THE CURRENTS DEFINED BY ANALYTIC VARIETIES

BY

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Introduction

This paper is concerned with integration on complex analytic spaces and the (De Rham) currents defined by such integration. It contains results about continuity of fibering and intersection of such currents. It also states sufficient conditions for a current to be defined by integration over a complex subvariety.

The methods of proof involve extensive use of the theory relating to the integral currents of H. Federer and W. Fleming, as it is developed in Federer's treatise *Geometric measure theory* [7]. Since this theory is unfamiliar to most complex analysts, we have stated necessary results in Chapter Two and in Section 5.1. Few theorems are needed for the fibering result, but Chapter Five uses the theory in more detail.

Each chapter has a brief introduction, but we will state the main results here.

The fibering theorem of Section 3.3 is stated for holomorphic maps $f\colon X^m \to Y^n$ between complex analytic spaces X^m and Y^n, where Y^n is locally irreducible and the complex dimension of the fibers $f^{-1}(y)$ equals $m-n$ for all y in Y. If u is a continuous $(2m-2n)$ form with compact support on X^m, u defines a current $[u]$ and for any holomorphic f the current $f_*[u]$ on Y is defined. If f satisfies the hypothesis above, we prove that $f_*[u] = [\lambda]$, the current defined by a continuous function λ on Y; moreover, $\lambda(y) = \langle [X], f, y\rangle (u)$, where $\langle [X], f, y\rangle$ is a current defined by integration on $f^{-1}(y)$, with suitable multiplicities.

The motivation for this theorem was the study of algebraic or analytic cycles (in this paper called holomorphic cycles to avoid confusion with real analytic sets) by means of De Rham cohomology and currents in [11], where it was necessary to study intersection and fibering. Similar results have been obtained by others: the proof which inspired the one in this paper is that of H. Federer [8], who treats complete projective varieties. If X is a manifold and Y is normal, the result may be found in W. Stoll [23], which includes a

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more general theorem for forms u with certain singularities. Another theorem of the same nature for algebraic varieties is contained in a paper of A. Andreotti and F. Norguet [3].

In Section 5.2 we prove that a closed positive locally integral current on a complex manifold (or complex space) is the current defined by integration on a complex subvariety (with multiplicities). This is used in Section 5.4 to prove two theorems of B. Shiffman on extension of analytic varieties [19], [20], and [21].

In Section 5.3 we draw the same conclusion for a closed positive current in an open set in $\mathbf{C}^n$ if the Lelong number at each point (or almost every point in the appropriate Hausdorff measure) of the support is a positive integer. This gives a partial answer to a question of Lelong [14], p. 2–07.

The result in 5.2 is a very special and tractable case of a more general problem in the theory of integral currents, that of regularity of minimal currents (i.e., solutions of Plateau's problem.) A closed positive rectifiable current in a Kähler manifold is minimal, and in this case Theorem 5.3.1 gives a precise statement about the nature of the singular locus.

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Partial list of notation

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$J_X A^{p,q}(V), d, d', d''$

1.2. $\mathcal{D}'_r(M), \mathcal{E}'_r(V), \mathcal{D}'_{p,q}(X)$, etc.

1.3. $[u], [c]$

1.4. $f_*, \mathcal{D}'_r(M)_f, b$, integration along the fiber, closed and exact currents, $T \llcorner u, T_1 \times T_2$

2.1. currents of order zero or representable by integration,
$\|u\|(x), \nu_K(u), \mathbf{M}(T); N^{\mathrm{loc}}_r(M), R_r(V), F^{\mathrm{loc}}_r(M), I_r(X)$, etc.

2.2. good coordinate cover $\mathcal{U}$, F and $\mathcal{F}$ topologies

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4.1. $S\cdot T$, $i(S, T, E)$

4.2. Lelong number $n(x, T)$

5.1. (ϕ, k) rectifiable set, $\mathrm{Tan}^k(H^k \llcorner E, x)$, $J_m f$, $\theta^k(\|S\|, x)$, $H^k \llcorner B) \wedge \eta$.

Forms and currents

This section recalls the definitions of the standard mappings of forms and currents on manifolds and also on complex analytic spaces. General references for this section are the books of Schwartz [18], Chapter IX, and De Rham [15] for currents and that of Weil [28] for complex manifolds. Complex analytic spaces are studied in Gunning and Rossi [9].

Notation. M and N will denote oriented real C^∞ manifolds without boundary. V and W will be complex manifolds, and X and Y will be (reduced) complex analytic spaces (complex spaces for short). We will denote that M (V or X) is of real (complex) dimension k by writing M^k (V^k or X^k).

1.1. Forms

Let $A^r(M)$ be the space of *complex valued* r-forms on M and let $A^r_c(M) \subset A^r(M)$ be the r-forms with compact support. We write write $A^r(V) = \sum_{p+q=r} A^{p,q}(V)$, where $A^{p,q}(V)$ is the space of (p, q) forms, and

$$A_c^{p,q}(V) = A^{p,q}(V) \cap A_c^{p+q}(V).$$

The operator $d: A^r(M) \to A^{r+1}(M)$ is the usual exterior differentiation; and on V, $d = d' + d''$, where d': $A^{p,q}(V) \to A^{p+1,q}(V)$ and d'': $A^{p,q}(V) \to A^{p,q+1}(V)$. If f: $M \to N$ is a C^∞ map, f^*: $A^r(N) \to A^r(M)$ is defined as usual; if f: $V \to W$ is holomorphic, $f^*(A^{p,q}(W)) \subset A^{p,q}(V)$. The maps d and f^* commute.

The corresponding spaces of continuous forms will be written as $CA^r(M)$, $CA_c^{p,q}(V)$, etc.

If X is a complex analytic space, we wish to define $A^{p,q}(X)$ as it is done in Bloom–Herrera [4].

Let $S(X)$ be the singular locus of X and $R(X) = X - S(X)$ be the manifold of regular points.

Suppose first that X is a subvariety of a complex manifold V. Let $J_X A^{p,q}(V) = \{u \in A^{p,q}(V): i^*u = 0 \text{ in } A^{p,q}(R(X))\}$, where i: $R(X) \to V$ is the inclusion map. Then we define $A^{p,q}(X)$ to be $A^{p,q}(V)/J_X A^{p,q}(V)$. It can be shown that $A^{p,q}(X)$ is independent of the imbedding; therefore, $A^{p,q}(X)$ may be defined for any complex analytic space X by using the local imbeddings in $\mathbf{C}^n$ given in the definition. (To be precise, we should define a sheaf of germs of (p, q) forms on X and take global sections.) We also define $A^r(X)$, $A_c^{p,q}(X)$, $CA^{p,q}(X)$, $CA_c^{p,q}(X)$ in an analogous manner.

If f: $X \to Y$ is a holomorphic map between complex spaces, a map f^*: $A^{p,q}(Y) \to A^{p,q}(X)$ is defined. Since $d'(J_X A^{p,q}(V)) \subset J_X A^{p+1,q}(V)$, etc., the operators d: $A^r(X) \to A^{r+1}(X)$, d': $A^{p,q}(X) \to A^{p+1,q}(X)$, and d'': $A^{p,q}(X) \to A^{p,q+1}(X)$ are defined; f^* commutes with d, d', d''.

1.2. Currents

The spaces $A^r(M)$ and $A^r_c(M)$ can be given their usual topologies, which make them locally convex topological vector spaces. We will recall the definition of convergence for sequences in these spaces.

A sequence (or net) $u_1, u_2, \ldots$ in $A^r(M)$ is said to converge to a form u in $A^r(M)$ if for every coordinate neighborhood the sequences of component functions of the sequence $\{u-u_i\}$ and also the sequences of higher derivatives of these component functions converge to zero uniformly on each compact set of the coordinate neighborhood. If $u, u_1, u_2, \ldots$ are in $A^r_c(M)$, $u_i \to u$ in $A^r_c(M)$ if there is a compact set $K \subset M$ such that supp $u_i \subset K$ for all i (supp u = support of u) and $u_i \to u$ in $A^r_c(M)$.

Definition 1.2.1. A (complex valued) *current of dimension* r on M is a complex-linear functional T: $A^r_c(M) \to \mathbf{C}$ which is continuous in the topology described above. The space of these currents will be denoted by $\mathcal{D}'_r(M)$.

If $T \in \mathcal{D}'_r(M)$, the *support* of T, supp T, is the closed subset of M with the following property: $x \notin$ supp T if and only if there is an open set $U \subset M$ containing x such that for $u \in A^r_c(M)$, supp $u \subset U$ implies $T(u) = 0$.

Definition 1.2.2. $\mathcal{E}'_r(M)$ is the space of continuous complex-linear functionals T: $A^r(M) \to \mathbf{C}$.

Since the inclusion $A^r_c(M) \to A^r(M)$ is a continuous map onto a dense subspace of $A^r(M)$, every $T \in \mathcal{E}'_r(M)$ induces a unique $\varrho(T) \in \mathcal{D}'_r(M)$ by restriction. It can be shown that the image of this restriction map is the subspace of currents in $\mathcal{D}'_r(M)$ with compact support; we will often identify $\mathcal{E}'_r(M)$ with this space (but not with the induced topology).

Sometimes it will be convenient to speak of the *degree* of a current $T \in \mathcal{D}'_r(M^m)$; this is defined to be dimension M^m − dimension $T = m - r$; we write $\mathcal{D}'^{m-r}(M^m) = \mathcal{D}'_r(M^m)$ and $\mathcal{E}'^{m-r}(M^m) = \mathcal{E}'_r(M^m)$.

We can define $\mathcal{D}'_r(X)$, when X is a subvariety of a complex manifold V, as the subspace of $\mathcal{D}'_r(V)$ which annihilates $J_X A^r(V)$: $T \in \mathcal{D}'_r(X) \subset \mathcal{D}'_r(V)$ if and only if $T(u) = 0$ for any $u \in J_X A^r(V)$. (N.b. $T \in \mathcal{D}'_r(X)$ implies supp $T \subset X$ but not conversely; see Example 1–3 (*c*).)

As in the case of $A^r(X)$, we can extend the definition of $\mathcal{D}'_r(X)$ to any complex space X. The support of a current is defined as before, and $\mathcal{E}'_r(X)$ is the subspace of currents in $\mathcal{D}'_r(X)$ with compact support.

Definition 1.2.3. If X is a complex space of dimension n (perhaps a complex manifold), then we define currents of type (p, q): $\mathcal{D}'_{p,q}(X) = \{T \in \mathcal{D}'_{p+q}(X)$: $T(u) = 0$ for $u \in A^{r,s}_c(X)$, $r \neq p\}$

$$\mathcal{E}'_{p,q}(X) = \mathcal{D}'_{p,q}(X) \cap \mathcal{E}'_{p+q}(X)$$

$$\mathcal{D}'^{\,n-p,\,n-q}(X) = \mathcal{D}'_{p,q}(X)$$

$$\mathcal{E}'^{\,n-p,\,n-q}(X) = \mathcal{E}'_{p,q}(X).$$

1.3. Examples

(a) $CA^r(M^m)$ can be identified with a subset of $\mathcal{D}'^r(M^m)$. For $u \in CA^r(M^m)$ we define the current $[u]$ by setting $[u](v) = \int_M u \wedge v$ for $v \in A_c^{m-r}(M^m)$. Since $\operatorname{supp} u = \operatorname{supp}[u]$, $[u] \in \mathcal{E}'^r(M)$ when $u \in CA_c^r(M)$. Also, $u \in CA^{p,q}(X)$ implies $[u] \in \mathcal{D}'^{p,q}(X)$.

(b) Singular chains define currents. If c is a locally finite C^∞ singular chain of dimension r, we define $[c](u) = \int_c u$ for $u \in A_c^r(M)$. An oriented submanifold may be considered to be a singular chain; thus it defines a current by integration. An important example of this is the *Dirac δ-function* at x, δ_x, for $x \in M$; this element of $\mathcal{E}_0'(M)$ is defined by $\delta_x(f) = f(x)$ for $f \in A^0(M)$. (c) A final example is the current $T \in \mathcal{E}_0'(\mathbf{R})$ defined by $T(f) = f'(0)$. Currents of this kind will be excluded from the spaces studied in Section 2.

1.4. Mappings of spaces of currents

Several important mappings of currents are defined as adjoints of mappings of forms.

Definition 1.4.1. If $f\colon M \to N$ is a C^∞ map, the *direct image map* $f_*\colon \mathcal{E}_r'(M) \to \mathcal{E}_r'(N)$ is a linear map defined as the adjoint of $f^*\colon A^r(N) \to A^r(M)$, i.e., $f_*T(u) = T(f^*u)$ for $u \in A^r(N)$ and $T \in \mathcal{E}_r'(M)$. If $f\colon X \to Y$ is holomorphic, $f_*\colon \mathcal{E}_{p,q}'(X) \to \mathcal{E}_{p,q}'(Y)$ is defined in the same way.

If $f\colon M \to N$ is proper (f^{-1}(a compact) is compact), $f^*(A_c^r(N)) \subset A_c^r(M)$; and f_* can be extended to a map $f_*\colon \mathcal{D}_r'(M) \to \mathcal{D}_r'(N)$. More generally, if $f|\operatorname{supp} T\colon \operatorname{supp} T \to N$ is proper, for $T \in \mathcal{D}_r'(M)$, then $f_*T \in \mathcal{D}_r'(N)$ may be defined by $f_*T(u) = T(f^*u)$ for $u \in A_c^r(N)$, since $\operatorname{supp} T \cap \operatorname{supp} f^*u$ is compact ([18], p. 364).

Definition 1.4.2. If $f\colon M \to N$ is a C^∞ mapping, let $\mathcal{D}_r'(M)_f$ be $\{T \in \mathcal{D}_r'(M)\colon f|\operatorname{supp} T$ is proper$\}$. By the remark above, $f_*\mathcal{D}_r'(M)_f \to \mathcal{D}_r'(N)$ is well-defined. Similarly, if $f = X \to Y$ is holomorphic, we may define $\mathcal{D}_{p,q}'(X)_f$ and extend f_* to this space. In general for any subspace $L \subset \mathcal{D}_r'(M)$ or $\mathcal{D}_r'(X)$, we shall set $L_f = \{T \in L\colon f|\operatorname{supp} T$ is proper$\}$—for example, $A^k(M)_f$ or $CA^k(X)_f$.

Example 1.4.3

(a) If $[c] \in \mathcal{E}_r'(M)$ is given by the singular chain c, $f_*[c]$ is the current defined by the chain f_*c.

(b) If $f\colon M^m \to N^n$ is any C^∞ map, $f_*(A^r(M)_f) \subset f_*(\mathcal{D}'^r(M)_f) \subset \mathcal{D}'^{r-k}(N)$, where $k = m - n$. However, if the rank of the differential Df_x equals n, the dimension of N, for all $x \in M$, then actually $f_*(A^r(M)_f) \subset A^{r-k}(N)$. In this case the map f_* is called *integration along the fiber*. For each $y \in N$, $f^{-1}(y) = M_y$ is an oriented k-dimensional submanifold of M; if $r = k$ and $u \in A^r(M)_f$, the function $f_*u \in A^0(N)$ is given by $f_*u(y) = \int_{M_y} u$ ([18], pp. 390–391]. Integration along the fiber for complex spaces will be discussed in Section 3.3.

