambda)$ significant components of f. A rectangle $B=B_\sigma(z_0)$ $(z_0\in\mathfrak{A}^*_\sigma)$ is called *admissible* (with respect to quasi-conformal modification) if each component of $B-\cup\mathscr{D}_i$ meets $\partial\mathscr{D}_i$ for at most two indices i.

If B is not admissible, then for some i a point ζ of $B\cap\partial\mathscr{D}_i$ would be adjacent in B to $\mathscr{D}_j$ for two different indices j $(\neq i)$. The set B^* in (3.16) has been chosen so that this cannot happen if $B\subset\mathfrak{A}^*$ (see Lemma 3.3, (3.21) and (3.22)) or in particular if $z_0\in\mathfrak{A}^*_\sigma$.

In §§4.2 and 4.3, we show that quasi-conformal modifications may always be constructed in admissible rectangles. This development has two purposes. First, it displays in a simpler setting the main ideas which are needed in Chapter 5 to construct modifications of a more global nature. In addition, two results which are important for later work will come from our local study.

The most compelling conclusion is obtained in § 4.7 (Lemma 4.11): the set $\mathcal{F}^{\#}$ of components of $\mathfrak{A}^*-(\cup \mathcal{D}_i)$ (i.e. those that are not in $\mathcal{F}(i, j)$ where $i \neq j$) cannot meet $\mathfrak{A}^*_\sigma$. This information makes it possible to construct our main quasi-conformal modifications (§ 5.2) in a simple way, since we will need only consider regions F whose closure meets ∂D_i for exactly two indices i.

In the final section, § 4.8, we will obtain Lemma 4.13 which is needed in Lemma 7.2 to show that integrals over certain B_σ's are negligible.

The principle of this chapter is that we may treat $\log f(z)$ (after quasi-conformal modification) as an analytic function in its own right (see (4.44)).

4.2. *First encounter with quasi-conformal modifications.* The importance of admissibility (introduced in § 4.1) is seen in the proof of

LEMMA 4.1. *Let $B=B_\sigma(z_0)$ be a rectangle contained in $\mathfrak{A}^*_\sigma$, where $\mathfrak{A}^*_\sigma$ is defined in* (4.2).

Then if n is sufficiently large in (1.9), *we may construct a quasi-conformal modification*

$$H(z)=\omega(f(z)) \quad (z \in B) \tag{4.3}$$

such that (for $1 \leq i \leq q$)

$$H(z)=(f(z)-a_i)^{\pm 1} \quad (z \in B \cap \mathcal{D}_i) \tag{4.4}$$

and

$$|\mu_H(z)| \equiv \left|\frac{H_{\bar{z}}}{H_z}(z)\right| < \eta \quad (z \in B). \tag{4.5}$$

Proof. Choose a fixed component D_0 of $B \cap \{\cup \mathcal{D}_i\}$ with, say $D_0 \subset \mathcal{D}_i$, and let, for example,

$$H(z)=\frac{1}{f(z)-a_i} \quad (z \in D_0 \cap B). \tag{4.6}$$

Formula (4.6) thus determines H on *zero-stage* D's.

Let $\{F\}$ be the components of $B-\cup \mathcal{D}_i$. *First-stage* F's are those whose closure in

B meets ∂D_0. If F_1 is any first-stage F, the condition that B is admissible ensures that there is at most one $j \neq i$ ($j=j(F_1)$) such that the closure of F_1 meets $\partial \mathcal{D}_j$ in B. In an analogous manner, components of $B \cap \{\cup \mathcal{D}_i\}$ (other than D_0) whose closures meet first-stage F's are called first-stage D's. In turn, their closures determine second-stage F's, and by continuing this process we assign a stage to each F and each D of $B \cap \{\cup \mathcal{D}_i\}$.

We next extend (4.6) to first-stage F's. Let F_1 be such a region, thus $\{\partial F_1 \cap B\} \subset \{\partial \mathcal{D}_i \cup \partial \mathcal{D}_j\}$ where $j=j(F_1) \neq i$ (we are assured that such j exists, although it will not be unique when $F_1 \in \mathcal{F}(i)$). Then F_1 is divided into $(F_1)_\infty$ and $(F_1)_0$ by

$$(F_1)_\infty = F_1 \cap \left\{ \left| \frac{f-a_j}{f-a_i} \right| > 1 \right\},$$

$$(F_1)_0 = F_1 \cap \left\{ \left| \frac{f-a_j}{f-a_i} \right| \leq 1 \right\}.$$

In order to apply Lemma 1.1, take $M'=1$ and $M=M_{jk}$ so large that all ω_{jk} ($1 \leq j, k \leq q$) may be constructed as in Lemma 1.1 with $\|\mu_{\omega_{jk}}\|_\infty < \frac{1}{2}\eta$ and such that (1.23) holds for all j and k.

We then choose n in (1.9) so large that the ε_n of (1.24) satisfy $\varepsilon < M_{jk}^{-1}$ for all j and k (this is consistent with the condition $\varepsilon < \eta$ in (1.24); note from (1.21) that $\eta \log M \geq C$). Once this is done, we can and do define

$$H(z) = \begin{cases} \omega_{ij}(f(z)) & (z \in (F_1)_\infty) \\ [\omega_{ji}(f(z))]^{-1} & (z \in (F_1)_0) \end{cases}. \tag{4.7}$$

A check of (1.23) ensures that H is continuous in the F_1's, and on $\partial D_0 \cap \partial F_1$, and, further,

$$H(z) = \begin{cases} f(z)-a_j & (z \in \partial F_1 \cap \partial \mathcal{D}_j) \\ [f(z)-a_i]^{-1} & (z \in \partial F_1 \cap \partial \mathcal{D}_i) \end{cases}. \tag{4.8}$$

Formula (4.8) makes it clear how to extend H to first-stage regions: $H(z)=[f(z)-a_i]^{-1}$ ($z \in D \subset \mathcal{D}_i$) or $H(z)=f(z)-a_j$ (if $z \in D \subset \mathcal{D}_j, j \neq i$). This places us exactly in the situation we confronted when attempting to extend (4.6) to first-stage F's, and so now H may be extended to second-stage F's so that formulas such as (4.8) hold at points in the closure of second-stage D's. By continuing this process, H becomes defined in all of B.

It is clear that H is of the form (4.3), where we take $\omega(w)=w-a_i$ or $(w-a_i)^{-1}$ when $w=f(z)$ with $z \in \cup \mathcal{D}_i \cap B$, and $\omega=\omega_{ij}((w-a_j)/(w-a_i))$ ($w=f(z), z \in (F_1)_\infty$) or $\omega=\{\omega_{ij}((w-a_i)/(w-a_j))\}^{-1}$ ($w=f(z), z \in (F_1)_0$).

Assertions (4.4) and (4.5) may be readily verified. For example, (4.4) follows from formulas of the nature (4.6) and (4.8). Also, (4.6) and (4.8) show that $\mu_H=0$ for $z\in\{\bigcup \mathcal{D}_i\}\cap B$. If z is in an F, then since $|\mu_{\omega\circ f}(z)|=|\mu_\omega(f(z))|$, (4.5) follows from (1.18).

Remarks. If $B\cap\{\bigcup \mathcal{D}_i\}=\emptyset$, we take H to be any of the functions $(f(z)-a_i)/(f(z)-a_j)$, for any distinct i and j. Note that once a component D_0 is chosen, and H is defined as in (4.6), then H is uniquely defined on all of B if $\mathcal{F}^{\#}\cap B=\emptyset$. By this we mean that there is no subdomain of $B-\bigcup \mathcal{D}_i$ whose boundary consists of portions of $\mathcal{G}$ and arcs of $\partial\mathcal{D}_i$ for $r\neq 2$ indices i [see Remark 2 of § 3.2 which follows the statement of Lemma 3.2].

4.3. *A nearly-equivalent meromorphic function.* Standard methods allow the function H of Lemma 4.4 to be replaced by a nearly equivalent (genuinely) meromorphic function; this depends on solving a Beltrami equation.

LEMMA 4.2. *Let H and B be as in Lemma* 4.1. *Then there exists a homeomorphism of the plane $z=\psi(\zeta)$, such that if $\zeta=\varphi(z)$, then the composition*

$$G(\zeta)=H(\varphi(\zeta))\quad (\zeta\in\psi^{-1}(B)) \tag{4.9}$$

is meromorphic in $\psi^{-1}(B)$. Further, by choosing A_n and γ appropriately (consistent with (1.24)) *we may arrange that*

$$\left|\frac{\varphi(\zeta)}{\zeta}-1\right|\leq\gamma^{10}\quad ((30A)^{-1}\varrho<|z|<30A\varrho). \tag{4.10}$$

G has 0 *and* ∞ *as near-Picard values in $\psi^{-1}(B)$ in the sense that*

$$n(0,G,\psi^{-1}(B))+n(\infty,G,\psi^{-1}(B))\leq C(A')^{-2h}T(\varrho). \tag{4.11}$$

Each set $\psi^{-1}(B_\sigma(z_0))$ contains the disc $\Delta(z_0,\tfrac{2}{3}\sigma)$.

Proof. Define ν in B by

$$\nu(z)=\frac{H_{\bar z}(z)}{H_z(z)}=\frac{\omega_{\bar w}}{\omega_w}(f(z))\frac{\overline{f'(z)}}{f'(z)}\quad (z\in B) \tag{4.12}$$

(the right equality used (4.3)) and extend ν to the full plane by taking

$$\nu(z)\equiv 0\quad (z\notin B). \tag{4.13}$$

Thus $\|\nu\|_\infty<\eta$.

Consider the Beltrami equation

$$\psi_{\bar{z}}(z) = \nu(z)\,\psi_z(z) \quad (|z| < \infty) \tag{4.14}$$

where ψ is to fix 0, ϱ and ∞. Then [2, Chapter 5] there is a unique solution, $\zeta=\psi(z)$, which is a homeomorphism. The function $\varphi(\zeta)$, which is our real interest, is the inverse function to ψ. Our assumptions ensure that both ψ and φ are $(1+\eta)/(1-\eta)$ quasiconformal mappings of the plane.

If G is defined by (4.9), a computation [2, p. 9] using the chain rule shows that $G_{\bar{\xi}}\equiv 0$ a.e. and so G is meromorphic.

We achieve (4.10) in an elementary manner. Consider the family of functions $\Phi(\zeta)=\varrho^{-1}\varphi(\varrho\zeta)$ as B and ν vary, subject to B meeting the hypotheses of Lemma 4.1 and $\|\nu\|_\infty\leq\eta$. Then if the $\eta_n\to 0$ sufficiently rapidly, and $\gamma_n\to 0$ and $A_n\to\infty$ sufficiently slowly (cf. (1.24)) we may apply normal family considerations to the family $\{\Phi\}$, and deduce (4.10) since the Φ's tend to the identity map.

Formula (4.9) shows that (4.11) depends on a similar bound for the zeros and poles of H in B itself. However in the components of $B\cap\{\bigcup\mathcal{D}_i\}$, $H(z)=[f(z)-a_i]^{\pm 1}$, so the zeros and poles of H for such z are controlled by (2.13). Since B is admissible and satisfies (3.18) (with $\Omega=B$), the components F of $B-\bigcup\mathcal{D}_i$ may be apportioned to classes $\mathcal{F}(\emptyset)$, $\mathcal{F}(i)$, $\mathcal{F}(i,j)$ relative to B, much as described after the statement of Lemma 3.2, depending on the indices j such that the closure of $F\cap B$ meets $\partial\mathcal{D}_j$. If, in this classification, $F\in\mathcal{F}(i,j)$ (with $i\neq j$) then (4.7) shows that the zeros and poles of H in F are among the a_i and a_j-values of f in F: thus (3.19) gives the bound (4.11) in this case. If, however, $F\in\mathcal{F}^{\#}=\{\mathcal{F}(\emptyset),\bigcup\mathcal{F}(i)\}$ then (3.20) applies and so does (4.11).

That $\Delta(z_0,\frac{2}{3}\sigma)\subset\varphi^{-1}(B_\sigma(z_0))$ follows from (4.10).

4.4. *Local logarithmic means.* For each $z_0\in\mathfrak{A}^*_\sigma$, Lemmas 4.1 and 4.2 produce a modification $G(\zeta)$ in each disc $\Delta(z_0,\frac{2}{3}\sigma)$. We now consider of the functions $\log|G(\zeta)|$ on circles $C(z_0,t)$, where $\sigma^3<t<\frac{1}{2}\sigma$.

We begin by making some calculations concerning the behaviour of f in the various $\mathcal{D}_i$, where the $\mathcal{D}_i$ are from Lemma 2.3. These computations play an essential role in our work but the reader may pass over this section to § 4.5, and refer back to results here when needed, as this section is very technical. Our analysis uses a (non-euclidean) proximity function

$$m_{z_0}(t,f_i)=\frac{1}{2\pi}\int_{C(z_0,t)}\log|f_i(z)|\,d\arg(z-z_0) \tag{4.15}$$

where

$$f_i(z)=\begin{cases} f(z) & (z\in\mathcal{D}_i) \\ 1 & (z\notin\mathcal{D}_i)\end{cases}. \tag{4.16}$$

In these calculations we use the local polar coordinates

$$z-z_0=t|z_0|\,e^{i\varphi},\quad \zeta-\zeta_0=u|\zeta_0|\,e^{iv}.$$

LEMMA 4.3. *Let* $z_0\in\mathfrak{A}$. *Then for* $1\leqslant i\leqslant q$

$$m_{z_0}(t,f_i)\leqslant CA^{\lambda+1}T(\varrho)\quad (\tfrac{1}{2}\sigma^3<t<1). \tag{4.17}$$

Proof. Let $z\in D\cap C(z_0,t)$ with D a component of $\{|f(z)-a_i|<\varepsilon\}$ and $D\in\mathcal{D}_i$. We study f_i in D by the formula (2.32), with Ω the annulus $\{(A')^{-1}\varrho<|z|<R\}$ and R chosen with $20A\varrho<R<30A\varrho$ and such that (cf. (2.10))

$$\sup_z \sum_1^q |\log|f(z)-a_i||\leqslant CT(40A\varrho)\leqslant CA^{\lambda+1}T(\varrho)\quad (|z|=R). \tag{4.18}$$

Then $(A\varrho)(\sigma^3|z_0|)^{-1}\leqslant CA^2\sigma^{-3}$, so (1.24), (2.13), (2.28), (2.36) and (4.18) (with R in place of $A'\varrho$ and the trivial bound $\omega\leqslant 1$) give for $z\in D\subset\mathcal{D}_i$ that

$$\begin{aligned}\int_{C(z_0,t)}\log|f_i(z)|\,dv &\leqslant \log\frac{1}{\varepsilon}+C(A')^{-5h}\,T(\varrho)\log(CA^2\sigma^{-3}) \\ &\quad +CA^{\lambda+1}T(\varrho)\sup_{z\in D\cap C(z_0,t)}\omega(z,I_R) \\ &\leqslant CA^{\lambda+1}T(\varrho)\quad (z\in D\cap C(z_0,t),\sigma^3<t<1),\end{aligned} \tag{4.19}$$

and (4.17) follows on summing over all $Q(\lambda)$ D's of $\mathcal{D}_i$.

LEMMA 4.4. *Let* $z_0\in\mathfrak{A}^*_\sigma$, *and let* $G(\zeta)$ *be the function of Lemma* 4.2, *which is therefore meromorphic in* $\Delta(z_0,t)$ *for all* $t\leqslant\frac{2}{3}\sigma$. *Then*

$$\begin{aligned}&\left|\int_{\Delta(z_0,s_1,s_2)}|\log|G(\zeta)||\,u^{-1}\,du\,dv-\sum\int_{s_1}^{s_2}m_{z_0}(t,f_i)\,t^{-1}\,dt\right| \\ &\leqslant CA^{\lambda+1}T(\varrho)\left\{\log\frac{s_2}{s_1}\right\}^{1/2}\{(A')^{-3h}+\gamma^2+C(\eta)\}\quad (\sigma^3<s_1\leqslant s_2\leqslant Cs_1\leqslant\tfrac{1}{2}\sigma)\end{aligned} \tag{4.20}$$

where in (4.20) *and below,* $C(\eta)$ *is a generic function with*

$$C(\eta)\leq C\eta^{\varepsilon_0} \tag{4.21}$$

for some $\varepsilon_0>0$.

Proof. Inequality (4.20) follows from two similar inequalities which use the function $H(z)$ of (4.3). We first prove the stronger assertion that for each t, $\frac{1}{2}\sigma^3<t<\frac{2}{3}\sigma$,

$$\left|\int_{C(z_0,t)}|\log|H(z)||\,d\varphi-\sum_{i=1}^{q}m_{z_0}(t,f_i)\right|\leq C(A')^{-3h}\,T(\varrho)\quad(\tfrac{1}{2}\sigma^3\leq t\leq\tfrac{2}{3}\sigma). \tag{4.22}$$

In the various $\mathscr{D}_i$ this is immediate by, for example, (4.6), for $\log|H|=\pm\log|f_i|$ in the $\mathscr{D}_i$. Consider now a region $\mathscr{F}(\emptyset)$, $\mathscr{F}(i)$ or $\mathscr{F}(i,j)$ of $B_\sigma(\zeta_0)$; in such a region, H is defined by formulae such as (4.7) (where always $j\neq i$). In these $\mathscr{F}$'s, we have that either $\varepsilon<|H(z)|<\varepsilon^{-1}$ or that $\log|H(z)|=\pm\log^+\{1/|f(z)-a_i|\}$ or $\pm\log^+\{1/|f(z)-a_j|\}$. The integrals over the sets where $|\log|H||<\log\varepsilon^{-1}$ may readily be absorbed into (4.22), since (1.24) shows that $(\log T(\varrho_n))^{-1}<\varepsilon_n<(A_n')^{-1}$. Otherwise we have that z is in a component $D\notin\bigcup\mathscr{D}_i$, and so (2.31) allows these contributions to $\int\log|H|$ to be absorbed into (4.22).

More interesting is that if $\sigma^3<s_1\leq s_2\leq Cs_1\leq\frac{1}{2}\sigma$, then

$$\begin{gathered}\left|\int_{\Delta(\zeta_0,s_1,s_2)}|\log|H(z)||\,t^{-1}\,dt\,d\varphi-\int_{\Delta(\zeta_0,s_1,s_2)}|\log|G(\zeta)||\,u^{-1}\,du\,dv\right|\\ \leq C\left\{\log\frac{s_2}{s_1}\right\}^{1/2}\{\gamma^2+C(\eta)\}\,A^{\lambda+1}T(\varrho)\end{gathered} \tag{4.23}$$

where now, unlike in (4.22), we need an area integral. In order to prove (4.23), we need further information about ψ than we have used to obtain Lemma 4.2 (recall that $\psi=\varphi^{-1}$). The Jacobian of ψ is [2, pp. 33, 27].

$$J=|\psi_z|^2-|\psi_{\bar z}|^2=(1+C(\eta))\,|\psi_z|^2 \tag{4.24}$$

where we are using (4.14), and $C(\eta)$ is as in (4.21). According to [2, p. 92], the function

$$h(z)\equiv\psi_z-1 \tag{4.25}$$

is in $L_p(\mathbf{C})$ and

$$\|h(z)\|_p\leq A_p\,\varrho^{2/p}\eta^{1/2}\quad(1<p<\infty) \tag{4.26}$$

where A_p is a constant which depends only on p (this is because the theory of [2] applies to the normalized function $\varrho^{-1}\psi(\varrho^{-1}z)$).

When changing local variables, we have that

$$u\,du\,dv = Jt\,dt\,d\varphi \tag{4.27}$$

where J is as in (4.24). Now (4.10) implies that $|\psi(z)-\psi(z_0)|\geq\frac{1}{2}|z-z_0|\geq\frac{1}{4}\sigma^3|z_0|$ if $z\in\Delta(\frac{1}{2}\sigma^3,\frac{1}{2})$, so we find using (4.10) and (1.24) that

$$\begin{aligned}\left|\frac{z-z_0}{\psi(z)-\psi(z_0)}-1\right| &= \frac{1}{|\psi(z)-\psi(z_0)|}\left|z\left\{1-\frac{\psi(z)}{z}\right\}-z_0\left\{1-\frac{\psi(z_0)}{z_0}\right\}\right| \\ &\leq \frac{1}{|\psi(z)-\psi(z_0)|}|(1+\varepsilon_1(z))-(1+\varepsilon_2(z))|\,(|z|+|z_0|) \\ &\leq C\sigma^{-3}\gamma^{10} = C\gamma^7 \quad (z\in\Delta(z_0,\tfrac{1}{2}\sigma^3,1))\end{aligned} \tag{4.28}$$

where ε_1 and ε_2 are functions which are of order γ^{10}. Since $t|z_0|=|z-z_0|$ and $u|\psi(z_0)|=|\psi(z)-\psi(z_0)|$, we have

$$\begin{aligned}\int_\Delta \log|G(\zeta)|\,u^{-1}\,du\,dv &= \int_\Delta |\log|H(\psi(\zeta))||\,u^{-2}u\,du\,dv \\ &= \int_{\varphi(\Delta)} |\log|H(z)||\left\{\frac{t|\psi(z_0)|}{|\psi(z)-\psi(z_0)|}\right\}^2 t^{-2}Jt\,dt\,d\varphi \\ &= \int_{\varphi(\Delta)} |\log|H(z)||\left|\frac{\psi(z_0)}{z_0}\right|^2\left|\frac{z-z_0}{\psi(z)-\psi(z_0)}\right|^2 Jt^{-1}\,dt\,d\varphi.\end{aligned}$$

According to (1.24), (4.10), (4.24)–(4.26), (4.28) and the convention (4.21), this means

$$\begin{aligned}&\left|\int_\Delta |\log|G(\zeta)||\,u^{-1}\,du\,dv-\int_{\varphi(\Delta)} |\log|H(z)||\,t^{-1}\,dt\,d\varphi\right| \\ &\leq [C\gamma^7+C(\eta)]\int_{\varphi(\Delta)} |\log|H(z)||\,t^{-1}\,dt\,d\varphi+C\int_{\varphi(\Delta)} (|h|+|h^2|)\,|\log|H(z)||\,t^{-1}\,dt\,d\varphi.\end{aligned} \tag{4.29}$$

Thus, the discrepancy in (4.23) is due to two factors:

(I) the right side of (4.29) is not zero

and

(II) $\varphi(\Delta)$ is different than Δ.

We first show that if $\alpha>\frac{1}{2}\sigma^3$ and $\alpha<\beta<C\alpha<\frac{1}{2}\sigma$, then

$$\int_{\Delta(z_0,\alpha,\beta)} \left\{ \sum_{i=1}^{q} |\log|f_i(z)|| + |\log|H(z)|| \right\} \{C\gamma^7 + C(\eta) + C(|h|+|h^2|)\} t^{-1}\, dt\, d\varphi$$
$$\leqslant C\left\{\log\frac{\beta}{\alpha}\right\}^{1/2} \{\gamma^7 + C(\eta)\} A^{\lambda+1} T(\varrho). \tag{4.30}$$

There is no problem to obtain this bound for the terms $C\gamma^7$ and $C(\eta)$ since the means of $|\log|f_i||$ and $|\log|H||$ satisfy (4.17) and (4.22). However, we do not have a uniform bound ($p=\infty$) for the functions h and h^2 in (4.30), so we need L_p forms of (4.17) and (4.22) with p near 1; we then take p' near ∞ and use the fact from (1.24) that η in (4.26) will overwhelm all A's, γ's and σ's.

