THE COHOMOLOGY OF THE SPECTRUM OF A MEASURE ALGEBRA

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Let G be a locally compact abelian group, $M(G)$ the measure algebra on G, and Δ the spectrum or maximal ideal space of $M(G)$. It is common knowledge that, for non-discrete G, $M(G)$ is an extremely complicated Banach algebra with a very large spectrum which cannot be satisfactorily described. In fact, much of the research in the area has consisted of constructing measures in $M(G)$ which demonstrate that Δ fails to have a property one might have hoped for. For example, $M(G)$ is non-symmetric and, in fact, Δ contains infinite dimensional analytic structure (cf. [28], [18], [8], [19]); also, $M(G)$ has a proper Shilov boundary which is not the closure in Δ of the dual group of G (cf. [20], [12]). By contrast, one encouraging result on $M(G)$ is Cohen's Idempotent Theorem, which characterizes the idempotents in $M(G)$ (cf. [4]).

The purpose of this paper is to show that there is one sense in which $M(G)$ is surprisingly simple; specifically, the cohomology groups of its spectrum can be quite readily computed. In fact, to compute the cohomology groups of Δ one needs only to investigate the spectra of the algebras $L^1(G')$, where G' ranges over all l.c.a. groups which are continuously isomorphic to G. In degree zero this result is just Cohen's Idempotent Theorem. In degree one it leads to a characterization of those invertible measures in $M(G)$ which have logarithms in $M(G)$.

The class of algebras $M(G)$ is a subclass of the class of all convolution measure algebras. This larger class also contains the algebras $M(S)$ for S a locally compact topological semigroup and $L^1(G)$ for G a locally compact group. Our main results apply to any commutative, semi-simple convolution measure algebra.

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Certain subalgebras of a convolution measure algebra will be group algebras in a sense we shall make precise in § 2. Each such group algebra is contained in a unique maximal group algebra. Our main theorem states that the cohomology of the spectrum of a convolution measure algebra is determined solely by the maximal group algebras it contains.

THEOREM A. *Let $\mathfrak{M}$ be a commutative, semi-simple convolution measure algebra with a normalized identity δ. Let $\{\mathfrak{N}_\alpha\}$ be the collection of maximal group algebras in $\mathfrak{M}$, and for each α set $\mathfrak{N}'_\alpha=\mathfrak{N}_\alpha$ if $\delta\in\mathfrak{N}_\alpha$ and $\mathfrak{N}'_\alpha=\mathfrak{N}_\alpha+\mathbf{C}\delta$ if $\delta\notin\mathfrak{N}_\alpha$. For each α, let Δ_α be the spectrum of $\mathfrak{N}'_\alpha$ and let Δ be the spectrum of $\mathfrak{M}$. Then for any coefficient group K, there is a map $i^*: \sum_\alpha \oplus H^p(\Delta_\alpha, K)\to H^p(\Delta, K)$ of cohomology, which is an isomorphism for $p>0$ and is onto for $p=0$.*

This theorem will be proved in § 3. It applies, in particular, when $\mathfrak{M}=M(G)$. In this case the maximal group algebras have the form $L^1(G_\alpha)$, where G_α ranges over those l.c.a. groups whose group is G and whose topology is at least as strong as that of G. Hence, each Δ_α will be the one point compactification of the dual group $\hat{G}_\alpha$ of G_α or $\hat{G}_\alpha$ itself if G_α is discrete. Thus, for the most interesting groups G, the expression $H^p(\Delta, K)$ will be quite computable. For example, if $G=R$ (the group of reals) then $H^0(\Delta, Z)=Z$, $H^1(\Delta, Z)=Z\oplus R$, and $H^p(\Delta, Z)=\Lambda^p R$ (the p-fold exterior product of R over Z) for $p>1$ (Corollary 3.15).

Theorem A has several consequences. For example, the Shilov Idempotent Theorem (cf. [17], [13]) implies that $H^0(\Delta, Z)$ is the additive subgroup of $\mathfrak{M}$ generated by the idempotents of $\mathfrak{M}$. This along with Theorem A yields the following generalization of Cohen's Idempotent Theorem:

THEOREM B. *With $\mathfrak{M}$ as in Theorem A, each idempotent $\mu\in\mathfrak{M}$ has the form $\mu=n_1\mu_1+\dots+n_k\mu_k$, where, for each i, $n_i\in Z$ and μ_i is a group character multiplied by the Haar measure in some group algebra contained in $\mathfrak{M}$.*

In the case where $\mathfrak{M}=M(G)$, each μ_i in the above theorem must have the form $d\mu_i=\chi_i d\nu_i$, where χ_i is a character on G and ν_i is Haar measure on some compact subgroup of G (cf. Corollary 4.1).

The Arens-Royden Theorem (cf. [1], [15]) implies that $H^1(\Delta, Z)$ is isomorphic to $\mathfrak{M}^{-1}/\exp(\mathfrak{M})$—the group of invertible elements of $\mathfrak{M}$ modulo the subgroup consisting of elements which have logarithms. Hence, applying Theorem A in the case $p=1$ leads to:

THEOREM C. *With $\mathfrak{M}$ as in Theorem A, each $\mu\in\mathfrak{M}^{-1}$ has a factorization of the form $\mu=\mu_1*\mu_2*\dots*\mu_k*e^\nu$, where $\nu\in\mathfrak{M}$ and each μ_i is an invertible element of an algebra $\mathfrak{N}'_i$ with $\mathfrak{N}_i$ a maximal group algebra in $\mathfrak{M}$.*

In case $\mathfrak{M}=M(G)$, each μ_i in the above theorem will be an element of $L^1(G_i)+\mathbf{C}\delta_0$ for some l.c.a. group G_i formed from G by possibly strengthening its topology (Corollary 4.2). If $G=R$ then the only such groups are R itself and R_d (R with the discrete topology). The cohomology groups of the spectra of $L^1(R)+\mathbf{C}\delta$ and $L^1(R_d)=M_d(G)$ are known. Hence, in this case, we can strengthen Theorem C and obtain:

COROLLARY 4.3. *If $\mu\in M(R)^{-1}$ then μ has a factorization of the form $\mu=\eta^k*\delta_x*e^\nu$ for some $k\in Z$, $x\in R$, and $\nu\in M(R)$; here η may be chosen to be any element of $L^1(R)+\mathbf{C}\delta_0$ whose Fourier transform has winding number one about zero as a function on $R\cup\{\infty\}$.*

The above corollary can be used to characterize the spectrum of each Wiener-Hopf operator W_μ, where $W_\mu f(x)=\int_0^\infty f(t)\,d\mu(x-t)$ for $f\in L^1(R^+)$, $x\geqslant 0$, and $\mu\in M(R)$ (cf. [5]).

Theorems B and C and their corollaries are proved in § 4. In § 1 we discuss terminology and background information, while § 2 is devoted to a characterization of the group algebras that occur in a given $\mathfrak{M}$. The critical Lemma of the paper (Lemma 5.1) is proved in § 5; this proof is delayed until the end of the paper because it involves specialized techniques that are not used elsewhere.

We apologize to the reader for the fact that the paper is far from being self-contained. The machinery developed in [19], [21], [22], and [23] is not common knowledge, but is used here in an essential way. Also, we assume throughout that the reader is familiar with basic commutative Banach algebra theory and harmonic analysis on l.c.a. groups (cf. [13], [9], [16]). In § 3 we assume the reader has some familiarity with basic sheaf theory as presented in the first two chapters of [3].

We are indebted to our colleague, R. G. Douglas, for raising the question (when does $\mu\in M(R)^{-1}$ have a logarithm?) which led to this research.

1. Preliminaries

This section is devoted to establishing certain terminology and reviewing portions of the theory of convolution measure algebras.

A convolution measure algebra $\mathfrak{M}$ is a partially ordered complex Banach algebra which, as a partially ordered Banach space, is an L-space (cf. [14], [19]) and whose multiplication $(\mu,\nu)\to\mu*\nu$ satisfies certain conditions; specifically, if $\mu\geqslant 0$ and $\nu\geqslant 0$ then $\mu*\nu\geqslant 0$, $\|\mu*\nu\|=\|\mu\|\,\|\nu\|$, and the closed convex hull of $\{\mu_1\in\mathfrak{M}: 0\leqslant\mu_1\leqslant\mu\}*\{\nu_1\in\mathfrak{M}: 0\leqslant\nu_1\leqslant\nu\}$ is $\{\omega\in\mathfrak{M}: 0\leqslant\omega\leqslant\mu*\nu\}$ (cf. [19], [24]).

If X is a locally compact Hausdorff space, then $M(X)$ will denote the complex Banach space of all finite, inner regular, Borel measures on X. An L-subspace $\mathfrak{M}$ of $M(X)$ is a closed subspace such that $\mu\in\mathfrak{M}$, $\nu\in M(X)$, and ν absolutely continuous with respect to μ, imply that $\nu \in \mathfrak{M}$. Any L-subspace of $M(X)$ is a complex L-space. Conversely, a complex L-space can always be represented (in many ways) as an L-subspace of $M(X)$ for a locally compact Hausdorff space X (cf. [14]). The notions of total variation, absolute continuity, and mutual singularity can be defined purely in terms of the norm and order relation in $\mathfrak{M}$ and are independent of any representation of $\mathfrak{M}$ as an L-subspace of $M(X)$.

If $\mathfrak{M}$ is an L-space we shall use the notation "$\mu < < \nu$" for "μ is absolutely continuous with respect to ν", "$\mu \perp \nu$" for "μ and ν are mutually singular", and "$|\mu|$" for "the total variation measure for μ".

We shall use the term "CM-algebra" to abbreviate "convolution measure algebra".

An L-subspace (L-subalgebra; L-ideal) of a CM-algebra $\mathfrak{M}$ is a closed subspace (subalgebra; ideal) $\mathfrak{N}$ such that $\mu \in \mathfrak{N}$, $\nu \in \mathfrak{M}$, and $\nu < < \mu$, imply that $\nu \in \mathfrak{N}$. An L-subalgebra of a CM-algebra is also a CM-algebra (cf. [19], [24]).

If S is a locally compact topological semigroup, then the measure algebra $M(S)$ is a CM-algebra (cf. [19], [24]), as is any L-subalgebra of $M(S)$. In particular, if G is an l.c.a. group then $M(G)$ and $L^1(G)$ are CM-algebras.

The techniques we use to study algebras such as $M(G)$ often involve taking an L-subalgebra of $M(S)$ for a semigroup S (possibly G) and representing it as an L-subalgebra of $M(T)$ for another semigroup T. It is this frequent change of space that makes it necessary to define and study the category of abstract CM-algebras. The following definition describes the morphisms in this category:

Definition 1.1. (cf. [19], Def. 2.5). If $\mathfrak{M}$ and $\mathfrak{N}$ are CM-algebras, then a CM-map φ: $\mathfrak{M} \to \mathfrak{N}$ is a bounded algebraic homomorphism such that if $0 \leqslant \mu \in \mathfrak{M}$, then $\varphi\mu \geqslant 0$, $\|\varphi\mu\| = \|\mu\|$, and $\varphi\{\nu \in \mathfrak{M}\colon 0 \leqslant \nu \leqslant \mathfrak{M}\} = \{\omega \in \mathfrak{N}\colon 0 \leqslant \omega \leqslant \varphi\mu\}$.

If S and T are locally compact semi-groups and α: $S \to T$ is a continuous homomorphism, then the map $\mu \to \mu \circ \alpha^{-1}$: $M(S) \to M(T)$ is a CM-map (cf. [19], Lemma 1.2).

If φ: $\mathfrak{M} \to \mathfrak{N}$ is a CM-map, then the image of φ in $\mathfrak{N}$ is an L-subalgebra of $\mathfrak{N}$; this is easily proved from the conditions in Definition 1.1. Also, if a CM-map is one to one, then it is an order preserving isomorphism-isometry onto its image (cf. [19], Corollary to Theorem 1.2).

The main representation theorem for CM-algebras is the following:

PROPOSITION 1.1. (cf. [19], Theorems 2.2 and 2.3). *If $\mathfrak{M}$ is a commutative* CM-*algebra, then there is a compact topological semigroup S and a* CM-*map $\mu \to \mu_S$: $\mathfrak{M} \to M(S)$ such that:*

(1) *the image of $\mu \to \mu_S$ is weak-* dense in $M(S)$;*

(2) *the set $\hat{S}$ of all continuous semicharacters on S (non-trivial, multiplicative, complex valued functions on S) separates the points of S; and*

(3) *each complex homomorphism of $\mathfrak{M}$ has the form $\mu \to \int_S f d\mu_S$ for a unique $f \in \hat{S}$.*

Note that if $\mathfrak{M}$ is semisimple then (2) and (3) above imply that $\mu \to \mu_S$ is an order preserving isomorphism-isometry. In any case, the image of $\mu \to \mu_S$ is an L-subalgebra of $M(S)$ which is isomorphic to $\mathfrak{M}$ modulo its radical.

