DIOPHANTINE APPROXIMATION OF COMPLEX NUMBERS

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Introduction

Several attempts have been made during the last 100 years to develop for complex numbers an algorithm having the properties that the regular continued fraction algorithm is known to possess in the real case, due to results of Euler, Lagrange, Legendre, Gauss, Galois, Serret, Markoff, and Hurwitz.

The most significant of such previous attempts are those of A. Hurwitz [18], J. Hurwitz [20], A. Auric [1], L. R. Ford [12], J. W. S. Cassels, W. Ledermann, K. Mahler [5], W. J. Leveque [22, 23], G. Poitou [28], and A. L. Schmidt [33]. However, it is a common feature of all these approaches that only very few of the nice properties of regular continued fractions are carried over.

It is the purpose of the present paper to develop in the Gaussian case two new kinds of algorithm, *regular chains* and *dually regular chains*, based on the concepts of *Farey sets* and *dual Farey sets* to be presented in chapter 1. As is pointed out, Farey sets appear to be a natural extension of the well-known circles and mesh triangles of L. R. Ford [13]. For this chapter we presuppose some knowledge of Farey triangles (sections 2–5 of [33]).

It is shown in chapter 2 that the representation of complex irrational numbers ξ by regular and dually regular chains—ch ξ and ch* ξ—is essentially unique, and that the theorem of Serret about equivalence extends in a natural way. Also every fair approximant p/q (with $p, q \in \mathbf{Z}[i]$) of ξ will appear as a convergent of ch ξ and of ch* ξ, thus extending a theorem of Legendre.

The classical theorems of Euler, Lagrange, and Galois about periodic and purely periodic regular continued fractions are extended in chapter 3 and used to give an effective solution of the complex Pellian equation.

Chapter 4 is devoted to a study of ch ξ_0 and ch* ξ_0 for $\xi_0 = \frac{1}{2}(1+i\alpha_0)$, where $\alpha_0 \in \mathbf{R}\setminus\mathbf{Q}$, and of the corresponding *C-regular* and *C-dually regular* continued fractions (4.2) and (4.2*) of α_0. These continued fractions, though sharing also the more subtle properties of regular continued fractions (an intrinsic characterization of convergents, a theory of C-approximation constants <2 and an ergodic theory in complete analogy with the corresponding theorems for regular continued fractions by Legendre, Markoff–Hurwitz and C. Ryll-Nardzewski), seem, somehow, to have escaped earlier notice.

In chapter 5 the theory of C-minimum of real binary quadratic forms and the corresponding theory of C-approximation constants is treated in complete analogy with the exposition by J. W. S. Cassels [4] on the Markoff chain.

In a separate paper I shall deduce the properties of the C-minimal forms g_Λ or h_M of chapter 5, by putting these in 1–1 correspondance to Markoff-symbols representing periods of even or odd lengths, respectively.

Chapter 6 contains first of all a reduction theory of complex binary quadratic forms similar to that of C. F. Gauss [16] for real indefinite binary quadratic forms. Apparently the only reduction theory of complex binary quadratic forms existing is the rather crude one contained in the famous paper of P. G. L. Dirichlet [10] on what is now known as "Dirichlet fields".

An important application of this reduction theory is the complete determination—in Theorem 6.6—of all complex binary quadratic forms Φ with $\sqrt{|D|}/\mu < 2$, where D and μ denote the discriminant and minimum of Φ.

Also chapter 6 contains the complete determination—in Theorem 6.7—of all complex irrationals ξ with approximation constant $C(\xi) < 2$.

Contributors to the early development of these theorems are L. R. Ford [13] and O. Perron [25, 26], who both determined the first minimum. Later J. W. S. Cassels [3] and A. L. Schmidt [33] proved the isolation of the first minimum, and J. W. S. Cassels [3] also indicated the forms G, $\bar{G}$ of Theorem 6.6. Very recently L. Ya. Vulakh [36] determined the minimum of the forms G_Λ of Theorem 6.6 (in an equivalent form), and gave also a brief indication of a proof of the symmetric case of that theorem, without, however, being explicit about the isolation technique involved.

In a separate paper I shall use the machinery of chapter 6 also to study the minimum of complex ternary and quaternary quadratic forms.

Also I announce a forthcoming paper on "Hurwitzian chains", including (for $a, b \in \mathrm{N}$)

$$\operatorname{ch} \exp\,[1/(a-ib)] = V_3\, \overrightarrow{V_n^{2bn+b-2} E_{n+2}\, V_n^{2an+a-1}\, V_{n+2}\, C}\,\Big|_{n=0}^{\infty},$$

$$\operatorname{ch} \exp\,[1/(-ib)] = V_3\, \overrightarrow{V_3^{2bn+b-2}\, C^2}\,\Big|_{n=0}^{\infty},$$

which extend the classical formula of Euler (cf. [27]),

$$\exp\,[1/a] = [\overrightarrow{1, 2an+a-1, 1}]_{n=0}^{\infty}.$$

Finally I announce a paper on the approximation of complex numbers by numbers from the field $\mathbf{Q}(\sqrt{-11})$. By combining the methods of the present paper with those of [33], I shall prove that the set of approximation constants has

$$((1\,588\,626+30\,690\sqrt{1\,085})/1\,205\,821)^{1/2}=1.4682\ \ldots$$

as the smallest limit point. There are infinitely many approximation constants below this point, the four smallest being

$$\frac{1}{2}\sqrt{5},\ \sqrt{2},\ \sqrt{\frac{1\,085}{509}},\ \sqrt{\frac{36\,860}{17\,099}}.$$

Chapter 1

Farey sets

1.1. Some basic notation

Let

$$\mathcal{J}=\{z=x+iy\,|\,y\geqslant 0\}\cup\{\infty\},$$
$$\mathcal{J}^*=\{z=x+iy\,|\,0\leqslant x\leqslant 1,\ y\geqslant (x-x^2)^{1/2}\}\cup\{\infty\}.$$

The sets $\mathcal{J}$, $\mathcal{J}^*$ and their subdivisions into

$$\mathcal{J}=\mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{V}_3\cup\mathcal{E}_1\cup\mathcal{E}_2\cup\mathcal{E}_3\cup\mathcal{C},$$
$$\mathcal{J}^*=\mathcal{V}_1^*\cup\mathcal{V}_2^*\cup\mathcal{V}_3^*\cup\mathcal{C}^*,$$

are shown in Fig. 1, Fig. 1*, respectively. All regions are bounded by straight lines or circles with radii 1/2 (or part thereof), and are supposed to be closed; $\infty\in\mathcal{V}_1,\ \mathcal{E}_2,\ \mathcal{E}_3,\ \mathcal{V}_1^*$.

The matrices V_1, V_2, V_3, E_1, E_2, E_3, C, S, I are defined as follows,

$$V_1=\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix},\qquad V_2=\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix},\qquad V_3=\begin{pmatrix}1-i & i\\ -i & 1+i\end{pmatrix},$$

$$E_1=\begin{pmatrix}1 & 0\\ 1-i & i\end{pmatrix},\qquad E_2=\begin{pmatrix}1 & -1+i\\ 0 & i\end{pmatrix},\quad E_3=\begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix},$$

$$C=\begin{pmatrix}1 & -1+i\\ 1-i & i\end{pmatrix},\quad S=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix},\qquad I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

For any invertible Matrix M,

$$M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

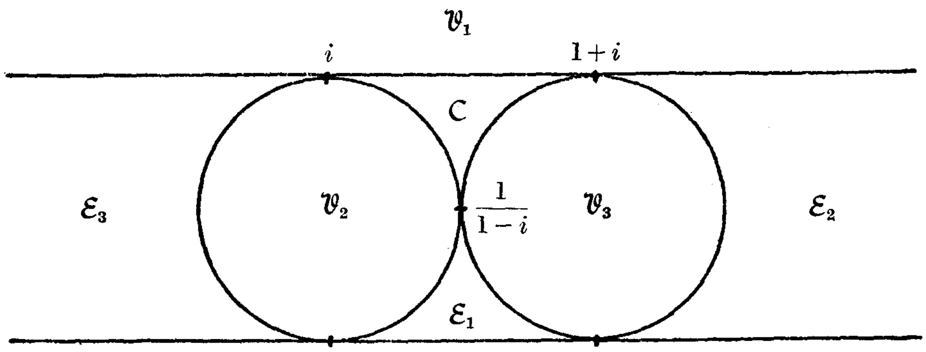

Fig. 1

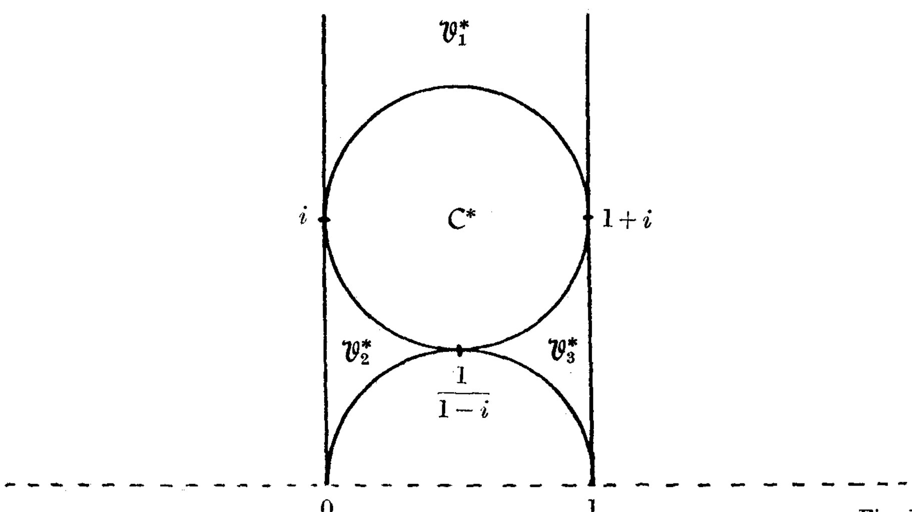

Fig. 1*

with elements in $\mathbf{C}$, let m denote the corresponding homographic map

$$m\colon z \mapsto \frac{az+b}{cz+d}.$$

Finally

$$\varkappa\colon z \mapsto \bar{z}$$

denotes complex conjugation of $\mathbf{C}$. For any subset S of $\mathbf{C}$ we write also $\overline{S}$ instead of $\varkappa(S)$. For any matrix M we write $\overline{M}$ for the complex conjugate of M.

We collect in the following lemma a number of simple relations between the matrices, maps and regions introduced above. Here and further on we use the convention that the index j ranges through $\{1, 2, 3\}$, and accordingly $j+1$ has to be reduced modulo 3.

LEMMA 1.1.

(i) $S^3 = I$,

(ii) $V_{j+1} = SV_jS^{-1}, E_{j+1} = SE_jS^{-1}, C = SCS^{-1}$,

(iii) $\det V_j = 1, \det E_j = i, \det C = -i$,

(iv) $V_j^{-1} = \overline{V_j}, E_j^{-1} = \overline{E_j}, C^{-1} = -\overline{C}$,

(v) $v_j^{-1} = \varkappa \circ v_j \circ \varkappa, \quad e_j^{-1} = \varkappa \circ e_j \circ \varkappa, \quad c^{-1} = \varkappa \circ c \circ \varkappa$,

(vi) $\mathfrak{V}_j = v_j(\mathfrak{I}), \mathcal{E}_j = e_j(\mathfrak{I}^*), \mathfrak{C} = c(\mathfrak{I}^*)$,

(vii) $\mathfrak{V}_j^* = v_j(\mathfrak{I}^*), \mathfrak{C}^* = c(\mathfrak{I})$,

(viii) $\mathfrak{V}_{j+1} = s(\mathfrak{V}_j), \mathcal{E}_{j+1} = s(\mathcal{E}_j), \mathfrak{C} = s(\mathfrak{C})$,

(ix) $\mathfrak{V}_{j+1}^* = s(\mathfrak{V}_j^*), \mathfrak{C}^* = s(\mathfrak{C}^*)$.

Proof. Obvious by inspection. Notice that s is the non-euclidean rotation to an angle $2\pi/3$ around $\frac{1}{2}(1+i\sqrt{3})$ if $\mathfrak{I}$ is considered as the non-euclidean plane in the Poincaré model.

A homographic map m: $z \mapsto (az+b)/(cz+d)$ is called *unimodular* if $a, b, c, d \in Z[i]$ (the ring of Gaussian integers), and $\det m = ad - bc \in \mathcal{U} = \{\pm 1, \pm i\}$. Notice that the determinant of a unimodular map is defined only as an element in the quotient group $\mathcal{U}/\mathcal{U}^2$ consisting of the two elements $\{\pm 1\}$, $\{\pm i\}$. A unimodular map m is called *properly unimodular* if $\det m = \{\pm 1\}$, and *improperly unimodular* if $\det m = \{\pm i\}$.

As usual $\xi, \eta \in \mathbf{C}$ are called *equivalent* if there exists a unimodular map m with $\eta = m(\xi)$. Also $\xi, \eta \in \mathbf{C}$ are called *properly* (*improperly*) *equivalent* if there exists a properly (improperly) unimodular map m with $\eta = m(\xi)$. Obviously each equivalence class of complex numbers consists of either one or two proper equivalence classes.

An immediate consequence of these definitions and Lemma 1.1 (iii) (vi) (vii) is that every boundary point occurring in Fig. 1 is properly equivalent to a real number, and that every boundary point occurring in Fig. 1* is improperly equivalent to a real number.

1.2. Farey sets

Let G denote the group of all unimodular maps. To any $m \in G$ we associate a *Farey set* $F(m)$ as follows,

(1) if m is properly unimodular, we define

$$F(m) = \begin{cases} m(\mathfrak{I}) \\ m(\overline{\mathfrak{I}}) \end{cases},$$

where we make the unique choice between the two sets $m(\mathfrak{J})$, $m(\overline{\mathfrak{J}})$ such that $F(m)$ becomes either a circular disc or a halfplane of the form

$$\{z=x+iy \mid y \geqslant b_0\},\ b_0 \in \mathbf{Z}; \tag{1.1}$$

(2) if m is improperly unimodular, we define

$$F(m) = \begin{cases} m(\mathfrak{J}^*) \\ m(\overline{\mathfrak{J}^*}) \end{cases},$$

where the upper possibility is chosen if $m(\mathfrak{J})$ is a circular disc or a halfplane of the form

$$\{z=x+iy \mid x \leqslant a_0\},\ a_0=0,\ -1,\ -2,\ \dots \tag{1.2}$$

or

$$\{z=x+iy \mid x \geqslant a_0\},\ a_0=1,\ 2,\ 3,\ \dots, \tag{1.3}$$

while the lower possibility is chosen if $m(\overline{\mathfrak{J}})$ is a circular disc or a halfplane of the form (1.2) or (1.3).

It is easily shown that by (1), (2), the Farey set $F(m)$ is well-defined for all $m \in G$, and we say that $F(m)$ is of *circular* (*triangular*) *type*, respectively. Notice that in Fig. 1 the sets $\mathfrak{J}$, $\mathfrak{V}_j$ are Farey sets of circular type, while the sets $\mathcal{E}_j$, $\mathcal{C}$ are Farey sets of triangular type.

Let $\mathfrak{F}$ be the set of all Farey sets, i.e.

$$\mathfrak{F} = \{F(m) \mid m \in G\}.$$

A consequence of a previous remark about the boundary points of Fig. 1, Fig. 1* is that any $z \in \partial F$, where $F \in \mathfrak{F}$, is properly equivalent to a real number.

For any $F \in \mathfrak{F}$, say $F=F(m)$, we define

$$\varrho(F) = ((N(c)+N(d)+N(c+d))^2 - 2(N^2(c)+N^2(d)+N^2(c+d)))^{1/2},$$

where $m: z \mapsto (az+b)/(cz+d)$ and N is the Gaussian norm. Apparently $\varrho(F)$ depends not only upon F but also upon m. It is a matter of simple trigonometry however, to show that $1/\varrho(F)$ is the radius of the circle $m(\mathbf{R})$, which is the circumscribed circle of F (notice that $\varrho(F)=0 \Leftrightarrow m(\mathbf{R})$ is a line). We shall prove later in this chapter that

$$\varrho(F) \in 2\mathbf{N}_0 = \{0, 2, 4, \dots\} \quad \text{for all } F \in \mathfrak{F}.$$

The following lemma describes an important relation between Farey sets and Farey triangles (cf. [33] for the definition of a Farey triangle FT and $N(FT)$).

Lemma 1.2. *For any $F \in \mathfrak{F}$, where F is assumed to be of circular type though not a half plane (i.e. F is a circular disc), either*

(i) *there is precisely one acute angled Farey triangle FT_0 (say) inscribed in F,*

or

(ii) *there is no acute angled Farey triangle inscribed in F, but in return precisely two right* *angled Farey triangles FT_1, FT_2 (say) both inscribed in F; the Farey triangles FT_1 and FT_2 are congruent, in particular $N(FT_1)=N(FT_2)$.*

Proof. Let $F=F(m)$ (say), and assume without restriction that $F=m(\mathfrak{J})$; otherwise we could replace the map $m: z\mapsto(az+b)/(cz+d)$ by $z\mapsto(bz+a)/(dz+c)$. The assumption means that the points a/c, b/d, $(a+b)/(c+d)$ define a positive cyclic ordering of ∂F. Equivalently the assumption means that $-d/c\in\mathbf{C}\setminus\mathfrak{J}$.

Since $FT(a/c, b/d, (a+b)/(c+d))$ is a Farey triangle inscribed in F, it follows that an arbitrary Farey triangle $FT=FT(p_1/q_1, p_2/q_2, (p_1+p_2)/(q_1+q_2))$ inscribed in F with p_1/q_1, p_2/q_2, $(p_1+p_2)/(q_1+q_2)$ defining a positive cyclic ordering of ∂F is given by

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}$$

where $r_1, r_2, s_1, s_2\in\mathbf{Z}$, $r_1s_2-r_2s_1=1$.

Consequently

$$-\frac{q_2}{q_1}=\frac{s_2\left(-\dfrac{d}{c}\right)-r_2}{-s_1\left(-\dfrac{d}{c}\right)+r_1},$$

whence $-q_2/q_1\in\mathbf{C}\setminus\mathfrak{J}$, since $-d/c\in\mathbf{C}\setminus\mathfrak{J}$ and $s_2r_1-s_1r_2=1$.

Since FT is similar to the triangle with vertices $0, 1, -q_2/q_1$, it follows that FT has all its angles $\leqslant\pi/2$ if and only if $-q_2/q_1\in\bar{\mathfrak{J}}^*\setminus\{0, 1, \infty\}$.

However, the set $\bar{\mathfrak{J}}^*\setminus\{0,1,\infty\}$ consists of three copies of the fundamental domain (six copies of its boundary) of the modular group Γ consisting of all maps $\varphi: z\mapsto(r_1z+r_2)/(s_1z+s_2)$, where $r_1, r_2, s_1, s_2\in\mathbf{Z}$, $r_1s_2-r_2s_1=1$, and operating on the set $\mathbf{C}\setminus\mathfrak{J}$. Since each FT inscribed in F occurs for three different φ's, corresponding to the cyclic permutations of the three vertices of FT, the result follows readily.

For any Farey set F we define its *norm* $N(F)$, as follows,

(1a) if F is a halfplane of the form (1.1), we put $N(F)=2$;

(1b) if F is a circular disc, we put $N(F)=N(FT_0)$ or $N(F)=N(FT_1)=N(FT_2)$, depending on whether case (i) or case (ii) of Lemma 1.2 occurs;

(2a) if $F=F(m)$ (say) is of triangular type, and $m(\mathfrak{J})$ is a halfplane of either of the forms (1.2), (1.3), we put $N(F)=2$;

(2b) if F is of triangular type, but not of the form (2a), the three vertices of F are the vertices of a Farey triangle FT, and we put $N(F)=N(FT)$.

Example. $N(\mathcal{J})=N(\mathcal{V}_1)=N(\mathcal{E}_2)=N(\mathcal{E}_3)=2$, $N(\mathcal{V}_2)=N(\mathcal{V}_3)=N(\mathcal{E}_1)=N(\mathcal{C})=4$.

1.3. A generating procedure of Farey sets

We consider the set

$$\mathfrak{F}_n=\{F(m_n)\,|\,M_n=T_0T_1\ldots T_n\},\ n\in\mathbf{N}_0,$$

where

$$T_0=V_1^{b_0},\ b_0\in\mathbf{Z};\ T_1\neq V_1,$$
$$T_\nu\in\{V_j, E_j, C\}\quad\text{if }\det T_0T_1\ldots T_{\nu-1}=\pm1,$$
$$T_\nu\in\{V_j, C\}\quad\text{if }\det T_0T_1\ldots T_{\nu-1}=\pm i,$$

for $1\leqslant\nu\leqslant n$.

Thus $\mathfrak{F}_0$ is the set of all halfplanes (1.1), and $\mathfrak{F}_1$ consists of $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{C}$ and all their translates by $b_0 i$, $b_0\in\mathbf{Z}$.

It follows easily by induction on n that

$$F(m_n)=m_n(\mathcal{J})\quad\text{if }\det m_n=\{\pm1\},\tag{1.4}$$

$$F(m_n)=m_n(\mathcal{J}^*)\quad\text{if }\det m_n=\{\pm i\}.\tag{1.5}$$

Also every Farey set of circular type $F(m_n)=m_n(\mathcal{J})\in\mathfrak{F}_n$ is divided into seven Farey sets in $\mathfrak{F}_{n+1}$ (cf. Fig. 2), three of which are of circular type, namely $F(m_n\circ v_j)=(m_n\circ v_j)(\mathcal{J})=m_n(\mathcal{V}_j)$, while four are of triangular type, namely $F(m_n\circ e_j)=(m_n\circ e_j)(\mathcal{J}^*)=m_n(\mathcal{E}_j)$ and $F(m_n\circ c)=(m_n\circ c)(\mathcal{J}^*)=m_n(\mathcal{C})$. Similarly every Farey set of triangular type $F(m_n)=m_n(\mathcal{J}^*)\in\mathfrak{F}_n$ is divided into four Farey sets in $\mathfrak{F}_{n+1}$ (cf. Fig. 2*), three of which are of triangular type, namely $F(m_n\circ v_j)=(m_n\circ v_j)(\mathcal{J}^*)=m_n(\mathcal{V}_j^*)$, while one is of circular type, namely $F(m_n\circ c)=(m_n\circ c)(\mathcal{J})=m_n(\mathcal{C}^*)$. Of course Fig. 1 is a special case of Fig. 2; however Fig. 1*, though similar to Fig. 2*, is not a special case of Fig. 2*, since $\mathcal{J}^*$ is not a Farey set.

It follows easily by induction that each parallel strip $\{z=x+iy\,|\,b_0\leqslant y\leqslant b_0+1\}$, $b_0\in\mathbf{Z}$, is tessellated into Farey sets from $\mathfrak{F}_n$, $n\geqslant1$, and indeed into $2\cdot5^{n-1}$ Farey sets of circular type, and $4\cdot5^{n-1}$ Farey sets of triangular type.

In Fig. 2, Fig. 2*, the Farey set $F(m_n\circ v_j)$ lies at *vertex* number j, and $F(m_n\circ e_j)$ lies at *edge* number j, while $F(m_n\circ c)$ lies *centrally*; this explains the use of the symbols V_j, E_j, C.

LEMMA 1.3. *Let $F(m_n)$ be an arbitrary Farey set in $\mathfrak{F}_n$, $n\geqslant0$, and put (cf. the short-hand notation in Fig. 2, Fig. 2*, where the points p_j/q_j, p_j'/q_j' are represented simply by j, j')*

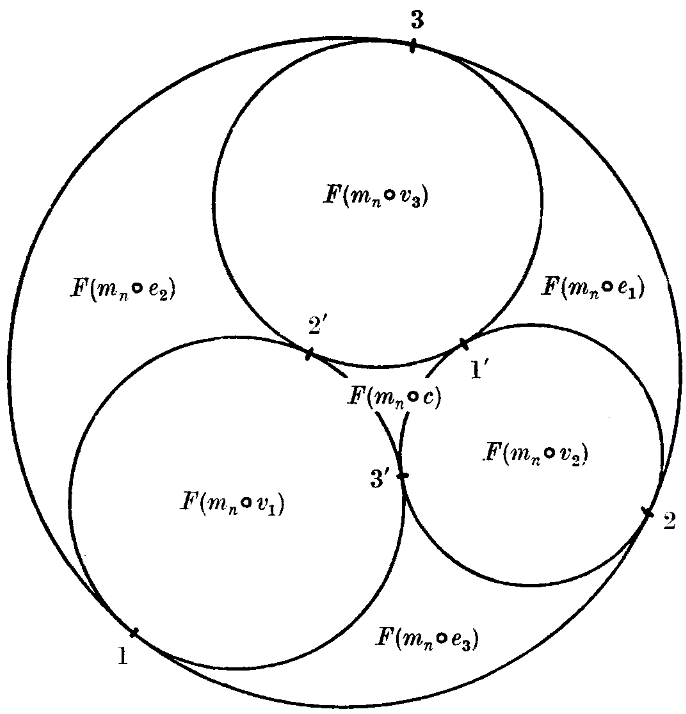
3
$F(m_n \circ v_3)$
$F(m_n \circ e_2)$
$F(m_n \circ e_1)$
2′
1′
$F(m_n \circ c)$
$F(m_n \circ v_2)$
3′
$F(m_n \circ v_1)$
2
1
$F(m_n \circ e_3)$

Fig. 2

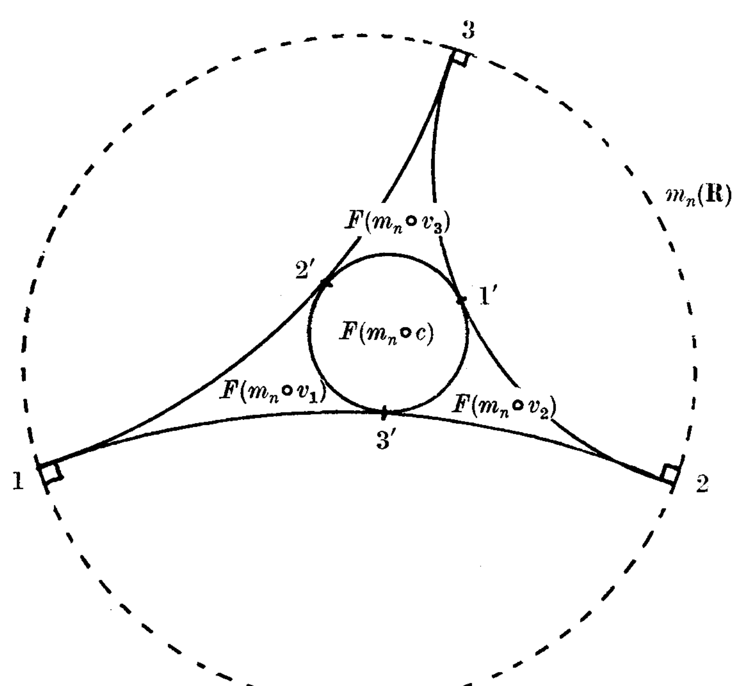
3
$m_n(\mathbf{R})$
$F(m_n \circ v_3)$
2′
1′
$F(m_n \circ c)$
$F(m_n \circ v_1)$
$F(m_n \circ v_2)$
3′
1
2

Fig. 2*

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} = M_n \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} p_1' & p_2' & p_3' \\ q_1' & q_2' & q_3' \end{pmatrix} = M_n C \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Also put

$$N_j = N(q_j),\ N_j' = N(q_j'),\ N = N_1 + N_2 + N_3,$$
$$N' = N_1' + N_2' + N_3',\ N^{(2)} = N_1^2 + N_2^2 + N_3^2.$$

(i) *In the subdivisions of* $F(m_n)$ *the following norm relations are valid,*

$$N_1 + N_1' = N_2 + N_2' = N_3 + N_3',$$
$$N' = 2N + 3(N^2 - 2N^{(2)})^{1/2}.$$

(ii) *If* $\det m_n = \{\pm 1\}$, $n \geq 1$, *then* $FT(p_1/q_1, p_2/q_2, p_3/q_3)$ *is acute angled or right angled.*

(iii) $N(F(m_n)) = N = N(FT(p_1/q_1, p_2/q_2, p_3/q_3))$.

(iv) *If* $\det m_n = \{\pm 1\}$, *then* (*cf. Fig.* 2)

$$\min\,(N(F(m_n \circ v_j)),\ N(F(m_n \circ e_j)),\ N(F(m_n \circ c))) \geq N(F(m_n)),$$

with equality if and only if $N(F(m_n)) = 2$.

(iv*) *If* $\det m_n = \{\pm i\}$, *then* (*cf. Fig.* 2*)

$$\min\,(N(F(m_n \circ v_j)),\ N(F(m_n \circ c))) \geq N(F(m_n)),$$

with equality if and only if $N(F(m_n)) = 2$.

(v)
$$\mathfrak{J} = \bigcup_{n=0}^{\infty} \mathfrak{J}_n,$$

where the union is a disjoint one.

Proof. The norm relations of (i) are simply the norm relations (36), (65) of [33].

We prove (ii) by induction on n. If $\det m_1 = \{\pm 1\}$ we must have $M_1 = V_1^{b_0} V_2$ or $M_1 = V_1^{b_0} V_3$; it follows easily that $FT(p_1/q_1, p_2/q_2, p_3/q_3)$ is right angled. If $\det m_{n+1} = \{\pm 1\}$, either

(a) $\det m_n = \{\pm 1\}$ and $m_{n+1} = m_n \circ v_j$ (cf. Fig. 2),

or

(b) $\det m_n = \{\pm i\}$ and $m_{n+1} = m_n \circ c$ (cf. Fig. 2*).

Case (a). Suppose without restriction that

$$N_1 \leq N_2 \leq N_3 \leq N_1 + N_2,$$

the last inequality being the inductive assumption. Then by the norm relations of (i),

$$N_1' \geqslant N_2' \geqslant N_3',$$
$$N_3' = N + (N^2 - 2N^{(2)})^{\frac{1}{2}} - N_3 \geqslant N - N_3 = N_1 + N_2 \geqslant N_3,$$
$$N_2' \leqslant N_3' + N_1, N_1' \leqslant N_3' + N_2, N_1' \leqslant N_2' + N_3;$$

however this proves the assertion (cf. Fig. 2).

Case (b). Suppose without restriction that $N_1' = \max N_j'$. Then by the norm relations of (i),

$$\begin{aligned} N_1' &= N + (N^2 - 2N^{(2)})^{1/2} - N_1 \leqslant N + 2(N^2 - 2N^{(2)})^{1/2} + N_1 \\ &= (N + (N^2 - 2N^{(2)})^{1/2} - N_2) + (N + (N^2 - 2N^{(2)})^{1/2} - N_3) = N_2' + N_3', \end{aligned}$$

thus proving the inductive step.