Definition 1.4.4. The *boundary operator* b: $\mathcal{D}'_{r+1}(M) \to \mathcal{D}'_r(M)$ is the adjoint of the exterior derivative d: $A^r_c(M) \to A^{r+1}_c(M)$: for $T \in \mathcal{D}'_{r+1}(M)$ and $u \in A^r_c(M)$, $bT(u) = T(du)$.

If T is the current $[v]$ defined by a form $v \in A^{m-r-1}(M^m)$, $b[v] = (-1)^{m-r}[dv]$, since $b[v](u) = [v](du) = \int_M v \wedge du = (-1)^{m-r}\int_M dv \wedge u + \int_M d(v \wedge u) = (-1)^{m-r}\int_M dv \wedge u = (-1)^{m-r}[dv](u)$ by the Stokes Theorem. Therefore, b: $\mathcal{D}'^s(M) \to \mathcal{D}'^{s+1}(M)$ is an extension of $(-1)^{s+1}d$: $A^s(M) \to A^{s+1}(M)$, and we can define d: $\mathcal{D}'^s(M) \to \mathcal{D}'^{s+1}(M)$ by $d = (-1)^{s+1}b$, which gives an extension of the usual d.

For complex manifolds V or complex spaces X, b and d can be defined as above, although the fact that $b[v] = (-1)^{s+1}[dv]$ for $v \in A^s(X)$ depends on Theorem 3.1.1. In these cases d': $\mathcal{D}'^{p,q}(X) \to \mathcal{D}'^{p+1,q}(X)$ and d'': $\mathcal{D}'^{p,q}(X) \to \mathcal{D}'^{p,q+1}(X)$ can be defined by $d = d' + d''$.

Since $df^*u = f^*du$, the operator b commutes with f_*. A current T is *closed* if $bT = 0$ and *exact* if $T = bS$ for some S.

Definition 1.4.5. If $T \in \mathcal{D}'_r(M)$ and $u \in CA^s(M)$, we may define the product $T \llcorner u \in \mathcal{D}'_{r-s}(M)$ by $T \llcorner u(v) = T(u \wedge v)$. This definition also applies to currents and forms on a complex space X. ($T \llcorner u$ is often denoted by $T \wedge u$).

We observe that if w is a form and $T = [w]$, $[w] \llcorner u = [w \wedge u]$. Also, if $T \in \mathcal{D}'_{p,q}(X)$ and $u \in A^{s,t}(X)$, $T \llcorner u \in \mathcal{D}_{p-s,q-t}(X)$.

Definition 1.4.6. If $T_i \in \mathcal{D}'_{r_i}(M_i)$, $i = 1, 2$, the *Cartesian product* $T_1 \times T_2$ of T_1 and T_2 is the unique current in $\mathcal{D}'_{r_1+r_2}(M_1 \times M_2)$ such that $T_1 \times T_2(p_1^*(u_1) \wedge p_2^*(u_2)) = T_1(u_1)\,T_2(u_2)$, where $p_i = M_1 \times M_2 \to M_i$ are the projections and $u_i \in A^{s_i}_c(M_i)$. ([7], p. 360.)

2. Geometric currents

This section discusses certain subsets of the whole space of currents which will be useful for studying integration on complex spaces.

Integral currents are a group of currents T such that T and bT are generalized singular chains with integer coefficients (*rectifiable currents*). These are contained in the space of *normal currents*—currents T such that T and bT are vector-valued measures. A still larger space is the space of *flat currents*, which includes the currents $[u]$ defined by locally integrable forms.

Section 2.1 defines these currents and states the *Support Theorem*, which says that a flat current supported on a submanifold comes from a current on the submanifold. Section 2.2 is devoted to the *Compactness Theorem* which makes possible normal families type arguments for normal and integral currents. Section 2.3 studies the fibering of these currents and includes the *Slicing Theorem*, which defines fibers almost everywhere. Finally, Section

2.4 defines Hausdorff measure and states a *Measure Support Theorem* and other useful results involving Hausdorff measure.

The general reference for all this material is Federer's *Geometric Measure Theory* [7]. A brief study of the subject is given in Almgren [2]. Most of the theorems stated here are found in a more general form in [7], but the full strength of these results will not be needed in this paper.

2.1. Normal, flat, and integral currents

The space $CA^r(M)$ may be given a topology by defining convergence to be uniform convergence on compact sets of component functions in any local coordinate system. With this topology the inclusion map i: $A^r(M) \to CA^r(M)$ is a continuous map with dense image; consequently, the dual space of continuous linear functionals on $CA^r(M)$ is mapped injectively by the adjoint of i onto a subspace of $\mathcal{E}'_r(M)$.

The space $CA^r_c(M)$ can be topologized so that a sequence $u_1, u_2, \ldots$ is convergent whenever there is a compact set $K \subset M$ with $\operatorname{supp} u_j \subset K$ for all j and the sequence converges in the topology on $CA^r(M)$.

The dual space of $CA^r_c(M)$ may be mapped injectively onto a subspace of $\mathcal{D}'_r(M)$. These maps will be used as identification, as stated in the following definition:

Definition 2.1.1. A current $T \in \mathcal{D}'_r(M)$ is said to be of *order zero* or *representable by integration* if it can be extended to a continuous linear functional (necessarily unique) on $CA^r_c(M)$.

The term representable by integration is appropriate since by the Riesz representation theorem such a current is a vector-valued measure. If $T \in \mathcal{E}'_r(M)$ is of order zero, T defines a continuous linear functional on $CA^r(M)$. ([18], pp. 24–25, 89.)

In studying such currents it is useful to define certain norms. Let us suppose that M is given a Riemannian metric. If γ is a multivector at $x \in M$, we denote by $|\gamma|_x$ the length of γ given by the metric.

Definition 2.1.2. If $u \in CA^r(M)$, $\|u\|$ is a continuous function on M defined by

$$\|u\|(x) = \sup \{|u(\gamma)|: \gamma \text{ is a decomposable } r\text{-vector at } x \text{ and } |\gamma|_x \leqslant 1\}.$$

For a set $K \subset M$ the *comass* of u on $K, \nu_K(u) = \sup \{\|u\|(x): x \in K\}$. If $T \in \mathcal{D}'_r(M)$ the *mass* of T is defined as $\mathbf{M}(T) = \sup \{|T(u)|: u \in A^r_c(M) \text{ and } \nu_M(u) \leqslant 1\}$. We say T has *finite mass* if $\mathbf{M}(T) < +\infty$.

Notice that the topology on $A^r(M)$ given by the seminorms ν_K, for K compact, is the one already introduced. Therefore, if $T \in \mathcal{E}'_r(M)$ is of order zero, $\mathbf{M}(T) = \sup \{|T(u)|:$

$\nu_{\operatorname{supp} T}(u) \leqslant 1\} < \infty$. If $T \in \mathcal{D}'_r(M)$ is representable by integration, then T has locally finite mass, i.e., for each $x \in M$, there is a $T_x \in \mathcal{D}'_r(M)$ such that $x \notin \operatorname{supp}(T - T_x)$ and $\mathbf{M}(T_x) < \infty$.

Definition 2.1.3. The space of *locally normal currents* of dimension r on M, $N_r^{\mathrm{loc}}(M)$, is the set of $T \in \mathcal{D}'_r(M)$ such that T and bT are both respresentable by integration. The *normal currents* $N_r(M) = N_r^{\mathrm{loc}}(M) \cap \mathcal{E}'_r(M)$.

If $T \in N_r(M)$, then $\mathbf{M}(T) + \mathbf{M}(bT) < \infty$.

Examples. The (a) and (b) examples of 1.3 are representable by integration; (c) is not. If $v \in A^r(M^m)$, $[v] \in N_{m-r}^{\mathrm{loc}}(M)$ but not if v is merely continuous. Example (b) is also locally normal.

We also wish to define a subgroup of $N_r^{\mathrm{loc}}(M)$ which consists of generalized singular chains with integer coefficients.

Recall that a map $f: A \to B$ between two metric spaces is Lipschitzian if and only if there is a number $C > 0$ such that dist $(f(a), f(b)) \leqslant C$ dist (a, b) for all $a, b \in A$. If $P \in \mathcal{E}'_r(U)$ is a finite integral polyhedral r-chain in some open $U \subset \mathbf{R}^s$ (i.e., P is the current defined by a finite sum of oriented linear simplices) and if f is a Lipschitzian map from U to a Riemannian manifold N, then we can define a current $f_*(P) \in \mathcal{E}'_r(N)$ by approximating f by C^1 functions f_i and taking a limit of $f_{i*}(P)$ (See Federer, [7], pp. 370–371, for details).

Definition 2.1.4. If $K \subset N$ is compact, we define $R_{r,K}(N)$, the rectifiable r-currents in K as follows: $T \in R_{r,k}(N)$ if and only if $T \in \mathcal{E}'_r(N)$ and for every $\varepsilon > 0$ there is an open set U of some $\mathbf{R}^s$, a Lipschitzian map $f: U \to N$ and a finite integral polyhedral r-chain P with $f(\operatorname{supp} P) \subset K$ such that $\mathbf{M}(T - f_* P) < \varepsilon$. (Thus the rectifiable currents are a completion of the group of Lipschitz chains.) We define $R_r(N)$, the *rectifiable r-curents* in N as $\bigcup R_{r,K}(N)$, where the union is taken over all compact $K \subset N$. The *locally rectifiable r-currents* in M, $R_r^{\mathrm{loc}}(N)$, is $\{T \in \mathcal{D}'_r(N)$: for each $x \in N$ there is a $T_x \in R_r(N)$ such that $x \in \operatorname{supp}(T - T_x)\}$, i.e., T agrees with T_x near x.

Rectifiable currents are representable by integration, but in general this is not true of their boundaries. We define a subgroup which has this property.

Definition 2.1.5. The group of *locally integral currents* $I_r^{\mathrm{loc}}(N) = \{T \in R_r^{\mathrm{loc}}(N)$: $bT \in R_{r-1}^{\mathrm{loc}}(N)\}$. We define the group of *integral currents* $I_r(N) = I_r^{\mathrm{loc}}(N) \cap \mathcal{E}'_r(N)$.

Definition 2.1.6. The space of *locally flat currents* of dimension r, $F_r^{\mathrm{loc}}(M) \subset \mathcal{D}'_r(M)$, is the completion of $N_r^{\mathrm{loc}}(M)$ in the F topology described in Section 2.2 (or see [7], pp. 367–368) and the *flat currents* $F_r(M) = F_r^{\mathrm{loc}}(M) \cap \mathcal{E}'_r(M)$.

Examples of locally flat currents are given by $I_r^{\mathrm{loc}}(M) \subset R_r^{\mathrm{loc}}(M) \subset F_r^{\mathrm{loc}}(M)$ and $N_r^{\mathrm{loc}}(M) \subset F_r^{\mathrm{loc}}(M)$. Also, the boundary of a current in $F_r^{\mathrm{loc}}(M)$ is in $F_r^{\mathrm{loc}}(M)$ and

$T \llcorner u \in F_{r-k}^{\text{loc}}(M)$ if $T \in F_r^{\text{loc}}(M)$ is representable by integration and u is a locally bounded Borel measurable k-form ([7], p. 374).

Definition 2.1.7. If X is a complex space, we define $N_r^{\text{loc}}(X)$, $R_r^{\text{loc}}(X)$, $I_r^{\text{loc}}(X)$, and $F_r^{\text{loc}}(X)$ to be subspaces of $\mathcal{D}'_r(X)$ described locally by $N_r^{\text{loc}}(X) = N_r^{\text{loc}}(V) \cap \mathcal{D}'_r(X)$, etc., if X is a subvariety of V. Also $N_r(X)$, $R_r(X)$, $I_r(X)$, and $F_r(X)$ are the corresponding spaces of currents with compact support.

If $f: M \to N$ is a C^∞ map or $g: X \to Y$ is holomorphic, f_* and g_* map rectifiable, integral, normal, and flat currents into rectifiable, integral, normal, and flat currents, respectively.

Remark. Two questions concerning currents on complex spaces arise at once. First, if V is a complex manifold, $\mathcal{D}'_r(V)$ and its various subspaces are defined two ways if we also consider V to be a complex space; we ask if these definitions agree. That the answer is yes can be seen as follows: if $i: V \to W$ imbeds (not immerses) V as a submanifold and $r: W \to V$ is a retraction ($r \circ i =$ identity on V), then if $T \in \mathcal{D}'_r(V)$ as a subvariety $i_* r_* T = T$ since $(u - r^* i^* u) \in J_V(W)$. Since such an r always exists locally, the desired result follows.

The second question is under what conditions does supp $T \subset X \subset V$, for $T \in \mathcal{D}'_r(V)$, imply $T \in \mathcal{D}'_r(X)$. This condition is necessary, but Example 1.3.(c) shows that it is not sufficient in general. The second corollary to the theorem below shows that this is sufficient if T is flat and of order zero (thus if normal or integral). This fact is the important property that makes these currents useful in geometry.

THEOREM 2.1.8. (Support Theorem) *If f and g are C^∞ maps from M to N, if $T \in F_r^{\text{loc}}(M)_f$, and if $f \mid \text{supp}\, T = g \mid \text{supp}\, T$, then $f_* T = g_* T$.*

COROLLARY. (i) *If $M \subset N$ is an imbedded submanifold and i is the inclusion map, $i_* F_r^{\text{loc}}(M) = \{T \in F_r^{\text{loc}}(N)\colon \text{supp}\, T \subset M\}$. Furthermore, if $T = i_* T'$ and T is locally normal, rectifiable, or integral, then T' is also.*

(ii) *If $X \subset V$ is a subvariety, $T \in F_r^{\text{loc}}(V)$ is representable by integration, and* supp $T \subset X$, $T \in F_r^{\text{loc}}(X)$.

Proof. The theorem is found in [7], pp. 372–373, for f and g Lipschitz. Part (i) of the corollary is proved as in the Remark above, except that we see $i_* r_* T = T$ because $i \circ r$ and the identity map on N agree on supp T. Part (ii) is proved by induction on dimension. We have $X = R(X) \cup S(X)$, where $R(X)$ is a manifold and dim $S(X) <$ dim X. Let λ be the characteristic function of $V-S(X)$; by the remark following 2.1.6 the current $T\llcorner\lambda$ is a locally flat current representable by integration if T is also. Since $R(X)$ is a submanifold of $V-S(x)$, it is immediate from part (i) that $T\llcorner\lambda(u)=0$ if $u\in J_X(V)$. But supp $(T-T\llcorner\lambda)\subset S(X)$, so $(T-T\llcorner\lambda)(u)=0$ for $u\in J_X(V)$, since $J_X(V)\subset J_{S(X)}(V)$, by the induction hypothesis (if dim $X=0$, X is a manifold). $\square$

The following result will be used later in conjunction with the support theorem.