We thus claim that there is a $p_0>1$ such that if $1\leqslant p\leqslant p_0$,

$$\left\{\int_{C(z_0,t)} |\log|H(z)||^p\, d\varphi\right\}^{1/p} + \sum_i \left\{\int_{C(z_0,t)} |\log|f_i(z)||^p\, d\varphi\right\}^{1/p} \leqslant CA^{\lambda+1}T(\varrho)$$
$$(\tfrac{1}{2}\sigma^3 < t < \tfrac{1}{2}\sigma), \tag{4.31}$$

where C does not depend on p. To see this, we first observe that it is possible to bound $H(z)$ by

$$|\log|H(z)|| \leqslant C\log\frac{1}{\varepsilon} + CA^{\lambda+1}T(\varrho) + \sum^{M} g^{\Omega}(z, z_\nu) \tag{4.32}$$

where $\Omega=\{(A')^{-1}\varrho<|z|<R\}$, R is from (4.18), and such that the number of Green functions which appear is $M\leqslant C(A')^{-2h}T(\varrho)$; this bound on M follows from (4.11). For let $z\in\Delta(z_0,t)$ with $z\in\mathcal{D}_i\cup\mathcal{F}(\emptyset)\cup\mathcal{F}(i)\cup\mathcal{F}(i,j)$ for some i and j, with $i\neq j$. Then if $|\log|H(z)||>\log\varepsilon^{-1}$, it follows that $H(z)=(f(z)-a_i)^{\pm1}$ or $(f(z)-a_j)^{\pm1}$. Thus we use (4.18) and find that

$$|\log|H(z)|| \leqslant \log\frac{1}{\varepsilon} + CA^{\lambda+1}T(\varrho) + \log^+|f_i| + \log^+|f_j| + \sum_{\nu=1}^{M'} g^{\Omega}(z, z_\nu)$$
$$(z\in\Delta\cap[\mathcal{D}_i\cup\mathcal{F}(\emptyset)\cup\mathcal{F}(i)\cup\mathcal{F}(i,j)]) \tag{4.33}$$

where $M'\leqslant C(A')^{-2h}T(\varrho)$. Since (2.32) and (4.18) give for each i that

$$|\log|f_i(z)|| \leqslant \log\frac{1}{\varepsilon} + CA^{\lambda+1}T(\varrho) + \sum_{(z\in\Omega\cap\mathcal{D}_i)}^{M''} g^{\Omega}(z, z_\nu) \tag{4.34}$$

with $M''\leqslant C(A')^{-2h}T(\varrho)$, we deduce (4.32) from (4.33) and (4.34).

The L_p-norms of the first two terms on the right side of (4.32) and (4.34) are clear. Since we may take $r=\frac{1}{2}\sigma^3|z_0|>\frac{1}{2}\sigma^3 A^{-1}\varrho$ and $R=CA\varrho$ in (2.29), we find from (2.29) and our bounds on M and M'' in (4.32) and (4.34) that

$$\sum^{M+M'}\left[\int_{C(z_0,t)} g(z,z_\nu)^p\,d\varphi\right]^{1/p} \leqslant C(A')^{-2h}\,T(\varrho)\,\Omega_p(A^2\sigma^{-3}) \leqslant (A')^{-h}\,T(\varrho) \quad (\tfrac{1}{2}\sigma^3<t\leqslant 1), \tag{4.35}$$

where the last inequality is a consequence of the understanding in (1.24) that $(A')^{-1}$ tends to zero as rapidly as desired compared to σ and A^{-1} (one could avoid this by making estimates of Ω_p in (2.29), but this is not needed).

The bound (4.31) follows from (4.32), (4.33), (4.34) and (4.35).

Before we estimate the L_p-norm for h, where h is defined in (4.25), we relate local coordinates in the various B_σ and $\Delta(z_0,h)$ to those of a standard polar (r,θ)-system (centered at zero). Suppose $\sigma<\frac{1}{2}$ and $|\zeta_0|=s_0$. For the moment, write $z=re^{i\theta}$. Then our systems are related by: $z-z_0=t|\zeta_0|\,e^{i\varphi}=r_0\,t\,e^{i\varphi}$, with Jacobian r_0^{-2}. Thus

$$t^{-1}\,dt\,d\varphi = t^{-2}\{t\,dt\,d\varphi\} = t^{-2}\{r_0^{-2}\,r\,dr\,d\theta\} = (r/tr_0)^2\,\{r^{-1}\,dr\,d\theta\}. \tag{4.36}$$

Now we consider the bounds on h. We first use (4.26), (4.36) and the assumption that $\alpha>\frac{1}{2}\sigma^3$ to deduce that

$$\begin{aligned}\int_{\Delta(z_0,\alpha,\beta)} |h(z)|^{p'}\,t^{-1}\,dt\,d\varphi &\leqslant C\sigma^{-6}r_0^{-2}\int |h(z)|^{p'}\,t\,dt\,d\varphi\\ &= C\sigma^{-6}\varrho^2 r_0^{-2}\int |h(Z)|^{p'}\,dX\,dY \quad (Z=X+iY=\varrho^{-1}z=\varrho^{-1}(w+z_0))\\ &\leqslant C\sigma^{-6}A^2(A_{p'}\,\eta^{1/2})^{p'},\end{aligned} \tag{4.37}$$

and a similar estimate will follow for h^2 since $\|h^2\|_p=\|h\|_{2p}^2$. (The penultimate term in (4.37) simply reduces h to the normalized form covered in [2].)

Inequality (4.30) follows from (4.31) and (4.37). For example, we may choose $1<p_0<2$, $1<p<p_0$ and $p'=p/(p-1)<\infty$. Then we obtain from (1.24) that

$$\int_{\Delta(z_0,\alpha,\beta)} |\log|H(z)||\,|h(z)|\,t^{-1}\,dt\,d\varphi \leqslant \left(\int |\log|H||^p\,t^{-1}\,dt\,d\varphi\right)^{1/p}\left(\int |h|^{p'}\,t^{-1}\,dt\,d\varphi\right)^{1/p'}$$

$$\leqslant CA^{\lambda+1}T(\varrho)\left\{\log\frac{\beta}{\alpha}\right\}^{1/p}(\sigma^{-6/p'}A^{2/p'}A_{p'}\eta^{1/2}) \tag{4.38}$$

$$\leqslant C(\eta)\,T(\varrho)\left\{\log\frac{\beta}{\alpha}\right\}^{1/2}\quad(\tfrac{1}{2}\sigma^3\leqslant\alpha<\beta<C\alpha<\tfrac{1}{2}\sigma).$$

This shows how we handle (I). To resolve the complications in (II), we note from (4.10) that $(\varphi(\Delta)-\Delta)\cup(\Delta-\varphi(\Delta))$ is contained in narrow annuli $s_1'<s_1<s_1''$, $s_2'<s_2<s_2''$, with

$$\log\frac{s_2''}{s_2'}+\log\frac{s_1''}{s_1'}\leqslant C\gamma^{10}. \tag{4.39}$$

Thus we may use computations such as (4.29) above, with $\alpha=s_1$ or s_2' and $\beta=s_1''$ or s_2''. Then $\alpha>\frac{1}{2}s_1>\frac{1}{2}\sigma^3$ and the factor $\{\log(\beta/\alpha)\}^{1/2}$ in (4.30) is at most $C\gamma^5$, so this may be absorbed in (4.20). We omit the details.

We record from (4.17) and (4.20) that

$$\int_{\Delta(z_0,\alpha,\beta)}|\log|G(\zeta)||\,u^{-1}\,du\,dv\leqslant CA^{\lambda+1}T(\varrho)\left\{\log\frac{\beta}{\alpha}\right\}^{1/2}\quad(\tfrac{1}{2}\sigma^3\leqslant\alpha\leqslant\beta\leqslant C\alpha\leqslant\tfrac{1}{2}\sigma). \tag{4.40}$$

4.5. *Taking logarithms.* We next introduce an "important" error term; it is an error term because of (4.43) below, but important since it allows us to take a logarithm in (4.44).

Thus, let $P(\zeta)$ be a canonical product

$$P(\zeta)=\frac{\pi(1-\zeta/b_n)}{\pi(1-\zeta/a_n)} \tag{4.41}$$

whose poles include the zeros of G and whose zeros contain the poles of G. We assume for all n that

$$(20A)^{-1}\varrho<|a_n|,\quad |b_n|<20A\varrho \tag{4.42}$$

and that

$$n(0,P)+n(\infty,P)\leqslant C(A')^{-h}\,T(\varrho) \tag{4.43}$$

((4.11) shows this is possible).

LEMMA 4.5. *Let P be as in* (4.41)–(4.43) *and let*

$$L(\zeta)=\frac{1}{T(\varrho)}(\log G(\zeta)P(\zeta)),\quad \zeta\in\Delta=\Delta(z_0,\tfrac{1}{2}\sigma) \tag{4.44}$$

where the branch is chosen such that

$$|\arg G(\zeta^*)|+|\arg P(\zeta^*)|\leqslant 4\pi \tag{4.45}$$

for some point ζ^* of $\Delta(z_0,\frac{1}{3}\sigma)$.

Then L is regular in Δ and

$$\int_{C(z_0,u)}|T(\varrho)(\operatorname{Re}L(\zeta))-\log|G(\zeta)||\,dv\leqslant C(A')^{-h/2}\,T(\varrho)\quad (c\sigma^3<u<\tfrac{1}{2}\sigma). \tag{4.46}$$

In particular,

$$|\operatorname{Re}L(\zeta)|\leqslant CA^{\lambda+1}\quad (\zeta\in\tfrac{2}{3}\Delta=\Delta(z_0,\tfrac{1}{3}\sigma)) \tag{4.47}$$

and

$$|L(\zeta)|\leqslant CA^{\lambda+1}\quad (\zeta\in\tfrac{1}{2}\Delta=\Delta(z_0,\tfrac{1}{4}\sigma)). \tag{4.48}$$

Remark. Although $A\to\infty$, the convention (1.24) allows this to occur arbitrarily slowly; thus (4.48) will be good enough to let us treat the various $L(\zeta)\equiv L(\zeta,z_0)$, as z_0 varies, as forming a normal family.

Proof. According to (4.44),

$$|T(\varrho)\operatorname{Re}L(\zeta)-\log|G(\zeta)||\leqslant|\log|P(\zeta)||\quad (\zeta\in C(z_0,\tfrac{1}{2}\sigma)\cup\Delta(z_0,\tfrac{1}{2}\sigma)).$$

It is not hard to combine (1.24), (4.42) and (4.43) with Lemma 2.1 of [18] to show that

$$\int_{C(z_0,u)}|\log|P(\zeta)||\,dv\leqslant C(A')^{-h}\,T(\varrho)\log\left(\frac{A}{\sigma^3}\right)\leqslant C(A')^{-h/2}\,T(\varrho),\quad (C\sigma^3<s\leqslant 1). \tag{4.49}$$

Thus (4.46) follows from (4.44) and (4.49). Using also (4.40), we see that

$$\int_{C(z_0,u)}|\operatorname{Re}L(\zeta)|\,dv<CA^{\lambda+1}$$

for some u such that $11/12<u<1$. This and the Poisson integral formula

$$L(z)=\frac{1}{2\pi}\int_{C(z_0,u)}\operatorname{Re}L(u\,e^{iv})\left(\frac{u\,e^{iv}+z}{u\,e^{iv}-z}\right)dv+\operatorname{Im}L(z_0) \tag{4.50}$$

together with (4.45) lead to (4.48).

4.6. *Normal families*. The next result is a simple consequence of the theory of normal families. We first need a definition which is modelled on the classical Boutroux–Cartan Lemma ([3], [23, § V.5]).

Definition. A subset E of the plane has *Cartan span* h if

$$h=\inf\sum r_i$$

where the inf is taken over all coverings of E by discs C_i $(1\leq i)$ of radius r_i.

This is related to one-dimensional Hausdorff measure, but we do not demand that the radii of the covering discs be uniformly small.

LEMMA 4.6. *Let Δ be the unit disc and let $k\Delta=\{|z|<k\}$. Let $k<1$, $N<\infty$, $\varepsilon^*>0$ and $\delta_N>0$ be given. Then there exist $\delta_1,\delta_2,\dots,\delta_{N-1}$ such that if g is holomorphic in Δ* with $\|g\|_\infty\leq 1$, *and if for any m $(2\leq m\leq N)$ we have*

$$|g(z)|\leq\delta_{m-1}\quad(z\in k\Delta\cap\gamma) \tag{4.51}$$

where $\gamma\cap k\Delta$ has Cartan span greater than ε^*, then

$$|g(z)|\leq\delta_m\quad(z\in k\Delta). \tag{4.52}$$

Proof. By taking δ_{N-1} sufficiently small, we find that (4.51) implies (4.52) when $m=N$. For otherwise, there would be a sequence δ_n $(=\delta_{N-1,n})$ tending to zero, and g_n such that (4.51) holds for g_n, δ_n on a $\gamma=\gamma_n$. As $n\to\infty$, a subsequence of the g_n tend to a bounded holomorphic function g_0, but (4.51) implies that g_0 is zero on a sequence with an accumulation point, so $g_0\equiv 0$.

Now that δ_{N-1} is known, this argument may be repeated with δ_{N-1} in place of δ_N, and we obtain δ_{N-2} so that (4.51) yields (4.52) with $m=N-1$. By continuing this process, all δ's are produced.

The particular normal family of interest here is $\{L(\zeta)\}=\{L(\zeta,z_0)\}$, where L is as in (4.44) and based on the modification $G(\zeta)=H(z)$, with $z\in B_\sigma(z_0)$. Our next result shows that the polynomial factors P of (4.44) can often play a negligible role in normal-family considerations.

Let Δ_1 and Δ_2 be disks with $\Delta=\Delta(z_0, 5\tau_0)\subset\Delta_1\cap\Delta_2$, where

$$\tau_0 > c\sigma^3. \tag{4.53}$$

Suppose we have modifications L_1 and L_2 in Δ_1 and Δ_2 where $\varphi(\Delta_i)\subset B_\sigma(z_i)$ ($i=1,2$), with φ from Lemma 4.2, and, as in (4.44),

$$L_i(\zeta)\,T(\varrho) = \log(G_i(\zeta)P_i(\zeta)) \quad (\zeta\in\Delta_i, i=1,2). \tag{4.54}$$

Finally, we assume that

$$\{B_{0.8\sigma}(z_1)\cap B_{0.8\sigma}(z_2)\}\cap\mathscr{F}^{\#}=\varnothing. \tag{4.55}$$

where $\mathscr{F}^{\#}$ is determined relative to $B_\sigma(z_1)\cap B_\sigma(z_2)$ in accord with Remark 2 which follows the statement of Lemma 3.2 in § 3.2. Thus there is no subregion of $[B_{0.8\sigma}(z_1)\cap B_{0.8\sigma}(z_2)]-\bigcup\mathscr{D}_i$ whose boundary consists of portions of $\mathscr{G}$ and arcs of $r\neq 2$ of the $\mathscr{D}_i$.

LEMMA 4.7. *The functions $G_i=H_i(\varphi_i(\zeta))$ (cf. (4.9)) may be chosen so that*

$$G_2(\zeta) = G_1(\Phi(\zeta)) \quad (\zeta\in\Delta) \tag{4.56}$$

where

$$\Phi(\zeta) = \varphi_1^{-1}(\varphi_2(\zeta)) = \psi_1(\varphi_2(\zeta)). \tag{4.57}$$

When (4.56) is known, then

$$R(\zeta) \equiv \{\log P_2(\zeta)-\log P_1(\Phi(\zeta))\}\,(T(\varrho))^{-1} \tag{4.58}$$

is holomorphic in Δ and (if $\zeta_0=\psi(z_0)$)

$$|R(\zeta)-\operatorname{Im}(R(\zeta_0))| \leqslant C(A')^{-h/2} \quad (\zeta\in\Delta(\zeta_0,\tau_0)). \tag{4.59}$$

Proof. Note from (4.55) and the remark at the end of § 4.2 that the only possible ambiguity in the construction of the H_i in $B_{0.8\sigma}(z_1)\cap B_{0.8\sigma}(z_2)$ is that H_i may be replaced by its reciprocal, so we may arrange that

$$H_2(z) = H_1(z) \quad (z\in B_{0.8\sigma}(z_1)\cap B_{0.8\sigma}(z_2)). \tag{4.60}$$

Thus if $\varphi=\varphi_i$ is as in (4.9) and Φ as in (4.57), we see from (4.60) that if $\zeta\in\Delta_1\cap\Delta_2$, and in particular if $\zeta\in\Delta$, then

$$G_2(\zeta)=H_2(\varphi_2(\zeta))=H_1(\varphi_2(\zeta))=H_1(\varphi_1\circ\varphi_1^{-1}\circ\varphi_2(\xi))=G_1(\Phi(\zeta))\quad(\zeta\in\Delta)$$

which is (4.56).

According to (4.54) and (4.56), for $\zeta\in\Delta$,

$$\begin{aligned}T(\varrho)\,L_2(\zeta)&=\log\{G_2(\zeta)\,P_2(\zeta)\}\\&=\log\{G_1(\Phi(\zeta))\,P_2(\zeta)\}\\&=T(\varrho)\,L_1(\Phi(\zeta))+\log P_2(\zeta)-\log P_1(\Phi(\zeta))\\&=T(\varrho)\,\{L_1(\Phi(\zeta))+R(\zeta)\},\end{aligned}\tag{4.61}$$

where R is defined by (4.58).

Since (4.60) is known, we recall that equations (4.14) which determine ψ_1 and ψ_2 (inverse to φ_1 and φ_2) agree on $B_{0.8\sigma}(z_1)\cap B_{0.8\sigma}(z_2)$. Thus the chain rule [2, p. 9] shows that $\Phi=\psi_1\circ\varphi_2$ is holomorphic in Δ, and so is R in (4.61).

We obtain (4.59) from the Boutroux-Cartan Lemma ([3], [23, § V.5]) and the bounds (4.10). According to the Boutroux-Cartan Lemma, if P is any product as in (4.41)–(4.43), then there exists a network of circles C_j of radius r_j such that

$$\sum r_j\leqslant C(A')^{-2}A^{-2}\varrho,\tag{4.62}$$

and outside the C_j we have that

$$|\log|P(\zeta)||\leqslant C(A')^{-h}\,T(\varrho)\log(CA^3(A')^2)<c(A')^{-h/2}\,T(\varrho)\quad(\zeta\notin C_j)\tag{4.63}$$

(we are using (1.24) to obtain the last inequality; also we mention that this lemma gives lower bounds for $-\Sigma\log|1-\zeta/b_n|$ and upper bounds for $-\Sigma\log|1-\zeta/a_n|$ in (4.41), but the corresponding upper and lower bounds are very elementary).

We apply (4.62) and (4.63) to P_2 and P_1 in (4.58); note that Φ is holomorphic and (4.10), (4.57), (1.24) and Cauchy's estimates give that

$$|(\Phi'(\zeta)-1)|=\frac{1}{2\pi}\left|\int_{C(\zeta_0,\frac12)}\frac{(\Phi(\zeta_1)-\zeta_1)}{(\zeta_1-\zeta)^2}\,d\zeta_1\right|\leqslant C\sigma^{10}\quad(\zeta\in\Delta(z_0,2\tau_0)),\tag{4.64}$$

so that Φ is uniformly Lipschitz. Thus, the image of the Cartan circles C_2 of (4.62) for

P_1, after composition by Φ, may still be enclosed in a similar network C_2' such that (4.62) remains true.

If C_1 is the Cartan-network for P_1, then (4.63) holds for P_1 and P_2 outside $C_1 \cup C_2$, and thus by (1.24) and (4.53) there is a region Ω, with

$$\Delta(z_0, 2\tau_0) \subset \Omega \subset \Delta(z_0, 3\tau_0)$$

such that (4.63) applies to both P_1 and P_2 on $\partial\Omega$. By the maximum principle for harmonic functions, $|\operatorname{Re} R(\zeta)| \leq C(A')^{-h/2}\, T(\varrho)$ in $\Delta(z_0, 2\tau_0)$, and so (4.59) follows from the Borel-Carathéodory inequality.

4.7 *The set $\mathcal{F}^{\#}$ disappears.* The next two lemmas show that the bound (4.48) may be significantly improved when $\mathcal{F}^{\#} \cap \frac{1}{10} B_\sigma(z_0) \neq \emptyset$, where

$$z_0 \in \mathfrak{A}_\sigma^* \cap \{(3A)^{-1}\varrho < |\zeta| < 3A\varrho\},$$

and $\mathcal{F}^{\#}$ is any region in $\mathfrak{A}_\sigma^* - \{\bigcup \mathcal{D}_i\}$ whose boundary consists of the grid $\mathcal{G}$ and portions of $\partial\mathcal{D}_i$ for $r+2$ choices of i. ($\mathfrak{A}_\sigma^*$ is defined in (4.2).)

LEMMA 4.8. *Suppose $z_0 \in \mathfrak{A}_\sigma^* \cap \{(3A)^{-1}\varrho < |\zeta| < 3A\varrho\}$ with*

$$B_\sigma(z_0) \cap \{\bigcup \mathcal{D}_i\} = \emptyset. \tag{4.65}$$

Then if H, G, P and L are constructed as in §§4.1–4.5, we have that

$$|L(\zeta)| \leq C\gamma^2 \quad (\zeta \in \tfrac{1}{3}\Delta = \Delta(z_0, \tfrac{1}{6}\sigma)). \tag{4.66}$$

Proof. Since $\log|G|$ is harmonic in $\Delta(z_0, \frac{1}{2}\sigma)$, estimate (4.66) is a consequence of (4.44), (4.45), (4.49), (4.50) and the estimate

$$\int_{\Delta'} |\log|G(\zeta)||\, u^{-1}\, du\, dv \leq C\gamma^2 T(\varrho),$$

where $\Delta' = \Delta(z_0, \frac{1}{4}\sigma, \frac{1}{2}\sigma)$. However, (4.20) and (1.24) imply that this estimate for $|\log|G||$ follows from

$$\int_{\frac{1}{4}\sigma}^{\frac{1}{2}\sigma} m_{z_0}(t, f_i)\, t^{-1}\, dt \leq C(A')^{-3h}\, T(\varrho) \quad (1 \leq i \leq q),$$

where the f_i are from (4.16). Assumption (4.65) ensures that $m_{z_0}(t, f_i) \equiv 0$ for $t \leq \frac{1}{2}\sigma$, so (4.66) follows.

The next results presents another situation in which we may obtain a conclusion much like (4.66).

LEMMA 4.9. *Let* $z_0 \in \mathfrak{A}_\sigma^* \cap \{(3A)^{-1}\varrho < |\zeta| < 3A\varrho\}$ *such that*

$$\{\partial\mathcal{F}^{\#} \cap (\bigcup \partial\mathcal{D}_i)\} \cap \tfrac{1}{10}B_\sigma(z_0) \neq \emptyset, \tag{4.67}$$

where $\mathcal{F}^{\#}$ *is as described at the beginning of this section. Then there is a set* γ^* *in* $\Delta(z_0, 0.8\sigma)$ *with Cartan span at least* $(2/3)\sigma|z_0|$ *such that the function* L *of* (4.44) *satisfies*

$$|L(\zeta)| \leq C(A')^{-h/2} \quad (\zeta \in \gamma^*) \tag{4.68}$$

in addition to (4.48).