We shall call S the structure semigroup of $\mathfrak{M}$.

Proposition 1.2. *If $\mathfrak{M}$ and $\mathfrak{N}$ are commutative* CM-*algebras with structure semigroups S and T respectively, then each* CM-*map φ: $\mathfrak{M} \to \mathfrak{N}$ induces a continuous homomorphism α: $S \to T$ such that the diagram*

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\mu \to \mu_S"] \arrow[d, "\varphi"] & M(S) \arrow[d, "\tilde{\alpha}"] \\ \mathfrak{N} \arrow[r, "\nu \to \nu_T"] & M(T) \end{tikzcd}$$

commutes, where $\tilde{\alpha}\mu = \mu \circ \alpha^{-1}$.

If the adjoint φ^*: $\mathfrak{N}^* \to \mathfrak{M}^*$ maps the spectrum of $\mathfrak{N}$ onto the spectrum of $\mathfrak{M}$, then α is one to one.

Proof. Let $A \subset \mathfrak{M}^*$ and $B \subset \mathfrak{N}^*$ be the norm closed linear spans of the sets of complex homomorphisms of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Now $\mathfrak{M}^*$ and $\mathfrak{N}^*$ each have a natural commutative C^*-algebra structure such that A and B are closed *-subalgebras isomorphic to $C(S)$ and $C(T)$ respectively (cf. [19], § 2). Since φ: $\mathfrak{M} \to \mathfrak{N}$ is a homomorphism, φ^*: $\mathfrak{N}^* \to \mathfrak{M}^*$ maps B into A. Since φ is a CM-map, φ^*: $\mathfrak{N}^* \to \mathfrak{M}^*$ is a *-algebra homomorphism which preserves the identity (cf. [19], Theorem 1.2); hence $\varphi^*|_B$: $B \to A$ has the same property. Since A and B are isomorphic to $C(S)$ and $C(T)$ respectively, we conclude that there is a continuous map α: $S \to T$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varphi^*"] \arrow[d, leftrightarrow] & A \arrow[d, leftrightarrow] \\ C(T) \arrow[r, "f \to f \circ \alpha"] & C(S) \end{tikzcd}$$

commutes. The isomorphism $A \to C(S)$ maps the spectrum of $\mathfrak{M}$ onto $\hat{S}$ and, similarly, $B \to C(T)$ maps the spectrum of $\mathfrak{N}$ onto $\hat{T}$ (cf. [19], § 2). It follows that $f \circ \alpha \in \hat{S} \cup \{0\}$ for every $f \in \hat{T}$. Since semicharacters separate points in S and T (Proposition 1.1 (2)), we conclude that α is a semigroup homomorphism.

The isomorphism $F \to f$: $A \to C(S)$ is characterized by the fact that $F(\mu) = \int f d\mu_S$ for $\mu \in \mathfrak{M}$ (cf. [19], § 2). This, with the construction of α, implies that $\tilde{\alpha}(\mu_S) = (\varphi\mu)_T$ where $\tilde{\alpha}(\mu_S) = \mu_S \circ \alpha^{-1}$.

If φ^* maps the spectrum of $\mathfrak{N}$ onto the spectrum of $\mathfrak{M}$, then $f \to f \circ \alpha$: $C(T) \to C(S)$ maps $\hat{T}$ onto $\hat{S}$. Since $\hat{S}$ separates points, it follows that α is one to one.

One consequence of the above proposition is that the structure semigroup S and the embedding $\mu \to \mu_S$ are unique up to the obvious equivalence.

Throughout the paper we shall work with CM-algebras $\mathfrak{M}$ which are semisimple. Proposition 1.1 allows us to consider such an algebra to be an L-subalgebra of $M(S)$, where S is the structure semigroup of $\mathfrak{M}$. Except in certain cases we shall automatically do this.

If $\mu \in \mathfrak{M}$ then we set

$$\mu\hat{\ }(f) = \int_S f d\mu$$

for $f \in \hat{S}$. If we give $\hat{S}$ the weak (Gelfand) topology generated by the family of functions $\{\mu\hat{\ }: \mu \in \mathfrak{M}\}$, then Proposition 1.1(3) allows us to identify $\hat{S}$ with the spectrum of $\mathfrak{M}$ and $\mu \to \mu\hat{\ }$ with the Gelfand transform. Warning! Convergence in the weak topology does not imply pointwise convergence on S.

The value of Proposition 1.1 is that it allows us to identify the spectrum of $\mathfrak{M}$ with a function space $\hat{S}$ which has a very rich structure. For example, if f, $g \in \hat{S}$ and $fg \not\equiv 0$ then $fg \in \hat{S}$; if $f \in \hat{S}$ then its complex conjugate $\bar{f}$ and absolute value $|f|$ are also in $\hat{S}$. In fact, each $f \in \hat{S}$ has a unique polar decomposition $f = |f| h$, where $h \in \hat{S}$, $|h|^2 = |h|$, and h is zero exactly on the interior of the zero set of f (cf. [19], § 3).

If S has an identity e, then $f(e) = g(e) = f(e)g(e) = 1$ for f, $g \in \hat{S}$. Hence, $fg \not\equiv 0$ and $fg \in \hat{S}$. It follows that, in this case, $\hat{S}$ is a semigroup under pointwise multiplication. If $\mathfrak{M}$ has a normalized identity δ, then S has an identity e and $\delta = \delta_e$ is the unit point mass at e (cf. [19], Theorem 3.1).

Unfortunately, the above operations on $\hat{S}$ do not generally behave well with respect to the weak topology on $\hat{S}$. Conjugation is clearly continuous. However, multiplication is separately, but not always jointly, continuous and the map $f \to |f|$ is not generally continuous. For this reason, in § 3 we shall sometimes use a stronger topology on $\hat{S}$—a topology under which all the above operations are continuous.

2. Group algebras

In this section we give an abstract characterization of those CM-algebras which are CM-isomorphic to group algebras. We also identify, for a given CM-algebra $\mathfrak{M}$, an L-subalgebra $\mathfrak{M}_1$ of $\mathfrak{M}$ which is made up of the group algebras which $\mathfrak{M}$ contains.

Definition 2.1. Let $\mathfrak{N} \neq (0)$ be a commutative, semisimple CM-algebra. If $\mathfrak{N}$ contains no nonzero proper L-ideal, then $\mathfrak{N}$ will be called an (abstract) group algebra. If $\mathfrak{N}$ contains an L-ideal $\mathfrak{N}_0 \neq (0)$ such that $\mathfrak{N}_0$ is a group algebra and $\mathfrak{N}/\mathfrak{N}_0$ is a radical algebra, then $\mathfrak{N}$ will be called an almost group algebra.

Note that if G is an l.c.a. group and $L^1(G)$ denotes the algebra of absolutely continuous measures on G, then a closed proper ideal I of $L^1(G)$ is annihilated by some character of G (cf. [16], 7.2). If I is a proper L-ideal, then $\mu \in I$ implies $\nu \in I$ for ν of the form $d\nu = \chi d\mu$ $(\chi \in \hat{G})$. It follows that every $\chi \in \hat{G}$ annihilates I and, hence, $I = 0$. Thus, the concrete group algebra $L^1(G)$ is an abstract group algebra in the sense of Definition 2.1.

If $\operatorname{Rad} L^1(G)$ is the intersection of all maximal ideals of $M(G)$ containing $L^1(G)$, then $\operatorname{Rad} L^1(G)$ is an L-ideal in $M(G)$ (cf. [21], Lemma 1). If $\mathfrak{N}$ is an L-subalgebra of $M(G)$ such that $L^1(G) \subset \mathfrak{N} \subset \operatorname{Rad} L^1(G)$, then $\mathfrak{N}$ contains $L^1(G)$ as an L-ideal and $\mathfrak{N}/L^1(G)$ is a radical algebra. Hence, any such $\mathfrak{N}$ is an almost group algebra in the sense of Definition 2.1. We shall prove that every almost group algebra has this form and that every (abstract) group algebra has the form $L^1(G)$.

PROPOSITION 2.1. *If $\mathfrak{N}$ is an almost group algebra, then the L-ideal $\mathfrak{N}_0$ of Definition 2.1 contains every L-subalgebra of $\mathfrak{N}$ which is a group algebra. Also, $\mathfrak{N}_0$ is contained in every nonzero L-ideal of $\mathfrak{N}$.*

Proof. If $\mathfrak{N}_1$ is an L-subalgebra of $\mathfrak{N}$ which is a group algebra, then $\mathfrak{N}_1 \cap \mathfrak{N}_0$ is an L-ideal of $\mathfrak{N}_1$. Hence, either $\mathfrak{N}_1 \cap \mathfrak{N}_0 = (0)$ or $\mathfrak{N}_1 \subset \mathfrak{N}_0$. However, if $\mathfrak{N}_1 \cap \mathfrak{N}_0 = (0)$ then $\mathfrak{N} \to \mathfrak{N}/\mathfrak{N}_0$ maps $\mathfrak{N}_1$—which is semisimple—isomorphically into a radical algebra. Since this is impossible, we conclude that $\mathfrak{N}_1 \subset \mathfrak{N}_0$.

If $\mathfrak{N}_2$ is an L-ideal of $\mathfrak{N}$, then $\mathfrak{N}_2 \cap \mathfrak{N}_0$ is an L-ideal of $\mathfrak{N}_0$. As above, we conclude that $\mathfrak{N}_0 \subset \mathfrak{N}_2$ or $\mathfrak{N}_2 = (0)$.

The following proposition is the key to this section:

PROPOSITION 2.2. *Let $\mathfrak{N}$ be a commutative, semisimple* CM-*algebra considered as an L-subalgebra of $M(S)$, where S is its structure semigroup. Then the following statements are equivalent:*

(1) $\mathfrak{N}$ *is an almost group algebra;*

(2) S *is a group;*

(3) *there is an l.c.a. group* G, *with* S *as its Bohr compactification, and an* L-*subalgebra* $\mathfrak{N}' \subset M(G)$, *with* $L^1(G) \subset \mathfrak{N}' \subset \operatorname{Rad} L^1(G)$, *such that the map* $i: G \to S$ *induces a* CM-*isomorphism* $\mu \to \mu \circ i^{-1}: M(G) \to M(S)$ *which maps* $\mathfrak{N}'$ *onto* $\mathfrak{N}$.

Proof. The equivalence of (2) and (3) is the hard part of the proposition. However, this has already been proved in Theorem 2 of [23]. Obviously (3) implies (1). Hence, we shall prove that (1) implies (2).

To prove that S is a group, we need only show that $|f| \equiv 1$ for every $f \in \hat{S}$. This follows from the fact that $\hat{S}$ separates points in S (Proposition 1.2). However, if $f \in \hat{S}$ then $K_r = \{s \in S: |f(s)| \leqslant r\}$ is a compact ideal in S for each $r \in [0, 1]$. Since the set $\mathfrak{N}^r$ of measures in $\mathfrak{N}$ which are concentrated on K_r forms an L-ideal of $\mathfrak{N}$, (1) implies that either $\mathfrak{N}^r = (0)$ or $\mathfrak{N}_0 \subset \mathfrak{N}^r$ for each r. Note that $\mathfrak{N}^r \cdot \mathfrak{N}^t \subset \mathfrak{N}^{r \cdot t}$, so that if $\mathfrak{N}^r \neq (0)$ for some $r \in (0, 1)$ then $\mathfrak{N}^t \neq (0)$ for all $t \in (0, 1)$. In this case, $\mathfrak{N}_0 \subset \bigcap_{r>0} \mathfrak{N}^r = \mathfrak{N}^0$. However, this implies that $\mu \to \int f d\mu$ is a nonzero complex homomorphism of $\mathfrak{N}$ which vanishes on $\mathfrak{N}_0$—contradicting the fact that $\mathfrak{N}/\mathfrak{N}_0$ is a radical algebra. We conclude that $\mathfrak{N}^r = (0)$ for all $r < 1$. This means that $|f| = 1$ a.e./μ for each $\mu \in \mathfrak{N}$. Since f is continuous and $\mathfrak{N}$ is weak-* dense in $M(S)$, we conclude that $|f| \equiv 1$. This completes the proof.

COROLLARY 2.3. *A* CM-*algebra* $\mathfrak{N}$ *is a group algebra in the sense of Definition 2.1 if and only if it is isomorphic via a* CM-*map to a concrete group algebra* $L^1(G)$.