PROPOSITION 2.1.9. *If $T\in\mathcal{D}'_m(M^m)=\mathcal{D}'^0(M^m)$ and $bT=0$, T is the current defined by a locally-constant function; if $T\in I^{\mathrm{loc}}_m(M)$ this function is integer valued. If V^n is a complex manifold and $T\in\mathcal{D}'_{2n}(V^n)=\mathcal{D}'^0(V^n)$ with $d''T=0$, T is the current defined by a holomorphic function.*

Proof. This is a standard result about elliptic operators; see [18] p. 143 and [7] p. 385. $\square$

2.2. The compactness theorem

In this section we will give topologies for spaces of locally normal and locally integral currents and then state a compactness theorem resembling Ascoli's theorem for functions. This theorem will furnish a kind of Montel theorem for complex analytic sets. We will first state the theorem, then define the terms used and finally give a reference for the proof.

THEOREM 2.2.1. (Compactness Theorem) *A bounded subset of $N^{\mathrm{loc}}_r(M)$ or $N^{\mathrm{loc}}_r(X)$ is relatively compact in $N^{\mathrm{loc}}_r(M)$ or $N^{\mathrm{loc}}_r(X)$ with the F topology. A bounded subset of $I^{\mathrm{loc}}_r(M)$ or $I^{\mathrm{loc}}_r(X)$ is relatively compact in $I^{\mathrm{loc}}_r(M)$ or $I^{\mathrm{loc}}_r(X)$ with the $\mathcal{F}$ topology.*

Let M be a manifold; we will call U a *good coordinate cover* if $\mathcal{U}=\{U_i\}_{i\in I}$ is a covering of M by open sets U_i and for each U_i there has been chosen a larger open set $U'_i\supset\bar{U}_i\supset U_i$ and a coordinate map $\phi_i\colon U'_i\to\mathbf{R}^m$ such that $\phi_i(U_i)=B_1$, the unit ball.

If X is a complex space, we will call $\mathcal{U}$ a *good coordinate cover* if $U'_i\supset\bar{U}_i\supset U_i$ are as above, but the coordinate maps $\phi_i\colon U'_i\to\mathbf{C}^n$ map U'_i as a subvariety of an open set in $\mathbf{C}^n$ and map U_i onto a subvariety of Δ_1, the unit polydisc.

If E is a relatively compact (i.e. has compact closure) open subset of a Riemannian manifold N, then we define F_E and $\mathcal{F}_E$ as follows: if $T\in\mathcal{D}'_r(N)$,

$$F_E(T)=\inf\{\mathbf{M}(R)+\mathbf{M}(S)\colon\ R\in\mathcal{D}'_r(N),\ S\in\mathcal{D}'_{r+1}(N),\ \text{and}\ \operatorname{supp}(T-R-bS)\subset N-\bar{E}\}.$$

If $T\in I^{\mathrm{loc}}_r(N)$,

$$\mathcal{F}_E(T)=\inf\{\mathbf{M}(R)+\mathbf{M}(S)\colon R\in R^{\mathrm{loc}}_r(N),\ S\in R^{\mathrm{loc}}_r(N)\ \text{and}\ \operatorname{supp}(T-R-bS)\subset N-\bar{E}\}.$$

F_E is not always finite for a general current but it defines a seminorm of $N^{\mathrm{loc}}_r(N)$. $\mathcal{F}_E$ defines a pseudometric on $I^{\mathrm{loc}}_r(N)$.

We observe that $F_E(T) \leqslant \mathbf{M}(T)$, $F_E(T) \leqslant \mathfrak{F}_E(T) \leqslant \mathbf{M}(T)$ when defined. Also, $F_E(bT) \leqslant F_E(T)$ and $\mathfrak{F}_E(bT) \leqslant \mathfrak{F}_E(T)$.

These pseudometrics can be used to topologize the locally normal and integral currents as described below:

Definition 2.2.2. Let $\mathfrak{U}$ be a good coordinate cover on M; the *$F(\mathfrak{U})$ topology on $N_r^{\mathrm{loc}}(M)$* is defined by taking finite intersections of sets of the form $\{T \in N_r^{\mathrm{loc}}(M)\colon F_{U_i}(T) < \varepsilon\}$ as a basis for the open sets. In other words, convergence is convergence in each F_{U_i}.

Definition 2.2.3. Let $\mathfrak{U}$ be a good coordinate cover on X; the *$F(\mathfrak{U})$ topology on $N_r^{\mathrm{loc}}(X)$* is defined by a basis of finite intersections of sets of the form $\{T \in N_r^{\mathrm{loc}}(X)\colon F_{\Delta_1}(\phi_{i*}(T \mid U_i')) < \varepsilon\}$.

Definition 2.2.4. The *$\mathfrak{F}(\mathfrak{U})$ topologies* on $I_r^{\mathrm{loc}}(M)$ and $I_r^{\mathrm{loc}}(X)$ are defined in an analogous manner.

Note. If $\mathfrak{U}$ and $\mathfrak{U}'$ are two good coordinate covers, the $F(\mathfrak{U})$ and $F(\mathfrak{U}')$ topologies and also the $\mathfrak{F}(\mathfrak{U})$ and $\mathfrak{F}(\mathfrak{U}')$ topologies are the same for M or X; hence, we will just refer to the *F and the $\mathfrak{F}$ topologies*. The topologies on M can be defined directly by the pseudometrics F_E and $\mathfrak{F}_E$ for all relatively compact open $E \subset M$. It is less obvious how to do this for X; and rather than discussing this at length, we have adopted the definition used here, which is adequate for our purposes.

Definition 2.2.4. A subset $A \subset N_r^{\mathrm{loc}}(M)$ is said to be *bounded* if for every relatively compact open $E \subset M$ there is a constant C_E such that $\mathbf{M}(T \mid E) + \mathbf{M}(bT \mid E) < C_E$ for all $T \in A$. A subset $A \subset N_r^{\mathrm{loc}}(X)$ is bounded if for every coordinate map $\phi\colon U \to \mathbf{C}^n$ which imbeds an open $U \subset X$ as a subvariety of an open set $D \subset \mathbf{C}^n$ the subset $\phi_*(A) = \{\phi_*(T \mid U)\colon T \in A\} \subset N_r^{\mathrm{loc}}(D)$ is bounded. It clearly suffices to check boundedness for a single coordinate covering.

Proof of Theorem 2.2.1. The theorem is proved for N_r and I_r with support in some fixed compact Lipschitz neighborhood retract K in Federer [7], pp. 411–415. To get the theorem for $N_r^{\mathrm{loc}}(\mathbf{R}^m)$ we can use partitions of unity and this with Tychonoff's theorem (since the F topology is induced from product topology) implies the result for $N_r^{\mathrm{loc}}(M)$ and $N_r^{\mathrm{loc}}(X)$. To show the theorem for I_r^{loc} we must also show that a limit point in the $\mathfrak{F}$ topology of a set of locally integral currents is locally integral. This is a local question, and the argument of Federer which proves the theorem for I_r proves this result for $I_r^{\mathrm{loc}}(M)$ and $I_r^{\mathrm{loc}}(X)$ as well. $\square$

2.3. Slicing (fibering) currents

If $f\colon M^m \to N^n$ is a smooth map of maximal rank and $u \in A_c^{m-n}(M)$. we saw in Example 1.4.3.(b) that $f_*[u]$ is the current defined by the smooth function of y, $\int_{M_y} u = [M](u)$.

We call $[M_y]=\langle [M], f, y\rangle$ the slice of $[M]$ at y, for $y \in N$. In this section we list results which say that for a general flat current T, $\langle T, f, y\rangle$ is defined for almost all $y \in N$ and the current $f_*(T\llcorner u)$ is defined by the measurable function $\langle T, f, y\rangle(u)$.

In $\mathbf{R}^n$ the notions of a set of Lebesgue n-measure zero and of locally Lebesgue integrable functions are invariant under differentiable coordinate changes; therefore, it makes sense to speak of sets of (Lebesgue) measure zero and locally integrable functions on a real n-manifold N.

If λ is a locally integrable function defined almost everywhere on N, it defines a current $[\lambda] \in \mathcal{D}'_n(N)$ by $[\lambda](u)=\int_N \lambda u$. Since the values of λ may be changed on a set of measure zero without changing $[\lambda]$, the value $\lambda(y)$ is not unambiguously determined for all $y \in N$ by $[\lambda]$. However, a theorem of Lebesgue says that a definite value may be assigned to almost all y by a process of differentiation.

Suppose N is given a Riemannian metric and that W is the volume form on N. Then for a fixed $y \in N$, if $C_1, C_2, \ldots$ is a regular sequence of closed sets converging to y ([17], p. 106), we may ask if the limit

$$a_y=\lim_{n\to\infty} \int_{C_n} \lambda W \Big/ \int_{C_n} W$$

exists and is independent of the regular sequence chosen. If for some $y \in N$ this is the case, we define $\tilde{\lambda}(y)=a_y$; if not, set $\tilde{\lambda}(y)=+\infty$. The theorem of Lebesgue states that $\tilde{\lambda}=\lambda$ almost everywhere ([17], p. 118). If $\tilde{\lambda}(y) \neq +\infty$, we say that λ has a *well-defined value* at y.

The definition of $\tilde{\lambda}$ is clearly local and independent of the choice of W (any continuous nonvanishing n-form would do). If $g\colon N\to N'$ is a diffeomorphism and $C_1, C_2, \ldots$ is a regular sequence of closed sets tending to $g(y)$; it is evident that

$$\widetilde{\lambda \circ g^{-1}}=\tilde{\lambda}\circ g^{-1}.$$

We will now apply these ideas to finding the values of certain currents defined by fibering maps.

THEOREM 2.3.1. *If* $T \in F_m^{\mathrm{loc}}(M^m)$, $T=[\lambda]$ *for some locally integrable function* λ.

Proof. [7], p. 376. $\square$

If $f\colon M^m \to N^n$ is a C^∞ map and $T \in N_{n+r}^{\mathrm{loc}}(M)$ and $u \in CA^r(M)$ are such that $f|\operatorname{supp} T \cap \operatorname{supp} u$ is proper (that is, $T\llcorner u \in N_n^{\mathrm{loc}}(M)_f$), then $f_*(T\llcorner u) \in F_n^{\mathrm{loc}}(M)$ (see remark after 2.1.6) and from previous theorem is the current $[\lambda_u]$ for some function λ_u.

Definition 2.3.2. If $y \in N$ and $\dot{\lambda}_u$ has a well-defined value at y for all such $u \in A^r(M)$ (it suffices to check for all $u \in A^r_c(M)$), then the current $\langle T, f, y\rangle \in \mathcal{D}'_r(M)$ defined by $\langle T, f, y\rangle(u)=\lambda_u(y)$ is called the *slice of* T at y by f.

Example. If T is the current defined by integration over some $(n+r)$-dimensional oriented submanifold $M' \subset M$ and rank $Df_x = n$ for all $x \in M'$, then $\langle T, f, y\rangle$ is the current defined by integration over the oriented submanifold $f^{-1}(y) \cap M' \subset M$. If $M' = M$ this is integration along the fiber as described in Example (b) after 1.4.2. Notice that in this example $\langle T, f, y\rangle(u)$ is actually a continuous function of y if $u \in CA_c^r(M)$ and is smooth if $u \in A_c^r(M)$.

THEOREM 2.3.3. *If $f: M^m \to N^n$ is a C^∞ map and $S, T \in F_{n+r}^{\mathrm{loc}}(M)$, the following statements are true.*

(1) *If $h: N \to N'$ is an orientation-preserving diffeomorphism and $\langle T, f, y\rangle$ exists, then $\langle T, h \circ f, h(y)\rangle$ exists and equals $\langle T, f, y\rangle$.*

(2) *If $u \in A^s(M)$, $s \leqslant r$, and $\langle T, f, y\rangle$ exists, then $\langle T \llcorner u, f, y\rangle$ exists and equals $\langle T, f, y\rangle \llcorner u$.*

(3) *If $\langle S, f, y\rangle$ and $\langle T, f, y\rangle$ exist, then $\langle S+T, f, y\rangle$ exists and equals $\langle S, f, y\rangle + \langle T, f, y\rangle$.*

(4) *If $x \in M$ we say that T slices at x if there is an $S \in F_{n+r}^{\mathrm{loc}}(M)$ such that $\langle S, f, f(x)\rangle$ exists and $x \notin$ supp $(T-S)$. If for every $x \in f^{-1}(y)$, T slices at x, then $\langle T, f, y\rangle$ exists (i.e., slicing is local in M).*

(5) *If $\langle T, f, y\rangle$ exists,* supp $\langle T, f, y\rangle \subset$ supp $T \cap f^{-1}(y)$.

(6) *If $r > 0$ and $\langle T, f, y\rangle$ exists, then so does $\langle bT, f, y\rangle$, and $\langle bT, f, y\rangle = b\langle T, f, y\rangle$.*

Proof. The remark in the fourth paragraph in this section shows (1) and (4) is clear. The rest is in Federer [7] pp. 436–438, or in the earlier paper [8]. We are actually using the definition found in this paper rather than [7] because it behaves better under change of coordinate (i.e., (1) holds). □

The following theorem is the fundamental result about the existence of slices.

THEOREM 2.3.4. (Slicing Theorem) *If $f: M \to N^n$ is a C^∞ map, M is a manifold with a countable basis for the open sets, and $T \in F_{n+r}^{\mathrm{loc}}(M)$ (resp. $N_{n+r}^{\mathrm{loc}}(M)$, $I_{n+r}^{\mathrm{loc}}(M)$), then for almost all $y \in N$ (in Lebesgue n-measure) $\langle T, f, y\rangle$ exists and is in $F_r^{\mathrm{loc}}(M)$ (resp. $N_r^{\mathrm{loc}}(M)$, $I_r^{\mathrm{loc}}(M)$).*

Proof. The theorem for a single coordinate neighborhood is in [7], p. 438 and p. 443; since a countable union of sets of measure zero has measure zero, the theorem follows immediately. □

2.4. Hausdorff measure

To measure sets of intermediate dimension, we define p-dimensional Hausdorff measure, H^p. For a p-dimensional submanifold M of a Riemannian manifold, $H^p(M)$ equals the volume of M.

Let A be a subset of a metric space B, and let $\delta(A)$ denote the diameter of A. We write $\delta^p(A)=\delta(A)^p\beta(p)$ if $p>0$ and $\delta^0(A)=1$ if $A\neq\emptyset$, $\delta^0(A)=0$ if $A=\emptyset$. ($\beta(p)=2^{-p}\alpha(p)^{-1}$, where $\alpha(p)$ is the volume of the unit ball in $\mathbf{R}^p$.)

Definition 2.4.1. If $p\geq 0$, $\varepsilon>0$, set $H^p_\varepsilon(A)=\inf\{\sum_i\delta^p(A_i);\, A\subset\bigcup A_i \text{ and } \delta(A_i)<\varepsilon\}$. Then the *Hausdorff p-measure* of A,

$$H^p(A)=\lim_{\varepsilon\to 0+}H^p_\varepsilon(A).$$

H^p is a regular metric outer measure, so Borel sets are measurable. If $H^p(A)<\infty$, $H^{p+a}(A)=0$ for any $a>0$. If f is a Lipschitzian map with Lipschitz constant K, $H^p(f(A))\leq K^pH^p(A)$. (See [25] pp. 21–22).

