# AN OPTIMAL STOPPING PROBLEM WITH LINEAR REWARD

BY

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## 1. Introduction

A gambler is to make one of two decisions: quitting or pursuing the game purely on the basis of present information. Both decisions are not equally favorable, but their efficiency depends on the unknown outcome of the game. A reward function measures this efficiency: the higher its value, the better the situation from the gambler's viewpoint. Hence he has to decide whether the future gain will outweigh the loss due to stopping or further unfavorable moves. We aim at studying the decision rules or strategies yielding the best possible average gains.

Such problems involve the time $t$ and a position $x$, meaning the "state of affairs". We deal only with the case in which $x$ moves according to a brownian motion. This is not so special a problem as it may seem; it is in fact typical and most of the problem studied to date either exhibit a brownian behavior, genuine or transformed, or can be so approximated by a suitable scaling, provided only that the number of trials is very large.

The game is now specified by fixing the "reward function" $g = g(x, t)$ with the following meaning: if the gambler decides to quit at time $t$ when his state of affairs is $x$, his reward will be $g(x, t)$. Contrariwise, if he decides to play for a possibly random period of time $\tau$, his average reward would be

$$Eg\,(x + x_\tau, t + \tau)$$

where $x_t$ is the brownian motion starting at $x_0 = 0$. Notice that $(x + x_s, t + s)$ is the customary space-time brownian motion starting at $(x, t)$; it is the graph of the usual brownian motion.

Of course, it is not permitted to the gambler to foresee the future. This is built in by allowing only "stopping times" $\tau$, by which you will understand that the event $\tau \leqslant t$ de-

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pends upon the brownian motion up to time $t$ only and not upon what it does afterwards. This paper deals only with games whose playing time is bounded above by some $T < \infty$ (games with a finite horizon).

The optimal reward $\hat{g}(x, t)$ starting at time $t$ and with a state of affairs $x$ is now obtained by maximizing

$$Eg\,(x + x_\tau, t + \tau)$$

over all stopping time $\tau \leqslant T - t$. The optimal policy or strategy is the stopping time $\tau$ achieving this maximum. This maximum is attained for some stopping time, provided $g$ satisfies a growth condition, reminiscent of Tychonov's condition for the uniqueness of the solution to the heat equation. This optimal stopping time consists of stopping as soon as you hit the region where $\hat{g} = g$. This region is called the stopping region, i.e. the region where no policy is better than just pulling out of the game. In its complement, the so-called continuation region (where $\hat{g} > g$), it pays to go on playing the game. The boundary $x = s(t)$ separating these two regions is the optimal (stopping) boundary. The problem also has a potential theoretical version and a physical interpretation in terms of Stefan's ice-melting problem. Proofs and additional information can be found in P. van Moerbeke [30]. Section 2 contains a succinct exposition of these basic facts.

The purpose of this paper is to give a description of the optimal boundary for a class of rewards where a gambler loses one per unit time as long as he plays and is only rewarded at the horizon $T$ with an amount $h(y)$, if the brownian motion reaches the horizon at $y$. The highest expected reward of a gambler starting at $(x, t)$ is obtained by maximizing

$$-E\,(\tau; \tau < T - t) + E(h(x + x_{T-t}); \tau = T - t), \tag{1}$$

over all stopping times $\tau \leqslant T - t$. The reward function $g$ is somewhat disguised in this formulation, but from section 2.2 it will appear that

$$\begin{aligned} g &= T - t \qquad t < T \\ &= h(x) \geqslant 0 \quad t = T \end{aligned} \tag{2}$$

is the reward function corresponding to (1). Theorem 1 aims at giving a description of the continuation region for a compact final gain $h(x)$, more precisely:

THEOREM 1: *Let*

$$\begin{aligned} g &= T - t \qquad t < T \\ &= h(x) > 0 \quad t = T, \; -a < x < a \\ &= 0 \qquad\quad\;\; t = T, \; |x| \geqslant a. \end{aligned}$$

*Let $h(x)$ be $C^3$ in $[-a, a]$, subject to the lateral conditions $(\partial/\partial x)h(\pm a) = 0$ and $\frac{1}{2}(\partial^2/\partial x^2)h(\pm a)$*

$=1$. *Moreover assume that* $\frac{1}{2}(\partial^2/\partial x^2)h$ *changes sign at most twice in* $[-a, a]$ *and* $\frac{1}{2}(\partial^2/\partial x^2)h-1$ *at most a finite number of times.*

Then the continuation region is a (simply) connected bounded region $C$, whose boundary $x=s(t)$ has a continuous derivative ($|\dot{s}|<\infty$) except for one critical point $t_\infty$, the latter corresponds to the lowest level of the continuation region (see figure 1). Moreover, the number of zeros of $\dot{s}$ cannot exceed the number of zeros of $-\frac{1}{2}(\partial^2/\partial x^2)h+1$.

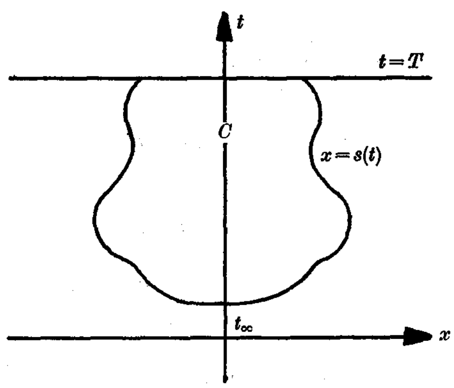


Fig. 1

The proof of this theorem which will be presented in section 3 goes through rather involved arguments. The main difficulty arises from a lack of monotonicity inherent to this problem, opposed to Stefan's ice-melting problem where the water-ice interface only moves in one direction; the relation of the optimization problem with a modified Stefan problem will be explained in sections 2.5 and 4.1.

The final section will be devoted to the study of the boundary near the critical point $t_\infty$. This discussion aims at a number of heuristic arguments in favor of the fact that the boundary either exhibits a cusp at the critical point, or behaves flatter than

$$\sqrt{t-t_\infty} \quad \text{for } t\downarrow t_\infty$$

and not flatter than

$$(1+\varepsilon)\sqrt{2(t-t_\infty)\log\log\frac{1}{t-t_\infty}}$$

for any $\varepsilon>0$ and for $t\downarrow t_\infty$.

I wish to express my deep gratitude to my teacher, Professor Henry P. McKean, who most generously guided my path throughout this work (and much more). I am grateful to Professor Mark Kac for a number of fruitful conversations with regard to some crucial steps in this proof.

## 2. Background

### 2.1. The potential theory for the space-time brownian motion

This potential theory was developed by J. L. Doob [8]. The reader will find a short description of this theory in Ito-McKean [15].

A real-valued function $f$ defined on $R^2$ is called *excessive* for the space-time brownian motion in an open domain $D$ of $R^2$, if

(i) $f$ is bounded below;

(ii) $Ef(x+x_\tau, t+\tau) \leq f(x, t)$ for every stopping time $\tau$ not exceeding the first exit time $\tau_D$ from $D$.

(iii) $Ef(x+x_{\tau_n}, t+\tau_n) \uparrow f(x, t)$ for every sequence of stopping times $\tau_n < \tau_D$ such that $P(\tau_n \downarrow 0) = 1$.

This definition is equivalent to another in which (ii) and (iii) are replaced by

(ii') $Ef(x+x_{s\wedge\tau_D}, t+s\wedge\tau_D) \leq f(x, t)$ for every $s \geq 0$.

(iii') $Ef(x+x_{s\wedge\tau_D}, t+s\wedge\tau_D) \uparrow f(x, t)$ when $s \downarrow 0$.

Note that (iii') is superfluous if $f$ is continuous. In such a case, excessivity is a local property, indeed, a continuous function which is bounded below is excessive as soon as

$$Ef(x+x_{\tau_n}, t+\tau_n) \leq f(x, t)$$

for all $(x, t) \in D$ and large $n$, where $\tau_n$ is the first exit time from the disc of radius $1/n$ centered at $(x, t)$. (Blumenthal-Getoor [4], p. 93).

Consider now functions bounded from below with at least two continuous derivatives in $x$ and one in $t$. Assume

$$E\int_0^{\tau_D} \frac{\partial f}{\partial x}(x+x_s, t+s)^2\, ds < \infty.$$

Then Ito's lemma tells you that

$$Ef(x+x_\tau, t+\tau) - f(x, t) = E\int_0^\tau H(x+x_s, t+s)\, ds \tag{3}$$

where $H = \partial f/\partial t + \frac{1}{2}\partial^2 f/\partial x^2$. For a proof of this fact, see section 2.3.5 of "Stochastic Integrals" by H. P. McKean [19]. Formula (3) gives an alternative characterization of excessivity for sufficiently differentiable functions: excessivity in the domain $D$ is the same as $H \leq 0$ for all $(x, t)$ in $D$.

A function $f$ is called *parabolic* in a domain $D \subset R^2$ if

(i) $f$ is excessive in $D$,

(ii) $Ef(x+x_{\tau_U}, t+\tau_U) = f(x, t)$, where $\tau_U$ is the first exit time from any open set $U$ with compact closure in $D$. Parabolic functions are $C^\infty$ and enjoy the property

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}=0.$$

Conversely any function bounded from below satisfying this equation is automatically parabolic in view of (3).

Finally a boundary point $(x, t)$ will be called *regular* for an open region $D$ if

$$P(x+x_s, t+s \in D, s \downarrow 0) = 0.$$

## 2.2. Definition and further characterizations of the optimal reward

Throughout section 2, definition and lemmas will be stated for more general rewards than is actually needed in Theorem 1. Assume that the reward $g$ is bounded below, is $C^\infty$ in $t<T$ and has limits $g(x, T-)$ for $t \uparrow T$; a discontinuity at $t=T$ is permitted, but $g(x, T)$ itself is to be continuous. The notation

$$h(x) = g(x, T)-g(x, T-)$$

is often employed and $h$ will be called the *final gain*. You may assume $h(x)\geqslant 0$; the motivation is that, if $h(x)<0$ in an interval, it is more favorable to stop the game a little before hitting the final horizon $t=T$.

Henceforth we impose upon $g$ a growth condition, called the *Tychonov condition*. This condition will be useful in matters like the continuity of $\hat{g}$, the existence of an optimal strategy, etc., ... If the functions

$$g, \frac{\partial g}{\partial t}, \frac{\partial g}{dx}, \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial t}, \frac{\partial^3 g}{\partial x^3}$$

and

$$h, \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x^2} \text{ and } \frac{\partial^3 h}{\partial x^3}$$

are bounded by $e^{o(x^2)}$ when $|x|$ tends to $\infty$ uniformly in any strip of finite depth $[t, T]$, then $g$ is said to satisfy the Tychonov condition.

The *optimal reward* is the supremum

$$\sup Eg(x+x_\tau, t+\tau) \equiv \hat{g}(x, t)$$

over all stopping times $\tau \leqslant T-t$. The *stopping regions* $S$ is defined as the set of points where quitting is best, i.e. where $\hat{g}=g$. Its complement $C$ is called the *continuation region*; this is where it pays to play the game, i.e., where $\hat{g} > g$. The *optimal boundary* separates the two regions.

As already observed by J. L. Snell [25] in the context of martingales, $\hat{g}$ can be regarded as the smallest excessive function exceeding $g$ and the backwards induction used by several authors in the discrete case is adapted to the present case in the form

$$\hat{g}(x, t) = \sup_n g_n(x, t)$$

where $g_0 = g$ and
$$g_n(x, t) = \sup_{0 \leq s \leq T-t} E g_{n-1}(x + x_s, t+s).$$

Moreover, from Ito's lemma,

$$(\hat{g} - g)(x, t) = \sup_{\tau \leq T-t} \left[ E \int_0^\tau H(x + x_s, t + s -)\, ds + E(h(x + x_{T-t}); t + \tau = T) \right], \tag{4}$$

where
$$H \equiv \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \qquad t < T$$

can be interpreted as the *payoff-rate*. From (4), it is obvious that maximizing (1) is the same as maximizing

$$Eg(x + x_\tau, t + \tau)$$

over all stopping times $\tau \leq T - t$, where $g$ is given by (2). Also as a consequence of (4) one never stops at a point where $H > 0$, as proceeding a little while will improve one's gain.

### 2.3. Analytical characterizations of the optimal reward $\hat{g}$

We state here a few lemmas without proof, to be used in the sequel:

LEMMA 1: (E. B. Dynkin [9]) *$\hat{g}$ is continuous throughout and is parabolic in the continuation region, i.e.*

$$\left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \hat{g} = 0 \quad in \quad C.$$

LEMMA 2: (H. M. Taylor [28]). *Stopping at $\hat{g} = g$ is the best policy.*

LEMMA 3: (J. A. Bather [1]). *At a boundary point $(x, t)$ regular for the continuation region $C$,*

$$\frac{\partial \hat{g}}{\partial x} = \frac{\partial g}{\partial x}.$$

LEMMA 4: (Gevrey [13]), *Let $u$ satisfy the equation*

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = H(x, t)$$

*with continuous $H$ in a region $C$ with zero boundary data. Whenever its boundary is Lipschitz continuous, the derivative $(\partial u/\partial x)$ is continuous up to the boundary and its limit equals its value at the boundary.*

The next lemma will play an important role in the proof of Theorem 1.