Definition 2.1 and Corollary 2.3 give an alternate characterization of group algebras to that of Rieffel in [14].

PROPOSITION 2.4. *If* $\mathfrak{N}$ *is a group algebra,* $\mathfrak{M}$ *a semisimple* CM-*algebra, and* $\varphi: \mathfrak{N} \to \mathfrak{M}$ *a* CM-*map, then* $\varphi(\mathfrak{N})$ *is a group algebra.*

Proof. If $\varphi: \mathfrak{N} \to \mathfrak{M}$ is a CM-map and $\mathfrak{N}$ is a group algebra, then the image of φ is an L-subalgebra of $\mathfrak{M}$, which we may as well assume is $\mathfrak{M}$ itself. If $\mathfrak{M}'$ is an L-ideal of $\mathfrak{M}$, then $\varphi^{-1}(\mathfrak{M}')$ is an L-ideal of $\mathfrak{N}$. Hence, $\varphi^{-1}(\mathfrak{M}') = (0)$ or $\varphi^{-1}(\mathfrak{M}') = \mathfrak{N}$ and, then, $\mathfrak{M}' = (0)$ or $\mathfrak{M}' = \mathfrak{M}$. Since $\mathfrak{M}$ is semisimple, we conclude that it is a group algebra.

If $\mathfrak{M}$ is a semisimple CM-algebra, then by a group algebra in $\mathfrak{M}$ we shall mean an L-subalgebra of $\mathfrak{M}$ which is a group algebra.

The support of a subspace $\mathfrak{N} \subset \mathfrak{M}$ will be the smallest closed subset of the structure semigroup S of $\mathfrak{M}$ on which each measure in $\mathfrak{N}$ is concentrated.

PROPOSITION 2.5. *If $\mathfrak{M}$ is a commutative, semisimple* CM-*algebra and $\mathfrak{N}$ is a group algebra in $\mathfrak{M}$, then*

(a) *every complex homomorphism of $\mathfrak{N}$ extends to a complex homomorphism of $\mathfrak{M}$; and*

(b) *the support of $\mathfrak{N}$ in S is a group which is isomorphic to the structure semigroup of $\mathfrak{N}$.*

Proof. Part (a) follows from the fact that a group algebra is symmetric and, hence, has Shilov boundary equal to its spectrum (cf. [16]; and [13], Corollary 3.3.26).

For part (b) let T be the structure semigroup of $\mathfrak{N}$ and $\mathfrak{N}_T$ the canonical image of $\mathfrak{N}$ in $M(T)$. By Proposition 1.2, the injection i: $\mathfrak{N}\to\mathfrak{M}$ induces a semigroup homomorphism α: $T\to S$ such that the map $\mathfrak{N}_T\to\mathfrak{N}\subset\mathfrak{M}$ is just $\mu\to\mu\circ\alpha^{-1}$. It follows that $\alpha(T)$ is the support of $\mathfrak{N}$. Note that T is a group by Proposition 2.2. That α is an isomorphism onto $\alpha(T)$ follows from part (a) and Proposition 1.2.

If K is a group in a compact semigroup S, then K is contained in a unique maximal group which is necessarily compact (cf. [27]).

PROPOSITION 2.6. *Let $\mathfrak{M}$ be a commutative, semisimple* CM-*algebra with structure semigroup S. Then each maximal group in S is either the support of a unique maximal group algebra in $\mathfrak{M}$ or is a set of measure zero for each $\mu\in\mathfrak{M}$. Furthermore, each group algebra in $\mathfrak{M}$ is contained in a unique maximal group algebra in $\mathfrak{M}$ which necessarily has a maximal group as support.*

Proof. If $K\subset S$ is a maximal group, let $\mathfrak{N}=\{\mu\in\mathfrak{M}$: μ is concentrated on $K\}$. If $\mathfrak{N}\neq(0)$ it follows from Lemma 2.1.C of [23] that $\mathfrak{N}$ is an almost group algebra with structure semigroup K. Hence, if $\mathfrak{N}_0$ is the L-ideal of Definition 2.1, then $\mathfrak{N}_0$ is a group algebra with support K. If $\mathfrak{N}_0\subset\mathfrak{N}_1$ and $\mathfrak{N}_1$ is a group algebra in $\mathfrak{M}$, then the support of $\mathfrak{N}_1$ is a group containing K. Since K is maximal, it follows that $\mathfrak{N}_1$ has support K and, hence, $\mathfrak{N}_1\subset\mathfrak{N}$. Proposition 2.1 then implies that $\mathfrak{N}_1=\mathfrak{N}_0$. Hence, $\mathfrak{N}_0$ is maximal.

If $\mathfrak{N}$ is any group algebra in $\mathfrak{M}$, then its support K is a group in S. If K^1 is the maximal group in S containing K, then K^1 is not a set of measure zero for $\mathfrak{M}$ and so K^1 supports a unique maximal group algebra $\mathfrak{N}^1$. We noted in the above paragraph that $\mathfrak{N}^1$ contains every group algebra with support in K^1. Hence, $\mathfrak{N}\subset\mathfrak{N}^1$.

PROPOSITION 2.7. *If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are group algebras in the semisimple* CM-*algebra $\mathfrak{M}$, then the closure of $\mathfrak{N}_1*\mathfrak{N}_2$ is also a group algebra in $\mathfrak{M}$.*

Proof. As in Proposition 2.2, let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be isomorphic to $L^1(G_1)$ and $L^1(G_2)$ respectively. Let α_1: $G_1\to K_1\subset S$ and α_2: $G_2\to K_2\subset S$ be the maps of G_1 and G_2 into their

Bohr compactifications. Then by Proposition 2.2, $\mu \to \mu \circ \alpha_1^{-1}$ and $\nu \to \nu \circ \alpha_2^{-1}$ are the isomorphisms $L^1(G_1) \to \mathfrak{N}_1 \subset \mathfrak{M}$ and $L^1(G_2) \to \mathfrak{N}_2 \subset \mathfrak{M}$ respectively. Let β: $G_1 \times G_2 \to S$ be defined by $\beta(g_1, g_2) = \alpha_1(g_1) \cdot \alpha_2(g_2)$. Note that β is a continuous homomorphism. It follows that $\omega \to \omega \circ \beta^{-1}$: $L^1(G_1 \times G_2) \to \mathfrak{M}$ is a CM-map which maps $\mu \times \nu$ to $\mu' * \nu'$, where $\mu' = \mu \circ \alpha_1^{-1}$ and $\nu' = \nu \circ \alpha_2^{-1}$ for $\mu \in L^1(G_1)$, $\nu \in L^1(G_2)$. Hence, the image of $\omega \to \omega \circ \beta^{-1}$ is the closure of $\mathfrak{N}_1 * \mathfrak{N}_2$. It is a group algebra by Proposition 2.4.

Definition 2.2. If $\mathfrak{M}$ is a commutative, semisimple CM-algebra, let $\mathfrak{M}_1$ denote the closed linear span in $\mathfrak{M}$ of the group algebras in $\mathfrak{M}$.

PROPOSITION 2.8. *The space $\mathfrak{M}_1$ above is an L-subalgebra of $\mathfrak{M}$ such that:*

(1) *the structure semigroup S_1 of $\mathfrak{M}_1$ is a union of groups, and inversion in these groups induces an involution on $\mathfrak{M}_1$ under which it is a symmetric algebra;*

(2) *every complex homomorphism of $\mathfrak{M}_1$ extends to a complex homomorphism of $\mathfrak{M}$;*

(3) *the injection i: $\mathfrak{M}_1 \to \mathfrak{M}$ induces an isomorphism of S_1 onto the support of $\mathfrak{M}_1$ in $\mathfrak{M}$.*

Proof. That $\mathfrak{M}_1$ is an L-subalgebra of $\mathfrak{M}$ follows from Proposition 2.7.

If S_1 is the structure semigroup of $\mathfrak{M}_1$ and we consider $\mathfrak{M}_1$ to be embedded in $M(S_1)$, then each group algebra of $\mathfrak{M}_1$ is concentrated on a group in S_1 by Proposition 2.5. Since $\mathfrak{M}_1$ is the closed linear span of its group algebras, it follows that the union of the groups in S_1 is dense. However, it is also closed (cf. [27]). It follows that S_1 is a union of groups.

Now inversion in the groups of S_1 defines a continuous map $s \to s'$ of S_1 onto S_1 with $s'' = s$ (cf. [27]). If we set $\mu^*(E) = \bar{\mu}(E')$ for E a Borel set of S_1, then $\mu \to \mu^*$ is an involution on $M(S_1)$ (cf. [19], § 3). Furthermore, $\mu \to \mu^*$ maps $\mathfrak{M}_1$ into itself. In fact, if $K \subset S_1$ is a maximal group and $\mathfrak{N}$ is the group algebra in $\mathfrak{M}_1$ supported on K, then K is the Bohr compactification of G, where $G \to K$ induces the isomorphism $L^1(G) \to \mathfrak{N}$. It follows that $L^1(G) \to \mathfrak{N}$ is a *-isomorphism (preserves involution). Hence, each group algebra in $\mathfrak{M}_1$ is closed under *and so is $\mathfrak{M}_1$. Now Theorem 3.4 of [19] implies that $\overline{\hat{\mu}} = (\mu^*)\hat{}$ for $\mu \in \mathfrak{M}_1$ and $\mathfrak{M}_1$ is a symmetric algebra. This proves part (1).

Part (2) follows from the facts that a symmetric Banach algebra has Shilov boundary equal to its spectrum and that every element of the Shilov boundary of $\mathfrak{M}_1$ extends to $\mathfrak{M}$ (cf. [13], Corollary 3.3.26). Then part (3) follows as in Proposition 2.5.

Definition 2.3. We shall call a CM-algebra $\mathfrak{M}$ small if there exists $\mu \in \mathfrak{M}$ such that $\nu << \mu$ for every $\nu \in \mathfrak{M}$.

It is easy to see that in an arbitrary CM-algebra the small L-subalgebras are directed upward and have union all of $\mathfrak{M}$. However, in § 3 we shall need the stronger result that this is also true of the small L-subalgebras $\mathfrak{N}$ whose group algebras are exactly the intersections with $\mathfrak{N}$ of the group algebras of $\mathfrak{M}$.

PROPOSITION 2.9. *Let $\mathfrak{M}$ be a commutative, semisimple* CM-*algebra. Let $\mathfrak{F}$ be the family of small L-subalgebras $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N}_1 = \mathfrak{N} \cap \mathfrak{M}_1$. Then $\mathfrak{F}$ is directed upward under inclusion and $\bigcup \mathfrak{F} = \mathfrak{M}$.*

Proof. First note that if $\mathfrak{N}$ is a group algebra and $\mathfrak{N}'$ is an L-subalgebra of $\mathfrak{N}$, then $\mathfrak{N}'$ is also a group algebra if and only if it is closed under the natural involution on $\mathfrak{N}$. In fact, if $\mathfrak{N} = L^1(G)$ and $0 \neq \mathfrak{N}' \subset \mathfrak{N}$ with $(\mathfrak{N}')^* = \mathfrak{N}'$, then the support of $\mathfrak{N}'$ in G must be a closed subgroup of positive measure—hence, an open subgroup—and $\mathfrak{N}'$ must consist of all elements of $\mathfrak{N}$ supported on this subgroup. It follows that if $\mathfrak{N} \subset \mathfrak{M}$ is an L-subalgebra, then $\mathfrak{N}_1 = \mathfrak{N} \cap \mathfrak{M}_1$ if and only if $\mathfrak{N} \cap \mathfrak{M}_1$ is closed under the involution on $\mathfrak{M}_1$.

Let $\mu \in \mathfrak{M}$ with $\mu \geqslant 0$ and $\|\mu\| = 1$. If $\nu = \sum 2^{-n}\mu^n$ then $\{\omega \in \mathfrak{M}: \omega < < \nu\}$ is a small L-subalgebra of $\mathfrak{M}$ containing μ. It follows that every element of $\mathfrak{M}$ is contained in a small L-subalgebra.

Now if $\{\mathfrak{N}^i\}$ is a nested sequence of small L-subalgebras with $\mathfrak{N}^i = \{\nu \in \mathfrak{M}: \nu < < \mu_i\}$ for $0 \leqslant \mu_i \in \mathfrak{M}$ with $\|\mu_i\| = 1$, then $\nu = \sum 2^{-i}\mu_i$ is a measure such that $\mathfrak{N} = \{\omega \in \mathfrak{M}: \omega < < \nu\}$ is a small L-subalgebra containing each $\mathfrak{N}^i$ with $\bigcup \mathfrak{N}^i$ dense in $\mathfrak{N}$. Similarly, if $\mathfrak{N}^1$ and $\mathfrak{N}^2$ are two small L-subalgebras of $\mathfrak{M}$, then the L-subalgebra generated by their union is also small.