For the rest of this section we will assume that all manifolds are Riemannian manifolds with a countable basis for the open sets; H^p is induced from the Riemannian metric.

THEOREM 2.4.2. (Measure Support Theorem) *If* $T\in F_r^{\text{loc}}(M)$ *and* $H^r(\operatorname{supp} T)=0$, *then* $T=0$.

Proof. [7], p. 378 and p. 173. □

In particular, if supp T is contained in a manifold or analytic set of real dimension less than r, $T=0$.

THEOREM 2.4.3. *Let* $f: M\to N^n$ *be a* C^∞ *map. If* $A\subset M$ *and* $H^{n+a}(A)=0$, $H^a(A\cap f^{-1}(y))=0$ *for almost all* $y\in N$ *(in Lebesgue, or equivalently, Hausdorff n measure.)*

Proof. [7] p. 188. □

The next theorem gives a tool for choosing good coordinate systems in $\mathbf{C}^n$.

THEOREM 2.4.4. *Let A be a subset of $\mathbf{C}^n$ and let $a>0$. If $H^{2k+a}(A)=0$, then, for almost all complex $(n-k)$ planes L through 0, $H^a(A\cap L)=0$. In particular if A is a complex subvariety of an open subset of $\mathbf{C}^n$, the complex dimension of $A\cap L<a/2$ for almost all L. Also, when A is closed in a neighborhood of 0 and π_L is a linear map from $\mathbf{C}^n$ to $\mathbf{C}^k$ with kernel L, $H^1(A\cap L)=0$ implies that there is a neighborhood U of 0 such that the restriction $\pi|A\cap U$ is a proper map.*

Proof. See Shiffman [19], p. 114, Lemma 2, including the remark following; also p. 118. The last part uses the fact that dimension $A\cap L=0$ if $H^1(A\cap L)=0$. □

3. Fiber integration for holomorphic maps

In 3.1 we discuss integration on a complex space and the group of holomorphic cycles. The principal goal of this section is the Fibering Theorem in 3.3, which says that if $f: X \to Y$ is a holomorphic q-fibering and Y is locally irreducible, the current of integration along the fiber $f_*[u]$ for any $u \in CA_c^{2q}(X)$ is given by a continuous function. This function is actually given by $\int_{f^{-1}(y)} u$ for all $y \in Y$ if $f^{-1}(y)$ is given suitable multiplicities. Under additional hypotheses on u, which are given in 3.3 the function $f_*[u]$ will be constant, holomorphic, or plurisubharmonic.

In Section 3.2 there are lemmas needed for 3.3 on the choosing of coordinates and on the volume of the fibers.

3.1. Holomorphic cycles

If M^m is an oriented submanifold of a Riemannian manifold N and $u \in A^m(N)$, then $\int_M u$ can be defined even when M is not closed if the volume of M is locally finite. Integration over M defines the locally rectifiable current $[M]$.

THEOREM 3.1.1. *If X is a k-dimensional complex subvariety of a complex manifold V and $u \in A_c^{2k}(V)$, the integral $\int_{R(X)} i^* u$ is defined, where i: $R(X) \to V$ is the inclusion map. If we define $[X](u) = \int_X u = \int_{R(X)} i^* u$, $[X]$ is a closed current in $I_{2k}^{\mathrm{loc}}(V)$.*

Proof. This is proved directly in Lelong [13] or Stolzenberg [25], Chapter I. One shows that $R(X)$ has locally finite volume by choosing clear coordinates (see Section 3.2). This implies that $[X]$ is defined. It is clearly in $R_{2k}^{\mathrm{loc}}(V) \subset F_{2k}^{\mathrm{loc}}(V)$; therefore, $b[X] \in F_{2k-1}^{\mathrm{loc}}(V)$. But is is evident that $\operatorname{supp} b[X] \subset S(X)$. Since $H^{2k-1}(S(X)) = 0$, $b[X] = 0$ by the Measure Support Theorem (2.4.2) and $[X]$ is closed, hence in $I_{2k}^{\mathrm{loc}}(V)$. $\square$

If X is a subvariety of Y, it follows that integration on X defines a current $[X] \in I_{2k}^{\mathrm{loc}}(Y)$.

Definition 3.1.2. If Y is a complex space, the group of *holomorphic k-cycles* $Z_k(Y) = \{T \in I_{2k}^{\mathrm{loc}}(Y)$: $T = \sum_{i \in I} n_i [X_i]\}$, where X_i is a k-dimensional subvariety of Y, n_i is a nonzero integer. The sum may be infinite; but $X = \bigcup_{i \in I} X_i$ is a subvariety of Y_1 for if the union were not locally finite the sum T would not exist.

The following proposition will be of help in recognizing holomorphic cycles.

PROPOSITION 3.1.3. *If X is a k-dimensional complex subvariety of Y, for any closed current $T \in F_{2k}^{\mathrm{loc}}(Y)$ with* supp *$T \subset X$, T is of the form $\sum a_i [X_i]$, where the X_i are the global irreducible components of $X = \bigcup X_i$. If $T \in I_{2k}^{\mathrm{loc}}(Y)$ as well, the a_i are integers and $T \in Z_k(Y)$.*

Proof. If we restrict T to $Y - S(X)$, by the Support Theorem Corollary 2.1.8.(i) and

Proposition 2.1.9, this restriction agrees with $T' = \Sigma a_i[X_i]$ for some unique $a_i \in \mathbf{C}$, since the X_i are connected manifolds. But then $T - T' \in F_{2k}^{\mathrm{loc}}(Y)$ and supp $(T - T') \subset S(X)$; therefore, $T = T'$ by the Measure Support Theorem (2.4.2). If $T \in I_{2k}^{\mathrm{loc}}(Y)$, the a_i are integers by 2.1.9. □

If W^k is a complex submanifold (not necessarily closed) of a hermitian manifold V and w is the Kähler form of the metric on V, then the volume of W equals $(1/k!)\int_W w^k$, where $w^k = w \wedge \ldots \wedge w$ k times.

PROPOSITION 3.1.4. *If* $T \in Z_k(V)$, *V a hermitian manifold with Kähler form w*, $\mathbf{M}(T) = (1/k!)\, T(w)$.

Proof. This follows from the remark above, which is proved in [25] pp. 6–8, or [7] p. 40 from Wirtinger's inequality.

Definition 3.1.5. A current $T \in \mathcal{D}'_{k,k}(V)$, V a complex manifold, is said to be *positive* if $T \llcorner u$ is positive for any $u \in A^{k,k}(V)$ of the form $(i/2)^k \lambda\, dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^k \wedge d\bar{z}^k$, where the z^i are local coordinates and $\lambda \geqslant 0 \in A_c^0(V)$ (a current $S \in \mathcal{D}'_{0,0}(V)$ is positive if $S(\mu) \geqslant 0$ for any $\mu \geqslant 0$ in $A_c^0(N)$).

It is clear that $[X]$ is positive if $X^k \subset V$ is a subvariety.

3.2. Lemmas on the volume of fibers

In this section we prove a lemma in local analytic geometry which gives a volume estimate needed in the next section.

Notation for this section. $I = (i_1, \ldots, i_r)$ is an r-ruple of integers, $i_1 \leqslant \ldots \leqslant i_r$, $|I| = \max_{1 \leqslant j \leqslant r} \{i_j\}$. If $|I| \leqslant n$, π_I: $\mathbf{C}^n \to \mathbf{C}^r$ is given by $\pi_I(z^1, \ldots, z^n) = (z^{i_1}, \ldots, z^{i_r})$; L_I is the linear subspace $\pi_I^{-1}(0)$. I and the s-tuple J are disjoint if they have no common element; the $(r+s)$-tuple $I \cup J$ is defined to be $(i_1, \ldots, i_r, j_1, \ldots, j_s)$ rearranged in the obvious way if I and J are disjoint.

Definition 3.2.1. Let X be the germ of a pure k-dimensional complex subvariety at $0 \in \mathbf{C}^n$. Linear coordinates $(z^1, \ldots, z^n)$ are called *clear coordinates* for X if for any k-tuple $I = (i_1, \ldots, i_k)$, π_I: $\mathbf{C}^n \to \mathbf{C}^k$ is a finite map of X at 0 (i.e., π_I makes X an analytic cover over $\mathbf{C}^k$ at 0, [9] p. 101).

Definition 3.2.2. Let π: $\mathbf{C}^n \to \mathbf{C}^d$, $k \geqslant d$, be a linear map given by linear functionals $w^i(z^1, \ldots, z^n)$, $i = 1, \ldots, d$. If X is as above, the linear coordinates $(z^1, \ldots, z^n)$ are called *clear π-coordinates* for X if they are clear coordinates for X and $w^i = z^i$, $i = 1, \dots, d$ (i.e. $\pi = \pi_I$, $I = (1, \dots, d)$).

Notice that the existence of clear π-coordinates implies that $\dim (X \cap \pi^{-1}(0)) = k - d$ at 0.

It is known ([26], p. 311) that a dense subset of the set of all coordinates, considered as points on a Stiefel manifold, are clear. This also follows immediately from 2.4.4. We wish to prove a slightly more general result about the existence of clear π-coordinates.

LEMMA 3.2.3. *If X is a purely k-dimensional complex analytic set in some open set $U \subset \mathbf{C}^n$ and the linear map $\pi = (w^1, \dots, w^d)$: $\mathbf{C}^n \to \mathbf{C}^d$ is such that $\dim \pi^{-1}(0) \cap X = k - d \geqslant 0$, then for almost all linear maps $(z^{d+1}, \dots, z^n)$: $\mathbf{C}^n \to \mathbf{C}^{n-d}$, the coordinates $(w^1, \dots, w^d, z^{d+1}, \dots, z^n)$ are clear π-coordinates for the germ of X at 0.*

Remark. What is meant above by almost all linear maps is that the statement is true on the complement of a set of measure zero in the space of matrices $\mathbf{C}^{n(n-d)}$. In the proof we will use the fact that if Y is the germ of a purely r-dimensional complex subvariety at $0 \in \mathbf{C}^n$, then $\dim (Y \cap p^{-1}(0)) = r - s$ for almost all linear p: $\mathbf{C}^n \to \mathbf{C}^s$. (2.4.4 and [9] p. 115). If $r = s$, to say that p is finite on Y at 0 is equivalent to saying that 0 is an isolated point of $Y \cap p^{-1}(0)$.

Proof of lemma. In the proof all subvarieties of $\mathbf{C}^n$ will be considered to be germs of varieties at 0. By hypothesis, $\dim (X \cap \pi^{-1}(0)) = k - d$. By the remark above, for almost all linear $(z^{d+1}, \dots, z^k)$: $\mathbf{C}^n \to \mathbf{C}^{k-d}$ the map is finite on $X \cap \pi^{-1}(0)$. Therefore the map $(w^1, \dots, w^d, z^{d+1}, \dots, z^k)$: $\mathbf{C}^n \to \mathbf{C}^k$ is finite on X. Fix these z^i and call the map π_k.

We do a proof by induction. We assume that we have a linear map π_r: $(w^1, \dots, w^d, z^{d+1}, \dots, z^r) = (z^1, \dots, z^r)$, $k \leqslant r < n$, of rank r such that for any k-tuple $I = (i_1, \dots, i_k)$, $|I| \leqslant r$, $\dim (X \cap L_I) = 0$. This implies that for any $(k-1)$-tuple $J = (j_1, \dots, j_{k-1})$, $|J| \leqslant r$, $\dim (X \cap L_J) = 1$.

Let $Y = \bigcup_J (X \cap L_J)$, where the union is over all $(k-1)$-tuples J with $|J| \leqslant r$. Since $\dim Y = 1$, for almost all linear functionals z: $\mathbf{C}^n \to \mathbf{C}$, $\pi_r \times z$: $\mathbf{C}^n \to \mathbf{C}^{r+1}$ has rank $r+1$ and $\dim (Y \cap z^{-1}(0)) = 0$. If we fix such a z and set $\pi_{r+1} = (w^1, \dots, w^d, z^{d+1}, \dots, z^r, z) = (z^1, \dots, z^{r+1})$, then for any k-tuple I with $|I| \leqslant r+1$, $\dim (X \cap L_I) = 0$.

Thus by induction we can construct a π_n, which gives clear π-coordinates by the properties of π_r. At each stage we choose any z outside of a set of measure zero in $\mathbf{C}^{n*}$, so almost all choices of $(z^{d+1}, \dots, z^n)$ give clear π-coordinates. $\square$

Suppose $(z^1, \dots, z^n)$ are clear π-coordinates for a purely k-dimensional subvariety X at 0. Then since π_I is finite on X at 0 for each k-tuple I, there is a relatively compact polydisc $\Delta_I \subset \mathbf{C}^n$ such that $\pi_I | X \cap \Delta_I$ is a proper map of $X \cap \Delta_I$ onto $\pi_I(\Delta_I)$ with n_I sheets.

Definition 3.2.4. Let Δ be any polydisc contained in $\bigcap_I \Delta_I$, where the Δ_I are as described above; such a Δ is called a *normal π-polydisc* for X at 0.

PROPOSITION 3.2.5. *If X is a purely k-dimensional subvariety of an open set $U \subset \mathbf{C}^n$ and Δ is a normal π-polydisc for X at 0 (where $\pi: \mathbf{C}^n \to \mathbf{C}^d$ is a linear map as before), then there is a constant C such that the $2(k-d)$ volume $(X \cap \Delta \cap \pi^{-1}(y)) < C$ for all $y \in \mathbf{C}^d$.*

Proof. It $\mathbf{C}^n$ has the usual metric, the Kähler form $w = (i/2) \sum_{i=1}^n dz^i \wedge d\bar{z}^i$ and $(w^{k-d}/(k-d)!) = \sum_J w^J$, where J is a $(k-d)$-tuple $(j_1, \ldots, j_{k-d})$ and

$$w^J = (i/2)^{k-d} dz^{j_1} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge dz^{j_{k-d}} \wedge d\bar{z}^{j_{k-d}}.$$

We recall from the previous section that

$$\text{volume } (X \cap \Delta \cap \pi^{-1}(y)) = \frac{1}{(k-d)!} \int_{X \cap \Delta \cap \pi^{-1}(y)} w^{k-d} = \sum_J \int_{X \cap \Delta \cap \pi^{-1}(y)} w^J.$$

In the π-coordinates, $\pi = \pi_{I_0}$, where $I_0 = (1, \ldots, d)$. If a given $(k-d)$-tuple J is not disjoint from I_0, clearly

$$\int_{X \cap \Delta \cap \pi^{-1}(y)} w^J = 0.$$

If J is disjoint from I_0, we set $J' = I_0 \cup J$. The map $\pi_{J'}$ maps $X \cap \Delta_{J'}$ onto $\pi_{J'}(\Delta_{J'})$ as an $n_{J'}$-sheeted analytic cover. Therefore, π_J maps $X \cap \Delta_{J'} \cap \pi^{-1}(y)$ onto the $(k-d)$ polydisc $\pi_J(\Delta_{J'})$ as a cover with sheets numbering $\leqslant n_{J'}$. Thus

$$E_J = \int_{X \cap \Delta_{J'} \cap \pi^{-1}(y)} w^J \leqslant n_{J'} \int_{\pi_J(\Delta_{J'})} w^J$$

(w^J equals π_J^* applied to the volume form on $\mathbf{C}^{k-d}$, which by abuse of notation we will also denote by w^J; so $w^J = \pi_J^* w^J = \pi_{J'}^* w^J$).