The truth of (iii) follows now immediately from the very definition of $N(F(m_n))$, (1.4), (1.5) and (ii).

(iv) and (iv*) are proved by straightforward applications of the norm relations of (i).

Finally, (v), which essentially tells that any $F \in \mathfrak{F}$ is contained in some $\mathfrak{F}_n$, $n \geqslant 0$, is proved by an easy induction on $N(F)$.

In view of the disjoint decomposition in Lemma 1.3 (v) any $F \in \mathfrak{F}$ lies in a unique $\mathfrak{F}_n$, $n \geqslant 0$. We call n the *order* of F and write accordingly $n = \operatorname{ord} F$.

LEMMA 1.4. *Let $F \in \mathfrak{F}$ be an arbitrary Farey set. Then*

(i) $N(F) \in \{2, 4, 6, \ldots\} = 2\mathbf{N}$,

(ii) $\varrho(F) \in \{0, 2, 4, \ldots\} = 2\mathbf{N}_0$,

(iii) $\operatorname{diam} F \leqslant 4(N(F))^{-1/2}$ *if* $N(F) > 2$.

Proof. Properties (i), (ii) are proved simultaneously by induction on ord F, starting with ord $F = 0$, where $N(F) = 2$ and $\varrho(F) = 0$. Assume for the inductive step that $F = F(m_n) \in \mathfrak{F}_n$ satisfies conditions (i), (ii), then we have to show the validity of (i), (ii) also for each of the seven (or four) Farey sets $F(m_n \circ t_{n+1})$ in the subdivision of $F = F(m_n)$.

With the notation of Lemma 1.3 we know that $N(F) = N \equiv 0 \pmod 2$, $\varrho(F) = (N^2 - 2N^{(2)})^{1/2} \equiv 0 \pmod 2$ (that $\varrho(F)$ is integral follows from the second norm relation of Lemma 1.3(i)), hence by the norm relations

$$N_j + N_j' = N + (N^2 - 2N^{(2)})^{1/2} \equiv 0 \pmod 2,$$

which proves the inductive step as regards (i) (cf. Fig. 2, Fig. 2*). The inductive step concerning (ii) follows when applying the second norm relation to $F(m_n \circ t_{n+1})$.

In the proof of (iii) we use again the notation of Lemma 1.3 with $F=F(m_n)$ and $FT=FT(p_1/q_1, p_2/q_2, p_3/q_3)$. In case $\det m_n=\{\pm 1\}$ it follows from Lemma 1.3 (ii) that

$$\operatorname{diam} F \leqslant 2/3^{1/2} \operatorname{diam} FT;$$

in case $\det m_n=\{\pm i\}$ we have simply (also when FT is obtuse angled)

$$\operatorname{diam} F = \operatorname{diam} FT.$$

In order to estimate diam FT we may assume without restriction that $N_1 \geqslant N_2 \geqslant N_3$, and then

$$\begin{aligned}\operatorname{diam} FT &= |p_2/q_2 - p_3/q_3| \leqslant |p_1/q_1 - p_2/q_2| + |p_1/q_1 - p_3/q_3| \\ &= (N_1 N_2)^{-1/2} + (N_1 N_3)^{-1/2} \leqslant 2N_1^{-1/2} \leqslant 2(N/3)^{-1/2}.\end{aligned}$$

Altogether, this proves inequality (iii).

Finally, I want to point out that the circles and mesh triangles of L. R. Ford [14] are special Farey sets, as described in the following

LEMMA 1.5. (i) *The circles of Ford are precisely the Farey sets* $F(m_n)$, *where either* $n=0$ *and* $M_0=V_j$ $(j=2, 3)$ *or* $n \geqslant 1$ *and* $M_n=E_j T_1 \ldots T_{n-1} C$ *with* $T_1, \ldots, T_{n-1} \in \{V_k, V_l\}$, (j, k, l) *being any permutation of* $(1, 2, 3)$.

(ii) *The mesh triangles of Ford are precisely the Farey sets* $F(m_n)$, *where* $n \geqslant 1$ *and* $M_n=E_j T_1 \ldots T_n$ *with* $T_1, .., T_n \in \{V_k, V_l\}$, (j, k, l) *being any permutation of* $(1, 2, 3)$.

The proof of Lemma 1.5 follows readily by the well-known interrelation between Farey fractions and regular continued fractions.

1.4. Dual Farey sets

To any unimodular map m we associate a *dual Farey set* $F^*(m)$ as follows,

(1) if m is properly unimodular, we define

$$F^*(m) = \begin{cases} m(\mathfrak{J}^*) & \text{if } F(m) = m(\mathfrak{J}) \\ m(\overline{\mathfrak{J}^*}) & \text{if } F(m) = m(\overline{\mathfrak{J}}); \end{cases}$$

(2) if m is improperly unimodular, we define

$$F^*(m) = \begin{cases} m(\mathfrak{J}) & \text{if } F(m) = m(\mathfrak{J}^*) \\ m(\overline{\mathfrak{J}}) & \text{if } F(m) = m(\overline{\mathfrak{J}^*}). \end{cases}$$

Further we let $\mathfrak{F}^*$ denote the set of all dual Farey sets, which are contained in the parallel strip

$$\{z=x+iy \mid 0 \leqslant x \leqslant 1\}.$$

The following result is easily deduced from Lemma 1.3 (v).

LEMMA 1.6. *The set $\mathfrak{F}^*$ can be described as*

$$\mathfrak{F}^* = \bigcup_{n=0}^{\infty} \mathfrak{F}_n^*,$$

where the union is a disjoint one, and

$$\mathfrak{F}_n^* = \{F^*(m_n) \mid M_n = T_0 T_1 \dots T_n\},\ n \geqslant 0,$$

where

$$T_0 = V_1^{b_0},\ b_0 \in \mathbf{Z};\ T_1 \neq V_1,$$
$$T_\nu \in \{V_j, C\} \quad \textit{if } \det T_0 T_1 \dots T_{\nu-1} = \pm 1,$$
$$T_\nu \in \{V_j, E_j, C\} \quad \textit{if } \det T_0 T_1 \dots T_{\nu-1} = \pm i$$

for $1 \leqslant \nu \leqslant n$.

Obviously the full content of sections 1.2 and 1.3 carry over to dual Farey sets. Notice in particular that $\mathfrak{J}^*$, $\mathfrak{V}_1^*$, $\mathfrak{V}_2^*$, $\mathfrak{V}_3^*$ are dual Farey sets of triangular type, while $\mathfrak{C}^*$ is a dual Farey set of circular type.

Chapter 2

Regular and dually regular chains

2.1. Representation of complex numbers

DEFINITION 2.1. *A regular chain is an infinite product*

$$T_0 T_1 \dots T_n \dots,$$

where

$$T_0 = V_1^{b_0},\ b_0 \in \mathbf{Z};\ T_1 \neq V_1,$$

and

$$T_n \in \{V_j, E_j, C\} \quad \textit{if } \det T_0 T_1 \dots T_{n-1} = \pm 1,$$
$$T_n \in \{V_j, C\} \quad \textit{if } \det T_0 T_1 \dots T_{n-1} = \pm i$$

for $n \geqslant 1$, with the additional restriction that no $n_0 \in \mathrm{N}$, $j \in \{1, 2, 3\}$, exist, such that $T_n = V_j$ for all $n \geqslant n_0$.

DEFINITION 2.1*. *A dually regular chain is an infinite product*

$$T_0 T_1 \dots T_n \dots,$$

where

$$T_0 = V_1^{b_0},\ b_0 \in \mathbf{Z};\ T_1 \neq V_1,$$

and

$$T_n \in \{V_j, C\} \quad \text{if } \det T_0 T_1 \dots T_{n-1} = \pm 1,$$
$$T_n \in \{V_j, E_j, C\} \quad \text{if } \det T_0 T_1 \dots T_{n-1} = \pm i$$

for $n \geq 1$, *with the additional restriction that no* $n_0 \in \mathbf{N}$, $j \in \{1, 2, 3\}$, *exist, such that* $T_n = V_j$ *for all* $n \geq n_0$.

In connection with these definitions we introduce some standard notation (valid in both cases),

$$M_n = T_0 T_1 \dots T_n,$$
$$\begin{pmatrix} p_1^{(n)} & p_2^{(n)} & p_3^{(n)} \\ q_1^{(n)} & q_2^{(n)} & q_3^{(n)} \end{pmatrix} = M_n \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$
$$m_n\colon z \mapsto \frac{p_1^{(n)} z + p_2^{(n)}}{q_1^{(n)} z + q_2^{(n)}}.$$

For a regular chain we put $F_n = F(m_n)$, and we call the sequence F_n, $n \in \mathbf{N}_0$, a *regular chain of Farey sets.*

For a dually regular chain we put $F_n^* = F^*(m_n)$, and we call the sequence F_n^*, $n \in \mathbf{N}_0$, a *dually regular chain of dual Farey sets.*

THEOREM 2.1. *For any regular chain*

$$T_0 T_1 \dots T_n \dots$$

we have

$$F_0 \supset F_1 \supset \dots \supset F_n \supset \dots,$$

and

$$\bigcap_{n=0}^{\infty} F_n = \{\xi\},$$

where $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$. *Further*

$$\lim_{n\to\infty} p_j^{(n)}/q_j^{(n)} = \xi \quad \text{for } j = 1, 2, 3$$

THEOREM 2.1*. *For any dually regular chain*

$$T_0 T_1 \dots T_n \dots$$

we have

$$F_0^* \supset F_1^* \supset \ldots \supset F_n^* \supset \ldots,$$

and

$$\bigcap_{n=0}^{\infty} F_n^* = \{\xi\},$$

where $\xi \in \{z = x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i)$. *Further*

$$\lim_{n\to\infty} p_j^{(n)}/q_j^{(n)} = \xi \quad \textit{for } j = 1, 2, 3.$$

Proof. From section 1.3 follows immediately that the sequence F_n, $n \geqslant 0$, is a decreasing sequence of Farey sets, and since each F_n is a compact set on the Riemann sphere, it follows that

$$\bigcap_{n=0}^{\infty} F_n \neq \emptyset.$$

By definition a regular chain is not of the form

$$V_1^{b_0} E_k \overrightarrow{V_1}, (k=2, 3),$$

where the arrow denotes periodicity. Let $n_0 \in \mathbf{N}$ be the smallest integer such that

$$T_0 T_1 \ldots T_{n_0} \neq V_1^{b_0} E_k V_1^{n_0-1}, (k=2,3),$$

then clearly

$$N(F_0) = N(F_1) = \ldots = N(F_{n_0-1}) = 2 < N(F_{n_0}) = 4.$$

Using Lemma 1.3 (iv) (iv*) we find that

$$4 = N(F_{n_0}) < N(F_{n_0+1}) < \ldots,$$

and hence by Lemma 1.4 (i) (iii)

$$\lim_{n\to\infty} \operatorname{diam} F_n = 0.$$

Altogether this proves Theorem 2.1, except for the assertion that $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$. Notice that $q_j^{(n)} \neq 0$ for $n \geqslant n_0$.

It is a simple geometric fact (cf. [33], Lemma 3), using Lemma 1.3 (ii) in case $\det m_n = \{\pm 1\}$, that

$$F_n \subset \bigcup_{j=1}^{3} B(p_j^{(n)}/q_j^{(n)}; (\sqrt{2}|q_j^{(n)}|^2)^{-1}), \quad n \geqslant n_0,$$

where

$$B(z_0; r) = \{z \in \mathbf{C} \mid |z - z_0| \leqslant r\}.$$

Hence it follows that the inequality

$$|\xi - p_j^{(n)}/q_j^{(n)}| \leqslant (\sqrt{2}|q_j^{(n)}|^2)^{-1}$$

is satisfied for infinitely many pairs (j, n), $j \in \{1, 2, 3\}$, $n \geqslant n_0$.

However, the restriction put on a regular chain not to be periodic with period V_j clearly has the effect that this inequality has infinitely many different solutions $p_j^{(n)}/q_j^{(n)}$; hence by a standard argument $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$.

Theorem 2.1* follows immediately from Theorem 2.1 by the very definition of $F^*(m_n)$ in section 1.4.

In connection with Theorems 2.1, 2.1* we introduce some further notation.

First of all we say that the regular (dually regular) chain $T_0 T_1 \dots T_n \dots$ *represents* ξ or *converges to* ξ, and we indicate this in writing

$$\xi = [T_0 T_1 \dots T_n \dots].$$

Further for any regular (dually regular) chain $T_0 T_1 \dots T_n \dots$ we notice that

$$T_n T_{n+1} T_{n+2} \dots, \; n \in \mathbf{N}_0,$$

is either a regular chain or a dually regular chain; in either case we put

$$\xi_n = [T_n T_{n+1} \dots], \; n \in \mathbf{N}_0, \tag{2.1}$$

which we call the *n'th complete quotient* of the given regular (dually regular) chain. It follows from Theorems 2.1, 2.1*, (1.4), (1.5), Lemma 1.3 (ii) and the proof of Lemma 1.2, that for all $n \in \mathbf{N}_0$,

$$\xi_n = t_n(\xi_{n+1}), \tag{2.2}$$

$$\xi_{n+1} \in \mathfrak{J}, \; -q_2^{(n)}/q_1^{(n)} \in \overline{\mathfrak{J}^*} \quad \text{if } \det M_n = \pm 1, \tag{2.3}$$

$$\xi_{n+1} \in \mathfrak{J}^*, \; -q_2^{(n)}/q_1^{(n)} \in \overline{\mathfrak{J}} \quad \text{if } \det M_n = \pm i. \tag{2.4}$$

Using (2.2) repeatedly we get

$$\xi = \xi_0 = m_n(\xi_{n+1}), \; n \in \mathbf{N}_0, \tag{2.5}$$

whence for $n \geqslant n_0$ (to secure that $q_j^{(n)} \neq 0$),

$$\xi_0 - p_1^{(n)}/q_1^{(n)} = -\varepsilon_n (q_1^{(n)}(q_1^{(n)}\xi_{n+1} + q_2^{(n)}))^{-1}, \tag{2.6}$$

$$\xi_0 - p_2^{(n)}/q_2^{(n)} = \varepsilon_n \xi_{n+1} (q_2^{(n)}(q_1^{(n)}\xi_{n+1} + q_2^{(n)}))^{-1}, \tag{2.7}$$

$$\xi_0 - p_3^{(n)}/q_3^{(n)} = \varepsilon_n (\xi_{n+1} - 1)(q_3^{(n)}(q_1^{(n)}\xi_{n+1} + q_2^{(n)}))^{-1}, \tag{2.8}$$

where

$$\varepsilon_n = p_1^{(n)} q_2^{(n)} - p_2^{(n)} q_1^{(n)} \in \{\pm 1, \pm i\}.$$

Taking absolute values we obtain the following approximation formulae (for $n \geqslant n_0$),

$$c_1^{(n)} = |\xi_{n+1} + q_2^{(n)}/q_1^{(n)}|, \tag{2.9}$$

$$c_2^{(n)} = |\xi_{n+1}^{-1} + q_1^{(n)}/q_2^{(n)}|, \tag{2.10}$$

$$c_3^{(n)} = |(\xi_{n+1} - 1)^{-1} + q_1^{(n)}/q_3^{(n)}|, \tag{2.11}$$

where we have put for abbreviation

$$c_j^{(n)} = (|q_j^{(n)}||q_j^{(n)}\xi_0 - p_j^{(n)}|)^{-1}.$$

Theorem 2.2. *For any $\xi \in \mathbf{C}$ not properly equivalent to a real number, there is precisely one regular chain* $\mathrm{ch}\,\xi$ *representing ξ. For any $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$, which is properly equivalent to a real number, there are precisely two regular chains* $\mathrm{ch}_1\,\xi$, $\mathrm{ch}_2\,\xi$ *representing ξ.*

Theorem 2.2*. *For any $\xi \in \{z = x+iy \mid 0<x<1\}$ not improperly equivalent to a real number, there is precisely one dually regular chain* $\mathrm{ch}^*\,\xi$ *representing ξ. For any $\xi \in \{z = x+iy \mid 0<x<1\}$, which is improperly equivalent to a real number, there are precisely two dually regular chains* $\mathrm{ch}_1^*\,\xi$, $\mathrm{ch}_2^*\,\xi$ *representing ξ. For any $\xi \in \mathbf{C}\setminus\mathbf{Q}(i)$ with* $\mathrm{Re}\,\xi \in \{0, 1\}$ *there is precisely one dually regular chain* $\mathrm{ch}^*\,\xi$ *representing ξ.*

Proof. In case $\xi \in \mathbf{C}$ is not properly equivalent to a real number, and hence $\xi \notin \partial F$ for any $F \in \mathfrak{F}$, it follows from the results in section 1.3 that for any $n \in \mathbf{N}_0$, there is a unique $F_n \in \mathfrak{F}_n$ with $\xi \in F_n$. Put $F_n = F(t_0 \circ t_1 \circ \ldots \circ t_n)$, $n \in \mathbf{N}_0$, then it is obvious that $T_0 T_1 \ldots T_n \ldots$ is a well-defined regular chain representing ξ, and clearly also the only one.

In case $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ it follows from the results in section 1.3 that for any $n \in \mathbf{N}_0$, there is a unique $F_n^{(1)}$ with $\alpha \in F_n^{(1)} \subseteq \mathfrak{J}$, and a unique $F_n^{(2)}$ with $\alpha \in F_n^{(2)} \subseteq \overline{\mathfrak{J}}$; of course $F_n^{(2)} = \varkappa(F_n^{(1)}) = \overline{F_n^{(1)}}$. Put $F_n^{(k)} = F(t_0^{(k)} \circ t_1^{(k)} \circ \ldots \circ t_n^{(k)})$ for $n \in \mathbf{N}_0$, $k \in \{1, 2\}$, then it is obvious that $T_0^{(k)} T_1^{(k)} \ldots T_n^{(k)} \ldots$ $(k = 1, 2)$ are well-defined regular chains representing α, and clearly also the only ones.

Let $\alpha = [a_0, a_1, a_2, \ldots]$ (regular continued fraction), then it is a simple matter to show that

$$\mathrm{ch}_1\,\alpha = E_3\, V_1^{-a_0-1}\, V_2\, V_1^{a_1-1}\, V_2^{a_2}\, V_1^{a_3}\, V_2^{a_4} \ldots \quad \text{if } \alpha < 0, \tag{2.12}$$

$$\mathrm{ch}_1\,\alpha = E_1\, V_2^{a_1-1}\, V_3^{a_2}\, V_2^{a_3}\, V_3^{a_4} \ldots \quad \text{if } 0 < \alpha < 1, \tag{2.13}$$

$$\mathrm{ch}_1\,\alpha = E_2\, V_1^{a_0-1}\, V_3^{a_1}\, V_1^{a_2}\, V_3^{a_3}\, V_1^{a_4} \ldots \quad \text{if } \alpha > 1, \tag{2.14}$$

and that

$$\mathrm{ch}_2\,\alpha = V_1^{-1} E_3\, V_1^{-a_0-1}\, V_3\, V_1^{a_1-1}\, V_3^{a_2}\, V_1^{a_3}\, V_3^{a_4} \ldots \quad \text{if } \alpha < 0, \tag{2.15}$$

$$\mathrm{ch}_2\,\alpha = V_1^{-1} C\, V_3^{a_1-1}\, V_2^{a_2}\, V_3^{a_3}\, V_2^{a_4} \ldots \quad \text{if } 0 < \alpha < 1, \tag{2.16}$$

$$\mathrm{ch}_2\,\alpha = V_1^{-1} E_2\, V_1^{a_0-1}\, V_2^{a_1}\, V_1^{a_3}\, V_2^{a_3}\, V_1^{a_4} \ldots \quad \text{if } \alpha > 1. \tag{2.17}$$

Now let $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ be properly equivalent to a real number, say $\xi = m(\eta)$, where m is properly unimodular and $\eta \in \mathbf{R} \setminus \mathbf{Q}$. By Lemma 1.3 (v) there is a smallest integer $n_0 \in \mathbf{N}_0$ such that $\xi \in \partial F$ for some $F \in \mathfrak{F}_{n_0}$.

In case $n_0=0$, then $\xi=\alpha+ib_0$ with $\alpha\in\mathbf{R}\setminus\mathbf{Q}$, $b_0\in\mathbf{Z}$, and clearly

$$\mathrm{ch}_1\,\xi=V_1^{b_0}\,\mathrm{ch}_1\,\alpha,\ \mathrm{ch}_2\,\xi=V_1^{b_0}\,\mathrm{ch}_2\,\alpha. \tag{2.18}$$

In case $n_0\geq 1$, then $F_{n_0}\in\mathfrak{F}_{n_0}$ is uniquely determined by requiring that $\xi\in F_{n_0}$, and that F_{n_0} be of circular type. Let $F_{n_0}=F(t_0\circ t_1\circ\ \dots\ \circ t_{n_0})=F(m_{n_0})$ be its standard representation (cf. section 1.3). Then $\xi=m_{n_0}(\alpha)$ with $\alpha\in\mathbf{R}\setminus\mathbf{Q}$, and we put $A_1=m_{n_0}(]0,1[)$, $A_2=m_{n_0}(]1,\infty[)$, $A_3=m_{n_0}(]-\infty,0[)$, and $\mathrm{ch}_1\,\alpha=U_0U_1U_2\ldots$ (given by (2.12), (2.13), (2.14)). Clearly

$$\mathrm{ch}_1\,\xi=T_0T_1\ \dots\ T_{n_0}U_0U_1U_2\ \dots \tag{2.19}$$

is a regular chain representing ξ. Also it is easily verified that there is precisely one other regular chain $\mathrm{ch}_2\,\xi$ representing ξ, namely

$$\mathrm{ch}_2\,\xi=T_0T_1\ \dots T_{n_0-1}\hat{T}_{n_0}\lambda(U_0)\,\lambda(U_1)\,\lambda(U_2)\ \dots, \tag{2.20}$$

where λ is the permutation

$$\lambda:\begin{pmatrix} V_1 & V_2 & V_3 & E_1 & E_2 & E_3 & C\\ V_1 & V_3 & V_2 & E_1 & E_3 & E_2 & C\end{pmatrix},$$

and

$$\hat{T}_{n_0}=\begin{cases} V_k & \text{if } \xi\in A_k \quad \text{and } T_{n_0}=C\\ E_k & \text{if } \xi\in A_k \quad \text{and } T_{n_0}=V_j\ (j\neq k).\\ C & \text{if } \xi\in A_k \quad \text{and } T_{n_0}=V_k\end{cases}$$

This proves the second part of Theorem 2.2.

The proof of Theorem 2.2* is easily derived from Theorem 2.2.

2.2. Equivalence

The relationship between regular (dually regular) chains for two numbers $\xi,\eta\in\mathbf{C}\setminus\mathbf{Q}(i)$ ($\{z=x+iy\,|\,0\leq x\leq 1\}\setminus\mathbf{Q}(i)$) that are equivalent (cf. section 1.1) is described in the following theorems.

THEOREM 2.3. *$\xi,\eta\in\mathbf{C}\setminus\mathbf{Q}(i)$ are properly equivalent if and only if ξ,η have regular chains of the form*

$$\mathrm{ch}\,\xi=T_0T_1\ \dots\ T_gT_{g+1}T_{g+2}\ \dots\ T_{g+n}\ \dots,$$
$$\mathrm{ch}\,\eta=U_0U_1\ \dots\ U_hU_{h+1}U_{h+2}\ \dots\ U_{h+n}\ \dots$$

with

(i) $\det t_0\circ t_1\circ\ \dots\ \circ t_g=\det u_0\circ u_1\circ\dots\circ u_h$

and

(ii) $U_{h+n} = S^j T_{g+n} S^{-j}$, $n \in \mathbf{N}$,

where $j \in \{1, 2, 3\}$ *is a fixed integer.*

THEOREM 2.3*. $\xi, \eta \in \{z = x+iy \mid 0 < x < 1\} \setminus \mathbf{Q}(i)$ *are properly equivalent if and only if* ξ, η *have dually regular chains of the form*

$$\mathrm{ch}^* \xi = T_0 T_1 \dots T_g T_{g+1} T_{g+2} \dots T_{g+n} \dots,$$
$$\mathrm{ch}^* \eta = U_0 U_1 \dots U_h U_{h+1} U_{h+2} \dots U_{h+n} \dots$$

with conditions (i), (ii) *of Theorem* 2.3 *satisfied.*

THEOREM 2.4. $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$, $\eta \in \{z = x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i)$ *are improperly equivalent if and only if* ξ, η *have chains of the form*

$$\mathrm{ch}\, \xi = T_0 T_1 \dots T_g T_{g+1} T_{g+2} \dots T_{g+n} \dots,$$
$$\mathrm{ch}^* \eta = U_0 U_1 \dots U_h U_{h+1} U_{h+2} \dots U_{h+n} \dots$$

with

$$\det t_0 \circ t_1 \circ \dots \circ t_g = \{\pm i\} \det u_0 \circ u_1 \circ \dots \circ u_h$$

and satisfying condition (ii) *of Theorem* 2.3.

Proof. We restrict ourselves to Theorem 2.3, since the two other theorems are proved similarly.

Suppose first that ξ, η have regular chains satisfying conditions (i), (ii). Then by (2.5)

$$\xi = t_0 \circ t_1 \circ \dots \circ t_g(\xi_{g+1}),$$
$$\eta = u_0 \circ u_1 \circ \dots \circ u_h(\eta_{h+1}),$$

and by condition (ii),

$$\eta_{h+1} = s^j(\xi_{g+1}).$$

Hence

$$\eta = u_0 \circ u_1 \circ \dots \circ u_h \circ s^j \circ (t_0 \circ t_1 \circ \dots \circ t_g)^{-1}(\xi);$$

by condition (i) and since $\det s = \{\pm 1\}$, this shows that ξ, η are properly equivalent.

Conversely, suppose that $\xi, \eta \in \mathbf{C} \setminus \mathbf{Q}(i)$ are properly equivalent, say $\eta = m(\xi)$, where $m: z \mapsto (az+b)(cz+d)^{-1}$ is properly unimodular, and let

$$\mathrm{ch}\, \xi = T_0 T_1 \dots T_n \dots$$

be a regular chain representing ξ. Put $F_n = F(t_0 \circ t_1 \circ \dots \circ t_n)$, then it follows that $m(F_n)$ is a Farey set of the same type as F_n and containg η, provided $-d/c \notin F_n$. However, this condition is certainly satisfied for all $n \geq n_0$ (say), since $-d/c \neq \xi \in \mathbf{C} \setminus \mathbf{Q}(i)$, and $\lim \operatorname{diam} F_n = 0$.

Case 1. We assume there exists a $g \geq n_0$ such that F_g (and hence also $m(F_g)$) is of triangular type. Let $m(F_g) = F(u_0 \circ u_1 \circ \ldots \circ u_h)$ be the representation of section 1.3, then in particular $\det t_0 \circ t_1 \circ \ldots \circ t_g = \det u_0 \circ u_1 \circ \ldots \circ u_h$ $(= \{\pm i\})$. Also we put

$$\begin{pmatrix} p_1^{(g)} & p_2^{(g)} & p_3^{(g)} \\ q_1^{(g)} & q_2^{(g)} & q_3^{(g)} \end{pmatrix} = T_0 T_1 \ldots T_g \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} P_1^{(h)} & P_2^{(h)} & P_3^{(h)} \\ Q_1^{(h)} & Q_2^{(h)} & Q_3^{(h)} \end{pmatrix} = U_0 U_1 \ldots U_h \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since m maps ∂F_g onto $\partial m(F_g)$ with preservation of orientation, it follows by (1.5) that

$$m(p_k^{(g)}/q_k^{(g)}) = P_{k+j}^{(h)}/Q_{k+j}^{(h)}$$

for $k \in \{1, 2, 3\}$ with some fixed $j \in \{1, 2, 3\}$; of course $k+j$ has to be reduced modulo 3. Now define U_{h+n}, $n \in N$, by condition (ii), then clearly $U_0 U_1 \ldots U_h U_{h+1} U_{h+2} \ldots U_{h+n} \ldots$ is a regular chain representing η.

Case 2. We assume for all $n \geq n_0$ that F_n (and hence also $m(F_n)$) is of circular type, whence in particular $T_n \in \{V_j\}$ for all $n > n_0$. Let u_n be the maximal angle of $FT(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, then $\pi/3 \leq u_n \leq \pi/2$ by Lemma 1.3 (ii).

In case u_n does not converge to $\pi/2$ for $n \to \infty$, we proceed as in case 1 using the conformal mapping property of m together with Lemma 1.2 (i) and Lemma 1.3 (ii).

In case u_n converges to $\pi/2$ for $n \to \infty$, it is a simple consequence of the norm relations of Lemma 1.3 (i) that there exists a fixed (n_1, k) with $n_1 > n_0$, $k \in \{1, 2, 3\}$, such that $T_n \in \{V_1, V_2, V_3\} \setminus \{V_k\}$ for all $n \geq n_1$. Hence $\xi \in \partial F^*(t_0 \circ t_1 \circ \ldots \circ t_{n_1})$, and since $F^*(t_0 \circ t_1 \circ \ldots \circ t_{n_1})$ is a dual Farey set of triangular type, the proof is completed as in case 1.

2.3. Approximation theorems

For any $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ $(\{z = x + iy \mid 0 \leq x \leq 1\} \setminus \mathbf{Q}(i))$ we shall call a reduced fraction p/q with $p, q \in \mathbf{Z}[i]$, $q \neq 0$, a *convergent* (*dual convergent*) of ξ if $p/q = p_j^{(n)}/q_j^{(n)}$ for some $n \in \mathbf{N}_0$, $j \in \{1, 2, 3\}$ corresponding to any regular (dually regular) chain of ξ.

As in [33] we define the *approximation constant* of a $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ as

$$C(\xi) = \limsup \left(|q| \, |q\xi - p| \right)^{-1},$$

the lim sup being taken over all $p, q \in \mathbf{Z}[i]$, $q \neq 0$.

THEOREM 2.5. *Let* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ $(\{z = x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i))$; *then any irreducible fraction* p/q *with* $p, q \in \mathbf{Z}[i]$, $q \neq 0$, *and satisfying*

$$\left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{(1+1/\sqrt{2})|q|^2}$$

is a convergent (*dual convergent*) *of* ξ.

Proof. The proof of Theorem 3 of [33] applies with some obvious changes.

THEOREM 2.6. *Let* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ $(\{z = x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i)$ *have the regular* (*dually regular*) *chain* $T_0 T_1 T_2 \ldots$; *then* (*with the notation of section* 2.1)

$$C(\xi) = \limsup \left(|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|\right)^{-1},$$

the lim sup *being taken over all* $(n, j) \in \mathbf{N}_0 \times \{1, 2, 3\}$.