Let $\mathfrak{N}$ be any small L-subalgebra of $\mathfrak{M}$. We let $\mathfrak{N}^1$ be the L-subalgebra generated by $\mathfrak{N}$ and $(\mathfrak{N} \cap \mathfrak{M}_1)^*$. We define a sequence $\{\mathfrak{N}^i\}$ inductively by letting $\mathfrak{N}^{i+1}$ be the L-subalgebra generated by $\mathfrak{N}^i$ and $(\mathfrak{N}^i \cap \mathfrak{M}_1)^*$. Note that $\mathfrak{N} \subset \mathfrak{N}^i \subset \mathfrak{N}^{i+1}$ and $(\mathfrak{N}^i \cap \mathfrak{M}_1)^* \subset \mathfrak{N}^{i+1}$ for each i. Furthermore, each $\mathfrak{N}^i$ is small. If $\mathfrak{N}^\infty$ is the closure of $\bigcup \mathfrak{N}^i$, then $\mathfrak{N}^\infty$ is a small L-subalgebra containing $\mathfrak{N}$ and such that $(\mathfrak{N}^\infty \cap \mathfrak{M}_1)^* \subset (\mathfrak{N}^\infty \cap \mathfrak{M}_1)$. It follows that $\mathfrak{N}_1^\infty = \mathfrak{N}^\infty \cap \mathfrak{M}_1$ and $\mathfrak{N}^\infty \in \mathfrak{F}$.

We conclude from the above that each $\mu \in \mathfrak{M}$ is contained in some member of $\mathfrak{F}$ and that for any pair $\mathfrak{N}^1$, $\mathfrak{N}^2 \in \mathfrak{F}$ there is a small L-subalgebra $\mathfrak{N}$ with $\mathfrak{N}^1$, $\mathfrak{N}^2 \subset \mathfrak{N}$ and an $\mathfrak{N}^3 \in \mathfrak{F}$ such that $\mathfrak{N} \subset \mathfrak{N}^3$. The proposition follows.

3. The main theorem

In the initial portion of this section (until Proposition 3.13) we shall use the blanket assumption that $\mathfrak{M}$ is a small, commutative, semisimple CM-algebra with a normalized identity. The structure semigroup of $\mathfrak{M}$ will be S, $\mathfrak{M}_1$ will be the closed linear span of the group algebras in $\mathfrak{M}$, and S_1 will be its structure semigroup. By Proposition 2.8, we may identify S_1 with the support of $\mathfrak{M}_1$ in S.

We let $i^\wedge: \hat{S} \to \hat{S}_1$ be defined by $i^\wedge f = f|_{S_1}$. Note that if we identify $\hat{S}$ and $\hat{S}_1$ with the spectra of $\mathfrak{M}$ and $\mathfrak{M}_1$ respectively, then $i^\wedge$ is the map between the spectrum of $\mathfrak{M}$ and the spectrum of $\mathfrak{M}_1$ induced by the injection $i: \mathfrak{M}_1 \to \mathfrak{M}$. By Proposition 2.8, $i^\wedge$ is onto.

Since we have assumed that $\mathfrak{M}$ has a normalized identity, it follows that S has an identity e and that δ_e is the identity of $\mathfrak{M}$. Since $\mathbf{C}\delta_e$ is a group algebra, we have $e \in S_1$ and δ_e is an identity for $\mathfrak{M}_1$ as well. We also have that both $\hat{S}$ and $\hat{S}_1$ are semigroups and are compact spaces in their respective Gelfand topologies and that $i^\wedge: \hat{S} \to \hat{S}_1$ is a continuous homomorphism.

If K is a given coefficient group, we shall be studying the map $i^*: H^p(\hat{S}_1, K) \to H^p(\hat{S}, K)$ induced on cohomology by $i^\wedge: \hat{S} \to \hat{S}_1$. Here we are refering to sheaf cohomology relative to the constant sheaf with stalk K (cf. [3]). This is equivalent to Čech and to Alexander-Spanier cohomology in our situation (cf. [3], Chapter 3). Our object will be to show that i^* is an isomorphism for all p. We shall do this by proving that $i^{\wedge -1}(\{f\})$ is a connected set with trivial cohomology for each $f \in \hat{S}_1$ and then applying the Vietoris mapping theorem (cf. [3], II. 11.2).

Since we have assumed that $\mathfrak{M}$ is small, we can find $\mu \in \mathfrak{M}$, $\mu \geqslant 0$, $\|\mu\| = 1$ such that $\nu << \mu$ for each $\nu \in \mathfrak{M}$. We use the measure μ to define a metric on $\hat{S}$.

Definition 3.1. For $f, g \in \hat{S}$ set $d(f, g) = \int |f - g| d\mu$. We shall call the topology that d induces on $\hat{S}$ the strong topology.

PROPOSITION 3.1. *The strong topology on $\hat{S}$ is a complete, metric topology which dominates the weak (Gelfand) topology of $\hat{S}$. Furthermore, $(f, g) \to fg$: $\hat{S} \times \hat{S} \to \hat{S}$, $f \to |f|$: $\hat{S} \to \hat{S}$, and $f \to \bar{f}$: $\hat{S} \to \hat{S}$ are all continuous maps in the strong topology.*

Proof. If $f_n \to f$ strongly in $\hat{S}$, then $|f - f_n| \leqslant 2$ for each n and $\int |f - f_n| d\mu \to 0$. It follows that $f_n \to f$ in the weak-* topology of $L^\infty(\mu)$. Hence, $\nu^\wedge(f_n) = \int f_n d\nu \to \int f d\nu = \nu^\wedge(f)$ for every $\nu << \mu$, i.e., for every $\nu \in \mathfrak{M}$. Thus, the strong topology dominates the weak topology. The other assertions are obvious.

We denote $\{f \in \hat{S}: f \geqslant 0\}$ by $\hat{S}^+$. Note that if $f \geqslant g \in \hat{S}^+$, then $d(f, g) = \int (f - g) d\mu = \mu^\wedge(f) - \mu^\wedge(g)$.

The next proposition is the key to the section. Everything that follows is just technical manipulation. The main ingredient (Lemma 5.1) of the proof is delayed until § 5.

PROPOSITION 3.2. *If $f \in \hat{S}^+$ then there is a unique minimal $h \in \hat{S}^+$ such that $h|_{S_1} = f|_{S_1}$. Furthermore, $h^2 = h$ and f and h are joined by an arc φ: $[a, b] \to \hat{S}^+$ such that*

(a) $\varphi(x) \leqslant \varphi(y)$ *for* $x \leqslant y$;

(b) $\mu\hat{}(\varphi(x)) = x$ *for all* $x \in [a, b]$; *and*

(c) $\varphi(b) = f$ *and* $\varphi(a) = h$.

Proof. Recall that S_1 is a union of groups (Proposition 2.8). It follows that if $g \in \hat{S}_1^+$, then $g(s) = 0$ or 1 for each $s \in S_1$; i.e., $g^2 = g$. In particular, $(f|_{S_1})^2 = f|_{S_1}$.

The set $\{g \in \hat{S}^+ \colon g|_{S_1} = f|_{S_1}\} = (i\hat{})^{-1}(i\hat{}(f)) \cap \hat{S}^+$ is a weakly compact subsemigroup of $\hat{S}^+$. Since $g_1 g_2 \leqslant g_1$ and $g_1 g_2 \leqslant g_2$ it follows that this set is directed downward. We conclude that it has a minimum h. Since $h^2 \leqslant h$ and $h^2|_{S_1} = h|_{S_1}$ we have $h = h^2$.

We set $a = \mu\hat{}(h)$ and $b = \mu\hat{}(f)$. If $\varepsilon > 0$ and $c \in [a, b]$, let $\Phi(c, \varepsilon)$ be the set of all functions $\varphi\colon [c, b] \to \hat{S}^+$ such that φ is nondecreasing, $\varphi(b) = f$, $\mu\hat{}(\varphi(c)) = c$, and $|\mu\hat{}(\varphi(x)) - x| \leqslant \varepsilon$ for all $x \in [c, b]$. We shall prove that $\Phi(a, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$.

For fixed $\varepsilon > 0$ let $c_0 = \inf\{c\colon \Phi(c, \varepsilon) \neq \emptyset\}$ (note that $\Phi(b, \varepsilon) \neq \emptyset$ trivially). Choose sequences $\{c_i\}$ and $\{\varphi_i\}$ such that $c_i \downarrow c_0$ and $\varphi_i \in \Phi(c_i, \varepsilon)$. Since $\hat{S}^+$ is weakly compact, there is a function $\varphi\colon (c_0, b] \to \hat{S}^+$ such that on each $[c_j, b]$ the sequence $\{\varphi_i\}_{i \geqslant j}$ clusters pointwise (in the weak topology) to φ. Since each φ_i is nondecreasing, so is φ. Clearly, $\varphi(b) = f$ and $\lim_{c \to c_0^+} \mu\hat{}(\varphi(c)) = \lim_i \mu\hat{}(e_i(c_i)) = \lim_i c_i = c_0$. Since φ is nondecreasing, $g = \lim_{c \to c_0^+} \varphi(c)$ exists in $\hat{S}^+$ and $\hat{\mu}(g) = c_0$. Hence, we may extend φ to $[c_0, b]$ by setting $\varphi(c_0) = g$. The inequality $|\mu\hat{}(\varphi_i(x)) - x| \leqslant \varepsilon$ holds on $[c_i, b]$ for each i and, hence, $|\mu\hat{}(\varphi(x)) - x| \leqslant \varepsilon$ holds on $[c_0, b]$. Thus, we have proved that $\varphi \in \Phi(c_0, \varepsilon)$ and $\{c \in [a, b]\colon \Phi(c, \varepsilon) \neq \emptyset\}$ contains a minimum c_0.

Suppose that the number c_0 above is not a. Choose $\varphi \in \Phi(c_0, \varepsilon)$ and let $\varphi(c_0) = g$. Now $\hat{\mu}(g) = c_0 > a = \hat{\mu}(h)$. Hence, $g \nleqslant h$ and the set $U = \{s \in S\colon g(s) > h(s)\}$ is nonempty. Since $h^2 = h$, either $h(s) = 0$ or $h(s) = 1$ for each $s \in S$. It follows that $U = \{s \in S\colon g(s) > 0 \text{ and } h(s) = 0\}$ and U is an ideal in the subsemigroup $\{s \in S\colon g(s) > 0\}$. Furthermore, U contains no points of S_1 since $g \leqslant f$ and $f|_{S_1} = h|_{S_1}$. At this point we invoke Lemma 5.1, which implies that there is a $g' \in \hat{S}^+$ such that $g' \leqslant g$ and $0 < \mu\hat{}(g) - \mu\hat{}(g') < \min\{\varepsilon, c_0 - a\}$. If $\mu\hat{}(g') = c'$, we define φ': $[c', b]$ by $\varphi' = \varphi$ on $[c_0, b]$ and $\varphi'(x) = g'$ for $x \in [c', c_0)$. Clearly, $\varphi' \in \Phi(c', \varepsilon)$ and $c' < c_0$. This contradicts the minimality of c_0 and proves that $c_0 = a$.

We now have that $\Phi(a, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$. Clearly each $\Phi(a, \varepsilon)$ is compact in the topology of weak pointwise convergence and the $\Phi(a, \varepsilon)$ are nested downward. Hence, $\bigcap_\varepsilon \Phi(a, \varepsilon) \neq \emptyset$. If $\varphi \in \bigcap_\varepsilon \Phi(a, \varepsilon)$ then φ satisfies (a) and (b) of the proposition and $\varphi(b) = f$. We must show that $\varphi(a) = h$.

If $\varphi(a)|_{S_1} \neq f|_{S_1}$ then there is a maximal group K of positive μ-measure in S_1 such that $\varphi(a)|_K = 0$ and $f|_K = 1$. However, φ is nondecreasing and each $\varphi(x)$ is either one or zero on K. Hence, the function $\mu\hat{}(\varphi(x)) = x$ must have a jump discontinuity—a contradiction. We conclude that $\varphi(a)|_{S_1}=f|_{S_1}$. Since h is minimal with this property, we have $h\leqslant\varphi(a)$. However, $\mu^\wedge(h)=a=\mu^\wedge(\varphi(a))$. We conclude that $h=\varphi(a)$. This completes the proof.