The second integral is a finite constant $C_J > 0$ independent of y. The first integral E_J is larger than the integral over $X \cap \Delta \cap \pi^{-1}(y)$ since $\Delta \subset \Delta_{J'}$. Therefore volume $(X \cap \Delta \cap \pi^{-1}(y)) \leqslant \sum_J E_J \leqslant \sum n_{J'} C_J$. $\square$

Finally, we make an observation that will be needed in the next section. We use the same notation as in the previous proof.

LEMMA 3.2.6. *Take a $(k-d)$-tuple J disjoint from I_0, as above. Let λ be a smooth function $\geqslant 0$ with $\lambda(0) > 0$ in $A_c^0(\pi_{J'}(\Delta)) \subset A_c^0(C^k)$. Then $(\lambda \circ \pi_{J'}) w^J$ is in $A^{2k-2d}(\Delta)$ and $\pi_*([X] \llcorner (\lambda \circ \pi_{J'}) w^J)$ is a current on $\mathbf{C}^d$ given by a continuous function which is > 0 near 0.*

Proof. This is just integration along the fiber for a product. If $p = \mathbf{C}^k \to \mathbf{C}^d$ is the projection on the first d coordinates, $p \circ \pi_{J'} = \pi$. Therefore, for any form $u \in A^{2d}(\mathbf{C}^d)$,

$$\pi_*([X] \llcorner (\lambda \circ \pi_{J'}) w^J)(u) = [X](\pi_{J'}{}^*(\lambda w^J) \wedge \pi^* u) = [X](\pi_{J'}{}^*(\lambda w^J \wedge p^* u)$$
$$= \pi_{J'*}[X \cap \Delta_{J'}](\lambda w^J \wedge p^* u),$$

which is defined since $\pi_{J'}$ is proper on $X \cap \Delta_{J'}$.

Now the current $\pi_{J'*}[X \cap \Delta_{J'}]$ is a closed locally integral current in $\pi_{J'}(\Delta_{J'})$; therefore, it is the current defined by an integer (by 2.1.9) which is obviously $n_{J'}$. Thus we have seen that the current $T = \pi_*([X] \llcorner (\lambda \circ \pi_J) w^J)$ applied to u is the same as $n_{J'} \int_{\mathbf{C}^k} \lambda w^J \wedge p^* u$ (since the support of λ is compact) and this equals $n_{J'}[\lambda w^J](p^* u) = n_{J'} p_*[\lambda w^J](u)$, where $[\lambda w^J] \in \mathcal{E}'_{2d}(\mathbf{C}^k)$. Thus $T = n_{J'} p_*[\lambda w^J]$, and this is given by a continuous function as we saw in 1.4.3 (b). □

3.3. Continuity of fibering

This section contains a theorem on continuity of fiber integration. At the end of the section are conditions under which the integration of a form along the fiber yields a constant, holomorphic, or plurisubharmonic function.

Definition 3.3.1. A holomorphic map $f\colon X \to Y$ between two analytic spaces is called a q-fibering if $\dim f^{-1}(y) = q$ for all $y \in Y$ ($f^{-1}(y)$ may be empty).

THEOREM 3.3.2. (Fibering theorem) *Let X^m be a complex analytic space and let Y^d be a locally irreducible complex analytic space. Suppose that $T \in Z_k(X)$ and that $f\colon X \to Y$ is a holomorphic map whose restriction to* supp T *is a $(k-d)$-fibering of* supp T, *then*

(1) *there is a map $\Phi\colon Y \to Z_{k-d}(X)$, continuous in the $\mathfrak{F}$ topology such that $\Phi(y) = \langle T, f, y \rangle$ for all $y \in R(Y)$,*

(2) *if $u \in CA^{2k-2d}(X)$ is a form such that $f\,|$*supp $u \cap$ supp T *is proper (i.e. $T \llcorner u \in \mathcal{D}'_{2d}(X)_f$), then $\Phi(y)(u) = \langle T, f, y \rangle (u)$ is a continuous function of $y \in Y$ which defines the current $f_*(T \llcorner u)$, (that is, for any $v \in A^{2d}_c(Y)$, $f_*(T \llcorner u)(v) = \int_y (\Phi(y)(u))v$).*

This theorem has been proved by Stoll [22] where X is a manifold and Y is normal and for certain discontinuous u. When X is a singular projective algebraic variety, the theorem has been proved by Federer [8] and Andreotti–Norguet [3]. The proof given here takes the same approach as that of Federer but uses the local geometry developed in Section 3.2.

Before giving the proof, we wish to choose local coordinates in which the fibering is like that of Section 3.2.

Construction of normal coordinate neighborhoods

Suppose Y is an open set in $\mathbf{C}^d$ and $f: X \to Y$ is a $(k-d)$-fibering, where X has pure dimension k. Let $x \in X$; we choose coordinates in X by finding an open set $U' \subset X$ with $x \in U'$ and a holomorphic imbedding $\phi: U' \to D \subset \mathbf{C}^n$ as a subvariety of an open set D such that $f \circ \phi^{-1}: \phi(U') \to Y$ is the restriction of a holomorphic $F: D \to Y$.

Let $\Gamma(F): D \to D \times Y$ be the graph function on F (with $\Gamma(F)(s) = (s, F(s))$) and let π: $D \times Y \to Y$ be the projection. Since $\Gamma(F)$ is an imbedding, the function $\phi' = \Gamma(F) \circ \phi$: $U' \to D \times Y \subset \mathbf{C}^{n+d}$ is a new coordinate map and $f = \pi \circ (\Gamma(F) \circ \phi)$ on U'. Therefore, in local coordinates f is given by the linear function $\pi: \mathbf{C}^{n+d} \to \mathbf{C}^d$; this is the situation studied in the preceding section.

If Y^d is a complex space, choose at $f(x)$ a coordinate map ψ onto $\mathbf{C}^m$ and, if needed, a projection $p: \mathbf{C}^m \to \mathbf{C}^d$ so that $p \circ \psi$ is finite on Y; then proceed as above with $\psi \circ f$ or $p \circ \psi \circ f$.

Once these coordinates are chosen we can find a normal neighborhood Δ for $\phi'(U')$ about $\phi'(x)$ and set $U = \phi'^{-1}(\Delta)$; this set U, with the accompanying coordinate systems, will be called a *normal f-coordinate neighborhood for x.*

Definition 3.3.3. Let $f: X^k \to Y^d$ be a $(k-d)$ fibering, where Y is a complex space. The pair $(\mathcal{U}, \mathcal{D})$ will be called a *normal coordinate cover* for f if U is a good coordinate cover for X, $\mathcal{D}$ is a good coordinate cover for Y (see Section 2.2), and for each $U_i \in \mathcal{U}$ there is $P_{j_i} \in \mathcal{D}$ such that U_i is a normal f-coordinate neighborhood with respect to the coordinates on U_i' and P'_{j_i}.

We see from the construction above that such $(\mathcal{U}, \mathcal{D})$ always exist.

Now we will prove the Fibering Theorem by showing that almost all of the slices $\langle T, f, y\rangle$ are given by the set-theoretic fiber with the multiplicity given by T. The rest of the fibers are then filled in using the Compactness Theorem.

Proof of Theorem 3.3.2. We break the proof into several steps. First, observe that it is sufficient to prove the theorem for the case $k = m$ and supp $T = X$, since forms on X can be restricted to supp T. Thus we assume $X = \operatorname{supp} T$; this implies X has pure dimension k and $f: X \to Y$ is a $(k-d)$-fibering. Also, since $T = \sum n_i [X_i]$, where the X_i are the irreducible components of X, we may assume $X = X_i$, some i, and $T = [X]$. Therefore, given $u \in A^{2k}(X)_f$, $f_*([X] \llcorner u) = f_*(u)$.

Case A. Y is a manifold and X has a countable basis for the open sets.

Step A1: Good Slices. For almost all $y \in Y$ the following is true: $\langle [X], f, y\rangle$ exists and is the current $[f^{-1}(y)]$ defined by integration on $f^{-1}(y)$.

Proof of Step A1. Let $B \subset X = \{x \in R(X)\colon \operatorname{rank} Df(x) < 2d\} \cup S(X)$; B is a closed set with

Hausdorff measure (in local coordinates) $H^{2k-2+a}(B)=0$ for any $a>0$. Therefore, by the Slicing Theorem 2.3.4 and Theorem 2.4.3 there is a set $G\subset Y$ such that $Y-G$ has measure zero, and for any $y\in G$, $H^{2k-2d+1}(f^{-1}(y)\cap B)=0$ and $\langle[X],f,y\rangle\in I^{\text{loc}}_{2k-2d}(X)$ exists. By Theorem 2.3.3, $b\langle[X],f,y\rangle=\langle b[X],f,y\rangle=0$, so $\langle[X],f,y\rangle$ is a closed locally integral current with support in $f^{-1}(y)$ and by Proposition 3.1.3 is a current in $Z_{k-d}(X)$ given by a sum of the irreducible components of $f^{-1}(y)$.

To verify that $\langle[X],f,y\rangle=[f^{-1}(y)]$ for $y\in G$, we will check that they agree at points of $f^{-1}(y)-B$. Let $u\in A_c^{2k-2d}(X)$ with supp $u\cap M=\phi$. On $X-B$ f is a map on a manifold with maximal rank, so by 1.4.3(b), $f_*[u]$ is a continuous function whose value at y is $\int_{f^{-1}(y)}u$. If $y\in G$ this equals $\langle[X],f,y\rangle(u)$. We have shown that supp $(\langle[X],f,y\rangle-[f^{-1}(y)])\subset B\cap f^{-1}(y)$, a set whose Hausdorff $(2k-2d)$ measure is zero for $y\in G$; therefore, by the Measure Support Theorem 2.4.2 $\langle[X],f,y\rangle=[f^{-1}(y)]$ for $y\in G$.

Step A2: Boundedness. The set $\{\langle[X],f,y\rangle\colon y\in G\}\subset Z_{k-d}(X)$ is bounded.

Proof of Step A2. As observed in Section 2.2 it suffices to check boundedness with respect to a given coordinate cover. Let $(\mathcal{U},\mathcal{D})$ be a normal coordinate cover for f; if U_i is a normal f-coordinate neighborhood with coordinates $\phi_i\colon U_i\to\Delta\subset\mathbf{C}^n$, then by Proposition 3.2.5 there is a constant C_i such that $M(\phi_{i*}([f^{-1}(y)]\,|\,U_i))<C_i$. But since $\langle[X],f,y\rangle=[f^{-1}(y)]$ for $y\in G$, this implies Step A2 by definition.

Step A3. Filling in the gaps. Suppose $y\in Y$; choose any sequence $y_1,y_2,\ldots$ in G such that $y_i\to y$. By Step A2, the sequence $\langle[X],f,y_i\rangle$ is bounded and by the Compactness Theorem 2.2.1 is relatively compact in the $\mathfrak{F}$ topology. Thus the sequence has a convergent subsequence; we wish to show that for any such subsequence the limit is uniquely determined. We will define $\Phi(y)$ to be this limit and conclude that the theorem then holds.

Relabeling the sequence if necessary, suppose $\langle[X],f,y_i\rangle\to S\in I^{\text{loc}}_{2k-2d}(X)$. Then $bS=\lim_{i\to\infty}b\langle[X],f,y_i\rangle=0$ and supp $S\subset f^{-1}(y)$. Thus by Proposition 3.1.3, $S\in Z_{k-d}(X)$ and is the current defined by some sum of the irreducible components $f^{-1}(y)_j$ of $f^{-1}(y)$: $S=\sum n_j[f^{-1}(y)_j]$. Therefore, S is determined by the n_j, and the n_j are determined by the behavior of S at points in $R(f^{-1}(y))$.

In fact, all we need to show is that for each component $f^{-1}(y)_j$ there is a point $x_j\in R(f^{-1}(y)_j)$ and a form $u_j\in A_c^{2k-2d}(X)$ with the following properties: supp $u_j\cap f^{-1}(y)\subset R(f^{-1}(y)_j)$, $\int_{f^{-1}(y)}u_j=1$, and $f_*([X]\llcorner u_j)=f_*[u_j]$ is given by a continuous function.

This suffices, for

$$n_j=S(u_j)=\lim_{i\to\infty}\langle[X],f,y_i\rangle(u_j)=\lim_{i\to\infty}f_*[u_j](y_i)=f_*[u_j](y),$$

which is independent of the sequence y_i. This u_j is supplied by Lemma 3.2.6, so we are done.

Conclusion for Case A. We have constructed a function Φ: $Y \to Z_{k-d}(X)$, continuous in the $\mathfrak{F}$ topology, by setting $\Phi(y) = \lim \langle [X], f, y_i \rangle$ for any sequence $y_1, y_2, \ldots$ in G. By $\mathfrak{F}$ continuity, for any $u \in A_c^{2k-2d}(X)$, the measurable function of $y \langle [X], f, y \rangle (u) = \Phi(y)(u)$, which defines the current $f_*([X] \llcorner u)$, is continuous. This implies that $\langle [X], f, y \rangle$ is defined for all $y \in Y$ and equals $\Phi(y)$. The continuity for $u \in CA^{2k-2d}(X)_f$ follows by approximating u with smooth forms.

Case B. Y is a locally irreducible complex analytic space; X has a countable base for the open sets.

Step B1: Boundedness. The set $\{\langle [X], f, y \rangle$: $y \in R(Y)\}$ is a bounded set.

This set is bounded near $x \in X$ by Step A2 if $f(x) \in R(Y)$. If $f(x) \in S(Y)$ the result follows by an indirect application of Step A2. Let g be a proper finite map of a neighborhood of $f(x)$ onto a polydisc Δ in $\mathbf{C}^d$. Then the set $\{\langle [X], g \circ f, s \rangle$: $s \in \Delta\}$ is bounded because Δ is a manifold, by Step A2. Since $M(\langle [X], f, y \rangle \,|\, U) \leqslant M(\langle [X], g \circ f, g(y) \rangle \,|\, U)$ for any $y \in R(Y)$ and any open $U \subset X$, we have the desired result.

Step B2: Filling in the holes. By Case A, we have the theorem for $R(Y)$. We use the same kind of convergence argument as in Step A3 to show that if $y \in S(Y)$ and $y_1, y_2, \ldots$ is a sequence in $R(Y)$ converging to y there is a subsequence of $\langle [X], f, y_i \rangle$ converging to a current $S = \sum n_j [f^{-1}(y)_j] \in Z_{k-d}(X)$.

