SOME EXTREMAL PROBLEMS IN THE THEORY OF NUMERICAL RANGES

BY

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0. Introduction

Let $\mathfrak{A}$ be a complex unital Banach algebra with dual space $\mathfrak{A}'$. The *numerical range* of an element a of $\mathfrak{A}$, $V(\mathfrak{A}, a)$, is defined by

$$V(\mathfrak{A}, a) = \{f(a): f \in \mathfrak{A}', f(1) = 1 = \|f\|\}$$

and is a compact convex subset of the complex field $\mathbf{C}$. The *numerical radius* of a, $v(a)$, is then defined by

$$v(a) = \max \{|\lambda|: \lambda \in V(\mathfrak{A}, a)\}.$$

Wherever possible we shall follow the notation of Bonsall and Duncan [6] to which we refer the reader for a systematic account of the theory of numerical ranges. In this paper we shall consider several problems of the following nature. Suppose that the numerical range $V(\mathfrak{A}, a)$ is restricted in size and shape. What conditions are then implied on the algebra generated by 1 and a; for example, how large can $\|a^n\|$ be? Several results are known in this area. For example, if $v(a) = 1$, then

$$\|a^n\| \leqslant n! \left(\frac{e}{n}\right)^n \quad (n = 1, 2, 3, \ldots)$$

and these inequalities are best possible; see Bollobás [3], [4], Browder [8], Crabb [9], [10]. In particular, the power inequality

$$v(a^n) \leqslant v(a)^n \quad (n = 1, 2, 3, \ldots)$$

which is known to hold in B*-algebras, does not hold for arbitrary Banach algebras. On the other hand, if $V(\mathfrak{A}, a)$ is a subset of the real field $\mathbf{R}$, i.e. if a is *Hermitian*, then

$$\varrho(a) = v(a) = \|a\|$$

where $\varrho(a)$ is the spectral radius of a (see Browder [8], Sinclair [13]), so that in this case

$$v(a^n) = v(a)^n \quad (n = 1, 2, 3, \ldots).$$

In §1 we consider the restriction

$$V(\mathfrak{A}, a) \subset K$$

where K is an arbitrary compact convex subset of $\mathbf{C}$. Since $\mathrm{Sp}\,(\mathfrak{A}, a) \subset K$, we then have $G(a) \in \mathfrak{A}$ whenever G is analytic on a neighbourhood of K. We give upper bounds for $\|G(a)\|$ and $V(\mathfrak{A}, G(a))$, and these upper bounds are all attained in a single extremal Banach algebra $A(K)$. This result had previously been obtained by Bollobás [4], but we present a different construction of the algebra $A(K)$. In fact, the algebra $A(K)$ consists of all complex functions f on K of the form

$$f(z) = \int e^{z\lambda} d\mu(\lambda) \quad (z \in K)$$

where μ is a regular Borel measure on $\mathbf{C}$. The dual space of $A(K)$ is identified with the space of entire functions with a certain majorant. This sets up a correspondence between certain extremal problems in the theory of numerical ranges and the study of entire functions with a given majorant.

The case when K is a closed interval in $\mathbf{R}$, i.e. the case of Hermitian elements, is considered in more detail in § 2. Let h be Hermitian with $\|h\| = 1$. For functions of the form $e^{i\alpha t}\, p(t)$, where p is a polynomial and $e^{-i\alpha}\, p(-1) = e^{i\alpha}\, p(1)$, we show that $e^{i\alpha t}\, p(t) = \Sigma_{-\infty}^{\infty} c_n\, e^{2\pi i n t}$ implies $\exp(i\alpha h)\, p(h) = \Sigma_{-\infty}^{\infty} c_n \exp(2\pi i n h)$. This result is not available by simple functional calculus arguments. We deduce that $\varrho(p(h)) = \|p(h)\|$ for certain polynomials, thus generalizing a result of Sinclair [13]. The support functionals f at $\cos\alpha + ih \sin\alpha$ $(\alpha \neq n\pi)$ are shown to have the interesting property that $f(h^n)$ is constant for n even and for n odd. Finally we give a partial description of the maximal case of $V(\mathfrak{A}, h^2)$ subject to $V(\mathfrak{A}, h) \subset [-1, 1]$ and also $V(\mathfrak{A}, h) \subset [0, 1]$.

In § 3 the results of §1 are generalized to the case of joint numerical ranges and are applied in particular to the study of normal elements. An element u of $\mathfrak{A}$ is said to be *normal* if it is of the form $u = h + ik$ where h, k are commuting Hermitian elements. For such elements, $V(\mathfrak{A}, u)$ is the convex hull of the spectrum of u, so that $v(u) = \varrho(u)$. Crabb [11] gave an example with $\|u\| = \sqrt{2}\, \varrho(u)$. We give a formula for

$$k_n = \max \{\|u^n\| : \varrho(u) = 1,\ u \text{ normal}\}$$

in terms of functions of exponential type. We show that $k_1 = 2$ and also that there is $M > 0$ with $k_n \geqslant M n^{\frac{1}{2}}$.

In the final section we consider the extremal problem subject to the condition $v=\varrho$, or equivalently the numerical range of each element is the convex hull of the spectrum. This condition forces $\mathfrak{A}$ to be a function algebra under an equivalent norm; in fact $\varrho \leqslant \|\cdot\| \leqslant \frac{1}{2}e\varrho$ and these inequalities are best possible.

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1. The Banach algebra $A(K)$

Let $\mathfrak{A}$ be a complex unital Banach algebra and let $a \in \mathfrak{A}$. Given a non-empty compact convex subset K of $\mathbf{C}$ we wish to study the extremal properties of elements $a \in \mathfrak{A}$ with $V(\mathfrak{A}, a) \subset K$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by a and 1. Then

$$V(\mathfrak{B}, a) = V(\mathfrak{A}, a)$$

(see e.g. [6] Theorem 2.4). We may therefore restrict our attention to singly generated Banach algebras. As usual, we shall denote the numerical range more briefly by $V(a)$. Since

$$V(ra) = rV(a) \quad (r \geqslant 0)$$

we shall further suppose for convenience that

$$\sup \{|z| : z \in K\} = 1.$$

The main cases of interest occur when K is the closed unit disc, a semi-disc, or a line segment. The case when K is a singleton is trivial and we shall therefore suppose that K has at least two points. When K has interior we shall suppose without loss that $0 \in \operatorname{int} K$; when K has no interior we shall suppose without loss that $[0, 1] \subset K \subset [-1, 1]$.

Given $a \in \mathfrak{A}$ let $\mathfrak{E}_a$ denote the exponenential group

$$\mathfrak{E}_a = \{\exp(\lambda a) : \lambda \in \mathbf{C}\}.$$

The norm on $\mathfrak{E}_a$ completely determines the compact convex set $V(a)$ since

$$\max \operatorname{Re} V(e^{i\theta}a) = \sup_{r>0} \frac{1}{r} \log \|\exp(re^{i\theta}a)\| \quad (\theta \in \mathbf{R})$$

(see e.g. [6] Theorem 3.4). Let

$$\omega(re^{i\theta}) = \exp(r \max \operatorname{Re}(e^{i\theta}K)) \quad (r \geqslant 0, \theta \in \mathbf{R}).$$

Note that $V(a) \subset K$ if and only if

$$\|\exp(\lambda a)\| \leqslant \omega(\lambda) \quad (\lambda \in \mathbf{C}). \tag{1}$$

The function ω has the property

$$\omega(\lambda+\mu) \leqslant \omega(\lambda)\,\omega(\mu) \quad (\lambda, \mu \in \mathbf{C}). \tag{2}$$

In fact, let

$$u(z) = z \quad (z \in K)$$

and note that

$$\omega(\lambda) = \|\exp(\lambda u)\|_\infty \quad (\lambda \in \mathbf{C})$$

where $\|\cdot\|_\infty$ denotes the supremum norm for bounded functions on K.