Definition 3.2. If $g\in\hat{S}_1$ we set $\Omega_g=i^{\wedge -1}(\{g\})=\{f\in\hat{S}\colon f|_{S_1}=g\}$. If $h^2=h\in\hat{S}^+$ and $h|_{S_1}=|g|$, then we set $\Delta_g^h=\{f\in\Omega_g\colon |f|\leqslant h\}$.

A tree in Ω_g will be a set Λ such that $f\in\Lambda$, $f_1\in\Omega_g$, and $|f_1|\leqslant|f|$ imply that $f_1\in\Lambda$.

Our object is to show that Ω_g has trivial cohomology. Roughly speaking, we shall do this by first showing that there is a point $\tilde{g}\in\Omega_g$ such that every Δ_g^h is contractible to $\tilde{g}$ and then showing that Ω_g can be approximated in a strong sense by finite unions of the sets Δ_g^h.

PROPOSITION 3.3. *If k is the minimal element of the set $\{f\in\hat{S}^+\colon f|_{S_1}=|g|\}$, then $k\cdot\Omega_g=\{\tilde{g}\}$ where $\tilde{g}\in\Omega_g$ and $|\tilde{g}|\leqslant|f|$ for every $f\in\Omega_g$. Necessarily, $\tilde{g}$ is in every tree in Ω_g and, in particular, in every Δ_g^h.*

Proof. Since $k^2=k$ (Proposition 3.2) it follows that $U=\{s\in S\colon k(s)>0\}=\{s\in S\colon k(s)=1\}$ is an open-compact subsemigroup of S. Let G be its kernel (minimal ideal). We shall prove that $G\subset S_1$. Let G_1 be the kernel of $U\cap S_1$ and let p and p_1 be the idempotents in G and G_1 respectively. Since $G_1\subset U$, we have $pp_1=p$. If $p\neq p_1$ there exists $h\in\hat{S}$ such that $h(p)=0$ and $h(p_1)=1$, since $\hat{S}$ separates points in S. We may assume $h\in\hat{S}^+$. It follows that $h=1$ on G_1 and, hence, $h=1$ on $U\cap S_1$. Thus, $hk|_{S_1}=k|_{S_1}=|g|$ and $hk\leqslant k$. It follows that $hk=k$ by the choice of k. However, $k(p)=1$ while $h(p)=0$—a contradiction. We conclude that $p=p_1$ and $G=G_1\subset S_1$.

Now since $G\subset S_1$, we have $f_1|_G=f_2|_G=g|_G$ for every f_1, $f_2\in\Omega_g$. However, $sp\in G$ for every $s\in U$. Hence $f_1(s)=f_1(ps)=f_2(ps)=f_2(s)$ for f_1, $f_2\in\Omega_g$ and $s\in U$. Since $k=0$ on $S\setminus U$, we have $kf_1=kf_2$ for all f_1, $f_2\in\Omega_g$. Thus, the set $\{kf\colon f\in\Omega_g\}$ consists of a single element $\tilde{g}$. Since $kf|_{S_1}=|g|g=g$ for $f\in\Omega_g$, we have $\tilde{g}\in\Omega_g$. Clearly, $\tilde{g}$ is the unique element of minimal absolute value in Ω_g.

PROPOSITION 3.4. *Let Λ be a compact tree in Ω_g which is contained in Δ_g^h for some h. Then Λ is contractible to the point $\tilde{g}$. Hence, Λ is connected and $H^p(\Lambda, K)=0$ for $p>0$ and arbitrary coefficient group K.*

Proof. We have $h^2=h\in\hat{S}^+$ and $h|_{S_1}=|g|$. By Proposition 3.2, there is a map $\varphi\colon [a,b]\to\hat{S}^+$ such that $\varphi(a)=k$ (the minimal $f\in\hat{S}^+$ such that $f|_{S_1}=h|_{S_1}=|g|$) and $\varphi(b)=h$. Furthermore, φ is nondecreasing and $\mu^\wedge(\varphi(x))=x$. It follows that φ is strongly continuous. Consider the map $\Theta\colon [a,b]\times\Lambda\to\hat{S}$ defined by $\Theta(x,f)=\varphi(x)\cdot f$. We have $\Theta(a,f)=kf=\tilde{g}$ for all $f\in\Lambda$ and $\Theta(b,f)=hf=f$ since $\Lambda\subset\Delta_g^h$. Furthermore, $|\Theta(x,f)|=|\varphi(x)f|\leqslant h|f|\leqslant|f|$ and

$\Theta(x\, f)|_{S_1}=\varphi(x)f|_{S_1}=k|_{S_1}\cdot f|_{S_1}=|g|g=g$. Since Λ is a tree in Ω_g, it follows that $\Theta(x, f)\in\Lambda$ for each x.

To complete the proof that Θ is a homotopy, we must show that it is continuous. If $x_\alpha\to x\in[a, b]$ and $f_\alpha\to f$ weakly in Λ, then for each $\nu\in\mathfrak{M}$

$$\left|\int\Theta(x, f)\,d\nu-\int\Theta(x_\alpha, f_\alpha)d\nu\right|\leqslant\left|\int(f-f_\alpha)\,\varphi(x)\,d\nu\right|+\left|\int f_\alpha(\varphi(x)-\varphi(x_\alpha))\,d\nu\right|$$
$$\leqslant\left|\int(f-f_\alpha)\,\varphi(x)\,d\nu\right|+\int|\varphi(x)-\varphi(x_\alpha)|\,d\nu\to 0.$$

Hence, Θ is weakly continuous. This completes the proof.

COROLLARY 3.5. *Each tree in Ω_g is connected. In particular, Ω_g is connected.*

Proof. If Λ is a tree in Ω_g and $f\in\Lambda$, then $\lim_{n\to\infty}|f|^{1/n}=h$ is an idempotent in $\hat{S}$ such that $h|_{S_1}=|g|$. Also, $f\in\Delta_g^h\cap\Lambda$ which is a tree in Δ_g^h. By Proposition 3.4 $\Delta_g^h\cap\Lambda$ is contractible to $\{\tilde{g}\}\subset\Lambda$. It follows that Λ is connected.

Definition 3.3. Let $H_g=\{h\in\hat{S}^+\colon\ h^2=h$ and $h|_{S_1}=|g|\}$. If $h\in H_g$ and $\varepsilon>0$ let $\Delta_g^h(\varepsilon)=\{f\in\Omega_g\colon\int|f-fh|\,d\mu\leqslant\varepsilon\}=\{f\in\Omega_g\colon d(f, fh)\leqslant\varepsilon\}$.

PROPOSITION 3.6. *For $h\in H_g$ and $\varepsilon>0$, the set $\Delta_g^h(\varepsilon)$ is a compact tree. Furthermore, $\bigcap_{\varepsilon>0}\Delta_g^h(\varepsilon)=\Delta_g^h$.*

Proof. If $f\in\Delta_g^h(\varepsilon)$, $f_1\in\Omega_g$, and $|f_1|\leqslant|f|$, then $|f_1-f_1h|=|f_1|\,|1-h|\leqslant|f|\,|1-h|=|f-fh|$. Hence, $f_1\in\Delta_g^h(\varepsilon)$ and $\Delta_g^h(\varepsilon)$ is a tree. The statement $\int|f-fh|d\mu\leqslant\varepsilon$ is equivalent to $|\nu\hat{\ }(f)-\nu\hat{\ }(fh)|=|\int(f-fh)d\nu|\leqslant\varepsilon$ for all $\nu\in\mathfrak{M}$ with $|\nu|\leqslant\mu$. Thus, $\Delta_g^h(\varepsilon)$ is weakly closed, hence, compact. Note that for $f\in\Omega_g$, $f\in\bigcap_{\varepsilon>0}\Delta_g^h(\varepsilon)$ if and only if $f=fh$. Since $h^2=h$ it follows that $f=fh$ if and only if $|f|\leqslant h$. Hence, $\bigcap_{\varepsilon>0}\Delta_g^h(\varepsilon)=\Delta_g^h$.

PROPOSITION 3.7. *Let $\{\varepsilon_h\colon h\in H_g\}$ be an arbitrary collection of positive numbers indexed over H_g. Then there exists a finite set $\{h_1, \ldots, h_n\}\subset H_g$ such that*

$$\Omega_g=\bigcup_{i=1}^{n}\Delta_g^{h_i}(\varepsilon_{h_i})$$

Proof. Suppose that $\{\varepsilon_h\colon h\in H_g\}$ is a collection for which the proposition fails. Let $\mathfrak{A}$ be the directed set consisting of all finite subsets of H_g ordered by inclusion. By assumption, for each $\alpha=\{h_1, \ldots, h_n\}\in\mathfrak{A}$, we may choose $f_\alpha\in\Omega_g\setminus\bigcup_{i=1}^n\Delta_g^{h_i}(\varepsilon_{h_i})$.

For each α we choose $h_\alpha\in H_g$ such that $|f_\alpha|\leqslant h_\alpha$ (this can be done as in the proof of Corollary 3.5). Let $k\in\hat{S}^+$ be a weak cluster point of the net $\{h_\alpha\}$. Note that $k|_{S_1}=|g|$ since $h_\alpha|_{S_1}=|g|$ for each α. If $h=\lim_n k^{1/n}$ then $h\in H_g$. However, note that

$$(\int |f_\alpha - hf_\alpha| d\mu)^2 \leqslant \int |f_\alpha|^2 d\mu \int |h_\alpha - hh_\alpha|^2 d\mu \leqslant \int (h_\alpha - 2hh_\alpha + hh_\alpha)\, d\mu$$
$$= \int (h_\alpha - hh_\alpha)\, d\mu = \mu\hat{}(h_\alpha) - \mu\hat{}(hh_\alpha),$$

and $\{\mu\hat{}(h_\alpha) - \mu\hat{}(hh_\alpha)\}$ clusters to zero since $\{h_\alpha\}$ clusters to $k \leqslant h$. It follows that $f_\alpha \in \Delta_g^h(\varepsilon_h)$ for a cofinal set of α's, hence, for some α with $h \in \alpha$. This contradicts the choice of the net $\{f_\alpha\}$ and completes the proof.

At this point we begin our final assault on the proposition that Ω_g has trivial cohomology. We shall assume the reader is familiar with basic sheaf theory as presented in Chapters 1 and 2 of [3].

We shall define a class of sheaves on Ω_g containing the soft sheaves and the constant sheaves and satisfying a two out of three theorem. We then prove that each sheaf in this class is acyclic on Ω_g by an induction argument.

Definition 3.4. A sheaf $\mathcal{S}$ on Ω_g will be called semisoft if

(1) the restriction map $\Gamma(\Omega_g, \mathcal{S}) \to \Gamma(\Lambda, \mathcal{S})$ is onto for each compact tree $\Lambda \subset \Omega_g$; and

(2) $\mathcal{S}|\Lambda$ is acyclic if Λ is any compact tree contained in a set Δ_g^h for $h \in H_g$.

PROPOSITION 3.8. *Soft sheaves and constant sheaves on Ω_g are semisoft.*

Proof. Property (1) holds for all soft sheaves by definition. If $\mathcal{K}$ is the constant sheaf with stalk K, then $\Gamma(\Omega_g, \mathcal{K}) = K = \Gamma(\Lambda, \mathcal{K})$ for each compact tree, since Ω_g and Λ are connected. It follows that (1) holds for constant sheaves. Property (2) holds for constant sheaves by Proposition 3.4 and for soft sheaves by II.9.8 of [3].

PROPOSITION 3.9. *If $0 \to \mathcal{S} \xrightarrow{\alpha} \mathcal{S}' \xrightarrow{\beta} \mathcal{S}'' \to 0$ is a short exact sequence of sheaves on Ω_g and $\mathcal{S}$ is semisoft, then for each compact tree $\Lambda \subset \Omega_g$ we have $0 \to \Gamma(\Lambda, \mathcal{S}) \xrightarrow{\alpha^*} \Gamma(\Lambda, \mathcal{S}') \xrightarrow{\beta^*} \Gamma(\Lambda, \mathcal{S}'') \to 0$ is also exact.*

Proof. We must prove that β^* is onto. Note that if $h \in H_g$ then $\Lambda \cap \Delta_g^h$ is a tree contained in Δ_g^h. By definition, $\mathcal{S}|_{\Lambda\cap\Delta_g^h}$ is acyclic. Hence, $H^1(\Lambda \cap \Delta_g^h, \mathcal{S}) = 0$ and $\beta^*\colon \Gamma(\Lambda \cap \Delta_g^h, \mathcal{S}') \to \Gamma(\Lambda \cap \Delta_g^h, \mathcal{S}'')$ is onto.