To show the uniqueness of S we choose a $u_j \in A_c^{2k-2d}(X)$ such that supp $u_j \cap f^{-1}(y) \subset R(f^{-1}(y))$ and $\int_{f^{-1}(y)} u_j = 1$; then $n_j = S(u_j)$.

By Case A, $\langle [X], f, z \rangle (u_j)$ is a continuous function of $z \in R(Y)$. Since Y is a locally irreducible complex analytic space, by a topological lemma found in [1], pp. 326–327, the fact that the limit points of $\langle [X], f, y_i \rangle (u_j)$ for any sequence $y_i \to y$, $y_i \in R(Y)$, are a subset of the integers implies that there is a unique continuous extension of $\langle [X], f, z \rangle (u_j)$ to all of Y. Since n_j is the value of this extension at y, S is uniquely determined.

Thus in Case B we can define $\Phi(y) = S$ as before, and the rest of the theorem follows as before.

Case C: No restriction on X.

Proof of C. Even if X does not have a countable basis for the open sets the theorem is still true; for if $u \in A_c^{2k-2d}(X)$, there is an open set containing supp u which does have a countable basis. Since slicing is local (Theorem 2.3.3(4)), this is sufficient. □

Remark. The requirement that Y be locally irreducible is clearly necessary. If near the point y, $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_k$, where Y_i is an irreducible component, then set $X = Y$, $T = [Y_i]$, and let $u \in A^0(X)$ be a function with $u(y) = 1$.

THEOREM 3.3.4. *If $f: X \to V^d$ is a holomorphic map from a complex space X to a manifold V and $T \in F_r^{\mathrm{loc}}(X)$, $r \geqslant 2d$, and if $u \in A^{r-2d}(X)$ is such that $f|\operatorname{supp} T \cap \operatorname{supp} u$ is proper, then $f_*[T \llcorner u] = [\lambda]$ for some locally integrable function λ on V.*

(i) *If $bT = 0$, $du = 0$, then λ is locally constant.*

(ii) *If $d''T = 0$, $d''u = 0$, then λ is holomorphic on V.*

(iii) *If $T \in Z_k(X)$, u is real and the current $[id'd''u]$ is positive, then if λ is upper semicontinuous, λ is plurisubharmonic.*

Remark. If $f: X \to Y$ is holomorphic, we can apply the theorem to $R(Y)$ and if Y is normal, locally irreducible, etc., can apply extension theorems to conclude that λ is holomorphic, etc., on all of Y.

Proof. This theorem is essentially in [3] p. 71, and the proof is the same here. Under the hypotheses (i), (ii), (iii), $f_*[T \llcorner u] = [\lambda]$ satisfies $b[\lambda] = 0$, $d''\gamma[\lambda] = 0$, and $[\lambda]$ is real and $id'd''[\lambda]$ is positive. Therefore, by 2.1.9 or [12], p. 25, we are done.

4. Intersection and multiplicity

The Fibering Theorem can be used to define intersection of holomorphic cycles. In Section 4.1 we show that this definition coincides with the classical definition in Draper [6] and the homological definition in Borel–Haefliger [5].

In Section 4.2 the Lelong number of complex analytic set at a point is defined (the multiplicity in another guise) and is proved to be an integer using the Fibering Theorem.

4.1. Intersection theory

Using the Fibering Theorem we will define the intersection of two holomorphic cycles. We shall need the following theorem about iterated slicing.

PROPOSITION 4.1.1. *If L^l, M^m, and N^n are C^∞ manifolds, if $T \in F_{m+n+r}(L)$, and if $f: L \to M$ and $g: L \to N$ are C^∞ maps, then for almost all $(x, y) \in M \times N$ the following slices exist and are equal: $\langle T, h, (x, y)\rangle = \langle\langle T, f, x\rangle, g, y\rangle$, where $h = (f, g): L \to M \times N$ is the Cartesian product of f and g. $\langle T, (f, g), (x, y)\rangle = (-1)^{mn}\langle T, (g, f), (y, x)\rangle$.*

Proof. See [7], p. 441. □

COROLLARY. *Suppose that X, $Y_1^{n_1}$, $Y_2^{n_2}$ are complex spaces, Y_1 and Y_2 locally irreducible, and $f_i: X \to Y_i$, $i = 1, 2$, are holomorphic. Then if $h = (f_1, f_2): X \to Y_1 \times Y_2$ and $T \in Z_k(X)$ are such that* $\dim (h^{-1}(x, y) \cap \operatorname{supp} T) = k - n_1 - n_2$ *for all $(x, y) \in Y_1 \times Y_2$, the following slices exist and are equal for all $(x, y) \in Y_1 \times Y_2$: $\langle T, h, (x, y)\rangle = \langle\langle X, f_1, x\rangle, f_2, y\rangle \in Z_{k-n_1-n_2}(X)$.*

Proof. The two slices are continuous functions of (x, y) by Theorem 3.3.2. Since they agree on a dense set, they are equal. $\square$

In papers by Draper [6] and Borel–Haefliger [5] there are definitions of intersection theory. We will give a definition using slicing and show that this is the definition given in the other cases. Then we will see that if $f\colon V \to W$ has rank = dimension W, and $X \in Z_k(V)$, the intersection of $f^{-1}(y)$ and X is $\langle X, f, y\rangle$.

PROPOSITION 4.1.2. *Let W^n be a complex submanifold of the manifold V^m which is defined by $m-n$ global functions; i.e., there is a holomorphic function $f\colon V\to \mathbf{C}^{m-n}$ which has maximal rank at every point and $f^{-1}(0) = W$. Then if $T\in Z_k(V)$, $k \geqslant n$, and* $\dim(W\cap \operatorname{supp} T) =$ *$k+n-m$, we define the intersection of W and T to be $W \cdot T = \langle T, f, 0\rangle$, which is a well-defined element of $Z_{k+n-m}(V)$ independent of the defining functions.*

Proof. It is clear that $\langle T, f, 0\rangle \in Z_{k+n-m}(V)$ is defined; since $\dim (f^{-1}(0)\cap \operatorname{supp} T) =$ $k+n-m$, by the upper semicontinuity of the fiber dimension (see [9], p. 159) there is a neighborhood $U \subset V$ of A such that $\dim (f^{-1}(y) \cap \operatorname{Supp} T \cap U) = k+m-n$ for all $y \in \mathbf{C}^{m-n}$. Then apply Theorem 3.3.2.

Therefore, it remains to show that if $f_0\colon V\to \mathbf{C}^{m-n}$ and $f_1\colon V\to\mathbf{C}^{m-n}$ are holomorphic functions with maximal rank everywhere and $f_0^{-1}(0) = f_1^{-1}(0) = W$, then $\langle T, f_0, 0\rangle = \langle T, f_1, 0\rangle$. Since these currents are in $Z_{k+n-m}(V)$, to show that they are equal it suffices to show that for any $x_0 \in R(W\cap \operatorname{supp} T)$, the multiplicity of the component containing x_0 is the same for both currents. Let N_0, N_1 be the respective multiplicities.

Since $R(W\cap\operatorname{supp} T)$ is a manifold, we can find a neighborhood $U_0 \subset V$ of x_0 and a submanifold Y of U of dimension $2m-n-k$ such that $Y\cap W\cap \operatorname{supp} T = \{x_0\}$ and Y is transverse to $R(W\cap\operatorname{supp} T)$. If we choose U_0 small enough we can assume $Y = g^{-1}(0)$, where $g\colon\ U_0 \to \mathbf{C}^{k+n-m}$ is a holomorphic function with maximal rank everywhere, and $g\,|\,R(W\cap \operatorname{supp} T)\cap U$ has maximal rank (this comes from transversality). Therefore, $\langle\langle T, f_i, 0\rangle\,|\,U, g, 0\rangle = N_i[x_0] \in Z_0(V)$.

However, by the Corollary to Proposition 4.1.1 $\langle\langle T, f_i, 0\rangle\,|\,U, g, 0\rangle = \langle T\,|\,U, (f_i, g), (0, 0)\rangle$. If we substitute in the hypotheses of Proposition 4.2, U for V, $W\cap Y$ for W, and (f_i, g) for f, we see that we have reduced the question to proving the Proposition for the case $k = m-n$.

Let $k = m-n$; we have two maps f_0 and f_1, finite on supp T at x_0, both of rank k on V with $f_0^{-1}(0) = f_1^{-1}(0) = W$. If we had $T = [\operatorname{supp} T] = [X]$, we would wish to show that the analytic covers $f_0\,|\,X$ and $f_1\,|\,X$ have the same number of sheets. This result can be found in [6], p. 184, Lemma 3.2; and since T is the sum of currents of the form $[X]$, for some X, we are done.

A proof of this result can be given using the Fibering Theorem. We find a family of holomorphic maps f_z: $U_0 \to \mathbf{C}^k$ holomorphic in z, with f_0, f_1 the same as before (up to coordinate change) such that $\langle [X], f_z, 0 \rangle$ is a continuous function into $Z_0(\{x_0\})$, hence constant. $\square$

Definition 4.1.3. If V is a complex manifold of dimension m, W is a submanifold of dimension n, and $T \in Z_k(V)$, with $\dim (W \cap \operatorname{supp} T) = k+n-m$, then we define $W \cdot T \in Z_{k+n-m}(V)$ to be the current such that $W \cdot T \mid U = W \mid U \cdot T$ in any open set U in which W is defined by global equations of maximal rank, as defined in Proposition 4.1.2. By this Proposition $W \cdot T$ is well-defined, If $S \in Z_k(V)$ and $T \in Z_n(V)$ then $S \cdot T$ is the unique element of $Z_{k+n-m}(V)$ such that $\Delta_*(S \cdot T) = \Delta(V) \cdot S \times T$ where Δ: $V \to V \times V$ is the diagonal map. This is well-defined by Proposition 4.2. When $S \cdot T$ is defined and E is an irreducible component of supp $(S \cdot T) \subset \operatorname{supp} S \cap \operatorname{supp} T$, we define the *intersection multiplicity* $i(S, T, E)$ to be the multiplicity of the component E in the current $S \cdot T$, (i.e., the unique integer such that $E - \operatorname{supp}(S \cdot T - i(S, T, E)[E]) \neq \emptyset$).

It is not immediately clear from the definition that the intersection of cycles with cycles and manifolds with cycles is the same; but this is shown in the next proposition.

PROPOSITION 4.1.4. *If $S = [W] \in Z_n(V^m)$, the current defined by a submanifold, and $T \in Z_k(V^m)$, then $S \cdot T = [W] \cdot T = W \cdot T$ whenever* $\dim (W \cap \operatorname{supp} T) = k+n-m$.

Proof. Since the question is local, we can suppose $W = g^{-1}(0)$, where g has rank $m-n$ on V and $\Delta(V) = f^{-1}(0)$, where f has rank n on $V \times V$. Let p_1: $V \times V \to V$ be projection on the first factor; then $p_1 \circ \Delta$ is the identity map on V.

Let $u \in A_c^{2(k+n-m)}(V \times V)$. By definition $S \cdot T(u) = \langle S \times T, f, 0 \rangle (p_1^* u)$; but $S \times T = [W] \times T = \langle [V] \times T, g \circ p_1, 0 \rangle$ (this is clear since $g \circ p_1$ has rank $m-n$ on $V \times R(\operatorname{supp} T)$). So $S \cdot T = \langle \langle [V] \times T, f, 0 \rangle, g \circ p_1, 0 \rangle$ by the Corollary to Proposition 4.1.2, since slicing by (f, g) is independent of the order of f and g.

We see that $\langle [V] \times T, f, 0 \rangle = \Delta_* T$ because f has rank m on $V \times R(\operatorname{supp} T)$. Therefore, $S \cdot T(u) = \langle \Delta_* T, g \circ p_1, 0 \rangle (p_1^* u)$, which is the value at zero of the function defining the current $(g \circ p_1)_* (\Delta_* T \llcorner p_1^* u) = g_*((p_{1*} \Delta_* T) \llcorner u) = g_*(T \llcorner u)$.

But the value at zero of the function defining the latter current is $W \cdot T(u)$. $\square$

Remark. By further use of the Corollary to Proposition 4.1.2, we may show the other elementary properties of intersection. In particular, the intersection product is associative and commutative.

We have now defined intersection; we prove the following proposition to show that this definition agrees with others.

Definition 4.1.5. An *intersection theory* I for holomorphic cycles on complex manifolds assigns for any $S \in Z_r(V)$, $T \in Z_s(V)$, such that $\dim(\operatorname{supp} S \cap \operatorname{supp} T) = r + s - \dim V$ a unique cycle $I(S, T) \in Z_{r+s-\dim V}(V)$.

PROPOSITION 4.1.6. *Suppose I is an intersection theory satisfying the five properties below, then I is the intersection theory defined in Definition 4.1.3. (The use of the symbol $I(S, T)$ below assumes that* $\operatorname{supp} S \cap \operatorname{supp} T$ *has the right dimension. The symbol $\cdot$ denotes intersection as defined above.)*

(1) Supp $I(S, T) \subset \operatorname{supp} S \cap \operatorname{supp} T$.

(2) *If $[W_1]$ and $[W_2]$ are cycles defined by submanifolds which meet transversally (i.e., for all $x \in W_1 \cap W_2$, the tangent space $TV_x = TW_{1x} + TW_{2x}$), then $I([W_1], I([W_2], T)) = I([W_1 \cdot W_2], T)$ for any T.*

(3) *If* $\dim S + \dim T = \dim V$, $I(S, T) = S \cdot T$.

(4) $\Delta_*(I(S, T)) = I([\Delta(V)], S \times T)$.

(5) *If $U \subset V$ is an open set and if ϱ_U is the restriction map from currents on V to currents on U, then $I(\varrho_U S, \varrho_U T) = \varrho_U I(S, T)$.*

Proof. Let $I(S, T)$ be defined for $S \in Z_r(V^m)$, $T \in Z_s(V^m)$; to determine $I(S, T)$ it is sufficient to know the multiplicity at x for all $x \in R$ $(\operatorname{supp} S \cap \operatorname{supp} T)$. This is the same as the multiplicity n_x of $\Delta_*(I(S, T))$ at (x, x); by (4) $\Delta_*(I(S, T)) = I([\Delta(V)], S \times T)$.

Find an open neighborhood $U \subset V \times V$ of (x, x) and a submanifold $W \subset U$ of dimension $3m - r - s$, transversal to $R(\Delta(V) \cap \operatorname{supp} S \times T)$, with $W \cap \Delta(V) \cap \operatorname{supp} S \times T = \{(x, x)\}$. Then $I([W], \varrho_U I([\Delta(V)], S \times T)) = W \cdot \varrho_U I([\Delta(V)], S \times T) = n_x[(x, x)]$ by (3).