The extremal case in (1) occurs when

$$\|\exp(\lambda a)\| = \omega(\lambda) \quad (\lambda \in \mathbf{C}) \tag{3}$$

and then we have $K = \operatorname{co} \operatorname{Sp}(a)$, the convex hull of the spectrum of a, since

$$\max \operatorname{Re} \operatorname{Sp}(e^{i\theta}a) = \inf_{r>0} \frac{1}{r} \log \|\exp(re^{i\theta}a)\|$$

(see e.g. [6] Theorem 3.8). Since $\mathfrak{A}$ is generated by a, the maximum principle actually gives $\operatorname{Sp}(a) = K$. Under the Gelfand representation we have $\hat{a} = u$, and $\hat{\mathfrak{A}}$ is a subalgebra of the algebra $\mathring{A}(K)$ of all continuous functions on K that are analytic on the interior of K. For such algebras we wish to construct the maximal norm such that (3) holds (the minimal norm is clearly $\|\cdot\|_\infty$). It is clear that we wish the algebra $\mathfrak{A}$ to be generated by the exponential group $\mathfrak{E}_u$. The linear span of $\mathfrak{E}_u$ is an algebra, but it does not contain the element u. In fact, the algebra of absolutely convergent series of exponentials contains the element u and gives the required extremal Banach algebra. For the purposes of exposition we shall adopt a slightly different approach to the definition of the extremal algebra. We are indebted to Dr S. Kaijser for the suggestion and also for pointing out to us Theorem 1.9.

We write $M(\mathbf{C})$ for the Banach space of (finite) complex regular Borel measures on $\mathbf{C}$, and $M^\omega(\mathbf{C})$ for the weighted space

$$M^\omega(\mathbf{C}) = \{\mu \in M(\mathbf{C}) : \|\mu\| = \int \omega\, d|\mu| < \infty\}.$$

Given $\mu \in M^\omega(\mathbf{C})$ let f_μ be defined on K by

$$f_\mu(z) = \int e^{z\lambda} d\mu(\lambda) \quad (z \in K).$$

We say that μ *represents* f_μ. Clearly $f_\mu \in \mathring{A}(K)$.

Definition 1.1. $A(K)=\{f_\mu\colon \mu\in M^\omega(\mathbf{C})\}$. For $f\in A(K)$, we define

$$\|f\| = \inf\{\|\mu\|\colon \mu \text{ represents } f\}.$$

PROPOSITION 1.2. *$(A(K), \|\cdot\|)$ is a complex unital Banach algebra such that*

(i) $\|f\|_\infty \leqslant \|f\|$ $(f\in A(K))$,

(ii) $\|\exp(\lambda u)\| = \|\exp(\lambda u)\|_\infty$ $(\lambda\in\mathbf{C})$.

Proof. Let $f(z)=\int e^{z\lambda}\,d\mu(\lambda)$ $(z\in K)$. Then

$$|f(z)| \leqslant \int \|e^{\lambda u}\|_\infty\, d|\mu| = \|\mu\|.$$

This gives (i). The trivial representation gives

$$\|e^{\lambda u}\| \leqslant \|e^{\lambda u}\|_\infty \quad (\lambda\in\mathbf{C})$$

and hence (ii). It is straightforward to verify that $\|\cdot\|$ is an algebra norm on $A(K)$ (use (2)). Let $f_n\in A(K)$ with $\Sigma\|f_n\|<\infty$. Let μ_n represent f_n with

$$\|\mu_n\| \leqslant \|f_n\| + 2^{-n}.$$

Let
$$f(z)=\Sigma f_n(z) \quad (z\in K).$$

Clearly $\Sigma\mu_n$ represents f, so that $A(K)$ is complete, as required.

THEOREM 1.3. *The Banach algebra $A(K)$ is generated by u, and hence its maximal ideal space can be identified with K.*

Proof. Since
$$z=\frac{1}{2\pi i}\int_\Gamma e^{z\lambda}\frac{d\lambda}{\lambda^2}$$

where Γ is the unit circle, we have $u\in A(K)$. Since each $f\in A(K)$ can be approximated by exponentials it follows that u generates $A(K)$. Hence the maximal ideal space of $A(K)$ may be identified with $\mathrm{Sp}(u)=K$.

COROLLARY 1.4. *Let $f\in A(K)$ and let G be analytic on a neighbourhood of $f(K)$. Then $G\circ f\in A(K)$. In particular $1/f\in A(K)$ provided $f(z)\neq 0$ $(z\in K)$.*

Definition 1.5. $E(K)=\{\phi\colon \phi \text{ entire}, \|\phi\|=\sup\{|\phi(\lambda)|/\omega(\lambda)\colon \lambda\in\mathbf{C}\}<\infty\}$. Clearly $E(K)$ is a Banach space.

Given $f\in A(K)$, f represented by μ, and $\phi\in E(K)$ let

$$\langle f,\phi\rangle = \int \phi\, d\mu. \tag{4}$$

LEMMA 1.6. *The above pairing* (4) *is well-defined.*

Proof. Let $\int e^{z\lambda}\,d\mu(\lambda)=0$ $(z\in K)$. We must show $\int \phi\,d\mu=0$.

Case 1. $0\in\operatorname{int}K$. Let $\Delta=\{z:|z|<\delta\}\subset K$, and let $D=\operatorname{int}(\operatorname{co}\{1,\Delta\})$. Then $(0,1)\subset D$ and $DK\subset K$. Let

$$G(z)=\int \phi(z\lambda)\,d\mu(\lambda)\quad(z\in D)$$

so that G is analytic on D. Let $\phi(\lambda)=\sum_{n=0}^{\infty}c_n\lambda^n$. We may suppose $\|\phi\|\leqslant 1$, so that $|\phi(\lambda)|\leqslant e^{|\lambda|}$ and hence $|c_n|\leqslant(e/n)^n$. Therefore

$$\sum_{n=0}^{\infty}|c_n\lambda^n|\leqslant e^{e|\lambda|}\quad(\lambda\in\mathbf{C}).$$

Since $\omega(\lambda)\geqslant e^{\delta|\lambda|}$, it follows by dominated convergence that

$$G(z)=\sum_{n=0}^{\infty}c_n z^n\int\lambda^n d\mu(\lambda)\quad(|z|<\delta/e).$$

The hypothesis gives $\int\lambda^n\,d\mu(\lambda)=0$ $(n=0,1,2,\ldots)$ and then analytic continuation gives $G(t)=0$ $(0<t<1)$. Finally dominated convergence gives $G(1)=0$, as required.