Proof. The result follows readily from Theorem 2.5, since

$$1 + 1/\sqrt{2} < \sqrt{3} \leqslant C(\xi),$$

the last inequality being a consequence of a theorem of Ford [13].

THEOREM 2.7. *Suppose* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ *is represented by a purely periodic regular* (*dually regular*) *chain*,

$$\xi = [\overrightarrow{\overline{T_0 T_1 \ldots T_{k-1}}}]$$

(*cf. chapter* 3). *Then*

$$C(\xi) = \sqrt{|D|}/\mu,$$

where D *and* μ *are determined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T_0 T_1 \ldots T_{k-1},$$

$$f(X, Y) = cX^2 + (d-a)XY - bY^2,$$

$$D = (d-a)^2 + 4bc,$$

$$\mu = \min |f(p_j^{(l)}/q_j^{(l)})|,$$

the minimum being taken over all $(l, j) \in \{0, 1, \ldots, k-1\} \times \{1, 2, 3\}$.

Proof. Obviously $\xi = (a\xi + b)/(c\xi + d)$ is a root of $f(x, 1) = 0$, and hence

$$f(X, Y) = c(X - \xi Y)(X - \xi' Y). \tag{2.21}$$

By the pure periodicity

$$f(p_j^{(l+nk)}/q_j^{(l+nk)}) = (ad-bc)^{2n} f(p_j^{(l)}/q_j^{(l)}),$$

in particular

$$|f(p_j^{(l+nk)}/q_j^{(l+nk)})| = |f(p_j^{(l)}/q_j^{(l)})| \tag{2.22}$$

for $n \in \mathbf{N}_0$, $0 \leqslant l \leqslant k-1$.

Consequently by (2.21) and (2.22),

$$(|q_j^{(l+nk)}||q_j^{(l+nk)}\xi - p_j^{(l+nk)}|)^{-1} = \frac{|c||p_j^{(l+nk)}/q_j^{(l+nk)} - \xi'|}{|f(p_j^{(l)}/q_j^{(l)}|}$$

$$\to \frac{|c||\xi-\xi'|}{|f(p_j^{(l)}/q_j^{(l)})|} = \frac{\sqrt{|D|}}{|f(p_j^{(l)}/q_j^{(l)})|} \quad \text{for } n \to \infty.$$

This proves Theorem 2.7 in view of Theorem 2.6.

By Theorem 2.7 it is a finite procedure to calculate $C(\xi)$ for any $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ having a periodic regular (dually regular) chain. We shall give three examples of this.

Example 2.1. (Ford [13].) $\xi = \frac{1}{2}(1+i\sqrt{3})$ has the (unique) regular and dually regular chain

$$\operatorname{ch} \xi = \operatorname{ch}^* \xi = \overrightarrow{C}.$$

It follows that

$$f(X, Y) = (1-i)(X^2 - XY + Y^2),\ D = 6i,\ \mu = \sqrt{2},$$

hence

$$C(\xi) = \sqrt{3} = \sqrt{4 - 1/1^2} = 1.7320\ldots.$$

Example 2.2. $\xi = \frac{1}{2}(1 + i(2+\sqrt{99})/5)$ has the (unique) regular chain

$$\operatorname{ch} \xi = \overrightarrow{V_1 E_1 C E_1 V_1 C}.$$

It follows by a simple calculation that

$$f(X, Y) = 2i(5X^2 - (5+2i)XY + (6+i)Y^2),\ D = 396,\ \mu = 10,$$

hence

$$C(\xi) = \sqrt{4 - 1/5^2} = 1.9899\ldots.$$

Example 2.3. (Cassels [3].) $\xi_0 = (-1 + \sqrt{-15-12i})/(2+4i)$, which is a root of the quadratic form

$$(1+2i)X^2 + XY + (2-i)Y^2,$$

has the (unique) regular chain

$$\operatorname{ch} \xi_0 = \overrightarrow{CCE_2CV_1E_1V_1CE_3CV_2E_2V_2CE_1CV_3E_3V_3}$$

of period 18. Since ξ_0 is improperly equivalent to ξ_1, and hence $C(\xi_0)=C(\xi_1)$, we consider ξ_1, which has the purely periodic dually regular chain

$$\operatorname{ch}^* \xi_1 = \overrightarrow{CE_2CV_1E_1V_1CE_3CV_2E_2V_2CE_1CV_3E_3V_3}.$$

The calculation of $C(\xi_1)$ is much facilitated by the observation that in $\operatorname{ch}^* \xi_1$,

$$T_{n+6} = ST_nS^{-1} \quad \text{for } n \in \mathbf{N}_0,$$

by Lemma 1.1 (ii). Hence by a slightly modified version of Theorem 2.7 we get

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = CE_2CV_1E_1V_1S = \begin{pmatrix} 1+9i & -7-6i \\ 12+3i & -11+6i \end{pmatrix},$$

$$\tilde{f}(X, Y) = (4+i)(3X^2 - 3XY + (2+i)Y^2), \; \tilde{D} = (4+i)^2(-15-12i),$$

$$\tilde{\mu} = \min_{1 \leqslant j \leqslant 3,\, 0 \leqslant l \leqslant 6} |\tilde{f}(p_j^{(l)}/q_j^{(l)})| = \sqrt{17} \cdot \sqrt{5},$$

hence

$$C(\xi_0) = C(\xi_1) = \sqrt{|\tilde{D}|}/\tilde{\mu} = \sqrt{\tfrac{3}{5}\sqrt{41}} = 1.9599\ldots.$$

In chapter 6 we shall prove the result that the set of approximation constants <2 is

$$\{\sqrt{4 - 1/\Lambda^2} \mid \Lambda = 1, 5, 29, 65, \ldots\} \cup \{\sqrt{\tfrac{3}{5}\sqrt{41}}\}.$$

The three examples above thus represent the three lowest approximation constants.

Chapter 3

Periodic chains

3.1. Periodic chains

A regular (dually regular) chain

$$T_0T_1 \ldots T_n \ldots$$

is called *periodic* with *period* $k \geqslant 1$ if there exists an $h \geqslant -1$ such that

$$T_n = T_{n+k} \quad \text{for all } n \geqslant h+1,$$

in which case we write the chain as

$$T_0T_1 \ldots T_h \overrightarrow{T_{h+1}T_{h+2} \ldots T_{h+k}} \tag{3.1}$$

The chain (3.1) is called *purely periodic* if we may take $h=-1$. (Notice that in order that (say) $V_1^2 C \overrightarrow{V_1 V_1}$ be considered as a purely periodic regular (dually regular) chain, we have to deviate slightly from the notation used so far and thus allowing T_1 to be V_1).

THEOREM 3.1. *A periodic regular chain represents a* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ *which is quadratic over* $\mathbf{Q}(i)$.

THEOREM 3.1*. *A periodic dually regular chain represents a* $\xi \in \{z=x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i)$ *which is quadratic over* $\mathbf{Q}(i)$.

Proof. Suppose ξ has the regular (dually regular) chain (3.1), then by (2.2), (2.5)

$$\xi = \xi_0 = m_h(\xi_{h+1}),$$
$$\xi_{h+1} = t_{h+1} \circ t_{h+2} \circ \ldots \circ t_{h+k}(\xi_{h+k+1}),$$

with m_h and $t_{h+1} \circ t_{h+2} \circ \ldots \circ t_{h+k}$ unimodular. Since $\xi_{h+1} = \xi_{h+k+1}$ by the periodicity, the result follows immediately.

THEOREM 3.2. *Every* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ *which is quadratic over* $\mathbf{Q}(i)$ *has a periodic regular chain* (*two in case* ξ *is properly equivalent to a real number*).

THEOREM 3.2*. *Every* $\xi \in \{z=x+iy \mid 0 \leqslant x \leqslant 1\} \setminus \mathbf{Q}(i)$ *which is quadratic over* $\mathbf{Q}(i)$ *has a periodic dually regular chain* (*two in case* ξ *is improperly equivalent to a real number and* $\operatorname{Re} \xi \notin \{0, 1\}$).

Proof. Let ξ be quadratic over $\mathbf{Q}(i)$, then ξ is a root of a quadratic equation

$$Az^2 + Bz + C = 0$$

with $A, B, C \in \mathbf{Z}[i]$, and $D = B^2 - 4AC$ not a square in $\mathbf{Z}[i]$. Hence

$$\xi = \frac{-B \pm \sqrt{D}}{2A},$$

where we fix $\sqrt{D}$ such that $\arg \sqrt{D} \in [0, \pi[$. In any case $\xi = \xi_0$ has the form

$$\xi_0 = \frac{\sqrt{D} + P_0}{Q_0}$$

with $P_0, Q_0 \in \mathbf{Z}[i]$ and $Q_0 \mid D - P_0^2$ (in $\mathbf{Z}[i]$), since

$$D - P_0^2 = D - B^2 = -4AC, \quad Q_0 = \pm 2A.$$

Let

$$T_0 T_1 \ldots T_n \ldots$$

be any regular (dually regular) chain representing ξ_0.

As the first step of the proof we want to show that any complete quotient ξ_n of the chain above is of the form

$$\xi_n = \frac{\sqrt{D} + P_n}{Q_n}$$

with $P_n, Q_n \in \mathbf{Z}[i]$ and $Q_n \mid D - P_n^2$.

We proceed by induction. Since we know the result to be true for ξ_0, suppose it true for ξ_n and consider ξ_{n+1}, which by (2.2) is of the form

$$\xi_{n+1} = t_n^{-1}(\xi_n)$$

with

$$T_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \{V_j, E_j, C\}.$$

Hence

$$\xi_{n+1} = \frac{-d\xi_n + b}{c\xi_n - a} = \frac{-d(\sqrt{D} + P_n) + bQ_n}{c(\sqrt{D} + P_n) - aQ_n}$$
$$= \frac{(bc - ad)\, Q_n \sqrt{D} - cd(D - P_n^2) + Q_n((-ad - bc)\, P_n + abQ_n)}{c^2(D - P_n^2) + Q_n(2acP_n - a^2 Q_n)}$$

However, since $bc - ad = \varepsilon$ is a unit in $\mathbf{Z}[i]$ and $Q_n \mid D - P_n^2$, we have

$$P_{n+1} = \varepsilon^{-1}(-cd(D - P_n^2)/Q_n - (ad + bc)P_n + abQ_n),$$
$$Q_{n+1} = \varepsilon^{-1}(c^2(D - P_n^2)/Q_n + 2acP_n - a^2Q_n)$$

both lying in $\mathbf{Z}[i]$. That $Q_{n+1} \mid D - P_{n+1}^2$ does not follow from the unimodularity of T_n but is easily checked in each of the seven cases $T_n \in \{V_j, E_j, C\}$, e.g. in case $T_n = C$, we find that

$$P_{n+1} = (-1 + i)(D - P_n^2)/Q_n - 3P_n + (1 + i)Q_n,$$
$$Q_{n+1} = -2(D - P_n^2)/Q_n - (2 + 2i)P_n + iQ_n,$$

and hence

$$D - P_{n+1}^2 = Q_{n+1}(-i(D - P_n^2)/Q_n + (2 - 2i)P_n - 2Q_n).$$

This completes the first step of the proof.

We shall use a prime (′) to indicate taking conjugates in the field extension $\mathbf{Q}(i) \subset \mathbf{Q}(i, \sqrt{D})$; in particular

$$\xi_n' = \frac{-\sqrt{D} + P_n}{Q_n},$$

and hence

$$\xi_n - \xi'_n = \frac{2\sqrt{D}}{Q_n}, \tag{3.2}$$

$$\xi_n^{-1} - \xi_n'^{-1} = \frac{2Q_n\sqrt{D}}{D-P_n^2}, \tag{3.3}$$

$$(\xi_n - 1)^{-1} - (\xi'_n - 1)^{-1} = \frac{2Q_n\sqrt{D}}{D-(P_n-Q_n)^2}. \tag{3.4}$$

As the second step of the proof we want to show that the sequence of complete quotients $\xi_0, \xi_1, \ldots$ contains only finitely many different numbers, or equivalently that the two sequences of Gaussian integers P_n, Q_n $(n \geq 0)$ are both bounded. By (3.2) and (3.3) this will follow if we prove the existence of a constant $\delta > 0$ such that

$$|\xi_n - \xi'_n| > \delta, \; |\xi_n^{-1} - \xi_n'^{-1}| > \delta \quad \text{for } n \geq 0. \tag{3.5}$$

From the formulae (2.6), (2.7), (2.8) we obtain by conjugating and taking absolute values,

$$|\xi'_{n+1} + q_2^{(n)}/q_1^{(n)}| = (|q_1^{(n)}|^2 |\xi'_0 - p_1^{(n)}/q_1^{(n)}|)^{-1}, \tag{3.6}$$

$$|\xi_{n-1}'^{-1} + q_1^{(n)}/q_2^{(n)}| = (|q_2^{(n)}|^2 |\xi'_0 - p_2^{(n)}/q_2^{(n)}|)^{-1}, \tag{3.7}$$

$$|(\xi'_{n+1} - 1)^{-1} + q_1^{(n)}/q_3^{(n)}| = (|q_3^{(n)}|^2 |\xi'_0 - p_3^{(n)}/q_3^{(n)}|)^{-1}. \tag{3.8}$$

Since

$$|q_j^{(n)}| \to \infty \quad \text{and} \quad |\xi'_0 - p_j^{(n)}/q_j^{(n)}| \to |\xi'_0 - \xi_0| \neq 0 \quad \text{for } n \to \infty,$$

it follows that the expressions in (3.6), (3.7), (3.8) all tend to zero for $n \to \infty$; thus to prove the existence of δ to be used in (3.5) it suffices to prove the existence of a constant $\delta_1 > 0$ such that

$$|\xi_{n+1} + q_2^{(n)}/q_1^{(n)}| > \delta_1, \; |\xi_{n+1}^{-1} + q_1^{(n)}/q_2^{(n)}| > \delta_1 \tag{3.9}$$

for all $n \geq n_1$ (say).

However, it follows from (3.2), (3.3), (3.4), since $Q_n \mid D - P_n^2$ and $Q_n \mid D - (P_n - Q_n)^2$, that

$$|\xi_n - \xi'_n|, \; |\xi_n^{-1} - \xi_n'^{-1}|, \; |(\xi_n - 1)^{-1} - (\xi'_n - 1)^{-1}|$$

are all $\leq 2\sqrt{|D|}$ for $n \geq 0$. Hence using once more that the expressions in (3.6), (3.7), (3.8) tend to zero for $n \to \infty$, we can find a constant $K > 2\sqrt{|D|} \geq 2$ (e.g. $K = 2\sqrt{|D|} + 1$), such that

$$|\xi_{n+1} + q_2^{(n)}/q_1^{(n)}|, \; |\xi_{n+1}^{-1} + q_1^{(n)}/q_2^{(n)}|, \; |(\xi_{n+1} - 1)^{-1} + q_1^{(n)}/q_3^{(n)}| \tag{3.10}$$

are all $< K$ for all $n \geq n_1 = n_1(K)$.

We will prove that the inequalities (3.9) are valid with $\delta_1 = 1/(2K)$ for all $n \geqslant n_1 = n_1(K)$. More explicitly, the first inequality of (3.9) follows from the second and third inequalities of (3.10), while the second inequality of (3.9) follows from the first and third inequalities of (3.10).

In fact, for each $n \in \mathbf{N}_0$, either (2.3) or (2.4) is valid. Now suppose (to prove the first inequality of (3.9), say) that for some $n \geqslant n_1$,

$$|\xi_{n+1} + q_2^{(n)}/q_1^{(n)}| \leqslant \frac{1}{2K} < \frac{1}{4}, \tag{*}$$

then we have to obtain a contradiction.

From (*) it follows, independently of whether (2.3) or (2.4) is valid, that ξ_{n+1} and $-q_2^{(n)}/q_1^{(n)}$ are both in the same circular disc $B(0; 1/2)$ or $B(1; 1/2)$; by symmetry we may as well assume that (2.3) is valid, and that ξ_{n+1} and $-q_2^{(n)}/q_1^{(n)}$ are both in $B(0; 1/2)$. It follows then from (2.3) and (*), that

$$-q_2^{(n)}/q_1^{(n)} = re^{i(-\pi/2+u)},\ 0 < r < 1/(\sqrt{2}K),\ 0 \leqslant u < \pi/4,$$

so that

$$|\operatorname{Im}(-q_1^{(n)}/q_2^{(n)})| = r^{-1}\cos u > K.$$

Hence by (2.3)

$$|\xi_{n+1}^{-1} + q_1^{(n)}/q_2^{(n)}| \geqslant |\operatorname{Im}(\xi_{n+1}^{-1} + q_1^{(n)}/q_2^{(n)})| \geqslant |\operatorname{Im}(-q_1^{(n)}/q_2^{(n)})| > K$$

contradicting the second inequality of (3.10). This completes the second step of the proof.

The final step of the proof is now an easy one. Since there are only finitely many different numbers in the sequence $\xi_0, \xi_1, \ldots$, we can pick five different indices $n_1, \ldots, n_5$ such that $\xi_{n_1} = \ldots = \xi_{n_5}$. Among these indices we can pick three, say n_1, n_2, n_3, such that $\det m_{n_1-1} = \det m_{n_2-1} = \det m_{n_3-1}$ $(=\{\pm 1\}, \{\pm i\})$. Then the number $\xi_{n_1} = \xi_{n_2} = \xi_{n_3}$ has

$$T_{n_1} T_{n_1+1} \ldots,\ T_{n_2} T_{n_2+1} \ldots,\ T_{n_3} T_{n_3+1} \ldots$$

as regular (dually regular) chains. However, since at most two different regular (dually regular) chains can represent the same number, two of the three chains above must be identical. This proves the periodicity.

The proof above of Theorems 3.2, 3.2* follows very closely the classical proof of Lagrange for the corresponding theorem about the periodicity of regular continued fractions of (real) quadratic irrationals. The most notable deviation lies in the second step, and is in fact inavoidable because of the differences in the notions of reducedness (cf. section 3.2).

It is easy to give an alternative proof of Theorems 3.2, 3.2* using the idea of Charves (cf. [27, pp. 71–72]), however, I shall leave the details to the reader.

3.2. Purely periodic chains

DEFINITION 3.1. *A number* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$, *which is quadratic over* $\mathbf{Q}(i)$ *with conjugate* ξ', *is called reduced if*

$$\xi \in \mathfrak{J} \text{ and } \xi' \in \overline{\mathfrak{J}^*},$$

and is called dually reduced if

$$\xi \in \mathfrak{J}^* \quad \text{and} \quad \xi' \in \overline{\mathfrak{J}}.$$

THEOREM 3.3. *Every purely periodic regular (dually regular) chain represents a* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$, *which is quadratic over* $\mathbf{Q}(i)$ *and reduced (dually reduced).*

Proof. To be specific let

$$\overrightarrow{T_0 T_1 \dots T_{k-1}}$$

be a purely periodic regular chain. By Theorem 3.1 it follows that $[\overrightarrow{T_0 T_1 \dots T_{k-1}}] = \xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ and is quadratic over $\mathbf{Q}(i)$. By the pure periodicity

$$\xi = \xi_{nk} \quad \text{for } n = 0, 1, 2, \dots,$$

whence, since all complete quotients $\xi_1, \xi_2, \dots$ are in $\mathfrak{J}$ by (2.3), (2.4), it follows at once that $\xi \in \mathfrak{J}$.

Further by the pure periodicity

$$\det m_{2nk-1} = (\det m_{nk-1})^2 = \{\pm 1\}, \; n \in \mathbf{N},$$

and hence by (2.3),

$$-q_2^{(2nk-1)}/q_1^{(2nk-1)} \in \overline{\mathfrak{J}^*} \quad \text{for } n \in \mathbf{N}.$$

Finally, using that the expressions in (3.6) tend to zero for $n \to \infty$, we obtain

$$\xi_0' = \xi_{2nk}' = \lim_{n \to \infty} -q_2^{(2nk-1)}/q_1^{(2nk-1)} \in \overline{\mathfrak{J}^*},$$

since $\overline{\mathfrak{J}^*}$ is a closed set. Thus ξ is reduced.

The dual part of Theorem 3.3 is proved similarly.

THEOREM 3.4. *Every* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ *which is quadratic over* $\mathbf{Q}(i)$ *and reduced (dually reduced) has a unique purely periodic regular (dually regular) chain.*

Proof. As the first step of the proof we want to establish the uniqueness.

By the description of Farey sets in section 1.3, there is no Farey set F of circular type with $\partial F \cap \mathfrak{J}\setminus\{0, 1, \infty\} \neq \emptyset$, $\partial F \cap \overline{\mathfrak{J}^*}\setminus\{0, 1, \infty\} \neq \emptyset$. Consequently, a $\xi \in \mathbf{C}\setminus\mathbf{Q}(i)$, quadratic over $\mathbf{Q}(i)$ and reduced, cannot be properly equivalent to a real number, since otherwise there would exist a properly unimodular map m with

$$\xi, \xi' \in m(\mathbf{R}) = \partial F(m).$$

Hence in this case the uniqueness follows from Theorem 2.2.

Similarly, the only dual Farey sets F^* of circular type with

$$\partial F^* \cap \mathfrak{J}^*\setminus\{0, 1, \infty\} \neq \emptyset, \partial F^* \cap \overline{\mathfrak{J}}\setminus\{0, 1, \infty\} \neq \emptyset,$$

are $F^*(e_j)$, $j = 1, 2, 3$. Consequently for any $\xi \in \mathbf{C}\setminus\mathbf{Q}(i)$, quadratic over $\mathbf{Q}(i)$ and dually reduced, we must have $\xi \in \partial\mathfrak{J}^*$, and hence the uniqueness follows from Theorem 2.2*, except for

$$\xi \in \{z = x + iy \mid 0 < x < 1, y = (x - x^2)^{1/2}\}.$$

These special ξ have two different dually regular chains, one of the form

$$T_0 T_1 T_2 \ldots, \quad \text{with } T_n \in \{V_2, V_3\} \quad \text{for } n \geq 0, \tag{*}$$

the other of the form

$$V_1^{-1} C E_1 U_3 U_4 \ldots, \quad \text{with } U_n \in \{V_2, V_3\} \quad \text{for } n \geq 3. \tag{**}$$

Since chains of the form (**) are never purely periodic, we have shown the uniqueness also in the dually regular case.

Disregarding dually regular chains of the form (**), every $\xi \in \mathbf{C}\setminus\mathbf{Q}(i)$, quadratic over $\mathbf{Q}(i)$ and reduced (dually reduced), has a unique regular (dually regular) chain, which by Theorems 3.2, 3.2* is periodic, say of the form (3.1).

As the second step of the proof we want to show that every complete quotient ξ_n belonging to the chain (3.1) is also reduced or dually reduced.

To be specific let ξ be reduced and (3.1) be its unique regular chain. Then we claim, that $\xi_n (n \geq 1)$ is reduced if $\det m_{n-1} = \{\pm 1\}$, and dually reduced if $\det m_{n-1} = \{\pm i\}$.

We proceed by induction on n, starting with $n = 0$ with m_{-1} being the identity map of determinant $\{\pm 1\}$. For the inductive step we must distinguish between two cases according as $\det m_{n-1} = \{\pm 1\}$ or $\{\pm i\}$.

Case 1. $\det m_{n-1} = \{\pm 1\}$; then $\xi_n \in \mathfrak{J}$, $\xi'_n \in \overline{\mathfrak{J}^*}$ by the inductive assumption. There are three subcases.

(a) $T_n = V_j$; then $\det m_n = \{\pm 1\}$. Of course $\xi_{n+1} \in \mathfrak{J}$ by (2.3), but also

$$\xi'_{n+1} = v_j^{-1}(\xi'_n) \in v_j^{-1}(\overline{\mathfrak{J}^*}) = \varkappa \circ v_j \circ \varkappa(\overline{\mathfrak{J}^*}) = \overline{\mathfrak{V}_j^*} \subset \overline{\mathfrak{J}^*}$$

by (2.2) and Lemma 1.1 (v) (vii), hence ξ_{n+1} is reduced.

(b) $T_n = E_j$; then $\det m_n = \{\pm i\}$. Of course $\xi_{n+1} \in \mathfrak{J}^*$ by (2.4), but also

$$\xi'_{n+1} = e_j^{-1}(\xi'_n) \in e_j^{-1}(\overline{\mathfrak{J}^*}) = \varkappa \circ e_j \circ \varkappa(\overline{\mathfrak{J}^*}) = \overline{\mathcal{E}}_j \subset \overline{\mathfrak{J}}$$

by (2.2) and Lemma 1.1 (v) (vi), hence ξ_{n+1} is dually reduced.

(c) $T_n = C$; then $\det m_n = \{\pm i\}$. Of course $\xi_{n+1} \in \mathfrak{J}^*$ by (2.4), but also

$$\xi'_{n+1} = c^{-1}(\xi'_n) \in c^{-1}(\overline{\mathfrak{J}^*}) = \varkappa \circ c \circ \varkappa(\overline{\mathfrak{J}^*}) = \overline{\mathfrak{C}} \subset \overline{\mathfrak{J}}$$

by (2.2) and Lemma 1.1 (v) (vi), hence ξ_{n+1} is dually reduced.

Case 2. $\det m_{n-1} = \{\pm i\}$; then $\xi_n \in \mathfrak{J}^*$, $\xi'_n \in \overline{\mathfrak{J}}$ by the inductive assumption. The two subcases (a) $T_n = V_j$, (b) $T_n = C$ are treated as in case 1.

This completes the second step of the proof.

To finish the proof of Theorem 3.4 suppose that the regular chain (3.1) of ξ is not purely periodic, hence $T_h \neq T_{h+k} (h \geqslant 0)$. Then we have to obtain a contradiction. We have to distinguish between two cases according as $\det m_h = \{\pm 1\}$ or $\{\pm i\}$.

Case 1. $\det m_h = \{\pm 1\}$; then $\xi_{h+1} \in \mathfrak{J}$, $\xi'_{h+1} \in \overline{\mathfrak{J}^*}$ by the second step of the proof, and $T_h = V_j$ or C by the very definition of a regular chain.

(a) $T_h = V_j$; then $\det m_{h-1} = \{\pm 1\}$. By the second step of the proof ξ_h is reduced, in particular $\xi'_h = v_j(\xi'_{h+1}) \in \overline{\mathfrak{J}^*}$, and hence $\xi'_{h+1} \in v_j^{-1}(\overline{\mathfrak{J}^*}) = \varkappa \circ v_j \circ \varkappa(\overline{\mathfrak{J}^*}) = \overline{\mathfrak{V}}_j^*$.

(b) $T_h = C$; then $\det m_{h-1} = \{\pm i\}$. By the second step of the proof ξ_h is dually reduced, in particular $\xi'_h = c(\xi'_{h+1}) \in \overline{\mathfrak{J}}$, and hence $\xi'_{h+1} \in c^{-1}(\overline{\mathfrak{J}}) = \varkappa \circ c \circ \xi(\overline{\mathfrak{J}}) = \overline{\mathfrak{C}^*}$.

Using the first step of the proof, we obtain from (a), (b) that T_h is uniquely determined by the position of ξ'_{h+1}. However, applying the same argument to ξ_{h+2k} (also reduced or dually reduced), we conclude that $T_h = T_{h+2k} = T_{h+k}$, since $\det m_{h+2k} = \det m_h = \{\pm 1\}$ and $\xi'_{h+1} = \xi'_{h+2k+1}$ by (3.1). This is the required contradiction.

Case 2. $\det m_h = \{\pm i\}$; then $\xi_{h+1} \in \mathfrak{J}^*$, $\xi'_{h+1} \in \overline{\mathfrak{J}}$ by the second step of the proof. The three subcases (a) $T_h = V_j$, (b) $T_h = E_j$, (c) $T_h = C$ lead to a contradiction as in case 1.

The dual part of Theorem 3.4 is proved similarly.

3.3. Inverse periods

THEOREM 3.5. *If $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ has the purely periodic regular (dually regular) chain*

$$\overrightarrow{T_0 T_1 \dots T_{k-1}},$$

then $\overline{\xi}'$ has the purely periodic dually regular (regular) chain

$$\overrightarrow{T_{k-1} T_{k-2} \dots T_0}.$$

Proof. To be specific suppose that ξ has the purely periodic regular chain

$$\overrightarrow{T_0 T_1 \dots T_{k-1}}.$$

Then $\xi \in \mathfrak{J}$, $\xi' \in \overline{\mathfrak{J}^*}$ are the roots of the quadratic equation (with coefficients in $\mathbf{Z}[i]$, when reduced)

$$z = t_0 \circ t_1 \circ \dots \circ t_{k-1}(z).$$

Evidently

$$\overrightarrow{T_{k-1} T_{k-2} \dots T_0}$$

is then a purely periodic dually regular chain, which represents a number $\eta \in \mathbf{C} \setminus \mathbf{Q}(i)$, quadratic over $\mathbf{Q}(i)$ and dually reduced. Hence $\eta \in \mathfrak{J}^*$, $\eta' \in \overline{\mathfrak{J}}$ are the roots of the quadratic equation

$$\begin{aligned} z = t_{k-1} \circ t_{k-2} \circ \dots \circ t_0(z) &= (\varkappa \circ t_{k-1}^{-1} \circ \varkappa) \circ (\varkappa \circ t_{k-2}^{-1} \circ \varkappa) \circ \dots \circ (\varkappa \circ t_0^{-1} \circ \varkappa)(z) \\ &= \varkappa \circ (t_{k-1}{}^{-1} \circ t_{k-2}{}^{-1} \circ \dots \circ t_0^{-1}) \circ \varkappa(z), \end{aligned}$$

whence applying $t_0 \circ t_1 \circ \dots \circ t_{k-1} \circ \varkappa$ on both sides,

$$\bar{z} = t_0 \circ t_1 \circ \dots \circ t_{k-1}(\bar{z}).$$

It follows that $\bar{\eta} \in \overline{\mathfrak{J}^*}$, $\bar{\eta}' \in \mathfrak{J}$ are also roots of the equation

$$z = t_0 \circ t_1 \circ \dots \circ t_{k-1}(z).$$

However, since $\mathfrak{J} \cap \overline{\mathfrak{J}^*} = \{0, 1\}$, we must have $\overline{\eta'} = \xi$, hence $\eta = \overline{\xi'}$, which we had to prove.

The dual part of Theorem 3.5 is proved similarly.