But by (5) this is just $I(W, I(\varrho_U[\Delta(V)], \varrho_U(S \times T)) = I(\varrho_U[W \cdot \Delta(V)], \varrho_U(S \times T)$ by (2). This equals $\varrho_U(W \cdot (\Delta(V) \cdot S \times T))$ by the associativity of $\cdot$, and this equals $N_x[(x, x)]$, where N_x is the multiplicity at (x, x) of $\Delta(V) \cdot X \times Y = \Delta_*(X \cdot Y)$ which equals the multiplicity of $X \cdot Y$ at x. $\square$

The paper of Draper [6] explicitly gives these properties for his definition of intersection, which is a "classical" one. The homological definition of Borel–Haefliger [5] gives all these properties except (4), but gives the projection formula, which implies (4). Thus these three definitions of intersection are the same.

4.2. The Lelong number

In this section we will define the Lelong number at a point for a closed, positive current. Then we will prove the result of Thie [26] that for a holomorphic cycle T the Lelong number is always an integer. A converse to this will be proved in Section 5.

Let $T \in \mathcal{D}'_{2k}(U)$ be a closed positive current in an open set $U \subset \mathbf{C}^n$. Let $(z^1, \ldots, z^n)$ be linear coordinates: $|z| = (\sum_{j=1}^n |z^j|^2)^{\frac{1}{2}}$. If $\eta = \frac{1}{4} i\,(d'' - d')\,|z|^2$, then $w = d\eta = \frac{1}{2} i\, d'd''\,|z|^2$ is the usual Kähler form.

Since T is closed and positive, $T \in N_{2k}^{\text{loc}}(U)$, [14] p. 3; and $T \llcorner w^k$ is a positive measure. Let $\lambda_{r,x}$ be the characteristic function of the ball of radius r about x; $\lambda_r(z) = 1$ if $|z-x| \leq r$, $\lambda_r(z) = 0$ if $|z-x| > r$.

Definition 4.2.1. If $x \in \operatorname{supp} T$, T a closed positive current as above, we define $n(x, r, T) = (1/\pi^k r^{2k}) T(\lambda_{r,x} w^k)$ for $r > 0$ and $n(x, T) = \lim_{r \to 0+} n(x, r)$.

The limit exists because $n(x, r)$ is a monotone increasing function of r ([14] p. 4, or for any volume minimizing rectifiable current see [7], section 5.4.3(2)).

THEOREM 4.2.2. (Thie [26]). *For a closed positive current T, $T \in Z_k(U)$ implies $n\,(x, T)$ is a positive integer for all $x \in \operatorname{supp} T$.*

This integer may be interpreted in terms of the tangent cone of T at x; [26], p. 310. Draper has shown that the Lelong number equals the algebraic multiplicity; [6] p. 202.

We will prove a converse of this in the next section. For completeness, we will give a proof of this theorem using the methods of this paper; the proof follows that of Thie, but the machinery that we have makes the proof easier to write down. A proof of this fact is implicit in 4.3.18 of [1] as well.

Proof of 4.2.2. We can assume $x = 0$, and we write $\lambda_r = \lambda_{r,0}$. If we choose clear coordinates, one can get volume estimates as in 3.2.5 and find a constant C such that $\mathbf{M}(T \llcorner \lambda_r) \leq Cr^{2k}$ for small r. [25], p. 16. By choosing coordinates by a constant multiple, we can assume the inequality holds for $r \leq 1$, and that the unit ball B_1 is contained in U.

Let μ_r: $\mathbf{C}^n \to \mathbf{C}^n$ be dilation by r, i.e. $\mu_r(z) = rz$. Then $(1/r^{2k})\, T(\lambda_r w^k) = (\mu_{1/r*} T)(\lambda_1 w^k)$. Now the current $\mu_{1/r*} T$ defines by restriction a current T_r in $Z_k(B_1)$. Furthermore, the inequality above gives $M(T_r) \leq C$, so by the Compactness Theorem 2.2.1 there is a subsequence T_{r_i} which converges to a current $S \in I_{2k}^{\text{loc}}(B_1)$. Also, since $\operatorname{supp} S$ is contained in the limit of the supp T_{r_i} (as closed sets, [25], p. 23), which is clearly the tangent cone of supp T at 0 (intersected with B_1) [29], pp. 510–11. Therefore $S \in Z_k(B_1)$ is a sum of complex cones.

Actually the net T_r converges. This can be seen by constructing a current $\tilde{T} \in Z_{k+1}(B_1 \times \mathbf{C})$ so that if p: $B_1 \times \mathbf{C} \to \mathbf{C}$ is the projection, $\langle \tilde{T}, p, r \rangle = T_r \times [r]$. If $T = [X]$, $\tilde{T} = [\tilde{X}]$, where $\tilde{X}$ is the closure of the set $\{(x, r)$: $rx \in X$ and $r \neq 0\}$ in $B_1 \times \mathbf{C}$ (cf. [8], Section 4.10). By the Fibering Theorem 3.3.2, T_r is continuous in r and $T_r \to S$ as $r \to 0$, where $S \times [0] = \langle \tilde{T}, p, 0 \rangle$.

Since $S = \lim_{i\to\infty} T_{r_i}$, $n(0, T) = (1/\pi^k)\, S(\lambda_1 w^k)$, and the question is reduced to finding the Lelong number of a cone.

Since this is algebraic, by Chow's Theorem, the results of [8] prove the theorem, but we will give a brief proof using the Fibering Theorem.

Let $\tilde{\eta} = i/4\,(d'' - d')\log |z|^2 = \eta/|z|^2$ and $\tilde{w} = d\tilde{\eta}$. Then $\tilde{\eta}\wedge\tilde{w}^s = \eta\wedge w^s/|z|^{2s+2}$. If $p:\mathbf{C}^n - \{0\} \to \mathbf{P}^{n-1}(\mathbf{C})$ is the usual projection onto projective space, $\tilde{w}$ is $p^*(\omega)$, where ω is the usual Kähler form on $\mathbf{P}^{n-1}(\mathbf{C})$.

If we approximate λ_1 by a smooth form λ, which is $\equiv 1$ on $B_{1-\varepsilon}$ and $\equiv 0$ outside $B_{1+\varepsilon}$, then $S(\lambda w^k) = S(\lambda d(\eta\wedge w^{k-1})) = -S(d\lambda\wedge\eta\wedge w^{k-1}) = -S(|z|^{2k} d\lambda\wedge\tilde{\eta}\wedge\tilde{w}^{k-1})$, since S is closed, and this equals $-\int_{p(\operatorname{supp} S)}(\langle S, p, y\rangle(|z|^{2k} d\lambda\wedge\tilde{\eta}))\,\omega$ by definition, since $p|\operatorname{supp} d\lambda$ is proper.

Since supp S is a cone, we can apply the Fibering Theorem with $Y = R(p(S))$. Since p acts locally like a product and the fibers are complex lines, $\langle S, p, y\rangle(|z|^{2k} d\lambda\wedge\tilde{\eta})$ is just the integral of $|z|^{2k} d\lambda\wedge\tilde{\eta}$ on a line.

Thus we are reduced to evaluating

$$\iint_{\mathbf{C}} |z|^{2k}\, d\lambda\wedge\tilde{\eta} = -\iint_{\mathbf{C}} \lambda\, d|z|^{2k}\wedge\tilde{\eta}$$

which is approximately equal to $-\int_0^{2\pi}\int_0^1 d(r^{2k})\,d\theta/2 = -\pi$, since λ approximates the characteristic function of the unit disc.

Thus we see that $S(\lambda_1 w^k) = \pi\tilde{S}(w)$, where S is a holomorphic cycle supported on $p(\operatorname{supp} S)\subset\mathbf{P}^{n-1}(\mathbf{C})$. But $\tilde{S}(w)$ is known to be $\pi^{k-1}m$, where m is an integer [16]. (It is homologous to m times a hyperplane.) □

5. Characterizations of holomorphic cycles

Here we prove two related theorems giving sufficient conditions for a current to be a holomorphic cycle. In Section 5.2 we show that a closed, positive locally rectifiable current is a holomorphic cycle. This is used in 5.4 to give proofs of two extension theorems of Shiffman [19], [20], and [21].

In Section 5.3 we show that a $2k$-current in an open set in $\mathbf{C}^n$ is a holomorphic cycle if it is closed, positive, and has integral Lelong numbers H^{2k} almost everywhere.

Both theorems require further results about the structure of rectifiable currents, as found in [7]. Some of these are stated in 5.1. Others, needed only for 5.3, are noted in that section.

5.1. Structure of integral currents

In this section we state theorems from [7] giving the structure of rectifiable (hence integral) currents in more detail. Since the results are local, we will restrict our attention to subsets of $\mathbf{R}^n$ and $\mathbf{C}^n$ with the usual metrics.

Definition 5.1.1. A set $E \subset \mathbf{R}^n$ is *k-rectifiable* if there exists a Lipschitzian function mapping some bounded subset of $\mathbf{R}^k$ onto E. If ϕ is a measure there is a weaker notion: E is *(ϕ, k)-rectifiable* if $\phi(E) < \infty$ and for every $\varepsilon > 0$ there exists a k-rectifiable set F with $\phi(E - F) < \varepsilon$.

The (H^k, k)-rectifiable sets are a natural generalization of a smooth manifold because it can be shown that for H^k almost all $x \in E$, if E is (H^k, k)-rectifiable, there is a k-dimensional linear *tangent space* $\operatorname{Tan}^k(H^k \llcorner E, x)$ at x, [7], p. 256.

Notation. If μ is a measure and E is a set, $\mu \llcorner E$ is the measure such that $\mu \llcorner E(A) = \mu(E \cap A)$. If η is a k-vector field, $(\mu \llcorner E) \wedge \eta$ is the current whose value on $u \in A_c^k(U)$ is $\int \langle u, \eta \rangle d\mu \llcorner E$.

THEOREM 5.1.2. *If f: $U \to \mathbf{R}^m$ is a C^∞ map, for open $U \subset \mathbf{R}^n$, and $E \subset U$ is an (H^k, k)-rectifiable set $k \geq m$, then for any $H^k \llcorner E$ integral function g,*

$$\int_E g J_m f dH^k = \int_{\mathbf{R}^m} \int_{f^{-1}(y) \cap E} g dH^{k-m} dH^m(y),$$

where the integral on the right exists and $J_k f(x)$ is the norm of $\Lambda^m Df(x)$: $\Lambda^m \operatorname{Tan}^k(H^k \llcorner E, x) \to \Lambda^m \operatorname{Tan}^m(\mathbf{R}^m, f(x))$.

Proof. [7], p. 258. □

Definition 5.1.3. Let $U \subset \mathbf{R}^n$ be open; if $S \in \mathcal{D}_k'(U)$ is representable by integration, there is a *positive measure* $\|S\|$ on U such that $S(u) = \int_U \langle u, \vec{S} \rangle d\|S\|$, where $\vec{S}$ is an $\|S\|$ measurable k-vector field with $\|S(x)\| = 1$ for $\|S\|$ almost all $x \in U$ (this norm is the dual of the one defined in 2.1.2), [7], p. 348, p. 357. The k-dimensional density

$$\Theta^k(\|S\|, x) = \lim_{t \to 0+} a(k)^{-1} t^{-k} \|S\| (B_t),$$

where B_t is the ball of radius t about x and $\alpha(k)$ is the k-volume of the unit k-ball $(= \pi^r / r!$ if $k = 2r)$, whenever this limit exists.

THEOREM 5.1.4. *Let $U \subset \mathbf{R}^n$ be open, $S \in \mathcal{E}_k'(U)$, then the following are equivalent:*

(1) *S is a rectifiable current, i.e.,* $S \in R_k(U)$.

(2) *There exists an H^k-measurable and (H^k, k)-rectifiable subset B of* supp S *and an $H^k \llcorner B$ summable k-vectorfield η such that $S = (H^k \llcorner B) \wedge \eta$; and for H^k almost all $x \in B$, $\eta(x)$ is a simple k-vector, $|\eta(x)|$ is a positive integer, and the subspace* $\operatorname{Tan}^k(H^k \llcorner B, x)$ *is represented by $\eta(x)$.*

Moreover, $\|S\| = H^k \llcorner \theta^k(\|S\|; \cdot)$; and $|\eta(x)| = \theta^k(\|S\|, x)$ for H^k almost all x.

Proof. [7] pp. 384–386. □

Thus $S \in I_k^{\text{loc}}(U)$ if S and bS satisfy condition (2) locally.

THEOREM 5.1.5. *If $S \in R_k(U)$ (we use the notation of the preceding theorem) and f: $U \to R^m$ is a C^∞ map, $k \geqslant m$, then for H^m almost all $y \in R^m$, $\langle S, f, y \rangle = (H^{k-m} \llcorner f^{-1}(y)) \wedge \zeta$, $\zeta(x)$ is the simple $(k-m)$-vector representing the $(k-m)$ dimensional oriented linear space $\eta(x) \cap \ker(Df(x))$ with $|\zeta(x)| = |\eta(x)|$* (ker *$Df(x)$ is the vertical subspace of the tangent space at x).*

Proof. [7], p. 444. □

5.2. Integral currents and holomorphic cycles

We wish to prove the following theorem:

THEOREM 5.2.1. *Let $S \in R_{2k}^{\text{loc}}(W^n)$, where W is a complex manifold; suppose that $bS = 0$ and that S is a positive current, then $S \in Z_k(W^n)$.*

The method of proof is to write supp S locally as a finite "branched cover" in some general sense, and then to use the standard methods of local analytic geometry to find holomorphic functions vanishing on supp S.

Proof of 5.2.1. Since the result is local, assume $W^n = U$, an open set in $\mathbf{C}^n$. We shall use the notation of Theorem 5.1.4; $S = (H^{2k} \llcorner B) \wedge \eta$. We assume that the $(H^{2k}, 2k)$-rectifiable set B is chosen so that for all $x \in B$ the properties of 5.1.4 (2) hold and $|\eta(x)| = \theta^{2k}(\|S\|, x)$, which we will denote by $\theta(x)$ for the duration of the proof.

The fact that $S \in \mathcal{D}'_{k,k}(U)$ implies that $\eta(x)$ represents a complex subspace for H^{2k} almost all $x \in B$, since (k, k) currents are invariant under multiplication by $\sqrt{-1}$ in the tangent space. Since S is positive, this implies that S is volume minimizing ([7], p. 652), and so $H^{2k+\alpha}$ (supp S) $= 0$ for $\alpha > 0$ by [7], p. 628 (5.4.5 (2)), p. 181 (3), and p. 173. (For a general rectifiable S, the closure of B may be large.)

Therefore, given some $x \in \operatorname{supp} S$, which we will assume to be 0, we will choose clear coordinates as in 3.2.1 by Theorem 2.4.4. Thus for each coordinate projection π: $\mathbf{C}^n \to \mathbf{C}^k$ there is an open neighborhood $V = V_k \times V_{n-k}$ of 0 such that $\pi \mid V \cap \operatorname{supp} S$ is proper and π: $V \to V_k$ is the projection on the first coordinates. We let $(z_1, ..., z_k)$ be coordinates in V_k and $(w_1, ..., w_{n-k})$ in V_{n-k}.