Case 2. $K\subset[-1,1]$. Let $\phi\in L^2(i\mathbf{R})$. By Paley-Wiener (see e.g. [12] p. 387) there is $\varrho\in L^2(K)$ such that

$$\phi(\lambda)=\int_K e^{\lambda t}\varrho(t)\,dt\quad(\lambda\in\mathbf{C}).$$

By Fubini
$$\int\phi(\lambda)\,d\mu(\lambda)=\int_K\varrho(t)\int e^{t\lambda}d\mu(\lambda)\,dt=0.$$

Given arbitrary $\phi\in E(K)$ let

$$\phi_n(\lambda)=\phi((1-1/n)\lambda)\,\frac{\exp(\lambda/n)-1}{\lambda/n}\quad(\lambda\in\mathbf{C}).$$

Then $\phi_n\in L^2(iR)$ and dominated convergence now gives $\int\phi\,d\mu=0$.

Given $\phi\in E(K)$ let $\Phi_\phi(f)=\langle f,\phi\rangle$ $(f\in A(K))$.

THEOREM 1.7. *The map* $\phi\to\Phi_\phi$ *is an isometric isomorphism of* $E(K)$ *onto* $A(K)'$.

Proof. Clearly $\Phi_\phi\in A(K)'$ and $\|\Phi_\phi\|\leqslant\|\phi\|$. Since $\Phi_\phi(e^{\lambda u})=\phi(\lambda)$, it follows that

$$\|\Phi_\phi\|\geqslant\sup\{|\phi(\lambda)|/\|e^{\lambda u}\|:\lambda\in\mathbf{C}\}=\|\phi\|.$$

Given $\Phi\in A(K)'$ let $\phi(\lambda)=\Phi(e^{\lambda u})$ $(\lambda\in\mathbf{C})$. Then $\phi\in E(K)$ and

$$\Phi\left(\int e^{\lambda u}\,d\mu(\lambda)\right)=\int\Phi(e^{\lambda u})\,d\mu(\lambda)=\int\phi\,d\mu=\Phi_\phi\left(\int e^{\lambda u}\,d\mu(\lambda)\right).$$

The proof is complete.

Theorem 1.7. is the tool for the proof of the main theorem in this section; but it is also of interest that $A(K)$ is a dual space. We write

$$C_0^\omega(\mathbf{C})=\{\phi\in C(\mathbf{C}):|\phi(\lambda)|/\omega(\lambda)\to 0 \text{ as } |\lambda|\to\infty\}.$$

Definition 1.8. $E_0(K)=E(K)\cap C_0^\omega(\mathbf{C})$. Clearly $E_0(K)$ is a closed subspace of $E(K)$. Given $f\in A(K)$ let $F_f(\phi)=\langle f,\phi\rangle$ $(\phi\in E_0(K))$.

THEOREM 1.9. *The map $f\to F_f$ is an isometric isomorphism of $A(K)$ onto $E_0(K)'$.*

Proof. Clearly $F_f\in E_0(K)'$ for each $f\in A(K)$. $E_0(K)$ is a closed subspace of $C_0^\omega(\mathbf{C})$. Using the Riesz representation theorem we obtain $E_0(K)'\approx M^\omega(\mathbf{C})/E_0(K)^\perp$. If μ represents $F\in E_0(K)'$, let $f(z)=\int e^{z\lambda}\,d\mu(\lambda)$ $(z\in K)$. It is now sufficient to show that $F\to f$ is well-defined, i.e. $\mu\in E_0(K)^\perp$ implies $f=0$.

Case 1. $0\in\operatorname{int}K$. For $z\in\operatorname{int}K$, $\lambda\to e^{z\lambda}$ is in $E_0(K)$. Hence $\mu\in E_0(K)^\perp$ implies $f(z)=0$ $(z\in\operatorname{int}K)$ and so $f=0$.

Case 2. $K\subset[-1,1]$. For $t\in K$, $\lambda\to(e^{t\lambda}-1)/\lambda$ is in $E_0(K)$. Hence $\mu\in E_0(K)^\perp$ implies

$$\int\frac{e^{t\lambda}-1}{\lambda}\,d\mu(\lambda)=0\quad(t\in K)$$

and differentiation gives $f=0$.

Remarks 1. Case 1 of Lemma 1.6 may also be proved by using the representation

$$\phi(\lambda)=\frac{1}{2\pi i}\int_\Gamma H(w)\,e^{\lambda w}\,dw$$

where H is the Borel transform of ϕ and Γ a contour containing K.

2. An annihilator argument shows that a dense spanning subset of $E_0(K)$ is given by the polynomials if $0\in\operatorname{int}K$ and by the functions $\{\lambda\to(e^{t\lambda}-1)/\lambda: t\in K\}$ if $K\subset[-1,1]$.

3. A further annihilator argument shows that each $f\in A(K)$ has a *discrete* representing measure. It is then not difficult to show that $\|f\|$ is determined by discrete representing measures. This relates to [7] Theorem 3. Moreover the norm of polynomials is determined by representing measures such that $\int\exp(e|\lambda|)\,d|\mu|(\lambda)<\infty$.

4. The action of Φ_ϕ is described more usefully by

$$\Phi_\phi(f) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \phi^{(n)}(0)\, f^{(n)}(0) \quad (f \text{ a polynomial}).$$

Our construction of $A(K)$ leads to an alternative proof of the result of Bollobás [4]. Let $(\mathfrak{A}, p)$ be an arbitrary complex unital Banach algebra with $a \in \mathfrak{A}$ such that $V(\mathfrak{A}, a) \subset K$. Then $\mathrm{Sp}(a) \subset K$, and so $G(a) \in \mathfrak{A}$ whenever G is analytic on some neighbourhood of K. This is the notation for the following theorem.

THEOREM 1.10. (Bollobás).

(i) $\sup\{p(G(a)) : V(\mathfrak{A}, a) \subset K,\ (\mathfrak{A}, p)\ \textit{arbitrary}\} = \|G(u)\| = \sup\{|F_\phi(G(u))| : \phi \in E(K), \|\phi\| = 1\}$.

(ii) $V(\mathfrak{A}, G(a)) \subset V(A(K), G(u)) = \{F_\phi(G(u)) : \phi \in E(K), \phi(0) = \|\phi\| = 1\}$.

Proof. Let $f \in \mathfrak{A}'$, $\|f\| = 1$ and let $\phi(\lambda) = f(e^{\lambda a})$ $(\lambda \in \mathbf{C})$. Then $\phi \in E(K)$ and $\|\Phi_\phi\| \leqslant 1$. Let P be a polynomial. Given any contour Γ containing K define μ_P by $d\mu_P(\lambda) = (1/\lambda)\, P(1/\lambda)\, d\lambda$. Then

$$f(P(a)) = f\left(\int_\Gamma e^{\lambda a}\, d\mu_P(\lambda)\right) = \int_\Gamma f(e^{\lambda a})\, d\mu_P(\lambda) = \int_\Gamma \phi(\lambda)\, d\mu_P(\lambda) = \Phi_\phi(P(u)).$$

The continuity of the functional calculus gives $f(G(a)) = \Phi_\phi(G(u))$ and so $p(G(a)) \leqslant \|G(u)\|$. Theorem 1.7 gives the formula for $\|G(u)\|$. Restrict the above to f with $f(1) = 1$ to get $V(\mathfrak{A}, G(a)) \subset V(A(K), G(u))$. For the final formula note that $\Phi_\phi(1) = \phi(0)$.