3.4. Special quadratic surds. The Pellian equation

Suppose $D \in \mathbf{Z}[i]$, where D is not a Gaussian square, and let

$$\operatorname{ch} \sqrt{D} = T_0 T_1 \dots T_n \dots$$

be any regular chain of $\sqrt{D}$ ($\arg \sqrt{D} \in [0, \pi[$). With the notation in section 3.1

$$\sqrt{D} = \xi_0 = m_n(\xi_{n+1}) = \frac{p_1^{(n)}(\sqrt{D} + P_{n+1}) + p_2^{(n)} Q_{n+1}}{q_1^{(n)}(\sqrt{D} + P_{n+1}) + q_2^{(n)} Q_{n+1}}$$

whence, using that $(1, \sqrt{D})$ is an independent set over $\mathbf{Q}(i)$,

$$\begin{pmatrix} p_1^{(n)} & D q_1^{(n)} \\ q_1^{(n)} & p_1^{(n)} \end{pmatrix} = M_n \begin{pmatrix} 1 & P_{n+1} \\ 0 & Q_{n+1} \end{pmatrix},$$

$$\begin{pmatrix} p_2^{(n)} & Dq_2^{(n)} \\ q_2^{(n)} & p_2^{(n)} \end{pmatrix} = M_n \begin{pmatrix} 0 & (D-P_{n+1}^2)/Q_{n+1} \\ 1 & -P_{n+1} \end{pmatrix},$$

$$\begin{pmatrix} p_3^{(n)} & Dq_3^{(n)} \\ q_3^{(n)} & p_3^{(n)} \end{pmatrix} = M_n \begin{pmatrix} 1 & P_{n+1}+(D-P_{n+1}^2)/Q_{n+1} \\ 1 & Q_{n+1}-P_{n+1} \end{pmatrix}.$$

Finally, taking determinants, we obtain

$$p_1^{(n)2} - Dq_1^{(n)2} = \varepsilon_n Q_{n+1}, \tag{3.11}$$

$$p_2^{(n)2} - Dq_2^{(n)2} = -\varepsilon_n (D-P_{n+1}^2)/Q_{n+1}, \tag{3.12}$$

$$p_3^{(n)2} - Dq_3^{(n)2} = -\varepsilon_n (D-(P_{n+1}-Q_{n+1})^2)/Q_{n+1}, \tag{3.13}$$

where

$$\varepsilon_n = \det M_n = p_1^{(n)} q_2^{(n)} - p_2^{(n)} q_1^{(n)} \in \{\pm 1, \pm i\}.$$

It is an easy consequence of Theorem 2.5 that for any solution $(X, Y) \in \mathbf{Z}[i] \times (\mathbf{Z}[i] \setminus \{0\})$ of the Diophantine equation

$$X^2 - DY^2 = \varepsilon, \quad \varepsilon \in \mathcal{U} = \{\pm 1, \pm i\}, \tag{3.14}$$

either X/Y or $-X/Y$ is a convergent $p_j^{(n)}/q_j^{(n)}$ of $\sqrt{D}$. Hence by (3.11), (3.12), (3.13) the complete solution $(X, Y) \in \mathbf{Z}[i] \times (\mathbf{Z}[i] \setminus \{0\})$ of (3.14) consists of

$$\{\varepsilon'(\pm p_j^{(n)}, q_j^{(n)}) \mid \varepsilon' \in \mathcal{U}, (j, n) \in S\}, \tag{3.15}$$

where

$$S = \{(1,n) \mid Q_n \in \mathcal{U}\} \cup \{(2,n) \mid (D-P_{n+1}^2)/Q_{n+1} \in \mathcal{U}\} \cup \{(3,n) \mid (D-(P_{n+1}-Q_{n+1})^2)/Q_{n+1} \in \mathcal{U}\}.$$

To solve equation (3.14) it is not restriction to assume that $D = a + ib$, $a, b \in \mathbf{N}_0$ (otherwise replace D by $-D$, $\bar{D}$ or $-\bar{D}$), and hence that $\sqrt{D} = \alpha + i\beta$ with $\alpha \geq \beta \geq 0$. It is then an easy consequence of Theorem 3.4, that $\sqrt{D}$ (except for $D = i, 1+i, 1+2i, 2+i, 3i, 4i$) has a unique regular chain of the form

$$\operatorname{ch} \sqrt{D} = V_1^{b_0} E_2 \overrightarrow{T_2 T_3 \ldots T_k T_{k+1}}, \tag{3.16}$$

where k denotes the shortest period with

$$\det t_2 \circ t_3 \circ \ldots \circ t_k \circ t_{k+1} = \{\pm 1\}.$$

It follows easily that a *fundamental solution* of **the *Pellian equation***

$$X^2 - DY^2 = \pm 1, \tag{3.17}$$

is given by

(a) $(p_{j+1}^{(l+1)}, q_{j+1}^{(l+1)})$, $j = 1, 2$, in case $k = 3l$ and $T_{n+l} = S^j T_n S^{-j}$ for $2 \leq n \leq 2l+1$, and otherwise by

(b) $(p_1^{(k+1)}, q_1^{(k+1)})$.

Also the calculation of (3.16) gives an effective method to decide whether the *non-Pellian equation*

$$X^2 - DY^2 = \pm i \tag{3.18}$$

has solutions, and to determine a *fundamental solution* if this happens to be the case.

This method of solving equation (3.14) is illustrated by the following examples.

Example 3.1. $D = 7+2i$; ch $\sqrt{7+2i}$ is of type (a) with $b_0 = 0$, $k = 3\cdot 13$, $j=1$ and

$$T_2 \ldots T_{14} = V_1 C V_1 E_3 C V_3 V_1 V_2 E_3 V_1 V_2 V_2 V_2.$$

Hence a fundamental solution of the Pellian equation is

$$(p_2^{(14)}, q_2^{(14)}) = (p_2^{(11)}, q_2^{(11)}) = (17-63i, 3-24i),$$

while the non-Pellian equation has no solutions.

Example 3.2. $D = 7+3i$; ch $\sqrt{7+3i}$ is of type (a) with $b_0 = 0$, $k = 3\cdot 16$, $j=2$ and

$$T_2 \ldots T_{17} = V_1 C V_1 V_3 V_1 V_1 E_2 V_1 C V_3 E_1 C E_1 V_3 V_3 V_3.$$

Hence a fundamental solution of the Pellian equation is

$$(p_3^{(17)}, q_3^{(17)}) = (p_3^{(13)}, q_3^{(13)}) = (99-98i, 28-42i),$$

while the non-Pellian equation has no solutions.

Example 3.3. $D = 7+4i$; since

$$\text{ch}\sqrt{7+4i} = V_1^0 E_2 \overrightarrow{\overline{V_1 C E_2 C E_2 V_1 V_1 V_1}}$$

is of type (b) with $b_0 = 0, k = 8$, a fundamental solution of the Pellian equation is

$$(p_1^{(9)}, q_1^{(9)}) = (p_1^{(5)}, q_1^{(5)}) = (-7+4i, -2+2i),$$

while the non-Pellian equation has no solutions.

Example 3.4. $D = 8+7i$; since

$$\text{ch}\sqrt{8+7i} = V_1 E_2 \overrightarrow{\overline{V_1 C V_2 V_2 V_2 V_2 V_2 E_1 V_2 C V_1 E_2 V_1 V_1 V_1 V_1}}$$

is a type of (b) with $b_0 = 1, k = 16$, a fundamental solution of the Pellian equation is

$$(p_1^{(17)}, q_1^{(17)}) = (p_1^{(11)}, q_1^{(11)}) = (-13+16i, -2+6i),$$

while a fundamental solution of the non-Pellian equation is

$$(p_2^{(8)}, q_2^{(8)}) = (p_2^{(3)}, q_2^{(3)}) = (-3-i, -1).$$

Chapter 4

C-regular and C-dually regular continued fractions

4.1. Introduction

We consider a complex number

$$\xi_0 = \tfrac{1}{2}(1+i\alpha_0), \quad \alpha_0 \in \mathbf{R}\setminus\mathbf{Q},$$

lying on the line $\operatorname{Re} z = \frac{1}{2}$, which is an axis of symmetry of Fig. 1 and Fig. 1*. It follows by Theorem 2.2 that ξ_0 has precisely one regular chain

$$\operatorname{ch} \xi_0 = T_0 T_1 T_2 \ldots,$$

and that

$$T_n \in \{V_1, E_1, C\}, \quad n \in \mathbf{N}_0.$$

Hence, collecting powers of V_1, we have

$$\operatorname{ch} \xi_0 = V_1^{a_0-1} U_1 V_1^{a_1-1} C V_1^{a_2-1} U_3 V_1^{a_3-1} C V_1^{a_4-1} \ldots, \tag{4.1}$$

where $\qquad a_0 \in \mathbf{Z}, a_n \in \mathbf{N}, U_{2n-1} \in \{E_1, C\}$ for $n \in \mathbf{N}$.

Put

$$\xi_n = [T_n T_{n+1} \ldots] = \tfrac{1}{2}(1+i\alpha_n),$$

then the relation $\xi_n = t_n(\xi_{n+1})$ is equivalent to

$$\alpha_n = \begin{cases} 2+\alpha_{n+1} & \text{if } T_n = V_1. \\ 1 - \dfrac{2}{1+\alpha_{n+1}} & \text{if } T_n = E_1. \\ 1 + \dfrac{2}{1+\alpha_{n+1}} & \text{if } T_n = C. \end{cases}$$

Hence corresponding to formula (4.1) we have

$$\alpha_0 = 2a_0 - 1 + \frac{2\varepsilon_1|}{|2a_1} + \frac{2|}{|2a_2} + \frac{2\varepsilon_3|}{|2a_3} + \frac{2|}{|2a_4} + \ldots, \tag{4.2}$$

with

$$\varepsilon_{2n-1} = \begin{cases} +1 & \text{if } U_{2n-1} = C, \\ -1 & \text{if } U_{2n-1} = E_1, \end{cases} \quad n \in \mathbf{N}.$$

Similarly, by Theorem 2.2*, ξ_0 has precisely one dually regular chain

$$\operatorname{ch}^* \xi_0 = V_1^{b_0-1} C V_1^{b_1-1} U_2 V_1^{b_2-1} C V_1^{b_3-1} U_4 V_1^{b_4-1} \ldots, \tag{4.1*}$$

where $b_0 \in \mathbf{Z}$, $b_n \in \mathbf{N}$, $U_{2n} \in \{E_1, C\}$ for $n \in \mathbf{N}$, and correspondingly

$$\alpha_0 = 2b_0 - 1 + \frac{2|}{|2b_1} + \frac{2\varepsilon_2|}{|2b_2} + \frac{2|}{|2b_3} + \frac{2\varepsilon_4|}{|2b_4} + \ldots, \tag{4.2*}$$

with

$$\varepsilon_{2n} = \begin{cases} +1 & \text{if } U_{2n} = C, \\ -1 & \text{if } U_{2n} = E_1, \end{cases} \qquad n \in \mathbf{N}.$$

A continued fraction (4.2) with $a_0 \in \mathbf{Z}$, $a_n \in \mathbf{N}$, $\varepsilon_{2n-1} \in \{-1, +1\}$ for $n \in \mathbf{N}$ is called *C-regular*. Similarly, a continued fraction (4.2*) with $b_0 \in \mathbf{Z}$, $b_n \in \mathbf{N}$, $\varepsilon_{2n} \in \{-1, +1\}$ for $n \in \mathbf{N}$ is called *C-dually regular*.

By the correspondences (4.1), (4.2) and (4.1*), (4.2*) the following results from Theorems 2.1, 2.1*, 2.2, 2.2*, 3.1, 3.1*, 3.2, 3.2*.

THEOREM 4.1. *Any C-regular (C-dually regular) continued fraction converges to some* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.

THEOREM 4.2. *Any* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *has precisely one C-regular (C-dually regular) continued fraction expansion.*

THEOREM 4.3. *A periodic C-regular (C-dually regular) continued fraction converges to an* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *which is quadratic over* $\mathbf{Q}$.

THEOREM 4.4. *Every* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *which is quadratic over* $\mathbf{Q}$ *has a periodic C-regular (C-dually regular) continued fraction expansion.*

An $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, quadratic over $\mathbf{Q}$ and with conjugate α', is called *C-reduced* if $\alpha > 0$, $\alpha' < -1$, and is called *C-dually reduced* if $\alpha > 1$, $\alpha' < 0$.

A C-regular (C-dually regular) continued fraction (4.2) ((4.2*)) is called *purely periodic* if the sequences $a_0, a_1, a_2, \ldots$ and $\varepsilon_1, \varepsilon_3, \varepsilon_5, \ldots$ ($b_0, b_1, b_2, \ldots$ and $\varepsilon_2, \varepsilon_4, \varepsilon_6, \ldots$) are both purely periodic.

Theorems 3.3, 3.4, 3.5 then specialize as follows.

THEOREM 4.5. *An* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, *quadratic over* $\mathbf{Q}$, *is C-reduced (C-dually reduced) if and only if the C-regular (C-dually regular) continued fraction of* α *is purely periodic.*

THEOREM 4.6. *If* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *has the purely periodic C-regular continued fraction* (4.2) *with*

$$a_0, a_1, a_2, \ldots = \overrightarrow{a_0, a_1, \ldots, a_{2k-1}},$$

$$\varepsilon_1, \varepsilon_3, \varepsilon_5, \ldots = \overrightarrow{\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2k-1}},$$

then $-\alpha'$ (*where* α' *is the conjugate of* α *in* $\mathbf{Q}(\alpha)$ *over* $\mathbf{Q}$) *has the purely periodic C-dually regular continued fraction* (4.2*) *with*

$$b_0, b_1, b_2, \ldots = \overrightarrow{a_{2k-1}, a_{2k-3}, \ldots, a_0},$$

$$\varepsilon_2, \varepsilon_4, \varepsilon_6, \ldots = \overrightarrow{\varepsilon_{2k-1}, \varepsilon_{2k-3}, \ldots, \varepsilon_1},$$

and conversely.

4.2. C-equivalence and C-duality

A (real) unimodular map $\varphi: t \mapsto (at+b)(ct+d)^{-1}$ with $a, b, c, d \in \mathbf{Z}$, $ad-bc=\pm 1$, is called *C-unimodular* if either $a \equiv d \equiv 0 \pmod 2$, $b \equiv c \equiv 1 \pmod 2$ or $a \equiv d \equiv 1 \pmod 2$, $b \equiv c \equiv 0 \pmod 2$. It is well known that the set Γ_C of all C-unimodular maps is a group, which is generated by the maps $\varphi_0: t \mapsto t+2$, $\varphi_1: t \mapsto t^{-1}$.

Correspondingly, $\alpha, \beta \in \mathbf{R} \cup \{\infty\}$ are called *C-equivalent* ($\alpha \approx \beta$) if there exists a map $\varphi \in \Gamma_C$ with $\beta = \varphi(\alpha)$.

Notice that $\mathbf{Q}$ consists of two C-equivalence classes $\mathbf{Q}_0$, $\mathbf{Q}_1$, where (for $j=0, 1$)

$$\mathbf{Q}_j = \{p/q \mid (p, q) \in \mathbf{Z} \times \mathbf{N}, \gcd(p, q) = 1, p+q \equiv j \pmod 2\}.$$

For any $\alpha \in \mathbf{R} \cup \{\infty\}$, the *C-dual* α^* of α is defined as $\alpha^* = (\alpha+1)/(\alpha-1)$.

Notice that $(\alpha^*)^* = \alpha$ and that $\alpha \approx \beta \Leftrightarrow \alpha^* \approx \beta^*$ for all $\alpha, \beta \in \mathbf{R} \cup \{\infty\}$. Notice also that $\alpha \approx \beta$ if and only if $\frac{1}{2}(1+i\alpha)$, $\frac{1}{2}(1+i\beta)$ are properly equivalent, and that $\alpha^* \approx \beta$ if and only if $\frac{1}{2}(1+i\alpha)$, $\frac{1}{2}(1+i\beta)$ are improperly equivalent.

Now Theorem 2.3 specializes as follows.

THEOREM 4.7. $\alpha, \beta \in \mathbf{R} \setminus \mathbf{Q}$ *are C-equivalent if and only if the C-regular continued fractions of* α, β *are of the form*

$$\alpha = 2a_0 - 1 + \frac{2\varepsilon_1|}{|2a_1} + \frac{2|}{|2a_2} + \frac{2\varepsilon_3|}{|2a_3} + \frac{|2}{|2a_4} + \ldots,$$

$$\beta = 2c_0 - 1 + \frac{2\delta_1|}{|2c_1} + \frac{2|}{|2c_2} + \frac{2\delta_3|}{|2c_3} + \frac{2|}{|2c_4} + \ldots,$$

with (*for suitable* $h, k \in \mathbf{N}$)

$$a_{2h-1+n} = c_{2k-1+n}, \; \varepsilon_{2h-1+2n} = \delta_{2k-1+2n} \quad \text{for all } n \in \mathbf{N}.$$

Of course, Theorems 2.3*, 2.4 have similar specializations.

4.3. C-convergents and C-dual convergents

For any $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ we write its C-regular continued fraction (4.2) reduced form as

$$\alpha = 2a_0 - 1 + \frac{\varepsilon_1|}{|a_1} + \frac{1|}{|2a_2} + \frac{\varepsilon_3|}{|a_3} + \frac{1|}{|2a_4} + \dots, \tag{4.3}$$

and we call the convergents p_n/q_n, $n \in \mathbf{N}_0$, of (4.3) the *C-convergents* of α.

Similarly we write its C-dually regular continued fraction (4.2*) in reduced form as

$$\alpha = 2b_0 - 1 + \frac{1|}{|b_1} + \frac{\varepsilon_2|}{|2b_2} + \frac{1|}{|b_3} + \frac{\varepsilon_4|}{|2b_4} + \dots, \tag{4.3*}$$

and we call the convergents p_n^*/q_n^*, $n \in \mathbf{N}_0$, of (4.3*) the *C-dual convergents* of α.

The following result is an immediate consequence of the recursion formulas of continued fractions.

THEOREM 4.8. *For any $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ with C-convergents p_n/q_n and C-dual convergents p_n^*/q_n^*, the following is valid:*

(i) $p_{2n}/q_{2n}, p_{2n}^*/q_{2n}^* \in \mathbf{Q}_0$, $p_{2n+1}/q_{2n+1}, p_{2n+1}^*/q_{2n+1}^* \in \mathbf{Q}_1$ *for* $n \in \mathbf{N}_0$,

(ii) $p_n q_{n+1} - p_{n+1} q_n = \pm 1$, $p_n^* q_{n+1}^* - p_{n+1}^* q_n^* = \pm 1$ *for* $n \in \mathbf{N}_0$,

(iii) $1 = q_0 < 2q_1 < q_2 < 2q_3 < q_4 < 2q_5 < \dots$,

(iii*) $1 = q_0^* < q_1^* < q_2^* < q_3^* < \dots$,

(iv) $|q_0 \alpha - p_0| > |q_1 \alpha - p_1| > |q_2 \alpha - p_2| > \dots$,

(iv*) $|q_0^* \alpha - p_0^*| > 2|q_1^* \alpha - p_1^*| > |q_2^* \alpha - p_2^*| > 2|q_3^* \alpha - p_3^*| > \dots$.

The following theorems generalize classical theorems of Legendre on regular continued fractions, and can be proved similarly.

THEOREM 4.9. *The sequence p_n/q_n, $n \in \mathbf{N}_0$, of C-convergents of $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ is characterized as the maximal sequence p_n/q_n satisfying conditions* (i), (iii), (iv) *of Theorem* 4.8.

The sequence p_n^*/q_n^*, $n \in \mathbf{N}_0$, of C-dual convergents of $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ is characterized as the maximal sequence p_n^*/q_n^* satisfying conditions (i), (iii*), (iv*) *of Theorem* 4.8.

THEOREM 4.10. *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. Any $p/q \in \mathbf{Q}_j$ $(j = 0, 1)$ satisfying the inequality*

$$|\alpha - p/q| < (c_j q^2)^{-1}, \quad (c_0 = \tfrac{3}{2}, c_1 = 2),$$

is a C-convergent of α.

Any $p/q \in \mathbf{Q}_j$ $(j=0, 1)$ *satisfying the inequality*

$$|\alpha - p/q| < (c_j^* q^2)^{-1}, \quad (c_0^* = 2,\, c_1^* = \tfrac{3}{2}),$$

is a C-dual convergent of α.

4.4. The C-approximation constant

For any $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ we define

$$d_j(\alpha) = \limsup_{p/q \in \mathbf{Q}_j} (q\,|q\alpha - p|)^{-1},\ (j = 0, 1),$$

$$d(\alpha) = \max\,(d_0(\alpha), \tfrac{1}{2}\, d_1(\alpha)),$$

and we call $d(\alpha)$ the *C-approximation constant* of α.

It was proved by W. T. Scott [34], that $d_0(\alpha) \geqslant 1$ for all $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, and by R. M. Robinson [31], that $d_1(\alpha) \geqslant 2$ for all $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. See also L. C. Eggan [11].

The following result is an easy consequence of Theorems 4.8, 4.9.

THEOREM 4.11. *For any* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ *with C-regular continued fraction* (4.2) *and C-dually regular continued fraction* (4.2*), *we have*

$$d_0(\alpha) = \limsup_n d_{2n} = \limsup_n d_{2n}^*,$$

$$d_1(\alpha) = \limsup_n d_{2n-1} = \limsup_n d_{2n-1}^*,$$

where

$$d_n = (q_n |q_n \alpha - p_n|)^{-1},\ d_n^* = (q_n^* |q_n^* \alpha - p_n^*|)^{-1},$$

are given by

$$2d_{2n} = 2a_{2n+1} + \frac{2|}{|2a_{2n+2}} + \frac{2\varepsilon_{2n+3}|}{|2a_{2n+3}} + \frac{2|}{|2a_{2n+4}} + \dots$$
$$+ \frac{2\varepsilon_{2n+1}|}{|2a_{2n}} + \frac{2|}{|2a_{2n-1}} + \frac{2\varepsilon_{2n-1}|}{|2a_{2n-2}} + \dots + \frac{2\varepsilon_3|}{|2a_2} + \frac{2|}{|2a_1},$$

$$d_{2n-1} = 2a_{2n} + \frac{2\varepsilon_{2n+1}|}{|2a_{2n+1}} + \frac{2|}{|2a_{2n+2}} + \frac{2\varepsilon_{2n+3}|}{|2a_{2n+3}} + \dots$$
$$+ \frac{2|}{|2a_{2n-1}} + \frac{2\varepsilon_{2n-1}|}{|2a_{2n-2}} + \frac{2|}{|2a_{2n-3}} + \dots + \frac{2\varepsilon_3|}{|2a_2} + \frac{2|}{|2a_1},$$

$$2d^*_{2n} = 2b_{2n+1} + \frac{2\varepsilon_{2n+2}|}{|2b_{2n+2}} + \frac{2|}{|2b_{2n+3}} + \frac{2\varepsilon_{2n+4}|}{|2b_{2n+4}} + \dots$$
$$+ \frac{2|}{|2b_{2n}} + \frac{2\varepsilon_{2n}|}{|2b_{2n-1}} + \frac{2|}{|2b_{2n-2}} + \dots + \frac{2|}{|2b_2} + \frac{2\varepsilon_2|}{|2b_1},$$

$$d^*_{2n-1} = 2b_{2n} + \frac{2|}{|2b_{2n+1}} + \frac{2\varepsilon_{2n+2}}{|2b_{2n+2}} + \frac{2|}{|2b_{2n+3}} + \dots$$
$$+ \frac{2\varepsilon_{2n}|}{|2b_{2n-1}} + \frac{2|}{|2b_{2n-2}} + \frac{2\varepsilon_{2n-2}|}{|2b_{2n-3}} + \dots + \frac{2|}{|2b_2} + \frac{2\varepsilon_2|}{|2b_1}.$$

COROLLARY. *For any* $\alpha, \beta \in \mathbf{R} \setminus \mathbf{Q}$,

$$\alpha \approx \beta \Rightarrow d_0(\alpha) = d_0(\beta),\ d_1(\alpha) = d_1(\beta),\ d(\alpha) = d(\beta),$$
$$\alpha^* \approx \beta \Rightarrow 2d_0(\alpha) = d_1(\beta),\ d_1(\alpha) = 2d_0(\beta),\ d(\alpha) = d(\beta).$$

Example 4.1. It follows from Theorem 4.11 that for

$$\sqrt{3} = 1 + \overrightarrow{\frac{2|}{|2} + \frac{2|}{|2}}, \quad (a_n = \varepsilon_{2n+1} = 1 \quad \text{for } n \in \mathbf{N}_0),$$

we have

$$d_0(\sqrt{3}) = \sqrt{3},\ d_1(\sqrt{3}) = 2\sqrt{3},\ d(\sqrt{3}) = \sqrt{3}.$$

Example 4.2. It follows from Theorem 4.11 that for

$$-1 + \sqrt{2} = 1 + \overrightarrow{\frac{-2|}{|2} + \frac{2|}{|2}}, \quad (a_n = -\varepsilon_{2n+1} = 1 \quad \text{for } n \in \mathbf{N}_0),$$

we have

$$d_0(-1+\sqrt{2}) = \sqrt{2},\ d_1(-1+\sqrt{2}) = 2\sqrt{2},\ d(-1+\sqrt{2}) = \sqrt{2}.$$

Example 4.3. It follows from Theorem 4.11 and its corollary that for

$$e = 3 + \frac{-1|}{|3} + \frac{1|}{|2} + \frac{-1|}{|5} + \frac{1|}{|2} + \frac{-1|}{|7} + \frac{1|}{|2} + \frac{-1|}{|9} + \dots,$$

we have

$$d_0(e) = \infty,\ d_1(e) = 2,\ d_0((e+1)/(e-1)) = 1,\ d_1((e+1)/(e-1)) = \infty.$$

In chapter 5 we shall extend the results of Markoff–Hurwitz by finding all C-approximation constants <2.

4.5. Ergodic theory

Let

$$X_1 =]0,2[\setminus \mathbf{Q}, \quad X_2 =]1,3[\setminus \mathbf{Q},$$

and let X be the disjoint union of X_1 and X_2. A normalized measure μ on X is defined by

$$\mu(]0,x_1[) = \frac{1}{2\log 3}\log(1+x_1), \quad \text{for } x_1 \in X_1,$$

$$\mu(]1,x_2[) = \frac{1}{2\log 3}\log x_2, \quad \text{for } x_2 \in X_2.$$

Further let $T\colon X \to X$ be given by

$$x_1 = 1 + \frac{2\varepsilon_1(x_1)|}{|2a_1(x_1)} + \frac{2|}{|2a_2(x_1)} + \frac{2\varepsilon_3(x_1)|}{|2a_3(x_1)} + \dots \mapsto T(x_1) = 1 + \frac{2|}{|2a_2(x_1)} + \frac{2\varepsilon_3(x_1)|}{|2a_3(x_1)} + \dots \in X_2,$$

$$x_2 = 1 + \frac{2|}{|2b_1(x_2)} + \frac{2\varepsilon_2(x_2)|}{|2b_2(x_2)} + \frac{2|}{|2b_3(x_2)} + \dots \mapsto T(x_2) = 1 + \frac{2\varepsilon_2(x_2)|}{|2b_2(x_2)} + \frac{2|}{|2b_3(x_2)} + \dots \in X_1.$$

A simple computation shows that T acts as a *measure preserving* transformation on (X,μ), i.e.

$$\mu(T^{-1}(]0,x_1[)) = \mu(]0,x_1[) \quad \text{for all } x_1 \in X_1,$$

$$\mu(T^{-1}(]1,x_2[)) = \mu(]1,x_2[) \quad \text{for all } x_2 \in X_2.$$

Further the argument used by C. Ryll-Nardzewski [32] shows that T is an *indecomposable* transformation on (X,μ), i.e. any measurable subset $E \subseteq X$ with $T^{-1}(E) = E$ has μ-measure 0 or 1.

Hence by the *individual ergodic theorem* we get the following analogue of a theorem of C. Ryll-Nardzewski [32]; also Corollaries 2, 3 represent analogues of well-known results of P. Lévy and A. Khintchine.

THEOREM 4.12. *For any $f\colon X \to \mathbf{R}$, $f \in L^1(X,\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) = \frac{1}{2\log 3}\left\{\int_0^2 \frac{f(x_1)}{1+x_1}\,dx_1 + \int_1^3 \frac{f(x_2)}{x_2}\,dx_2\right\}$$

for almost all $x \in X$ (with respect to μ).

COROLLARY 1. *For almost all $x_2 \in X_1$, $x_1 \in X_2$, we have*

$$p(\varepsilon_{2n-1}(x_1) = -1) = p(\varepsilon_{2n}(x_2) = -1) = (\log 2)/(\log 3),$$
$$p(\varepsilon_{2n-1}(x_1) = 1) = p(\varepsilon_{2n}(x_2) = 1) = (\log 3 - \log 2)/(\log 3),$$

where $p(c_n = c)$ denotes the frequency of c in the sequence (c_n), $n \in \mathbf{N}$.

Proof. Apply Theorem 4.12 for $f = \chi_{A_k}(k = 1, 2)$, where χ_{A_k} is the indicator function of A_k, and

$$A_1 =]0, 1[\subseteq X_1, \; A_2 =]1, 2[\subseteq X_1.$$

COROLLARY 2. *For almost all $x_1 \in X_1$, $x_2 \in X_2$, and all $m \in \mathbf{N}$ we have*

$$p(a_{2n-1}(x_1) = m) = p(a_{2n}(x_1) = m) = p(b_{2n-1}(x_2) = m)$$
$$= p(b_{2n}(x_2) = m) = \frac{1}{\log 3} \log \frac{(2m+1)^2}{(2m-1)(2m+3)}.$$

Proof. Apply Theorem 4.12 for $f = \chi_{A_k}$, $(k = 1, 2)$, where

$$A_1 = \left]1 - \frac{1}{m}, 1 - \frac{1}{m-1}\right[\cup \left]1 + \frac{1}{m+1}, 1 + \frac{1}{m}\right[\subseteq X_1,$$

$$A_2 = \left]1 + \frac{2}{2m+1}, 1 + \frac{2}{2m-1}\right[\subseteq X_2.$$

COROLLARY 3. *For almost all $x_1 \in X_1$ $x_2 \in X_2$, we have*

$$\lim_{N\to\infty} (a_1(x_1)\, a_3(x_1) \ldots a_{2N-1}(x_1))^{1/N}$$
$$= \lim_{N\to\infty} (a_2(x_1)\, a_4(x_1) \ldots a_{2N}(x_1))^{1/N}$$
$$= \lim_{N\to\infty} (b_1(x_2)\, b_3(x_2) \ldots b_{2N-1}(x_2))^{1/N}$$
$$= \lim_{N\to\infty} (b_2(x_2)\, b_4(x_2) \ldots b_{2N}(x_2))^{1/N}$$
$$= \prod_{m=1}^{\infty} \left(1 + \frac{4}{(2m-1)(2m+3)}\right)^{\log m/\log 3}.$$

Proof. Apply Theorem 4.12 for $f = f_k$, $(k = 1, 2)$, where

$$f_1 \colon x_1 \mapsto \log a_1(x_1), \; x_2 \mapsto 0,$$
$$f_2 \colon x_1 \mapsto 0, \; x_2 \mapsto \log b_1(x_2).$$

Theorem 4.12 and its corollaries can be refined by the method of C. de Vroedt [35].