The closed current $\pi_* S \in I_{2k}^{\text{loc}}(V_k)$ equals $m[V_k]$ for some nonnegative integer m, by 2.1.9. By 5.1.5, for almost all $y \in V_k$, $\langle S, \pi, y\rangle$ exists and equals

$$\sum_{x \in \pi^{-1}(y) \cap B} \theta(x)[x] \in Z_0(V), \qquad \text{where} \sum_{x \in \pi^{-1}(y) \cap B} \theta(x) = \langle S, \pi, y\rangle (1) = m.$$

Denote the subset of y in V_k for which this is true by $G \subset V_k$.

We wish to construct a holomorphic function $P_j(z, w) = w_j^m + a_{ij}(z) w_j^{m-1} + ... + a_{mj}(z)$, where the $a_{ij}(z_1, ..., z_k)$ are holomorphic functions on V_k, so that $P_j(x) = 0$ for all $x = (z, w) \in \pi^{-1}(G) \cap B$. We do this by adapting the method of Bishop described in [25], pp. 30–33; it follows a construction in local analytic geometry [9], Chapter III.

Let $P_j(z, W)$ with indeterminate W be given by $P_j(z, W) = \prod_{\{x_r\}_z} (W - w_j(x_r))^{\theta(x_r)}$, where $\{x_r\}_z = \{x \in \pi^{-1}(z) \cap B\}$, for $z \in G$. For each such z this is a monic polynomial of degree m, and the coefficient $a_{ij}(z)$ is given by the ith elementary symmetric function in m variables, σ_i, applied to the m-tuple $\mathbf{w}_j(z) = (w_j(x_1), ..., w_j(x_1), w_j(x_2), ..., w_j(x_q))$, where each $w_j(x_r)$ appears $\theta(x_r)$ times.

But the polynomial $\sigma_i(t_1, ..., t_m)$ can be expressed by a polynomial in the power sums $S_p = \sum_i t_i^p$, $p = 0, ..., i$ [27], p. 81. Furthermore, the function of z, $S_p(\mathbf{w}_j(z)) = \sum_{\{x_r\}} \theta(x_r) w_j^p(x_r) = \langle S, \pi, z\rangle (w_j^p)$, which by 3.3.4 is a holomorphic function ($d''S = 0$, since $bS = 0$ and $S \in \mathcal{D}'_{k,k}$). Hence $a_{ij}(z)$ is a holomorphic function also. If we redefine G by removing a set of H^{2k} measure 0, we have $P_j(z, w_j) = 0$ for any $x = (z, w) \in \pi^{-1}(G) \cap B$.

If we set $X = \{x \in U\colon P_j(x) = 0,\ j = 1, ..., k - n\}$, this is a subvariety of dimension k (possibly $\varnothing$ if $m = 0$) and $\pi^{-1}(G) \cap B \subset X$. We are not through, however, for *a priori* there might be vertical components of B. Therefore, for each k-tuple I, we do the above construction for the coordinate projection $\pi_I = \mathbf{C}^n \to \mathbf{C}^k$ and get V_I, G_I, X_I, etc. for each I.

Let $V = \bigcap V_I$ and $X = \bigcup X_I$; we wish to show that H^{2k} almost all of $B \cap U$ is contained in X.

Let $B_I = \{x \in B\colon J_{2k}\pi_I(x) \neq 0\}$ (i.e., $D\pi_I$ has rank $2k$ on $\eta(x)$). Let $C_I = B \cap V - \pi_I^{-1}(G_I)$. Then $H^{2k}(B_I \cap C_I) = 0$, for by 5.1.2

$$\int_{B_I \cap C_I} J_{2k}\pi_I(x)\, dH^{2k}(x) = \int_{V_{k_I} - G_I} \left(\int_{\pi_I^{-1}(y) \cap B} g\, dH^0 \right) dH^{2k}(y) = 0,$$

where g is the characteristic function of $B_I \cap C_I$. Since $J_{2k}\pi_I > 0$ on $B_I \cap C_I$, $H^{2k}(B_I \cap C_I) = 0$.

Since the π_I range over each projection onto a k-plane in these clear coordinates, $B = \bigcup B_I$. Now $(B \cap V) \cap (\bigcup \pi_I^{-1}(G_I)) \subset X$, but $(B \cap V) - \bigcup_I \pi_I^{-1}(G_I) = \bigcap_I (B \cap V - \pi_I^{-1}(G_I)) \subset \bigcap_I [(B - B_I) \cup (B_I \cap C_I)] = [\bigcap_I (B - B_I)] \cup D = D$, where D is the union of the remaining intersections. Since $D \subset \bigcup_I (B_I \cap C_I)$, $H^{2k}(D) = 0$.

Thus we have shown $H^{2k}(B \cap V - X) = 0$ and $S | V = (H^{2k} \llcorner B \cap X) \wedge \eta$. Since X is closed in U, supp $S \cap V \subset X$. But by 3.1.3 this implies $S | V \subset Z_k(V)$. Since we chose any $x \in \operatorname{supp} S$ to be 0, this shows $S \subset Z_k(U)$. □

5.3. The Lelong number and holomorphic cycles

The following result is a partial answer to a question raised by Lelong [14] p. 7; a more general result in codimension one has been obtained by E. Bombieri [30], [31]. The method of proof will be to show that S satisfies (2) of Theorem 5.1.4; then Theorem 5.2.1 will give the desired conclusion.

THEOREM 5.3.1. *If $U \subset \mathbf{C}^n$ is an open set and $S \in \mathcal{D}'_{2k}(U)$ is a closed positive current with Lelong number $n(x, S)$ equal to a positive integer for H^{2k} almost all $x \in$ supp S, then $S \in Z_k(U)$.*

Proof. As we observed in 4.2, since S is closed and positive, $S \in N_{2k}^{\mathrm{loc}}(U)$. Since the theorem is local, we may assume—shrinking U if necessary that $\mathbf{M}(S) < \infty$.

Let $\omega = w^k/k!$, where w is the Kähler form. As noted in 3.1.4

$$\|S\|(U) = \mathbf{M}(S \mid U) \geqslant (S \mid U)(\omega).$$

Therefore, comparing Definitions 5.1.3 and 4.2.1 we see that $\theta^{2k}(\|S\|, x) \geqslant n(x, S) \geqslant 1$ for H^{2k} almost all $x \in \operatorname{supp} S$.

Now using a result about densities, ([7], p. 181, 2.10.19 (3) and p. 171, 2.10.2)

$$H^{2k}(A) \leqslant \|S\|(A) \quad \text{for any } A \subset \operatorname{supp} S. \tag{*}$$

Thus $H^{2k}(\operatorname{supp} S) < \infty$.

We also have, by a more refined version of the Measure Support Theorem, 2.4.2,

$$\|S\|(A) = 0 \quad \text{if } \mathcal{G}_1^{2k}(A) = 0 \text{ for any } A \subset V. \tag{**}$$

(See [7], p. 410.) The integral geometric measure $\mathcal{G}_1^{2k}$ is a measure $\leqslant cH^{2k}$ for a constant $c > 0$ ([7] p. 173); therefore, $H^{2k}(A) = 0$ implies $\|S\|(A) = 0$.

Now (**) implies that $\theta^{2k}(\|S\| \text{ supp } S, x) > 0$ for $\|S\|$ almost all $x \in \operatorname{supp} S$. This implies ([7], p. 299, 3.3.15 and p. 171, 2.10.2) that supp $S = B \cup C$ where B is $(\|S\|, 2k)$ rectifiable and $\mathcal{G}^k(C)=0$. Since by (**) $\|S\|(C)=0$, supp S is $(\|S\|, 2k)$ rectifiable. By (*) supp S is also $(H^{2k}, 2k)$ rectifiable.

Then (by [7], p. 255, 3.2.18) there exist compact subsets $K_1, K_2, \ldots$ of $\mathbf{C}^k$ and Lipschitzian maps $f_1, f_2, \ldots$ of $\mathbf{C}^k$ into $V \subset \mathbf{C}^n$ such that $f_1(K_1), f_2(K_2), \ldots$ are disjoint subsets of supp S with $H^{2k}(\operatorname{supp} S - \bigcup_{i=1}^{\infty} f_i(K_i))=0$. Moreover, each f_i is one-to-one, and the Lipschitz constants of f_i and $(f_i|K_i)^{-1}$ are ≤ 2.

By (**) $S=\sum_{i=1}^{\infty} S \llcorner f_i(K_i)$. We can extend $(f_i|K_i)^{-1}$ to a Lipschitzian map $g_i: V \to \mathbf{C}^k$ ([7] p. 201). The current $S_i = S \llcorner f_i(K_i) \in F_{2k}(V)$ as noted in the discussion after 2.1.6; therefore, by the Support Theorem for Lipschitzian maps, [7] p. 373, (cf. 2.1.8) $S_i = f_{i*} g_{i*} S_i$ and this equals $(H^{2k} \llcorner f_i(K_i)) \wedge \eta$ (by 2.3.1 and [7] p. 383, 4.1.25), where $\eta(x)$ is a simple $2k$-vector associated with $\operatorname{Tan}^{2k}(H^{2k} \llcorner f_i(K_i), x)$ for H^{2k} almost all $x \in f_i(K_i)$.

Thus we have that $S=(H^{2k} \llcorner \operatorname{supp} S) \wedge \eta$ and for H^{2k} almost all $x, \eta(x)$ is a simple $2k$-vector associated with the $2k$-dimensional real linear subspace $\operatorname{Tan}^{2k}(H^{2k} \llcorner \operatorname{supp} S, x)$ [7], p. 254).

Since $H^{2k}(\operatorname{supp} S) < \infty$, (**) and the Radon–Nikodym Theorem imply that $S=\|S\| \wedge \vec{S}$ $=(H^{2k} \llcorner \operatorname{supp} S) \wedge \lambda \vec{S}$, where λ is a $H^{2k} \llcorner \operatorname{supp} S$ summable function. Since $\lambda(x)\vec{S}(x) = \eta(x)$, a simple vector (for $H^{2k} \llcorner \operatorname{supp} S$ almost all x), and $S \in \mathcal{D}'_{k,k}(V)$, by the argument in the proof of 5.2.1, $\vec{S}(x)$ represents a complex linear subspace. Since $|\vec{S}(x)|=1$, $\langle \omega(x), \vec{S}(x)\rangle = 1$, for these x and $\|S\|(U)=\int_U \langle \omega, \vec{S}\rangle \, d\|S\| = S \llcorner U(\omega)$ for any Borel set U.

Thus we see that $\theta^{2k}(\|S\|, x)=n(x, S)$ for all x. Now the measure $\|S\|=(H^{2k} \llcorner \operatorname{supp} S) \wedge \lambda$; by Lebesgue's Theorem ([7], p. 156, 2.9.8, cf. Section 2.3) for H^{2k} almost all x,

$$\lambda(x)=\lim_{t \to 0+} \frac{\|S\|(B_t)}{H^{2k} \llcorner \operatorname{supp} S(B_t)}.$$

But this limit equals $$\theta^{2k}(\|S\|, x)/\theta^{2k}(H^{2k} \llcorner \operatorname{supp} S, x),$$

which equals $\theta^{2k}(\|S\|, x)$ since the denominator equals 1 (for H^{2k} almost all x), [7], p. 256. Then $\lambda(x)=|\eta(x)|=n(x, S)$ is an integer for H^{2k} almost all x and by Theorem 5.1.4, $S \in I^{\text{loc}}(U)$. $\square$

5.4. Application to extension theorems

We will apply Theorem 5.2.1 to give proofs of two theorems of Shiffman about extension of analytic sets. Since the proof of 5.2.1 resembles Shiffman's proof of 5.4.1, there is little new in this proof; but the proof of 5.2.2 avoids certain estimates in the original proof. However, 5.2.2 gives only "half" of Shiffman's theorem.

THEOREM 5.4.1. (Shiffman). *Let U be open in $\mathbf{C}^n$ and let E be closed in U. Let X be a pure*

k-dimensional complex analytic set in $U-E$, and let X be the closure of $\bar{X}$ in U. If $H^{2k-1}(E)=0$, then $\bar{X}$ is a pure k-dimensional analytic set in U.

Proof. See [19] for original proof. We wish to show that the volume of X is locally bounded in U, for then X will define a current $[X]\in R_{2k}^{\text{loc}}(U)$. Furthermore, by the Measure Support Theorem 2.4.2, since clearly supp $b[X]\subset E$, $b[X]=0$. Then applying 5.2.1, we see that $[X]\in Z_k(U)$; and this implies the theorem.

To see that the volume of X is locally bounded, we observe that $H^{2k+a}(X\cup E)=0$ for $a>0$ and use 2.4.4 to get clear coordinates at each point. As in Section 3.2 a local upper bound for the volume is given by the number of sheets of the coordinate projections π: $(X\cup E)\cap U_k\times U_{n-k}\to U_k$. In this case, since we cannot assume $X\cup E$ is analytic, it is less clear that there is an upper bound for the number of sheets. However, let $F=S(X)\cup\{x\in R(X)\colon \text{rank } D\pi(x)<2k\}$; then $H^{2k-1}(E\cup F)=0$ and $H^{2k-1}(\pi(E\cup F))=0$. By [10] p. 104, therefore, the (topological) dimension of $\pi(E\cup F)\leqslant 2k-2$ and $U'_k=(U_k-\pi(E\cup F))$ is connected ([10] p. 48). Then $\pi\,|\,X\cap\pi^{-1}(U'_k)$ is a proper map with maximal rank, hence a covering space with a finite number of sheets. □

The other theorem deals with extension through $\mathbf{R}^n$. The theorem in [21] includes all dimensions, but the most interesting case is dimension one, given in [20], also which is included here.

Let ν: $\mathbf{C}^n\to\mathbf{C}^n$ be the complex conjugation map, $\nu(z_1, ..., z_n)=(\bar{z}_1, ..., \bar{z}_n)$. The set $\mathbf{R}^n\subset\mathbf{C}^n$ is left fixed by ν; also, for any complex analytic subvariety X of an open set $U\subset\mathbf{C}^n$, $\nu(X)$ is a subvariety of $\nu(U)$.

THEOREM 5.4.2. *Let $U\subset\mathbf{C}^n$ be an open set and let X be a complex subvariety of $U-\mathbf{R}^n$ of pure dimension k such that $X=\nu(X)$. Then if the volume of X is finite and k is odd, $\bar{X}$ the closure of X in U is a subvariety of pure dimension k.*

Proof. In this case the volume assumption says that X defines a current $[X]\in R_{2k}^{\text{loc}}(U)$; the question is whether $b[X]=0$. Now $\nu(X)=X$ as sets, but since ν is conjugate linear, the map ν_* reverses the orientation of the odd-dimensional manifold $R(X)$. Thus $\nu_*[X]=-[X]$, and so $b\nu_*[X]=\nu_*b[X]=-b[X]$.

But $b[X]\subset\mathbf{R}^n$ clearly and $b[X]\in F_{2k-1}^{\text{loc}}(U)$ since it is the boundary of a locally flat current, thus by the Support Theorem 2.1.8, $\nu_*b[X]=b[X]$. Thus $b[X]=0$ and we apply 5.2.1. □

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Added in proof:

[30]. BOMBIERI, E., Algebraic values of meromorphic maps. *Invent. Math.*, 10 (1970), 267–287.
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