COROLLARY 1.11. (Bollobás [3], Browder [8], Crabb [10]). *Let* $K = \{z : |z| \leqslant 1\}$.

(i) $\sup\{p(a^n) : v(a) = 1\} = \|u_n\| = n!\,(e/n)^n$.

(ii) *Given* $v(a) = 1$, $V(\mathfrak{A}, a^n) \subset V(A(K), u^n) = \{z : |z| \leqslant c_n\}$
where $c_n = \sup\{|\phi^{(n)}(0)| : \phi \in E(K), \phi(0) = \|\phi\| = 1\}$.

Proof. Given $v(a) = 1$ we have $p(a^n) \leqslant n!\,(e/n)^n$. Let $\phi(\lambda) = (e\lambda/n)^n$ $(\lambda \in \mathbf{C})$. Then $\|u^n\| \geqslant |\phi^{(n)}(0)| = n!\,(e/n)^n$, and this gives (i). For $\phi \in E(K)$ with $\phi(0) = \|\phi\| = 1$, $\theta \in \mathbf{R}$, let $\psi(\lambda) = \phi(\lambda e^{i\theta})$. Then $\psi \in E(K)$, $\psi(0) = \|\psi\| = 1$, and (ii) now follows.

For the case $K = \{z : |z| \leqslant 1\}$ it is not difficult to obtain the following information about the functions in $A(K)$ in terms of their Taylor expansions. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| \leqslant 1,\ \Sigma n^{\frac{1}{2}} |a_n| < \infty.$$

Then $f \in A(K)$. On the other hand there exists $g \in A(K)$ such that $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ and $\Sigma n^{\frac{1}{2}} |b_n|$ diverges.

Consider now the general case with $0 \in \text{int } K$. Given G analytic on a neighbourhood of K we have $G(u) \in A(K)$. For any representation $G(z) = \int e^{z\lambda}\, d\mu(\lambda)$ we have $G(a) = \int e^{\lambda a}\, d\mu(\lambda)$. To see this let $0 < t < 1$, $\Gamma = t^{-1}\partial K$. Then

$$G(ta) = \frac{1}{2\pi i}\int_\Gamma G(tz)\,(z-a)^{-1}\,dz = \int \frac{1}{2\pi i}\int_\Gamma e^{t\lambda z}(z-a)^{-1}\,dz\,d\mu(\lambda) = \int e^{t\lambda a}\,d\mu(\lambda).$$

Now let $t \to 1$. This technique does not apply when $K \subset [-1, 1]$; the Hermitian case requires the finer technique of the next section.

2. Some properties of Hermitian elements

Let $\mathfrak{A}$ be a complex unital Banach algebra. Let H denote the set of Hermitian elements of $\mathfrak{A}$, i.e. the set of elements with real numerical range. When H is closed under squaring, $H + iH$ is a B^*-algebra with H as the set of self-adjoint elements (see e.g. [6] § 6) and then many properties of H follow from the well developed theory of B^*-algebras. On the other hand there are many examples when H is not closed under squaring (see e.g. [4], [6], [8]) and then the techniques of B^*-algebras are not available to give results about Hermitian elements. For example the proof of Sinclair's theorem that

$$\varrho(h) = \|h\| \quad (h \in H)$$

seems to depend on Bernstein's theorem on entire functions of exponential type (though under an elementary disguise in [5]). We prove here some results in similar vein.

We give first a method for deriving expressions for polynomials in h in terms of the linear span of $\{e^{ith}: t \in \mathbf{R}\}$. To be precise, if h is Hermitian with $\|h\| = 1$, and p is a polynomial with $|p(1)| = |p(-1)|$, choose $\alpha \in \mathbf{R}$ such that $e^{-i\alpha}\,p(-1) = e^{i\alpha}\,p(1)$. If the Fourier series for $e^{i\alpha t}\,p(t)$ on $(-1, 1)$ is $\Sigma_{-\infty}^{\infty} c_n\, e^{in\pi t}$, then

$$p(h) = \sum_{-\infty}^{\infty} c_n \exp\,((n\pi - \alpha)\,ih).$$

The results show that for a large class of polynomials $p(h)$, the spectral radius of $p(h)$ already coincides with the norm of $p(h)$ in the extremal Banach algebra $A[-1, 1]$. For these polynomials we obtain an explicit representation in $A[-1, 1]$ (by discrete measures) in which the norm is attained. Such a representation is a special case of the concept of minimal extrapolation as introduced by Beurling [1].

We give first the proof for a linear polynomial; the argument may be extended to deal with an arbitrary polynomial p as above. The method is similar to Sinclair's proof [13] that $\|h + \alpha\| = \varrho(h + \alpha)$. We write $\mathbf{Z}$ for the set of all integers.

THEOREM 2.1. *Let $h \in H$ with $\|h\| \leqslant 1$, and let $\alpha \in \mathbf{R} \setminus \pi\mathbf{Z}$. Then*

$$\cos \alpha + ih \sin \alpha = \sin^2 \alpha \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{(n\pi + \alpha)^2} \exp\left((n\pi + \alpha)\, ih\right). \tag{1}$$

Proof. Suppose first that $\|h\| < 1$. Let $g \in \mathcal{H}'$ with $\|g\| = 1$, and let

$$f(z) = g(\exp(izh)) \quad (z \in \mathbf{C}).$$

Then f is entire with $|f(z)| \leqslant e^{|z|}$ $(z \in \mathbf{C})$, $|f(x)| \leqslant 1$ $(x \in \mathbf{R})$. Note that

$$g(\cos \alpha + ih \sin \alpha) = f(0) \cos \alpha + f'(0) \sin \alpha.$$

We now apply the theory of Boas [2] § 11.2 with

$$L[f(z)] = f(z) \cos \alpha + f'(z) \sin \alpha$$

$$\lambda(t) = \cos \alpha + it \sin \alpha.$$

Recall that the function $\exp(-i\alpha t)\, \lambda(t)$ has the absolutely convergent Fourier series on $(-1, 1)$ given by

$$\sin^2 \alpha \sum_{n \in \mathbf{Z}} \frac{(-1)^n}{(n\pi + \alpha)^2} \exp(in\pi t) = \sum_{n \in \mathbf{Z}} c_n \exp(in\pi t).$$

Thus, when f is of the form

$$f(z) = \int_{-1}^{1} \exp(izt)\, d\mu(t) \tag{2}$$

where μ is of bounded variation, Theorem 11.2.6 of [2] gives

$$f(x) \cos \alpha + f'(x) \sin \alpha = \sum_{n \in \mathbf{Z}} c_n f(x + \alpha + n\pi) \quad (x \in \mathbf{R}). \tag{3}$$

For arbitrary f consider first f_δ defined by

$$f_\delta(z) = \frac{\sin \delta z}{\delta z} f(z) \quad (z \in \mathbf{C})$$

for $0 < \delta < 1 - \|h\|$. Then f_δ has an integral representation as in (2) so that (3) holds with f replaced by f_δ. Since

$$\sum_{n \in \mathbf{Z}} |c_n| < \infty \text{ and } \left|\frac{\sin \delta x}{\delta x}\right| \leqslant 1 \quad (x \in \mathbf{R}),$$

we may then let $\delta \to 0$ to obtain (3) for arbitrary f. Put $x = 0$ to give

$$g(\cos\alpha + ih\sin\alpha) = \sum_{n\in\mathbf{Z}} c_n\, g(\exp((n\pi+\alpha)\,ih)).$$

Since $g\in\mathfrak{A}'$ is arbitrary (of norm 1) we deduce that

$$\cos\alpha + ih\sin\alpha = \sin^2\alpha \sum_{n\in\mathbf{Z}} \frac{(-1)^n}{(n\pi+\alpha)^2} \exp((n\pi+\alpha)\,ih)).$$

Since the above series converges absolutely, the case $\|h\| = 1$ follows by a routine continuity argument.