Chapter 5

C-minimum of binary quadratic forms

5.1. Indefinite quadratic forms

Let

$$f\colon (x, y) \mapsto f(x, y) = \alpha x^2 + \beta xy + \gamma y^2, \quad \alpha, \beta, \gamma \in \mathbf{R},$$

be a quadratic form with $\alpha \neq 0$ and discriminant $\delta = \delta(f) = \beta^2 - 4\alpha\gamma > 0$. The *first* and *second roots* $\vartheta_1(f)$ and $\vartheta_2(f)$ of f are defined as follows,

$$\vartheta_1(f) = (-\beta + \sqrt{\delta})/(2\alpha), \quad \vartheta_2(f) = (-\beta - \sqrt{\delta})/(2\alpha);$$

notice that

$$\vartheta_1(-f) = \vartheta_2(f), \quad \vartheta_2(-f) = \vartheta_1(f). \tag{5.1}$$

In the sequel we shall only consider indefinite forms f with both roots $\vartheta_1(f), \vartheta_2(f) \in \mathbf{R} \setminus \mathbf{Q}$.

Let Γ'_C denote the group of all matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; a, b, c, d \in \mathbf{Z}, ad - bc = \pm 1,$$

with either $a \equiv d \equiv 0 \pmod 2$, $b \equiv c \equiv 1 \pmod 2$ or $a \equiv d \equiv 1 \pmod 2$, $b \equiv c \equiv 0 \pmod 2$.

Two forms f, g are called *C-equivalent* ($f \approx g$) if there exists a matrix $M \in \Gamma'_C$ with $g(x, y) = f(ax + by, cx + dy)$; notice that (with the notation of section 4.2)

$$f \approx g \Rightarrow \vartheta_1(f) \approx \vartheta_1(g), \vartheta_2(f) \approx \vartheta_2(g). \tag{5.2}$$

For any form f the *C-dual form* f^* is defined by $f^*(x, y) = \frac{1}{2} f(x + y, x - y)$; notice that

$$\vartheta_1(f^*) = (\vartheta_1(f))^*, \quad \vartheta_2(f^*) = (\vartheta_2(f))^*. \tag{5.3}$$

Using the special notation (for $j = 0, 1$)

$$\mathbf{Z}_j^2 = \{(x, y) \in \mathbf{Z}^2 \setminus \{(0, 0)\} \mid \gcd(x, y) = 1, x + y \equiv j \pmod 2\},$$

we define

$$\nu_j = \nu_j(f) = \inf_{(x, y) \in \mathbf{Z}_j^2} |f(x, y)|, \quad (j = 0, 1),$$

$$\nu = \nu(f) = \min(\nu_0(f), 2\nu_1(f)),$$

and we call $\nu(f)$ the *C-minimum* of f.

LEMMA 5.1. *Suppose that* $f \approx g$, $\lambda \in \mathbf{R}$. Then

(i) $(f^*)^* = f$, $f^* \approx g^*$,

(ii) $(\lambda f)^* = \lambda f^*$, $\lambda f \approx \lambda g$,

(iii) $\delta(f^*) = \delta(f)$, $\delta(f) = \delta(g)$, $\delta(\lambda f) = \lambda^2 \delta(f)$,

(iv) $\nu_j(\lambda f) = |\lambda| \nu_j(f)$, $\nu(\lambda f) = |\lambda| \nu(f)$,

(v) $\nu_j(f) = \nu_j(g)$, $\nu(f) = \nu(g)$,

(vi) $\nu_0(f) = 2\nu_1(f^*)$, $2\nu_1(f) = \nu_0(f^*)$, $\nu(f) = \nu(f^*)$.

Proof. (i)–(iv) are obvious; (v) follows from the fact that for $M \in \Gamma'_C$, $(x, y) \mapsto (ax+by, cx+dy)$ maps $\mathbf{Z}_j^2$ onto $\mathbf{Z}_j^2$, $(j=0, 1)$. Similarly (vi) follows from the fact that $(x, y) \mapsto (x+y, x-y)$ maps $\mathbf{Z}_0^2$ onto $2\mathbf{Z}_1^2$ and $\mathbf{Z}_1^2$ onto $\mathbf{Z}_0^2$.

LEMMA 5.2. *Suppose that* $f(a, c) = \alpha' > 0$, $(a, c) \in \mathbf{Z}_1^2$. *Then there exist* b, $d \in \mathbf{Z}$ *with* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'_C$, *such that*

$$f(ax+by, cx+dy) = \alpha' x^2 + \beta' xy + \gamma' y^2,$$

and

$$2\alpha' \leqslant \beta' \leqslant 4\alpha'.$$

Proof. Analogous to that of Lemma 1, chapter II of [4].

LEMMA 5.3. *Let* ϑ *be any of the (irrational) roots of* f. *Then*

A. $d(\vartheta) \leqslant \delta^{1/2}(\nu(f))^{-1}$.

B. $d(\vartheta) = \delta^{1/2}(\nu(f))^{-1}$ if $\alpha, \beta, \gamma \in \mathbf{Q}$;

in this case f *attains its C-minimum, i.e.* $\exists (x, y) \in \mathbf{Z}_0^2$ *with* $|f(x, y)| = \nu(f) \vee \exists (x, y) \in \mathbf{Z}_1^2$ *with* $|f(x, y)| = \frac{1}{2}\nu(f)$.

C. *If, in addition to B,* f *has the following property,*

$$(*)\ [\exists (x, y) \in \mathbf{Z}_0^2 \textit{ with } f(x, y) = \nu(f) \vee \exists (x, y) \in \mathbf{Z}_1^2 \textit{ with } f(x, y) = \tfrac{1}{2}\nu(f)]$$
$$\wedge\ [\exists (x, y) \in \mathbf{Z}_0^2 \textit{ with } f(x, y) = -\nu(f) \vee \exists (x, y) \in \mathbf{Z}_1^2 \textit{ with } f(x, y) = -\tfrac{1}{2}\nu(f)],$$

then either

$$|\vartheta - p/q| < (d(\vartheta) q^2)^{-1} \quad \textit{for infinitely many } p/q \in \mathbf{Q}_0,$$

or

$$|\vartheta - p/q| < (2d(\vartheta) q^2)^{-1} \quad \textit{for infinitely many } p/q \in \mathbf{Q}_1.$$

Proof. Analogous to that of Lemma 4, chapter II of [4]; notice that our $d(\vartheta)$, as defined in section 4.4, corresponds to Cassels' $(\nu(\vartheta))^{-1}$.

THEOREM 5.1. (Isolation theorem.) *Suppose that* $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$, *with* $\alpha, \beta, \gamma \in \mathbf{Q}$ *and irrational roots, satisfies condition* (*) *of Lemma* 5.3 C. *Then there exist* $\nu' < \nu(f)$ *and* $\varepsilon_0 > 0$, *depending only on* α, β, γ *such that*

$$\nu(g) < \nu'$$

for all forms g, *not proportional to* f, *and of the shape*

$$g(x, y) = \alpha' x^2 + \beta' xy + \gamma' y^2$$

with

$$|\alpha - \alpha'| < \varepsilon_0, \ |\beta - \beta'| < \varepsilon_0, \ |\gamma - \gamma'| < \varepsilon_0.$$

Proof. Analogous to that of Theorem I, chapter II of [4].

5.2. A diophantine equation

We consider the following system of equations

$$x_1 + x_2 = 2y_1 y_2, \ 2x_1 x_2 = y_1^2 + y_2^2, \tag{5.4}$$

first studied by L. Ya. Vulakh [36] in the equivalent form

$$y_1^2 + y_2^2 + 2x^2 = 4y_1 y_2 x. \tag{5.5}$$

A solution $(x_1, x_2; y_1, y_2) = (\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2) \in \mathbf{N}^4$ of (5.4) is called *singular* if $\Lambda_1 = \Lambda_2$ or $\mathrm{M}_1 = \mathrm{M}_2$. A solution $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ is said to have *height* $h = \Lambda_1 + \Lambda_2$. Two solutions $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$, $(\Lambda_1^*, \Lambda_2^*; \mathrm{M}_1^*, \mathrm{M}_2^*)$ are considered to be *equal* if $\{\Lambda_1, \Lambda_2\} = \{\Lambda_1^*, \Lambda_2^*\}$, $\{\mathrm{M}_1, \mathrm{M}_2\} = \{\mathrm{M}_1^*, \mathrm{M}_2^*\}$. Two different solutions $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$, $(\Lambda_1^*, \Lambda_2^*; \mathrm{M}_1^*, \mathrm{M}_2^*)$ are called *neighbours* if $\{\Lambda_1, \Lambda_2\} \cap \{\Lambda_1^*, \Lambda_2^*\} \neq \emptyset \wedge \{\mathrm{M}_1, \mathrm{M}_2\} \cap \{\mathrm{M}_1^*, \mathrm{M}_2^*\} \neq \emptyset$.

The following lemmas are easily proved.

LEMMA 5.4. *There is precisely one singular solution, namely* (1, 1; 1, 1); *this solution has precisely one neighbouring solution* (1, 5; 1, 3).

LEMMA 5.5. *Every non-singular solution* $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ *with* (*say*) $\Lambda_1 < \Lambda_2$, $\mathrm{M}_1 < \mathrm{M}_2$, *has precisely four different neighbouring solutions*

$$(\Lambda_i, \Lambda_{ij}; \mathrm{M}_j, \mathrm{M}_{ij}), \ i, j \in \{1, 2\},$$

where

$$\Lambda_{ij} = \Lambda_i(8\mathrm{M}_j^2 - 1) - (\Lambda_1 + \Lambda_2), \ \mathrm{M}_{ij} = 4\Lambda_i \mathrm{M}_j - \mathrm{M}_k,$$

with $k \in \{1, 2\} \setminus \{j\}$. *The corresponding heights* $h = \Lambda_1 + \Lambda_2$, $h_{ij} = \Lambda_i + \Lambda_{ij}$, *satisfy the inequalities*

$$h_{11} < h < h_{ij}, \quad (i, j) \neq (1, 1).$$

The following result is now obvious by Lemmas 5.4, 5.5.

THEOREM 5.2. *The complete solution* $\in \mathbf{N}^4$ *of* (5.4) *is obtained from the singular solution* $(1, 1; 1, 1)$ *by successive adjunction of neighbouring solutions of bigger heights. For any solution* $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$, *we have*

$$\gcd(\Lambda_1, \Lambda_2) = \gcd(\mathrm{M}_1, \mathrm{M}_2) = 1.$$

The tree of solutions of (5.4) is thus of the form

$$\begin{array}{ccc} & (1,1;1,1) & \\ & | & \\ & (1,5;1,3) & \\ (1,65;3,11) & (5,349;3,59) & (5,29;1,17) \end{array}$$

$$\{\Lambda\} = \{1, 5, 29, 65, 169, 349, 901, 985, 4\,549, 11\,521, \ldots\},$$

$$\{\mathrm{M}\} = \{1, 3, 11, 17, 41, 59, 99, 153, 339, 571, 577, \ldots\}.$$

Following an idea of G. Frobenius [15], we associate to any non-singular solution $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ of (5.4) with $\Lambda_1 < \Lambda_2$, $\mathrm{M}_1 < \mathrm{M}_2$ a quintuple $(\varepsilon; l_1, l_2; m_1, m_2)$ as follows,

$$(1, 5; 1, 3) \mapsto (+1; 0, 2; 1, 1);$$

if

$$(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2) \mapsto (\varepsilon; l_1, l_2; m_1, m_2),$$

then for $(i, j) = (1, 2), (2, 1), (2, 2)$,

$$(\Lambda_i, \Lambda_{ij}; \mathrm{M}_j, \mathrm{M}_{ij}) \mapsto (\varepsilon_{ij}; l_i, l_{ij}; m_j, m_{ij}),$$

where

$$\varepsilon_{12} = \varepsilon_{21} = \varepsilon, \quad \varepsilon_{22} = -\varepsilon,$$
$$l_{ij} = (\Lambda_{ij} m_j - \varepsilon_{ij} \mathrm{M}_{ij}) / \mathrm{M}_j,$$
$$m_{ij} = (\mathrm{M}_{ij} l_i + \varepsilon_{ij} \mathrm{M}_j) / \Lambda_i.$$

By induction with respect to the tree of solutions we obtain easily

LEMMA 5.6. *Let* $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ *with* $\Lambda_1 < \Lambda_2$, $\mathrm{M}_1 < \mathrm{M}_2$ *be any non-singular solution of* (5.4), *and* $(\varepsilon; l_1, l_2; m_1, m_2)$ *its associated quintuple. Then* $\varepsilon = \pm 1$ *and* l_1, l_2, m_1, m_2 *are positive integers satisfying*

(i) $\Lambda_1 l_2 - \Lambda_2 l_1 = 2\varepsilon \mathrm{M}_1^2$,

(ii) $\Lambda_1 m_1 - \mathrm{M}_1 l_1 = \varepsilon(4\Lambda_1 \mathrm{M}_1 - \mathrm{M}_2)$,

(iii) $\Lambda_1 m_2 - \mathrm{M}_2 l_1 = \varepsilon \mathrm{M}_1$,

(iv) $\Lambda_2 m_1 - \mathrm{M}_1 l_2 = \varepsilon \mathrm{M}_2$,

(v) $\Lambda_2 m_2 - \mathrm{M}_2 l_2 = -\varepsilon \mathrm{M}_1$,

(vi) $\mathrm{M}_1 m_2 - \mathrm{M}_2 m_1 = -2\varepsilon \Lambda_1$,

(vii) $l_i^2 + 1 \equiv 0 \pmod{\Lambda_i}$, $(i = 1,2)$,

(viii) $m_i^2 + 2 \equiv 0 \pmod{\mathrm{M}_i}$, $(i = 1, 2)$,

(ix) $l_i \equiv 0 \pmod 2$, $m_i \equiv 1 \pmod 2$, $(i = 1, 2)$,

(x) $0 \leqslant l_i \leqslant \Lambda_i$, $0 \leqslant m_i \leqslant \mathrm{M}_i$, $(i = 1, 2)$.

COROLLARY. *Any* Λ *is of the form* $\Lambda = x^2 + y^2$ *with* x *even and* y *odd; any* M *is of the form* $\mathrm{M} = x^2 + 2y^2$ *with* x *odd.*

5.3. A chain of C-minimal forms

For any non-singular solution $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$, $\Lambda_1 < \Lambda_2$, $\mathrm{M}_1 < \mathrm{M}_2$, of (5.4) with associated quintuple $(\varepsilon; l_1, l_2; m_1, m_2)$, we put

$$\lambda_i = (l_i^2 + 1)/\Lambda_i, \ \mu_i = (m_i^2 + 2)/\mathrm{M}_i, \quad (i = 1, 2); \tag{5.6}$$

notice that $\lambda_i, \mu_i \in \mathbf{N}$ by Lemma 5.6 (vii) (viii). We define quadratic forms g_Λ, h_M $(\Lambda = \Lambda_i, \mathrm{M} = \mathrm{M}_i)$ by

$$\Lambda g_\Lambda(x, y) = \Lambda x^2 + (4\Lambda - 2l)xy + (\lambda - 4l)y^2,$$
$$\mathrm{M} h_\mathrm{M}(x, y) = \mathrm{M} x^2 + (4\mathrm{M} - 2m)xy + (\mu - 4m)y^2.$$

Notice that by (5.6) we have the following discriminants,

$$\delta(\Lambda g_\Lambda) = 16\Lambda^2 - 4, \quad \delta(\mathrm{M} h_\mathrm{M}) = 16\mathrm{M}^2 - 8; \tag{5.7}$$

hence g_Λ, h_M are indefinite binary quadratic forms with rational coefficients and irrational roots. Notice also that by Lemma 5.6 (x), the forms g_Λ, h_M satisfy the reduction condition in Lemma 5.2.

Identically

$$\Lambda^2 g_\Lambda(x, y) = \varphi_\Lambda(y, z), \quad M^2 h_M(x, y) = \psi_M(y, w), \tag{5.8}$$

where

$$\varphi_\Lambda(y, z) = y^2 + 4\Lambda yz + z^2, \; \psi_M(y, w) = 2y^2 + 4Myw + w^2, \tag{5.9}$$

and

$$z = \Lambda x - ly, \quad w = Mx - my. \tag{5.10}$$

By the *chain of forms* g_Λ, h_M we shall understand a tree of quadruples $(g_{\Lambda_1}, g_{\Lambda_2}; h_{M_1}, h_{M_2})$ corresponding to the tree of solutions $(\Lambda_1, \Lambda_2; M_1, M_2)$ of (5.4). The simplest forms occurring in this chain are

$$1g_1 = (1, 4, 1), \qquad 1h_1 = (1, 2, -1),$$
$$5g_5 = (5, 16, -7), \qquad 3h_3 = (3, 10, -3),$$
$$29g_{29} = (29, 92, -43), \qquad 11h_{11} = (11, 38, -11),$$
$$65g_{65} = (65, 224, -67), \qquad 17h_{17} = (17, 54, -25),$$
$$349g_{349} = (349, 1\,124, -491), \qquad 59h_{59} = (59, 190, -83).$$

LEMMA 5.7. *For any non-singular solution* $(\Lambda_1, \Lambda_2; M_1, M_2)$, $\Lambda_1 < \Lambda_2$, $M_1 < M_2$, *of* (5.4) *with associated quintuple* $(\varepsilon; l_1, l_2; m_1, m_2)$, *we have*

(i) $g_{\Lambda_2}(l_2, \Lambda_2) = g_{\Lambda_2}(l_2 - 4\Lambda_2, \Lambda_2) = 1$ *for any* ε,

(iia) $g_{\Lambda_2}(m_2, M_2) = g_{\Lambda_2}(m_1 - 4M_1, M_1) = -2$ *if* $\varepsilon = +1$,

(iib) $g_{\Lambda_2}(m_2 - 4M_2, M_2) = g_{\Lambda_2}(m_1, M_1) = -2$ *if* $\varepsilon = -1$,

(iii) $h_{M_2}(m_2, M_2) = h_{M_2}(m_2 - 4M_2, M_2) = 2$ *for any* ε,

(iva) $h_{M_2}(l_1, \Lambda_1) = -1$, $h_{M_2}(m_1 - 4M_1, M_1) = -2$ *if* $\varepsilon = +1$,

(ivb) $h_{M_2}(l_1 - 4\Lambda_1, \Lambda_1) = -1$, $h_{M_2}(m_1, M_1) = -2$ *if* $\varepsilon = -1$.

Proof. Analogous to that of Lemma 8, chapter II of [4].

COROLLARY. *Let* $f(x, y) = x^2 + \beta xy + \gamma y^2$ *and suppose that* f *satisfies some of the following inequalities,*

(i) $f(l_2, \Lambda_2) \geq 1$, $f(l_2 - 4\Lambda_2, \Lambda_2) \geq 1$,

(iia) $f(m_2, M_2) \leq -2$, $f(m_1 - 4M_1, M_1) \leq -2$,

(iib) $f(m_2 - 4M_2, M_2) \leq -2$, $f(m_1, M_1) \leq -2$,

(iii) $f(m_2, M_2) \geq 2$, $f(m_2 - 4M_2, M_2) \geq 2$,

(iva) $f(l_1, \Lambda_1) \leq -1$, $f(m_1 - 4M_1, M_1) \leq -2$,

(ivb) $f(l_1 - 4\Lambda_1, \Lambda_1) \leq -1$, $f(m_1, M_1) \leq -2$.

Then the following holds,

$$f = g_{\Lambda_2} \quad \textit{in case } \varepsilon = +1 \textit{ and } f \textit{ satisfies (i), (iia),}$$

$$f = g_{\Lambda_2} \quad \textit{in case } \varepsilon = -1 \textit{ and } f \textit{ satisfies (i), (iib),}$$

$$f = h_{M_2} \quad \textit{in case } \varepsilon = +1 \textit{ and } f \textit{ satisfies (iii), (iva),}$$

$$f = h_{M_2} \quad \textit{in case } \varepsilon = -1 \textit{ and } f \textit{ satisfies (iii), (ivb).}$$

Proof. Analogous to that of Lemma 8, Corollary, chapter II of [4].

LEMMA 5.8. *For any form g_Λ, we have $g_\Lambda^* \approx -g_\Lambda$; for any form h_M, we have $h_M \approx -h_M$.*

Proof. Using Lemmas 5.1, 5.6, 5.7, the proof is analogous to that of Lemma 9, chapter II of [4].

COROLLARY. *For any form g_Λ, we have $(\vartheta_1(g_\Lambda))^* \approx \vartheta_2(g_\Lambda)$; for any form h_M, we have $\vartheta_1(h_M) \approx \vartheta_2(h_M)$.*

Proof. This follows from Lemma 5.8 together with (5.1), (5.2), (5.3).

LEMMA 5.9. *For any forms g_Λ, h_M, we have*

$$|g_\Lambda(x, y)| \geqslant 2 \quad \textit{for all } (x, y) \in \mathbf{Z}_0^2,$$

$$|g_\Lambda(x, y)| \geqslant 1 \quad \textit{for all } (x, y) \in \mathbf{Z}_1^2,$$

$$|h_M(x, y)| \geqslant 2 \quad \textit{for all } (x, y) \in \mathbf{Z}_0^2,$$

$$|h_M(x, y)| \geqslant 1 \quad \textit{for all } (x, y) \in \mathbf{Z}_1^2.$$

Proof. Analogous to that of Lemma 10, chapter II of [4].

COROLLARY 1. *For any forms g_Λ, h_M, we have*

$$\nu_0(g_\Lambda) = 2,\ \nu_1(g_\Lambda) = 1,\ \nu(g_\Lambda) = 2,$$

$$\nu_0(h_M) = 2,\ \nu_1(h_M) = 1,\ \nu(h_M) = 2.$$

COROLLARY 2. *All forms g_Λ, h_M satisfy condition (*) of Lemma 5.3 C, hence Lemma 5.3 C and Theorem 5.1 can be applied.*

Proof. Both corollaries follow from Lemmas 5.7, 5.9.

LEMMA 5.10. *Let* $f(x, y) = x^2 + \beta xy + \gamma y^2$; *then (with the notation of Lemma* 5.7, *Corollary)*

(1a) $\delta(f) \geqslant 16 - 4/\Lambda_2^2 = \delta(g_{\Lambda_2})$ *if* $\varepsilon = +1$ *and* (iia) *holds,*

(1b) $\delta(f) \geqslant 16 - 4/\Lambda_2^2 = \delta(g_{\Lambda_2})$ *if* $\varepsilon = -1$ *and* (iib) *holds,*

(2a) $\delta(f) \geqslant 16 - 8/M_2^2 = \delta(h_{M_2})$ *if* $\varepsilon = +1$ *and* (iva) *holds,*

(2b) $\delta(f) \geqslant 16 - 8/M_2^2 = \delta(h_{M_2})$ *if* $\varepsilon = -1$ *and* (ivb) *holds.*

Proof. Analogous to that of Lemma 11, chapter II of [4].

LEMMA 5.11. *Let* $f(x, y) = x^2 + \beta xy + \gamma y^2$; *then*

$$\delta(f) \geqslant 16 + 4/\Lambda_2^2 \quad \text{if } f(l_2, \Lambda_2) \leqslant -1,\ f(l_2 - 4\Lambda_2, \Lambda_2) \leqslant -1,$$

$$\delta(f) \geqslant 16 + 8/M_2^2 \quad \text{if } f(m_2, M_2) \leqslant -2,\ f(m_2 - 4M_2, M_2) \leqslant -2.$$

Proof. Analogous to that of Lemma 12, chapter II of [4].

LEMMA 5.12. *Let* $f(x, y) = x^2 + \beta xy + \gamma y^2$, *where* $2 \leqslant \beta \leqslant 4$ *and* $0 < \delta(f) = \beta^2 - 4\gamma < 16$.

Suppose that

$$|f(x, y)| \geqslant 2 \quad \text{for all } (x, y) \in \mathbf{Z}_0^2,$$

$$|f(x, y)| \geqslant 1 \quad \text{for all } (x, y) \in \mathbf{Z}_1^2.$$

Then f *is either a* g_Λ *or a* h_M.

Proof. We shall use Cassels' notation,

$$P(x, y)\colon f(x, y) > 0,\ N(x, y)\colon f(x, y) < 0.$$

If $P(-1, 1)$, then $1 - \beta + \gamma \geqslant 2$, a contradiction to $2 \leqslant \beta \leqslant 4$ and $\beta^2 - 4\gamma > 0$. Hence $N(-1, 1)$, i.e. $1 - \beta + \gamma \leqslant -2$ or

$$\beta \geqslant \gamma + 3. \tag{5.11}$$

If $P(0, 1)$, then $\gamma \geqslant 1$, which together with (5.11) gives $\beta \geqslant 4$. Since $2 \leqslant \beta \leqslant 4$, we must have $\beta = 4$ and consequently $\gamma = 1$; hence $f = (1, 4, 1) = g_1$.

In the sequel we assume that $f \neq g_1$, hence $N(0, 1)$, i.e.

$$\gamma \leqslant -1. \tag{5.12}$$

If $P(-3, 1)$, then $9 - 3\beta + \gamma \geqslant 2$, which together with (5.12) gives $\beta \leqslant 2$. Since $2 \leqslant \beta \leqslant 4$, we must have $\beta = 2$ and consequently $\gamma = -1$; hence $f = (1, 2, -1) = h_1$.

In the sequel we assume that $f \neq g_1, h_1$, hence

$$N(0, 1) \quad \text{and } N(-3, 1). \tag{5.13}$$

The proof now follows by induction. Let $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$, $\Lambda_1 < \Lambda_2$, $\mathrm{M}_1 < \mathrm{M}_2$, be any non-singular solution of (5.4) with associated quintuple $(\varepsilon, l_1, l_2; m_1, m_2)$, and suppose that we have

$$N(l_1, \Lambda_1) \quad \text{and } N(m_1 - 4\mathrm{M}_1, \mathrm{M}_1) \quad \text{in case } \varepsilon = +1, \tag{5.14 a}$$

$$N(l_1 - 4\Lambda_1, \Lambda_1) \quad \text{and } N(m_1, \mathrm{M}_1) \quad \text{in case } \varepsilon = -1. \tag{5.14 b}$$

Notice that the inductive hypothesis for the solution $(1, 5; 1, 3)$ of (5.4), which has $\varepsilon = +1$, is precisely (5.13), and thus is satisfied if $f \neq g_1, h_1$.

From the inductive hypothesis for $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ we want to deduce that $f = g_{\Lambda_2}$ or h_{M_2} or to prove the inductive hypothesis for at least one of the three neighbouring solutions of $(\Lambda_1, \Lambda_2; \mathrm{M}_1, \mathrm{M}_2)$ of bigger height. By Lemma 5.5 these solutions are of the form

$$\begin{aligned}
&(\Lambda_1, -; \mathrm{M}_2, -) \quad \text{with quintuple } (\varepsilon; l_1, -; m_2, -),\\
&(\Lambda_2, -; \mathrm{M}_1, -) \quad \text{with quintuple } (\varepsilon; l_2, -; m_1, -),\\
&(\Lambda_2, -; \mathrm{M}_2, -) \quad \text{with quintuple } (-\varepsilon; l_2, -; m_2, -).
\end{aligned}$$

If $P(m_2, \mathrm{M}_2)$ and $P(m_2 - 4\mathrm{M}_2, \mathrm{M}_2)$, then $f = h_{\mathrm{M}_2}$ by Lemma 5.7, Corollary, since either (iii) (iva) or (iii) (ivb) are valid.

Otherwise, either $N(m_2, \mathrm{M}_2)$ or $N(m_2 - 4\mathrm{M}_2, \mathrm{M}_2)$, in which case we distinguish between four subcases:

(1) $\varepsilon = +1$ and $N(m_2, \mathrm{M}_2)$. If $P(l_2, \Lambda_2)$ and $P(l_2 - 4\Lambda_2, \Lambda_2)$, then $f = g_{\mathrm{M}_2}$ by Lemma 5.7, Corollary, since (i) (iia) are valid. Otherwise either $N(l_2, \Lambda_2)$ or $N(l_2 - 4\Lambda_2, \Lambda_2)$; in case $N(l_2, \Lambda_2)$ the inductive hypothesis (5.14a) is satisfied for $(\Lambda_2, -; \mathrm{M}_1, -)$, in case $N(l_2 - 4\Lambda_2, \Lambda_2)$ the inductive hypothesis (5.14b) is satisfied for $(\Lambda_2, -; \mathrm{M}_2, -)$.

(2) $\varepsilon = +1$ and $N(m_2 - 4\mathrm{M}_2, \mathrm{M}_2)$. Then the inductive hypothesis (5.14a) is satisfied for $(\Lambda_1, -; \mathrm{M}_2, -)$.

(3) $\varepsilon = -1$ and $N(m_2, \mathrm{M}_2)$. Then the inductive hypothesis (5.14b) is satisfied for $(\Lambda_1, -; \mathrm{M}_2, -)$.

(4) $\varepsilon = -1$ and $N(m_2 - 4\mathrm{M}_2, \mathrm{M}_2)$. If $P(l_2, \Lambda_2)$ and $P(l_2 - 4\Lambda_2, \Lambda_2)$, then $f = g_{\Lambda_2}$ by Lemma 5.7, Corollary, since (i) (iib) are valid. Otherwise either $N(l_2, \Lambda_2)$ or $N(l_2 - 4\Lambda_2, \Lambda_2)$; in case $N(l_2, \Lambda_2)$ the inductive hypothesis (5.14a) is satisfied for $(\Lambda_2, -; \mathrm{M}_2, -)$, in case $N(l_2 - 4\Lambda_2, \Lambda_2)$ the inductive hypothesis (5.14b) is satisfied for $(\Lambda_2, -; \mathrm{M}_1, -)$.

Consequently, if f were not a g_Λ or a h_M, it would have to satisfy either (5.14a) or

(5.14b) for an infinite sequence $(\Lambda_1^{(r)}, \Lambda_2^{(r)}; M_1^{(r)}, M_2^{(r)})$, $r \in \mathbf{N}$, with heights $h^{(r)} \to \infty$ for $r \to \infty$. It would then follow from Lemma 5.10 ((2a) or (2b) apply) that

$$\delta(f) \geq 16 - 8/M_2^{(r)2} \quad \text{for } r \in \mathbf{N}.$$

However, since $M_2^{(r)} \to \infty$ for $r \to \infty$, this would imply that $\delta(f) \geq 16$, a contradiction. This proves the lemma.