If $\eta'' \in \Gamma(\Lambda, \mathcal{S}'')$ then for each $h \in H_g$ choose $\eta'_h\Lambda \in \Gamma(\cap \Delta_g^h, \mathcal{S}')$ such that $\beta^*\eta'_h = \eta''|_{\Lambda\cap\Delta_g^h}$. We may assume that $\eta'_h \in \Gamma(U_h, \mathcal{S}')$ and $\beta^*\eta'_h = \eta''|_{U_h}$ for some weak neighborhood U_h of $\Lambda \cap \Delta_g^h$ in Λ (cf. [3], II.10.4(e)). It follows from Proposition 3.5 that there is an $\varepsilon_h > 0$ such that $\Lambda \cap \Delta_g^h(\varepsilon_h) \subset U_h$. By Proposition 3.6 we may choose $\{h_1, ..., h_n\} \subset H_g$ such that $\Omega_g = \bigcup_{i=1}^n \Delta_g^{h_i}(\varepsilon_{h_i})$. Let $K_i = \Delta_g^{h_i}(\varepsilon_{h_i}) \cap \Lambda$ and $\Lambda_j = \bigcup_{i=1}^j K_i$ and note that each Λ_j is a compact tree in Λ and $\Lambda_n = \Lambda$.

We proceed by induction and suppose that we have defined an element $\eta'_j \in \Gamma(\Lambda_j, \mathcal{S}')$

such that $\beta^*\eta_j'=\eta''|_{\Lambda_j}$. Let $\xi_j'=\eta_j'|_{\Lambda_j}-\eta'_{h_{j+1}}|_{\Lambda_j\cap K_{j+1}}$ and note that $\beta^*\xi_j'=0$. Hence, there is an element $\xi_j\in\Gamma(\Lambda_j\cap K_{j+1}, \mathcal{S})$ such that $\alpha^*\xi_j=\xi_j'$. Since $\mathcal{S}$ is semisoft and $\Lambda_j\cap K_{j+1}$ is a compact tree, we can extend ξ_j to an element $\bar{\xi}_j\in\Gamma(\Omega_g, \mathcal{S})$. Note that $\eta_j'=\eta'_{h_{j+1}}+\alpha^*\bar{\xi}_j$ on $\Lambda_j\cap K_{j+1}$ and, hence, there is a section $\mathfrak{N}'_{j+1}\in\Gamma(\Lambda_{j+1}, \mathcal{S}')$ such that $\eta'_{j+1}|_{\Lambda_j}=\eta_j'$ and $\eta'_{j+1}|_{K_j}=\eta'_{h_{j+1}}+\alpha^*\bar{\xi}_j$. Clearly, $\beta^*\eta'_{j+1}=\eta''|_{\Lambda_{j+1}}$. By induction, there is an $\mathfrak{N}'\in\Gamma(\Lambda, \mathcal{S}')$ such that $\beta^*\eta'=\eta''$. Hence, β^* is onto.

PROPOSITION 3.10. *If* $0\to\mathcal{S}\xrightarrow{\alpha}\mathcal{S}'\xrightarrow{\beta}\mathcal{S}''\to 0$ *is a short exact sequence of sheaves on* Ω_g *and* $\mathcal{S}$ *and* $\mathcal{S}'$ *are semisoft, then* $\mathcal{S}''$ *is also semisoft.*

Proof. That $\mathcal{S}''$ is acyclic on each compact tree in Δ_g^h follows trivially. If Λ is a compact tree in Ω_g and $\eta''\in\Gamma(\Lambda, \mathcal{S}'')$, then Proposition 3.9 implies that there exists $\eta'\in\Gamma(\Lambda, \mathcal{S}')$ such that $\beta^*\eta'=\eta''$. Since $\mathcal{S}'$ is semisoft there is a section $\xi'\in\Gamma(\Omega_g, \mathcal{S}')$ such that $\xi'|_\Lambda=\eta'$. If $\xi''=\beta^*\xi'$ then $\xi''\in\Gamma(\Omega_g, \mathcal{S}'')$ and $\xi''|_\Lambda=\eta''$. Hence, $\mathcal{S}''$ is also semisoft.

PROPOSITION 3.11. *Each semisoft sheaf on* Ω_g *is acyclic.*

Proof. Let $\mathcal{S}$ be semisoft and let $\mathcal{S}\xrightarrow{\varepsilon}\mathcal{C}^0\xrightarrow{\delta_0}\mathcal{C}^1\xrightarrow{\delta_1}\dots$ be a soft resolution of $\mathcal{S}$ (cf. [3], II.1.1.). We set $\mathcal{K}^i=\operatorname{Ker}\delta_i$ for $i=0, 1, \dots$. Note that $0\to\mathcal{K}^i\to\mathcal{C}^i\to\mathcal{K}^{i+1}\to 0$ is exact for $i=0, 1, \dots$ and $\mathcal{S}\xrightarrow{\varepsilon}\mathcal{K}^0$ is an isomorphism. Since $\mathcal{S}$ is semisoft and each $\mathcal{C}^i$ is semisoft, we conclude by induction that each $\mathcal{K}^i$ is semisoft. Hence, by Proposition 3.9 we have $0\to\Gamma(\Omega_g, \mathcal{K}^i)\to\Gamma(\Omega_g, \mathcal{C}^i)\to\Gamma(\Omega_g, \mathcal{K}^{i+1})\to 0$ is exact for each i. It follows that $0\to\Gamma(\Omega_g, \mathcal{S})\to\Gamma(\Omega_g, \mathcal{C}^0)\to\Gamma(\Omega_g, \mathcal{C}^1)\to\dots$ is exact. Hence, $\mathcal{S}$ is acyclic (cf. [3], II.4.1.).

PROPOSITION 3.12. *If K is any coefficient group, then the map* $i^\wedge\colon \hat{S}\to\hat{S}_1$ *induces a map* $i^*\colon H^p(\hat{S}_1, K)\to H^p(\hat{S}, K)$ *which is an isomorphism for each* p.

Proof. By Corollary 3.5 and Proposition 3.11, we have that for each $g\in\hat{S}_1$ the set $i^{\wedge -1}(\{g\})=\Omega_g$ is connected and has trivial cohomology for any constant sheaf. That $i^*\colon H^p(\hat{S}_1, K)\to H^p(\hat{S}, K)$ is an isomorphism in each degree now follows from the Vietoris mapping theorem (cf. [3], II.11.2).

Our next step is to remove the assumption that $\mathfrak{M}$ is small. We do this by using Proposition 2.9.

PROPOSITION 3.13. *Let $\mathfrak{M}$ be a commutative, semisimple* CM-*algebra with a normalized identity. Let $\mathfrak{M}_1$ be the closed linear span of the group algebras in $\mathfrak{M}$. Let S and S_1 be the structure semigroups of $\mathfrak{M}$ and $\mathfrak{M}_1$ respectively. If* $i^\wedge\colon \hat{S}\to\hat{S}_1$ *is the map induced by* $i\colon \mathfrak{M}_1\to\mathfrak{M}$, *then the map* $i^*\colon H^p(\hat{S}_1, K)\to H^p(\hat{S}_1, K)$, *induced by* $i^\wedge$, *is an isomorphism for each p and each coefficient group K.*

Proof. Let $\{\mathfrak{N}^\alpha\}_{\alpha\in\mathfrak{U}}$ be the collection of algebras which are in the set $\mathfrak{F}$ of Proposition 2.9 and contain the identity, indexed by a directed set in such a way that $\alpha \leqslant \beta$ implies $\mathfrak{N}^\alpha \subset \mathfrak{N}^\beta$. For each $\beta \leqslant \alpha$ let $j_{\alpha\beta}: \mathfrak{N}^\alpha \to \mathfrak{N}^\beta$ and $j_\alpha: \mathfrak{N}^\alpha \to \mathfrak{M}$ be the injections and $j\hat{}_{\alpha\beta}: \hat{S}^\alpha \to \hat{S}^\beta$ and $j\hat{}_\alpha: \hat{S} \to \hat{S}^\alpha$ the corresponding induced maps on the spectra. We let $k_{\alpha\beta}$, $\hat{k}_{\alpha\beta}$, k_α, and $\hat{k}_\alpha$ be the corresponding maps for $\mathfrak{M}_1$ and $\mathfrak{N}_1^\alpha = \mathfrak{M}_1 \cap \mathfrak{N}^\alpha$ and their spectra $\hat{S}_1$ and $\hat{S}_1^\alpha$. Note that $\{\hat{S}^\alpha, j\hat{}_{\alpha\beta}\}$ and $\{\hat{S}_1^\alpha, k\hat{}_{\alpha\beta}\}$ are inverse limit systems of compact Hausdorff spaces with $\varprojlim \hat{S}^\alpha = \hat{S}$ and $\varprojlim \hat{S}_1^\alpha = \hat{S}_1$, since $\bigcup \mathfrak{N}^\alpha = \mathfrak{M}$. Furthermore, the diagram

$$\begin{tikzcd}
\hat{S} \arrow[r, "j\hat{}_\alpha"] \arrow[d, "i\hat{}"'] & \hat{S}^\alpha \arrow[d, "i\hat{}_\alpha"] \\
\hat{S}_1 \arrow[r, "k\hat{}_\alpha"] & \hat{S}_1^\alpha
\end{tikzcd}$$

commutes for each α since $\mathfrak{N}_1^\alpha = \mathfrak{M}_1, \cap \mathfrak{N}^\alpha$. If we pass to cohomology we have $H^*(\hat{S}, K) = \varinjlim H^*(\hat{S}^\alpha, K)$ and $H^*(\hat{S}_1, K) = \varinjlim H^*(\hat{S}_1^\alpha, K)$ (cf. [3], II.14.4). Also, the diagram

$$\begin{tikzcd}
H^*(\hat{S}, K) & H^*(\hat{S}^\alpha, K) \arrow[l, "j_\alpha^*"'] \\
H^*(\hat{S}_1, K) \arrow[u, "i^*"] & H^*(\hat{S}_1^\alpha, K) \arrow[l, "k_\alpha^*"'] \arrow[u, "i_\alpha^*"']
\end{tikzcd}$$

commutes. Since each $\mathfrak{N}_\alpha$ is small, we have each i_α^* is an isomorphism by Proposition 3.12. It follows that i^* is also an isomorphism.

The above proposition reduces our main theorem (Theorem A) to the case where $\mathfrak{M} = \mathfrak{M}_1$ is a sum of maximal group algebras. We need one final proposition before proving Theorem A.

PROPOSITION 3.14. *Let A be a commutative Banach algebra with identity e and spectrum Δ. Suppose $A = A_1 \oplus A_2$, where A_1 is a subalgebra containing e and A_2 is an ideal. Let $A_2' = A_2 + \mathbf{C}e$. If Δ_1 and Δ_2 are the spectra of A_1 and A_2' respectively, and $i\hat{}_1: \Delta \to \Delta_1$ and $i\hat{}_2: \Delta \to \Delta_2$ are the maps induced by the injections $i_1: A_1 \to A$ and $i_2: A_2' \to A$, then $i\hat{}_1$ and $i\hat{}_2$ induce maps i_1^* and i_2^* of cohomology such that $i_1^* + i_2^*: H^p(\Delta_1, K) \oplus H^p(\Delta_2, K) \to H^p(\Delta, K)$ is an isomorphism for each $p > 0$ and surjective for $p = 0$.*

Proof. Since $A = A_1 \oplus A_2$ with A_2 an ideal, the injection i_1: $A_1 \to A$ has a left inverse π: $A \to A_1$ with kernel A_2. Hence, $i_1^{\wedge}$: $\Delta \to \Delta_1$ has a right inverse $\pi^{\wedge}$: $\Delta_1 \to \Delta$. If we identify Δ_1 with $\pi^{\wedge}(\Delta_1) \subset \Delta$, then $i^{\wedge}: \Delta \to \Delta_1 \subset \Delta$ is a retract of Δ onto Δ_1. It follows that the cohomology exact sequence for the pair (Δ, Δ_1) splits into short exact sequences

$$0 \to H^p(\Delta, \Delta_1; K) \to H^p(\Delta, K) \overset{\pi^*}{\to} H^p(\Delta_1, K) \to 0$$

with i_1^*: $H^p(\Delta_1, K) \to H^p(\Delta, K)$ a right inverse for π^* in each degree (cf. [6], p. 50).