LEMMA 5: *Let the optimal boundary $x=s(t)$ be continuously differentiable ($|\dot{s}|<\infty$) in an interval $(t_0, t_1)$; then in this interval,*

(a) *$(\partial/\partial x)\,(\hat{g}-g)$ and $(\partial/\partial t)\,(\hat{g}-g)$ both are continuous across and they vanish at the boundary.*

(b) *$\frac{1}{2}(\partial^2/\partial x^2)(\hat{g}-g)$ is continuous in $C$ up to the boundary and equals $-H$ at the boundary.*

(c) *$(\partial^2/\partial x\partial t)(\hat{g}-g)$ is continuous in $C$ up to the boundary and equals $2H(s(t), t)\,\dot{s}(t)$ at the boundary.*

(d) *$(\partial^2/\partial t^2)(\hat{g}-g)$ $(s(t), t)$ exists and equals $-2H(s(t), t)\,\dot{s}^2(t)$ at a boundary point where $\dot{s}>0$.*

*Proof.* Thanks to Lemma 1, $\hat{g}-g$ is continuous everywhere and satisfies

$$\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)(\hat{g}-g)=-H \text{ in } C \text{ with } (\hat{g}-g)\,(s(t), t)=0. \tag{5}$$

By assumption, $s(t)$ is differentiable between $t_0$ and $t_1$. Therefore, using Lemma 4, $(\partial/\partial x)(\hat{g}-g)$ exists and is continuous up to the boundary and according to Lemma 3,

$$\frac{\partial(\hat{g}-g)}{\partial x}(s(t), t)=0 \quad t_0<t<t_1 \tag{6}$$

which is half of (a). Now $(\partial/\partial x)\,(\hat{g}-g)$ is continuous in $(t_0, t_1)$ and satisfies

$$\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}(\hat{g}-g)=-\frac{\partial H}{\partial x} \quad \text{in } C\cap(t_0, t_1)$$

and, at the boundary $$\frac{\partial}{\partial x}(\hat{g}-g)\,(s(t), t)=0.$$

According to Lemma 4 again, $(\partial^2/\partial x^2)(\hat{g}-g)$ exists everywhere and is continuous up to the boundary. Since $\hat{g}$ satisfies the backward heat equation this automatically implies that $(\partial/\partial t)(\hat{g}-g)$ exists and is continuous up to the boundary. Differentiating $(\hat{g}-g)(s(t), t)$ with respect to $t$ and taking into account (6) yields

$$\frac{\partial}{\partial t}(\hat{g}-g)\,(s(t), t)=0 \qquad t_0<t<t_1 \tag{7}$$

which is the other half of (a).

From (7) it develops that

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x^2}(s(t),t) = -H(s(t),t), \quad t_0 < t < t_1 \tag{8}$$

which establishes (b). Next observe that

$$u = \frac{\partial}{\partial t}(\hat{g}-g)$$

is continuous and satisfies

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = -\frac{\partial H}{\partial t} \quad \text{in } C \cap (t_0, t_1)$$

and, at the boundary, $$u(s(t),t) = 0.$$

Again Lemma 4 applied to $u$ leads to the continuity of $(\partial^2/\partial x\partial t)(\hat{g}-g)$ up to the boundary. Its value at the boundary is given by differentiating $(\partial/\partial x)(\hat{g}-g)(s(t),t)$ with respect to $t$:

$$\dot{s}(t)\frac{\partial^2}{\partial x^2}(\hat{g}-g)(s(t),t) + \frac{\partial^2}{\partial x\partial t}(\hat{g}-g)(s(t),t) = 0$$

which combined with (8) completes the proof of point (c).

Finally, as to (d), if $\dot{s} > 0$, then

$$\frac{\partial}{\partial t}(\hat{g}-g)(s(t),t) = 0$$

implies
$$\begin{aligned} 0 &= \frac{1}{\delta}\left(\frac{\partial(\hat{g}-g)}{\partial t}(s(t+\delta),t+\delta) - \frac{\partial(\hat{g}-g)}{\partial t}(s(t),t)\right) \\ &= \frac{s(t+\delta)-s(t)}{\delta}\,\frac{\dfrac{\partial(\hat{g}-g)}{\partial t}(s(t+\delta),t+\delta) - \dfrac{\partial(\hat{g}-g)}{\partial t}(s(t),t+\delta)}{s(t+\delta)-s(t)} \\ &\quad + \frac{\dfrac{\partial(\hat{g}-g)}{\partial t}(s(t),t+\delta) - \dfrac{\partial(\hat{g}-g)}{\partial t}(s(t),t)}{\delta}. \end{aligned}$$

Hence, by making $\delta \downarrow 0$,

$$\frac{\partial^2(\hat{g}-g)}{\partial t^2} = -\dot{s}(t)\frac{\partial^2(\hat{g}-g)}{\partial x\partial t}(s(t),t) = -2\dot{s}(t)^2 H(s(t),t),$$

using (c) in the last equality. This establishes (d).

We are now in a position to state

PROPOSITION 1. *Assume the continuation region $C = (\hat{g} > g)$ to be bounded by one or more continuous curves $x = s(t)$, which are once continuously differentiable, except possibly* *for a number of isolated singular points (i.e. where $s(t)$ does not exist or where $|\dot{s}(t)| = \infty$). Then*

$$\frac{\partial \hat{g}}{\partial t} + \frac{1}{2}\frac{\partial^2 \hat{g}}{\partial x^2} = 0 \quad in\ C \tag{9}$$

$$\hat{g}(s(t), t) = \lim_{(x,t)\to(s(t),t)} \hat{g}(x, t) = g(s(t), t) \tag{10}$$

$$\frac{\partial \hat{g}}{\partial x}(s(t), t) = \lim_{(x,t)\to(s(t),t)} \frac{\partial \hat{g}}{\partial x}(x, t) = \frac{\partial g}{\partial x}(s(t), t) \quad where\ |\dot{s}| < \infty. \tag{11}$$

$$\hat{g}(x, T) = g(x, T). \tag{12}$$

Recall from section 2.2 that $H \leqslant 0$ in the region $\hat{g} = g$.

The boundary conditions (10) and (11) are called the "smooth fit" relations. The problem determined by (9), (10), (11) and (12) constitutes a "free boundary problem". The converse of proposition 1 is also true; more precisely

PROPOSITION 2. *Let $u$ be a Tychonov-type function defined in $t \leqslant T$. Let $C$ be an open set in that region with boundary curve $x = s(t)$ which is differentiable, except possibly for a finite number of isolated points where $\dot{s}$ blows up. If $(\partial/\partial t)u + \frac{1}{2}(\partial^2/\partial x^2)u = 0$ in $C$,*

$$u = g \ at\ (x, t) = (s(t), t),$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial x} \ at\ (x, t) = (s(t), t) \ \ if\ |\dot{s}| < \infty,$$

$$u(x, T) = g(x, T),$$

$$u > g \ in\ C \ and\ u = g \ elsewhere,$$

*and*

$$H \leqslant 0 \ in\ R^2 \backslash C,$$

*then $u$ is actually $\hat{g}$ and $s(t)$ the optimal stopping boundary.*

## 2.4. The integral equation for the optimal boundary

For conversation's sake, let the continuation region be bounded on either side by two curves $x = s_i(t)$, $i = 1, 2$, with $\dot{s}_i(t)$ continuous, bounded and $s_1 < s_2$. Also $\hat{g}$ is assumed to have at least 4 derivatives in $x$ between and up to $s_1$ and $s_2$. Then from Lemma 5 it develops that the auxiliary function

$$v = \frac{\partial}{\partial t}(\hat{g} - g)$$

satisfies the following free boundary problem

$$\frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}=-\frac{\partial H}{\partial t}(x,t) \text{ in } C \tag{13}$$

$$v(s_i(t),t)=0 \tag{14}$$

$$\frac{\partial v}{\partial x}(s_i(t),t)=2H(s_i(t),t)\,\dot{s}_i(t) \tag{15}$$

and
$$v(x,T)=-\frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x)-H(x,T^-),\ s_1(T)\leqslant x\leqslant s_2(T). \tag{16}$$

Green's theorem applied to this boundary value problem yields a system of two integral equations in $s_i(t)$:

$$\begin{aligned}H(s_i(t),t)\,\dot{s}_i(t)=&\int_0^{T-t}dt\int_{s_1(T-\tau)}^{s_2(T-\tau)}\frac{\partial K}{\partial x}(s_i(t),T-t;\xi,\tau)\frac{\partial H}{\partial t}(\xi,T-\tau)\,d\xi\\&+\int_0^{T-t}\frac{\partial K}{\partial x}(s_i(t),T-t;s_2(T-\tau),\tau)\,H(s_2(T-\tau),T-\tau)\,\dot{s}_2(T-\tau)\,d\tau\\&-\int_0^{T-t}\frac{\partial K}{\partial x}(s_i(t),T-t;s_1(T-\tau),\tau)\,H(s_1(T-\tau),T-\tau)\,\dot{s}_1(T-\tau)\,d\tau\\&+\int_{s_1(T)}^{s_2(T)}\frac{\partial K}{\partial x}(s_i(t),T-t;\xi,0)\,v(\xi,T)\,d\xi\end{aligned} \tag{17}$$

where $v(\xi, T)$ is given by (16).

### 2.5. Relation to Stefan's ice melting problem

A natural time-reversal shows that $w(x,\tau)=-v(x,T-\tau)$ and $\sigma_i(\tau)=s_i(T-\tau)$ satisfy

$$\frac{\partial w}{\partial \tau}-\frac{1}{2}\frac{\partial^2 w}{\partial x^2}=-\frac{\partial H}{\partial t}(x,T-\tau) \quad \text{in } C$$

$$w(\sigma_i(\tau),\tau)=0$$

$$\frac{\partial w}{\partial x}(\sigma_i(\tau),\tau)=2H(\sigma_i(\tau),T-\tau)\,\dot{\sigma}_i(\tau)$$

and
$$w(x,0)=\frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x)+H(x,T^-\tau), \quad \sigma_1(0)\leqslant x\leqslant \sigma_2(0). \tag{18}$$

This free boundary problem is at least formally Stefan's ice melting problem: $w$ satisfies the heat equation with a possible heat in- or output, the temperature at the interface vanishes and the heat flux is proportional to the rate of melting (because $H<0$), with a variable heat-capacity; $(\frac{1}{2}(\partial^2/\partial x^2)h(x)+H(x,T^-))$ plays the role of the initial temperature. If you will visualize the interval $\sigma_1(0) \leqslant x \leqslant \sigma_2(0)$ as filled with water at temperature $w(x, 0)$ and $x \leqslant \sigma_1(0)$ and $x \geqslant \sigma_2(0)$ with ice at zero temperature, then the boundary $x = \sigma_1(\tau)$ is the curve described by the interface between water and ice.

### 2.6. A local theorem

The optimal boundary is continuously differentiable for a little while starting from the horizon. Such is the object of Proposition 3. This statement plays an essential role in Theorem 1. Its proof can be found in P. van Moerbeke [30].

PROPOSITION 3. *Let $g$ be a $C^5$(1) reward function in $t < T$ and let the final gain satisfy the conditions*

(i) *$h(x)$ is positive and $C^3$ in $x_1 < x < x_2$ and vanishes outside,*

(ii) $\dfrac{\partial h}{\partial x}(x_i) = 0 \qquad i = 1, 2,$

(iii) $\dfrac{1}{2}\dfrac{\partial^2 h}{\partial x^2}(x_i) = -H(x_i, T^-) > 0.$

*Moreover let $H < 0$ in $x \leqslant x_1$ and $x \geqslant x_2$.*

*Then the continuation region is bounded on either side by two continuously differentiable curves $s_i(t)$, starting at $s_i(T) = x_i$, in a small time interval $(T-\varepsilon, T]$, whose length depends only upon*

$$M = \max_{i=1,2} \left| \frac{\dfrac{1}{2}\dfrac{\partial^3 h}{\partial x^3}(x_i) + \dfrac{\partial H}{\partial x}(x_i, T^-)}{H(x_i, T-)} \right| \tag{19}$$

*and upon the supnorm on $D$ of the functions*

$$H, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial t}, \frac{\partial^2 H}{\partial x \partial t} \text{ and } \frac{\partial^3 h}{\partial x^3}, \tag{20}$$

*where* $$D = \{x_1 - M(T-t) < x < x_2 + M(T-t), \quad t > 0\}.$$

## 3. Proof of Theorem 1

We shall spell out the proof to Theorem 1, under the simplifying assumption that $h(x)$ is an *even function*; then the continuation region is symmetric in $x$. The extension of the proof to the general case requires no major changes but rather clumsier notations.

(1) a $C^n$-function $g$ has bounded and continuous partials $\dfrac{\partial^{r+s} g}{\partial x^r t^s}$ with $r + 2s \leqslant n$.

Outline of the proof: from Proposition 3 we know that, starting from the finite horizon, $\dot{s}$ does not blow up for a little while. Extend the boundary down to a point $t_0$ where $\dot{s}$ ceases to exist. If $s(t_0)=0$, you would have reached the bottom of $C$ [see Fig. 1] and since $h$ is even, the proof would be finished. Therefore the main point is to prove that, if $s(t_0)>0$, then $s(t)$ can be extended a bit below $t_0$ in a continuously differentiable way. An argument involving Proposition 3 shows that this is so as soon as $|\dot{s}|$ is bounded in $(t_0, T]$.

Therefore all the effort of Theorem 1 is put into proving that $|\dot{s}|$ is bounded in $(t_0, T]$. The proof of the latter is achieved through a sequence of lemmas, from 6 to 14. The reader familiar with Kac's [16] "principle of not feeling the curvature" knows that a brownian observer close enough to the boundary will feel it only as a straight line, to a first approximation. This idea is now adapted to the space-time brownian motion so that the boundary can actually be replaced by a tangent line near the point under consideration. The estimates are then performed in this simplified region.

*Remark*: Theorem 1 proves the smoothness of the boundary in the half open interval $(t_\infty, T]$; it remains an open question to establish smoothness of $s(t)$ as a curve at the critical point. In section 4.2 we discuss the shape of the boundary near $t_\infty$.

Theorem 1 refers to a linear reward in $t<T$, which is the easiest situation to handle. For general $g$, one expects a statement of the following nature: the boundary is differentiable except at a finite number of critical points, provided some conditions are imposed upon the level lines of $H$, as will transpire from Lemma 9. The present theorem can be generalized to the case where $g$ is merely concave in $x$, below $t=T$, with a considerable loss of simplicity. Further extensions get more and more delicate and the answers are unsatisfactory.

The lateral conditions for $h$ at the points $\pm\ a$ are essential to give the boundary a nice start at $t=T$. If $(\partial/\partial x)h(\pm a)=0$, but if $\frac{1}{2}(\partial^2/\partial x^2)h(\pm a)$ would be different from 1, the boundary would get off the horizon approximately as a parabola $x=\alpha\sqrt{T-t}$, whose opening $\alpha$ can be computed using the methods of [29] or [30]. If, however, $(\partial/\partial x)h(+a)<0$ and $(\partial/\partial x)h(-a)>0$, then the boundary may behave in a much more singular way.