Proof. According to (4.67), there is a $z_1 \in \frac{1}{10}B_\sigma(z_0)$ with $z_1 \in \partial\mathcal{D}_i$. Let $B = (99/100)\,B_\sigma(z_0)$. We refer to the four sides of B as the horizontal sides (on which $|z|$ is constant but $\arg z$ varies) and vertical sides (on which $\arg z$ is constant and $|z|$ varies). It follows that there is an arc γ of $\partial\mathcal{D}_i \cap \partial\mathcal{F}^{\#}$ which joins z_1 to a point z_2 of $\partial\mathcal{D}_i \cap \partial B$, and is contained in B otherwise. With no loss of generality, we assume that z_2 is on one of the vertical sides, so that $\arg z$ varies by at least $\frac{7}{8}\sigma|z_0|$ on γ.

On γ, $|f(z)-a_i| = \varepsilon$, and since $\gamma \subset \partial\mathcal{F}^{\#}$, (3.20) and (3.43) ensure that $\arg(f-a_i)$ is nearly constant on γ: there exists a complex number c with $|\mathrm{Re}\, c| = |\log\varepsilon|$ such that

$$|\log(f(z)-a_i) - c| \leq C(A')^{-2h}\, T(\varrho) \quad (z \in \gamma).$$

Thus (4.4) implies that if H is as in Lemma 4.1, then

$$|\log H(z) - c'| \leq C(A')^{-2h}\, T(\varrho) \quad (z \in \gamma) \tag{4.69}$$

where, by (4.45), $|c'| \leq |\log\varepsilon| + 2\pi$.

Estimate (4.69) is almost what we need, but has the disadvantage that the branch of $\log H$ is taken on γ, and since there is no *a priori* regularity of γ, there is no obvious way to directly control $\arg P$ on all of γ, where P is from (4.41).

We deduce a bound for $|\log P(z)|$, similar to that of (4.69), on a substantial subset γ_1 of γ. We have assumed that γ "ends" on one of the vertical sides of ∂B. Let γ^* be one of the *horizontal* sides on ∂B, say the one on which $|z|$ is smaller, and for each $\zeta \in \gamma^*$, let $z(\zeta)$ be the point of γ which may be "seen" from ζ: $\arg z(\zeta) = \arg\zeta$ and the segment

$$\Gamma(\zeta) = \{te^{i\alpha};\, \alpha = \arg\zeta,\, |\zeta| \leq t < |z(\zeta)|\}$$

is disjoint from γ. This collection of these points $z(\zeta)$ forms a subcomponent γ_1 of γ, and also has Cartan span at least $\frac{7}{8}\sigma|z_0|$, since $z(\zeta)$ exists for $\zeta \in$ an arc of ∂B which lies below z_1 and z_2. Since $\gamma_1 \subset \gamma$, (4.69) persists on γ.

Finally, let $\{C_j\}$ be the Cartan circles of (4.62) and (4.63) which correspond to the factor P in (4.44), and let $\gamma_2 = \gamma_1 - \{C_j\}$, so that (by 4.62)) γ_2 has Cartan span at least $\frac{4}{5}\sigma|z_0|$. Of course, (4.63) implies that $|\log|P||$ is uniformly small on γ_2. We claim, further, that if $z = z(\zeta) \in \gamma_2$, then

$$|\arg z - \arg z_2| \leq C(A')^{-h} T(\varrho) \quad (z \in \gamma_2). \tag{4.70}$$

This is proved by first constructing a path $\Gamma_{z(\zeta)}$ from z_2 to $z = z(\zeta)$, which consists of a portion of the vertical side of ∂B which contains z_2, a portion of γ^* and $\Gamma(\zeta)$. There is another path which connects this pair of points: the portion of γ (which we call $\gamma(\zeta)$) between z_2 and $z(\zeta)$. Then $\Gamma \equiv \gamma(\zeta) \cup \Gamma_{z(\zeta)}$ is a closed curve, so the argument principle and (4.43) yield that $|\Delta_\Gamma \arg P(z)| \leq C(A')^{-h} T(\varrho)$. Since $\Gamma_{z(\zeta)}$ is composed of three noneuclidean line segments, we also have from (4.43) that $|\Delta_{\Gamma_{z(\zeta)}} \arg P(z)| \leq C(A')^{-h} T(\varrho)$, and these two estimates yield (4.70).

Choose $\zeta \in \gamma_2$. Then from definitions (4.44), (4.45) of L and (4.63), (4.69) and (4.70) (twice) we have

$$\begin{aligned} |L(\zeta)| &\leq T(\varrho)^{-1} \{|\log G(\zeta)| + |\log|P(\zeta)|| + |\arg P(\zeta)|\} \\ &\leq C((A')^{-2h} + (A')^{-h/2} + (A')^{-h}) T(\varrho) \quad (\zeta \in \gamma_2) \end{aligned}$$

which is (4.68).

We now use Lemmas 4.6–4.9 to show that if $\mathcal{F}^{\#} \cap \mathfrak{A}^*_\sigma \neq \emptyset$, then the functions L must be uniformly small on large subsets of $\mathfrak{A}$.

LEMMA 4.10. *Suppose* $\zeta_1 = \psi(z_1)$ *with* $z_1 \in \mathfrak{A}^* \cap \{\partial\mathcal{F}^{\#} \cup (\bigcup \partial \mathcal{D}_i)\}$ *as in* (4.67), *with*

$$B_\sigma(z_1) \subset \mathfrak{A}^* \cap \{(3A)^{-1}\varrho < |z| < 3A\varrho\}.$$

Let $\Delta_1, \ldots, \Delta_N$ *be a chain of discs* $\Delta_i = \Delta(\zeta_i, (20)^{-1}\sigma)$, $\zeta_i = \psi(z_i)$, $\zeta_i \in \mathfrak{A}^*_\sigma$, $N \leq C\sigma^{-2}$,

$$\zeta_i \in \tfrac{1}{2}\Delta_{i+1} \quad (i \geq 1). \tag{4.71}$$

Then if n is sufficiently large in (1.9), *we may arrange that the functions* $L_i = L(\zeta, \zeta_i)$ *of* (4.44) *satisfy.*

$$|L(\zeta)| \leq C\sigma^2 \quad (\zeta \in 0.9\Delta_i, 1 \leq i \leq N). \tag{4.72}$$

Proof. Recall the terminology of Lemma 4.6 and choose

$$\delta_N=\sigma^2 A^{-(\lambda+1)}, \quad k=0.9, \quad \varepsilon^*=10^{-3},$$

and let $\delta_1, \dots, \delta_{N-1}$ be the constants obtained from Lemma 4.6. Now consider modifications of f in rectangles $B_i \equiv B_\sigma(z_i)$.

Let δ_1 be determined from Lemma 4.6, and choose n so large that

$$CA^{\lambda+1}(A')^{-h/2}+CA^{\lambda+1}\gamma^2 \leqslant \delta_1; \tag{4.73}$$

(1.24) shows that this is possible.

By hypothesis, $\partial\mathcal{F}^{\#}$ meets the "center" of B_1, so (4.67) and (4.68) hold relative to the rectangle B_1. Thus $|L_1(z)| \leqslant C(A')^{-h}$ on a subset of Δ_1 which has Cartan span at least $(2/3)^{-1}(10)^{-1}\sigma|z_1|=15^{-1}\sigma|z_1|$; hence Lemma 4.6 and (4.48) yield for $i=2$ that

$$|L_{i-1}(z)| \leqslant (CA^{\lambda+1})\,\delta_i \quad (\zeta \in \tfrac{2}{3}\Delta_{i-1}). \tag{4.74}$$

We claim that if (4.74) holds for some $i<N$, it holds for $i+1$. There are two cases. Suppose first that

$$[0.8B_{i-1} \cap 0.8B_i] \cap \{\mathcal{F}^{\#}\} \neq \varnothing, \tag{4.75}$$

and consider the various rectangles B_s contained in $[0.8B_{i-1} \cap 0.8B_i]$ with, say $s=2^{-6}\sigma$. By (4.75), we see that some such B_s is contained in some $\mathcal{D}_i$ (and so satisfies (4.65)) or else meets some $\partial\mathcal{D}_i \cap \partial\mathcal{F}^{\#}$, so that (4.67) holds. It then follows that (4.74) holds with $i+1$ in place of i: if (4.65) holds, we see this at once by (4.66), and if (4.67) holds, we use (4.68) and Lemma 4.6, in fact (4.74) then holds with δ_1 in place of δ_1 (since obviously $\delta_i > \delta_{i-1} > \dots > \delta_1$ in Lemma 4.6).

Next, suppose that (4.75) is not satisfied. Then hypothesis (4.55) holds. We now imagine L_i as an "almost" analytic continuation of L_{i-1}, in the sense of Lemma 4.7. Choose a disc $\Delta=\Delta(\zeta^*, 5\tau)$ such that

$$\Delta \subset (0.8\Delta_i \cap 0.8\Delta_{i-1} \quad \text{and} \quad \Phi_i(\Delta) \subset (0.8\Delta_i \cap 0.8\Delta_{i-1}$$

with $\Phi_i=\varphi_{i-1}^{-1} \circ \varphi_i$ (Compare with (4.57)). Because of (4.10) and (4.71), we may take $\tau=(10)^{-2}\sigma$. According to Lemma 4.7, we may arrange L_i and L_{i-1} so that (cf. (4.61))

$$\begin{aligned} T(\varrho)\,L_i(\zeta) &= \log\{G_i(\zeta)\,P_i(\zeta)\} = \log\{G_{i-1}(\Phi_i(\zeta))\,P_i(\zeta)\} \\ &= T(\varrho)\,\{L_{i-1}(\Phi_i(\zeta))+R_i(\zeta)\} \quad (\zeta \in \Delta) \end{aligned} \tag{4.76}$$

where (4.59) holds. Choose Ω such that $\partial\Omega$ is disjoint from the Cartan network for P_i and $P_{i-1}(\Phi_i)$ and $\Delta(\zeta^*, 3\tau)\supset\Omega\supset\Delta(\zeta^*, 2\tau)$. Then (1.24), (4.45) and (4.63) show that $|R_i(\zeta_i)|\leqslant C(A')^{-h/2}T(\varrho)$. Thus (4.76), (4.74) and (4.59) yield that

$$|L_i(\zeta)|\leqslant (CA^{\lambda+1})\,\delta_i+C(A')^{-h/2}\leqslant C(A^{\lambda+1})\,\delta_i \quad (\zeta\in\Delta) \tag{4.77}$$

by (4.73). Relative to $0.9\Delta_i$, Δ has Cartan span at least $10^{-3}\sigma|\zeta_i|$. Thus (4.77) and Lemma 4.6 yield (4.7) with i in place of $i-1$, even when (4.75) fails. By repeating this process, we get $|L(\zeta)|\leqslant CA^{\lambda+1}\delta_N$ for $\zeta\in\bigcup 0.9\Delta_i$, and by the original choice of δ_N, this implies (4.72).

It is now possible to prove

LEMMA 4.11. *Let $\mathfrak{A}^*_\sigma$ be as in* (4.2). *Then $\mathcal{F}^{\#}\cap\mathfrak{A}^*_\sigma=\emptyset$.*

Proof. Suppose the lemma false, and choose $z_1\in\mathcal{F}^{\#}\cap\mathfrak{A}^*_\sigma$, with $z_1\in\mathcal{F}^{\#}$. Let $\zeta_1=\psi(z_1)$. Since the set B^* of (3.16) is a union of $Q(\lambda)$ small rectangles, we take $\zeta_1=\psi(z_1)$ and construct a chain of discs $\Delta(\zeta_j, \frac{1}{20}\sigma)$ $(1\leqslant j\leqslant N\leqslant C\sigma^{-2})$ with $\zeta_j\in\mathfrak{A}^*_\sigma$, so that (4.71) holds. Further, we may arrange that each point ζ with $(1-\frac{1}{4}\sigma)\,s_0<|\zeta|<(1+\frac{1}{4}\sigma)\,s_0$ is covered by the union of the annuli $\Delta_j'=\Delta(\zeta_j, \frac{1}{4}\sigma, \frac{1}{2}\sigma)$; here s_0 is some number with $\varrho<s_0<2\varrho$. Lemma 4.9 implies that (4.72) holds in each of the Δ_j.

We recall the family $m_\zeta(t,f_i)$ of (4.15) and first prove that for $\zeta=\zeta_j$

$$\sum_{i=1}^{q}\int_{\frac{1}{2}\sigma}^{\sigma} m_\zeta(t,f_i)\,t^{-1}\,dt\equiv\sum_i\int_{\Delta_j'}\log|f_i(z)|\,t^{-1}\,dt\,d\varphi<C\sigma^{3/2}T(\varrho). \tag{4.78}$$

Indeed, we find from (4.20), (4.46), (4.72) and the convention (1.24) that $(\zeta=\zeta_j, \Delta'=\Delta_j')$

$$\begin{aligned}\sum_i\int_{\frac{1}{2}\sigma}^{\sigma} m_\zeta(t,f_i)\,t^{-1}\,dt &\leqslant \left|\int_{\Delta'}|\log|G(\zeta)||\,u^{-1}\,du\,dv-\sum\int_{\frac{1}{2}\sigma}^{\sigma} m_\zeta(t,f_i)\,t^{-1}\,dt\right|\\ &\quad+\int_{\Delta'}|\log|G(\zeta)||\,u^{-1}\,du\,dv\\ &\leqslant C\sigma^2T(\varrho)+\int_{\Delta'}|\log|G(\zeta)||\,u^{-1}\,du\,dv\\ &\leqslant C\sigma^2T(\varrho)+T(\varrho)\int_{\Delta'}|\mathrm{Re}\,L(\zeta)|\,u^{-1}\,du\,dv\\ &\quad+\int_{\Delta'}|T(\varrho)\,(\mathrm{Re}\,L(\zeta))-\log|G(\zeta)||\,u^{-1}\,du\,dv\end{aligned}$$

$$\leqslant C\sigma^2 T(\varrho)+C\sigma^2 T(\varrho)+C(A')^{-h/2}\,T(\varrho)$$
$$\leqslant C\sigma^2 T(\varrho).$$

which is (4.78). We add (4.78) over the $O(\sigma^{-2})$ annuli Δ_j', and use (4.36) where now $\frac{1}{8}\sigma<t<\frac{1}{4}\sigma$, $CA^{-1}\varrho<r<A\varrho$. Thus, using (1.24)

$$\sum_j \int_{\Delta_j'} \log|f_i(r\,e^{i\theta})|\,d\theta r^{-1}dr \leqslant CA^4\sigma^2 \sum_j \int_{\frac{1}{4}\sigma}^{\frac{1}{2}\sigma} m_{\zeta_j}(t,f_i)\,t^{-1}\,dt \leqslant C\sigma^4 T(\varrho)\,\sigma^{-2},$$

and so

$$\sum_i \int_{s_0(1-\frac{1}{8}\sigma)}^{s_0(1+\frac{1}{8}\sigma)} m(r,a_i,\mathscr{D}_i)\,r^{-1}\,dr \leqslant C\sigma^{3/2} T(\varrho). \tag{4.79}$$

However, this contradicts (1.3) and (2.31), for since $s_0>\varrho$ and the annulus $s_0(1-\frac{1}{8}\sigma)<|z|<s_0(1+\frac{1}{8}\sigma)$ has logarithmic length comparable to σ, we find from (1.24), (2.31) and (4.79) that

$$\begin{aligned}
C\sigma T(\varrho) \leqslant C\sigma T(s_0) &\leqslant \sum \int_{s_0(1-\frac{1}{8}\sigma)}^{s_0(1+\frac{1}{8}\sigma)} m(r,a_i)\,r^{-1}\,dr \\
&\leqslant \sum \int m(r,a_i,\mathscr{D}_i)\,r^{-1}\,dr \\
&\quad +\sum \int [m(r,a_i)-m(r,a_i,\mathscr{D}_i)]\,r^{-1}\,dr \\
&\leqslant \sum \int m(r,a_i,\mathscr{D}_i)\,r^{-1}\,dr + C\sigma(A')^{-5h}\,T(\varrho) \\
&\leqslant C\sigma^2 T(\varrho).
\end{aligned}$$

Since $\sigma\to 0$, we have a contradiction, and the lemma is proved.

Remark. Lemma 4.10 clarifies one question raised at the end of Section 4.2: how unique is each G (or H)? There are two ambiguities in constructing H. In (4.6), we made the initial choice that H be large in $D_0\cap B$; we might as well have taken $H(z)=f(z)-a_i$ in $D_0\cap B$. Also, in extending H from a component $D\cap B$ to an F, the choice of j (cf. (4.7)) will not be unique unless $F\in\mathscr{F}(i,j)$. However, we now know that all F's which meet $\mathfrak{A}_\sigma^*$ are in $\cup\,\mathscr{F}(i,j)$. Thus the only ambiguity in the definition of G is whether we use H or H^{-1} and the particular choice of maps φ. This observation is the basis of (6.24).

4.8. *Proof of Lemmas* 4.12 *and* 4.13. In this final section, we give a situation in which the bound (4.48) may be significantly improved. Inequality (4.89) will be needed in § 7.2 to show that the removal of B^* from $\mathfrak{A}$ in (3.16) and (3.17) does not seriously affect our estimates.

LEMMA 4.12. *Let $\tau_0<1$ be chosen, and suppose that $B=B_\sigma(z_1)$ is a rectangle with*

$$z_1 \in \mathfrak{A} \cap (\partial \mathcal{D}_i). \tag{4.80}$$

Then, in the terminology of (4.15) *and* (4.16), *we have*

$$m_{z_1}(t, f_i) \leq C\sigma^{\tau_0} T(\varrho) \quad (1 \leq i \leq q;\ \tfrac{1}{3}\sigma \leq t \leq \tfrac{1}{2}\sigma). \tag{4.81}$$

Remark. In (4.80) it is not essential that $z_1 \in \mathfrak{A}_\sigma^*$ of (4.2); in fact, the most convenient application of this lemma will be when $z_1 \in B^*$ (cf. (3.16)).

Proof. We start with the first inequality of (4.19), and Ω as in the proof of Lemma 4.3, but use the estimate in (2.35) for harmonic measure, where now $\theta(t)$ is the angular measure of $D \cap C(z_1, t)$. Let j satisfy $1 \leq j \leq q$. Then assumption (4.80), the obvious bound $\theta_j(t) \leq 2\pi$ and (1.24) give for $1 \leq j \leq q$ that

$$\begin{aligned} m_{z_1}(t, f_j) &\leq \log \frac{1}{\varepsilon} + C(A')^{-5h} T(\varrho) \log\left(\frac{CA\varrho}{\sigma^3 |z_1|}\right) + CA^{\lambda+1} T(\varrho) \exp\{-\pi \int_t^A (t\theta_j(t))^{-1}\, dt\} \\ &\leq CA^{\lambda+1} (t/A)^{1/2}\, T(\varrho) \quad (\sigma^3 \leq t < 1), \end{aligned} \tag{4.82}$$

and we use this standard estimate for $t=\sigma^3$ with (1.24) to obtain that

$$m_{z_1}(\sigma^3, f_j) \leq C\sigma^{3/2} A^{\lambda+1/2} T(\varrho) < C\sigma T(\varrho) \quad (1 \leq j \leq q). \tag{4.83}$$

The trivial bound $\theta_j(t) \leq 2\pi$ in (4.82) is not adequate, however, when t is near σ. However, according to (3.8)–(3.10) and (3.14), we may find a region Ω with $\partial\Omega \subset \mathcal{G}$ and $\Delta(z_1, \sigma) \subset \Omega \subset \Delta(z_1, 2\sigma)$. For each j, $1 \leq j \leq q$, define p_j so that

$$S_{\mathcal{D}_j}(\Omega) = \sigma^{p_j} T(\varrho) \tag{4.84}$$

(and take $p_j=\infty$ when $\partial \mathcal{D}_j \cap \Omega = \emptyset$). Order the indices j so that $p_1 \leq p_2 \leq \dots$. For this proof only, let $2\eta = 1-\tau_0$ and suppose that

$$p_1 \geq 1-\eta = \tau_0 + \eta \tag{4.85}$$

(thus (4.85) also holds for all $j \geq 1$). Then (1.24), (2.13) and (2.40) (applied to discs centered at z_1) yield from (4.84) and (4.85) that

$$
\begin{aligned}
m(t,f_j)-m(\sigma^3,f_j) &= \int_{\sigma^3}^{t} u\frac{d}{du}m(u,f_j)\,u^{-1}\,du \leq C\log\varepsilon^{-1} \\
&\quad + \int_{\sigma^3}^{t} S_{\mathscr{D}_j}(t,\Omega)\,t^{-1}\,dt + C\delta T(\varrho)\log\frac{t}{\sigma^3} \\
&\leq C\sigma^{\tau_0+\eta}\log\frac{1}{\sigma}T(\varrho) < \sigma^{\tau_0}\log T(\varrho) \quad (1\leq j\leq q,\ \sigma^3\leq t\leq 1).
\end{aligned}
\tag{4.86}
$$

and thus (4.81) would follow from this and (4.83). Thus we need only show (4.84).

Note that the situation $p_1 \leq 1-\eta$ and $p_2 \geq 1-\frac{1}{2}\eta$ is impossible from (3.44), since there are only $Q(\lambda)$ classes $\mathscr{F}(j,k)$ in Ω. Indeed, (3.44) and (1.24) ensure that $p_2=p_1+o(1)$. We will prove that

$$
p_1+p_2 \geq 2-o(1). \tag{4.87}
$$

Let us grant (4.87) for the moment. Then (4.87) gives (4.85) and so (4.81). This would prove Lemma 4.11.

We now prove (4.87). Once again, we use (1.24), (2.13) and (2.40) as in (4.86). Thus, since $\Omega \subset \Delta(z_1, 2\sigma)$,

$$
\begin{aligned}
S_{\mathscr{D}_j}(\Omega) \leq S_{\mathscr{D}_j}(\Delta(z_1,2\sigma)) &\leq C\int_{2\sigma}^{4\sigma} S_{\mathscr{D}_j}(t)\,t^{-1}\,dt \\
&\leq C(A')^{-5h}T(\varrho) + C[m_{z_1}(4\sigma,f_j)-m_{z_1}(2\sigma,f_j)] + O(\log\frac{1}{\varepsilon}) \\
&\leq C(A')^{-5h}T(\varrho) + Cm_{z_1}(4\sigma,f_j),
\end{aligned}
$$

and an upper bound for $m_{z_1}(4\sigma,f_j)$ may be obtained from (1.24) and the first inequality of (4.82). Thus, for $1 \leq j \leq q$.

$$
S_{\mathscr{D}_j}(\Omega) \leq C(A')^{-4h}T(\varrho) + CA^{\lambda+1}T(\varrho)\exp\{-\pi\int_{2\sigma}^{A}(t\theta_j(t))^{-1}\,dt\},
$$

and so, in the notation of (4.84), we have from (1.24) that

$$
-p_j\log\sigma \sim -p_j\log\left(\frac{2\sigma}{A}\right) \geq \{1+o(1)\}\,\pi\int_{2\sigma}^{A}(t\theta_j(t))^{-1}\,dt.
$$

Hence (4.87) follows from the inequality [15, p. 108]

$$\pi\left(\frac{1}{\theta_1(t)}+\frac{1}{\theta_2(t)}\right)\geq 4\pi(\theta_1(t)+\theta_2(t))^{-1}\geq 2,$$

since we may use this to give lower bounds for the p_j of (4.84).

COROLLARY. *Let $z_0\in\mathfrak{A}^*_\sigma$ and suppose that*

$$d(z_0,z_1)\leq C_1\sigma \tag{4.88}$$

(d=noneuclidean distance) where z_1 satisfies (4.80). *Then*

$$\int_{\frac{1}{3}\sigma}^{\frac{1}{2}\sigma} m_{z_0}(t,f_i)\,t^{-1}\,dt\leq C\sigma^{\tau_0}T(\varrho)\quad (1\leq i\leq q), \tag{4.89}$$

where C may depend on C_1.