Now the map $i_2^{\wedge}$: $\Delta \to \Delta_2$ is one to one on $\Delta \setminus \Delta_1$ and maps Δ_1 to a single point p. It follows that i_2^*: $H^p(\Delta_2, \{p\}; K) \to H^p(\Delta, \Delta_1; K)$ is an isomorphism. However, $H^p(\Delta_2, \{p\}; K) \to H^p(\Delta_2, K)$ is an isomorphism for $p > 0$ and injective for $p = 0$. The proposition now follows from inspection of the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^p(\Delta_2,\{p\}; K) \arrow[r] \arrow[d, "i_2^*"] & H^p(\Delta_2, K) \arrow[r] \arrow[d, "i_2^*"] & H^p(\{p\}, K) \arrow[r] \arrow[d, "i_2^*"] & 0 \\
0 \arrow[r] & H^p(\Delta, \Delta_1; K) \arrow[r] & H^p(\Delta, K) \arrow[r, shift left, "\pi^*"] & H^p(\Delta_1, K) \arrow[l, shift left, "i_1^*"] \arrow[r] & 0
\end{tikzcd}$$

THEOREM A. *Let $\mathfrak{M}$ be a commutative, semisimple* CM-*algebra with normalized identity δ and spectrum Δ. Let $\{\mathfrak{N}_\alpha\}$ be the collection of all maximal group algebras in $\mathfrak{M}$. If $\delta \in \mathfrak{N}_\alpha$ set $\mathfrak{N}'_\alpha = \mathfrak{N}_\alpha$ and if $\delta \notin \mathfrak{N}_\alpha$ set $\mathfrak{N}'_\alpha = \mathfrak{N}_\alpha + \mathrm{C}\delta$. Let Δ_α be the spectrum of $\mathfrak{N}'_\alpha$ and $i_\alpha^{\wedge}$: $\Delta \to \Delta_\alpha$ the map induced by the injection i_α: $\mathfrak{N}'_\alpha \to \mathfrak{M}$. If, for each α and each coefficient group K, i_α^*: $H^p(\Delta_\alpha, K) \to H^p(\Delta, K)$ is the corresponding map of cohomology, then the map i^*: $\sum_\alpha \oplus H^p(\Delta_\alpha, K) \to H^p(\Delta, K)$ induced by $\{i_\alpha^*\}$ is an isomorphism for $p > 0$ and onto for $p = 0$.*

Proof. By proposition 3.13, we may assume without loss of generality that $\mathfrak{M}$ is the closed linear span of its maximal group algebras, i.e., that $\mathfrak{M} = \mathfrak{M}_1$.

Since $\mathfrak{M} = \mathfrak{M}_1$ the structure semigroup S of $\mathfrak{M}$ is the union of its maximal groups. Each such maximal group is uniquely determined by the idempotent it contains. If $p_1, \ldots, p_n$ are finitely many idempotents, then they generate a finite subsemigroup of S. It follows that any finite collection of maximal groups is contained in a subsemigroup of S which is the union of finitely many maximal groups. Hence, any finite collection of maximal group algebras of $\mathfrak{M}$ is contained in an L-subalgebra of $\mathfrak{M}$ which is a sum of finitely many maximal group algebras (Proposition 2.6). It follows that the L-subalgebras of $\mathfrak{M}$ which are finite sums of maximal group algebras form a directed (upward) set whose union is dense in $\mathfrak{M}$. We conclude by a limiting argument, as in Proposition 3.13, that Theorem A is true if it is true when $\mathfrak{M}$ contains only finitely many maximal group algebras. We shall complete the proof by induction.

The group algebra $\mathbf{C}\delta$ may or may not be maximal. We induct on the number n of maximal group algebras in $\mathfrak{M}$ which are different from $\mathbf{C}\delta$. If $n=1$ then either $\mathfrak{M}=\mathfrak{N}$ is a (discrete) group algebra or $\mathfrak{M}=\mathfrak{N}+\mathbf{C}\delta$ for a group algebra $\mathfrak{N}$ with $\delta \notin \mathfrak{N}$. In either case, $\mathfrak{N}'=\mathfrak{M}$ and the theorem is trivially true. To complete the induction, we will show that if $n>1$ then $\mathfrak{M}=\mathfrak{M}_1 \oplus \mathfrak{M}_2$ with $\mathfrak{M}_2$ an ideal and each of $\mathfrak{M}_1$ and $\mathfrak{M}_2'=\mathfrak{M}_2+\mathbf{C}\delta$ a sum of fewer than n maximal group algebras different from $\mathbf{C}\delta$. In view of Proposition 3.14, this will complete the proof.

If $\mathfrak{M}=\mathfrak{N}_1 \oplus \mathfrak{N}_2$ with $\mathfrak{N}_1$ and $\mathfrak{N}_2$ maximal group algebras and $\delta \in \mathfrak{N}_1$, then necessarily $\mathfrak{N}_2$ is an ideal and we automatically have the required decomposition of $\mathfrak{M}$. If this is not the case, but $n>1$, then $\mathfrak{M}$ contains at least two maximal group algebras which do not contain δ. It follows that there are idempotents $p \neq q$ in S with $p \neq e \neq q$ (e is the identity of S and $\delta=\delta_e$). If $f \in \hat{S}$ separates p and q, then $U=\{s \in S: |f(s)|=1\}$ is an open-closed subsemigroup of S with $S \setminus U=\{s \in S: |f(s)|=0\}$ an ideal. Each of U and $S \setminus U$ contains at least one maximal group of S different from $\{e\}$ (U contains either p or q but not both). It follows that $\mathfrak{M}_1=\{\mu \in \mathfrak{M}: \mu$ is concentrated on $U\}$ and $\mathfrak{M}_2=\{\mu \in \mathfrak{M}: \mu$ is concentrated on $S \setminus U\}$ yields the required decomposition of $\mathfrak{M}$. This completes the proof of Theorem A.

We should mention that the cohomology groups of the spectrum of an algebra $\mathfrak{N}'=\mathfrak{N}+\mathbf{C}\delta$, for $\mathfrak{N}$ a group algebra, are quite computable. In fact, if $\mathfrak{N}=L^1(G)$ then the spectrum of $\mathfrak{N}'$ is the one point compactification of the dual group $\hat{G}$. By the structure theorem for l.c.a. groups (cf. [9], [16]), $\hat{G}$ has an open-closed subgroup of the form $R^n \times H$, where H is compact. Hence, the cohomology of $\hat{G} \cup \{\infty\}$ can easily be computed if one knows the cohomology of H. However, the cohomology of any compact abelian group is computed in [10].

If $\mathfrak{M}=M(G)$ for an l.c.a. group G, then it follows from Lemma 2.4 of [23] that each maximal group algebra in $\mathfrak{M}$ has the form $L^1(G')$, where G' is an l.c.a. group continuously isomorphic to G. For the real line R, the only such groups are R itself and R_d (R with the discrete topology). Hence, the only maximal group algebras in $M(R)$ are $L^1(R)$ and $M_d(R)=L^1(R_d)$. The dual group of R is R and its one point compactification is the circle. The dual group of R_d is the Bohr compactification of R. The integral cohomology groups of the Bohr compactification of R are just the groups $\Lambda^p R$ (the p-fold exterior product of R considered as a Z-module) by Hofmann's results in [10]. Hence, we have the following Corollary to Theorem A:

COROLLARY 3.15. *If Δ is the spectrum of $M(R)$, then $H^0(\Delta, Z)=Z$, $H^1(\Delta, Z)=Z \oplus R$, and $H^p(\Delta, Z)=\Lambda^p R$ for $p>1$.*

4. Applications

In this section $\mathfrak{M}$ will denote a commutative, semisimple CM-algebra with a normalized identity δ. The collection of maximal group algebras in $\mathfrak{M}$ will be denoted $\{\mathfrak{N}_\alpha\}_{\alpha\in\mathfrak{A}}$, and for each α, $\mathfrak{N}'_\alpha$ will be $\mathfrak{N}_\alpha$ if $\delta\in\mathfrak{N}_\alpha$ and $\mathfrak{N}_\alpha+\mathbf{C}\delta$ if $\delta\notin\mathfrak{N}_\alpha$. The spectrum of $\mathfrak{N}'_\alpha$ will be Δ_α and the spectrum of $\mathfrak{M}$ will be Δ.

Recall from Theorem A that the injections i_α: $\mathfrak{N}'_\alpha\to\mathfrak{M}$ induce maps i^*_α: $H^p(\Delta_\alpha, K)\to H^p(\Delta, K)$, for each coefficient group K, such that the resulting map i^*: $\sum\oplus H^p(\Delta_\alpha, K)\to H^p(\Delta, K)$ is an isomorphism for $p>0$ and is onto for $p=0$. If we apply this when $K=Z$ and $p=0$ we obtain a generalization of Cohen's idempotent theorem (cf. [4], [16]).

THEOREM B. *If $\mu\in\mathfrak{M}$ is an idempotent, then $\mu=n_1\mu_1+\ldots+n_k\mu_k$, where for each i, $n_i\in Z$ and $d\mu_i=\chi_i d\nu_i$, where ν_i is the Haar measure in a group algebra in $\mathfrak{M}$ and χ_i is a character on the corresponding group.*

Proof. The group $H^0(\Delta, Z)$ is naturally isomorphic to the additive group of continuous integer valued functions on Δ. If $\mu^2=\mu\in\mathfrak{M}$ then $\mu\hat{}$ is such a function. By Theorem A, $\hat{\mu}=i^*(f_1\oplus\ldots\oplus f_j)=i^*_{\alpha_1}f_1+\ldots+i^*_{\alpha_j}f_j$ for a set $f_1, \ldots, f_j$ with each f_i a continuous integer valued function on Δ_{α_i}. By the Shilov idempotent theorem (cf. [17], [13]) each f_i is $\hat{\nu_i}$ for some $\nu_i\in\mathfrak{N}'_{\alpha_i}$ with ν_i a linear combination, with integer coefficients, of idempotents in $\mathfrak{N}'_{\alpha_i}$. It follows that Theorem B is true if it is true for each algebra of the form $L^1(G)+\mathbf{C}\delta$. However, this is trivial (cf. [16], Chapter 3).

COROLLARY 4.1. (Cohen's Idempotent Theorem). *If G is an l.c.a. group, then each idempotent in $M(G)$ has the form $n_1\mu_1+\ldots+n_k\mu_k$, where for each i, $n_i\in Z$ and $d\nu_i=\chi_i d\mu_i$, where ν_i is the Haar measure of a compact subgroup of G and χ_i is a character of G.*

Proof. It follows from Lemma 2.4 of [23] that each maximal group algebra in $M(G)$ has the form $L^1(G')$, where G' is an l.c.a. group whose group is G and whose topology is at least as strong as the topology of G. Since a compact subgroup of G' is also a compact subgroup of G, the corollary follows from Theorem B.

We obtain another application by applying Theorem A in the case $p=1$. We denote the group of invertible elements of $\mathfrak{M}$ by $\mathfrak{M}^{-1}$.

THEOREM C. *If $\mu\in\mathfrak{M}^{-1}$ then $\mu=\mu_1*\mu_2*\ldots*\mu_n*e^\nu$, where $\nu\in\mathfrak{M}$ and each $\mu_i\in(\mathfrak{N}'_{\alpha_i})^{-1}$ for some α_i.*

Proof. By the Arens-Royden Theorem (cf. [1], [15]), $H^1(\Delta, Z)$ is naturally isomorphic to $\mathfrak{M}^{-1}/\exp(\mathfrak{M})$. Hence, there is a natural onto homomorphism γ: $\mathfrak{M}^{-1}\to H^1(\Delta, Z)$ with

Ker $\gamma = \exp(\mathfrak{M})$. Similarly, there are maps γ_α: $(\mathfrak{N}'_\alpha)^{-1} \to H^1(\Delta_\alpha, Z)$ with Ker $\gamma_\alpha = \exp(\mathfrak{N}'_\alpha)$. The naturality of the Arens-Royden isomorphism implies that

$$\begin{tikzcd} \mathfrak{N}_\alpha^{-1} \arrow[r, "i_\alpha"] \arrow[d, "\gamma_\alpha"] & \mathfrak{M}^{-1} \arrow[d, "\gamma"] \\ H^1(\Delta_\alpha, Z) \arrow[r, "i_\alpha^*"] & H^1(\Delta, Z) \end{tikzcd}$$

is a commutative diagram for each α. It now follows from Theorem A that there are indices $\alpha_1, \ldots, \alpha_k$ and elements $\mu_i \in (\mathfrak{N}'_{\alpha_i})^{-1}$ such that $\mu * (\mu_1 * \ldots * \mu_k)^{-1} \in \operatorname{Ker} \gamma = \exp(\mathfrak{M})$. This completes the proof.