LEMMA 6. *Assume the conditions of Theorem* 1. *Let the boundary $x=\pm s(t)$ of the continuation region be continuously differentiable in $(t_0, T]$ and pick a point $x=s(t)$ on the boundary with $t_0<t<T$ chosen so that $\dot{s}(t)>0$. Then the following expansion is valid*

$$(\hat{g}-g)\,(x,t+\delta)=\frac{\partial^2}{2}\frac{\partial^2}{\partial t^2}(\hat{g}-g)\,(x,t)+o(\delta^2).$$

*Proof.* If a function $f(t)$ is continuously differentiable on the real line and if the second derivative exists at a point $t$, then we have the following Taylor expansion:

$$f(t+\delta) = f(t) + \delta f(t) + \frac{\delta^2}{2} f''(t) + o(\delta^2).$$

Lemma 5 affirms that $\hat{g}-g$ and $(\partial/\partial t)(\hat{g}-g)(x,t)$ vanish at the boundary; they are continuous near the boundary and $(\partial^2/\partial t^2)(\hat{g}-g)(s(t),t)$ exists whenever $\dot{s}>0$, whence

$$(\hat{g}-g)(x,t+\delta) = (\hat{g}-g)(x,t) + \delta\frac{\partial}{\partial t}(\hat{g}-g)(x,t) + \frac{\delta^2}{2}\frac{\partial^2}{\partial t^2}(\hat{g}-g) + o(\delta^2)$$
$$= \frac{\delta^2}{2}\frac{\partial^2}{\partial t^2}(\hat{g}-g) + o(\delta^2).$$

LEMMA 7. *Let $D$ be the rectangle $-b<x<b$, $0<t<\infty$. Let $u$ be continuous in $\bar{D}$ and satisfy*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \text{ in } D$$

$$u(x,0) = \phi(x) \text{ where } \phi(x) < 0 \text{ for } x<0 \text{ and } \phi(x)>0 \text{ for } x>0.$$
$$u(-b,t) = \psi_1(t) < 0$$
$$u(b,t) = \psi_2(t) > 0$$

*with infinitely differentiable $\psi_1$ and $\psi_2$ (cf. figure 2).*

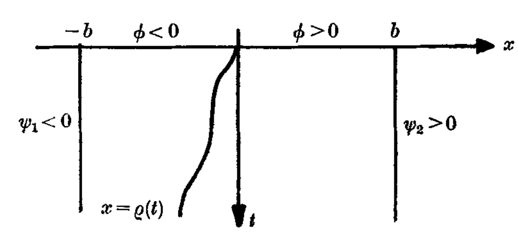


Fig. 2

*Then the set $\{u(x,t)=0\}$ is a continuous curve $x=\varrho(t)$ with $\varrho(0)=0$, and $-b<\varrho(t)<b$.*

*Proof.* Since $u$ is $C^\infty$, for almost all $\varepsilon>0$, $\{u=\varepsilon\}$ does not contain points where $((\partial/\partial x)u, (\partial/\partial t)u)=(0,0)$ (A. B. Brown [6]). Hence $\{u=\varepsilon\}$ is a continuous arc in the $(x,t)$ plane for almost all $\varepsilon$. Choose two such

$$\varepsilon, \varepsilon' < \min(\min|\psi_1|, \min|\psi_2|).$$

There is a continuous arc $\mathcal{A}_\varepsilon$, where $u=\varepsilon$ (resp. $\mathcal{A}_{-\varepsilon'}$ where $u=-\varepsilon'$) starting somewhere between $-b$ and $b$, which inside $|x|<b$ extends as far as you like, by choosing $\varepsilon$ ($\varepsilon'$) small enough. We now prove that any horizontal line $t=t_0$ cuts this curve $\mathcal{A}_\varepsilon$ (resp. $\mathcal{A}_{-\varepsilon'}$) in exactly one point. If $t=t_0$ would cut $\mathcal{A}_\varepsilon$ in more than one point, then perhaps, after a slight rechoos-

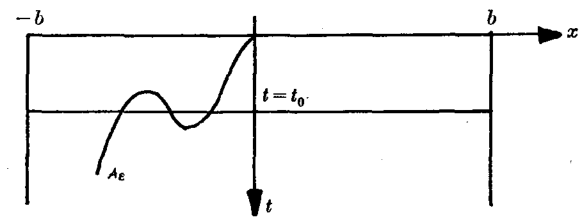


Fig. 3

ing of $t_0$, a piece of the arc $\mathcal{A}_\varepsilon$ would lie above an interval of the line $t=t_0$; see figure 3. But, because of the maximum principle, $u=\varepsilon$ along this interval. This contradicts the fact that $u(x, t)$, a solution of the heat equation, is analytic in $x$. Hence any horizontal line $t=t_0$ cuts $\mathcal{A}_\varepsilon$ in exactly one point. Therefore $\mathcal{A}_\varepsilon$ and $\mathcal{A}_{-\varepsilon'}$ can be represented by two continuous curves $\varrho_\varepsilon^+(t)$ and $\varrho_{\varepsilon'}(t)$; they constitute part of the boundary of $\{u>\varepsilon\}$ and $\{u<-\varepsilon'\}$. Moreover, it is always possible to pick

$$\varrho_{\varepsilon_1}^+(t)>\varrho_{\varepsilon_2}^+(t) \quad \text{for all t,}$$

whenever $\varepsilon_1>\varepsilon_2$. With this choice, $\varrho_\varepsilon^+(t)$ decreases to $\varrho_0^+(t)$ which is upper semi-continuous as a decreasing limit of continuous functions, while $\varrho_{\varepsilon'}^-(t)$ increases to a lower semi-continuous function $\varrho_0^-(t)$, and

$$\varrho_0^+(t)\geqslant\varrho_0^-(t).$$

Assume that
$$\varrho_0^+(t)>\varrho_0^-(t),$$

i.e., pick $x$ such that $\varrho_0^+(t)>x>\varrho_0^-(t)$. In Figure 2, the heat is flowing downwards, i.e., in the $t$-direction, while the corresponding brownian motion is running upwards. If $\tau$ is the first hitting time of $\varrho_{\varepsilon'}$ or $\varrho_\varepsilon^+$ or $t=0$, then, since $u$ is parabolic,

$$Eu(x+x_\tau, t+\tau)=u(x, t)$$

which can be made arbitrarily small, by choosing $\varepsilon$ and $\varepsilon'$ small. Because, for all $\varepsilon$ and $\varepsilon'$

$$\varrho_{\varepsilon'}^-(t)<\varrho_0^-(t)<\varrho_0^+(t)<\varrho_\varepsilon^+(t),$$

$u$ must vanish everywhere between $\varrho_0^-(t)$ and $\varrho_0^+(t)$, which is absurd in view of $u$'s analyticity. Hence $\varrho_0^-(t)=\varrho_0^+(t)$ and its common value $\varrho_0(t)$ is a continuous curve. Because of the maximum principle again you deduce that $u$ does not vanish either in $(-b, \varrho_0(t))$ nor in $(\varrho_0(t), b)$, which implies that $\varrho_0(t)$ is the only place where $u$ vanishes. For a different proof of this theorem originally conjectured by Sturm [27], see Pólya [21](1).

(1) Private communication by M. Schreiber.

LEMMA 8. *Let $g$ be as in Theorem 1. Then $\dot{s}(t)$ changes sign only a finite number of times in $(t_0, T]$.*

*Proof.* The pertinent expression

$$u=\frac{\partial}{\partial t}(\hat{g}-g)=-\frac{1}{2}\frac{\partial^2\hat{g}}{\partial x^2}-\frac{\partial g}{\partial t}=-\frac{1}{2}\frac{\partial^2\hat{g}}{\partial x^2}+1$$

is even in $x$ and at the horizon it coincides with $-\frac{1}{2}(\partial^2/\partial x^2)h+1$, which has a finite number of zeros. Starting at $t=T$, the zeros of $u$ trace out continuous curves, at least for a little while. We now establish that such a curve can be continued as long as it does not intersect any other such curve. That these curves have limits, when $t$ converges to a point $t_1$, is obvious, because otherwise $u(x, t_1)$ would vanish in an interval, which would contradict $u$'s analyticity. At $t_1$, consider a box $D$ as in Figure 4. The maximum principle is responsible for the fact that $u>0$ or $u<0$ inside the respective regions bounded by the curves where

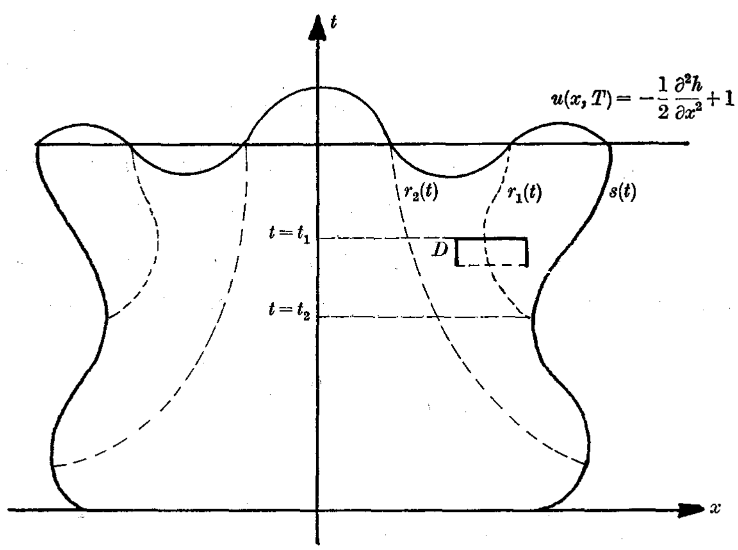


Fig. 4

$u=0$. Hence on the top $t=t_1$ of the box, $u<0$ to the left and $u>0$ to the right of the point where $u=0$. Moreover the fact that $u$ is continuous implies that $u$ is negative on one side and positive on the other side for a little while. Lemma 7 tells you that the set $u=0$ consists of exactly one curve in the box $D$ as long as $u$ does not vanish on the sides.

We conclude that each "root curve" is continuous and can be extended up to the point where it meets another such. The maximum principle also implies that two curves which have intersected each other now stop and have no continuation below. Therefore at each level $t$, the number of roots of $u$ never exceeds the (initial) number of roots of $-\frac{1}{2}(\partial^2/\partial x^2)h+1$.

Observe that each time a "root" curve meets the boundary $\pm s(t)$ (let it be in $t_2$), the slope of the boundary $\dot{s}(t)$ changes sign. For example, in Figure 4, $u>0$ in the region between the root curve $r_1(t)$ and the boundary $s(t)$ ahead of $t_2$; so at the boundary $(\partial/\partial x)u<0$ and since

$$\frac{\partial u}{\partial x}(s(t),t)=\frac{\partial^2}{\partial x\partial t}(\hat{g}-g)=-2\dot{s},$$

$\dot{s}$ is positive there. Furthermore $u<0$ between the next root curve $r_2(t)$ and the boundary below $t_2$. Hence $\dot{s}<0$ for $t<t_2$ down to the point where $r_2(t)$ meets $s(t)$. As a straightforward by-product, the number of zeros of $\dot{s}$ will never exceed the number of zeros of $-\frac{1}{2}(\partial^2/\partial x^2)h+1$.

COROLLARY. *If the hypotheses of Theorem 1 are satisfied and if $s(t)$ is continuously differentiable for $t>t_0$, then lim $s(t)$, $t\downarrow t_0$ exists.*

*Proof.* Since $s(t)$ is continuous in $(t_0, T]$ and is monotone except for a finite number of sign changes of $\dot{s}$, lim $s(t)$ exists when $t$ converges to $t_0$.

LEMMA 9. *Consider $g$ as in Theorem 1 and let $s$ be continuously differentiable for $t>t_0$. Then $(\partial^2/\partial x^2)\hat{g}>0$ in a strip of fixed width around the boundary extending down to $t_0$.*

*Proof.* From Lemma 5 we know that $\frac{1}{2}(\partial^2/\partial x^2)\hat{g}$ is continuous in $-s(t)\leqslant x\leqslant s(t)$ for $t_0<t\leqslant T$, satisfies the backward heat equation in the continuation region between $t_0$ and $T$ and, at the boundary

$$\frac{1}{2}\frac{\partial^2\hat{g}}{\partial x^2}(\pm s(t),t)=1.$$

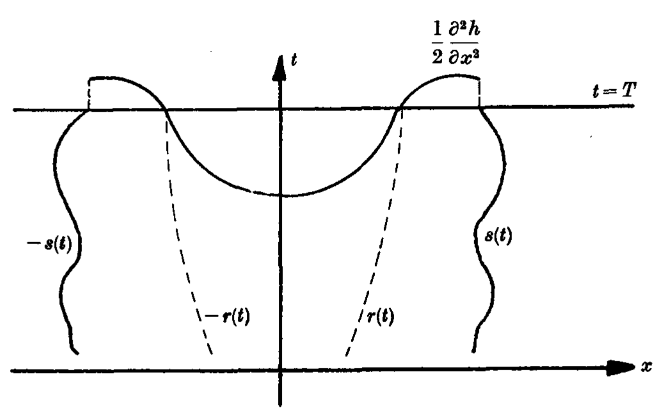


Fig. 5

Since $\hat{g}$ fits smoothly with $g$, we also have a "heat" conservation law

$$\int_{-s(t)}^{s(t)} \frac{1}{2}\frac{\partial^2 \hat{g}}{\partial x^2} = \frac{1}{2}\left(\frac{\partial \hat{g}}{\partial x}(s(t), t) - \frac{\partial \hat{g}}{\partial x}(-s(t), t)\right) = 0. \tag{21}$$

Using an argument similar to the one used in Lemma 8, there are at most two curves where $u = \frac{1}{2}(\partial^2/\partial x^2)\hat{g} = 0$.