Proof. Let $z-z_0=t|z_0|e^{i\varphi}, z-z_1=t'|z_1|e^{i\varphi'}$. Then, as in (4.36), we have that

$$t^{-1}\,dt\,d\varphi=t^{-2}t\,dt\,d\varphi=t^{-2}|z_0|^{-2}|dz|^2=\left(\frac{t'}{t}\right)^2\left|\frac{z_1}{z_0}\right|^2(t')^{-1}\,dt'\,d\varphi'$$

and in $\Delta(z_0,\frac{1}{3}\sigma,\frac{1}{2}\sigma)$, (4.88) ensures that $C\leq(t'|z_1|)/(t|z_0|)\leq C$ where C depends only on C_1. Thus (4.89) follows from (4.81).

LEMMA 4.13. *Let $\tau_1<1$ be given, and let $B=B_\sigma(z_0)$ be a logarithmic rectangle with $z_0\in\mathfrak{A}^*_\sigma$, and such that for some fixed $C_1>0$*

$$B_{C_1\sigma}(z_0)\cap(\cup\,\partial\mathfrak{D}_i)\neq\emptyset. \tag{4.90}$$

Then the function $L(\zeta)$ of (4.44) *satisfies*

$$|L(\zeta)|\leq C\sigma^{\tau_1}\quad (\zeta\in\Delta(z_0,\tfrac{1}{4}\sigma)). \tag{4.91}$$

Proof. Hypothesis (4.90) allows (4.89) to be applied, where we take $\tau_1<\tau_0<1$. Choose $s_1=\frac{1}{3}\sigma$ and $s_2=\frac{1}{2}\sigma$ in (4.20). Then (4.20), (4.89) and (1.24) yield that

$$\begin{aligned}\int_{\Delta(z_0,\frac{1}{3}\sigma,\frac{1}{2}\sigma)}|\log|G(\zeta)||\,u^{-1}\,du\,dv&\leq\sum\int_{\frac{1}{3}\sigma}^{\frac{1}{2}\sigma}m_{z_0}(t,f_i)\,t^{-1}\,dt+CA^{\lambda+1}T(\varrho)\{(A')^{-3h}+\sigma^2+C(\eta)\}\\&\leq C\sigma^{\tau_1}T(\varrho).\end{aligned}$$

Since the correction term P in (4.44) always satisfies (4.49), (4.91) follows from these estimates for the harmonic function $\operatorname{Re} L(\zeta)$ and (4.45).

5. Division of the annulus $\mathfrak{A}$; The main quasi-conformal modification

5.1. *Subdividing the annulus.* In this section we use the set B^* of (3.16) to divide the annulus $\mathfrak{A}$ of (1.10) into $Q(\lambda)$ overlapping concentric annuli $A_m^\circ = A_{m,n}^\circ$ as suggested in § 1.2.

Let

$$\mathcal{P} \tag{5.1}$$

be those numbers r such that $r=|\zeta|$ where ζ is the center of any of the $Q(\lambda)$ rectangles in B^*. Arrange the numbers x of $\mathcal{P} \cup A^{-1}\varrho \cup A\varrho$ in increasing order so that $A^{-1}\varrho = x_1 < x_2 < \ldots < x_k = A\varrho$, where $k=Q(\lambda)$. Set $i_1=1$, and for $m \geq 2$, define i_m to be the first index such that $i_m > i_{m-1}$ and

$$x_i > x_{i-1} \exp(15\sigma). \tag{5.2}$$

Then set $M = \sup m$. If $i \leq i_m$, $1 \leq m \leq M$, we take s_m to be so that $\{|z| = s_m\}$ is in the grid $\mathcal{G}$ and

$$x_i \exp(-5\sigma) < s_m < x_i \exp(-4\sigma). \tag{5.3}$$

Also, we define a sequence s_m' by taking $i = i_{m+1} - 1$ for $1 \leq m < M$ and $i=k$ for $m=M$, and then determining s_m' with $\{|z| = s_m'\}$ in the grid such that

$$x_i \exp(4\sigma) < s_m' < x_i \exp(5\sigma). \tag{5.4}$$

LEMMA 5.1. *For $1 \leq m \leq M$, consider the overlapping annuli*

$$A_m^\circ = \{s_m \leq |z| \leq s_{m+1}'\} = A_m^* \cup A_m^+ \cup A_m^-, \tag{5.5}$$

so that $A_m^\circ \cap A_{m+1}^\circ = \{s_{m+1} < |z| < s_{m+1}'\} = A_m^+ = A_{m+1}^-$. Then

$$\bigcup A_m^\circ \supset \mathfrak{A}. \tag{5.6}$$

Further, if $\mathcal{P}$ is as in (5.1) and if for $\sigma > 0$ we let

$$\mathcal{P}_\sigma = \{s; \left|\log \frac{s}{s_0}\right| < 4\sigma \text{ for some } s_0 \in \mathcal{P}\}, \tag{5.7}$$

then

$$\mathcal{P}_\sigma \subset \left[\left(\bigcup_m A_m^+\right) \cap \left(\bigcup_m A_m^-\right)\right] \tag{5.8}$$

and, as $n \to \infty$,

$$\sup_m \log \frac{s'_m}{s_m} < CQ\sigma \tag{5.9}$$

where $Q = Q(\lambda)$ *is an upper bound for the number of rectangles in* B^*.

Finally, we may arrange that

$$\log \frac{s'_m}{s_m} > C\sigma, \quad \log \frac{s_m}{s'_{m-1}} > C\sigma \quad (1 \leq m \leq M(n)) \tag{5.10}$$

for all large n.

The subannuli A^*, A^+, A^- are called respectively the *kernel, upper* and *lower* portions of $A^\circ = A^\circ_m$. The lemma asserts, in addition to (5.6), that none of these subannuli are too thin. Further, all circles $\{|z| = s\}$, with $s \in \mathcal{P}$ are well inside the upper or lower portions of the A°.

Proof. Clearly $M \leq k \leq Q(\lambda)$ and (5.6) and (5.7) hold. It follows from (5.2)–(5.4) that for $1 \leq m \leq M$ we have

$$s_{m+1} > s'_m \exp(4\sigma), \quad s'_m < s_m \exp[15(k+2)\sigma]$$

and

$$s'_m > s_m \exp(8\sigma),$$

which yields (5.9) and (5.10). This proves the lemma.

The function f will be considered in regions A_m modified from the A°. Let $B = B_\sigma(\zeta_0) \in B^*$ be such that $B \cap A^\circ \neq \emptyset$, and suppose for example that $\zeta_0 \in A^+$ (cf. (5.5)). We now define

$$\sigma^\# = \sigma^\#_m = \tfrac{1}{2} \log \frac{s'_{m+1}}{s_{m+1}};$$

then (5.9) and (5.10) imply that

$$C\sigma \leq \sigma^\# < C\sigma. \tag{5.11}$$

A rectangle $B_{\sigma^\#}$ will be deleted from A°, where $B_{\sigma^\#}$ is chosen so that $A^+ \supset B_{\sigma^\#} \supset B_\sigma(\zeta_0)$, and so that two sides of $B_{\sigma^\#}$ lie in ∂A^+. More exactly, $B_{\sigma^\#}$ is a logarithmic rectangle composed of arcs from $|z| = s'_{m+1}, |z| = s_{m+1}$, and rays $\theta^+(\zeta_0), \theta^-(\zeta_0)$ in $\{s_{m+1} < |z| < s'_{m+1'}\}$ which are also in $\mathcal{G}$, with

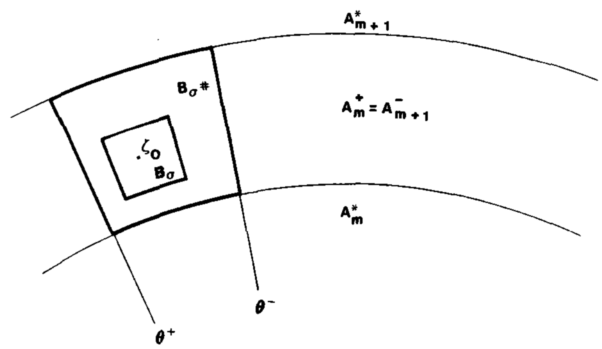

Fig. 1

$$|\theta^+ - [\arg \zeta_0 + \sigma^\#]| < 2\tau,$$
$$|\theta^- - [\arg \zeta_0 - \sigma^\#]| < 2\tau$$

where τ satisfies (3.10) (see Fig. 1). If $\zeta_0 \in A^-$, the only change is that $\partial B_\sigma[\zeta_0]$ includes arcs of $|z| = s_m$ and $|z| = s'_m$ as well as the corresponding segments of $\arg z = \theta^+(\zeta_0)$ and $\arg z = \theta^-(\zeta_0)$. Let

$$B^\# = \cup B_{\sigma^\#}[\zeta_0], \tag{5.12}$$

a union of $Q(\lambda)$ rectangles. Each rectangle B_σ of B^* (cf. (3.16)) is well inside $B^\#$. In particular, (5.7) and (5.8) ensure that $\mathfrak{A} - B^\#$ is in $\mathfrak{A}^*_\sigma$, where $\mathfrak{A}^*_\sigma$ has been introduced in (4.2). Thus any rectangle $B_\sigma(z_0)$ with $z_0 \in \mathfrak{A} - B^\#$ is admissible and no set $\mathscr{F}^\#$ may meet $\mathfrak{A} - B^\#$. However, we note from (5.11) that

$$d(\partial B^\#, \partial B^*) < C\sigma, \tag{5.13}$$

where d is non-euclidean distance.

We also record that if $A = A_m$ is defined by

$$A = A^\circ - B^\# \tag{5.14}$$

then A is doubly-connected,

$$\begin{gathered} \partial A \subset \mathscr{G}, \\ [A^\circ - A] \subset [A^+ \cup A^-], \end{gathered} \tag{5.15}$$

and if $\mathfrak{A}^*_\sigma$ is as in (4.2), then

$$A \subset \mathfrak{A}^*_\sigma. \tag{5.16}$$

5.2. *The main quasi-conformal modifications.* We can now produce the functions H_m which were mentioned in § 1.2. The idea is not very different from that of § 4.2, but since each A_m is doubly-connected rather than simply-connected, it will be necessary to make modifications of $f(z^2)$. For this reason, we introduce sets $\tilde{\mathscr{A}}_{m,n}$ which are related to the $\mathscr{A}_{m,n}$ of Lemma 5.1 by

$$A_{m,n} = (\tilde{\mathscr{A}}_{m,n})^2. \tag{5.17}$$

Each point of $A_{m,n}$ has two antipodal antecedents in $\tilde{\mathscr{A}}_{m,n}$. Let $\mathscr{D}[m]$ and $\mathscr{F}[m]$ be the components of $(\cup \mathscr{D}_i) \cap A_{m,n}$ and $A_{m,n} - (\cup \mathscr{D}_i)$. To limit notation, we will identify the $\mathscr{D}[m]$ and $\mathscr{F}[m]$ with their images in $\tilde{\mathscr{A}}_{m,n}$, and freely write $\mathscr{D}_i$, $\mathscr{F}(i,j)$ in $\tilde{\mathscr{A}}_{m,n}$. However, $T(r,f)$ will always refer to the characteristic of f relative to $\mathfrak{A}$.

LEMMA 5.2. *Let $\eta>0$ be given. Then if n is sufficiently large in* (1.8), *it is possible to define, for each (m,n), a quasi-conformal mapping ω on the Riemannian image of $\tilde{\mathscr{A}}_{m,n}$ of* (5.17) *such that*

$$H(z) = \omega(f(z^2)) \quad (z \in \tilde{\mathscr{A}}_{m,n}) \tag{5.18}$$

has

$$\|\mu_H(z)\|_\infty < \eta \quad (z \in \tilde{\mathscr{A}}_{m,n}) \tag{5.19}$$

and $(\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_{m,n})$

$$n(0, H, \tilde{\mathscr{A}}) + n(\infty, H, \tilde{\mathscr{A}}) \leq C(A')^{-2h}\, T(\varrho). \tag{5.20}$$

Proof. Most of the work has been done already. The components of $\mathscr{F}[m]$ would normally be classified as $\mathscr{F}(\emptyset)[m], \ldots, \mathscr{F}(i_1, \ldots, i_p)[m]$, but it is now easy to see, using Lemma 4.11 and (5.16), that they are all $\mathscr{F}(i,j)[m]$, with $i \neq j$. For let γ be a component of $\tilde{\mathscr{A}} \cap \{\cup \partial \mathscr{D}_i\}$ with, say, $\gamma \subset \partial \mathscr{D}_i$. Then each $B_\sigma(\zeta), \sigma \in \partial \mathscr{D}_i$, is admissible since (5.16) holds; recall the definition of admissability from § 4.1. Thus, if F is the component of $\tilde{\mathscr{A}} - \{\cup \partial \mathscr{D}_i\}$, whose boundary contains γ, then (5.16) and Lemma 4.11 imply that $F \cap B_\sigma(\zeta) \cap \partial \mathscr{D}_j \neq \emptyset$ for some $j \neq i$. If also $F \cap B_\sigma(\zeta) \cap \partial \mathscr{D}_k \neq \emptyset$ for some $k \neq i,j$ and some $\zeta \in \gamma$, then according to the construction of § 3.3, there would be a rectangle $B_\sigma(\zeta)$, with $\zeta \in \gamma$, which belongs to the set B^* of (3.16). Thus, γ could not be connected relative to $\tilde{\mathscr{A}}$.

Hence, $\tilde{\mathscr{A}}$ itself is admissible for quasi-conformal modification. Now choose some component D of $\cup \mathscr{D}[m]$, say $D \in \mathscr{D}_i[m]$, which meets both components of $\partial A_{m,n}$; this is possible from (2.31), since significant components of each $\mathscr{D}_i (1 \leq i \leq q)$

meet $\{|z|=A^{-1}\varrho\}$. Then D corresponds to two antipodal components D_0 and D_0' in $\mathscr{A}^{\sim}$.

We start with D_0 in $\mathscr{A}^{\sim}$ and slightly modify the ideas of Lemma 4.1. Let $E(D_0)$ be the union of D_0 with the components of $\mathscr{A}^{\sim}-D_0$ which do not contain D_0'. Then D_0 is a zero-stage component of $E(D_0)$ which bounds various first-stage F's of $E(D_0)$. (According to Lemma 4.11, these F's are $\mathscr{F}(i,j)$'s).... Then H is defined in $E(D_0)$ by (4.6), (4.7), (4.8).

Now $\mathscr{A}^{\sim}-D_0$ is connected and we move along $\mathscr{A}^{\sim}$ anti-clockwise from $E(D_0)$. We first encounter an $F\in\mathscr{F}(i,j)$, and use (4.7) there. If $E(F)$ is the union of F with components of $\mathscr{A}^{\sim}-F$ which do not contain D_0', we may extend H to $E(F)$ by reflecting from F.

We continue in this manner anti-clockwise in $\mathscr{A}^{\sim}$ until we encounter D_0'. If we are led to defining $H(z)=(f(z)-a_i)^{-1}$ in D_0' we have that $H(-z)=H(z)$ in $E(D_0')$; if $H(z)=(f(z)-a_i)$ in D_0', then $H(-z)=H(z)^{-1}$ for $z\in E(D_0')$, and the relevant identity persists as H is continued from D_0' to D_0 through $\mathscr{A}^{\sim}$.

Note that no pair $\{z,-z\}$ can be in an $E(D)$ or $E(F)$. For example, suppose this happened with $\{z,-z\}\in E(D)$ with $D\neq D_0$. Then z could be joined to $-z$ by a curve γ which would fail to meet one of D_0 or D_0', and so γ^2 would correspond to a curve which surrounded the origin but failed to meet D. Since this is impossible, it shows that H is well-defined.

5.3. *Back to meromorphic functions.* We have already seen, in the proof of Lemma 4.2 in § 4.3, how to replace H by a nearly equivalent meromorphic function G.

For each pair m,n, define $\nu=\nu_{m,n}$ on $\mathscr{A}^{\sim}$ by

$$\nu_{m,n}(z)=\mu(H_{m,n}(z))=\frac{(H_{m,n})_{\bar z}}{(H_{m,n})_z}\quad (z\in\mathscr{A}^{\sim}). \tag{5.21}$$

We extend ν to be zero outside $\mathscr{A}^{\sim}$ so that ν is defined in the plane with

$$\|\nu\|_\infty<\eta. \tag{5.22}$$

With ν now defined on all of C, we solve the associated Beltrami equation:

$$\begin{gathered}\psi_{\bar z}=\nu(z)\,\psi(z),\\ \psi(0)=0,\ \psi(\infty)=\infty,\quad \psi(\varrho^{1/2})=\varrho^{1/2}.\end{gathered} \tag{5.23}$$

Again, as in § 4.3, there exists [2, Chapter 5] a unique homeomorphic solution $\zeta=\psi(z)$

to this equation. As in § 4.3, our interest is in the inverse function

$$z=\varphi(\zeta). \tag{5.24}$$

The notations A^+, A^-, A°, A^* from Lemma 5.1 now translate to the $\psi(z)$-plane. Thus $\mathscr{A}^\circ, \mathscr{A}^*$ are the largest annuli centered at the origin with

$$(\varphi(\mathscr{A}^\circ))^2 \subset A^\circ, \quad (\varphi(\mathscr{A}^*))^2 \subset A^*, \tag{5.25}$$

and we write

$$\mathscr{A}^\circ = \{s_m \leq |z| \leq s'_{m+1}\}, \quad \mathscr{A}^* = \{s'_m \leq |z| \leq s_{m+1}\}, \tag{5.26}$$

as suggested by Lemma 5.1. Similarly, $Q(\lambda)$ logarithmic rectangles $B_{\sigma^*}[\zeta]$ are removed from $\mathscr{A}^\circ - \mathscr{A}^*$, and are chosen so that they reach the boundary of $\mathscr{A}^\circ$, and are minimal with respect to the property that their image under φ^2 contains the union $B^\#$ of (5.12). Finally, let

$$\mathscr{A} = \mathscr{A}^\circ - \bigcup B_{\sigma^*}[\zeta]. \tag{5.27}$$

It then follows from the chain rule that if

$$G(\zeta) = \omega f(\varphi(\zeta)^2) = H(\varphi(\zeta)) \quad (z \in \mathscr{A}) \tag{5.28}$$

then

$$G \textit{ is meromorphic in } \mathscr{A}. \tag{5.29}$$

We may now make the choice of parameters $\eta = \eta_n$ of (1.24) more precise; namely η is taken so small that

$$\left| \frac{\varphi(\zeta)}{\zeta} - 1 \right| \leq \gamma^{10} \quad ((30A)^{-1/2} \varrho^{1/2} < |\zeta| < (30A)^{1/2} \varrho^{1/2}). \tag{5.30}$$

This may be ensured from normal family considerations, and on retarding the rate at which $A \to \infty$ in (1.10); compare with (4.10).

Given $G(z)$ as in (5.28), let

$$K(\zeta) = G(\zeta) \frac{\pi_\infty(\zeta)}{\pi_0(\zeta)} = G(\zeta) P(\zeta) \quad (z \in \mathscr{A}) \tag{5.31}$$

where the canonical products are taken over zeros and poles of G in $\mathscr{A}$ (compare with definitions (4.41) and (4.44)). Unlike in (4.44), we hesitate to take a logarithm, since $\mathscr{A}$ is not simply-connected, but K is free of zeros and poles in $\mathscr{A}$). The logarithmic derivative of K will be intensively studied in §6.3 and afterwards.

Note from (5.20) that

$$n(0, G, \mathscr{A})+n(\infty, G, \mathscr{A})<C(A')^{-2h}\,T(\varrho), \tag{5.32}$$

so the factor P in (5.31) will be an error term, as in Chapter 4.

6. The logarithmic derivative and quasi-conformal modification

6.1. *The logarithmic derivative.* If g is meromorphic in the plane, the behaviour of its logarithmic derivative, g'/g, plays a central role in R. Nevanlinna's work. Indeed, the key step in his proof of the second fundamental theorem is the "lemma of the logarithmic derivative", which says that $\int \log^+|g'/g|\,d\theta \equiv 2\pi m(r, g'/g)$ is an error term.

Later, W. H. J. Fuchs [13] showed that $|g'/g|$ itself satisfies

$$r\int_0^{2\pi}\left|\frac{g'}{g}(re^{i\theta})\right| d\theta \leqslant CT(2r, g), \tag{6.1}$$

and he has used (6.1), and refinements thereof, several times (cf. [14], [16]). (Inequality (6.1) fails on an exceptional set, but we will use area integrals in this work.)

Our use of (6.1) is in this spirit but we will want to apply it to $K(z)$ or, what is essentially the same, to $G(z)$, where K and G are related by (5.31). If we apply the chain rule to definitions (5.28) and (5.31), we find that derivatives of the mapping φ must be estimated, and this will require us to use the calculus of §4.4.

The first section of this Chapter is independent of all earlier work. We suppose $g(z)$ is meromorphic in the plane, and will study means of g'/g over an annulus

$$\mathscr{K}=\mathscr{K}(r, A)=\{A^{-1}r<|z|<Ar\} \quad (A>2). \tag{6.2}$$

The prototype result is

LEMMA 6.1. *Let g be meromorphic in the plane and $\mathscr{K}$ the annulus* (6.2). *Then for* $1\leqslant p\leqslant\frac{3}{2}$,

$$\left\|\frac{g'}{g}\right\|_p^* \equiv \left\|\frac{g'}{g}\right\|_{\mathscr{K}(r,A),p}^* \equiv \left\{\int\!\!\int_{\mathscr{K}}\left|\frac{g'}{g}(te^{i\theta})\right|^p dt\,d\theta\right\}^{1/p} \leqslant CA^{1+1/p}r^{1/p-1}T(4Ar, g). \tag{6.3}$$

Remarks. (1) Inequality (6.3) concerns an L_p-norm with respect to logarithmic measure. We write $\|\cdot\|_p^*$ for the L_p-norm with respect to this measure and let $\|\cdot\|_p$ be the norm with respect to planar measure. When the context makes confusion unlikely, the dependence on the set $\mathscr{K}$ or r and A will be ignored.

(2) We have required $p\leqslant\frac{3}{2}$ so that all estimates in (6.3) are independent of p, but (6.3) holds for each fixed $p\in[1,2)$. It cannot hold when $p=2$, for near a zero or pole, $|g'/g|\sim c|z-z_0|^{-1}$ with $c\neq 0$.

(3) The powers of A in (6.3) and later formulas are not crucial, only that they are of polynomial growth as $A\to\infty$.