LEMMA 5.13. *There are $2^{\aleph_0}$ different forms*

$$f(x, y) = x^2 + \beta xy + \gamma y^2 \quad \textit{with}$$

$$2 \leq \beta \leq 4, \ \delta(f) = \beta^2 - 4\gamma = 16, \ \nu(f) = 2.$$

Proof. Analogous to that of Lemma 14, chapter II of [4].

5.4. The main theorem on the C-minimum of forms

THEOREM 5.3. *Let $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$, $\delta(f) > 0$.*

A. *If*

$$\sqrt{\delta(f)}/\nu(f) < 2, \tag{5.15}$$

then f is C-equivalent to a multiple of some g_Λ or h_M.

B. *Conversely* (5.15) *holds for all forms C-equivalent to a multiple of some g_Λ or h_M; specifically*

$$\sqrt{\delta(g_\Lambda)}/\nu(g_\Lambda) = \sqrt{4 - 1/\Lambda^2}, \quad \sqrt{\delta(h_M)}/\nu(h_M) = \sqrt{4 - 2/M^2}.$$

C. *There are $2^{\aleph_0}$ forms f, none of which are C-equivalent to a multiple of any other, such that*

$$\sqrt{\delta(f)}/\nu(f) = 2.$$

Proof. Part B follows by Lemma 5.1, (5.7) and Lemma 5.9, Corollary 1.

Part C follows from Lemma 5.13, since any C-equivalence class of forms is denumerable.

To prove part A we notice, that by Lemma 5.1 we may as well assume that

$$0 < \delta(f) < 16, \quad \nu(f) = 2, \tag{5.16}$$

hence by the definition of C-minimum either $\nu_1(f) = 1$ or $\nu_0(f) = 2$.

Case 1. $\nu_1(f) = 1$. Then for any $\varepsilon > 0$, there are $(a, c) \in \mathbf{Z}_1^2$ such that

$$1 = \nu_1(f) \leq |f(a, c)| = \alpha' < 1 + \varepsilon.$$

Hence by Lemma 5.2 we have

$$\pm f \approx f' = (\alpha', \beta', \gamma'), \tag{5.17}$$

where

$$1 \leqslant \alpha' < 1+\varepsilon, \quad 2\alpha' \leqslant \beta' \leqslant 4\alpha', \ \delta(f') = \delta(f).$$

If $\alpha' = 1$, then f' is some g_Λ, h_M by Lemma 5.12, and the conclusion follows from (5.17). Otherwise we can find an infinite sequence of forms

$$f_n = (\alpha_n, \beta_n, \gamma_n), \quad n \in \mathbf{N},$$

with

$$\lim \alpha_n = 1, \ 2\alpha_n \leqslant \beta_n \leqslant 4\alpha_n, \ \delta(f_n) = \delta(f),$$

each f_n being C-equivalent to $\pm f$. By a simple compactness argument, we may as well assume that

$$\beta_n \to \beta_0, \ \gamma_n \to \gamma_0 \quad \text{for } n \to \infty.$$

Then

$$f_0 = (1, \beta_0, \gamma_0)$$

has

$$2 \leqslant \beta_0 \leqslant 4, \ \delta(f_0) = \delta(f).$$

Further since

$$|f_0(x,y)| = \lim_{n\to\infty} |f_n(x,y)|$$

we have

$$|f_0(x,y)| \geqslant 1 \quad \text{for all } (x,y) \in \mathbf{Z}_1^2,$$
$$|f_0(x,y)| \geqslant 2 \quad \text{for all } (x,y) \in \mathbf{Z}_0^2.$$

Hence by Lemma 5.12, f_0 is some g_Λ, h_M. However, by Lemma 5.9, Corollary 2, we may apply Theorem 5.1 to $f_0(=g_\Lambda$ or $h_M)$, and consequently for some sufficiently large n, $f_n = af_0(=ag_\Lambda$ or $ah_M)$. Since also $f \approx \pm f_n$, this proves A in this case.

Case 2. $\nu_0(f) = 2$. Then by Lemma 5.1,

$$\nu_1(f^*) = 1, \ \nu(f^*) = 2, \ 0 < \delta(f^*) = \delta(f) < 16,$$

and hence by case 1, we have $f^* \approx ag_\Lambda$ or ah_M; also $a = \pm 1$, since

$$\nu_1(ag_\Lambda) = \nu_1(ah_M) = |a|$$

by Lemma 5.1 and Lemma 5.9, Corollary 1. Finally

$$2\nu_1(f)=\nu_0(f^*)=\nu_0(\pm g_\Lambda)=\nu_0(\pm h_M)=2,$$

and thus also $\nu_1(f)=1$, and hence f belongs to case 1 also. This ends the proof of part A.

5.5. The main theorem on the C-approximation constant

THEOREM 5.4. *Let* $\vartheta \in \mathbf{R}\setminus\mathbf{Q}$.

A. *If*

$$d(\vartheta)<2, \tag{5.18}$$

then ϑ *is C-equivalent to a root in some* g_Λ *or* h_M.

B. *Conversely* (5.18) *holds if* ϑ *is C-equivalent to a root in some* g_Λ *or* h_M*; specifically*

$$d(\vartheta)=\sqrt{4-1/\Lambda^2}, \quad \textit{when} \ g_\Lambda(\vartheta,1)=0,$$

$$d(\vartheta)=\sqrt{4-2/M^2}, \quad \textit{when} \ h_M(\vartheta,1)=0.$$

For any g_Λ, *we have* $(\vartheta_1(g_\Lambda))^*\approx\vartheta_2(g_\Lambda)$; *for any* h_M, *we have* $\vartheta_1(h_M)\approx\vartheta_2(h_M)$.

C. *There are* $2^{\aleph_0}$ *different C-equivalence classes of irrationals* ϑ, *such that* $d(\vartheta)=2$.

Proof. To prove part A notice that since $d(\vartheta)=\max(d_0(\vartheta), \frac{1}{2}d_1(\vartheta))$, either $d_0(\vartheta)=d(\vartheta)<2$ or $d_1(\vartheta)=2d(\vartheta)<4$.

Case 1. $d_1(\vartheta)=2d(\vartheta)<4$. We consider the form

$$f(x,y)=2d(\vartheta)\,x(\vartheta x-y).$$

By the definition of $d(\vartheta)$: $\forall\varepsilon>0\ \exists Y_0=Y_0(\varepsilon)>0$, such that

$$|f(x,y)|>2-\varepsilon \quad \text{for all } (x,y)\in\mathbf{Z}_0^2 \quad \text{with } |\vartheta x-y|<Y_0(\varepsilon),$$

$$|f(x,y)|>1-\varepsilon \quad \text{for all } (x,y)\in\mathbf{Z}_1^2 \quad \text{with } |\vartheta x-y|<Y_0(\varepsilon).$$

Further, since $d_1(\vartheta)=2d(\vartheta)$, there is a sequence $(a_n,c_n)\in\mathbf{Z}_1^2$ such that

$$|f(a_n,c_n)|\to 1,\ a_n\to\infty,\ |\vartheta a_n-c_n|\to 0 \quad \text{for } n\to\infty.$$

It is now clear that the proof of Theorem III, chapter II of [4] carries over with obvious modifications; this gives the conclusion in case 1.

Case 2. $d_0(\vartheta)=d(\vartheta)<2$. Then by Theorem 4.11, Corollary,

$$d_1(\vartheta^*)=2d_0(\vartheta)=2d(\vartheta)=2d(\vartheta^*)<4,$$

hence by case 1, $\vartheta^* \approx \vartheta'$, where ϑ' is a root of some g_Λ or h_M. But then by Theorem 4.11, Corollary, Lemma 5.1, Lemma 5.3 B, Lemma 5.9, Corollary,

$$d_1(\vartheta) = 2d_0(\vartheta^*) = 2d_0(\vartheta') = 2d(\vartheta') = 2d(\vartheta^*) = 2d(\vartheta),$$

and thus ϑ belongs to case 1 also. This ends the proof of part A.

Part B follows directly from Theorem 5.3 B, Lemma 3 B and Lemma 5.8, Corollary.

Proof of C. By Theorem 5.3 C together with Lemma 5.3 A, there are $2^{\aleph_0}$ different C-equivalence classes of irrationals ϑ such that $d(\vartheta) \leqslant 2$. By Theorem 5.4 A there are only enumerably many different C-equivalence classes of irrationals ϑ such that $d(\vartheta) < 2$.

5.6. Other methods

Instead of the proof of Theorems 5.3, 5.4 given above, which follows closely the proof of J. W. S. Cassels [4] for the Markoff chain, we could have extended either (1) the proof of A. Markoff [24] as presented by L. E. Dickson [9] or (2) the proof of C. G. Lekkerkerker [21].

The extension of Markoff's method is based on the formula for $d(\vartheta)$ in Theorem 4.11, and a similar formula for $\nu(f)$ obtained by developing a theory of *C-reduced* and *C-dually reduced* quadratic forms.

Lekkerkerker's method is extended as follows: For any basis $(\mathbf{e}_1, \mathbf{e}_2)$ of $\mathbf{R}^2$ over $\mathbf{R}$, let

$$L = L(\mathbf{e}_1, \mathbf{e}_2) = \{x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \mid x_1, x_2 \in \mathbf{Z}\},$$

$$L_j = L_j(\mathbf{e}_1, \mathbf{e}_2) = \{x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \mid (x_1, x_2) \in \mathbf{Z}_j^2\}, \quad (j = 0, 1).$$

For

$$S_0 = \{(\xi_1, \xi_2) \mid |\xi_1\xi_2| < 1\}$$

L is called *C-admissible* for S_0 if

$$L_0 \cap S_0 = L_1 \cap \frac{1}{\sqrt{2}} S_0 = \varnothing.$$

For example

$$L((1/\sqrt{2}, 1/\sqrt{2}), (-1 + 1/\sqrt{2}, 1 + 1/\sqrt{2})) \quad \text{with } \det L = \sqrt{2},$$

$$L((1/\sqrt{2}, 1/\sqrt{2}), (-\sqrt{3/2}, \sqrt{3/2})) \quad \text{with } \det L = \sqrt{3},$$

are both C-admissible for S_0.

Essentially all C-admissible L's for S_0 of $\det L < 2$ are constructed by the procedure of Lekkerkerker, however using the matrices

$$C_0 = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \ D_0 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \ \hat{D}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix},$$

which satisfy the relations

$$C_0 D_0 = \hat{D}_0 K C_0,\ \hat{D}_0 C_0 = C_0 K D_0,\ C_0 \hat{D}_0 = D_0 C_0,$$

where

$$K = \begin{pmatrix} -1 & 8 \\ 0 & -1 \end{pmatrix}.$$

For example

$$A = D_0 C_0 \hat{D}_0 C_0 = 4\begin{pmatrix} 12 & 19 \\ 5 & 8 \end{pmatrix},$$

$$B = D_0 C_0^2 = 2\begin{pmatrix} 7 & 11 \\ 3 & 5 \end{pmatrix},$$

and hence we have the following associated forms,

$$f_{C_0} = (1, 1-1, -3) = (1, 0, -3) \approx g_1,$$

$$f_{D_0} = 1, 1-3, -1) = (1, -2, -1) \approx h_1,$$

$$f_A = (5, 8-12, -19) = (5, -4, -19) \approx 5g_5,$$

$$f_B = (3, 5-7, -11) = (3, -2, -11) \approx 3h_3.$$

I shall confine myself with this hint of the extension of Lekkerkerker's method.

Also recent work of H. Cohn [6], [7] and M. Hall [17] on the Markoff chain have extensions to the present situation.

Chapter 6

Complex binary quadratic forms

6.1. Reduction theory

We consider binary quadratic forms $\Phi = (A, B, C)\colon \mathbf{C}^2 \to \mathbf{C}$, given by

$$\Phi\colon (X, Y) \mapsto AX^2 + BXY + CY^2,$$

where $A, B, C \in \mathbf{C}$ satisfy the following condition,

$$A \neq 0,\ D = B^2 - 4AC \neq 0,\quad \xi, \eta \in \mathbf{C} \setminus \mathbf{Q}(i). \tag{6.1}$$

Here ξ, η are the *roots* of Φ, i.e. of $\Phi(x, 1) = 0$.

For any complex unimodular matrix M, we let $\tilde{M}\colon (X, Y) \mapsto (aX + bY, cX + dY)$ be the corresponding linear map, and (as usual) m be the corresponding homographic map.

Notice that if Φ satisfies (6.1), then also any Ψ *equivalent* to Φ, i.e. of the form $\Psi=\Phi\circ\tilde{M}$ where M is unimodular. In fact, this follows from the formulae,

$$D_\Psi=(\det \mathrm{M})^2 D_\Phi=\pm D_\Phi, \tag{6.2}$$

$$\xi_\Psi=m^{-1}(\xi_\Phi),\ \eta_\Psi=m^{-1}(\eta_\Phi) \tag{6.3}$$

$$A_\Psi=\Phi(a,c),\ C_\Psi=\Phi(b,d),\ A_\Psi+B_\Psi+C_\Psi=\Phi(a+b,c+d). \tag{6.4}$$

Notice also that an *integral* form (A, B, C), i.e. a form with $A, B, C\in\mathbf{Z}[i]$, satisfies (6.1) precisely when $D=B^2-4AC$ is not a square in $\mathbf{Z}[i]$.

DEFINITION 6.1. *A binary quadratic form satisfying* (6.1) *is called reduced if* (*with suitable notation*)

$$\xi\in\mathfrak{J} \quad \textit{and} \quad \bar{\eta}\in\mathfrak{J}^*,$$

and is called dually reduced if (*with suitable notation*)

$$\xi\in\mathfrak{J}^* \quad \textit{and} \quad \bar{\eta}\in\mathfrak{J}.$$

Notice that if (A, B, C) is reduced (dually reduced), then $(\bar{A}, \bar{B}, \bar{C})$ is dually reduced (reduced), (C, B, A) is dually reduced (reduced) and $(\bar{C}, \bar{B}, \bar{A})$ is reduced (dually reduced).

Notice also that if Φ is reduced (dually reduced), then $\lambda\Phi$, $\lambda\in\mathbf{C}\diagdown\{0\}$, is reduced (dually reduced) and $\Phi\circ\tilde{S}$, $\Phi\circ\tilde{S}^{-1}$ are both reduced (dually reduced), the last statement being a consequence of Lemma 1.1 (viii) (ix).

THEOREM 6.1. *Suppose that* $\Phi=(A, B, C)$ *satisfying* (6.1) *represents primitively a number* A' *with*

$$0<|A'|\leqslant\sqrt{|D|/2}.$$

Then there exists a form $\Phi'=(A', B', C')$, *which is equivalent to* Φ, *and such that* Φ' *is either reduced or dually reduced.*

Proof. By a suitable choice between the two values of $\sqrt{D}$ and $\sqrt{-D}$ we can make sure that either

(a) $\arg(\sqrt{D}/A')\in[\pi/4, 3\pi/4]$

or

(b) $\arg(\sqrt{-D}/A')\in[\pi/4, 3\pi/4]$.

By assumption there exist $a_0, c_0\in\mathbf{Z}[i]$ with $\gcd(a_0, c_0)=1$ and $\Phi(a_0, c_0)=A'$. Then determine $b_0, d_0\in\mathbf{Z}[i]$, such that $a_0d_0-b_0c_0=1$ in case (a) and $a_0d_0-b_0c_0=i$ in case (b), and put

$$M_0=\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0\end{pmatrix},\ M_k=M_0\begin{pmatrix} 1 & k \\ 0 & 1\end{pmatrix}, k\in\mathbf{Z}[i].$$

In either case the forms

$$\Phi_k = \Phi \circ \tilde{M}_k, \quad k \in \mathbf{Z}[i],$$

are equivalent to Φ, and $\Phi_k(1, 0) = \Phi(a_0, c_0) = A'$; hence we need only prove that Φ_k is reduced or dually reduced for a suitable $k \in \mathbf{Z}[i]$.

However, the roots ξ_k, η_k (suitably enumerated) of Φ_k satisfy the following relations,

$$\xi_k = \xi_0 - k, \quad \eta_k = \eta_0 - k \quad \text{for all } k \in \mathbf{Z}[i], \tag{6.5}$$

$$\xi_k - \eta_k = \sqrt{D}/A' \qquad \text{in case (a) for all } k \in \mathbf{Z}[i], \tag{6.6 a}$$

$$\xi_k - \eta_k = \sqrt{-D}/A' \quad \text{in case (b) for all } k \in \mathbf{Z}[i]. \tag{6.6 b}$$

Hence it follows from (6.6a), (6.6b) that

$$\arg(\xi_k - \eta_k) \in [\pi/4, 3\pi/4] \quad \text{for all } k \in \mathbf{Z}[i]. \tag{6.7}$$

Also, it follows from (6.6a), (6.6b) and the assumption of the theorem, that

$$|\xi_k - \eta_k| \geqslant \sqrt{2} \quad \text{for all } k \in \mathbf{Z}[i]. \tag{6.8}$$

By (6.5) we choose $k \in \mathbf{Z}[i]$, such that

$$\eta_k \in [0, 1] \times [0, -i].$$

If

$$\eta_k \in [0, 1] \times \left[-\frac{i}{2}, -i\right] \subseteq \mathfrak{J}^*,$$

then by (6.7), (6.8), $\xi_k \in \mathfrak{J}$, and Φ_k is reduced. If

$$\eta_k \in [0, 1] \times \left[0, -\frac{i}{2}\right] \subseteq \bar{\mathfrak{J}},$$

then by the same argument, either Φ_k or Φ_{k-1} or Φ_{k+1} is dually reduced.

COROLLARY *Every form Φ satisfying* (6.1) *is equivalent to a reduced form and also equivalent to a dually reduced form.*

Proof. The result follows from Theorem 6.1 together with a well-known result of O. Perron [26], that every form satisfying (6.1) will represent primitively (in the Gaussian case) a number A' with $0 < |A'| \leqslant \sqrt{|D|/3}$, and finally the remark, that if (A, B, C) is reduced (dually reduced), then (C, B, A), which is equivalent to (A, B, C), is dually reduced (reduced).

THEOREM 6.2. *Suppose that (A, B, C) is reduced (dually reduced) form satisfying* (6.1). *Then at least one of the following inequalities is satisfied,*

$$|AC| < 4|D|, \quad |A(A+B+C)| < 4|D|, \quad |C(A+B+C)| < 4|D|.$$

Proof. Assuming that (A, B, C) is reduced (otherwise we consider (C, B, A)), and that

$$|AC| \geqslant 4|D| \quad \text{and} \quad |A(A+B+C)| \geqslant 4|D|, \tag{*}$$

we have to derive a contradiction.

Since

$$B^2 - D = 4AC, \ (B+2A)^2 - D = 4A(A+B+C),$$

it follows from (*) that

$$|B| \geqslant \sqrt{15}\sqrt{|D|} \quad \text{and} \quad |B+2A| \geqslant \sqrt{15}\sqrt{|D|}. \tag{**}$$

Also (with a suitable determination of $\sqrt{D}$)

$$\xi = (-B+\sqrt{D})/(2A) \in \mathfrak{J}, \ \eta = (-B-\sqrt{D})/(2A) \in \overline{\mathfrak{J}}^*,$$

and hence at least one of the following cases occur,

(a) $\arg \xi \in [0, \pi]$ and $\arg \eta \in [-\pi/2, -\pi/4]$,

(b) $\arg(\xi - 1) \in [0, \pi]$ and $\arg(\eta - 1) \in [-3\pi/4, -\pi/2]$.

However, for

$$\frac{\xi}{\eta} - 1 = \frac{-2\sqrt{D}}{B+\sqrt{D}}$$

we get by (**)

$$\left|\frac{\xi}{\eta} - 1\right| \leqslant \frac{2}{\sqrt{15}-1} < \frac{1}{\sqrt{2}};$$

hence

$$\arg \frac{\xi}{\eta} \in]-\pi/4, \pi/4[,$$

contradicting (a).

Similarly, for

$$\frac{\xi-1}{\eta-1} - 1 = \frac{-2\sqrt{D}}{B+2A+\sqrt{D}}$$

we get by (**)

$$\left|\frac{\xi-1}{\eta-1} - 1\right| \leqslant \frac{2}{\sqrt{15}-1} < \frac{1}{\sqrt{2}};$$

hence

$$\arg((\xi-1)/(\eta-1)) \in]-\pi/4, \pi/4[,$$

contradicting (b).

This proves the theorem.

COROLLARY. *For any given $D \in \mathbf{Z}[i]$, where D is not a Gaussian square, there are only finitely many integral forms with discriminant D which are either reduced or dually reduced.*

In analogy with a well-known procedure of C. F. Gauss [16] (cf. also L. E. Dickson [8]), we associate to any form Φ_0 with roots ξ_0, η_0, which is

(a) reduced with $\xi_0 \in \mathfrak{J}$ and $\bar{\eta}_0 \in \mathfrak{J}^*$,

or

(b) dually reduced with $\xi_0 \in \mathfrak{J}^*$ and $\bar{\eta}_0 \in \mathfrak{J}$,

a *double chain* of forms

$$\ldots, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Phi_1, \Phi_2, \ldots \tag{6.9}$$

In fact, let

$$\operatorname{ch} \xi_0 = T_0 T_1 \ldots; \ \operatorname{ch}^* \bar{\eta}_0 = T_{-1} T_{-2} \ldots, \tag{6.10a}$$

$$\operatorname{ch}^* \xi_0 = T_0 T_1 \ldots; \ \operatorname{ch} \bar{\eta}_0 = T_{-1} T_{-2} \ldots, \tag{6.10b}$$

respectively. [If ξ_0, η_0 are not equivalent to real numbers, the chains in (6.10a), (6.10b) are uniquely determined by Theorems 2.2, 2.2*. Also, if Φ_0 is integral, the chains in (6.10a), (6.10b) become uniquely determined by requiring these to be purely periodic by Theorem 3.4. In any case we require that $T_0 \neq V_1^{-1}$.] Then we define Φ_n, $n \in \mathbf{Z}$, recursively by

$$\Phi_{n+1} = \Phi_n \circ \tilde{T}_n, \ n \in \mathbf{Z}. \tag{6.11}$$

By (6.11) the roots of Φ_n and Φ_{n+1} are related as follows,

$$\xi_{n+1} = t_n^{-1}(\xi_n), \ \eta_{n+1} = t_n^{-1}(\eta_n) = \varkappa \circ t_n \circ \varkappa(\eta_n),$$

and hence

$$\xi_n = t_n(\xi_{n+1}), \ \bar{\eta}_{n+1} = t_n(\bar{\eta}_n), \ n \in \mathbf{Z}. \tag{6.12}$$

By (6.10a), (6.10b), (6.12) it follows that

$$\xi_n = [T_n T_{n+1} \ldots], \ \bar{\eta}_n = [T_{n-1} T_{n-2} \ldots], \ n \in \mathbf{Z}. \tag{6.13}$$

By the very definition of (T_n) it follows, that one of the chains in (6.13) is regular and the other is dually regular. Consequently, each Φ_n, $n \in \mathbf{Z}$, is either reduced or dually reduced.

Notice that in case Φ_0 is reduced (dually reduced) and integral, the two-way infinite sequence (T_n) defined by (6.10a) or (6.10b) is periodic by Theorem 3.5.

The actual calculation of a double chain of forms is facilitated by the following table:

Φ	(A, B, C)
$\Phi\circ\tilde{S}$	$(C, -B-2C, A+B+C)$
$\Phi\circ\tilde{S}^{-1}$	$(A+B+C, -2A-B, A)$
$\Phi\circ\tilde{V}_1$	$(A, 2iA+B, -A+iB+C)$
$\Phi\circ\tilde{V}_2$	$(A-iB-C, B-2iC, C)$
$\Phi\circ\tilde{V}_3$	$(-2iA-(1+i)B-C, (2+2i)A+3B+(2-2i)C, -A+(-1+i)B+2iC)$
$\Phi\circ\tilde{E}_1$	$(A+(1-i)B-2iC, iB+(2+2i)C, -C)$
$\Phi\circ\tilde{E}_2$	$(A, (-2+2i)A+iB, -2iA-(1+i)B-C)$
$\Phi\circ\tilde{E}_3$	$(-A, iB, C)$
$\Phi\circ\tilde{C}$	$(A+(1-i)B-2iC, (-2+2i)A+3iB+(2+2i)C, -2iA-(1+i)B-C)$

We shall illustrate the computation of double chains of forms in the following examples, which correspond to Examples 2.1, 2.2, 2.3.

Example 6.1. The integral form

$$\Phi_0=(1, -1, 1)$$

is both reduced and dually reduced with

$$\operatorname{ch}\xi_0=\operatorname{ch}^*\xi_0=\overrightarrow{C}.$$

Since

$$\Phi_1=\Phi_0\circ\tilde{C}=(-\mathrm{i}, i, -i)=-i\Phi_0,$$

we have

$$\Phi_n=(-i)^n\Phi_0,\quad n\in\mathbf{Z}.$$

Example 6.2. The integral form

$$\Phi_0=(5, -5-2i, 6+i)$$

is both reduced and dually reduced with

$$\operatorname{ch}\xi_0=\overrightarrow{V_1E_1CE_1V_1C},\quad \operatorname{ch}^*\xi_0=\overrightarrow{CE_1CE_1CC}.$$

Thus precisely two double chains pass through $\Phi_0=\Psi_0$, namely (Φ_n), where

$$\Phi_1=\Phi_0\circ\tilde{V}_1=(5, -5+8i, 3-4i),$$
$$\Phi_2=\Phi_1\circ\tilde{E}_1=(7i, 6-7i, -3+4i),$$
$$\Phi_3=\Phi_2\circ\tilde{C}=(7, -7+6i, 4-3i),$$

$$\Phi_4 = \Phi_3 \circ \tilde{E}_1 = (5i,\ 8-5i,\ -4+3i),$$

$$\Phi_5 = \Phi_4 \circ \tilde{V}_1 = (5i,\ -2-5i,\ 1+6i),$$

$$\Phi_6 = \Phi_5 \circ \tilde{C} = (5,\ -5-2i,\ 6+i) = \Phi_0,$$

$$\Phi_{n+6} = \Phi_n \text{ for all } n \in \mathbf{Z};$$

and (Ψ_n), where

$$\Psi_1 = \Psi_0 \circ \tilde{C} = (-9i,\ 6+9i,\ -3-4i),$$

$$\Psi_2 = \Psi_1 \circ \tilde{E}_1 = (7,\ -7-8i,\ 3+4i),$$

$$\Psi_3 = \Psi_2 \circ \tilde{C} = (-7i,\ 8+7i,\ -4-3i),$$

$$\Psi_4 = \Psi_3 \circ \tilde{E}_1 = (9,\ -9-6i,\ 4+3i),$$

$$\Psi_5 = \Psi_4 \circ \tilde{C} = (-5i,\ 2+5i,\ -1-6i),$$

$$\Psi_6 = \Psi_5 \circ \tilde{C} = (-5,\ 5+2i,\ -6-i) = -\Psi_0,$$

$$\Psi_{n+6} = -\Psi_n \quad \text{for all } n \in \mathbf{Z}.$$

Example 6.3. The integral form

$$\Phi = (1+2i,\ 1,\ 2-i)$$

is not reduced but dually reduced. However, the form

$$\Phi_0 = \Phi \circ \tilde{C} = (-3i,\ 3i,\ 1-2i)$$

is both reduced and dually reduced with

$$\operatorname{ch} \xi_0 = \overrightarrow{V_2 E_2 V_2 CE_1 CV_3 E_3 V_3 CE_2 CV_1 E_1 V_1 CE_3 C},$$

$$\operatorname{ch}^* \xi_0 = \overrightarrow{CE_2 CV_1 E_1 V_1 CE_3 CV_2 E_2 V_2 CE_1 CV_3 E_3 V_3}.$$

Thus precisely two double chains pass through $\Phi_0 = \Psi_0$, namely (Φ_n), where

$$\Phi_1 = \Phi_0 \circ \tilde{V}_2 = (2-i,\ -4+i,\ 1-2i),$$

$$\Phi_2 = \Phi_1 \circ \tilde{E}_2 = (2-i,\ -3+2i,\ 2+i)$$

$$\Phi_3 = \Phi_2 \circ \tilde{V}_2 = (2+i,\ -1-2i,\ 2+i),$$

$$\Phi_4 = \Phi_3 \circ \tilde{C} = (1-4i,\ 2+5i,\ -1-2i),$$

$$\Phi_5 = \Phi_4 \circ \tilde{E}_1 = (4+i,\ -3-4i,\ 1+2i),$$

$$\Phi_6 = \Phi_5 \circ \tilde{C} = (-3i,\ 3i,\ 1-2i) = \Phi_0 \circ \tilde{S}^{-1},$$

$$\Phi_{n+6} = \Phi_n \circ \tilde{S}^{-1} \quad \text{for all } n \in \mathbf{Z};$$

and (Ψ_n), where

$$\Psi_1=\Psi_0\circ\tilde{C}=(-1-2i,\,3+4i,\,-4-i),$$
$$\Psi_2=\Psi_1\circ\tilde{E}_2=(-1-2i,\,2+5i,\,1-4i),$$
$$\Psi_3=\Psi_2\circ\tilde{C}=(-2-i,\,1+2i,\,-2-i),$$
$$\Psi_4=\Psi_3\circ\tilde{V}_1=(-2-i,\,3-2i,\,-2-i),$$
$$\Psi_5=\Psi_4\circ\tilde{E}_1=(1-2i,\,-4+i,\,2-i),$$
$$\Psi_6=\Psi_5\circ\tilde{V}_1=(1-2i,\,3i,\,-3i)=\Psi_0\circ\tilde{S}^{-1},$$
$$\Psi_{n+6}=\Psi_n\circ\tilde{S}^{-1}\quad\text{for all } n\in\mathbf{Z}.$$

6.2. Minimum of quadratic forms

Let $\Phi=(A,\,B,\,C)$ be a complex quadratic form. Then

$$\mu(\Phi)=\inf|\Phi(X,\,Y)|,$$

the infimum being taken over all $(X,\,Y)\in\mathbf{Z}[i]^2\setminus\{(0,\,0)\}$, is called the *minimum* of Φ.

If Φ does not satisfy (6.1), it is obvious that $\mu(\Phi)=0$.