It is to be proved that the two root curves $\pm r(t)$ of $\frac{1}{2}(\partial^2/\partial x^2)\hat{g}$ cannot intersect the boundary $\pm s(t)$ as long as $t_0 \leqslant t \leqslant T$; see figure 5. Suppose the contrary; for example, let $r(t)$ meet the boundary—say—at time $t_1 \geqslant t_0$. Then on the one hand, since the interval $(r(t), s(t))$ becomes smaller and smaller when $t$ decreases to $t_1$, the total amount of (positive) heat contained therein would decrease to zero, by the maximum principle. By symmetry, the same is true for the interval $(-s(t), -r(t))$. But on the other hand $\frac{1}{2}(\partial^2/\partial x^2)\hat{g}$ would remain negative in the interval $(-r(t), r(t))$. Hence the net amount of "heat" in the interval $(-s(t), s(t))$ would become negative, when $t$ approaches $t_1$; this violates the conservation law (21). Now $r(t)$ cannot intersect $s(t)$ in the time interval $[t_0, T]$ and consequently $\frac{1}{2}(\partial^2/\partial x^2)\hat{g}$ remains positive in a strip around the boundary.

LEMMA 10. *Let $g$ be as in Theorem 1 and let $s$ be continuously differentiable for $t > t_0$. Then*

$$\dot{s} \geqslant \inf \frac{1}{4}\frac{\partial^3 h}{\partial x^3} \quad in\ (t_0, T].$$

*Proof.* Consider the situation pictured in Figure 6. Let $T_1$, $T_2$, $T_3$, etc. be the values of $t$ where $\dot{s}$ changes sign. The boundary $s(t)$ satisfies the integral equation (17) which simplifies to

$$\begin{aligned} -\dot{s}(t) = &-\int_0^{T-t} \frac{\partial K}{\partial x}(s(t), T-t; s(T-\tau), \tau)\, \dot{s}(T-\tau)\, d\tau \\ &-\int_0^{T-t} \frac{\partial K}{\partial x}(s(t), T-t;\ -s(T-\tau), \tau)\, \dot{s}(T-\tau)\, d\tau \\ &+\int_{-a}^{a} K(s(t), T-t; \xi, 0)\, v'(\xi, T)\, d\xi, \end{aligned} \tag{22}$$

where $$v = \frac{\partial}{\partial t}(\hat{g} - g) \quad \text{and} \quad v'(\xi, T) = -\frac{1}{2}\frac{\partial^3 h}{\partial \xi^3}.$$

In the region (i) where $\dot{s} \leqslant 0$, the integrands of the first two integrals of (22) are both non-negative and its left-hand side is nonnegative. This leads to the inequality

$$-\dot{s}(t) \leqslant \int_{-a}^{a} K(s(t), T-t; \xi, 0)\left(-\frac{1}{2}\right)\frac{\partial^3 h}{\partial \xi^3}(\xi)\, d\xi.$$

Hence in region (i)

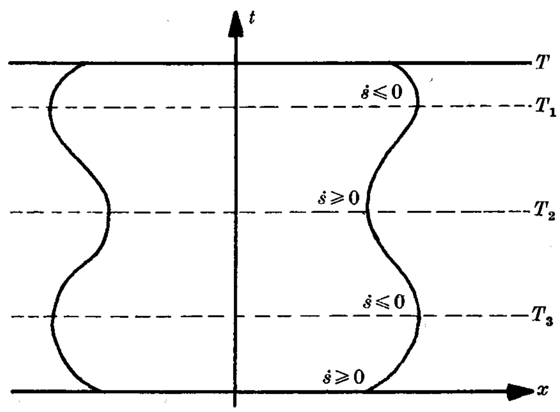


Fig. 6

$$-\dot{s}(t) \leqslant \sup_{|x|\leqslant a}\left(-\frac{1}{2}\frac{\partial^3 h}{\partial x^3}\right)\int_{-a}^{a} K(s(t), T-t; \xi, 0)\, d\xi$$

$$= \sup_{|x|\leqslant a}\left(-\frac{1}{2}\frac{\partial^3 h}{\partial x^3}\right)\int_{\frac{-a-s(t)}{\sqrt{T-t}}}^{\frac{a-s(t)}{\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}}\, d\eta\, e^{-\eta^2/2}.$$

Since both $(a-s(t))/\sqrt{T-t}\leqslant 0$ and $(-a-s(t))/\sqrt{T-t}\leqslant 0$, the latter integral does not exceed $\frac{1}{2}$, and

$$-\dot{s}(t)\leqslant \frac{1}{2}\sup_{|x|\leqslant a}\left(-\frac{1}{2}\frac{\partial^3 h}{\partial x^3}\right). \tag{23}$$

Moreover $$\left(\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\left(\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}\right)=0 \quad \text{in } |x|<s(t), t<T,$$

so the maximum principle tells you that $\frac{1}{2}\,(\partial^2/\partial x\,\partial t)(\hat{g}-g)$ reaches its maximum along the boundary $x=\pm s(t)$ or at the final horizon $t=T$. But since $\frac{1}{2}(\partial^2/\partial x\,\partial t)(\hat{g}-g)(s(t),t)=-\dot{s}(t)$ and since $\frac{1}{2}(\partial^2/\partial x\,\partial t)(\hat{g}-g)(x,T)=\frac{1}{2}(-\frac{1}{2}(\partial^3/\partial x^3)h)$,

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}\leqslant \sup_{\substack{T_1<t\leqslant T\\ |x|\leqslant a}}\left(-\dot{s}(t), \frac{1}{2}\left(-\frac{1}{2}\frac{\partial^3 h}{\partial x^3}\right)\right). \tag{24}$$

Since already in strip (i)

$$-\dot{s}(t)\leqslant \frac{1}{2}\sup_{|x|\leqslant a}\left(-\frac{1}{2}\frac{\partial^3 h}{\partial x^3}\right) \tag{25}$$

(24) becomes

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}\leqslant -\inf_{|x|\leqslant a}\frac{1}{4}\frac{\partial^3 h}{\partial x^3}. \tag{26}$$

In the strip $[T_2, T_1]$ labeled by (ii) in Fig. 6, $\dot{s}\geqslant 0$. Again by the maximum principle

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}\bigg|_{t=T_2} \leqslant \max\left(\sup_{|x|\leqslant s(T_1)}\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}(x, T_1), \sup_{T_2\leqslant t\leqslant T_1} -\dot{s}(t)\right)$$

$$\leqslant \max\left(\sup_{|x|\leqslant s(T_1)}\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}(x, T_1), 0\right)$$

and using inequality (26)

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t}\bigg|_{t=T_2} \leqslant \max\left(-\inf_{|x|\leqslant a}\frac{1}{4}\frac{\partial^3 h}{\partial x^3}, 0\right). \tag{27}$$

Notice that the zero may be deleted on the right-hand side of the latter inequality, because $(\partial^2/\partial x^2)h$ must have a minimum from the assumption on $h$. The same inequality holds for $T_2$ replaced by $T_4$, by $T_6$, etc., with $T_{2n} > t_0$. Lemma 8 assures there are only a finite number of $T_{2n} > t_0$. Hence in the continuation region between $t_0$ and $T$,

$$\frac{1}{2}\frac{\partial^2(\hat{g}-g)}{\partial x\,\partial t} \leqslant -\frac{1}{4}\inf_{|x|\leqslant a}\frac{\partial^3 h}{\partial x^3}$$

and, in particular,

$$\dot{s} \geqslant \inf_{|x|\leqslant a}\frac{1}{4}\frac{\partial^3 h}{\partial x^3}.$$

which establishes Lemma 10.

LEMMA 11. *Let $g$ be as in Theorem 1 and let $s$ be continuously differentiable for $t > t_0$. Consider a boundary point $(x, t)$ where $\dot{s} > 0$ and consider the optimal stopping time $\tau$ viewed from $(x, t+\delta)$ for a positive, sufficiently small increment $\delta$. The following inequality will now be established:*

$$-H\dot{s}^2 + o(1) \leqslant \frac{1}{\delta}E\left(\frac{h(x+x_{T-t-\delta})}{T-t} - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x+x_{T-t-\delta}) + o(1); t+\delta+\tau = T\right). \tag{28}$$

*Both quantities $o(1)$ tend to zero with $\delta$ and the one under the expectation sign does so uniformly over the brownian paths.*

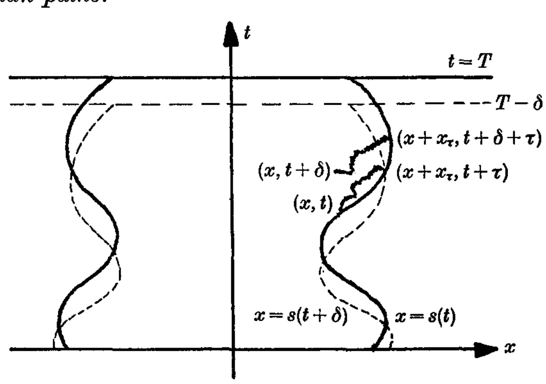


Fig. 7

*Proof.* If $\tau$ is the optimal stopping time viewed from the point $(x, t+\delta)$, denote by $\tau_\delta$ the same stopping time but for paths starting at $(x, t)$, so that in fact $\tau_\delta = \tau$ (cf. figure 7). Since $(x, t)$ is a stopping point

$$Eg(x+x_{\tau_\delta}, t+\tau_\delta) \leqslant \hat{g}(x, t) = g(x, t) \tag{29}$$

Multiply both sides by $g(x, t+\delta)/g(x, t)$. The expectation in (29) can be split into two parts

$$E\left[\frac{g(x+x_{\tau_\delta}, t+\tau_\delta)}{g(x, t)}\, g(x, t+\delta); t+\tau_\delta < T-\delta\right]$$

$$+E\left[\frac{\hat{g}(x+x_{T-t-\delta}, T-\delta)}{g(x, t)}\, g(x, t+\delta); t+\tau_\delta = T-\delta\right] \leqslant g(x, t+\delta) \tag{30}$$

and
$$\hat{g}(x, t+\delta) = Eg(x+x_\tau, t+\delta+\tau) \tag{31}$$

can be decomposed similarly. Subtracting (30) from (31) we obtain

$$\hat{g}(x, t+\delta) - g(x, t+\delta)$$

$$\leqslant E\left[g(x+x_\tau, t+\delta+\tau) - \frac{g(x+x_{\tau_\delta}, t+\tau_\delta)}{g(x, t)}\, g(x, t+\delta); t+\delta+\tau < T\right]$$

$$+E\left[h(x+x_{T-t-\delta}) - \frac{\hat{g}(x+x_{T-t-\delta}, T-\delta)}{g(x, t)}\, g(x, t+\delta); t+\delta+\tau = T\right]. \tag{32}$$

Notice that, since $g(x, t) = T-t$ for $t < T$,

$$g(x+x_\tau, t+\delta+\tau) - \frac{g(x+x_{\tau_\delta}, t+\tau_\delta)}{g(x, t)}\, g(x, t+\delta) = -\tau\,\frac{\delta}{T-t} \leqslant 0 \tag{33}$$

while

$$h(x+x_{T-t-\delta}) - \frac{\hat{g}(x+x_{T-t-\delta}, T-\delta)}{g(x, t)}\, g(x, t+\delta)$$

$$= (h(x+x_{T-t-\delta}) - \hat{g}(x+x_{T-t-\delta}, T-\delta)) + \frac{\delta}{T-t}\,\hat{g}(x+x_{T-t-\delta}, T-\delta)$$

$$= \delta\left(\frac{\delta\hat{g}}{\partial t}\,(x+x_{T-t-\delta}, T) + o(1)\right) + \frac{\delta}{T-t}\,(h(x+x_{T-t-\delta}) + o(1)), \tag{34}$$

where $o(1)$ tends to zero with $\delta$, uniformly over all brownian paths, because both $\hat{g}$ and $(\partial\hat{g}/\partial t)$ are continuous in $|x| \leqslant s(t)$, $T \geqslant t \geqslant t_0$. Since

$$\frac{\partial\hat{g}}{\partial t}\,(y, T) = -\frac{1}{2}\frac{\partial^2 g}{\partial y^2}\,(y, T) = -\frac{1}{2}\frac{\partial^2 h}{\partial y^2}\,(y),$$

(34) equals
$$\delta\left[\frac{h(x+x_{T-t-\delta})}{T-t} - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\,(x+x_{T-t-\delta}) + o(1)\right].$$

According to Lemmas 5 and 6,

$$\frac{(\hat{g}-g)\,(x, t+\delta)}{\delta^2} = -H\dot{s}^2 + o(1). \tag{35}$$

The inequality (32) with (33), (34) and (35) put in yields (28), which establishes Lemma 6.

Lemma 12. *Let $u$ satisfy*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

*in the shaded wedge (see Figure 8). It vanishes at the boundary $x = \psi(t) = b(t - t_0)$, and at the final horizon $T$, it assumes the value $\phi(x)$, where $\phi$ is a continuous function vanishing beyond $N$ and far to the left, with $N < M = b(T - t_0)$. Then*

$$\left|\frac{\partial u}{\partial t}\,(0, t_0) - b^2 \int \frac{1}{\sqrt{2\pi(T-t_0)}}\, e^{-\frac{y^2}{2(T-t_0)}}\, \phi(y)\, dy\right| \leqslant \textit{constant} \;\times \|\phi\|_\infty$$

*for all $b$, provided $T - t_0$ and $M - N$ remain bounded away from zero.*

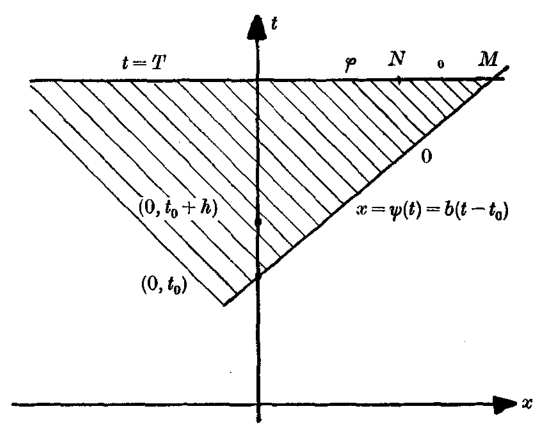


Fig. 8

(For the proof, see the Appendix)

Lemma 13. *Let $g$ be as in Theorem 1 and let $s$ be continuously differentiable for $t > t_0$. Consider a boundary point $(x, t)$ where not only $\dot{s}(t) \geqslant 0$, but also $\dot{s}(t) \geqslant \dot{s}(u)$ for all $u$ between $t$ and $T$. Then at such a point $\dot{s}$ satisfies the inequality*[1]

$$\dot{s}^2(t)\left[\frac{T - t - Eh\,(x + x_{T-t})}{T-t} + \frac{1}{2}\,E\left(\frac{\partial^2 h}{\partial x^2}\right)^{-}(x + x_{T-t})\right] \leqslant \textit{constant} \;\times \left\|\frac{h}{T-t} - \frac{1}{2}\left(\frac{\partial^2 h}{\partial y^2}\right)^{-}\right\|_\infty.$$

*provided $\dot{s}$ is large enough and $T - t$ bounded away from zero.*

[1] ( )$^-$, resp. ( )$^+$ means take negative, resp. positive part.