The case $\alpha = \frac{1}{2}\pi$ in (1) gives the explicit representation of h in the closed convex hull of $\exp(iH)$ which was also obtained by F. F. Bonsall.

In a similar way we can derive the following theorem.

THEOREM 2.2. *Let $h\in H$ with $\|h\|\leqslant 1$, and let $\alpha\in\mathbf{R}\setminus\pi\mathbf{Z}$.*

(a) $$h = \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\sin((n+\tfrac{1}{2})\pi h).$$

(b) $$h^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi h.$$

(c) $$\cos\alpha + ih^3\sin\alpha = \sum_{n\in\mathbf{Z}} \frac{3(-1)^n\sin\alpha}{(n\pi+\alpha)^4}\{(n\pi+\alpha)^2\sin\alpha + 2(n\pi+\alpha)\cos\alpha - 2\sin\alpha\}$$
$$\times\exp((n\pi+\alpha)\,ih).$$

COROLLARY 2.3. *Let $h\in H$ with $\|h\| = 1$.*

(i) (Sinclair [13]) $\|\cos\alpha + ih\sin\alpha\| = 1$ $(\alpha\in\mathbf{R})$.

(ii) $\|\cos\alpha + ih^3\sin\alpha\| = 1$ $(\alpha\in\mathbf{R},\ |\alpha - \frac{1}{2}\pi| < \pi/9)$.

(iii) $\|h^2 - t\| = 1 - t = \varrho(h^2 - t)$ $(0\leqslant t\leqslant\frac{1}{3})$.

Proof. (i) Put $h = 1$ in (a) and we see that the sum of the moduli of the coefficients is one. Hence $\|\cos\alpha + ih\sin\alpha\|\leqslant 1$. Also, since $\varrho(h) = \|h\| = 1$, we have

$$\|\cos\alpha + ih\sin\alpha\| \geqslant \varrho(\cos\alpha + ih\sin\alpha) = 1.$$

(ii) For $|\alpha - \frac{1}{2}\pi| < \pi/9$, $n\in\mathbf{Z}$, we have

$$(n\pi+\alpha)^2\sin\alpha + 2(n\pi+\alpha)\cos\alpha - 2\sin\alpha \geqslant 0.$$

Now argue as in (i), using (c) above.

(iii) For $0\leqslant t\leqslant\frac{1}{3}$, we deduce from (b) above that

$$\|h^2 - t\| \leqslant \tfrac{1}{3} - t + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 - t = \varrho(h^2 - t).$$

THEOREM 2.4. *Let $h \in H$ with $\|h\| = 1$, let $\alpha \in \mathbf{R} \setminus \pi\mathbf{Z}$, and let $f \in \mathfrak{A}'$ with $f(\cos \alpha + ih \sin \alpha) = 1 = \|f\|$. Then there exists $c \in [-1, 1]$ such that*

$$f(h^n) = \begin{cases} -i \sin \alpha + c \cos \alpha & \text{if } n = 1, 3, 5, \ldots \\ \cos \alpha - ic \sin \alpha & \text{if } n = 0, 2, 4, \ldots . \end{cases}$$

Proof. Using the representation for $\cos \alpha + ih \sin \alpha$ from Theorem 2.1, we deduce that $f(\exp((n\pi + \alpha)\, ih)) = (-1)^n$ $(n \in \mathbf{Z})$. Let

$$F(z) = f(\exp((\pi z + \alpha)\, ih)) - \cos \pi z \quad (z \in \mathbf{C}).$$

Then $F(z) = 0$ $(z \in \mathbf{Z})$, and $|F(z)| \leqslant M e^{|\mathrm{Im}\, \pi z|}$ for some constant M. Hence by Liouville, $F(z) = a \sin \pi z$ $(z \in \mathbf{C})$ for some constant a. Then

$$f(\exp((\pi z + \alpha)\, ih)) = \cos \pi z + a \sin \pi z$$

and since $|f(\exp((\pi z + \alpha)\, ih))| \leqslant 1$ $(z \in \mathbf{R})$, we must have $a = ic$ where $c \in [-1, 1]$. Therefore

$$f(\exp(zih)) = \cos(z - \alpha) + i\, c \sin(z - \alpha).$$

The proof is completed by comparing coefficients of z^n in this last equation.

COROLLARY 2.5. *Let $h \in H$, $f \in \mathfrak{A}'$ with*

$$f(1) = 1 = \|f\| = \|h\|, \quad f(h) = \pm 1.$$

Then f is multiplicative on the algebra generated by 1 and h.

The above theorem was also obtained by B. Bollobás; the corollary by A. M. Sinclair (private communication).

THEOREM 2.6. *Let $h \in H$ with $\|h\| = 1$. Then*

(i) $V(h^2) \subset \{z: \mathrm{Re}\, z \geqslant 0,\ |z - \frac{1}{3}| \leqslant \frac{2}{3}\}$,

(ii) $\sup\{|t|: t \in \mathbf{R},\ it \in V(h^2)\} \leqslant \left\{\dfrac{4}{3\pi^2} - \dfrac{1}{45}\right\}^{\frac{1}{2}} < .336.$

Proof. Let $f \in \mathfrak{A}'$, $f(1) = 1 = \|f\|$. Then

$$\mathrm{Re}\, f(\cos th) \leqslant 1 \quad (t \in \mathbf{R})$$

and so

$$1 - \tfrac{1}{2}t^2\, \mathrm{Re}\, f(h^2) + \phi(t) \leqslant 1 \quad (t \in \mathbf{R})$$

where $\phi(t) = O(t^4)$ as $t \to 0$. Therefore $\mathrm{Re}\, f(h^2) \geqslant 0$. By Corollary 2.3 (iii) we have $\|h^2 - \frac{1}{3}\| = \frac{2}{3}$ and so

$$|f(h^2)-\tfrac{1}{3}| = |f(h^2-\tfrac{1}{3})| \leqslant \tfrac{2}{3}.$$

This proves part (i).

Suppose now that $f(h^2) \in i\mathbf{R}$. By considering the inequality

$$|f(\cos th)| \leqslant 1 \quad (t \in \mathbf{R})$$

for small t, we deduce that

$$|f(h^2)|^2 \leqslant -\tfrac{1}{3} \operatorname{Re} f(h^4).$$

The method of Theorem 2.2 may be used to establish that

$$\left\| h^4 - 2\left(1-\frac{6}{\pi^2}\right) h^2 \right\| \leqslant \frac{4}{\pi^2} - \frac{1}{15}.$$

Since $\operatorname{Re} f(h^2) = 0$, we now have

$$-\operatorname{Re} f(h^4) \leqslant \frac{4}{\pi^2} - \frac{1}{15}.$$

Therefore

$$|f(h^2)|^2 \leqslant \frac{4}{3\pi^2} - \frac{1}{45}$$

and the proof is complete.