Proof of Lemma 6.1. According to the differentiated Poisson-Jensen formula,

$$\frac{g'}{g}(z)=\frac{1}{2\pi}\int_0^{2\pi}\log|g(Re^{i\varphi})|\frac{2Re^{i\varphi}\,d\varphi}{(Re^{i\varphi}-z)^2}+\sum_a\pm\frac{R^2-|a|^2}{(R^2-\bar{a}z)(z-a)}\quad(|z|<R),\tag{6.4}$$

where the a's are the zeros and poles of g in $\{|z|<R\}$; a $+$sign is used when a is a zero of g, and a $-$sign when $g(a)=\infty$. Let $z\in\mathcal{K}$ and choose $R=2Ar$; then routine estimates give

$$\begin{aligned}\left\|\int_0^{2\pi}\log|g(Re^{i\varphi})|\frac{2Re^{i\varphi}\,d\varphi}{(R\,e^{i\varphi}-z)^2}\right\|^*_{\mathcal{K},p}&\leqslant 4\left\|\frac{r}{(R-r)^2}[T(R,g)+O(1)]\right\|^*_{\mathcal{K},p}\\&\leqslant C\|Ar^{-1}T(R,g)\|^*_{\mathcal{K},p}\\&\leqslant CT(4Ar,g)\,A^{1/p+1}r^{1/p-1}.\end{aligned}\tag{6.5}$$

Also since $R=2Ar$, each term in the summation of the zeros and poles is $O(|z-a|)^{-1}$. Now if $|a|\geqslant(2A)^{-1}r$, we have

$$\begin{aligned}\iint_{|z-a|\leqslant\frac{1}{2}|a|}\left|\frac{1}{z-a}\right|^p dt\,d\theta&\leqslant\frac{2}{|a|}\iint_{|z-a|<\frac{1}{2}|a|}\frac{1}{|z-a|^p}\,t\,dt\,d\theta=\frac{4\pi}{a}\int_0^{\frac{1}{2}|a|}t^{1-p}\,dt\\&=O(|a|)^{1-p}=O\left(\frac{A}{r}\right)^{p-1}\end{aligned}\tag{6.6}$$

since $1\leqslant p\leqslant\frac{3}{2}$. On the rest of $\mathcal{K}$, we have $|z-a|\geqslant\frac{1}{2}|a|$, and in particular if $|a|<r/(2A)$ we have for $z\in\mathcal{K}$ that $|z-a|\geqslant r/(4A)$. Hence

$$\|(z-a)^{-1}\|^*_{\mathcal{K},p}\leqslant CA^{1+1/p}r^{1/p-1},\tag{6.7}$$

and there are $n(2Ar,0,g)+n(2Ar,\infty,g)<CT(4Ar,g)$ terms in the sum. This, (6.5) and (6.6) give (6.3).

COROLLARY 1. *Under the hypotheses of Lemma* 6.1, *we have that*

$$\left\|\frac{g'}{g}\right\|^*_{\mathcal{K},p} \leq C\sigma^{2/p-1}r^{1/p-1}T(4Ar, g) \tag{6.8}$$

when $A=1+\sigma$ *with* $\sigma \leq \frac{1}{2}$.

Proof. We leave to the reader the task of adapting (6.5) to this context, and concern ourselves here only with the at most $CT(4Ar, g)$ terms in the sums of zeros and poles in (6.4).

Let a be one of the zeros or poles of g, and $\mathcal{K}_1 \subset \mathcal{K}$ be the subset on which $|z-a|>4\sigma r$.

Then

$$\int_{\mathcal{K}_1} \frac{1}{|z-a|^p}\, dt\, d\theta \leq C(Ar^{-1})^p\, \sigma r;$$

otherwise, we find that

$$\int_{\mathcal{K}-\mathcal{K}_1} \frac{1}{|z-a|^p}\, dt\, d\theta \leq \frac{2}{|a|} \int |z-a|^{-p} t\, dt\, d\theta$$

$$\leq \frac{CA}{r} \int_0^{4\sigma r} t^{1-p}\, dt = CA\sigma^{2-p} r^{1-p}.$$

COROLLARY 2. *Let* $\mathcal{K}$ *be a region* $\frac{1}{4}r<|z|<4r$, $\theta_0<\arg z<\theta_0+\sigma$. *Then*

$$\left[\int_{\mathcal{K}} \left|\frac{g'}{g}\right|^p dt\, d\theta\right]^{1/p} \leq C\sigma^{2/p-1} r^{1/p-1} T(4Ar, g). \tag{6.9}$$

The proof is exactly that of Corollary 1.

6.2. *Lemma* 6.1 *after quasi-conformal modification.* In this section we show that estimates of the nature (6.3) are preserved under quasiconformal modification. It is possible to do this in considerable generality, but we need this only for the situation described in § 5.3, and trust that the interested reader will be able to adapt our arguments to more general situations if necessary. The principal simplification here is that all dilatations are uniformly small as in (5.22) rather than just having norm <1. Thus, in our situation, Neumann series such as in [2, p. 92, equation (5)] will converge for each fixed $p \in (1, \infty)$ if $\|\mu\|_\infty$ is sufficiently small; when we only assume that $\|\mu\|_\infty \leq k < 1$, then p must be very near two [2, p. 91].

Thus, suppose that g is meromorphic in the plane, $\mathcal{K}$ is the annulus $\{A^{-1/2}r<|z|<A^{1/2}r\}$, there are quasi-conformal maps $\varphi(\zeta)$ of the plane and $\omega(w)$ ($w=g(z), z\in\mathcal{K}$) with $\|\mu_\varphi\|_\infty$ and $\|\mu_\omega\|_\infty$ small, and such that

$$G(\zeta)=\omega\circ g\circ\varphi(\zeta)=H\circ\varphi(\zeta)\quad(\zeta\in\mathcal{K}) \tag{6.10}$$

is meromorphic in $\mathcal{K}$. In the notions of (5.18) and (5.28), we are taking $g(z)=f(z^2)$, so that $T(r,g)=2T(r^2,f)$. The mapping φ is inverse to ψ, where ψ satisfies (5.23) (with $r=\varrho^{1/2}$), and φ and ψ are rigid in the sense (5.30). Finally, we suppose that

$$|\omega_w(w)|\leq C|w|^{-1}|\omega(w)| \tag{6.11}$$

(according to (1.22), (6.11) is satisfied in our application).

LEMMA 6.2. *Let G be defined by* (6.10). *Then for each $p\leq\frac{3}{2}$, we have*

$$\left\|\frac{G'}{G}\right\|^*_{\mathcal{K},p}\leq\varkappa(A)\,r^{1/p-1}T(4A^{1/2}r,g)\leq\varkappa(A)\,r^{1/p-1}T(4A\varrho,f) \tag{6.12}$$

where $\varkappa$ is a generic function of at most polynomial growth in A as $A\to\infty$.

Proof. Since φ^{-1} and $H=\omega\circ g$ satisfy the same Beltrami equation and g is meromorphic ($g_{\bar z}\equiv 0$), we find from (6.10) that

$$\begin{aligned}G'(\zeta)&=(\omega\circ g)_z\,\varphi_\zeta+(\omega\circ g)_{\bar z}\,\bar\varphi_\zeta\\&=\omega_w\,g_z\,\varphi_\zeta+\omega_{\bar w}\,\bar g_z\,\varphi_\zeta+\omega_w\,g_{\bar z}\,\bar\varphi_\zeta+\omega_{\bar w}\,\bar g_{\bar z}\,\bar\varphi_\zeta\\&=\omega_w(g\circ\varphi)\,g'(\varphi)\,\varphi_\zeta+\omega_{\bar w}(g\circ\varphi)\,\overline{g'(\varphi)}\,\bar\varphi_\zeta\end{aligned}$$

so (5.22) and (6.11) imply that

$$\begin{aligned}\frac{G'}{G}(\zeta)&=(\omega(w))^{-1}\left[\omega_w(w)\,g'(\varphi)\,\varphi_\zeta+\omega_{\bar w}(w)\,\overline{g'}(\varphi)\,\bar\varphi_\zeta\right]\\&=\{1+C(\eta)\}\,\omega^{-1}\omega_w\,g'\varphi_\zeta=B(\zeta)\frac{g'}{g}(\varphi(\zeta))\,\varphi_\zeta\quad(w=g\circ\varphi(\zeta))\end{aligned} \tag{6.13}$$

with $C(\eta)$ as in (4.21), and $B(\zeta)$ in L_∞.

We first show that if $1\leq p<\frac{3}{2}$, then ($\zeta=ue^{v}$)

$$\left\{\int\!\!\int_{\mathcal{K}}\left|\frac{g'}{g}(\varphi(\zeta))\right|^p du\,dv\right\}^{1/p}\leq\varkappa(A)\,r^{1/p-1}T(4r,g). \tag{6.14}$$

Let $z=\varphi(\zeta)$, and choose $q>1$ so that $pq<\frac{3}{2}$. Then (4.24), (5.30), (6.10) and (6.11) give that (J=Jacobian, $\mathcal{K}'=\varphi(\mathcal{K})$, $\psi=\varphi^{-1}$)

$$\begin{aligned}\left\{\int\!\!\int_{\mathcal{K}}\left|\frac{g'}{g}(\varphi(\zeta))\right|^p du\,dv\right\}^{1/p} &= \left\|\frac{g'}{g}(\varphi)\right\|^*_{\mathcal{K},p}\\ &= \left\{\int\!\!\int_{\mathcal{K}}\left|\frac{g'}{g}(\varphi(\zeta))\right|^p \frac{u\,du\,dv}{u}\right\}^{1/p}\\ &= \left\{\int\!\!\int_{\mathcal{K}'}\left|\frac{g'}{g}(z)\right|^p J\left|\frac{r}{\psi(z)}\right| dr\,d\theta\right\}^{1/p}\\ &\leq C\left\|\frac{g'}{g}\right\|^*_{\mathcal{K}',pq}\{\|J\|^*_{\mathcal{K}',q'}\}^{1/p}\\ &\leq C\left\|\frac{g'}{g}\right\|^*_{\mathcal{K}',pq}\left\{\|\psi_z^2\|^*_{\mathcal{K}',q'}\right\}^{1/p}.\end{aligned} \tag{6.15}$$

According to (4.25), (4.26), (5.30) and the conventions (4.21) and (1.24),

$$\begin{aligned}\|\psi_z-1\|^*_{\mathcal{K}',q'} &\leq \varkappa(A)\,r^{-1/q'}\|\psi_z-1\|_{q'}\\ &\leq \varkappa(A)\,C(\eta)\,r^{1/q'} = C(\eta)\,r^{1/q'},\end{aligned} \tag{6.16}$$

so in particular

$$\begin{aligned}\|\psi_z^2\|^*_{\mathcal{K}',q'} &\leq \|(\psi_z-1)^2\|^*_{\mathcal{K}',q'}+\|2\psi_z+1\|^*_{\mathcal{K}',q'}\\ &= \left\{\|\psi_z-1\|^*_{\mathcal{K}',2q'}\right\}^2+\|2\psi_z+1\|^*_{\mathcal{K}',q'} \leq Cr^{1/q'}.\end{aligned} \tag{6.17}$$

Recall that $1<q$ and $pq\leq\frac{3}{2}$ We use (6.3) in our computation and obtain in (6.15) that

$$\begin{aligned}\left\|\frac{g'}{g}(\varphi)\right\|^*_{\mathcal{K},p} &\leq \{\varkappa(A)\,r^{1/pq-1}T(4A^{1/2}r,g)\}\,\{Cr^{1/pq'}\}\\ &\leq \varkappa(A)\,r^{1/p-1}T(4A^{1/2}r,g).\end{aligned}$$

The function φ in (6.10) is the inverse function to ψ as in (5.24), so that φ solves the Beltrami equation

$$\varphi_{\bar\zeta}=\nu_1(\zeta)\,\varphi_\zeta \quad (\varphi(0)=0,\ \varphi(\varrho^{1/2})=\varrho^{1/2},\ \varphi(\infty)=\infty) \tag{6.18}$$

where [2, p. 9]

$$\nu_1(\zeta)=-\frac{\psi_{\bar z}}{\bar\psi_{\bar z}}(z).$$

Thus $\|\nu_1\|_\infty=\|\nu\|_\infty$, so that (4.24)–(4.26) and (6.16) apply to φ (except that in Chapter 4, φ and ψ were to fix $z=\varrho$, while now φ and ψ fix $z=\varrho^{1/2}$; see (5.23). Thus (6.13), (6.14) and (6.16) (using φ in place of ψ) give (6.12):

$$\begin{aligned}\left\|\frac{G'}{G}\right\|^*_{\mathcal{K},p} &\leqslant C\left\|\frac{g'}{g}(\varphi)\right\|^*_{\mathcal{K},pq}\|\varphi_\zeta\|^*_{\mathcal{K},pq'}\\ &\leqslant C\left\|\frac{g'}{g}(\varphi)\right\|^*_{\mathcal{K},pq}\|1+(\varphi_\zeta-1)\|^*_{\mathcal{K},pq'}\\ &\leqslant \varkappa(A)\,r^{1/p-1}T(4A^{1/2}r,g).\end{aligned}$$

COROLLARY. *Let $\mathcal{L}$ be a logarithmic region*

$$\mathcal{L}=\{\zeta;\alpha r<|\zeta|<\beta r\}$$

or

$$\mathcal{L}=\{\zeta;\tfrac{1}{3}r<|\zeta|<3r,\,\theta_0<\arg\zeta_0<\theta_0+\gamma\},$$

where

$$\log\frac{\beta}{\alpha}<C\sigma\quad\text{or}\quad\gamma<C\sigma. \tag{6.19}$$

Then

$$\left\|\frac{G'}{G}\right\|^*_{\mathcal{L},p}\leqslant\varkappa(A)\,\sigma^{2/p-1}r^{1/p-1}T(\varrho). \tag{6.20}$$

Proof. This just requires the more refined estimates (6.8) and (6.9) in (6.15), and the proof is omitted.

6.3. *Almost-analytic continuation revisited.* For $1\leqslant m\leqslant M(n)$, let the functions G and K be as in (5.28) and (5.31) ($G=G_m, K=K_m$). Each function K_m'/K_m is single-valued and regular in $\mathcal{A}_m$, and has a Laurent expansion in $\mathcal{A}_m^*$ (see (5.26)). We will prove in Lemma 7.1 that the coefficients of

$$\left(\frac{K_m'}{K_m}(\zeta)\right)^2=\Sigma\,\alpha_l(m)\,\zeta^l\quad(z\in\mathcal{A}_m^*) \tag{6.21}$$

are nearly independent of m. In the simplest case, when $f(z)=\exp(z^p)$, then $(K'/K)^2(\zeta)=T^2(\varrho)\,\varrho^{-2p}(\zeta)^{4p-2}$, so that $\alpha_{2p-2}=T^2(\varrho)\,\varrho^{-2p}$ and all other α's are zero.

Let us see why (6.21) is likely to be invariant. According to (5.27), (5.31) and (5.32), for each m,

$$\frac{K_m'}{K_m}=\frac{G_m'}{G_m}+\sum_{j=1}^{N_m}\pm\frac{1}{\zeta-c_{jm}}\quad (z\in\mathscr{A}_m) \tag{6.22}$$

where the c's include all zeros and poles of G in $\mathscr{A}_m$ and

$$N_m\leqslant C(A')^{-2h}T(\varrho)\quad (1\leqslant m\leqslant M(n)). \tag{6.23}$$

Let $\zeta_0\in\mathscr{A}_m$. Then $z^*=\varphi(\zeta_0)^2\in A_m\cap\mathfrak{A}_\sigma^*$ (cf. (5.16) and (5.17)) so there exists a local quasi-conformal modification in $B_\sigma(z^*)$ as described in §§4.2 and 4.3, say $G^*(\zeta)=H^*(\varphi^*(\zeta))$ in $\Delta(z^*,\frac{1}{2}\sigma)$. According to Lemma 4.11 (and the remark at the end of §4.7), any other local q.c. modification can only replace H^* by $(H^*)^{-1}$. Thus the expressions

$$(\log H)_z^2\equiv\left(\frac{H_z}{H}\right)^2,\quad (\log H)_{\bar z}\equiv\left(\frac{H_{\bar z}}{H}\right)^2 \tag{6.24}$$

are invariant. This suggests that $(G'/G)^2$ is almost invariant, and (6.22) and (6.23) will transfer this invariance to K. This is the motivation for the following result.

LEMMA 6.3. *Let $\mathscr{A}[m]=\mathscr{A}[m,n]=\mathscr{A}_{m,n}\cap\mathscr{A}_{m+1,n}$ (thus $\mathscr{A}[m]$ is what remains when $Q(\lambda)$ rectangles $B_{\sigma^*}[\zeta]$ disconnect a narrow annulus). Then K_m and K_{m+1} are holomorphic in $\mathscr{A}[m]$, with ($\zeta=ue^v$)*

$$\int_{\mathscr{A}[m]}\left|\left(\frac{K_m'}{K_m}\right)^2-\left(\frac{K_{m+1}'}{K_{m+1}}\right)^2\right|^{1/2}du\,dv<C\gamma^4T(\varrho). \tag{6.25}$$

Proof. Since

$$\left(\sum a_n\right)^\alpha\leqslant\sum a_n^\alpha\quad (a_n>0,0\leqslant\alpha\leqslant 1), \tag{6.26}$$

we find that

$$\begin{aligned}(\|(f^2-g^2)\|_{\mathscr{X},1/2}^*)^{1/2}&\equiv\int_{\mathscr{X}}|f^2-g^2|^{1/2}\,dr\,d\theta=(\|((g+(f-g))^2-g^2)\|_{\mathscr{X},1/2}^*)^{1/2}\\&=(\|(2g(f-g)+(f-g)^2\|_{\mathscr{X},1/2}^*)^{1/2}\\&\leqslant(\|2g(f-g)\|_{\mathscr{X},1/2}^*)^{1/2}+(\|(f-g)^2\|_{\mathscr{X},1/2}^*)^{1/2}\\&\equiv(\|2g(f-g)\|_{\mathscr{X},1/2}^*)^{1/2}+\|(f-g)\|_{\mathscr{X},1}^*\\&\leqslant 2^{1/2}(\|g\|_{\mathscr{X},1}^*)^{1/2}(\|f-g\|_{\mathscr{X},1}^*)^{1/2}+\|f-g\|_{\mathscr{X},1}^*\end{aligned} \tag{6.27}$$

This formalism means that when f and g are close in 1-norms, they are close in $\frac{1}{2}$-norms.

To control notation, we will write G, H and φ in place of $G_m(\zeta)=H_m(\varphi_m(\zeta))$, and K in place of K_m; and G_1, H_1 and φ_1 for $G_{m+1}(\zeta)=H_{m+1}(\varphi_{m+1}(\zeta))$ and K_1 for K_{m+1}.

Finally, much as in (4.57), we let

$$w=\Phi(\zeta)=\varphi_1^{-1}\circ\varphi(\zeta), \quad \zeta=\Psi(w). \tag{6.28}$$

Here are the key steps needed to obtain (6.25):

$$\int_{\mathcal{A}[m]}\left|\frac{K'}{K}-\frac{G'}{G}\right| du\,dv \leqslant C(A')^{-2h}T(\varrho) \tag{6.29}$$

(with similar inequality for K_1 and G_1);

$$\int_{\mathcal{A}[m]}\left|\frac{G'}{G}(\zeta)\,(1-(\varphi_1)_w)\right| du\,dv < C\gamma^9 T(\varrho), \tag{6.30}$$

$$\int_{\mathcal{A}[m]}\left|\frac{G_1'}{G_1}(w)\,(1-\varphi_\zeta)\right| du\,dv < C\gamma^9 T(\varrho), \tag{6.31}$$

and finally that

$$\int_{\mathcal{A}[m]}\left|\left[\left(\frac{G_1'}{G_1}(w)\right)\varphi_\zeta\right]^2-\left[\left(\frac{G'}{G}(\zeta)\right)(\varphi_1)_w\right]^2\right|^{1/2} du\,dv < (C(\eta)+C\gamma^9)\,T(\varrho) < C\gamma^9 T(\varrho). \tag{6.32}$$

Note that (1.24) and (6.12), (6.29)–(6.32) give (6.25), since by (6.26) and (6.27),

$$\begin{aligned}
\int_{\mathcal{A}[m]}\left|\left(\frac{K'}{K}\right)^2-\left(\frac{K_1'}{K_1}\right)^2\right|^{1/2} du\,dv &\leqslant \left\{\left\|\left(\frac{K'}{K}\right)^2-\left(\frac{G'}{G}\right)^2\right\|^*_{1/2}\right\}^{1/2} \\
&+\left\{\left\|\left(\frac{G'}{G}\right)^2-\left(\frac{G'}{G}(\varphi_1)_w\right)^2\right\|^*_{1/2}\right\}^{1/2}+\left\{\left\|\left(\frac{G'}{G}(\varphi_1)_w\right)^2-\left(\frac{G_1'}{G_1}\varphi_\zeta\right)^2\right\|^*_{1/2}\right\}^{1/2} \\
&+\left\{\left\|\left(\frac{G_1'}{G_1}\right)^2-\left(\frac{G_1'}{G_1}\varphi_\zeta\right)^2\right\|^*_{1/2}\right\}^{1/2}+\left\{\left\|\left(\frac{K_1'}{K_1}\right)^2-\left(\frac{G_1'}{G_1}\right)^2\right\|^*_{1/2}\right\}^{2} \\
&\leqslant C\left\{\left(\left\|\frac{G'}{G}\right\|^*_{\mathcal{K},1}\right)^{1/2}\left(\left\|\frac{K'}{K}-\frac{G'}{G}\right\|^*_{\mathcal{K},1}\right)^{1/2}+\left\|\frac{K'}{K}-\frac{G'}{G}\right\|^*_{\mathcal{K},1}\right\} \\
&+C\left\{\left(\left\|\frac{G'}{G}\right\|^*_{\mathcal{K},1}\right)^{1/2}\left(\left\|\frac{G'}{G}-\frac{G'}{G}(\varphi_1)_w\right\|^*_{\mathcal{K},1}\right)^{1/2}+\left\|\frac{G'}{G}(1-(\varphi_1)_w\right\|^*_{\mathcal{K},1}\right\}
\end{aligned}$$

$$+\int\left|\left(\frac{G'}{G}(\zeta)(\varphi_1)_w\right)^2-\left(\frac{G_1'}{G_1}(w)\varphi_\zeta\right)^2\right|^{1/2}du\,dv$$
$$+C\left\{\left(\left\|\frac{G_1'}{G_1}\right\|^*_{\mathcal{K},1}\right)^{1/2}\left(\left\|\frac{G_1'}{G_1}(w)(\varphi_\zeta-1)\right\|^*_{\mathcal{K},1}\right)^{1/2}+\left\|\frac{G_1'}{G}(\varphi_\zeta-1)\right\|^*_{\mathcal{K},1}\right\}$$
$$+C\left\{\left(\left\|\frac{G_1'}{G_1}\right\|^*_{\mathcal{K},1}\right)^{1/2}\left(\left\|\frac{G_1'}{G_1}-\frac{K_1'}{K_1}\right\|^*_{\mathcal{K},1}\right)^{1/2}+\left\|\frac{G_1'}{G_1}-\frac{K_1'}{K_1}\right\|^*_{\mathcal{K},1}\right\}$$
$$\leqslant CT(4A\varrho)\,\{\varkappa(A)\,(A')^{-h}+\gamma^{9/2}+\gamma^9\}<\gamma^4T(\varrho)$$

Proof of (6.29). This follows from (5.32), (6.7) (with $A=2$) and (6.22):

$$\int_{\mathcal{A}[m]}\left|\frac{K'}{K}-\frac{G'}{G}\right|du\,dv\leqslant\sum^N\int_{\mathcal{A}[m]}\left|\frac{1}{z-c}\right|du\,dv\leqslant C(A')^{-2h}T(\varrho).$$

Proof of (6.30). The functions Φ and Ψ of (6.28) are holomorphic, and are compositions of maps each factor of which satisfies the estimate (6.16). Thus (1.24), (5.30) and Cauchy's estimates show that

$$\begin{aligned}&|(\Phi'(\zeta)-1)|<C\gamma^{10}\quad((20A)^{-1/2}\varrho^{1/2}<|\zeta|<(20A)^{1/2}\varrho^{1/2}),\\&|(\Psi'(w)-1)|<C\gamma^{10}\quad((20A)^{-1/2}\varrho^{1/2}<|w|<(20A)^{1/2}\varrho^{1/2}),\end{aligned}\tag{6.33}$$

(compare with (4.64)).