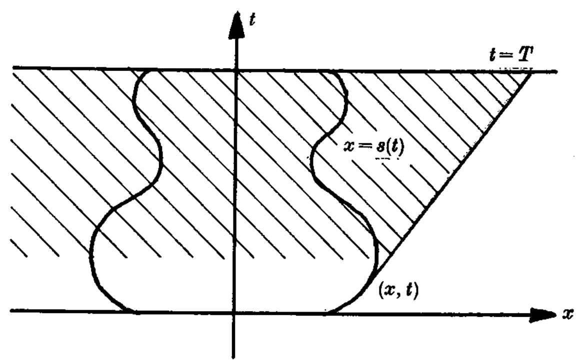


Fig. 9

*Proof.* This lemma is a consequence of Lemmas 11 and 12. In the inequality (28) you may as well replace $-\frac{1}{2}(\partial^2/\partial x^2)h$ and $o(1)$ by their positive part $-\frac{1}{2}((\partial^2/\partial x^2)h)^-$ and $o(1)^+$:

$$-H\dot{s}^2+o(1)\leqslant\frac{1}{\delta}E\left(\frac{h(x+x_{T-t-\delta})}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)^-(x+x_{T-t-\delta})+o(1)^+;t+\delta+\tau=T\right) \quad (36)$$

where $H=-1$.

In the next step replace the continuation region $-s(t)<x<s(t)$ by the wedge shaped by the lines $t=T$ and $y=x+b(u-t)$, as in figure 9, where $b=\dot{s}(t)$. By the assumption that $\dot{s}(t)\geqslant\dot{s}(u)$ for all $u$, $t\leqslant u\leqslant T$, the wedge contains the continuation region, at least ahead of $t$. Let $\tau_l$ be the first hitting time of the oblique line. In the inequality (36) you may replace the expectation by the overestimate

$$E\left(\frac{h(x+x_{T-t-\delta})}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)^-(x+x_{T-t-\delta})+o(1)^+;t+\delta+\tau_l=T\right). \quad (37)$$

This expression is the solution to a boundary value problem for the backwards heat equation vanishing at the oblique line $y=x+b(u-t)$ and taking on at $t=T$ the values

$$\begin{cases}\left(\dfrac{h(y)}{T-t}-\dfrac{1}{2}\left(\dfrac{\partial^2 h}{\partial y^2}\right)^-(y)+o(1)^+\right) & -s(T)\leqslant x\leqslant s(T)\\ 0 & |x|>s(T).\end{cases}$$

Hence the limit of $1/\delta$ times (37) when $\delta$ tends to zero is the $t$-slope of this solution at the boundary (i.e., at the oblique line).

Now we are in a position to use Lemma 12, which affirms that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}E\left(\frac{h(x+x_{T-t-\delta})}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)^-(x+x_{T-t-\delta})+o(1)^+;t+\delta+\tau_l=T\right)$$
$$\leqslant b^2\int_{-a}^{a}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(x-y)^2}{2(T-t)}}\left[\frac{h(y)}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial y^2}\right)^-(y)\right]dy$$
$$+\text{constant}\times\left\|\frac{h}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{dy^2}\right)^-\right\|_\infty. \qquad (38)$$

Now let $\delta$ tend to zero in (36) and combine it with (38). You get

$$\mathfrak{s}^2\leqslant \mathfrak{s}^2 E\left(\frac{h(x+x_{T-t})}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)^-(x+x_{T-t})\right)$$
$$+\text{constant}\times\left\|\frac{h}{T-t}-\frac{1}{2}\left(\frac{\partial^2 h}{\partial y^2}\right)^-\right\|_\infty$$

which completes Lemma 13.

LEMMA 14. *Consider*

$$g\begin{cases}=T-t & t<T\\ =h(x) & t=T\end{cases}$$

*and require that*

(i) $h(x)\leqslant(1+\eta)x^2$ in $x\leqslant 0$ *for some* $\eta>0$, *and* $h(x)=0$ *in* $x>0$;

(ii) $h(x)\geqslant bx^2$ *for* $-M\leqslant x\leqslant 0$, *for some* $0<b<1$.

*If, for some* $\delta_0>0$, *the optimal boundary* $s$ *is continuously differentiable above* $T-\delta_0$, *then*

$$\inf_{\substack{x=s(t)\\ T-\delta_1\leqslant t\leqslant T}}\frac{(T-t)-Eh\,(x+x_{T-t})}{T-t}=Q>0 \qquad (39)$$

*for some* $0<\delta_1<\delta_0$, *with constants* $Q$ *and* $\delta_1$ *depending on* $\eta$, $b$ *and* $M$ *only.*

*Proof. Step* I. For some $\delta_1<\delta_0$ depending only on $M$ and $b$, you have $s(t)\geqslant\alpha_2\sqrt{T-t}$, $0\leqslant T-t\leqslant\delta_1$ where $\alpha_2$ is the largest root of the equation:

$$\int_{-\alpha-\frac{M}{\sqrt{\delta_1}}}^{-\alpha}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}(z+\alpha)^2\,dz=\frac{1}{b}.$$

To begin with, the boundary must belong to the region

$$Eh(x+x_{T-t})\leqslant T-t;$$

namely in the stopping region, playing to the end must be worse than quitting. Since

$$h\geqslant bx^2\quad\text{in }[-M,0]$$

the boundary must *a fortiori* belong to the region where

$$E(b(x+x_{T-t})^2;\ -M \leqslant x+x_{T-t} \leqslant 0) \leqslant T-t,$$

i.e.,

$$b\int_{-M-x}^{-x} \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{y^2}{2(T-t)}}(x+y)^2\,dy \leqslant T-t.$$

Therefore the required portion of the boundary lies in the set

$$\left\{(\alpha\sqrt{T-t}, t);\, 0 \leqslant T-t \leqslant \delta_0 \text{ and } \int_{-\frac{M}{\sqrt{T-t}}-a}^{-\alpha} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}(\alpha+z)^2\,dz \leqslant \frac{1}{b}\right\}. \tag{40}$$

Choose a positive $\delta_1 < \delta_0$ small enough, such that

$$\int_{-\alpha-\frac{M}{\sqrt{\delta_1}}}^{-\alpha} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}(z+\alpha)^2\,dz = \frac{1}{b} \tag{41}$$

has two roots $\alpha_1 < \alpha_2$. Because of this choice of $\delta_1$, the set

$$\left\{(\alpha\sqrt{T-t}, t);\, 0 < T-t \leqslant \delta_1, \int_{-\alpha-\frac{M}{\sqrt{\delta_1}}}^{-\alpha} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}(z+\alpha)^2\,dz \leqslant \frac{1}{b}\right\}$$

consists of two disjoint sets $U_1$ and $U_2$, corresponding to the shaded region in Figure 10; $U_1$, resp. $U_2$, is bounded to the right, resp. left, by $x=\alpha_1\sqrt{T-t}$, resp. $x=\alpha_2\sqrt{T-t}$. Moreover $\alpha_2<0$, because for all $\alpha \geqslant 0$

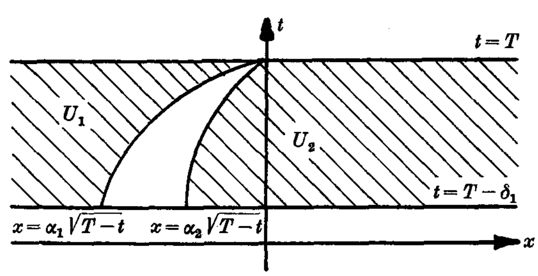


Fig. 10

$$\int_{-\alpha-\frac{M}{\sqrt{\delta_1}}}^{-\alpha} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}(z+\alpha)^2\,dz \leqslant \int_{-\infty}^{-\alpha} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}(z+\alpha)^2 dz \leqslant \frac{1}{2} < \frac{1}{b}.$$

So far we know that $s(t)$ belongs to the set $U_1 \cup U_2$. Since $s(0)=0$, $\dot{s} < \infty$ in $[T-\delta_1, T]$ and $\alpha_2 < 0$, the second component $U_2$ must contain the boundary. This is exactly the statement of Step I.

In *Step* II, we establish the inequality

$$\frac{(T-t)-Eh\,(x+x_{T-t})}{T-t} > \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-(\alpha-w)^2}\,dw \left(\frac{1}{2}-(1+\eta)\int_{-\infty}^{-w} \frac{1}{\sqrt{\pi}}\, e^{-v^2}(w+v)^2 dw\right)^+, \tag{42}$$

valid in the intersection of the stopping region and the strip $T-\delta_0<t<T$; $\alpha$ denotes $x/\sqrt{T-t}$.

Consider a game with the same reward, but allowing only three alternatives: not playing at all, playing for a period of time $(T-t)/2$, or playing until the end $T$. The stopping region for such a game is

$$\left\{(x,t)\,\middle|\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\frac{T-t}{2}}}e^{-\frac{(x-y)^2}{2\frac{T-t}{2}}}dy\left(\max\left(Eh\,(y+x_{(T-t)/2}),\frac{T-t}{2}\right)\right)<T-t\right\};\tag{43}$$

evidently, it contains the stopping region for the original game. A few elementary manipulations of the integral in (43) transforms the inequality into

$$Eh\,(x+x_{T-t})+\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi(T-t)}}e^{-\frac{(x-y)^2}{T-t}}dy\left(\frac{T-t}{2}-Eh\,(y+x_{(T-t)/2})\right)^+<T-t.\tag{44}$$

Recall that $h(x)\leqslant(1+\eta)\,x^2$ for $x\leqslant 0$, with $\eta>0$. It gives

$$\left[\frac{T-t}{2}-h(y+x_{(T-t)/2})\right]^+\geqslant\left[\frac{T-t}{2}-(1+\eta)\,((y+x_{(T-t)/2})^-)^2\right]^+.\tag{45}$$

Putting (45) into (44), you see that the set (43) is contained in

$$\left\{(x,t)\,\middle|\,Eh\,(x+x_{T-t})+\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi(T-t)}}e^{-\frac{(x-y)^2}{(T-t)}}dy\left[\frac{T-t}{2}-(1+\eta)\,E((y+x_{(T-t)/2})^-)^2\right]^+<T-t\right\};$$

the latter is the same as

$$\left\{\frac{(T-t)-Eh(x+x_{T-t})}{T-t}\right.$$
$$\left.>\frac{1}{T-t}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi(T-t)}}e^{-\frac{(x-y)^2}{(T-t)}}dy\left[\frac{T-t}{2}-(1+\eta)\int_{-\infty}^{-y}\frac{1}{\sqrt{\pi(T-t)}}e^{-\frac{z^2}{T-t}}(y+z)^2dz\right]^+\right\}.\tag{46}$$

Make a few changes in variables $z/\sqrt{T-t}=v$ and $y/\sqrt{T-t}=w$; call $x/\sqrt{T-t}=\alpha$. Then the set (46) becomes

$$\left\{\frac{(T-t)-Eh\,(x+x_{T-t})}{T-t}>\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-(\alpha-w)^2}dw\left[\frac{1}{2}-(1+\eta)\int_{-\infty}^{-w}\frac{1}{\sqrt{\pi}}e^{-v^2}(w+v)^2dw\right]^+\right\}\tag{47}$$

which settles Step II.

*Step* III finishes the proof of Lemma 14. The reader easily convinces himself that the right-hand side of the inequality in (47) is increasing in $\alpha$. According to Step I,

$\alpha = x/\sqrt{T-t} \geqslant \alpha_2$ in the stopping region, provided $T-\delta_1 \leqslant t \leqslant T$. Hence in the stopping region, the right-hand side of the inequality in (47) is bounded below by

$$Q = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(\alpha_2 - w)^2} dw \left[\frac{1}{2} - (1+\eta) \int_{-\infty}^{-w} \frac{1}{\sqrt{\pi}} e^{-v^2} (w+v)^2 dw\right]^+$$

which only depends on $\eta$, $b$, and $M$. Combining this with the result of Step II finishes the proof.

*Proof of Theorem* 1. According to Proposition 3, the boundary is continuously differentiable in a time interval $(t_0, T]$. Choose $t_0$ as low as possible, so that $\dot{s}$ ceases to exist there. Can the boundary be extended below $t_0$ under the assumption that the two boundaries have not yet intersected? The main point is to establish that $\dot{s}(t)$ is bounded in $(t_0, T]$; this is the content of (a), (b), (c), (d), and (e) below. The remaining task is to prove that $s(t)$ can be extended in a smooth way a bit below $t_0$; this contradicts $t_0$'s choice as the point where $\dot{s}$ ceases to exist.