If Φ satisfies (6.1), then by Theorem 6.1, Corollary, we may associate (in several ways) to Φ a double chain of forms (Φ_n), each Φ_n being equivalent to Φ, and thus a two-way infinite product

$$\prod T_n=\ldots.\;T_{-2}T_{-1}T_0T_1T_2\;\ldots,$$

satisfying the following two conditions,

(i) $T_n\in\{V_j,\,E_j,\,C\}$ for each $n\in\mathbf{Z}$,

(ii) if for $n_1<n_2$ we have

$$T_{n_1}=E_{j_1},\;T_{n_2}=E_{j_2}\quad\text{and } T_n\neq E_j,\quad (j=1,\,2,\,3),\quad\text{for } n_1<n<n_2,$$

then

$$\operatorname{card}\{n\in\mathbf{Z}\,|\,n_1<n<n_2,\;T_n=C\}\equiv 1 \pmod 2).$$

A two-way infinite product $\prod T_n$ satisfying conditions (i), (ii) is called a *regular double chain.*

Conversely, for any regular double chain $\prod T_n$ and any $d\in\mathbf{C}\setminus\{0\}$, we may define for $n\in\mathbf{Z}$,

$$\xi_n=[T_nT_{n+1}\ldots],$$
$$\bar{\eta}_n=[T_{n-1}T_{n-2}\ldots],$$
$$\Phi_n(X,\,Y)=\varepsilon_n d(\xi_n-\eta_n)^{-1}(X-\xi_nY)(X-\eta_nY),$$

where

$$\varepsilon_n = \begin{cases} \det\ T_0 T_1 \dots T_{n-1} & \text{if } n \geqslant 0 \\ (\det\ T_{-1} T_{-2} \dots T_{-(-n)})^{-1} & \text{if } n < 0. \end{cases}$$

Then it is easily seen, that (Φ_n) is a double chain of forms with discriminant $(\Phi_0) = d^2$, and $\prod T_n$ is thus a regular double chain associated to Φ_0.

For any regular double chain $\prod T_n$ we define

$$K = K(\prod T_n) = \sup_{n \in \mathbf{Z}} \{ |\xi_n - \eta_n|,\ |\xi_n^{-1} - \eta_n^{-1}|,\ |(\xi_n - 1)^{-1} - (\eta_n - 1)^{-1}| \}.$$

Also we define a *congruence* relation in the set of regular double chains by putting

$$\prod T_n \equiv \prod U_n$$

if

$$U_n = \pi(T_{\sigma(n)}) \quad \text{for all } n \in \mathbf{Z},$$

where $\sigma \colon \mathbf{Z} \to \mathbf{Z}$ is of one of the forms (with a fixed $k \in \mathbf{Z}$),

$$\sigma \colon n \mapsto n + k \quad \text{or } \sigma \colon n \mapsto -n + k,$$

and $\pi \colon \{V_j, E_j, C\} \to \{V_j, E_j, C\}$ is of the form

$$\pi \colon V_j \mapsto V_{\bar{\pi}(j)},\ E_j \mapsto E_{\bar{\pi}(j)},\ C \mapsto C,$$

where $\bar{\pi}$ is a permutation of $\{1, 2, 3\}$.

It is easily verified, that $\equiv$ is an equivalence relation in the set of regular double chains, and that the set

$$\{ |\xi_n - \eta_n|,\ |\xi_n^{-1} - \eta_n^{-1}|,\ |(\xi_n - 1)^{-1} - (\eta_n - 1)^{-1}| \mid n \in \mathbf{Z} \}$$

appearing in the definition of $K(\prod T_n)$ is invariant under $\equiv$. Consequently

$$\prod T_n \equiv \prod U_n \Rightarrow K(\prod T_n) = K(\prod U_n). \tag{6.14}$$

THEOREM 6.3. *Suppose that Φ is a form satisfying* (6.1) *with discriminant D and minimum μ, and that $\prod T_n$ is a regular double chain associated to Φ. Then*

A. $\sqrt{|D|}/\mu \geqslant K(\prod T_n)$.

B. *If, in addition, Φ is proportional to an integral form, then*

$$\sqrt{|D|}/\mu = K(\prod T_n).$$

Proof. The double chain (Φ_n) defined above by

$$\Phi_n(X, Y) = \varepsilon_n d(\xi_n - \eta_n)^{-1}(X - \xi_n Y)(X - \eta_n Y),$$

where $|\varepsilon_n|=1$ and $d^2=D$, has each Φ_n equivalent to Φ. Consequently for each $n\in\mathbf{Z}$, we have

$$\mu=\mu(\Phi_n)\leqslant \min(|\Phi_n(1,0)|,|\Phi_n(0,1)|,|\Phi_n(1,1)|)$$
$$=\sqrt{|D|}\min(|\xi_n-\eta_n|^{-1},|\xi_n^{-1}-\eta_n^{-1}|^{-1},|(1-\xi_n)^{-1}-(1-\eta_n)^{-1}|^{-1}).$$

This proves part A.

The proof of part B is essentially identical to that of Theorem 2.7.

COROLLARY. *If (Φ_n) with $\Phi_n=(A_n, B_n, C_n)$ is a periodic double chain of forms satisfying* (6.1), *then*

$$\mu(\Phi_0)=\min(|A_n|,\ |C_n|,\ |A_n+B_n+C_n|), \tag{6.15}$$

where it suffices to extend the minimum over a period.

Example 6.4. Using formula (6.15) for the forms Φ_0 occurring in Examples 6.1, 6.2, 6.3, we obtain.

$$\mu((1,-1,1))=1, \qquad \sqrt{|D|}/\mu=\sqrt{3}=1.7320\ldots,$$
$$\mu((5,-5+2i,6+i))=5, \quad \sqrt{|D|}/\mu=\sqrt{4-5^{-2}}=1.9899\ldots,$$
$$\mu((1+2i,1,2-i))=\sqrt{5}, \quad \sqrt{|D|}/\mu=\sqrt{\tfrac{3}{5}\sqrt{41}}=1.9599\ldots.$$

A *main problem* about the minimum of complex binary quadratic forms is that of determining all forms Φ with $\sqrt{|D|}/\mu<2$.

We shall solve this problem in two steps, using Theorem 6.3:

1°. In section 6.3 we describe rather precisely those regular double chains ΠT_n, which have $K(\Pi T_n)<2$.

2°. In section 6.4 we solve the remaining problems by means of Theorem 5.3.

6.3. Regular double chains $\prod T_n$ with $K<2$

Throughout this section we deduce restrictions on a regular double chain ΠT_n in order to satisfy the condition $K<2$, by the following two arguments:

(1) If $T_nT_{n+1}\ldots$ is a regular chain and $T_{n-1}T_{n-2}\ldots$ is a dually regular chain, then by Theorems 2.1, 2.1*,

$$\xi_n=[T_nT_{n+1}\ldots]\in F(T_nT_{n+1}\ldots T_{n+h-1}),$$
$$\bar{\eta}_n=[T_{n-1}T_{n-2}\ldots]\in F^*(T_{n-1}\ldots T_{n-k}),$$

for all $h, k \in \mathbf{N}_0$. [Notice, that we write $F(T_n \dots T_{n+h-1})$ instead of $F(t_n \circ \dots \circ t_{n+h-1})$, etc.]. However, since

$$|\xi_n - \eta_n| \leq \mathrm{K} < 2,$$

we must have

$$d(F(T_n T_{n+1} \dots T_{n+h-1}), \overline{F^*}(T_{n-1} \dots T_{n-k})) < 2$$

for all $h, k \in \mathbf{N}_0$; here d denotes the euclidean distance between the two sets involved.

Similarly, if $T_n T_{n+1} \dots$ is a dually regular chain and $T_{n-1} T_{n-2} \dots$ is a regular chain, then we must have

$$d(F(T_{n-1} \dots T_{n-k}), \overline{F^*}(T_n T_{n+1} \dots T_{n+h-1})) < 2$$

for all $h, k \in \mathbf{N}_0$.

(2) If, by argument (1), we have excluded a product $U_{n-k} \dots U_{n-1} U_n U_{n+1} \dots U_{n+h-1}$ from appearing as a subproduct in ΠT_n, then by (6.14) any product *congruent* to $U_{n-k} \dots U_{n-1} U_n U_{n+1} \dots U_{n+h-1}$, i.e. obtained by applying a permutation π: $\{V_j, E_j, C\} \to \{V_j, E_j, C\}$ induced by a permutation $\bar{\pi}$ of $\{1, 2, 3\}$, and possibly a reversion, is also excluded as a subproduct of ΠT_n.

We shall often refer to an application of argument (2) by the phrase: "using congruence ...".

The first four lemmas will deal with *V-subproducts* i.e. subproducts of ΠT_n consisting only of V_j's. By the very definition of a regular double chain, a *maximal* V-subproduct, if finite, will be of the form

$$U_1 V_{j_1}^r V_{j_2} \dots V_{j_m} U_2, \tag{6.16}$$

with $U_1, U_2 \in \{E_j, C\}$, and either $U_1 = C$ or $U_2 = C$.

LEMMA 6.1. *The regular double chain ΠT_n contains, modulo congruence, only the following V-subproducts,*

$$V_1,\ V_1 V_2,\ V_1 V_2 V_3.$$

Proof. Using congruence, we get the result concerning V-subproducts of length 2 from

$$d(F(V_1 V_1), \overline{F^*}(I)) = 2.$$

Using this result and congruence, we obtain the result concerning V-subproducts of length 3 by excluding $V_1 V_2 V_1$ as a subproduct. Hence assume that $V_1 V_2 V_1$ is a subproduct of ΠT_n, and assume, using congruence, that the maximal V-subproduct containing $V_1 V_2 V_1$ is either of the form (6.16) with $U_1 = C$ or is infinite to the left. Since

$$d(F(V_1 V_2 V_1), \overline{F^*}(V_3 T)) > 2 \quad \text{for } T = V_1, V_2, C,$$

and $V_3 V_3$ has already been excluded, it follows that ΠT_n either contains a subproduct congruent to $C V_1 V_2 V_1$ or a subproduct congruent to $\overleftarrow{V_1 V_2}$, the arrow indicating periodicity to the left. However, since

$$d(F(V_1 V_2 V_1, \overline{F^*}(C)) > 2,$$

$$d([\overrightarrow{V_1 V_2}], [\overrightarrow{V_2 V_1}]) = \sqrt{5} > 2,$$

both possibilities are excluded. This proves the result concerning V-subproducts of length 3.

Finally, if ΠT_n contains a V-subproduct of length $\geqslant 4$, then by the same argument as above, either ΠT_n contains a subproduct congruent to $C V_1 V_2 V_3 V_1$ or a subproduct congruent to $\overleftarrow{V_1 V_2 V_3}$. However, since

$$d(F(V_1 V_2 V_3 V_1), \overline{F^*}(C)) > 2,$$

$$d([\overrightarrow{V_1 V_2 V_3}], [\overrightarrow{V_3 V_2 V_1}]) = \sqrt[4]{20} > 2,$$

both alternatives are impossible. This excludes the existence of a V-subproduct of length $\geqslant 4$.

LEMMA 6.2. *The regular double chain ΠT_n contains no V-subproduct of length* 3.

Proof. Using congruence, we need only by Lemma 6.1 exclude $C V_1 V_2 V_3 U_1 U_2$ with $U_1 \in \{E_j, C\}$, as a subproduct of ΠT_n. However, this follows from

$$d(F(V_1 C) \setminus F(V_1 C V_1^2), \overline{F^*}(V_3 V_2 C)) > 2,$$

$$d(F(V_1 V_2 E_2), \overline{F^*}(V_3 C)) > 2,$$

$$d(F(V_1 V_2 V_3 E_2), \overline{F^*}(C)) > 2,$$

$$d(F(V_1 E_2 V_1) \setminus F(V_1 E_2 V_1 V_3^2), \overline{F^*}(V_3 V_2 C)) > 2,$$

$$d(F(V_1 E_2 V_2), \overline{F^*}(V_3 V_2 C)) > 2,$$

$$d(F(V_1 V_2 E_3 V_1), \overline{F^*}(V_3 C)) > 2,$$

$$d(F(V_1 V_2 E_3 C), \overline{F^*}(V_3 C)) > 2,$$

which excludes $U_1 = C$, $U_1 = E_3$, $U_1 = E_2$, $U_1 U_2 = E_1 V_3$, $U_1 U_2 = E_1 V_1$, $U_1 U_2 = E_1 V_2$, $U_1 U_2 = E_1 C$, respectively.

LEMMA 6.3. *The regular double chain ΠT_n contains no V-subproduct of length* 2.

Proof. Using congruence, we need only by Lemma 6.2 exclude $CV_1V_2U_1U_2U_3 \ldots$ with $U_1 \in \{E_j, C\}$, as a subproduct of ΠT_n. Since

$$d(F(V_1V_2E_2), \overline{F^*}(C)) = 2,$$
$$d(F(V_1C) \setminus F(V_1CV_1^2), \overline{F^*}(V_3C)) > 2,$$
$$d(F(V_1V_2E_3V_1), \overline{F^*}(C)) > 2,$$
$$d(F(V_1E_2V_1) \setminus F(V_1E_2V_1V_3^2), \overline{F^*}(V_3C)) > 2,$$
$$d(F(V_1E_2V_2) \setminus F(V_1E_2V_2V_3^2), \overline{F^*}(V_3C)) > 2,$$
$$d(F(V_1V_2E_3CU), \overline{F^*}(C)) > 2 \quad \text{for } U \in \{V_2, V_3, E_1, C\},$$
$$d(F(V_1E_2CV_3), \overline{F^*}(V_3C)) > 2,$$
$$d(F(V_1E_2CE_1) \setminus F(V_1E_2CE_1V_2^2), \overline{F^*}(V_3C)) > 2,$$
$$d(F(V_1E_2CE_2) \setminus F(V_1E_2CE_2V_1^2), \overline{F^*}(V_3C)) > 2,$$

we can exclude $U_1 = E_2$, $U_1 = C$, $U_1U_2 = E_3V_1$, $U_1U_2 = E_3V_2$, $U_1U_2 = E_3V_3$, $U_1U_2U_3 = E_3CU$ with $U \in \{V_2, V_3, E_1, C\}$, $U_1U_2U_3 = E_3CV_1$, $U_1U_2U_3 = E_3CE_2$, $U_1U_2U_3 = E_3CE_3$, respectively. Consequently $U_1 = E_1$ is the only possibility.

Since

$$d(F(V_1E_3U), \overline{F^*}(V_3C)) \quad \text{for } U \in \{V_3, V_1, C\},$$

we can exclude $U_2 \in \{V_1, V_2, C\}$. Consequently $U_1U_2 = E_1V_3$ is the only possibility.

Now $U_3 = V_3$ contradicts Lemma 6.1. Further $U_3 = V_2$ implies $U_4 = C$ by Lemma 6.2 and the definition of a regular double chain, hence $U_1U_2U_3U_4 = E_1V_3V_2C \equiv CV_1V_2E_3$, which was excluded above. Similarly, $U_3 = V_1$ implies $U_4 = C$, however, this is also impossible, since

$$d(F(V_1E_3V_2V_3) \setminus F(V_1E_3V_2V_3V_2^2), \overline{F^*}(V_3C)) > 2.$$

Consequently $U_1U_2U_3 = E_1V_3C$ is the only possibility.

Since

$$d(F(V_1E_2V_3V_2C), \overline{F^*}(CU)) > 2 \quad \text{for } U \in \{V_2, V_3, E_1, E_2, C\},$$
$$d(F(V_1E_3V_2CV_2), \overline{F^*}(V_3C)) > 2,$$

we can exclude $U_4 \in \{V_1, V_2, E_3, E_1, C\}$, $U_4 = V_3$, respectively. Consequently $U_1U_2U_3U_4 = E_1V_3CE_2$ is the only possibility.

Since

$$d(F(V_1E_3V_2CE_1U), \overline{F^*}(V_3C)) > 2 \quad \text{for } U \in \{V_1, V_2, C\},$$

we can exclude $U_5 \in \{V_2, V_3, C\}$. Consequently $U_1U_2U_3U_4U_5 = E_1V_3CE_2V_1$ is the only possibility.

If $U_6 = V_j$, then $CE_2V_1V_jC$ would be a subproduct of ΠT_n, however, this possibility has already been excluded in course of the proof. Consequently $U_1U_2U_3U_4U_5U_6 = E_1V_3CE_2V_1C$ is the only possibility. However, this is also excluded since

$$d(F(V_1E_2CV_3), \overline{F^*}(C)) > 2.$$

This proves Lemma 6.3.

LEMMA 6.4. *The regular double chain ΠT_n contains V_j only in the following combinations,*

$$CV_jE_jC, \quad CE_jV_jC, \quad CV_jE_jV_jC.$$

Proof. Using congruence, we need only by Lemma 6.3 consider subproducts of ΠT_n of the form $\ldots W_3W_2W_1CV_1U_1U_2U_3\ldots$, where $U_1 \in \{E_j, C\}$. Since

$$d(F(V_1C), \overline{F^*}(C)) = 2,$$

we may assume, using congruence, that either (a) $U_1 = E_2$ or (b) $U_1 = E_1$.

Case (a), $U_1 = E_2$. Since

$$d(F(V_1E_2V_1), \overline{F^*}(C)) = 2,$$
$$d(F(V_1E_2V_2), \overline{F^*}(C)) > 2,$$

either (aa) $U_2 = C$ or (ab) $U_2 = V_3$.

Subcase (aa), $U_1U_2 = E_2C$. Since

$$d(F(V_1E_2CU), \overline{F^*}(C)) \geqslant 2 \quad \text{for } U \in \{V_1, V_3, E_1, E_2, C\},$$
$$d(F(V_1), \overline{F^*}(CE_1)) = 2,$$

we must have $U_3 = E_3$.

Suppose that $U_4 = V_j$, then $U_5 = C$ by Lemma 6.3 and the definition of a regular double chain. However, $U_1U_2U_3U_4U_5 = E_2CE_3V_2C \equiv CV_1E_2CE_1$ was excluded above, hence $U_4 \neq V_2$, and since

$$d(F(V_1E_2CE_3V_jC), \overline{F^*}(C)) > 2 \quad \text{for } j \in \{1, 3\},$$

also $U_4 \notin \{V_1, V_3\}$. Consequently $U_4 = C$.

Since

$$d(F(E_3CE_1C), \overline{F^*}(V_2)) > 2,$$

this leads to a contradiction; thus subcase (aa) is excluded. We use this fact tacitly in the sequel.

Subcase (ab), $U_1U_2=E_2V_3$. Then $U_3=C$ by Lemma 6.3. Since

$$d(F(V_1E_2V_3C),\ \overline{F^*}(CW_1))>2 \quad \text{for } W_1\in\{V_3,\ E_1,\ C\},$$
$$d(F(V_1),\ \overline{F^*}(CV_3E_1))=2,$$

we can exclude $W_1\in\{V_3,\ E_1,\ C\}$, $W_1=V_2$, respectively. Consequently either (aba) $W_1=E_2$ or (abb) $W_1=V_1$ or (abc) $W_1=E_3$.

Subcase (aba), $W_1CV_1U_1U_2U_3=E_2CV_1E_2V_3C$. Then either $W_2=C$ or $W_3W_2=CV_2$ by Lemma 6.3 and the exclusion of subcase (aa). By symmetry $U_4\in\{E_2,\ V_3,\ E_1\}$, and in case $U_4=E_2$, either $U_5=C$ or $U_5U_6=V_2C$.

Since

$$d(F(V_1E_2V_3C),\ \overline{F^*}(CE_2V_2C))>2,$$

we can exclude $W_3W_2W_1=CV_2E_2$, and hence by symmetry also $U_4U_5U_6=E_2V_2C$. Since

$$d(F(V_1E_2V_3CU_4U_5U_6),\ \overline{F^*}(CE_2C))>2$$

for $U_4=V_3$, $U_4U_5=E_1C$, $U_4U_5U_6=E_1V_1C$, $U_4U_5=E_2C$, we can exclude $W_2W_1=CE_2$, and hence by symmetry also $U_4U_5=E_2C$. Thus subcase (aba) is excluded, and necessarily $U_4\in\{V_3,\ E_1\}$.

Subcase (abb), $W_1CV_1U_1U_2U_3=V_1CV_1E_2V_3C$. Since

$$d(F(V_1E_2V_3C),\ \overline{F^*}(CV_1E_1))>2,$$
$$d(F(V_1E_2V_3CU_4U_5U_6),\ \overline{F^*}(CV_1E_2V_3C))>2$$

for $U_4=V_3$, $U_4U_5=E_1C$, $U_4U_5U_6=E_1V_1C$, and

$$d(F(V_1C),\ \overline{F^*}(C))=2,$$

the only possibility remaining is $W_2=E_3$, and consequently $W_4W_3=CV_2$.

Since

$$K(\overleftrightarrow{CV_3E_1V_2CV_2E_3V_1CV_1E_2V_3})=\sqrt{\frac{21}{5}}>2,$$

we may assume, using congruence and $U_4\in\{V_3,\ E_1\}$, that $U_4=E_1$ and consequently $U_5=C$ or $U_5U_6=V_1C$. However, this leads to a contradiction, since

$$d(F(V_3E_2V_1CV_1E_3V_2C),\ \overline{F^*}(CE_1U_5U_6))>2$$

for $U_5=C$ and $U_5U_6=V_1C$. Thus subcase (abb) is excluded, and by symmetry also $U_4=V_3$ is excluded.

Subcase (abc), $W_1CV_1U_1U_2U_3U_4=E_3CV_1E_2V_3CE_1$. Then $W_2=C$ or $W_3W_2=CV_3$ and $U_5=C$ or $U_5U_6=V_1C$. However, all four combinations are impossible, since

$$d(F(V_3E_2V_1CE_3W_2W_3),\ \overline{F^*}(CE_1U_5U_6))>2$$

in all four cases. This excludes subcase (abc).

Consequently subcase (ab) is excluded, and thus case (a) is excluded.

Case (b), $U_1=E_1$. Then by Lemma 6.3 and the exclusion of subcase (aa), we conclude that either $U_2=C$ or $U_2U_3=V_1C$. This completes the proof of Lemma 6.4.

LEMMA 6.5. *The regular double chain* ΠT_n *contains* E_j *only in the following combinations,*

$$CE_jC,\quad CV_jE_jC,\quad CE_jV_jC,\quad CV_jE_jV_jC.$$

Proof. The result is an immediate consequence of Lemma 6.4 and the definition of a regular double chain.

LEMMA 6.6. *The regular double chain* ΠT_n *contains* $CV_jE_jV_jC$ *as a subproduct only if either*

$$\prod T_n=\overleftrightarrow{CV_1E_1V_1CE_2CV_3E_3V_3CE_1CV_2E_2V_2CE_3}$$

or

$$\prod T_n=\overleftrightarrow{CV_1E_1V_1CE_3CV_2E_2V_2CE_1CV_3E_3V_3CE_2}.$$

Conversely, for each of these regular double chains

$$K=K(\prod T_n)=\sqrt{\tfrac{3}{5}\sqrt{41}}=1.9599\ldots.$$

Proof. Using congruence, we may assume that ΠT_n contains the subproduct $\ldots W_3W_2W_1CV_1E_1V_1CU_1U_2U_3\ldots$. Since

$$d(F(V_1E_1V_1C),\ \overline{F^*}(CX))>2\quad \text{for } X\in\{V_2, V_3, E_1, C\},$$

$$d(F(V_1E_1V_1C),\ \overline{F^*}(CV_1E_1))=2,$$

we conclude that $U_1,\ W_1\in\{E_2, E_3\}$. Since also

$$d(F(E_3V_3CE_1C)\cup F(E_3V_3CE_1V_1C),\ \overline{F^*}(V_3CE_1C)\cup\overline{F^*}(V_3CE_1V_1C))>2,$$

we conclude by Lemma 6.5 that $U_1\neq W_1$. Hence, using congruence, we may assume that $U_1=E_2$, $W_1=E_3$.

Using Lemma 6.5, we obtain from

$$d(F(V_1E_1V_1CE_2C)\cup F(V_1E_1V_1CE_2V_2C),\ \overline{F^*}(CE_3V_3C))>2,$$

that $W_3 \neq V_3$; by symmetry, also $U_2 \neq V_2$. Hence by Lemma 6.5, it follows that $U_2 = W_2 = C$.

Since

$$d(F(V_1), \overline{F^*}(CE_1)) = 2,$$
$$d(F(C^2), \overline{F^*}(CE_1CV_3E_3V_3C)) > 2,$$
$$d(F(E_3CV_2E_2), \overline{F^*}(CV_2E_2V_2C)) > 2,$$
$$d(F(E_3CE_1C) \cup F(E_3CE_1V_1C), \overline{F^*}(CV_2E_2V_2C)) > 2,$$
$$d(F(E_2CV_1E_1V_1C), \overline{F^*}(CE_1C) \cup \overline{F^*}(CE_1V_1C)) > 2,$$
$$d(F(E_3CV_2E_2V_2C), \overline{F^*}(CE_3C) \cup \overline{F^*}(CE_3V_3C)) > 2,$$

we can exclude $U_3 = V_2$, $U_3 = C$, $U_3 = V_1$, $U_3 = E_3$, $U_3 = E_1$, $U_3 = E_2$, respectively, by Lemmas 6.4, 6.5. Consequently $U_3 = V_3$, and hence by Lemma 6.4, either $U_4U_5 = E_3C$ or $U_4U_5U_6 = E_3V_3C$. However, since

$$d(F(V_2E_2C), \overline{F^*}(CE_1CV_3E_3V_3C)) > 2,$$

we can exclude the first of these possibilities, and hence $U_4U_5U_6 = E_3V_3C$.

Repeating the argument used above in both directions, we conclude that necessarily

$$\Pi T_n \equiv \overleftrightarrow{CV_1E_1V_1CE_2CV_3E_3V_3CE_1CV_2E_2V_2CE_3}.$$

By (6.14), Theorem 6.3 B and Examples 6.3, 6.4, this yields the result.

Lemma 6.7. *The regular double chain* ΠT_n *contains* $\overleftarrow{C}$ *or* $\overrightarrow{C}$ *as a subproduct only if* $\Pi T_n = \overleftrightarrow{C}$. *Conversely*

$$K = K(\overleftrightarrow{C}) = \sqrt{3} = 1.7320 \ldots .$$

Proof. That $K(\overleftrightarrow{C}) = \sqrt{3}$ follows from Theorem 6.3 B and Examples 6.1, 6.4. Hence assuming by congruence, that $\Pi T_n = \overleftarrow{C}U_1U_2U_3 \ldots$ with $U_1 \in \{V_1, E_1\}$, we have to reach a contradiction.

By Lemmas 6.4, 6.5, 6.6, either (a) $U_1U_2 = E_1C$ or (b) $U_1U_2U_3 = E_1V_1C$ or (c) $U_1U_2U_3 = V_1E_1C$.

Case (a). This possibility is excluded at once by

$$d(F(C^4), \overline{F^*}(CE_1C)) > 2.$$

Case (b). Since $K(\overleftarrow{C}E_1V_1\overrightarrow{C}) = 2$, we may assume, that

$$\overleftarrow{C}U_1U_2U_3 \ldots = \overleftarrow{C}E_1V_1C^kX_1X_2X_3 \ldots,$$

where $X_1 \neq C$. By the definition of a regular double chain together with Lemmas 6.4, 6.5, it follows that $k = 2m - 1$, $m \in \mathbf{N}$.

However,

$$d([\overrightarrow{C}], \overleftarrow{F^*}(CE_1\, V_1\, C^{2m-1}X_1\, X_2\, X_3)) > d([\overrightarrow{C}], \overleftarrow{[CE_1\, V_1\, \overrightarrow{C}]}) = 2$$

for $X_1 = V_1$, $X_1 X_2 = E_2 C$, $X_1 X_2 X_3 = E_2 V_2 C$, $X_1 X_2 = E_3 C$, $X_1 X_2 X_3 = E_3 V_3 C$, and hence by Lemmas 6.4, 6.5, 6.6, we have $X_1' \notin \{V_1, E_2, E_3\}$.

Similarly

$$d([V_1 E_1 \overrightarrow{C}], \overleftarrow{F^*}(C^{2m-1}X_1 X_2 X_3)) > d([V_1 E_1 \overrightarrow{C}], [\overleftarrow{\overrightarrow{C}}]) = 2$$

for $X_1 X_2 X_3 = V_2 E_2 C$, $X_1 X_2 X_3 = V_3 E_3 C$, $X_1 = E_1$, and hence by Lemmas 6.4, 6.5, 6.6, we have $X_1 \notin \{V_2, V_3, E_1\}$.

This excludes case (b).

Case (c). This possibility is excluded similarly to case (b).

From Lemmas 6.4, 6.5, 6.6, 6.7 and the definition of a regular double chain we obtain the following structure theorem.

THEOREM 6.4. *Suppose that the regular double chain ΠT_n with $K(\Pi T_n) < 2$ is different from $\overleftrightarrow{C}$ and from*

$$\overleftrightarrow{CV_1 E_1\, V_1 CE_2 CV_3 E_3\, V_3 CE_1 CV_2 E_2\, V_2 CE_3}$$

and

$$\overleftrightarrow{CV_1 E_1\, V_1 CE_3 CV_2 E_2\, V_2 CE_1 CV_3 E_3\, V_3 CE_2}.$$

Then $\prod T_n$ is of the form

$$\ldots C^{2m_{-1}-1} T_1^{(-1)} \ldots T_{k_{-1}}^{(-1)} C^{2m_0-1} T_1^{(0)} \ldots T_{k_0}^{(0)} C^{2m_1-1} \ldots,$$

where $m_r \in \mathbf{N}$ for all $r \in \mathbf{Z}$ and $T_1^{(r)} \ldots T_{k_r}^{(r)}$ for each $r \in \mathbf{Z}$ is one of the nine products,

$$E_j,\ V_j E_j,\ E_j V_j \quad \text{with } j \in \{1, 2, 3\}.$$

LEMMA 6.8. *The regular double chain ΠT_n contains*

$$C^{2l-1} E_j C^{2m-1}$$

as a subproduct only if $(l, m) \in \{(1, 1), (2, 1), (1, 2)\}$.

Proof. Using congruence, we may assume that $l \geq m$ and $j = 1$. Then $l \geq 3$ is excluded by

$$d(F(C^{2l-2}), \overleftarrow{F^*}(CE_1 C)) \geq d(F(C^4), \overleftarrow{F^*}(CE_1 C)) > 2,$$

and $(l, m) = (2, 2)$ is excluded by

$$d(F(C^2), \overleftarrow{F^*}(CE_1 C^3)) > 2.$$

LEMMA 6.9. *The regular double chain* ΠT_n *contains* $C^3 E_j C$ *or* $CE_j C^3$ *as a subproduct only in the following combinations,*

$$E_j C^3 E_j CE_j \quad \text{or} \quad E_j CE_j C^3 E_j.$$

Proof. Using congruence, we get the result from Theorem 6.4 and Lemma 6.8, since

$$d(F(C^2), \overline{F^*}(CE_1 CX_1 X_2 X_3)) > 2,$$

$$d(F(C^2 X_1 X_2 X_3), \overline{F^*}(CE_1 C)) > 2,$$

are both satisfied for $X_1 = V_1$, $X_1 X_2 = E_k C$ $(k=2, 3)$, $X_1 X_2 X_3 = E_k V_k C$ $(k=2, 3)$, $X_1 X_2 X_3 = V_k E_k C$ $(k=2, 3)$.