Given $h \in H$ with $\|h\| = 1$, Theorem 1.10 shows that the maximal case of $V(h^2)$ is given by

$$V = \{\phi''(0) \colon \phi \text{ entire}, \phi(0) = 1, |\phi(z)| \leqslant e^{|\operatorname{Re} z|} \ (z \in \mathbf{C})\}.$$

Bollobás [4] shows that $V \supset \{z \colon |z - \tfrac{1}{2}| \leqslant \tfrac{1}{2}\}$. We also have $V \supset \{it \colon t \in [-\tfrac{1}{8}, \tfrac{1}{8}]\}$. To see this let ϕ be defined by

$$\phi(z) = 1 \pm \tfrac{1}{2} i(\cosh \tfrac{1}{2}z - 1) - \tfrac{1}{4}(\cosh \tfrac{1}{2}z - 1)^2.$$

Use the Phragmen-Lindelöf principle to get $|\phi(z)| \leqslant e^{|\operatorname{Re} z|}$, and note that $\phi''(0) = \pm i/8$.

The above results give an approximate description of $V(h^2)$ in the extremal algebra $A([-1, 1])$. Note also that in the extremal algebra $A([-1, 1])$ we have

$$\|e^{ish^2}\| \geqslant \frac{2}{M} s^{\frac{1}{2}} \quad (s > 0), \qquad \text{where } M = \sup_{a, b \in \mathbf{R}} \left| \int_b^a e^{-ix^2} dx \right|.$$

To see this let

$$\phi(z) = \int_{-1}^{1} e^{zt} e^{-ist^2} dt \quad (z \in \mathbf{C}).$$

Then $\phi \in E([-1,1])$ and $\Phi_\phi(e^{ish^2}) = 2$. By Phragmen–Lindelöf

$$\|\phi\| = \sup_{x \in \mathbf{R}} \left| \int_{-1}^{1} e^{ixt} e^{-ist^2} dt \right| = \sup_{x \in \mathbf{R}} s^{-\frac{1}{2}} \left| \int_{(-1-x/2s)s^{\frac{1}{2}}}^{(1-x/2s)s^{\frac{1}{2}}} e^{-it^2} dx \right| \leqslant M s^{-\frac{1}{2}}$$

and so $\|e^{ish^2}\| \geqslant 2s^{\frac{1}{2}}/M$. In particular the extremal norm on $A([-1, 1])$ is not equivalent to the supremum norm since $\|e^{ish^2}\|_\infty = 1$.

Now consider the extremal problem for $V(h^2)$ subject to the restriction $V(h)=[0, 1]$. It is easy to see that the maximal set for $V(h^2)$ again has non-empty interior, and also $\|e^{ish^2}\| \geqslant s^{\frac{1}{2}}/M$ in $A([0, 1])$. However we have the following result which contrasts with the situation for arbitrary Hermitian elements.

THEOREM 2.7. *Let k be a positive Hermitian element. Then*

$$V(k^2) \cap i\mathbf{R} \subset \{0\}.$$

Proof. We may suppose $V(k) \subset [0, 2]$. Let $h = k - 1$ and then $V(h) \subset [-1, 1]$. Let $f(1) = 1 = \|f\|$ and suppose $f(k^2) = ia$ with $a \in \mathbf{R}$. Then

$$ia = 1 + 2f(h) + f(h^2).$$

Note that $b = -1 - 2f(h) \in \mathbf{R}$ and $f(h^2) = b + ia$. We have $|f(\exp ith)| \leqslant 1$ $(t \in \mathbf{R})$ and so

$$\left|1 + itf(h) - (b + ia)\frac{t^2}{2} + O(t^3)\right| \leqslant 1.$$

Consideration of the above for small t gives

$$f(h)^2 \leqslant b = -1 - 2f(h)$$

and so $f(h) = -1$. By Theorem 2.4 we now have $f(h^2) = 1$ and so $f(k^2) = 0$.

3. The Banach algebra $A(\Xi)$

In this section we generalize the results of § 1 by considering extremal problems in which several elements are involved. In this case we impose a condition on the joint numerical range of several elements. Let $\mathfrak{A}$ be a complex unital Banach algebra and let $a_1, \ldots, a_n \in \mathfrak{A}$. Recall that the joint numerical range of $a_1, \ldots, a_n$, $V(\mathfrak{A}; a_1, \ldots, a_n)$ is defined by

$$V(\mathfrak{A}; a_1, \ldots, a_n) = \{(f(a_1), \ldots, f(a_n)) : f \in D(\mathfrak{A}, 1)\}$$

so that $V(\mathfrak{A}; a_1, \ldots, a_n)$ is a compact convex subset of $\mathbf{C}^n$. (For $x \in \mathfrak{A}$, $D(\mathfrak{A}, x) = \{f \in \mathfrak{A}' : f(x) = 1 = \|f\|\}$.) For brevity we write $\mathbf{a} = (a_1, \ldots, a_n)$ and

$$V(\mathfrak{A}; \mathbf{a}) = \{f(\mathbf{a}) : f \in D(\mathfrak{A}, 1)\}.$$

Given $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{C}^n$ we write

$$\boldsymbol{\lambda} \cdot \mathbf{a} = \lambda_1 a_1 + \ldots + \lambda_n a_n$$

$$\boldsymbol{\lambda} \cdot \mathbf{z} = \lambda_1 z_1 + \ldots + \lambda_n z_n.$$

Let Ξ be a compact convex subset of $\mathbf{C}^n$ and let $\|\cdot\|_\infty$ denote the supremum norm for bounded functions on Ξ. Let

$$u_r(\mathbf{z}) = z_r \quad (\mathbf{z} = (z_1, \ldots, z_n))$$

and let

$$\omega(\boldsymbol{\lambda}) = \|\exp(\boldsymbol{\lambda} \cdot \mathbf{u})\|_\infty \quad (\boldsymbol{\lambda} \in \mathbf{C}^n).$$

The method of § 1 generalizes to give

$$V(\mathfrak{A}; \mathbf{a}) \subset \Xi \Leftrightarrow \|\exp(\boldsymbol{\lambda} \cdot \mathbf{a})\| \leqslant \omega(\boldsymbol{\lambda}) \quad (\boldsymbol{\lambda} \in \mathbf{C}^n).$$

Let $A(\Xi)$ be the set of all $f\colon \Xi \to \mathbf{C}$ such that

$$f(\mathbf{z}) = \int \exp(\boldsymbol{\lambda} \cdot \mathbf{z})\, d\mu(\boldsymbol{\lambda}) \quad \text{where} \int \omega\, d|\mu| < \infty$$

and let $\|f\|$ be the infimum of $\int \omega\, d|\mu|$ over all such representations.

Let $E(\Xi)$ be the set of entire functions ϕ in n-variables such that $\|\phi\| = \sup\{|\phi(\boldsymbol{\lambda})|/\omega(\boldsymbol{\lambda})\colon \boldsymbol{\lambda} \in \mathbf{C}^n\} < \infty$. Then, as in § 1, $A(\Xi)$ is a complex unital commutative Banach algebra, the polynomials are dense in $A(\Xi)$ and the maximal ideal space may be identified with the joint spectrum of $u_1, \ldots, u_n$, i.e. Ξ. Moreover, $E(\Xi)$ is the dual of $A(\Xi)$ under the natural pairing. We state informally the generalization of Theorem 1.10.