(a) Lemma 10 provides us with the lower estimate

$$\dot{s}(t) \geqslant \inf_{-a \leqslant x \leqslant a} \frac{1}{4} \frac{\partial^3 h}{\partial x^3}, \quad t_0 < t \leqslant T.$$

(b) Supplying an upper estimate is a more delicate affair. We apply Lemmas 13 and 14 to the new reward

$$g_1(x, t) \begin{cases} = T_1 - t & \text{for } t < T_1 \\ = h_1(x) \equiv \hat{g}(x, T_1) - g(x, T_1) & \text{for } t = T_1 \end{cases}$$

for $t_0 < T_1 < T$. It is plain that $g_1$'s optimal boundary, denoted by $s_1(t)$, is the same as $s(t)$ below the level $T_1$. Moreover $g_1$ satisfies all the conditions of Theorem 1, with $a = s(T_1) = s_1(T_1)$, using Lemmas 5, 8 and 9. Also $g_1$'s symmetry is preserved.

Since $s_1$ is continuously differentiable for $t_0 < t < T_1$, we are now in a position to apply Lemma 13 to $g_1$ with the result that

$$\dot{s}_1(t)^2 \left(\frac{T_1 - t - Eh_1(x + x_{T_1 - t})}{T_1 - t} + \frac{1}{2} E\left(\frac{\partial^2 h_1}{\partial x^2}\right)^- (x + x_{T_1 - t})\right)$$
$$\leqslant \text{constant} \times \left\| \frac{h_1}{T_1 - t} - \frac{1}{2}\left(\frac{\partial^2 h_1}{\partial y^2}\right)^- \right\|_\infty. \tag{48}$$

This inequality yields an upper bound to $\dot{s}$, provided the supnorm on the right-hand side is bounded and the expression between brackets on the left is bounded away from zero, both irrespective of $T_1$. The former is obvious, if $T_1 - t$ exceeds a fixed positive number

$\delta_1/2$, say (cf. Lemma 14); the point is that both $h_1(x)=\hat{g}(x, T_1)-g(x, T_1)$ and $(\partial^2/\partial y^2)h_1=(\partial^2/\partial y^2)\hat{g}(y, T_1)$ are bounded in the continuation region between $t_0$ and $T$.

Lemma 14 takes care of the left-hand side. Therefore conditions (i) and (ii) must be verified. To see (i), you first observe that $h_1(x)$ vanishes outside $(-\infty, s(T_1))$ and that

$$h_1(x)=h_1(s(T_1))+\frac{\partial h_1}{\partial x}(s(T_1))(x-s(T_1))+\frac{\partial^2 h_1}{\partial x^2}(s(T_1))\frac{(x-s(T_1))^2}{2}$$
$$+\frac{\partial^3 h_1}{\partial x^3}(x+\theta(x-s(T_1)))\frac{(x-s(T_1))^3}{6} \text{ for some } \theta \text{ with } 0<\theta<1.$$

It simplifies to:

$$=(x-s(T_1))^2\left[1+(x-s(T_1))\frac{1}{6}\frac{\partial^3 h_1}{\partial x^3}(x+\theta(x-s(T_1)))\right]$$

by smooth fit; see Lemma 5. By application of the maximum principle to $\partial^3\hat{g}/\partial x^3$,

$$\frac{\partial^3 h_1}{\partial x^3}(x+\theta(x-s(T_1)))\geqslant \min\left(\inf_{-a\leqslant x\leqslant a}\frac{\partial^3 h}{\partial x^3}, \inf_{T_1\leqslant t\leqslant T}\frac{\partial^3\hat{g}}{\partial x^3}(s(t),t)\right)$$
$$=\min\left(\inf_{-a\leqslant x\leqslant a}\frac{\partial^3 h}{\partial x^3}, \inf_{T_1\leqslant t\leqslant T} 4\dot{s}(t)\right)\geqslant \inf_{-a\leqslant x\leqslant a}\frac{\partial^3 h}{\partial x^3}$$

by Lemma 10. Notice that this expression is negative. Since $x-s(T_1)<0$,

$$h_1(x)=(x-s(T_1))^2\left(1+(x-s(T_1))\frac{1}{6}\frac{\partial^3 h_1}{\partial x^3}(x+\theta(x-s(T_1)))\right)$$
$$\leqslant (x-s(T_1))^2\left(1+(x-s(T_1))\frac{1}{6}\inf\frac{\partial^3 h}{\partial x^3}\right)\leqslant (x-s(T_1))^2(1+\eta),$$

where $\eta>0$ is a number exceeding $-2\max_{t_0\leqslant t\leqslant T}s(t)\times\frac{1}{6}\inf(\partial^3/\partial x^3)h>0$. Of course, $\max s(t)<\infty$ because $\dot{s}$ is bounded below in $(t_0, T]$. It remains to prove (ii). By Lemma 9, $h_1(x)\geqslant b(x-s(T_1))^2$ for $-M<x-s(T_1)<0$, where $2b=\inf(\partial^2/\partial x^2)\hat{g}$ in the strip of fixed width $M$ around the boundary, which does the trick.

Lemma 14 may now be applied to prove

(c) $$\min_{\substack{x=s(t)\\ T_1-t\leqslant\delta_1}}\frac{(T_1-t)-Eh_1(x+x_{T_1-t})}{T_1-t}=Q>0$$

independently of $T_1$. Moreover

(d) $$0<-E\left(\frac{\partial^2 h_1}{\partial x^2}\right)^-(x+x_{T_1-t})\leqslant Q/2$$

by a new choice of $\delta_1$, if need be. This follows also from Lemma 9 and the boundedness of $(\partial^2/\partial x^2)\hat{g}$ in the continuation region.

These two facts (c) and (d) combined imply

$$\text{(e)} \qquad \inf_{\substack{x=s(t) \\ T_1-t\leqslant\delta_1}} \left[\frac{T_1-t-Eh_1(x+x_{T_1-t})}{T_1-t}+\frac{1}{2}E\left(\frac{\partial^2 h_1}{\partial x^2}\right)^-(x+x_{T_1-t})\right]\geqslant Q/2.$$

Putting the results achieved so far into (48),

$$\sup_{T_1-\delta_1\leqslant t\leqslant T_1-\delta_1/2} \dot{s}(t)^2 \leqslant \text{ constant } \times\frac{4}{Q\delta_1}\left(\|\hat{g}\|_\infty+\left\|\frac{1}{2}\frac{\partial^2\hat{g}}{\partial y^2}\right\|_\infty\right)$$

irrespective of $T_1$. This provides an upper estimate for $\dot{s}$ in the whole of interval $(t_0, T]$.

So far we have proven that $\dot{s}$ is bounded in the interval $(t_0, T]$. The final job is to show that then $s(t)$ can be extended in a smooth way a bit below $t_0$. The maxmum principle combined with the boundary estimate

$$\frac{\partial^3\hat{g}}{\partial x^3}(s(t), t)=4\dot{s}(t)$$

implies that $(\partial^3/\partial x^3)\hat{g}$ is bounded throughout the continuation region between $t_0$ and $T$. But now Proposition 3 applies to $g_1(x, t)$ and tells you that the optimal boundary is continuously differentiable in a time interval $(T_1-\varepsilon, T_1]$ whose length $\varepsilon$ is independent of $T_1$, namely

$$\sup_{-s(T_1)\leqslant x\leqslant s(T_1)}\left|\frac{\partial^3 h_1}{\partial x^3}\right| \leqslant \sup_{\substack{(x,t)\in C \\ t_0<t\leqslant T}}\left|\frac{\partial^3\hat{g}}{\partial x^3}(x, t)\right| < \infty,$$

is independent of $T_1$. Therefore, by choosing $T_1>t_0$ close enough to $t_0$, the optimal boundary is seen to be differentiable below $t_0$, as advertised.

But it remains to check that the optimal reward $\hat{g}$ associated with $s(t)$ exceeds $g$. Assume the contrary, namely that for some $x$ in $(-s(t), s(t))$, $\hat{g}(x, t)=g(x, t)$. From the assumptions on $h$, $(\partial/\partial x)h$ has exactly one zero in the interval $(-s(t), s(t))$. Moreover, since $(\partial/\partial x)(\hat{g}-g)$ satisfies the backward heat equation and since its total integral

$$\int_{-s(t)}^{s(t)}\frac{\partial}{\partial x}(\hat{g}-g)\,dx=0,$$

there is a unique zero curve, with $(\partial/\partial x)(\hat{g}-g)$ different from zero in the interior of the two regions separated by the zero curve (maximum principle). This implies that $\hat{g}-g$ has to be increasing and then decreasing when $x$ moves from $-s(t)$ to $s(t)$, so that it can only vanish at the sides $\pm s(t)$, whence the result. As a side-result, the continuation region is simply connected.

The continuation region must be bounded, because any point $(x, t)$ with

$$t \leqslant T - \sup_{-a \leqslant x \leqslant a} h(x)$$

is a stopping point. To see this, introduce the reward function

$$f \begin{cases} = \text{constant} = \sup\limits_{-a \leqslant x \leqslant a} h(x) & \text{for } t = T \\ = T - t & \text{for } t < T \end{cases}$$

with optimal reward

$$f \begin{cases} = \text{constant} = \sup\limits_{-a \leqslant x \leqslant a} h(x) & \text{for } t > T - \sup h(x) \\ = T - t & \text{for } t \leqslant T - \sup h(x) \end{cases}$$

and continuation region $T > t > T - \sup h(x)$. Since $f \geqslant g$, $g$'s continuation region must be contained in the region $T > t > T - \sup h(x)$. This fact combined with the lower estimate for $\dot{s}$, implies that $g$'s continuation region is bounded.

Finally the continuation region must close at its bottom, because if not, the optimal boundary would be extendible, contradicting the fact that $g$'s continuation region is bounded. Finally it is connected because if you would have another patch of continuation points having no point in common with the horizon, $g$'s excessivity ($H = -1$) and $\hat{g}$'s parabolicity would imply $\hat{g} < g$, which is absurd. This establishes Theorem 1.

## 4. Concluding remarks

### 4.1. Some further comments concerning the Stefan problem

According to section 2.5, $w(x, \tau) = -(\partial/\partial t)(\hat{g} - g)(x, T - \tau)$ with

$$\begin{aligned} g &= T - t \quad t < T \\ &= h(x) \quad t = T, \end{aligned}$$

chosen as in Theorem 1 and symmetric, and $\sigma(\tau) = s(T - \tau)$ satisfy

$$\frac{\partial w}{\partial \tau} = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \quad \text{in } [-\sigma(\tau), \sigma(\tau)],$$

$$w(\pm \sigma(\tau), \tau) = 0,$$

$$\frac{\partial w}{\partial x}(\pm \sigma(\tau), \tau) = \mp 2 \dot{\sigma}(\tau),$$

$$w(x, 0) = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} - 1.$$

The boundary $\pm \sigma(\tau)$ represents the position in time of the interface between water and ice, water at temperature $w(x, \tau)$ in the region $|x| < \sigma(\tau)$ and ice at zero temperature in the complementary region. Of course, you permit the temperature of the water to be negative, *i.e.*, *supercooled*, and positive in the different areas. As was remarked in Lemma 9, at each moment $\tau$, the total amount of heat equals the total amount of heat needed to freeze all the water, *i.e.*,

$$\int_{-\sigma(\tau)}^{\sigma(\tau)} w(x,\tau)\,dx = -2\,\sigma(\tau) \tag{49}$$

and, in particular,

$$\int_{-a}^{a} w(x,0)\,dx = -2\,a. \tag{50}$$

Both relations follow from the smooth fit conditions, and (50) would still be valid even if they were not true; then the initial temperature would contain $\delta$-functions but would still satisfy (50). Relation (49) follows merely from (50) and the heat conservation law.

That for optimal stopping problems with a compact final gain the total amount of heat equals the total amount of heat required to freeze the water is an important fact: it is responsible for the compactness of the continuation region and for the fact that the continuation region closes down at the bottom. It is instructive to see what occurs when the total amount of heat does not equal the total amount of heat necessary to freeze the water. This question was raised by B. W. Knight.[1]

(a) First assume that the total amount of heat would exceed the amount necessary to freeze the liquid:

$$\int_{-s(T)}^{s(T)} \frac{\partial^2 h}{\partial x^2} < 0,$$

i.e., the liquid would not be sufficiently supercooled to become completely solid (cf. figure 11). Then there always would be a residual amount of unfrozen water, and it is intuitively

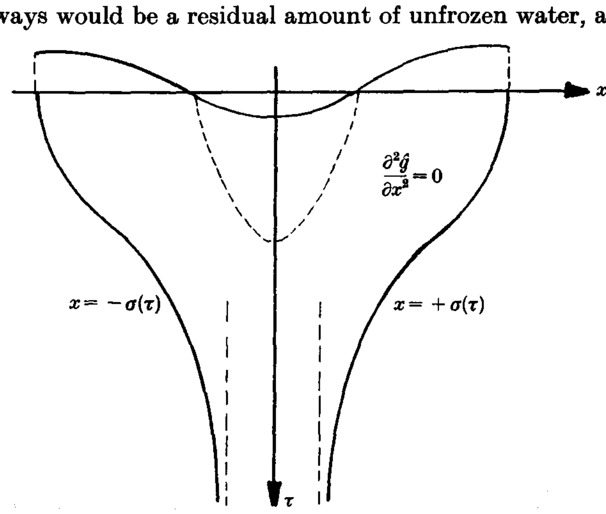


Fig. 11

[1] Private communication.

obvious that this residual amount equals the excess amount of heat which would have to be extracted from the system to achieve complete freezing. In view of Lemma 9, this case corresponds to the situation where the root curves of $(\partial^2/\partial x^2)\hat{g}$ meet each other before meeting the boundary. Then, the boundary can be extended indefinitely, as a result of a small modification of Theorem 1.