LEMMA 6.10. *The regular double chain* ΠT_n *does not contain any of the following products as a subproduct,*

$$CE_j CE_k V_k C, \quad CV_k E_k CE_j C \quad (j \neq k).$$

Proof. Using congruence, we obtain the result from

$$d(F(E_2 V_2 C), \overline{F^*}(CE_1 C)) > 2.$$

LEMMA 6.11. *The regular double chain* ΠT_n *does not contain any of the following products as a subproduct,*

$$CE_j CV_k E_k C, \quad CE_k V_k CE_j C \quad (j \neq k).$$

Proof. Using congruence, we need only exclude $CE_1 CV_2 E_1 CX_1 X_2 X_3$ as a subproduct. However, since

$$d(F(E_2 CX_1 X_2 X_3), \overline{F^*}(V_2 CE_1 C)) > 2$$

for $X_1 X_2 X_3 = V_k E_k C$ $(k=1, 2, 3)$, $X_1 X_2 = E_k C$ $(k=1, 2)$, $X_1 X_2 X_3 = E_k V_k C$ $(k=1, 2)$, $X_1 = C$, we conclude by Theorem 6.4 that $X_1 = E_3$. Consequently, by Theorem 6.4 and Lemma 6.10 only $X_2 X_3 = V_3 C$ remains as a possibility. However, using congruence, we can exclude this by

$$d(F(E_2 V_2 C), \overline{F^*}(CE_1 V_1 C)) > 2.$$

LEMMA 6.12. *The regular double chain* ΠT_n *does not contain any of the following products as a subproduct,*

$$CE_j CE_k C \quad (j \neq k).$$

Proof. Using congruence, we need only exclude $W_3 W_2 W_1 CE_1 CE_2 CU_1 U_2 U_3$ as a subproduct. Since

$$d(F(E_2 CU_1 U_2 U_3), \overline{F^*}(CE_1 C)) > 2$$

for $U_1U_2U_3=V_kE_kC$ $(k=1, 2, 3)$, $U_1U_2=E_kC$ $(k=1, 2)$, $U_1U_2U_3=E_kV_kC$ $(k=1, 2)$, and $U_1=C$ is excluded by Lemmas 6.8, 6.9, we conclude by Theorem 6.4 that $U_1=E_3$; analogously $W_1=E_3$. Since also

$$d(F(E_2CE_3V_3C), \overline{F^*}(CE_1CE_3C)\cup\overline{F^*}(CE_1CE_3V_3C))>2,$$

it follows by Theorem 6.4 that only $U_2=C$ and analogously $W_2=C$ remains as a possibility.

Repeating the argument in both directions, we find that $\Pi T_n\equiv\overleftrightarrow{CE_1CE_2CE_3}$ is the only possibility left. However, this is also excluded, since

$$K(\overleftrightarrow{CE_1CE_2CE_3})=\sqrt[4]{17}>2.$$

LEMMA 6.13. *The regular double chain ΠT_n does not contain any of the following products as a subproduct,*

$$E_jC^{2m-1}V_j,\quad V_jC^{2m-1}E_j\quad (m\in\mathbf{N}).$$

Proof. Using congruence, we need only by Theorem 6.4 exclude $E_1C^{2m-1}V_1E_1C^{2s-1}X_1X_2X_3$ with $X_1\neq C$ as a subproduct.

It is easily proved by means of Lemma 6.16 below that in order to have

$$d(F(V_1E_1C^{2s-1}X_1X_2X_3), \overline{F^*}(C^{2m-1}E_1))<2,$$

we must have $X_1\neq E_1$, $X_1X_2X_3\neq V_kE_kC$ $(k=2, 3)$ and $s<m$. By Theorem 6.4 we must have $X_1\notin\{E_1, V_2, V_3\}$.

For $1<s<m$ it follows similarly that in order to have

$$d(F(C^{2s-2}X_1X_2X_3), \overline{F^*}(CE_1V_1C^{2m-1}E_1))<2,$$

we must have $X_1\neq V_1$, $X_1X_2X_3\neq E_kV_kC$ $(k=2, 3)$. By Theorem 6.4 we must have $X_1\notin\{V_1, E_2, E_3\}$ in this case.

For $s=1$ it follows from

$$d(F(V_1), \overline{F^*}(CE_1))=2,$$

that $X_1\neq V_1$, and hence necessarily $X_1=E_k$ $(k=2, 3)$. By Lemma 6.10, it follows that $X_2\neq C$, and consequently by Theorem 6.4 we must have $X_1X_2X_3=E_kV_kC$ $(k=2, 3)$. However, this possibility is excluded by

$$d(F(E_kV_kC), \overline{F^*}(CE_1V_1C))>2\quad\text{for } k\in\{2, 3\}.$$

LEMMA 6.14. *The regular double chain ΠT_n does not contain any of the following products as a subproduct,*

$$V_jE_jC^{2m-1}E_kV_k\quad (j\neq k).$$

Proof. Using congruence, we need only by Theorem 6.4 exclude

$$CV_2E_2C^{2m-1}E_1V_1C^{2s-1}X_1X_2X_3$$

with $X_1 \neq C$ as a subproduct.

It is easily proved by means of Lemma 6.16 below that in order to have

$$d(F(C^{2m-2}E_2V_2C),\ \overline{F^*}(CE_1V_1C^{2s-1}X_1X_2X_3))<2,$$

we must have $X_1 \neq V_1$, $X_1X_2X_3 \neq E_kV_kC$ $(k=2, 3)$, and further $X_1X_2X_3 \neq V_kE_kC$ $(k=2, 3)$ in case $s \geqslant m-1$.

Similarly, in order to have

$$d(F(V_1E_1C^{2m-1}E_2V_2C),\ \overline{F^*}(C^{2s-1}X_1X_2X_3))<2,$$

we must have $X_1X_2X_3 \neq V_kE_kC$ $(k=2, 3)$ in case $s<m-1$.

Finally, $X_1 \neq E_1$ by Lemma 6.13, and $X_1X_2 \neq E_kC$ $(k=2, 3)$ by Lemmas 6.8, 6.9, 6.12.

Now the result follows by Theorem 6.4.

LEMMA 6.15. *The regular double chain ΠT_n does not contain any of the following products as a subproduct,*

$$E_jV_jC^{2m-1}V_kE_k \quad (j \neq k).$$

Proof. Using congruence, we need only by Theorem 6.4 exclude

$$CE_2V_2C^{2m-1}V_1E_1C^{2s-1}X_1X_2X_3$$

with $X_1 \neq C$ as a subproduct.

It is easily proved by means of Lemma 6.16 below that in order to have

$$d(F(V_1E_1C^{2s-1}X_1X_2X_3),\ \overline{F^*}(C^{2m-1}V_2E_2C))<2,$$

we must have $X_1 \neq E_1$, $X_1X_2X_3 \neq V_kE_kC$ $(k=2, 3)$. Since $X_1 \neq V_1$ by Lemma 6.13, it follows by Theorem 6.4 that either $X_1X_2=E_kC$ $(k=2, 3)$ or $X_1X_2X_3=E_kV_kC$ $(k=2, 3)$.

However, the first possibility is excluded by Lemmas 6.8, 6.9, 6.10, and the second possibility is excluded by Lemma 6.14.

LEMMA 6.16. *For $r \in \mathbf{N}$ we have*

$$C^r = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}^r = \frac{1}{\sqrt{12}}\left(\frac{1+i}{2}\right)^{r-2}\begin{pmatrix} \frac{1}{2}\alpha_r + i(\frac{1}{2}\alpha_r+\alpha_{r-1}) & -\alpha_r \\ \alpha_r & -\frac{1}{2}\alpha_r + i(\frac{1}{2}\alpha_r+\alpha_{r-1}) \end{pmatrix},$$

where

$$\alpha_r = (1+\sqrt{3})^r - (1-\sqrt{3})^r.$$

Proof. Since $\alpha_{r+1} = 2\alpha_r + 2\alpha_{r-1}$, we obtain the formula by induction on r.

LEMMA 6.17. $K(\overleftrightarrow{V_1E_1C^rV_3E_3C^r})>2$ *for all $r \in \mathbf{N}$.*

Proof. A computation of $V_1E_1C^rV_3E_3C^r$ by means of Lemma 6.16 shows that the regular double chain $\overleftarrow{V_1E_1C^rV_3E_3C^r}$ is associated to the quadratic form

$$\Phi=(-\tfrac{1}{2}\alpha_r^2+\alpha_r\alpha_{r-1}+\alpha_{r-1}^2-i(5\alpha_r^2+6\alpha_r\alpha_{r-1}+2\alpha_{r-1}^2),$$
$$-\tfrac{3}{2}\alpha_r^2-\alpha_r\alpha_{r-1}-\alpha_{r-1}^2+i(6\alpha_r^2+4\alpha_r\alpha_{r-1}),$$
$$+\tfrac{3}{2}\alpha_r^2-\alpha_r\alpha_{r-1}-\alpha_{r-1}^2-i(6\alpha_r^2+8\alpha_r\alpha_{r-1}+2\alpha_{r-1}^2)).$$

Hence by further computations,

$$D=D(\Phi)=\left(\frac{19}{2}\alpha_r^2+13\alpha_r\alpha_{r-1}+5\alpha_{r-2}^2\right)^2-9\cdot 4^{r+1},$$

$$\Phi(-1,i)=4\alpha_r^2+6\alpha_r\alpha_{r-1}+2\alpha_{r-1}^2+i(\tfrac{5}{2}\alpha_r^2+3\alpha_r\alpha_{r-1}+\alpha_{r-1}^2),$$
$$D-4N(\Phi(-1,i))=9\cdot 4^r.$$

Consequently it follows by Theorem 6.3 B, since $\mu=\mu(\Phi)\leqslant|\Phi(-1,i)|$, that

$$K(\overleftrightarrow{V_1E_1C^rV_3E_3C^r})\geqslant\sqrt{\frac{D}{N(\Phi(-1,i))}}=\sqrt{4+\frac{9\cdot 4^r}{N(\Phi(-1,i))}}>2.$$

LEMMA 6.18. *The regular double chain ΠT_n does not contain any of the following products as a subproduct,*

$$V_jE_jC^{2m-1}V_kE_k,\quad E_kV_kC^{2m-1}E_jV_j\quad (j\neq k).$$

Proof. Using congruence, we need only by Theorem 6.4 exclude

$$CV_3E_3C^{2m-1}V_1E_1C^{2s-1}X_1X_2X_3$$

with $X_1\neq C$ as a subproduct.

It is easily proved by means of Lemma 6.16 that in order to have

$$d(F(C^{2s-2}X_1X_2X_3),\ \overline{F^*}(CE_1V_1C^{2m-1}E_3V_3C))<2,$$

we must have $X_1\neq V_1$, $X_1X_2\neq E_kC$ $(k=2,3)$, $X_1X_2X_3\neq E_kV_kC$ $(k=2,3)$; hence by Theorem 6.4, $X_1\notin\{V_1,E_2,E_3\}$.

Similarly, in order to have

$$d(F(E_2V_2C^{2m-1}E_1V_1C),\ \overline{F^*}(C^{2s-1}Y_1Y_2Y_3))<2,$$

we must have $Y_1\neq V_3$, $Y_1Y_2\neq E_jC$ $(j=1,2)$, $Y_1Y_2Y_3\neq E_jV_jC$ $(j=1,2)$, and further $s\leqslant m$ in case $Y_1Y_2Y_3=V_1E_1C$. Using congruence, we conclude by Theorem 6.4 that $X_1\notin\{V_2,E_3,E_1\}$, and that $s\leqslant m$ in case $X_1X_2X_3=V_3E_3C$.

Repeating the argument above, we see that there exists a number $l \in \mathbf{N}$, $l \leqslant m$, such that

$$\prod T_n \equiv \dots \overline{V_1 E_1 C^{2l-1} V_3 E_3 C^{2l-\overrightarrow{1}}}.$$

However, this possibility is excluded by Lemma 6.17.

Combining Theorem 6.4 and Lemmas 6.8, 6.9, 6.10, 6.11, 6.12, 6.14, 6.15, 6.18, we obtain the following important improvement of Theorem 6.4.

THEOREM 6.5. *Suppose that the regular double chain ΠT_n satisfies the preliminary restrictions of Theorem* 6.4. *Then ΠT_n is congruent to*

$$\dots C^{2m_{-1}-1} T_1^{(-1)} \dots T_{k_{-1}}^{(-1)} C^{2m_0-1} T_1^{(0)} \dots T_{k_0}^{(0)} C^{2m_1-1} \dots,$$

where $m_r \in \mathbf{N}$ for all $r \in \mathbf{Z}$ and $T_1^{(r)} \dots T_{k_r}^{(r)}$ for each $r \in \mathbf{Z}$ is one of the three products

$$E_1,\ V_1 E_1,\ E_1 V_1.$$

6.4. The main theorem on the minimum of quadratic forms

THEOREM 6.6. *Let $\Phi = (A, B, C)$ be a complex quadratic form with discriminant $D = D(\Phi) \neq 0$ and minimum $\mu = \mu(\Phi)$.*

A. *If*

$$\sqrt{|D|}/\mu < 2, \tag{6.17}$$

then Φ is equivalent to a multiple of either

$$G = (-3i,\ 3i,\ 1-2i),\ \bar{G} = (3i,\ -3i,\ 1+2i)$$

or some G_Λ, where

$$G_\Lambda(X, Y) = g_\Lambda(2X - Y,\ iY),$$

the forms g_Λ being the C-minimal forms of chapter 5.

B. *Conversely* (6.17) *holds for all forms equivalent to a multiple of G, $\bar{G}$ or some G_Λ; specifically*

$$\sqrt{|D(G)|}/\mu(G) = \sqrt{|D(\bar{G})|}/\mu(\bar{G}) = \sqrt{\tfrac{4}{5}\sqrt{41}},$$

$$\sqrt{|D(G_\Lambda)|}/\mu(G_\Lambda) = \sqrt{4 - \Lambda^{-2}}.$$

C. *There are $2^{\aleph_0}$ forms Φ, none of which are equivalent to a multiple of any other, such that*

$$\sqrt{|D(\Phi)|}/\mu(\Phi) = 2.$$

Proof. To prove part A we notice, that Φ must have both roots inequivalent to real numbers, since otherwise $\sqrt{|D|}/\mu \geqslant \sqrt{5}$ by the Markoff-Hurwitz approximation theorem. Let ΠT_n by any regular double chain associated to Φ. By Theorem 6.3 A, we must have

$$K(\Pi T_n) \leqslant \sqrt{|D|}/\mu < 2,$$

and hence by Theorems 6.4, 6.5 either

$$\text{I. } \Pi T_n = \overleftrightarrow{CV_1E_1V_1CE_3CV_2E_2V_2CE_1CV_3E_3V_3CE_2},$$

$$\Pi T_n = \overleftrightarrow{CV_1E_1V_1CE_2CV_3E_3V_3CE_1CV_2E_2V_2CE_3},$$

or

$$\text{II. } \Pi T_n \equiv \Pi U_n, \text{ where } U_n \in \{V_1, E_1, C\} \text{ for all } n \in \mathbf{Z}.$$

Case I. (The asymmetric case.) It follows by Example 6.3 that Φ is equivalent to a multiple of $G, \bar{G}$, respectively. Notice that $G, \bar{G}$ are inequivalent, since $D(G) = 15+12i \neq \pm D(\bar{G}) = \pm(15-12i)$.

Case II. (The symmetric case.) Then Φ is equivalent to a form, where the roots ξ, η satisfy $\operatorname{Re}\xi = \operatorname{Re}\eta = \frac{1}{2}$ (cf. chapter 4), hence Φ is equivalent to a multiple of a form F, where

$$F(X, Y) = f(2X-Y, iY), \tag{6.18}$$

where $f=(\alpha, \beta, \gamma)$ is a real form with $\delta(f)>0$ and having irrational roots. Notice that F also satisfies (6.17), and that $D(F) = -4\delta(f)$.

If f is not C-equivalent to a multiple of some g_Λ or h_M, then $\sqrt{\delta(f)}/\nu(f) \geqslant 2$ by Theorem 5.3 A, hence either there exists $(x_0, y_0) \in \mathbf{Z}_0^2$, such that

$$\sqrt{\delta(f)}/|f(x_0,y_0)| > \sqrt{|D(F)|}/\mu(F) = 2\sqrt{\delta(f)}/\mu(F),$$

of there exists $(x_1, y_1) \in \mathbf{Z}_1^2$, such that

$$\sqrt{\delta(f)}/|2f(x_1,y_1)| > \sqrt{|D(F)|}/\mu(F) = 2\sqrt{\delta(f)}/\mu(F).$$

Consequently, either

$$\mu(F) > 2|f(x_0,y_0)| = |f((1+i)x_0, (1+i)y_0)| = \left|F\left(\frac{ix_0+y_0}{1+i}, (1-i)y_0\right)\right|,$$

or

$$\mu(F) > 4|f(x_1, y_1)| = |f(2x_1, 2y_1)| = |F(x_1 - iy_1, -2iy_1)|,$$

in contradiction to the definition of $\mu(F)$, since $(ix_0+y_0)/(1+i) \in \mathbf{Z}[i]$.

Suppose that $f \approx ch_{M_2}$ $(c \neq 0)$, where $(\Lambda_1, \Lambda_2; M_1, M_2)$ is any non-singular solution of equation (5.4) with $\Lambda_1 < \Lambda_2$, $M_1 < M_2$, and associated quintuple $(\varepsilon; l_1, l_2; m_1, m_2)$. Then $F \sim cH_{M_2}$, where $H_{M_2}(X, Y) = h_{M_2}(2X-Y, iY)$.

It follows easily by Lemma 5.6, that in case $\varepsilon = 1$, we have

$$|\mathrm{M}_2 H_{\mathrm{M}_2}(\tfrac{1}{2}(\Lambda_1 - 1 + l_1 \mathrm{i}), \Lambda_1)| = \Lambda_1 \sqrt{16\mathrm{M}_2^2 - 8}.$$

Also, it follows by Lemma 5.6 (vii) (ix), that Λ_1 and $1 - l_1 i$ have a common divisor ϱ (in $\mathbf{Z}[i]$) with $N(\varrho) = \Lambda_1$, in particular $(\Lambda_1 - 1 + l_1 i)/\varrho \in \mathbf{Z}[i]$; however, since $N((\Lambda_1 - 1 + l_1 i)/\varrho) = (\Lambda_1^2 - 2\Lambda_1 + 1 + l_1^2)/\Lambda_1 = \Lambda_1 - 2 + \lambda_1 \equiv 1 - 2 + 1 = 0 \pmod 4$ by Lemma 5.6 (vii), also $(\Lambda_1 - 1 + l_1 i)/(2\varrho) \in \mathbf{Z}[i]$.

Consequently, for $\mathrm{M} = \mathrm{M}_2$ and $\varepsilon = 1$, we have

$$\mu(\mathrm{M}H_{\mathrm{M}}) \leqslant \sqrt{16\mathrm{M}^2 - 8}. \tag{6.19}$$

In case $\mathrm{M} = \mathrm{M}_2$ and $\varepsilon = -1$, we can prove (6.19) similarly, using that

$$|\mathrm{M}_2 H_{\mathrm{M}_2}(\tfrac{1}{2}(\Lambda_1 - 1 + (l_1 - 4\Lambda_1)\, i), \Lambda_1)| = \Lambda_1 \sqrt{16\mathrm{M}_2^2 - 8}.$$

Finally, (6.19) is valid for $\mathrm{M} = 1$, since

$$H_1(X, Y) = h_1(2X - Y, iY) = 2(1 - i)((1 + i)X^2 - 2XY + Y^2).$$

Using (5.7) and (6.19), we see that in case $f \approx ch_{\mathrm{M}}$ (and hence $F \sim cH_{\mathrm{M}}$), we have

$$\sqrt{|D(F)|}/\mu(F) = \sqrt{|D(\mathrm{M}H_{\mathrm{M}})|}/\mu(\mathrm{M}H_{\mathrm{M}}) = 2\sqrt{\delta(\mathrm{M}h_{\mathrm{M}})}/\mu(\mathrm{M}H_{\mathrm{M}}) \geqslant 2.$$

This completes the proof of part A.

Part B. It follows from Example 6.4 that

$$\sqrt{|D(\bar{G})|}/\mu(\bar{G}) = \sqrt{|D(G)|}/\mu(G) = \sqrt{\tfrac{3}{8}\sqrt{41}}.$$

Also it follows from (5.7) that

$$\sqrt{|D(\Lambda G_\Lambda)|} = 2\sqrt{\delta(\Lambda g_\Lambda)} = 4\Lambda\sqrt{4 - \Lambda^{-2}},$$

hence to prove B we need only show that

$$\mu(\tfrac{1}{4}\Lambda G_\Lambda) = \Lambda. \tag{6.20}$$

By Lemma 5.6, the form

$$\begin{aligned}\tfrac{1}{4}\Lambda G_\Lambda(X, Y) &= \Lambda X^2 - (\Lambda - i(2\Lambda - l))\, XY + ((\Lambda - \lambda)/4 + l - i(\Lambda - l/2))\, Y^2 \\ &= \Lambda(X - \tfrac{1}{2}(1 + i\vartheta_1(g_\Lambda))\, Y)\,(X - \tfrac{1}{2}(1 + i\vartheta_2(g_\Lambda))\, Y)\end{aligned}$$

has coefficients in $\mathbf{Z}[i]$, and it follows easily from Theorem 5.3 B by considering the two cases $\operatorname{Re}(X/Y) = \tfrac{1}{2}$ and $\operatorname{Re}(X/Y) \neq \tfrac{1}{2}$ separately, that

$$\tfrac{1}{4}|D(\tfrac{1}{4}\Lambda G_\Lambda)| = \Lambda^2 - \tfrac{1}{4} \leqslant N(\mu(\tfrac{1}{4}\Lambda G_\Lambda)) \leqslant \Lambda^2 = N(\tfrac{1}{4}\Lambda G_\Lambda(1, 0)).$$

This yields (6.20), since $N(\mu(\tfrac{1}{4}\Lambda G_\Lambda)) \in \mathbf{Z}$.

Part C follows similarly from Theorem 5.3 C by considering forms F, where $F(X, Y) = f(2X - Y, iY)$, and f is any form with $\sqrt{\delta(f)}/\nu(f) = 2$.

6.5. The main theorem on the approximation constant

THEOREM 6.7. *Let* $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$.

A. *If*

$$C(\xi) < 2, \tag{6.21}$$

then ξ *is equivalent to a root of either* G, $\bar{G}$ *or some* G_Λ.

B. *Conversely if* ξ *is equivalent to a root of* G *or* $\bar{G}$, *then*

$$C(\xi) = \sqrt{\tfrac{3}{5}\sqrt{41}},$$

and if ξ *is equivalent to a root of* G_Λ, *then*

$$C(\xi) = \sqrt{4 - \Lambda^{-2}}.$$

C. *There are* $2^{\aleph_0}$ *different equivalence classes of* ξ *such that* $C(\xi) = 2$.

Remark. Since no isolation theorem (like Theorem I, chapter II of [4] or Theorem 5.1) is available for complex forms, Theorem 6.7 cannot be deduced from Theorem 6.6 (cf. the proofs of Theorem III, chapter II of [4] or Theorem 5.4).

Proof. Suppose $\xi \in \mathbf{C} \setminus \mathbf{Q}(i)$ with $C(\xi) \leqslant 2$. Then ξ is inequivalent to a real number, since otherwise $C(\xi) \geqslant \sqrt{5}$ by the theorem of Hurwitz ([19]). Hence by Theorem 2.2, $\xi = \xi_0$ has a unique regular chain

$$\operatorname{ch} \xi = T_1 T_0 \dots T_n \dots.$$

Also by Theorem 2.6

$$C(\xi) = \limsup c_j^{(n)},$$

where $c_j^{(n)}$ for $1 \leqslant j \leqslant 3$ is given by (2.9), (2.10), (2.11).

By putting

$$\zeta_{n+1} = -q_2^{(n)}/q_1^{(n)},$$

we may rewrite the $c_j^{(n)}$ as follows,

$$c_1^{(n)} = |\xi_{n+1} - \zeta_{n+1}|,\ c_2^{(n)} = |\xi_{n+1}^{-1} - \zeta_{n+1}^{-1}|,\ c_3^{(n)} = |(\xi_{n+1} - 1)^{-1} - (\zeta_{n+1} - 1)^{-1}|.$$

From

$$T_0 T_1 \dots T_n = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} \\ q_1^{(n)} & q_2^{(n)} \end{pmatrix}$$

we get by taking inverses and using Lemma 1.1 (iv), that

$$\begin{pmatrix} q_2^{(n)} & -p_2^{(n)} \\ -q_1^{(n)} & p_1^{(n)} \end{pmatrix} = \pm\,\varepsilon_n\,\overline{T_n}\ldots\overline{T_1}\,\overline{T_0}.$$

Consequently for any h, $k \in \mathbf{N}(k \leqslant n)$, we have

$$\xi_{n+1} = [T_{n+1}T_{n+2}\ldots] \in \begin{cases} F(T_{n+1}\ldots T_{n+h}) & \text{if } \varepsilon_n = \pm 1 \\ F^*(T_{n+1}\ldots T_{n+h}) & \text{if } \varepsilon_n = \pm i \end{cases},$$

$$\zeta_{n+1} = -q_2^{(n)}/q_1^{(n)} \in \begin{cases} \overline{F}(T_n\ldots T_{n-k}) & \text{if } \varepsilon_n = \pm i \\ \overline{F^*}(T_n\ldots T_{n-k}) & \text{if } \varepsilon_n = \pm 1 \end{cases}.$$

Because of the resemblance between the formulae for

$$C([T_0T_1\ldots T_n\ldots]) \quad \text{and } K(\Pi T_n),$$

it is obvious, that the content of section 6.3 can be modified to yield the following result:

If $C(\xi) < 2$, then either

I. $\operatorname{ch}\xi = T_0\ldots T_h\overrightarrow{CV_1E_1V_1CE_3CV_2E_2V_2CE_1CV_3E_3V_3CE_2},$

$\operatorname{ch}\xi = T_0\ldots T_h\overrightarrow{CV_1E_1V_1CE_2CV_3E_3V_3CE_1CV_2E_2V_2CE_3},$

or

II. $\xi \sim \eta = [U_0U_1\ldots U_n\ldots]$, where $U_n \in \{V_1, E_1, C\}$ for all $n \in \mathbf{N}_0$.

Case I. (The asymmetric case.) By Example 6.3 and Theorems 2.2, 2.4, ξ is necessarily equivalent to a root of either G or $\overline{G}$. Conversely if ξ is equivalent to a root of either G or $\overline{G}$, then by Theorem 2.7 and Theorem 6.6, we have

$$C(\xi) = \sqrt{\tfrac{3}{8}\sqrt{41}}.$$

Case II. (The symmetric case.) Then (cf. chapter 4) necessarily $\xi \sim \frac{1}{2}(1+i\alpha)$, where $\alpha \in \mathbf{R}\setminus\mathbf{Q}$. For convenience we put

$$C_1(\tfrac{1}{2}(1+i\alpha)) = \limsup\,(|q|\,|\tfrac{1}{2}(1+i\alpha)q-p|)^{-1},$$

the lim sup being extended over all $p, q \in \mathbf{Z}[i]$ $(q \neq 0)$ with $\operatorname{Re}(p/q) = \frac{1}{2}$.

Similarly we put

$$C_2(\tfrac{1}{2}(1+i\alpha)) = \limsup\,(|q|\,|\tfrac{1}{2}(1+i\alpha)q-p|)^{-1},$$

the lim sup being extended over all $p, q \in \mathbf{Z}[i]$ $(q \neq 0)$ with $\operatorname{Re}(p/q) \neq \frac{1}{2}$.

We claim that for any $\alpha\in\mathbf{R}\setminus\mathbf{Q}$, we have

$$C_1(\tfrac{1}{2}(1+i\alpha))=d(\alpha), \tag{6.22}$$

where $d(\alpha)$ is the C-approximation constant of α, and

$$C_2(\tfrac{1}{2}(1+i\alpha))\leqslant 2. \tag{6.23}$$

Suppose first that $\pi/\varrho\in\mathbf{Q}_0$ (cf. section 4.2), then

$$\tfrac{1}{2}(1+i(\pi/\varrho))=(\varrho+\pi i)/(2\varrho),$$

where

$$N(\varrho+\pi i)=\pi^2+\varrho^2\equiv 2 \pmod 4),\ N(2\varrho)=2\cdot 2\varrho^2.$$

By putting $p=(\varrho+\pi i)/(1-i)$ and $q=\varrho(1+i)$, we obtain, since $\gcd(\pi,\varrho)=1$, that

$$p,\ q\in\mathbf{Z}[i],\ \gcd(p,q)=1,\ \tfrac{1}{2}(1+i(\pi/\varrho))=p/q.$$

Also

$$(|q|\,|\tfrac{1}{2}(1+i\alpha)q-p|)^{-1}=(|\varrho|\,|\alpha\varrho-\pi|)^{-1}.$$

Suppose next that $\pi/\varrho\in\mathbf{Q}_1$, then

$$\tfrac{1}{2}(1+i(\pi/\varrho))=(\varrho+\pi i)/(2\varrho),$$

where

$$N(\varrho+\pi i)=\pi^2+\varrho^2\equiv 1 \pmod 4),\ N(2\varrho)=4\varrho^2.$$

By putting $p=\varrho+\pi i$ and $q=2\varrho$, we obtain, since $\gcd(\pi,\varrho)=1$, that

$$p,\ q\in\mathbf{Z}[i],\ \gcd(p,q)=1,\ \tfrac{1}{2}(1+(\pi/\varrho))=p/q.$$

Also

$$(|q|\,|\tfrac{1}{2}(1+i\alpha)q-p|)^{-1}=\tfrac{1}{2}(|\varrho|\,|\alpha\varrho-\pi|)^{-1}.$$

Altogether this proves (6.22).

If $\operatorname{Re}(p/q)\neq\frac{1}{2}$ and $p,q\in\mathbf{Z}[i]$, then obviously $p/q=a+ib$ with $a,b\in\mathbf{Q}$, and $|a-\frac{1}{2}|\geqslant 1/(2|q|^2)$. This proves (6.23).

It follows by (6.22) that if $\xi\sim\frac{1}{2}(1+i\alpha)$ has $C(\xi)<2$, then necessarily $d(\alpha)<2$. By Theorem 5.4 A, α must be a root of some g_Λ or h_M, hence $\frac{1}{2}(1+i\alpha)$ must be a root of some G_Λ or H_M. What remains to be proved of parts A and B follows thus by Theorems 2.6 and 6.6.

It follows from (6.22), (6.23) and Theorem 5.4 C that $C(\frac{1}{2}(1+i\alpha))=2$ for any α with $d(\alpha)=2$. This proves part C.

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