Let $h=\varphi_\zeta-1$ and $h_1=(\varphi_1)_w-1$, as suggested by (4.25). Estimate (6.16) applies to h and h_1. Thus (5.30), (6.28), (6.33) and the chain rule give that

$$\begin{aligned}|(\varphi_1)_w-1|&=|(\varphi_1)_\zeta\Psi'(w)-1|\\&\leqslant|(\varphi_1)_\zeta-1|+|(\varphi_1)_\zeta(\Psi'(w)-1)|\\&\leqslant|h_1|+C\gamma^{10}|1+h_1|,\end{aligned}\tag{6.34}$$

and we deduce using (1.8) [or (1.11) with $\varepsilon=1$], (6.12) (with A bounded), (6.16) and (6.34) that

$$\begin{aligned}\int_{\mathcal{A}[m]}\left[\left|\frac{G'}{G}(\zeta)(1-(\varphi_1)_w)\right|du\,dv\right.&\leqslant\left\|\frac{G'}{G}\right\|^*_{\mathcal{A},p}\{\|h_1\|^*_{\mathcal{A},p'}+C\gamma^{10}\|1+h_1\|^*_{\mathcal{A},p'}\}\\&\leqslant\{Cr^{1/p-1}T(8A\varrho)\}\,\{r^{1/p'}(C(\eta)+\gamma^{10})\}\\&\leqslant\varkappa(A)\,(C(\eta)+\gamma^{10})\,T(\varrho)=(C(\eta)+\gamma^9)\,T(\varrho)\\&\leqslant C\gamma^9T(\varrho)\end{aligned}$$

by the conventions (1.24) and (4.21).

Proof of (6.31). This is symmetric to that of (6.30), and is left to the reader.

Proof of (6.32). We use the definitions (5.28) for G and G_1 (where H and H_1 have small maximal dilatation) and the invariance suggested by (6.24) in $\mathscr{A}[m]$ to obtain that

$$G'(\zeta) = H_z \varphi_\zeta + H_{\bar z} \bar\varphi_\zeta = \{1+C(\eta)\}\, H_z \varphi_\zeta$$

and a similar formula for G_1'; thus, on taking (6.28) into account we find that

$$\begin{aligned}
\frac{G'}{G}(\zeta)^2 &= \{1+C(\eta)\}\left[\frac{H_z(\varphi)\,\varphi_\zeta}{H(\varphi)}\right]^2 \equiv \{1+C(\eta)\}\left[\frac{H_z(\varphi_1\,\Phi)\,\varphi_\zeta}{H(\varphi_1\,\Phi)}\right]^2 \\
&\equiv \{1+C(\eta)\}\left[\frac{(H_1)_z(\varphi_1\,\Phi)}{H_1(\varphi_1\,\Phi)}\varphi_\zeta\right]^2 \quad (!) \\
&= \{1+C(\eta)\}\left[\frac{(H_1)_z(\varphi_1(w))\,(\varphi_1)_w}{H_1(\varphi_1(w))}\,\frac{\varphi_\zeta}{(\varphi_1)_w}\right]^2 \quad (w=\Phi(\zeta)) \\
&\equiv \{1+C(\eta)\}\left[\frac{(H_1\circ\varphi_1)_w}{H_1\circ\varphi_1}\,\frac{\varphi_\zeta}{(\varphi_1)_w}\right]^2 \\
&= \{1+C(\eta)\}\left[\frac{G_1'}{G_1}(w)\,\frac{\varphi_\zeta}{(\varphi_1)_w}\right]^2.
\end{aligned}$$

Thus, if $\zeta\in\mathscr{A}[m]$ and $w=\Phi(\zeta)\in\mathscr{A}[m]$, we have that

$$\left|\left(\frac{G_1'}{G_1}(w)\,\varphi_\zeta\right)^2-\left(\frac{G'}{G}(\zeta)\,(\varphi_1)_w\right)^2\right|^{1/2} \leqslant C(\eta)\left|\frac{G'}{G}(\varphi_1)_w\right|$$

and so (1.8) (or 1.11) with $\varepsilon=1$), (6.12), (6.16) and (6.34) give that

$$\begin{aligned}
\int_{\mathscr{A}[m]\cap\Psi(\mathscr{A}[m])}\left|\left(\frac{G_1'}{G}\varphi_\zeta\right)^2-\left(\frac{G'}{G}(\varphi_1)_w\right)^2\right|^{1/2} du\,dv &\leqslant C(\eta)\left\|\frac{G'}{G}\right\|_p^*\|\varphi_1\|_{p'}^* \qquad (6.35)\\
&\leqslant C\varkappa(A)\,C(\eta)\,T(\varrho) < C(\eta)\,T(\varrho).
\end{aligned}$$

We next observe that $\mathscr{A}[m]\cap\Psi(\mathscr{A}[m])$ is nearly all of $\mathscr{A}[m]$. According to (5.11), (5.14) and (5.17), $\mathscr{A}[m]$ is a narrow annulus from which $Q(\lambda)$ rectangles $B_{\sigma^\#}$ have been removed:

$$\mathscr{A}[m] = \{\zeta;\, s_{m+1}<|\zeta|<s'_{m+1},\, \theta_j+\sigma^\# \leqslant \arg\zeta \leqslant \theta_{j+1}-\sigma^\#\} \qquad (6.36)$$

where, by (5.9), (5.10) and (5.11), $\log(s'_{m+1}/s_{m+1})$ and $\sigma^\#$ are comparable to σ. Thus

(5.30) and (6.28) ensure that there are numbers $r'<1<r''$ and $\theta_1'<\theta_1''<\ldots\theta_j'<\theta_j''<\theta_{j+1}'<\ldots\theta_N''<\theta_1'+2\pi$ $(N=Q(\lambda))$ such that

$$\log\frac{r''}{r'}+\sum_1^{Q(\lambda)}\theta_j''-\theta_j'\le C\gamma^{10} \tag{6.37}$$

(compare with (4.39)), and such that

$$\{\mathscr{A}[m]-\Phi(\mathscr{A}[m])\}\cup\{\Phi(\mathscr{A}[m])-\mathscr{A}[m]\}\subset\{r's_{m+1}<|\zeta|<r''s_{m+1}\}\cup\{r's_{m+1}'<|\zeta|<r''s_{m+1}''\}\bigcup_j\{\tfrac{1}{2}s_{m+1}<|\zeta|<2s_{m+1}';\theta_j'<\arg\zeta<\theta_j''\}. \tag{6.38}$$

Then (6.20) may be used to show that the contribution to (6.32) from the sets (6.38) is negligible. We choose p slightly larger than one in (6.20). For example, let $\mathscr{L}=\{r's_{m+1}<|\zeta|<r''s_{m+1}\}$. Then using (6.34), we find that

$$\begin{aligned}\int_{\mathscr{L}}\left|\frac{G'}{G}(\zeta)(\varphi_1)_w\right|du\,dv&\le\int_{\mathscr{L}}\left|\frac{G'}{G}\right|(1+|h_1|+C\gamma^{10}|h_1|)\,du\,dv\\&\le\left\{\int\!\!\int_{\mathscr{L}}\left|\frac{G'}{G}\right|^p du\,dv\right\}^{1/p}\left\{\int\!\!\int_{\mathscr{A}}[1+|h_1|+C\gamma^{10}|h_1|]^{p'}\,du\,dv\right\}^{1/p}\\&\le\varkappa(A)\,\gamma^{19/2}T(\varrho)<\gamma^9T(\varrho).\end{aligned}$$

Similarly,

$$\int_{\mathscr{L}}\left|\frac{G_1'}{G_1}\varphi_\zeta\right|du\,dv<\gamma^9T(\varrho),$$

and these manipulations may be applied on the remaining $Q(\lambda)$ choices of $\mathscr{L}$. This proves that the contribution from (6.38) may be absorbed in the right side of (6.32), so (6.32) follows from this and (6.35).

This proves the lemma.

7. Completion of proof

7.1. *Maximum-modulus and $H_{1/2}$-norms.* We begin with an elementary consequence of Cauchy's inequality:

LEMMA 7.1. *Let $y>0$ be fixed and for $k>0$, let $kB=kB(z_0)$ be the logarithmic rectangle*

$$\max\left(\left|\log\left(\frac{r}{r_0}\right)\right|,|\theta-\theta_0|\right)\le ky\quad(z_0=r_0e^{i\theta_0},z=re^{i\theta}).$$

If h is holomorphic in B with

$$\int_B |h|^{1/2}\, dr\, d\theta < p, \tag{7.1}$$

then

$$\max |h(z)| \leq \frac{Cp^2 e^{2y}}{y^4 r_0^2} \quad (z \in \tfrac{1}{2}B). \tag{7.2}$$

Proof. Since $|h|^{1/2}$ is a subharmonic function of $w=\log z$, we have

$$\begin{aligned}|h(w_1)|^{1/2} &\leq \frac{1}{\pi t^2}\int_{|w-w_1|\leq t} |h|^{1/2}|dw|^2 = \frac{1}{\pi t^2}\int |h|^{1/2}\left|\frac{dw}{dz}\right|^2 |dz|^2 \\ &= \frac{1}{\pi t^2}\int |h|^{1/2}\frac{dr\,d\theta}{r}\end{aligned}$$

so long as $|w-w_1|<t$ is contained in B. Let $w_1=\log z_1$, with $z_1 \in \frac{1}{2}B$. We may then take $t=\frac{1}{2}y$ and observe that $r \geq r_0 e^{-y}$. Thus

$$|h(w_1)|^{1/2} \leq \frac{Ce^y p}{y^2 r_0},$$

from which (7.2) follows.

7.2. *Analysis of Laurent coefficients.* Recall the functions $K=K_m$ and the Laurent expansions of (6.21), where $\alpha_l=\alpha_l(m)$. The next result indicates that these coefficients are nearly independent of m.

LEMMA 7.2. *Let $\tau_1, 0<\tau_1<1$ be given. Then the Laurent coefficients $\alpha_l(m)$ of* (6.21) *satisfy.*

$$|\alpha_l(m)-\alpha_l(m+1)|1 \leq C2^{|l|} s_m^{-(l+1)} \sigma^{\tau_1} \varrho^{-1/2} T^2(\varrho) \tag{7.3}$$

where s_m is given in (5.26).

Remark. In this chapter, we view the K's as the main object of study, and so write $K(z)$ $(z=re^{i\theta})$ in place of $K(\zeta)$.

Proof. The modified annuli $\mathscr{A}^\circ=\mathscr{A}^\circ_m$ and $\mathscr{A}=\mathscr{A}_m$ have been described in (5.25)–(5.27). The mappings ψ (cf. (5.23)) are rigid in the sense (5.30). Thus there exist $O(\sigma^{-1})$ rectangles $B_\sigma(z_i)\,(1\leq i\leq p)$ such that

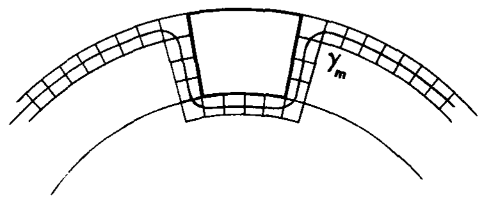

Fig. 2

$$B_\sigma(z_i) \subset \mathscr{A}_m \quad (1 \leqslant i \leqslant p, p \leqslant C\sigma^{-1}), \tag{7.4}$$

$\bigcup \frac{1}{10} B_\sigma(z_i)$ *is a continuum with winding number* 1 *about the origin* (7.5)

$$B_\sigma(z_i) \subset \mathscr{A}_m^* \cap \mathscr{A}_{m+1}^* \quad (i \notin \mathscr{U}) \tag{7.6}$$

where $\mathscr{U}$ is a set of at most $Q(\lambda)$ indices i. Let $\mathscr{B}_\sigma^+(m)$ be the union of these rectangles $B_\sigma(z_i)$. Similarly, we let $\mathscr{B}_\sigma^-(m+1)$ be rectangles as in (7.5) and (7.6), with $B\sigma(z_i) \subset \mathscr{A}_{m+1}^\circ$.

Next, let (for $j = m, m+1$) γ_j be a curve contained in $\mathscr{B}_\sigma^+(m)$ or $\mathscr{B}_\sigma^-(m+1)$ with $n(\gamma, 0) = 1$ (winding number) such that if $\gamma_j \cap B_\sigma(z_i) \neq \emptyset$, then $\gamma_j \cap B_\sigma(z_i)$ has length comparable to the side-length of $B_\sigma(z_i)$. We can assume that $\gamma_m = \gamma_{m+1} = \gamma$ for z in all but $Q(\lambda)$ rectangles of $\mathscr{B}_\sigma^+(m) \cup \mathscr{B}_\sigma^-(m+1)$. (See Fig. 2.) The interior of

$$\bigcup_m [(\gamma_m - \gamma) \cup (\gamma_{m+1} - \gamma)]$$

contains the $Q(\lambda)$ excluded rectangles B_{σ^*} of (5.27).

According to Cauchy's formula,

$$a_l(m) = \frac{1}{2\pi i} \int_{\gamma_m} \left(\frac{K_m'}{K_m}\right)^2 (z)\, z^{-(l+1)}\, dz, \tag{7.7}$$

with a similar formula for $a_l(m+1)$. We will use (7.7) to get (7.3).

Suppose (7.6) holds for $B_\sigma(z_i)$. Let

$$\int_{B_\sigma(z_i)} \left| \left(\frac{K_m'}{K_m}\right)^2 - \left(\frac{K_{m+1}'}{K_{m+1}}\right)^2 \right|^{1/2} dr\, d\theta = p_i;$$

then (6.25) shows that $\Sigma p_i \leqslant C\gamma^4 T(\varrho)$, and so $\Sigma (p_i)^2 \leqslant (\Sigma p_i)^2 \leqslant C\gamma^8 T^2(\varrho)$ (summation only over all $B_\sigma(z_i)$ which satisfy (7.6)). Because of (5.17), we have that the length of $\gamma_j \cap B_\sigma(z_i)$ is at most $C(A\varrho)^{1/2}\sigma$ and each B_σ lies in $|z| \geqslant (A^{-1}\varrho)^{1/2}$. When $i \notin \mathscr{U}$, $\gamma_{m+1} = \gamma_m = \gamma$, and since $\sigma < 1$, (1.24), Lemma 7.1 and (7.5) give that

$$\sum_{i\notin \mathcal{U}} \int_{\gamma\cap B_{\sigma/2}(z_i)} \left| \left(\frac{K'_m}{K_m}\right)^2 - \left(\frac{K'_{m+1}}{K_{m+1}}\right)^2 \right| |dz| \leq \varkappa(A)\, e^{2\sigma} \sum_{i\in\mathcal{U}} \frac{p_i^2}{\sigma^4\varrho}(\sigma\varrho^{1/2}) \leq \varkappa(A)\,(\sigma^3\varrho^{1/2})^{-1} \sum (p_i)^2$$
$$\leq \varkappa(A)\,(\sigma^3\varrho^{1/2})^{-1}\gamma^8 T^2(\varrho) = \varkappa(A)\,\gamma^7\varrho^{-1/2}T^2(\varrho), \tag{7.8}$$

where $\varkappa$ is a function of polynomial growth.

We now control the contributions from $\gamma_j \cap B_\sigma(B_i)$ with $i \in \mathcal{U}$.

Consider the rectangles $B_\sigma(z_i^2)$, where $z_i^2 \in \mathfrak{A}_\sigma^*$(cf. (4.2)). Then two modifications of f in $\Delta(z_i^2, \frac{1}{3}\sigma)$ are $L(w)$ and $(T(\varrho))^{-1}\log K(w^{1/2})$ $(w \in \Delta(z_i^2, \frac{1}{3}\sigma))$ where L is given by (4.44), and K by (5.31) (it is now possible to take a logarithm since the Δ's are simply-connected). Thus we are in the precise situation (4.54) with (4.55), and obtain from Lemma 4.7 and (6.24) that $(T(\varrho))^{-1}\log K(w^{1/2}) \equiv \pm L(\Phi(w))$, $w \in \Delta(z_i^2, \frac{1}{3}\sigma)$, where Φ is rigid as in (4.57) and (6.33); i.e. that

$$\frac{K'}{K}(z) = \pm 2zL'(\Phi(z^2))\,\Phi'(z^2)\,T(\varrho) \quad (z \in \Delta(z_i, \tfrac{1}{3}\sigma)). \tag{7.9}$$

The factor Φ' is well-controlled by (6.33), and (4.48) gives a trivial upper bound for L', which is *not* adequate for our purposes. What saves us is that we are able to apply (4.89) with $\frac{1}{2}<\tau_0<1$. In order to apply (4.89), it is necessary to check that (4.88) is satisfied, where $z_0=\varphi(z_i)^2$ and hence some history must be reviewed. The $B_\sigma(z_i^2)\,(i\in\mathcal{U})$ are contained in the rectangles $B^\#$ of (5.12) which have been removed as in (5.14). According to (5.11), (5.13) and (5.30), the noneuclidean distance from z_i^2 to B^* is comparable to σ. Finally, our construction (3.21), (3.22) of $B^\#$ implies that each B of B^* meets $\bigcup(\partial\mathcal{D}_i)$. Thus (5.30) implies that

$$d(\varphi(z_i)^2, (\bigcup\partial\mathcal{D}_i)) \leq C\gamma^5 + d(z_i^2, (\bigcup\partial\mathcal{D}_i)) \leq C\sigma$$

and so we have from (4.91) and Cauchy's estimates that

$$|L'(z^2)| \leq C\sigma^{\tau_0}T(\varrho)\,(\sigma|z_i^2|)^{-1} \quad (z \in B_{\frac{1}{3}\sigma}(z_i)).$$

This information is used in (7.9), and we conclude using (6.33) that

$$\begin{aligned}\int_{B_{\frac{1}{6}\sigma}(z_i)} \left|\frac{K'}{K}\right| dr\,d\theta &\leq CT(\varrho)\int_{B_{\frac{1}{3}\sigma}(z_i)} |zL'(z^2)|\,dr\,d\theta \\ &\leq C|z_i|\,\sigma^{\tau_0}T(\varrho)\,(\sigma|z_i^2|)^{-1}\,\sigma^2|z_i| \\ &\leq C\sigma^{1+\tau_0}T(\varrho).\end{aligned}$$

According to Lemma 7.1, this gives the uniform bound

$$\left|\frac{K'}{K}\right|^2 \leqslant \frac{\sigma^{2+2\tau_0}T^2(\varrho)}{\sigma^4|z_i|^2} \leqslant \varkappa(A)\sigma^{2\tau_0-2}\varrho^{-1}T^2(\varrho) \quad (z\in B_{\frac{1}{10}\sigma}(z_i)).$$

Finally, on summing over the $Q(\lambda)$ rectangles $B_\sigma(z_i)$ of $\mathscr{U}$ and recalling (7.5), we deduce that

$$\int_{\gamma_m-\gamma}\left|\frac{K_m'}{K_m}\right|^2|dz| \leqslant C\sigma^{2\tau_0-1}\varrho^{-1/2}T^2(\varrho); \tag{7.10}$$

of course, (7.10) also holds when m is replaced by $(m+1)$. We choose τ_0 in (4.98) so that $2\tau_0-1>\tau_1$, where τ_1 appears on the right side of (7.3).

In order to obtain (7.3) from (7.8) and (7.10), we consider the integrand in (7.7) and estimate $|z|^{-(l+1)}$ on the contour by: $|z|^{-(l+1)}\leqslant 2^{|l|+1}s_m^{-(l+1)}A$, and use (1.24) to absorb the factor $\varkappa(A)$ into σ^{τ_1}.

7.3. *Analytic continuation of the Laurent expansions.* There are only $C=Q(\lambda)$ modified subannuli $\mathscr{A}_m^*$, so if $A=A_n$ in (1.10) is sufficiently large, we may find $y^\#, y^\circ$, and y' with

$$\begin{gathered} A^{1/2}\varrho<(y')^2<\tfrac{1}{16}A\varrho \\ 4\varrho<(y^\circ)^2<100C\varrho \\ 16A^{-1}\varrho<(y^\#)^2<A^{-1/2}\varrho, \end{gathered} \tag{7.11}$$

such that

$$\mathscr{A}'=\left\{\tfrac14 y'\leqslant|z|\leqslant 4y'\right\}\subset\mathscr{A}_{j'}^* \tag{7.12}$$

$$\mathscr{A}^\circ=\left\{\tfrac14 y^\circ\leqslant|z|\leqslant 4y^\circ\right\}\subset\mathscr{A}_{j^\circ}^* \tag{7.13}$$

and

$$\mathscr{A}^\#=\left\{\tfrac14 y^\#\leqslant|z|\leqslant 4y^\#\right\}\subset\mathscr{A}_{j^\#}^* \tag{7.14}$$

for some j', j° and $j^\#$ (perhaps not all distinct).

We first record an $O(1)$ form of Lemma 7.2.

LEMMA 7.3. *The Laurent coefficients $a_l(j'), a_l(j^\circ)$ and $a_l(j^\#)$ satisfy for any fixed $\varepsilon>0$*

$$|a_l(j')| \leq C \frac{T^2(\varrho)}{\varrho} \left(\frac{(y')^2}{\varrho}\right)^{2\lambda+\varepsilon-1} 3^{-|l|}(y')^{-l} \tag{7.15}$$

$$|a_l(j^\circ)| \leq C \frac{T^2(\varrho)}{\varrho} 3^{-|l|} \varrho^{-l/2} \tag{7.16}$$

and

$$|a_l(j^\#)| \leq C \frac{T^2(\varrho)}{\varrho} \left(\frac{(y^\#)^2}{\varrho}\right)^{2\lambda-\varepsilon-1} 3^{-|l|}(y^\#)^{-l}. \tag{7.17}$$

Proof. According to (1.8), (6.7), (6.22), (6.23), (6.12) and (7.11)–(7.14),

$$\begin{aligned}\int_{\mathscr{A}'} \left(\left|\frac{K'_{j'}}{K_{j'}}\right|^2\right)^{1/2} dr\,d\theta &\equiv \int_{\mathscr{A}'} \left|\frac{K'_{j'}}{K_{j'}}\right| dr\,d\theta \leq \int_{\mathscr{A}'} \left|\frac{G'}{G}\right| dr\,d\theta + \sum \int_{\mathscr{A}'} \frac{1}{|z-c|}\,dr\,d\theta \\ &\leq CT((6y')^2) + CA^2(A')^{-2h}T(\varrho) \leq C\left(\frac{(y')^2}{\varrho}\right)^{\lambda+\varepsilon/2} T(\varrho),\end{aligned} \tag{7.18}$$

$$\int_{\mathscr{A}^\circ} \left(\left|\frac{K'_{j^\circ}}{K_{j^\circ}}\right|^2\right)^{1/2} dr\,d\theta \leq CT(\varrho)$$

and

$$\begin{aligned}\int_{\mathscr{A}^\#} \left(\left|\frac{K'_{j^\#}}{K_{j^\#}}\right|^2\right)^{1/2} dr\,d\theta &\leq CT((6y^\#)^2) + C(A')^{-2h}\, T(\varrho) \\ &\leq C\left(\frac{(y^\#)^2}{\varrho}\right)^{\lambda-\varepsilon/2} T(\varrho).\end{aligned}$$

These inequalities with Cauchy's estimates, (7.7) and (7.2) give (7.15)–(7.17).