(b) Next suppose that the total initial amount of heat would be less than the amount necessary to freeze the liquid:

$$\int_{-a}^{a} \frac{\partial^2 h}{\partial x^2} < 0,$$

i.e., the liquid would be too cold (cf. figure 12). Then one expects a swift freezing of the water and the smaller the distance between both pieces of ice, the smaller the temperature gets in view of the fact that an inescapable amount of cold (negative heat) will remain. But the maximum principle does not allow the water to decrease its temperature below the minimum of the initial temperature. In any case as long as the root curves of $(\partial^2/\partial x^2)\hat{g}$ do not intersect the boundary, the arguments of Theorem 1 show that the boundary can be extended downwards in a smooth way. At the very moment the root curves of $(\partial^2/\partial x^2)\hat{g}$ meet the boundary, the solution ceases to exist, because the boundary condition

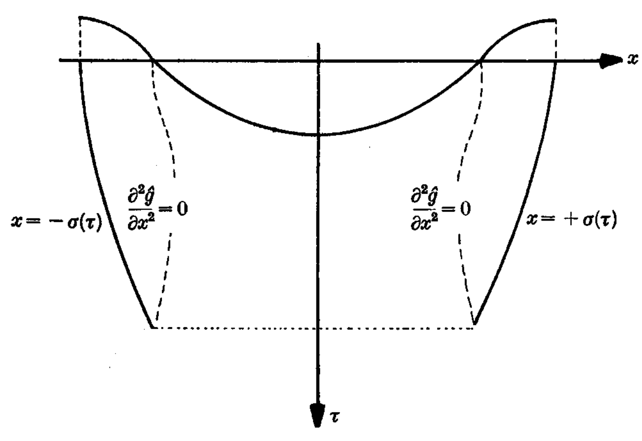


Fig. 12

$$\frac{1}{2}\frac{\partial^2 \hat{g}}{\partial x^2}(\pm\sigma(\tau), \tau) = 1$$

cannot be satisfied anymore; one may think of this as a sudden and instantaneous freezing of all the water.

These two arguments have shown that if the ice-melting picture is to come from an optimal stopping problem, you must actually have (50).

### 4.2. The shape of the boundary near critical points

Here the reward function $g$ is as in Theorem 1 and $h(x)$ even. Put the origin of the $(x, t)$-axes at the lowest point, i.e., the critical point. It will be shown heuristically that the boundary either exhibits a cusp at the critical point, or behaves flatter than $\varepsilon\sqrt{t}$ and not flatter than $(1+\varepsilon)\sqrt{2t \log\log(1/t)}$ for any $\varepsilon > 0$. It remains an open question whether both can exist or only the latter.

The continuation region is a bounded open set with continuously differentiable boundary $x = \pm s(t)$ for $t > 0$, given by the solution to the equation (17):

$$\begin{aligned}\dot{s}(t) = &-\int_0^{T-t} \frac{s(t)-s(T-\tau)}{\sqrt{2\pi(T-t-\tau)^3}}\, e^{-\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}\, \dot{s}(T-\tau)\, d\tau \\ &-\int_0^{T-t} \frac{s(t)+s(T-\tau)}{\sqrt{2\pi(T-t-\tau)^3}}\, e^{-\frac{(s(t)+s(T-\tau))^2}{2(T-t-\tau)}}\, \dot{s}(T-\tau)\, d\tau \\ &-\int_{-s(T)}^{sTt)} K(s(t), T-t; \xi, 0)\left(-\frac{1}{2}\frac{\partial^3 h}{\partial \xi^3}\right)(\xi)\, d\xi \quad 0 < t \leqslant T. \qquad (51)\end{aligned}$$

From Lemma 8 it follows that $s(t)$ must be non-decreasing in the neighborhood of the critical point. So we may as well put the final horizon $T$ a bit ahead of the critical point so as to make the boundary monotone for $0 \leqslant t \leqslant T$, as in Figure 13. Now we have two possibilities, either

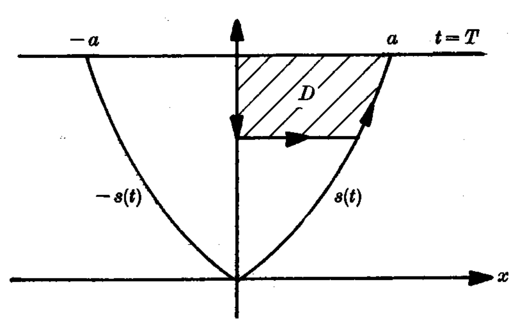


Fig. 13

(1) $\overline{\lim}\, \dot{s} < \infty$ or (2) $\overline{\lim}\, \dot{s} = \infty$, when $t \downarrow 0$.

Case (1) implies that $s(t)$ is Lipschitz continuous in the closed interval $[0, T]$; hence the integrals on the right-hand side of (51) would have limits when $t$ tends to zero. Therefore $\lim_{t\downarrow 0} \dot{s}(t)$ would exist and would be finite. When $t \downarrow 0$, the sum of the two first integrals on the right-hand side of (51) would cancel each other and the last integral would vanish because $(\partial^3/\partial x^3)h$ is an odd function. Hence $\dot{s}(0+) = 0$ and the continuation region would be cusp shaped down at the bottom. This fact is rigorous; the rest will be heuristic.

In case (2), we merely proceed by exclusion, by limiting the discussion to curves such that $s(t)$ is a power or a power times a slowly varying function.(1) We show that:

(a) The critical point is regular for the continuation region, under the assumption that $s(t)/\sqrt{t}$ is monotone near $t=0$. Indeed, since $\dot{s}\geqslant 0$, (51) yields the inequality

$$\dot{s}(t)\leqslant -\int_0^{T-t}\frac{s(t)-s(T-\tau)}{\sqrt{2\pi(T-t-\tau)^3}}\,e^{-\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}\,\dot{s}(T-\tau)\,d\tau$$

$$-\int_{-a}^{a}K(s(t),T-t;\xi,0)\left(-\frac{1}{2}\frac{\partial^2 h}{\partial\xi^3}\right)(\xi)\,d\xi.$$

Hence

$$\dot{s}(t)\leqslant \sup_{t\leqslant u\leqslant T}|\dot{s}(u)|\left[\int_0^{T-t}\frac{s(T-\tau)-s(t)}{\sqrt{2\pi(T-t-\tau)^3}}\,e^{-\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}\,d\tau\right]$$

$$+\left|\int_{-a}^{a}K(s(t),T-t;\xi,0)\frac{1}{2}\frac{\partial^3 h}{\partial\xi^3}(\xi)\,d\xi\right|. \tag{52}$$

If the critical point would be irregular (cf. section 2.1) and since $t^{-1/2}s(t)$ is decreasing, Kolmogorov's test (cf. Ito–McKean [15]) states that

$$\int_0^{T-}\frac{s(T-\tau)}{\sqrt{2\pi(T-\tau)^3}}\,e^{-\frac{s(T-\tau)^2}{2(T-\tau)}}\,d\tau<\infty. \tag{53}$$

$T>0$ can always be chosen so small that the integral (53) is smaller than 1. From this we expect that the integral

$$\int_0^{T-t}\frac{s(T-\tau)-s(t)}{\sqrt{2\pi(T-t-\tau)^3}}\,e^{\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}$$

gets below 1 when $t$ converges to zero. Hence it would follow from (52) that $\dot{s}$ is bounded; this contradicts the assumption that the critical point is irregular.

(b) $s(t)$ cannot behave like $t^{\varepsilon}$, $1>\varepsilon>1/2$ near $t=0$. Indeed, a standard integration by parts permits you to write the integral between brackets in (52) as

$$-2\int_0^{-\frac{s(t)+s(T)}{\sqrt{T-t}}}\frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}dz+2\int_0^{T-t}\frac{\dot{s}(T-\tau)}{\sqrt{2\pi(T-t-\tau)}}\,e^{-\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}\,d\tau.$$

Therefore

$$\dot{s}(t)\leqslant \sup_{t\leqslant u\leqslant T}|\dot{s}(u)|\,2\left[\int_0^{T-t}\frac{\dot{s}(T-\tau)}{\sqrt{2\pi(T-t-\tau)}}\,e^{-\frac{(s(t)-s(T-\tau))^2}{2(T-t-\tau)}}\,d\tau\right]+\text{Constant}. \tag{54}$$

(1) A slowly varying function $\omega(t)$ converges to zero or to infinity more slowly than any power of $t$, when $t\downarrow 0$.

If $s(t)$ would behave as announced under (b), the integral between the brackets in (54) would be bounded and could be made smaller than 1; hence $\dot{s}(t)$ would be bounded, which is absurd. By comparison, also $t^{\varepsilon}\omega(t)$, $1>\varepsilon>1/2$, with a slowly varying function $\omega(t)$, can be excluded.

(c) $s(t)$ cannot behave like $t^{1/2}$ for $t\downarrow 0$. We recall that $v=\partial(\hat{g}-g)/\partial t$ satisfies (13), (14), (15) and (16). Without loss of generality, you can assume $-\frac{1}{2}(\partial^2 h/\partial x^2)+1\geqslant 0$; cf. Lemma 8. Apply Green's theorem over the boundary of the shaded region $D$ in figure 13 to the form

$$2\,vw\,dx-(wv_x-vw_x)\,dt, \tag{55}$$

where

$$w=\frac{x}{t}\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}$$

is a solution of the heat eqution. On the one hand, the line integral of (55) equals

$$\iint\left(-\frac{\partial}{\partial x}\left(w\frac{\partial v}{\partial x}-v\frac{\partial w}{\partial x}\right)-\frac{\partial}{\partial t}(2vw)\right)dx\,dt=0,$$

and on the other hand it equals

$$\int_0^{s(t)}2v(x,t)\frac{x}{t}\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\,dx+\int_a^0 2v(x,T)\,\frac{x}{T}\frac{1}{\sqrt{2\pi T}}\,e^{-x^2/2T}\,dx$$
$$-\int_t^T\frac{s(\tau)}{\tau}\frac{1}{\sqrt{2\pi\tau}}\,e^{-\frac{s(\tau)^2}{2\tau}}\,v_x(s(\tau),\tau)\,d\tau+\int_T^t v(0,\tau)\frac{1}{\tau\sqrt{2\pi\tau}}\,d\tau.$$

Put in the boundary condition $v_x(s(\tau),\tau)=-2\dot{s}(\tau)$. Then

$$\int_t^T\frac{s(\tau)\dot{s}(\tau)}{\tau^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{s(\tau)^2}{2\tau}}\,d\tau$$
$$=\frac{1}{2\sqrt{2\pi}}\int_t^T\frac{v(0,\tau)}{\tau^{\frac{3}{2}}}\,d\tau-\int_0^{s(t)}v(x,t)\frac{x}{t}\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\,dx$$
$$+\int_0^a v(x,T)\,\frac{x}{T}\frac{1}{\sqrt{2\pi T}}\,e^{-x^2/2T}\,dx.$$

Moreover since $v\geqslant 0$, we get the following inequality:

$$\int_t^T\frac{s(\tau)\,\dot{s}(\tau)}{\tau^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{s(\tau)^2}{2\tau}}\,d\tau$$
$$\leqslant\frac{1}{2\sqrt{2\pi}}\int_t^T\frac{v(0,\tau)}{\tau^{\frac{3}{2}}}\,d\tau+\int_0^a v(x,T)\frac{x}{T}\frac{1}{\sqrt{2\pi T}}\,e^{-x^2/2T}\,dx. \tag{56}$$

Suppose $s(t)$ would behave like $\alpha\sqrt{t}$ near $t=0$. Then the inequality (56) would imply

$$\frac{\alpha^2}{2\sqrt{2\pi}}\,e^{-\alpha^2/2}\int_t^T\frac{d\tau}{\tau^{\frac{3}{2}}}\leqslant\frac{1}{2\sqrt{2\pi}}\int_t^T\frac{v(0,\tau)}{\tau^{\frac{3}{2}}}\,d\tau+\int_0^{s(T)}v(x,T)\frac{x}{T}\frac{1}{\sqrt{2\pi T}}\,e^{-x^2/2T}\,dx.$$

Because of this inequality and the fact that the left side blows up to infinity, as $t\downarrow 0, v(0,\tau)$ cannot tend to 0. Since the critical point is regular, any Dirichlet solution, e.g., $v(x,\tau)$, assumes its proper boundary value, whence we have a contradiction and $s(t)\simeq\alpha\sqrt{t}$ cannot be a valid assumption.

(d) $s(t)$ cannot behave like $\sqrt{t}\omega(t)$, where $\omega(t)$ is a slowly varying function, converging to zero with $t$.

The probability of not hitting a parabola $x=\pm\alpha\sqrt{t}$ before time $T$ behaves like $t^{\beta(\alpha)}$, where $-2\beta(\alpha)$ is the largest pole of the function

$$\frac{D_{-\lambda}(0)+D_\lambda(0)}{D_{-\lambda}(\alpha)+D_\lambda(\alpha)};$$

$D_\lambda$ and $D_{-\lambda}$ denote the parabolic cylinder functions. You find that

$$0<\beta(\alpha)<\infty,\quad \lim_{\alpha\uparrow\infty}\beta(\alpha)=0,\quad \lim_{\alpha\downarrow 0}\beta(\alpha)=\infty,\quad\text{and }\beta(1)=1.$$

This follows easily from a result by L. Breiman [5].

Assume $s(t)$ would behave like $\sqrt{t}\omega(t)$. By comparison, $u(0,t)$, alias the probability of not hitting the parabola before time $T$, must behave at least like $t^{\gamma(t)}$ where $\gamma(t)$ tends to infinity for $t\downarrow 0$. The most important contribution of the integrand on the left-hand side of (56) is of order $\omega^2(\tau)/\tau^{3/2}$ for small $\tau$, while on the right-hand side it is of order $\tau^{\gamma(t)}/\tau^{3/2}$. This implies that the inequality in (56) cannot hold under this assumption, whence the result under (d).

The upshot of the results in (a), (b), (c), and (d) is to substantiate (though not really to prove) the assertion under section 4.2.