THEOREM 3.1. *$A(\Xi)$ is the extremal Banach algebra subject to*

$$V(\mathfrak{A}, \mathbf{a}) \subset \Xi,\ a_j a_k = a_k a_j \ (j, k = 1, \ldots, n).$$

Remark. The mutual commuting of the elements is required to give

$$\exp(\boldsymbol{\lambda} \cdot \mathbf{a}) \exp(\boldsymbol{\mu} \cdot \mathbf{a}) = \exp(\boldsymbol{\lambda} + \boldsymbol{\mu}) \cdot \mathbf{a}).$$

The simplest case occurs when Ξ is a direct product, say $\Xi = K_1 \times K_2 \times \ldots \times K_n$ where each K_j is compact convex in $\mathbf{C}$. In this case it is easy to show that

$$\omega(\boldsymbol{\lambda}) = \omega_1(\lambda_1)\, \omega_2(\lambda_2) \ldots \omega_n(\lambda_n)$$

where ω_j is determined by K_j as in § 1. Given $f \in A(\Xi)$ of the form

$$f(\mathbf{z}) = f_1(z_1)\, f_2(z_2) \ldots f_n(z_n)$$

where $f_r \in (A(K_r), \|\cdot\|_r)$, $r = 1, \ldots, n$, it is straightforward to verify that

$$\|f\| = \|f_1\|_1 \|f_2\|_2 \cdots \|f_n\|_n.$$

Moreover the linear span of such functions is dense in $A(\Xi)$. In other words this case gives a rather trivial generalization of the one variable case; in fact $A(\Xi)$ is the projective tensor product of the $A(K_r)$.

For a non-trivial application of Theorem 3.1, recall that $x \in \mathfrak{A}$ is *normal* if $x = h + ik$ where h, k are Hermitian and $hk = kh$. For such elements it is known that $V(x) = \text{co Sp}(x)$ and $\|x\| \leqslant 2\varrho(x)$ (see [6] Theorem 5.14). When all the powers of h and k are Hermitian it follows from the Vidav-Palmer theorem (see e.g. [6] Theorem 5.9) that $\varrho(x) = \|x\|$. For general Banach algebras, Crabb [11] gave an example of a normal element x with $\varrho(x) = 1$, $\|x\| = \sqrt{2}$. We show below that

$$\max \{\|x\| : x \text{ normal}, \varrho(x) = 1\} = 2.$$

More generally, we consider the evaluation of

$$k_n = \max \{\|x^n\| : x \text{ normal}, \varrho(x) = 1\}. \tag{1}$$

It is easy to see that the problem corresponds to an application of Theorem 3.1 in the case

$$\Xi = \{(s, t) : s, t \in \mathbf{R}, s^2 + t^2 \leqslant 1\}.$$

We then have

$$\omega(\lambda_1, \lambda_2) = \sup \{\exp (s \operatorname{Re} \lambda_1 + t \operatorname{Re} \lambda_2) : s, t \in \mathbf{R}, s^2 + t^2 \leqslant 1\} = \exp (\{(\operatorname{Re} \lambda_1)^2 + (\operatorname{Re} \lambda_2)^2\}^{\frac{1}{2}}).$$

For the rest of this section, ω will have the above definition. We wish to evaluate

$$\sup \left\{ \left| n! \left(\frac{\partial}{\partial \lambda_1} + i \frac{\partial}{\partial \lambda_2} \right)^n \phi(0, 0) \right| : \phi \in E(\Xi), \|\phi\| = 1 \right\}.$$

LEMMA 3.2. *Let ϕ be an entire function of 2 variables such that*

(i) $|\phi(\lambda, \mu)| \leqslant \exp (|\lambda| + |\mu|)$ $(\lambda, \mu \in \mathbf{C})$,
(ii) $|\phi(\lambda, \mu)| \leqslant 1$ $(\lambda, \mu \in i\mathbf{R})$,
(iii) $|\phi(\lambda \cos \theta - \mu \sin \theta, \lambda \sin \theta + \mu \cos \theta)| = |\phi(\lambda, \mu)|$ $(\lambda, \mu \in \mathbf{C}, \theta \in \mathbf{R})$.

Then $|\phi(\lambda, \mu)| \leqslant \omega(\lambda, \mu)$ $(\lambda, \mu \in \mathbf{C})$.

Proof. Given $\mu \in i\mathbf{R}$ we have

$$|\phi(\lambda, \mu)| \leqslant e^{|\mu|} e^{|\lambda|} \quad (\lambda \in \mathbf{C}), \quad |\phi(\lambda, \mu)| \leqslant 1 \quad (\lambda \in i\mathbf{R})$$

and so the Phragmen-Lindelöf principle gives

$$|\phi(\lambda, \mu)| \leqslant \exp (|\operatorname{Re} \lambda|) \quad (\lambda \in \mathbf{C}).$$

Given arbitrary $\lambda, \mu \in \mathbf{C}$, choose $\theta \in \mathbf{R}$ such that $\lambda \sin \theta + \mu \cos \theta \in i\mathbf{R}$. Then

$$|\phi(\lambda, \mu)| = |\phi(\lambda \cos \theta - \mu \sin \theta, \lambda \sin \theta + \mu \cos \theta)| \leqslant \exp (|\operatorname{Re} (\lambda \cos \theta - \mu \sin \theta)|) \leqslant \omega(\lambda, \mu).$$

THEOREM 3.3. *Let k_n be as in* (1), *let*

$$\mathfrak{G}_n = \{g \in E([-1, 1]) : \|g\| = 1, g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0\}$$

and let $\mathfrak{P}_{n-1}$ be the real subspace of $A([-1, 1])$ consisting of the real polymonials in u of degree at most $n-1$. Then

$$k_n = 2^n \sup \{|g^{(n)}(0)| : g \in \mathfrak{G}_n\} = 2^n \operatorname{dist}(u^n, \mathfrak{P}_{n-1}).$$

Proof. Let $\mathfrak{F}_n$ be the set of functions $g \in \mathfrak{G}_n$ of the form

$$g(z) = \sum_{r=0}^{\infty} c_r z^{n+2r} \quad (z \in \mathbf{C}).$$

Note by Bernstein's theorem that $|c_r| \leqslant 1/(n+2r)!$ $(r = 0, 1, 2, \ldots)$. Given $g \in \mathfrak{F}_n$ define $\phi_g (\equiv \phi)$ on $\mathbf{C}^2$ by

$$\phi(\lambda, \mu) = \sum_{r=0}^{\infty} c_r (\lambda - i\mu)^{n+r} (\lambda + i\mu)^r.$$

Then ϕ is entire and $|\phi(\lambda, \mu)| \leqslant \exp(|\lambda| + |\mu|)$. Given $s, t \in \mathbf{R}$ let $y = (s^2 + t^2)^{\frac{1}{2}}$ and then

$$|\phi(is, it)| = |\sum_{r=0}^{\infty} c_r (iy)^{n+2r}| \leqslant 1.$$

Also
$$\phi(\lambda \cos\theta - \mu \sin\theta, \lambda \sin\theta + \mu \cos\theta) = e^{-in\theta} \phi(\lambda, \mu)$$

and so Lemma 3.2 gives $|\phi(\lambda, \mu)| \leqslant \omega(\lambda, \mu)$.