Lemma 7.3 may be used to estimate the terms of the Laurent series in (6.21) when $|l|$ is large. Thus (7.11), (7.15) and the Pólya peak inequality (1.8) (or the weak form (1.11)) show that if $t<y'$, then

$$\sum_L^\infty |a_l(j')|\, t^l \equiv \sum_L^\infty |a_l(j')|\,(y')^l \left(\frac{t}{y'}\right)^l \leq C\left(\frac{(y')^2}{\varrho}\right)^{2\lambda+\varepsilon-1} \left(\frac{(y')^2}{t^2}\right)^{-\frac{1}{2}L} \frac{T^2(\varrho)}{\varrho}, \tag{7.19}$$

where, if we use (1.11) instead of (1.8), it is possible that $C=C_\varepsilon$.

Since we may pass from $\mathscr{A}'$ to $\mathscr{A}^\circ$ or $\mathscr{A}^\#$ (cf. (7.12)–(7.14)) by passing through at most $Q(\lambda)$ of the $\mathscr{A}^*_j$, we use (7.19), (7.3) and (7.11) to obtain for example that if $|L-4\lambda|<5$ then

$$\sum_L^\infty |a_l(j^\circ)^l|(y^\circ)^l \leq \sum_L^\infty |a_l(j')|(y^\circ)^l + \sum_L^\infty \sum_m{}^* |a_l(j_{m+1}) - a_l(j_m)|(y^\circ)^l$$

$$\leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y')^2}{\varrho}\right)^{2\lambda+\varepsilon-1}\left(\left(\frac{y'}{y^\circ}\right)^2\right)^{-\frac{1}{2}L} + \sigma^{\tau_1}\sum_m{}^*\left\{2^{|l|}\left(\frac{y^\circ}{s_m}\right)^l\right\}\right] \tag{7.20}$$

$$\leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y')^2}{\varrho}\right)^{2\lambda+\varepsilon-1-\frac{1}{2}L} + \sigma^{\tau_1}\sum_m{}^*\left\{\sum_L^\infty 2^{|l|}\left(\frac{y^\circ}{s_m}\right)^l\right\}\right]$$

where Σ^* refers to a sum over $Q(\lambda)$ indices m which correspond to those s_m with $3y^\circ \leq s_m \leq (A\varrho)^{1/2}$ (note (7.13), which implies that $\mathscr{A}^\circ$ is well disjoint from the circles $\{|z|=s_m\}$). Thus, if $2\lambda+\varepsilon-1-\frac{1}{2}L\leq 0$, (7.11) and (7.20) yield that

$$\sum_L^\infty |a_l(j^\circ)|(y^\circ)^l \leq C\frac{T^2(\varrho)}{\varrho}\left[(A^{1/2})^{2\lambda+\varepsilon-1-\frac{1}{2}L} + C\sigma^{\tau_1}\right], \tag{7.21}$$

and similarly

$$\sum_L^\infty |a_l(j^\#)|(y^\circ)^l \leq C\frac{T^2(\varrho)}{\varrho}\left[(A^{1/2})^{2\lambda+\varepsilon-1-\frac{1}{2}L} + C\sigma^{\tau_1}\right]. \tag{7.22}$$

This argument is symmetric in the sense that we may start with the $a_l(j^\#)$ and work to the $a_l(j')$. Thus, start with (7.17) in place of (7.15), use (1.8) (or (1.11)), (1.24) and (7.11). Then, if $t>y^\#$ we have

$$\sum_{-\infty}^M |a_l(j^\#)|t^l \leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y^\#)^2}{\varrho}\right)^{2\lambda-\varepsilon-1}\sum^M\left(\frac{t}{3y^\#}\right)^l\right]$$

$$\leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y^\#)^2}{\varrho}\right)^{2\lambda-\varepsilon-1}\left(\left(\frac{y^\#}{t}\right)^2\right)^{-\frac{1}{2}M}\right],$$

Thus if $2\lambda-\varepsilon-1-\frac{1}{2}M\geq 0$ but $|M-4\lambda|<5$, we deduce from (7.3) and (7.11) that

$$\sum_{-\infty}^M |a_l(j^\circ)|(y^\circ)^l \leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y^\#)^2}{\varrho}\right)^{2\lambda-\varepsilon-1}\left(\frac{(y^\#)^2}{\varrho}\right)^{-\frac{1}{2}M} + \sigma^{\tau_1}\right]$$

$$\leq C\frac{T^2(\varrho)}{\varrho}\left[(A^{-1/2})^{2\lambda-\varepsilon-1-\frac{1}{2}M} + \sigma^{\tau_1}\right], \tag{7.23}$$

$$\sum_{-\infty}^M |a_l(j')|(y')^l \leq C\frac{T^2(\varrho)}{\varrho}\left[\left(\frac{(y^\#)^2}{\varrho}\right)^{2\lambda-\varepsilon-1}\left(\left(\frac{y^\#}{y'}\right)^2\right)^{-\frac{1}{2}M} + \sigma^{\tau_1}\right].$$

7.4. *Proof that* 2λ *is an integer.* In this section we obtain the first part of the theorem. We first need a result which is a modification of Fuchs's lemma 3.2 of [16]. We consider K'/K (where $K=K_{f^\circ}$) in the annulus $\mathscr{A}^\circ$ of (7.13), which is close to $\{|z|=\varrho^{1/2}\}$. For $c>0$, let $c\mathscr{A}^\circ=\{(4c)^{-1}y^\circ<|z|<4cy^\circ\}$, with similar definitions of $c\mathscr{A}'$ and $c\mathscr{A}^{\#}$ (these annuli were introduced in (7.12) and (7.14)).

LEMMA 7.4. *There is a constant c, which depends only on f, when n in* (1.8) *is large, such that*

$$\int_{\frac{1}{2}\mathscr{A}^\circ}\left|\frac{K'}{K}\right|^2 dr\,d\theta \geqslant c \geqslant c\varrho^{-1/2}T^2(\varrho). \tag{7.24}$$

Proof. Inequality (7.24) is a consequence of Schwarz's inequality and

$$\int_{\frac{1}{2}\mathscr{A}^\circ}\left|\frac{K'}{K}\right| dr\,d\theta \geqslant cT(\varrho), \tag{7.25}$$

since $\int_{\frac{1}{2}\mathscr{A}^\circ} dr\,d\theta \geqslant c\varrho$. We recall G from (5.28) and (6.22). According to (5.28) and formulae such as (4.6)–(4.8), each circle $|z|=r$ must have a point $re^{i\theta(r)}$ such that $|\log|G(re^{i\theta(r)})||>cT(r^2)>cT(\varrho)$. Also, since each circle $|z|=r$ must meet $\mathscr{D}_i$ for at least two indices i, we may find $\theta'(r)$ with $|\log|G(re^{i\theta'(r)})||=|\log\varepsilon|$. Choose r such that

$$\varrho^{1/2}<r<100C\varrho^{1/2}, \tag{7.26}$$

where $C=Q(\lambda)$ is the number of modified annuli $\mathscr{A}$. Then we have that

$$cT(r^2)<\int_0^{2\pi}\left|\frac{G'}{G}\right| r\,d\theta, \tag{7.27}$$

so, since T is increasing and (7.26) holds,

$$cT(\varrho)<\int_{\frac{1}{2}\mathscr{A}^\circ}\left|\frac{G'}{G}\right| dr\,d\theta.$$

This, (6.7) (with $A=4$), (6.22), (6.23) and (7.26) give (7.25), for

$$\begin{aligned}\int\left|\frac{K'}{K}\right| dr\,d\theta &\geqslant \int\left|\frac{G'}{G}\right| dr\,d\theta-\sum\int\left|\frac{1}{z-c}\right| dr\,d\theta\\ &\geqslant cT(r)-C(A')^{-2h}\,T(\varrho)>cT(\varrho).\end{aligned}$$

This proves the lemma.

We now apply (7.24) to prove assertion (1.4):

LEMMA 7.5. *Under the hypotheses* (1.2) *and* (1.3), *it follows that*

$$2\lambda \textit{ is a positive integer.} \tag{7.28}$$

Proof. The first goal is to show that

$$4\lambda \text{ is a positive integer,} \tag{7.29}$$

by using (7.22) and (7.23).

Choose integers M and L and $\varepsilon_0>0$ so that

$$M \leq 2(2\lambda-1-2\varepsilon_0) \leq 2(2\lambda-1+2\varepsilon_0) \leq L. \tag{7.30}$$

If $\Lambda=4\lambda-2$ is an integer, take $M=\Lambda-1, L=\Lambda+1$; otherwise, choose ε_0 so small that (7.30) is possible with $L=M+1$. Then $|L-4\lambda|<5$ and $|M-4\lambda|<5$, so (7.22) and (7.23) hold in $\mathscr{A}$, and thus $(K=K_{\rho})$.

$$\left|\left(\frac{K'}{K}\right)^2-\alpha_\Lambda z^\Lambda\right| \equiv |S(z)| \leq C\sigma^{\tau_1}\frac{T^2(\varrho)}{\varrho} \quad (z\in\tfrac{1}{2}\mathscr{A}) \tag{7.31}$$

where we take $\alpha_\Lambda=0$ if Λ is not an integer. Thus if $\alpha_\Lambda=0$ in (7.31), we would have from (7.24) and (7.31) that $(K=K_{\rho})$

$$C\varrho^{-1/2}T^2(\varrho) \leq \int_{\frac{1}{2}\mathscr{A}} \left|\frac{K'}{K}\right|^2 dr\,d\theta \leq C\sigma^{\tau_1}\varrho^{-1/2}T^2(\varrho),$$

which is a contradiction. This proves (7.29), and, moreover, α_Λ cannot be too small:

$$|\alpha_\Lambda| \geq cT^2(\varrho)\,\varrho^{-1-\frac{1}{2}\Lambda}. \tag{7.32}$$

Now we recall from (6.21) that the α's are coefficients of $(K'/K)^2$, where K'/K is single-valued in $\mathscr{A}$. It follows from (7.31) and (7.32) that (K'/K) is a non-vanishing in $\frac{1}{2}\mathscr{A}$. Choose the branch of square-root of K'/K so that

$$|K'/K-\alpha_\Lambda^{1/2}r^{\frac{1}{2}\Lambda}| \leq C\sigma^{\frac{1}{2}\tau_1}T(\varrho)\,\varrho^{-1/2}$$

in the intersection of $\frac{1}{2}\mathscr{A}$ with the positive axis and continue this branch through $\mathscr{A}$ to the negative axis in the positive and negative directions of rotation. If $\frac{1}{2}\Lambda$ is not an integer, then (7.31) and (7.32) yield that

$$\left|\left(\frac{K'}{K}\right)(re^{i\pi+})-\left(\frac{K'}{K}\right)(re^{i\pi-})\right| \geqslant 2\left|\alpha_\Lambda^{1/2}\right| r^{\frac{1}{2}\Lambda}-C\sigma^{\tau_1}T(\varrho)\,\varrho^{-1/2}$$
$$\geqslant cT(\varrho)\,\varrho^{-1/2},$$

which contradicts the fact that K'/K is single-valued. This proves the lemma.

COROLLARY. *Let $f(z)$ satisfy* (1.2) *and* (1.3). *Then there is a fixed integer $N\geqslant 2$ such that all Pólya peaks of $T(r,f)$ have order $\lambda=\frac{1}{2}N$. In particular,* (1.11) *holds for all large ϱ and r.*

Proof. Lemmas 7.4 and 7.5 show that all Pólya peaks must have order $\frac{1}{2}m$ with m an integer. But $\mathscr{P}=\{\lambda; T(r)$ has Pólya peaks of order$\}$ is connected [7], and hence some $m=N$ works for all peaks.

Since Weitsman [26] has shown that (1.2) and (1.3) imply that λ in (1.8) must always be at least one, it follows that $N\geqslant 2$.

7.5. *Growth of the functions G.* We apply the last results, in particular the corollary, to analyse the $G_m(z)$ which were constructed in Chapter 5 (cf. (5.28)). The next result is much like Lemma 4.4.

LEMMA 7.6. *Let $r_1<\frac{1}{2}r_2$ be such that the annulus*

$$\mathscr{A}\left(\tfrac{1}{4}r_1, 4r_2\right) \equiv \left\{\tfrac{1}{4}r_1<|z|<4r_2\right\} \subset \mathscr{A}_m \tag{7.33}$$

for some $\mathscr{A}_m$ of (5.27). *Let $G=G_m$ be associated to f by* (5.28). *Then if $\frac{1}{2}r_1\leqslant t_1\leqslant t_2\leqslant 2r_2$, with*

$$t_2>(1+\sigma)\,t_1 \tag{7.34}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{t_1}^{t_2} |\log|G(re^{i\theta})||\, r^{-1}\,dr\,d\theta \leqslant C\log\frac{t_2}{t_1}\,T((2t_2)^2). \tag{7.35}$$

Proof. Assumptions (5.30) and (7.34) guarantee that the image of the annulus $\{t_1<|z|<t_2\}$ under φ is contained in $\{(1-\sigma)\,t_1<|z|<(1+\sigma)\,t_2\}$. Let $w=\varphi(z)^2=se^{it}$, so that $z=\psi(w^{1/2})$, let $W=w^{1/2}$ and $s=|w|$. Then (5.28), (4.24), (5.29), and (1.24) give that

$$\int_0^{2\pi}\int_{t_1}^{t_2}|\log|G(re^{i\theta})||\,r^{-1}\,dr\,d\theta=\int_0^{2\pi}\int_{t_1}^{t_2}|\log|\omega(f(\varphi(z)^2))||\,r^{-2}r\,dr\,d\theta$$
$$\leq C\int_0^{2\pi}\int_{((1-\sigma)t_1)^2}^{((1+\sigma)t_2)^2}|\log|\omega(f(w))||\left(\frac{s}{|\psi|}\right)^2|\psi_w|^2s^{-1}\,ds\,dt$$
$$\leq C\int_0^{2\pi}\int_{((1-\sigma)t_1)^2}^{((1+\sigma)t_2)^2}|\log|\omega(f(w))||\,|\psi_w|^2s^{-1}\,ds\,dt. \tag{7.36}$$

Let $((1-\sigma)t_1)^2\leq s\leq((1+\sigma)t_2)^2$. Now $\int_E\log^+|1/(f(se^{it})-a)|\,dt$ is small unless $a\in\{a_i\}_1^q$ and E is contained in the sets $\bigcup\mathcal{D}_i(m)$ of Lemma 5.2, and the construction of H in Lemma 5.2 has the precise effect that if $E\subset\mathcal{D}_i(m)\cap\{|z|=s\}$ and $E^{1/2}$ is one of the two preimages of E in $\mathcal{A}'_m$, then

$$\int_{E^{1/2}}|\log|H(re^{i\theta})||\,d\theta=\int_E|\log|f(se^{it})-a_i||\,dt. \tag{7.37}$$

Thus (7.35) follows from (7.36) in a routine way, using (1.11) (compare with (4.38)).

7.6. *Asymptotic expansions of the functions G.* There are only $Q(\lambda)$ subannuli $\mathcal{A}^\circ$ in § 5.3 (see (5.25)) and $\bigcup\mathcal{A}^\circ$ has logarithmic length $>c\log A$. Thus there are subannuli $\mathcal{A}^\circ$ whose logarithmic length is large. Such subannuli are not necessarily centered at $\varrho^{1/2}$ (where ϱ is the Pólya peak), but now Lemma 7.5 (and its corollary) show that (1.11) holds, and this is good enough.

Thus, let

$$A_1=\log A. \tag{7.38}$$

It follows that we may choose pairs s_1,t_1, and s_2,t_2 with

$$A^{-1/4}\varrho^{1/2}<s_1<(A_1)^{-3}\varrho^{1/2},\quad A_1^3\varrho^{1/2}<s_2<A^{1/4}\varrho^{1/2},\quad t_i=A_1^2s_i, \tag{7.39}$$

and such that each annulus $\{(A_1)^{-2}s_i\leq r\leq A_1^2t_i\}$ $(i=1,2)$ is contained in one of the annuli $\mathcal{A}_m$ of (5.27). We choose the "center" u_i by

$$u_i^2=s_it_i\quad(i=1,2). \tag{7.40}$$

Choose $\gamma>0$ so that

$$2\lambda+\frac{1}{4}-(1-\gamma)(2\lambda+1)<-\tfrac{1}{2}; \tag{7.41}$$

this is equivalent if $\gamma<\frac{1}{4}(2\lambda+1)^{-1}$. We then define

$$\mathscr{B}_i=\{A_1^{-\gamma}u_i<|z|<A_1^{\gamma}u_i\}\quad(i=1,2) \tag{7.42}$$

and

$$\mathscr{C}_i=\{A_1^{-2}s_i\leq|z|\leq A_1^2t_i\}\quad(i=1,2). \tag{7.43}$$

Thus $\mathscr{B}_i$ is nested well inside $\mathscr{C}_i$.

We recall the definition (5.31) of $K=K_i=G_iP_i(i=1,2)$ which is valid in each $\mathscr{C}_i$. Each function K is holomorphic and zero-free in its $\mathscr{C}$ and has a "Laurent" expansion.

$$\log K(z)=\sum_{-\infty}^{\infty}\gamma_l z^l+\gamma^*\log\left(\frac{z}{u}\right)\quad(z\in\mathscr{C}_i); \tag{7.44}$$

since Re $\{\log K\}$ is single-valued, γ^* is real. We use weak estimates of these coefficients γ to get good asymptotic formulae in the $\mathscr{B}_i$,

LEMMA 7.7. *In each expansion* (7.44) (i=1 *or* 2) *we have* $(u=u_i)$

$$|\gamma^*|<(A_1)^{-2\lambda-1/4}T(u^2), \tag{7.45}$$

and

$$\left|\log K-\gamma^*\log\left(\frac{r}{u}\right)-\gamma_{2\lambda}z^{2\lambda}\right|<CA_1^{-1/2}T(u^2)\quad(z\in\mathscr{B}_i), \tag{7.46}$$

while

$$|\gamma_{2\lambda}|>cu^{-2\lambda}T(u^2). \tag{7.47}$$

Proof. We first show (7.45). Now since $K\neq0,\infty$ in $\mathscr{C}_i$, we have that

$$\gamma\equiv\frac{1}{2\pi i}\int_{|z|=r}\frac{K'}{K}(z)\,dz\quad(A_1^{-2}s_i<r\leq A_1^2t_i),$$

so that, in the notation (6.3), $|\gamma|\leq C\|K'/K\|^*_{B,1}$, where B is any annulus $\{s_0<|z|<2s_0\}$ which is contained in C. By taking s_0 close to $A_1^{-2}s_i$, we deduce from (1.11), (6.22), (6.23), (7.39), (7.40), (6.7) and (6.12) that

$$|\gamma|\leq C\left\{\left\|\frac{G'}{G}\right\|^*_{B,1}+\sum\|(z-c)^{-1}\|^*_{B,1}\right\}\leq CT((4A_1^{-2}S_i)^2)+C(A')^{-h/2}T(\varrho)$$

$$= CT((4A_1^{-3}u)^2)+C(A')^{-\frac{1}{2}h}T(\varrho)$$
$$= C(A_1^{-3})^{2\lambda-\varepsilon}T(u^2),$$

and this implies (7.45).

The next goal is to bound $|\mathrm{Re}\,|K||$ locally; this is much like the situation in Lemma 4.5, now that we know (7.45). In terms of the local coordinates in § 4.4 and the definitions (4.1), we find that if $\frac{1}{4}A_1^{-1}s_i \leq |z_0| \leq 4A_1 t_i$, then

$$\log|K(z_0)| = \int_{C(z_0,t)} \log|K(z)|\,d\varphi.$$

so that (5.31) and (4.49) give

$$\begin{aligned}|\log|K(z_0)|| &\leq C\int_{\Delta(z_0,\frac{1}{2},\frac{2}{3})} |\log|K(z)||\,d\varphi\, t^{-1}\,dt \\ &\leq C\int_{\Delta(z_0,\frac{1}{2},\frac{1}{3})} |\log|P||\,t^{-1}\,dt\,d\varphi + C\int_{\Delta(z_0,\frac{1}{2},\frac{1}{3})} |\log|G||\,t^{-1}\,dt\,d\varphi \\ &\leq C(A')^{-h}T(\varrho) + C\int_{\Delta(z_0,\frac{1}{2},\frac{1}{3})} |\log|G||\,t^{-1}\,dt\,d\varphi.\end{aligned}$$

The local (t,φ) coordinate system is related to the standard system by $(z=re^{i\theta})$: $z-z_0=t|z_0|\,e^{i\varphi}$. In $\Delta(z_0,\frac{1}{2},\frac{2}{3})$, we have $\frac{1}{2}<t<\frac{2}{3}$ and $\frac{1}{2}r_0<r<2r_0\,(r_0=|z_0|)$. Thus

$$t^{-1}\,dt\,d\varphi = t^{-2}t\,dt\,d\varphi = (tr_0)^{-2}\,r\,dr\,d\theta \leq Cr^{-1}\,dr\,d\theta,$$

and we may rewrite our last step as

$$\begin{aligned}|\mathrm{Re}\log K(z_0)| = |\log|K(z_0)|| &\leq C(A')^{-h}T(\varrho) + C\int_0^{2\pi}\int_{cr_0}^{Cr_0} |\log|G||\,r^{-1}\,dr\,d\theta \\ &\leq CT(r_0^2) \quad (A_1^{-2}s_i \leq |z_0| \leq A_1^{-2}t_i),\end{aligned} \tag{7.49}$$

where we use (1.11), (1.24) and (7.35) to obtain the last line. According to the Borel-Carathéodory inequality, (7.49) implies that

$$|\log K(z) - \log K(z_0)| \leq CT(r_0^2) \quad (z \in \Delta(z_0,\tfrac{1}{2})) \tag{7.50}$$

so long as $\frac{1}{4}A_1^{-1}s_i \leq |z_0| \leq 4A_1 t_i$.

We use (7.50) and (1.11) to prove that

$$\left|\log K(z)-\gamma^*\log\frac{r}{u}\right|\leqslant CA_1^{-2\lambda+1/4}T(u_i^2)\quad (|z|=\tfrac{1}{4}s_i,\, i=1,2) \tag{7.51}$$

and

$$\left|\log K(z)-\gamma^*\log\frac{r}{u}\right|\leqslant CA_1^{2\lambda+1/4}T(u_i^2)\quad (|z|=4t_i,\, i=1,2). \tag{7.52}$$

Because of (7.45), it suffices to prove (7.51) and (7.52) in the slit annuli $\{\frac{1}{4}s_i\leqslant|z|\leqslant 4t_i,\, |\arg z|<\pi\}$, and so we show that

$$|\log K(z)|\leqslant CA_1^{-2\lambda+1/4}T(u_i^2)\quad (|z|=\tfrac{1}{4}s_i,\, \arg z|<\pi), \tag{7.53}$$

$$|\log K(z)|\leqslant CA_1^{2\lambda+1/4}T(u_i^2)\quad (|z|=4t_i,\, |\arg z|<\pi). \tag{7.54}$$

Choose a point w_0 on $|z|=\frac{1}{4}s_i$, $|\arg w_0|<\pi$, and choose $|\arg K(w_0)|\leqslant\pi$. The circle $\{|z|=4s_i\}$ may be covered by at most C discs $\Delta(w_i,\frac{1}{4})\,(1\leqslant i\leqslant C)$ where $w_{i+1}\in\Delta(w_i,\frac{1}{5})$, so we may start with (7.49) $\left[\text{with } r_0=\frac{1}{4}s_i\right]$ and obtain (7.53) by using (1.11) and (7.50) on each of these circles.