## Appendix: Proof of Lemma 12

The proof of Lemma 12 will require a number of propositions:

PROPOSITION 1. (Sonine [26]). *Consider the equation:*

$$\int_0^y\frac{1}{\sqrt{y-\tau}}\,e^{-\frac{b^2}{2}(y-t)}\mu'(\tau)\,d\tau=f(y), \tag{57}$$

*where $\mu'(t)$ stands for the derivative of $\mu$ and $f(0)=0$. Its solution is*

$$\mu(t)=\mu(0)+\frac{1}{2\pi}\int_0^t dy\, f(y)\,K(t-y), \tag{58}$$

*where the kernel $K$ is given by*

$$K(s)=2b\int_0^{b\sqrt{s}} dx\, e^{-x^2/2}+\frac{2e^{-b^2 s/2}}{\sqrt{s}}.$$

This proposition will be used to solve the following Dirichlet problem:

PROPOSITION 2. *Let $u$ be a bounded solution of $(\partial/\partial t)u+\frac{1}{2}(\partial^2/\partial x^2)u=0$ in the shaded region of Figure 8 which vanishes at the boundary $x=\psi(t)=b(t-t_0)$ and, at the horizon $T$, agrees with a bounded continuous function $\phi$ vanishing beyond $N(N<M=b(T-t_0))$.*

*Then*

$$u(x,t)=\int_{-\infty}^{N}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(x-y)^2}{2(T-t)}}\phi(y)\,dy+\int_0^{T-t}\frac{1}{\sqrt{2\pi(T-t-\tau)}}e^{-\frac{(x-\psi(T-\tau))^2}{2(T-t-\tau)}}\mu'(\tau)\,d\tau \tag{59}$$

*where $\mu$ is given by* (58) *with*

$$f(s)=-\int_{-\infty}^{N}\frac{1}{\sqrt{s}}e^{-\frac{(M-bs-y)^2}{2s}}\phi(y)\,dy.$$

*Proof.* Since the right-hand side of (59) satisfies the heat equation, we may assume that $u$ has that form for some choice of $\mu'$. Besides the initial condition, which $u$ obviously satisfies, it ought to vanish at the boundary $x=b(t-t_0)$, i.e.,

$$0=u(b(t-t_0),t)=\int_{-\infty}^{N}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(b(t-t_0)-y)^2}{2(T-t)}}\phi(y)\,dy$$
$$+\int_0^{T-t}\frac{1}{\sqrt{2\pi(T-t-\tau)}}e^{-\frac{(b(t-t_0)-b(T-\tau-t_0))^2}{2(T-t-\tau)}}\mu'(\tau)\,d\tau,$$

which amounts to solving the integral equation

$$\int_0^{T-t}\frac{1}{\sqrt{T-t-\tau}}e^{-\frac{b^2}{2}(T-t-\tau)}\mu'(\tau)\,d\tau=-\int_{-\infty}^{N}\frac{1}{\sqrt{T-t}}e^{-\frac{(b(t-t_0)-y)^2}{2(T-t)}}\phi(y)\,dy=f(T-t).$$

Its solution $\mu'(t)$ is given by the derivative of (58),

$$\mu'(t)=\frac{1}{2\pi}\int_0^t f'(y)\,K(t-y)\,dy$$

which finishes Proposition 2.

PROPOSITION 3. *For the Dirichlet solution $u$ of Proposition 2,*

$$\left|\frac{\partial u}{\partial t}(0, t_0) + \frac{b^2}{\sqrt{2\pi}} f(T - t_0)\right| \leqslant \textit{constant} \times \|\phi\|_\infty$$

*provided $M - N$ and $T - t_0$ remain bounded away from zero.*

*Proof.* The derivative of (59) can be expressed as

$$\begin{aligned}\frac{\partial u}{\partial t}(0, t_0) = g(t_0) &- \frac{b^2}{2\pi}\int_0^{T-t_0} \frac{d\tau}{\sqrt{2\pi(T-t_0-\tau)}} e^{-\frac{b^2}{2}(T-t_0-\tau)} \int_0^\tau d\sigma f'(\sigma) K(\tau - \sigma) \\ &- \frac{1}{2\pi}\int_0^{T-t_0} \frac{d\tau}{\sqrt{2\pi(T-t_0-\tau)}} e^{-\frac{b}{2}(T-t_0-\tau)} \int_0^\tau d\sigma f''(\sigma) K(\tau - \sigma),\end{aligned} \tag{60}$$

where

$$g(t_0) = \frac{d}{dt_0}\int_{-\infty}^{N} \frac{1}{\sqrt{2\pi(T-t_0)}} e^{-\frac{y^2}{2(T-t_0)}} \phi(y)\, dy.$$

Add $(b^2/\sqrt{2\pi}) f$ to both sides of (60) and make the substitution $\xi^2 = b^2(T - t_0 - \tau)$ in the integrals. After some rearrangements, you obtain:

$$\frac{\partial u}{\partial t} + \frac{b^2}{\sqrt{2\pi}} f = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} + \text{(v)}, \text{ where} \tag{61}$$

(i) $\quad = g(t_0)$

(ii) $$= \frac{b^2}{2\pi}\left[-\int_0^{b\sqrt{T-t_0}} \frac{2\, d\xi}{b\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0-\frac{\xi^2}{b^2}} d\sigma f'(\sigma)\, 2b \int_0^{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}} dx\, e^{-\frac{x^2}{2}} + \int_0^\infty \frac{2}{b}\frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0} d\sigma f'(\sigma)\, 2b \int_0^\infty dx\, e^{-\frac{x^2}{2}}\right]$$

(iii) $$= -\frac{b^2}{2\pi}\int_0^{b\sqrt{T-t_0}} \frac{2}{b}\frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0-\frac{\xi^2}{b^2}} d\sigma f'(\sigma)\, 2\, \frac{e^{-\frac{b^2}{2}\left(T-t_0-\frac{\xi^2}{b^2}-\sigma\right)}}{\sqrt{\left(T-t_0-\frac{\xi^2}{b^2}-\sigma\right)}}$$

(iv) $$= -\frac{1}{2\pi}\int_0^{b\sqrt{T-t_0}} \frac{2}{b}\frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0-\frac{\xi^2}{b^2}} d\sigma f''(\sigma)\, 2b \int_0^{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}} dx\, e^{-\frac{x^2}{2}}$$

(v) $$= -\frac{1}{2\pi}\int_0^{b\sqrt{T-t_0}} \frac{2}{b}\frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0-\frac{\xi^2}{b^2}} d\sigma f''(\sigma)\, 2\, \frac{e^{-\frac{b^2}{2}\left(T-t_0-\frac{\xi^2}{b^2}-\sigma\right)}}{\sqrt{\left(T-t_0-\frac{\xi^2}{b^2}-\sigma\right)}}.$$

It is easy to see that $f$ and all its derivatives satisfy the bound

$$|f^{(n)}| \leqslant C_n |\phi\|_\infty$$

independently of $b$, provided $M = b(T - t_0) > N$; indeed this proviso implies that the whole region swept out by the line $x = b(t - t_0)$, when $b$ increases, remains bounded away from the support of $\phi$. The constants $C_n$ depend on the order $n$ of the derivative, the distance $M - N$ and the pivot $(0, t_0)$. From now on the letter $C$ is used to denote anyone of several constants that depend on the pivot $(0, t_0)$ and the distance $M - N$, only.

Bounding (61) (i) does not cause any trouble:

$$|g(t_0)| \leqslant \text{Constant } \times \|\phi\|_\infty.$$

The term (53) (ii) can be decomposed in the following fashion:

$$(\alpha) \qquad \frac{b^2}{2\pi} \int_0^{\frac{b\sqrt{T-t_0}}{2}} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{\frac{b-1}{b}\left(T-t_0-\frac{\xi^2}{b^2}\right)} d\sigma f'(\sigma)\, 2b \int_{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}^{\infty} dx\, e^{-\frac{x^2}{2}}$$

$$(\beta) \qquad + \frac{b^2}{2\pi} \int_{\frac{b\sqrt{T-t_0}}{2}}^{b\sqrt{T-t_0}} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{\frac{b-1}{b}\left(T-t_0-\frac{\xi^2}{b^2}\right)} d\sigma f'(\sigma)\, 2b \int_{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}^{\infty} dx\, e^{-\frac{x^2}{2}}$$

$$(\gamma) \qquad + \frac{b^2}{2\pi} \int_0^{b\sqrt{T-t_0}} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\frac{b-1}{b}\left(T-t_0-\frac{\xi^2}{b^2}\right)}^{T-t_0-\frac{\xi^2}{b^2}} d\sigma f'(\sigma)\, 2b \int_{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}^{\infty} dx\, e^{-\frac{x^2}{2}}$$

$$(\delta) \qquad + \frac{b^2}{2\pi} \int_0^{b\sqrt{T-t_0}} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{T-t_0-\frac{\xi^2}{b^2}}^{T-t_0} d\sigma f'(\sigma)\, 2b \int_0^{\infty} dx\, e^{-\frac{x^2}{2}}$$

$$(\varepsilon) \qquad + \frac{b^2}{2\pi} \int_{b\sqrt{T-t_0}}^{\infty} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{T-t_0} d\sigma f'(\sigma)\, 2b \int_0^{\infty} dx\, e^{-\frac{x^2}{2}}.$$

As to $(\alpha)$, since
$$0 \leqslant \xi \leqslant \frac{b\sqrt{T-t_0}}{2},$$

we have
$$T - t_0 - \frac{\xi^2}{b^2} \geqslant T - t_0 - \frac{b^2(T-t_0)}{4b^2} = \frac{3}{4}(T - t_0),$$

and since
$$0 \leqslant \sigma \leqslant \frac{b-1}{b}\left(T - t_0 - \frac{\xi^2}{b^2}\right),$$

we have
$$b\sqrt{T - t - \frac{\xi^2}{b^2} - \sigma} \geqslant \sqrt{b}\sqrt{\frac{3(T-t_0)}{4}}.$$

Then using the standard bound

$$\int_z^\infty dx\, e^{-\frac{x^2}{2}} \leqslant \frac{e^{-\frac{z^2}{2}}}{z},$$

it is seen that

$$|(\alpha)| \leqslant \frac{b^2}{2\pi} \int_0^{\frac{b\sqrt{T-t_0}}{2}} \frac{2}{b} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_0^{\frac{b-1}{b}\left(T-t_0-\frac{\xi^2}{b^2}\right)} d\sigma\, |f'(\sigma)|\, 2b \int_{\sqrt{b}\sqrt{\frac{3(T-t_0)}{4}}}^\infty dx\, e^{-\frac{x^2}{2}}$$

$$\leqslant \frac{b^2}{2\pi} \frac{2}{b} \frac{1}{2} (T-t_0) \sup |f'|\, 2b \frac{1}{\sqrt{b}\sqrt{\frac{3(T-t_0)}{4}}} e^{-\frac{3b(T-t_0)}{8}},$$

which is exponentially small, for large $b$.

The term $(\beta)$ is even easier to bound:

$$|(\beta)| \leqslant \frac{b^2}{2\pi} 4 \int_{\frac{b\sqrt{T-t_0}}{2}}^{b\sqrt{T-t_0}} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \sup |f'|\, (T-t_0) \tfrac{1}{2} \sqrt{2\pi},$$

which is also exponentially small for large $b$.

In $(\gamma)$, let $a = T - t_0 - (\xi^2/b^2)$. Then, using three consecutive changes in variables $\tau = a - \sigma$, $x/(b\sqrt{\tau}) = y$ and $\tau = u^2/(b^2 y^2)$, you find

$$\int_{\frac{b-1}{b}a}^{a} d\sigma \int_{b\sqrt{a-\sigma}}^\infty dx\, e^{-x^2/2} = \frac{2}{b^2} \int_1^\infty \frac{dy}{y^3} \int_0^{y\sqrt{a}\,b} u^2\, du\, e^{-u^2/2},$$

which is overestimated by

$$\frac{2}{b^2} \int_1^\infty \frac{dy}{y^3} \int_0^\infty u^2\, du\, e^{-\frac{u^2}{2}}.$$

Therefore

$$|\gamma| \leqslant \frac{b^2}{2\pi} \sup |f'|\, 4 \int_0^{b\sqrt{T-t}} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\frac{b-1}{b}\left(T-t_0-\frac{\xi^2}{b^2}\right)}^{T-t_0-\frac{\xi^2}{b^2}} d\sigma \int_{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}^\infty dx\, e^{-\frac{x^2}{2}}$$

$$\leqslant \frac{\sup |f'|}{2\pi} 8 \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_1^\infty \frac{dy}{y^3} \int_0^\infty u^2\, du\, e^{-\frac{u^2}{2}} \leqslant C \|\phi\|_\infty.$$

As to $(\delta)$,

$$|\delta| \leqslant \frac{b^2}{2\pi} \sqrt{2\pi} \int_0^{b\sqrt{T-t_0}} 2 \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{T-t_0-\frac{\xi^2}{b^2}}^{T-t_0} d\sigma\, |f'(\sigma)|$$

$$\leqslant \frac{b^2}{2\pi} \sup |f'| \sqrt{2\pi} \int_0^\infty 2 \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \frac{\xi^2}{b^2} \leqslant \frac{b^2}{2\pi} \frac{\sup |f'|}{b^2} 2 \int_0^\infty d\xi\, e^{-\frac{\xi^2}{2}} \xi^2 \leqslant C \|\phi\|_\infty,$$

while $(\varepsilon)$ is exponentially small. Summing up our efforts

$$|\text{ii}| \leqslant C\|\phi\|_\infty.$$

Consider now

$$|(\text{iii})| = \left|\frac{b^2}{2\pi}\int_0^{b\sqrt{T-t_0}}\frac{2}{b}\frac{d\xi}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}\int_0^{T-t_0-\frac{\xi^2}{b^2}}d\sigma f'(\sigma)\frac{4}{b}\frac{d}{d\sigma}\int_0^{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}dx\, e^{-\frac{x^2}{2}}\right|$$

$$= \left|\frac{8}{2\pi}\int_0^{b\sqrt{T-t_0}}\frac{d\xi}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}\int_0^{T-t_0-\frac{\xi^2}{b^2}}d\sigma f''(\sigma)\int_0^{b\sqrt{T-t_0-\frac{\xi^2}{b^2}-\sigma}}dx\, e^{-\frac{x^2}{2}}\right|,$$

using integration by parts and the fact that $f'(0)=0$. This is overestimated by a constant multiple of

$$\sup|f''| \leqslant C\|\phi\|_\infty$$

and similar estimates apply to (iv) and (v), so the proof of Proposition 3 is complete.

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