Given $\phi \in E(\Xi)$, $\|\phi\| = 1$, let $\psi_\phi (\equiv \psi)$ be defined on $\mathbf{C}^2$ by

$$\psi(\lambda, \mu) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \phi(\lambda \cos\theta - \mu \sin\theta, \lambda \sin\theta + \mu \cos\theta)\, d\theta.$$

Then ψ is entire and $|\psi(\lambda, \mu)| \leqslant \omega(\lambda, \mu)$. A routine calculation shows that $\psi = \phi_g$ for some $g \in \mathfrak{F}_n$.

If $a = u_1 + iu_2$ in $A(\Xi)$ then

$$\Phi_\phi(a^n) = \Phi_\psi(a^n) = 2^n g^{(n)}(0).$$

Theorem 3.1 now gives $k_n = 2^n \sup \{|g^{(n)}(0)| : g \in \mathfrak{F}_n\}$.

Given $g \in \mathfrak{G}_n$, let $h(z) = \frac{1}{2}\{g(z) + (-1)^n g(-z)\}$. Then $h \in \mathfrak{F}_n$ and $|h^{(n)}(0) = |g^{(n)}(0)|$. This proves the first part of the theorem.

Let $\mathfrak{H}_n = \{g \in \mathfrak{G}_n : g(\mathbf{R}) \subset \mathbf{R}\}$. Given $g \in \mathfrak{G}_n$ let $h(z) = \frac{1}{2}\{g(z) + \overline{g(z)}\}$ and we see that $k_n = 2^n \sup \{|g^{(n)}(0)| : g \in \mathfrak{H}_n\}$.

Finally note that $\Phi_\phi(a^n) = 2^n \Phi_g(u^n)$ and use Theorem 1.10.

COROLLARY 3.4.

(i) $k_1 = 2$.

(ii) *There is* $M > 0$ *such that* $k_n \geqslant Mn^{\frac{1}{3}}$ $(n = 1, 2, 3, \ldots)$.

Proof. (i) Take $g = \sinh$. In fact, using Theorem 2.4, one can show that g gives the *unique* functional Φ on $A(\Xi)$ with $\Phi(a) = 2$, $\|\Phi\| = 1$. (ii) Take $g(z) = J_n(iz)$, where J_n is the usual Bessel function. Since $\sup\{|J_n(x)| : x > 0\}$ is of order $n^{-\frac{1}{3}}$, the result follows.

It is easy to see, from Montel's theorem, that k_n is attained at some member of $\mathfrak{G}_n$; but it is not attained by J_n (nor by the slightly better functions $z \to (iz)^{\frac{1}{2}} J_{n-\frac{1}{2}}(iz)$). Note that the power inequality fails for some normal elements.

Finally we give a partial solution to a related problem.

THEOREM 3.5.

$$\max\{\|u\| : u \text{ normal}, \varrho(u) = 1, \operatorname{Re} u, \operatorname{Im} u \text{ positive}\} \leqslant \tfrac{1}{2}(1 + \sqrt{3}).$$

Proof. Let u be normal with $\varrho(u) = 1$, $\operatorname{Re} u$, $\operatorname{Im} u$ positive. Then $V(u) \subset \{z : |z| \leqslant 1, \operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0\}$, and for the extremal case we may assume equality. Let $e^{-i\pi/4} u = p + iq$ with p, q Hermitian.

Then
$$V(p) = [0, 1], \quad V(q) = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

and so
$$\|u\| = \|p + iq\| \leqslant \|p - \tfrac{1}{2}\| + \|iq + \tfrac{1}{2}\| = \tfrac{1}{2}(1 + \sqrt{3})$$

by Sinclair [13].

4. Complex Banach algebras with $v = \varrho$

Let $\mathfrak{A}$ be a complex unital Banach algebra with $v = \varrho$. By [6] Theorem 4.7, $\mathfrak{A}$ is commutative and ϱ is an algebra norm with $\varrho \leqslant \|\cdot\| \leqslant e\,\varrho$. In fact, a sharper estimate holds.

THEOREM 4.1. *Let* $\mathfrak{A}$ *be a complex unital Banach algebra with* $v = \varrho$. *Then* $\|a\| \leqslant \frac{1}{2} e \varrho(a)$ $(a \in \mathfrak{A})$.

Proof. Let $a \in \mathfrak{A}$ with $\varrho(a) < 1$ and let Γ be the unit circle. Then

$$-\frac{2}{e}\, a = \frac{1}{2\pi i} \int_\Gamma \exp\left(\frac{za + 1}{za - 1}\right) \frac{dz}{z^2}.$$

Note that $\operatorname{Re} \lambda \leqslant 0$ for $\lambda \in \operatorname{Sp}((za + 1)(za - 1)^{-1})$, $|z| = 1$. Since $v = \varrho$ we have $\|\exp(x)\| = \varrho(\exp(x))$ $(x \in \mathfrak{A})$. Therefore $2e^{-1}\|a\| \leqslant 1$ and the proof is complete.

Let K be as in § 1 and recall that $\mathring{A}(K)$ is the algebra of continuous functions on K that are analytic on the interior of K. Any norm on $\mathring{A}(K)$ with $v=\varrho$ gives $V(f)=\operatorname{co}\operatorname{Sp}(f)$. The Vidav-Palmer theorem shows that when $K\subset[-1, 1]$, $v=\varrho$ if and only if the norm is the spectral norm; this latter assertion fails if K has interior. Define $\|\cdot\|$ on $\mathring{A}(K)$ by

$$\|f\| = \inf\{\sum|c_n|\varrho(g_n): f=\sum c_n g_n, g_n\in \exp \mathring{A}(K)\}.$$

THEOREM 4.2. *$(\mathring{A}(K), \|\cdot\|)$ is the extremal singly generated complex unital Banach algebra with $V(u)\subset K$, $v=\varrho$.*

Proof. Straightforward by the methods of § 1.

PROPOSITION 4.3. *When K is the closed unit disc, $\|u\|=\frac{1}{2}e$.*

Proof. Let $\Phi(f)=f'(0)$ $(f\in\mathring{A}(K))$, so that $\Phi\in\mathring{A}(K)'$, $\Phi(u)=1$. It is enough to show that $\|\Phi\|\leqslant 2e^{-1}$, or equivalently,

$$|\exp(f(0))\, f'(0)|\leqslant 2e^{-1}\varrho(\exp f) \quad (f\in\mathring{A}(K)).$$

Let $g=f-f(0)$, so that $g\in\mathring{A}(K)$, $g(0)=0$. A classical exercise gives $|g'(0)|\leqslant 2$ when $\operatorname{Re} g\leqslant 1$ and so

$$|g'(0)|\leqslant 2e^{-1}\exp(\sup \operatorname{Re} g) \quad (g\in\mathring{A}(K), g(0)=0).$$

The result follows.

When K has interior, the Riemann mapping theorem gives a $\tau\in\mathring{A}(K)$ which is a home omorphism onto the closed unit disc Δ. This induces an isometric isomorphism between $\mathring{A}(K)$ and $\mathring{A}(\Delta)$ and gives $\varrho(\tau)=1$, $\|\tau\|=\frac{1}{2}e$ in $(\mathring{A}(K), \|\cdot\|)$.

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