The proof of (7.54) is similar, once we know that our branch of $\log K$ also satisfies

$$|\arg K(w_m)|\leqslant CA_1^{2\lambda+1/4}T(u_i^2) \tag{7.55}$$

for some w_m with $|w_m|=4t_i$. We obtain (7.55) from (7.50) and (1.11), by moving from w_0 to w_m by $m\leqslant C\log(t_i/s_i)$ discs $\Delta=\Delta(w_i,\frac{1}{4})$, and to do this efficiently, we arrange this so that $|w_i|\geqslant(11/10)|w_{i-1}|$, $i\geqslant C$. In this chain of discs, suppose that $w_h,\dots,w_m$ satisfy $|w_j|\geqslant u_i$. Then (1.11), (7.39) and (7.50) show that

$$\begin{aligned}
|\arg K(w_m)-\arg K(w_h)| &\leqslant \sum_h^{m-1}|\arg K(w_{j+1})-\arg K(w_j)|\\
&\leqslant C\sum_h^{m-1}T(|w_j|^2)\leqslant CT(u_i^2)\sum_h^{m}\left(\frac{w_j}{u_i}\right)^{2\lambda+1/4}\\
&= CT(u_i^2)\left(\frac{w_m}{u_i}\right)^{2\lambda+1/4}\sum_h^{m-1}\left(\frac{w_j}{w_m}\right)^{2\lambda+1/4}\\
&\leqslant C\left(\frac{w_m}{u_i}\right)^{2\lambda+1/4}T(u_i^2)\\
&\leqslant CA_1^{2\lambda+1/4}T(u_i^2),
\end{aligned}$$

and we control the contribution from the $|w_i|<u$ in an analogous manner. This proves (7.53) and (7.54), and thus (7.51) and (7.52).

We now observe from (7.41), (7.45), (7.52) and Cauchy's estimates that ($|z|=r$)

$$\begin{aligned}\left|\sum_{2\lambda+1}^{\infty}\gamma_l z^l\right| &\leqslant CA_1^{2\lambda+1/4}\,T(u_i^2)\sum_{2\lambda+1}^{\infty}\left(\frac{r}{A_1 u_i}\right)^l\\ &\leqslant CA_1^{2\lambda+1/4}\,T(u_i^2)\sum_{2\lambda+1}^{\infty}(A_1^{\gamma-1})^l\\ &\leqslant CA_1^{2\lambda+1/4-(1-\gamma)(2\lambda+1)}\,T(u_i^2)\\ &\leqslant CA_1^{-1/2}\,T(u_i^2)\quad (A_1^{-\gamma}u_i<r<A_1^{\gamma}u_i)\end{aligned}$$

and, similarly, using (7.51) with (7.41) and (7.45), we have that

$$\left|\sum_{-\infty}^{2\lambda-1}\gamma_l z^l\right| \leqslant CA_1^{-1/2}\,T(u_i^2)\quad (A_1^{-\gamma}u_i<r<A_1^{\gamma}u_i).$$

These estimates give (7.46).

Finally, $\log K$ is holomorphic in $\Delta(z_0,t)$ if $t<\frac{1}{2}$ so that when $\frac{1}{2}u_i\leqslant|z|=r\leqslant 2u_i$, we have that

$$\left|\frac{K'}{K}(z)\right| \leqslant \frac{1}{2\pi}\left|\int_{C(z,t)}\frac{\log K(\zeta)}{(\zeta-z)^2}\,d\zeta\right|$$

Thus (7.44)–(7.46) give that

$$\left|\frac{K'}{K}(z)\right| \leqslant 2\lambda|\gamma_{2\lambda}|\,r^{2\lambda-1}+CT(u_i^2)\,r^{-1}[A_1^{-1/2}+A_1^{-2\lambda-1/4}].$$

However, (7.27), (6.7) (with $A=4$), (6.22) and (6.23) now show that

$$\begin{aligned}cT(u_i^2) &\leqslant \int_{\frac{1}{2}u_i}^{2u_i}\int_0^{2\pi}\left|\frac{G'}{G}\right| r\,d\theta\,\frac{dr}{r}\\ &\leqslant \int\int\left|\frac{K'}{K}\right| dr\,d\theta+\sum^{N}\int\int\left|\frac{1}{z-c}\right| dr\,d\theta\\ &\leqslant C|\gamma_{2\lambda}|\,u_i^{2\lambda}+CA_1^{-1/2}\,T(u_i^2);\end{aligned}$$

since $A_1\to\infty$, this is a contradiction unless (7.47) holds.

COROLLARY. *For* $i=1,2$, *define* $\mathcal{B}_i$ *as in* (7.42). *Then each* G_i ($i=1,2$) *satisfies*

$$\log G_i(z) = \gamma_{2\lambda} z^{2\lambda} - \sum^{N} \pm \log\left(1-\frac{z}{c}\right) + S, \quad (z \in \mathcal{B}_i) \tag{7.56}$$

where the c's are the zeros and poles of G_i *in* $\mathcal{B}_i$ (*so that* $N \leq C(A')^{-2h} T(\varrho)$) *and*

$$|S(z)| \leq CA_1^{-1/2} T(r_0)^2 \quad (\tfrac{1}{2}r_0 < |z| < 2r_0). \tag{7.57}$$

The coefficient $\gamma_{2\lambda}$ *satisfies* (7.47).

Proof. This is just rewriting (7.44) using (7.45)–(7.47).

7.7. *Behaviour of f near the Pólya peak.* Recall the $\mathcal{B}_i$ ($i=1,2$) from (7.42). Let $\mathcal{B}$ be one of these annuli. Then (7.47), (7.56), (7.37) and the explicit relation between $G=G_j$ and $f(z^2)$ (described above in § 5.3, centering on (5.28)), determine $f(z)$ in each $(\mathcal{B})^2$. Thus, each circle $|z|=r$ ($A_1^{-\gamma} u_i < r < (A_1^{-\gamma} u_i, i=1,2)$) divides into 4λ intervals I_j, in which $\operatorname{Re}(G(z))$ is alternately small and large. According to the formulae in § 4.2 (especially (4.6)–(4.8) where $\omega\{a_i, a_j\}=\{0,\infty\}$) the 4λ intervals on which $|\log|G(z)||$ is large correspond (after the map $w=z^2$) to 2λ intervals I_j on which f is near the various deficient values. Further, (1.24) with (7.47), (7.56) and (7.57) shows that $|\int_{I_j} \log|G(re^{i\theta})|\,d\theta|$ is [within error $CA_1^{-1/2} T(u_i^2)$] independent of j. In fact

$$\int_{I_j} \log|G(r e^{i\theta})| \sim (2\lambda)^{-1} T(r^2) \sim \frac{1}{2} \int_{2I_j} \log^+ \frac{1}{|f(r^2 e^{i\theta}) - a_j|}\, d\theta. \tag{7.58}$$

Thus, we may choose ψ_1 and ψ_2 and define $\theta_{j,1}$ (relative to $(\mathcal{B}_1)^2$) and $\theta_{j,2}$ (relative to $(\mathcal{B}_2)^2$) with

$$\theta_{j,1} = \psi_1 + \pi j \lambda^{-1}, \quad \theta_{j,2} = \psi_2 + \pi j \lambda^{-1}$$

($0 \leq j \leq 2\lambda - 1$). For each $\tau > 0$ let

$$I_{j,1}(\tau) = \left\{ |\theta - \theta_{j,1}| < \frac{\pi}{2\lambda} - \tau \right\}, \quad I_{j,2}(\tau) = \left\{ |\theta - \theta_{j,2}| < \frac{\pi}{2\lambda} - \tau \right\},$$

where $I_{j,1}, I_{j,2}$ are used in place of $I_{j,1}(0)$ and $I_{j,2}(0)$. We deduce that if τ is fixed (but small) and n in (1.8) is large, then each $I_{j,i}(\tau) \cap \mathcal{B}_i^2$ is contained in some $D \in \bigcup \mathcal{D}_j$, where the $\bigcup \mathcal{D}_j$ are the significant components of f from Lemma 2.3, say $D = D_{j(i)}$.

According to (7.39), $\mathcal{B}_1^2$ is well inside the bounded component of the complement of $\mathcal{B}_2^2$. Let us correspond the integers k, $1 \leq k \leq 2\lambda$ to components $D = \Phi_1(k)$ of $\bigcup \mathcal{D}_i$ by:

$\Phi_1(k)=D$ if $I_{k,1}(\tau)\subset D$. Similarly, we may define $\Phi_2(k)$ by identifying the $D\in\bigcup\mathscr{D}_i$ with the integer k such that $I_{k,2}(\tau)\subset D$.

We call these D *strongly significant* in $\mathscr{B}_1^2$ or $\mathscr{B}_2^2$.

LEMMA 7.8. *Let $D\in\bigcup\mathscr{D}_i$ and suppose $\Phi_1(k_1)=\ldots=\Phi_1(k_r)=D$. Then there are r distinct integers k', $1\leq k'\leq 2\lambda$ with $\Phi_2(k')=D$.*

Proof. Suppose there were a component D^* of D in $\{A_1^{-2\gamma}u_1^2<|z|<A'\varrho\}$ such that

$$\partial D^*\subset\{|f-a_i|=\varepsilon\}\cup\{|z|=A_1^{-2\gamma}u_1^2\} \tag{7.59}$$

(thus D^* *does not meet* $\{|z|=A'\varrho\}$). Choose r^* with $A_1^{-2\gamma}u_1^2<r^*\leq 2A_1^{-2\gamma}u_1^2$ and such that [22, p. 25]

$$\sum_1^q\left|\log^*\frac{1}{|f(z)-a_i|}\right|\leq CT(2A_1^{-2\gamma}u_1^2)\quad(|z|=r^*). \tag{7.60}$$

Let $\Omega=\{r^*<|z|<A'\varrho\}\cap D^*$, and consider the estimate (2.32). We estimate the Green functions by (2.33) and the boundary integral by (7.60). Thus (1.11), (1.24) and (7.59) yield that

$$m(r,a_i,D^*)\leq C(A')^{-5h}T(\varrho)+CT(2A_1^{-2\gamma}u_1^2)\leq C(2A_1^{-2\gamma})^{(\lambda-\varepsilon)}T(u_1^2). \tag{7.61}$$

If we take r close to $A_1^{2\gamma}u_1^2$ and compare (7.61) with (7.58) and use (1.11) once more, we are led to the contradiction that

$$CA_1^{2\gamma(\lambda-\varepsilon)}T(u_1^2)\leq C\lambda^{-1}T(A_1^{2\gamma}u_1^2)\leq CA_1^{-2\gamma(\lambda-\varepsilon)}T(u_1^2). \tag{7.62}$$

Thus if $\Phi_1(k)=D$, then D may be joined to $\{|z|=A'\varrho\}$ from $\mathscr{B}_1^2\cap I_{k,1}$ without entering the other $I_{j,1}(\tau)$.

Our argument gives a little more. Not only does each component D^* of $D\cap\{|z|\geq A_1^{-2\gamma}u_1^2\}$ which is the range of Φ_1 meet $|z|=A'\varrho$ (and so pass through $\mathscr{B}_2^2$) but when it enters $\mathscr{B}_2^2$ it cannot be contained in the small sectors of $\mathscr{B}_2^2$ defined by $2\pi-\bigcup I_{j,2}(\tau)$. For if $D^*\cap\{|z|=r\}$ were contained in 2λ intervals of opening τ where $A_1^{-2\gamma}u_2^2<r<A_1^{2\gamma}u_2^2$ then the useful "small arcs" Lemma of Edrei-Fuchs [12, p. 322] and (1.11) would show that

$$\begin{aligned}m(r,a_i,D^*)&\leq C\tau\log\tau^{-1}T(A_2^{2\gamma}u_2^2)\\&\leq C\tau^{1/2}T(A\varrho)\leq C\tau^{1/2}A^{2(\lambda+\varepsilon)}T(A^{-1}\varrho)\\&\leq CA_1^{-(\lambda-\varepsilon)}T(u_1^2)\end{aligned} \tag{7.63}$$

so long as, using the notions of (1.24),

$$\tau_n < A_n^{-1} \tag{7.64}$$

(for example, we could choose $\tau_n = \sigma_n$), and we would reach a contradiction just as in our analysis of (7.62)).

Finally, suppose $\Phi_1(k_1) = \ldots = \Phi_1(k_p) = D$. We claim that p distinct components D^* of $D \cap \{|z| \geq A_1^{-\gamma} u_1\}$ meet $\mathscr{B}_2^2$. The simpler case is to assume that

$$k_l - k_{l'} \not\equiv 1 \pmod{2\lambda} \tag{7.65}$$

for any choice of l or l'. Then if our claim was false, then two D^* would have to merge before encountering $\mathscr{B}_2^2$. Hence some strongly significant component D^{**}, of $\bigcup \mathscr{D}_i \cap \{|z| \geq A_1^{-2\gamma} u_1^2\}$ in $\mathscr{B}_1^2$ is "trapped" by D^*, so that $D^{**} \cap \{A_1^{-2\gamma} u_1^2 < |z| < A'\varrho\}$ would have a component $D^{\#}$ such that (7.59) holds with $D^{\#}$ in place of D^*. Thus (7.58) and (7.61) would both hold for $D^{\#} \cap \mathscr{B}_1^2$, and this is a contradiction exactly as in (7.62).

It remains to show that (7.65) always is satisfied. Suppose (7.65) were false. Then for some k, k' with $k - k' \equiv 1 \pmod{2\lambda}$ we have $I_{k,1}(\tau) \subset D, I_{k',1}(\tau) \subset D$ for some $D \in \mathscr{D}_i$. Let Ω be the region $\mathscr{B}_1^2 \cap \{|\theta - \theta_0| \leq \tau\}$, where θ_0 is chosen so that $\bar{\Omega} \cap \bar{I}_{k,1}(\tau) \neq \emptyset$ and $\bar{\Omega} \cap \bar{I}_{k',1}(\tau) \neq \emptyset$. According to (5.28), (7.47), (7.56) and (7.57), we have ($l = k$ or k'):

$$\log \frac{1}{|f(z) - a_i|} = \gamma r^\lambda \cos \lambda(\theta - \theta_{l,1}) + S + \sum^{N} \pm \log \left| 1 - \frac{z}{a} \right| \quad (z \in I_{l,1}(\tau)), \tag{7.71}$$

where $N < C(A')^{-h} T(\varrho)$, $|S| \leq C A_1^{-1/2} T(\varrho)$ and $|\gamma| > c u^{-\lambda} T(u)$. Further, we may take γ to be real, since the $I_{l,1}$ are chosen so that the function G_i is small only near $\partial I_{l,1}$.

It is easy to see from (7.71) that the change in $\arg(f(z) - a_i)$ along $|\arg z - \theta_0| = \tau\,(-\tau)$ is negative (positive), and the contribution from $\{|\arg z - \theta_0| = \tau\} \cap \{r < |z| < 2r\}$ in absolute value is comparable to $T(r)$. Also, $m(r, a_i, \Omega)$ satisfies (7.63), and hence, by differentiating the proximity function (cf. (2.40)), we may ensure that

$$|\Delta \arg(f(z) - a_i)| \leq C A_1^{-(\lambda - \varepsilon)} T(u_1^2)$$

on $|z| = h$, with h nearly $A_1^{\pm 2\gamma} u_1^2$. Thus, by (1.24) and the argument principle we have

$$\begin{aligned} n(\infty, \Omega) - n(a_i, \Omega) &\geq C T(u_1^2) - C A_1^{-(\lambda - \varepsilon)} T(u_1^2) \\ &\geq C T(u_1^2), \end{aligned}$$

and similarly, $n(a_j,\Omega)-n(a_i,\Omega)\geq CT(u_1^2)$ for all $j\neq i$. Thus

$$\sum n(r,a_j,\Omega)\geq (q-1)\,S(r,\Omega)\quad (\tfrac{1}{2}A_1^{2\gamma}u_1^2\leq r\leq A_1^{2\gamma}u_1^2).$$

Since $(q-1)>(q-2)$, this and (2.14) contradict (1.3). Thus (7.65) cannot hold, and we have proved Lemma 7.8.

Remark. Lemma 7.8 and the representation (7.56) (subject to (7.57) and (7.47)) show that in the region $\{A_1^{-2\gamma}u_1^2<|z|<A_1^{2\gamma}u_2^2\}$, the only significant contribution to $m(r,a_i)$ come from components D of $\cup\,\mathcal{D}_i$ which are in the range of Φ_1 or Φ_2, since (7.64) holds. We summarize our results in

LEMMA 7.9. *There is a* $1-1$ *correspondence between strongly significant components in* $\mathcal{B}_1^2$ *and* $\mathcal{B}_2^2$, *so that the set* B^* *of* (3.16) *is disjoint from*

$$\mathcal{B}^*=\{A_1^{-3}\varrho_n\leq |z|\leq A_1^3\varrho_n\}\tag{7.71}$$

where, as in (7.38), $A_1=\log A$. *Thus* (7.56), (7.57) *and* (7.47) *hold for all* $z\in\mathcal{B}^*$. *In particular, the annulus* $\{\frac{1}{36}\varrho_n<|z|<36\varrho_n\}$ *is divided into* 2λ *disjoint sectors* $I_j=\{|\theta-\alpha_j|<\pi/2\lambda\}$ *such that*

$$\log\frac{1}{|f(re^{i\theta})-a_j|}=\left(\frac{r}{\varrho}\right)^{\lambda}\cos\lambda(\theta-\alpha_j)+R+\sum^{N}\pm\log\left|1-\frac{z}{a}\right|\quad (\theta\in I_j)\tag{7.72}$$

for some $a_j\in\{a_i\}_1^q$ *where* $N<C(A')^{-h}T(\varrho)$ *and* $|R|\leq CA_1^{-1/2}T(\varrho)$.

Let us check a few of these assertions. The bounds in (7.71) follow from (7.38) and (7.39). Next, consider the set B^* of (3.16), which is a union of rectangles from § 3.3. Since the strongly significant components link $\mathcal{B}_1^2$ to $\mathcal{B}_2^2$, it follows that $B^*\cap\mathcal{B}^*=\emptyset$. Thus in (7.39) we may take $s_1=A_1^{-1}\varrho^{1/2}$ and $t_1=A_1\varrho^{1/2}$, and observe that $\{A_1^{-2}s_1<r<A_1^2t_1\}$ is contained in some annulus $\mathcal{A}_m$ of (5.27). Since $\varrho=s_1t_1$ the representation (7.56) subject to (7.57) and (7.47) applies and gives (7.72) (cf. (7.71)).

7.8. *Proof of the theorem.* We have already shown (1.4), and (1.5) follows from the asymptotic expansions (7.58).

Finally, consider (1.6). Thus far, we have centered all attention on subannuli (1.10) of (1.9), where the $\{\varrho_n\}$ satisfy (1.8).

Now that (1.11) is known for all ε, we modify the original ϱ_n by

$$\varrho_n = 2^n, \tag{7.73}$$

and use (1.11) with $\varepsilon=\frac{1}{4}$. Then the developments of Chapters 2–5 apply to slowly expanding annuli centered at the ϱ_n, and for each n, a sequence $G_{m,n}$ may be constructed in the modified annuli $\mathcal{A}_{m,n}$ as in (5.25)–(5.29). Next, we obtain subannuli $\mathcal{B}_1(n)$, $\mathcal{B}_2(n)$ for n, as in (7.42), which lie well on opposite sides of $\{|z|=\varrho_n\}$. Lemma 7.4 (§ 7.4) is a global statement, and only it and (1.11) are used after § 7.4. Thus (7.56), subject to (7.57) and (7.47), (7.58) and Lemma 7.9 apply, and we deduce that (7.72) holds in each annulus

$$\{\tfrac{1}{36}\varrho_n < |z| < 36\varrho_n\}.$$

Since now ϱ_n satisfies (7.73), we see that these annuli overlap, and thus (1.6) follows at once. This completes the proof.

References

[1] AHLFORS, L. V., Zur Theorie der Überlagerungsflachen. *Acta Math.*, 64 (1935), 157–194.
[2] — *Lectures on quasiconformal mappings*. Van Nostrand, Princeton, 1967.
[3] BOAS, R. P., *Entire functions*. Academic Press, New York, 1954.
[4] DRASIN, D., Quasi-conformal modifications of functions having deficiency sum two. *Ann. of Math.*, 114 (1981), 493–518.
[5] — A note on functions with deficiency sum two. *Ann. Acad. Sci. Fenn. (A.I.)*, 2 (1976), 59—66.
[6] — The inverse problem of the Nevanlinna theory. *Acta Math.*, 138 (1977), 83—151.
[7] DRASIN, D. & SHEA, D. F., Pólya peaks and the oscillation of positive functions. *Proc. Amer. Math. Soc.*, 34 (1972), 403–411.
[8] DRASIN, D & WEITSMAN, A., Meromorphic functions with large sums of deficiencies. *Adv. in Math.*, 15 (1975), 93–126.
[9] DUREN, P. L., *Theory of H^p Spaces*, Academic Press, New York, 1970.
[10] EDREI, A. & FUCHS, W. H. J., On the growth of meromorphic functions with several deficient values. *Trans. Amer. Math. Soc.*, 93 (1959), 292–328.
[11] — Valeurs déficientes et valeurs asymptotiques des fonctions meromorphes. *Comment. Math. Helv.*, 33 (1959), 258–295.
[12] — Bounds for the number of deficient values of certain classes of meromorphic functions. *Proc. London Math. Soc. (3)*, 12 (1962), 315–344.
[13] FUCHS, W. H. J., A theorem on the Nevanlinna deficiencies of meromorphic functions of finite order. *Ann. of Math.*, 68 (1958), 203–209.
[14] — Proof of a conjecture of G. Pólya concerning gap series. *Illinois J. Math.*, 7 (1963), 661–667.
[15] — *Topics in the theory of functions of one complex variable*. Van Nostrand, Princeton, 1967.
[16] — *Topics in Nevanlinna theory*. Proc. NRL Conf. on Classical function theory. Naval Research Laboratory, Washington, 1970.
[17] GOLDBERG, A. A. & OSTROVSKII, I. V., *The Distribution of values of meromorphic functions* (in Russian). Nauka, Moscow, 1970.

[18] HAYMAN, W. K., *Meromorphic functions*. Oxford, 1964.
[19] HILLE, E., *Lectures on ordinary differential equations*. Addison Wesley, Reading (Mass.), 1969.
[20] MILES, J., A note on Ahlfors' theory covering surfaces. *Proc. Amer. Math. Soc.*, 21 (1969), 30–32.
[21] NEVANLINNA, F., Über eine Klass meromorpher Funktionen. *Comptes rendus de septième congrès des mathématiciens scandinaves tenu à Oslo 19–22 août 1929*. A. W. Brøggers boktrykkeri: A/S, Oslo, 1930.
[22] NEVANLINNA, R., *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthier Villars, Paris, 1929.
[23] — *Analytic functions*. Springer, New York, 1970.
[24] PFLUGER, A., Zur Defektrelation ganzer Funktionen endicher Ordnung. *Comment Math. Helv.*, 19 (1946), 91–104.
[25] RICKMAN, S., Solution to the problem of Picard's theorem for quasiregular mappings in dimension three. To appear.
[26] WEITSMAN, A., Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna. *Acta Math.*, 123 (1969), 115—139.
[27] — A growth property of the Nevanlinna characteristic. *Proc. Amer. Math. Soc.*, 26 (1970), 65–70.
[28] — A theorem on Nevanlinna deficiencies. *Acta Math.*, 128 (1972), 41–52.
[29] — *Some remarks on the spread of a Nevanlinna deficiency*. Report No. 7 (1977), Institut Mittag-Leffler, Djursholm (Sweden).
[30] WITTICH, H., *Neuere Untersuchungen über eindeutige analytische Funktionen*. Springer, Berlin, 1955.

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