COROLLARY 4.2. *If G is an l.c.a. group and $\mu \in M(G)^{-1}$, then $\mu = \mu_1 * \ldots * \mu_k * e^\nu$ with $\nu \in M(G)$ and each μ_i an element of $(L^1(G_i) + \mathbf{C}\delta_0)^{-1}$ for some l.c.a. group G_i whose group is G and whose topology is at least as strong as the topology of G.*

We are currently working on a paper in which we shall use Corollary 4.2 to develop methods for determining the spectrum of a measure in $M(G)$. We can use these methods, for example, to prove that if $\mu \in M(G)$ and the powers of μ are mutually singular and singular with respect to any group algebra in $M(G)$, then the spectrum of μ is exactly the unit disc.

COROLLARY 4.3. *If $\mu \in M(R)^{-1}$ then $\mu = \eta^k * \delta_y * e^\nu$ for some $\nu \in M(R)$, $y \in R$, and $k \in Z$, where η is any element of $L^1(R) + \mathbf{C}\delta_0$ such that $\eta\hat{}$ has winding number one about zero as a function on $R \cup \{\infty\}$.*

Proof. The only l.c.a. groups continuously isomorphic to R are R and R_d (R with the discrete topology). Hence, by Corollary 4.2, $\mu = \mu_1 * \mu_2 * e^\omega$ with $\mu_1 \in (L^1(R) + \mathbf{C}\delta_0)^{-1}$ and $\mu_2 \in M_d(R)^{-1}$. However, by results of Bohr (cf. [2]), $\mu_2 = \delta_y * e^{\omega_2}$ for some $y \in R$ and $\omega_2 \in M_d(R)$. Also, since the spectrum of $L^1(R) + \mathbf{C}\delta_0$ is the one-point compactification of the line, $(L^1(R) + \mathbf{C}\delta_0)/\exp(L^1(R) + \mathbf{C}\delta_0) \approx Z$ and is generated by any element whose Fourier transform has winding number one about zero. Since the measure η above has this property, we conclude that $\mu_1 = \eta^k * e^{\omega_1}$ for some $k \in Z$ and $\omega_1 \in L^1(R) + \mathbf{C}\delta_0$. If we set $\nu = \omega + \omega_1 + \omega_2$ the proof is complete.

COROLLARY 4.4. *If $\mu \in M(R)^{-1}$ then μ can be factored as $\mu = \mu_1 * \mu_2 * \eta^k * \delta_x$ with $k \in Z$, $x \in R$, η as in Corollary 4.3, and $\mu_1 \in M(R^+)^{-1}$, $\mu_2 \in M(R^-)^{-1}$.*

Proof. We write $\mu=\eta^k*\delta_x*e^\nu$ as in Corollary 4.3. If $\nu_1=\nu|_{R^+}$ and $\nu_2=\nu-\nu_1$, then $\mu_1=e^{\nu_1}$ and $\mu_2=e^{\nu_2}$ yields the above factorization.

If $\mu\in M(R)$ then the Wiener-Hopf operator $W_\mu: L^p(R^+)\to L^p(R^+)$ is defined by $W_\mu f(x)=\int_0^\infty f(t)\,d\mu(x-t)$ for $x\in R^+$, $f\in L^p(R^+)$. It is easy to see that W_μ will be invertible (for any $p\geqslant 1$) provided $\mu=\mu_1*\mu_2$ with $\mu_1\in M(R^+)^{-1}$ and $\mu_2\in M(R^-)^{-1}$. By Corollary 4.4, this will be true of $\mu\in M(R)^{-1}$ provided $k=0=x$. Since the k and x of Corollary 4.4 can be computed for a given μ, Corollary 4.4 yields strong information about the spectrum of a Wiener-Hopf operator. This is discussed more fully in [5]. Our conversation with R. G. Douglas concerning this problem led to the research presented here.

5. The key lemma

This section contains the core of the proof of Theorem A—the missing ingredient in the proof of Proposition 3.2. Throughout the section, $\mathfrak{M}$ will be a CM-algebra which is commutative, semisimple and has a normalized identity δ.

LEMMA 5.1. *Suppose there exists $\mu\in\mathfrak{M}$ with $\mu\geqslant 0$ and $f\in\hat{S}^+$ such that* $\sup\{\mu\hat{}(g)$*: $g\in\hat{S}^+$, $g\leqslant f$, $g\neq f\}<\mu\hat{}(f)$. Then $f^2=f$ and there is a group algebra in $\mathfrak{M}$ supported on the kernel of the open-closed subsemigroup $U=\{s\in S: f(s)=1\}$. Hence, the kernel of U is contained in S_1.*

The proof of Lemma 5.1 relies heavily on machinery developed in [22] which gives a relationship between spectral properties of n-tuples of measures in a CM-algebra and the existence of absolutely continuous measures. Unfortunately, the results of [22] do not apply directly, since in [22] it is always assumed that $\mathfrak{M}\subset M(G)$ for an l.c.a. group G. However, the techniques of [22] can be extended to cover our present situation.

Note that if $\sup\{\mu\hat{}(g): g\leqslant f,\ g\neq f,\ g\in\hat{S}^+\}<\mu\hat{}(f)$, then $f^2=f$; otherwise, $f^r\neq f$, $f^r\leqslant f$, and $f^r\in\hat{S}^+$ for $r>1$, and $\lim_{r\to 1+}f^r=f$, uniformly, so that $\lim_{r\to 1+}\mu\hat{}(f^r)=\mu\hat{}(f)$. It follows that we may as well assume that $f=1$ since, otherwise, we may replace $\mathfrak{M}$ by the L-subalgebra $\{\nu\in\mathfrak{M}: \nu \text{ is concentrated on } U\}$, where $U=\{s\in S: f(s)=1\}$. Also, we may as well assume μ is normalized. Hence, throughout the remainder of the section, we shall assume that $\mu\hat{}(g)\leqslant k<1=\mu\hat{}(1)$ for all $g\in\hat{S}^+$ with $g\neq 1$. Our object will be to prove that the kernel of S supports a group algebra in $\mathfrak{M}$.

We shall first identify an l.c.a. group G whose Bohr compactification is the kernel of S and whose group algebra, $L^1(G)$, stands a chance of being embedded in $\mathfrak{M}$.

PROPOSITION 5.2. *With the Gelfand topology on $\hat{S}$, the group $H=\{f\in\hat{S}: |f|=1\}$ is open in $\hat{S}$ and is an l.c.a. group.*

Proof. By the strong topology on $\hat{S}$, we shall mean the topology in which $f_\alpha \to f$ if and only if $\int |f - f_\alpha| d\nu \to 0$ for every positive $\nu \in \mathfrak{M}$. Note that the weak and strong topologies agree on H. In fact, if $f_\alpha \to f$ weakly in H and $\nu \in \mathfrak{M}^+$, then

$$\left(\int |f - f_\alpha| d\nu\right)^2 \leqslant \int |f - f_\alpha|^2 d\nu = \int (2 - f_\alpha \bar{f} - \bar{f}_\alpha f)\, d\nu = 2\nu\hat{\ }(1) - \nu\hat{\ }(f_\alpha \bar{f}) - \nu\hat{\ }(\bar{f}_\alpha f),$$

and $f_\alpha \bar{f} \to |f|^2 = 1$, $\bar{f}_\alpha f \to |f|^2 = 1$ weakly. Hence, $\int |f - f_\alpha| d\nu \to 0$ and $f_\alpha \to f$ strongly. Since multiplication in $\hat{S}$ is jointly strongly continuous and conjugation (which is inversion in H) is weakly and strongly continuous, we have that H is a topological group. To show that H is locally compact we shall show that it is open in $\hat{S}$—a compact space.

If $f \in H$ and $g \in \hat{S} \setminus H$, then we set $d\nu = \bar{f} d\mu$ and note that

$$\begin{aligned} |\nu\hat{\ }(f) - \nu\hat{\ }(g)| &= \left|\int (f-g)\bar{f} d\mu\right| \geqslant \int |f|^2 d\mu - \left|\int g\bar{f} d\mu\right| \geqslant \int |f|^2 d\mu - \int |g|\, d\mu \\ &= \mu\hat{\ }(1) - \mu\hat{\ }(g) \geqslant 1 - k > 0. \end{aligned}$$

It follows that H is weakly open in $\hat{S}$ and is an l.c.a. group.

Let G be the l.c.a. group whose dual group is H. Since the kernel of S is a compact group whose dual group is H with the discrete topology, it follows that the kernel of S is $\bar{G}$—the Bohr compactification of G (cf. [16], Chapter 1). Let α: $G \to \bar{G} \subset S$ be the natural map of G into its Bohr compactification.

We shall now construct a new semigroup S_0 and an embedding $\mathfrak{M} \to M(S_0)$ such that S_0 has most of the properties of the structure semigroup of $\mathfrak{M}$ except that it is only locally compact. This new semigroup will have G as its kernel rather than $\bar{G}$.

Consider the locally compact topological semigroup $S \times G$. We set $S_0 = \{(s, g) \in S \times G$: $ps = \alpha(g)\}$, where $p \in \bar{G}$ is the idempotent in the kernel of S. Since α: $G \to \bar{G}$ and $s \to ps$: $S \to \bar{G}$ are continuous homomorphisms, it follows that S_0 is a closed subsemigroup of $S \times G$. Hence, S_0 is a locally compact topological semigroup.

Note that the map $g \to (\alpha(g), g) \in \bar{G} \times G \subset S \times G$ is a topological isomorphism of G onto an ideal in S_0. It follows that we may identify G with the kernel of S_0 via this map. Also, the projection $(s, g) \to s$: $S \times G \to S$ is, when restricted to S_0, a continuous one to one homomorphism β: $S_0 \to S$ with the property that $\beta|_G$ is the map α: $G \to \bar{G}$. Furthermore, the image of β in S is exactly $\{s \in S$: $ps \in \alpha(G)\}$. We shall prove that every measure in $\mathfrak{M}$ is concentrated on this subset of S.

If $\nu \in \mathfrak{M}$ then $\delta_p * \nu \in M(\bar{G})$ and for $f \in H$, $(\delta_p * \nu)\hat{\ }(f) = f(p)\nu\hat{\ }(f) = \nu\hat{\ }(f)$. Since $\nu\hat{\ }$ is continuous on H, it follows that $\delta_p * \nu = \omega \circ \alpha^{-1}$ for some $\omega \in M(G)$ (cf. [16], 1.4.3). It follows that there is a countable set $\{K_i\}_{i=1}^\infty$ of compact subsets of G such that $\delta_p * \nu$ is concentrated on $\bigcup_{i=1}^\infty \alpha(K_i)$. Hence, ν is concentrated on $\bigcup_{i=1}^\infty L_i \subset \beta(S_0)$, where $L_i = \{s \in S$: $ps \in \alpha(K_i)\}$.

The sets L_i are not only in the image of β but $\beta^{-1}(L_i)=\{(s, g)\in S_0: g\in K_i\}$ is compact in S_0. It follows that each measure $\nu\in\mathfrak{M}$ has a unique pre-image ν' in $M(S_0)$ such that $\nu=\nu'\circ\beta^{-1}$. Hence, there is an L-subalgebra $\mathfrak{M}'\subset M(S_0)$ such that $\nu'\to\nu'\circ\beta^{-1}$ is an isomorphism of $\mathfrak{M}'$ onto $\mathfrak{M}$.

Now if F is a complex homomorphism of $\mathfrak{M}$, then $F(\nu)=\int_S f d\nu=\int_{S_0} f\circ\beta d\nu'$ for some $f\in\hat{S}$. Since $f\circ\beta$ is a bounded, continuous semicharacter on S_0, we have that each complex homomorphism of $\mathfrak{M}'$ is given by an element of $\hat{S}_0$, where $\hat{S}_0$ denotes the set of bounded, continuous semicharacters on S_0. If we identify $\mathfrak{M}$ with $\mathfrak{M}'$, we have proved:

PROPOSITION 5.3. *There is a locally compact topological semigroup S_0, with kernel G, such that $\mathfrak{M}$ may be identified with an L-subalgebra of $M(S_0)$ in such a way that each complex homomorphism of $\mathfrak{M}$ has the form $\nu\to\nu\hat{}(f)=\int_{S_0} f d\nu$ for some $f\in\hat{S}_0$.*

Note that the semicharacters on S_0 of the form $f\circ\beta$ for $f\in\hat{S}$ separate points in S_0. This follows since β is one to one and $\hat{S}$ separates points in S.

In passing from S to S_0 we may have lost, in addition to compactness, another property of the structure semigroup: conceivably $\mathfrak{M}$ may fail to be weak-* dense in $M(S_0)$. Hence, the functions $\nu\hat{}(f)=\int_{S_0} f d\nu$ for $\nu\in\mathfrak{M}$ may fail to separate points in $\hat{S}_0$. If f, $g\in\hat{S}_0$ we write $f\sim g$ if $\nu\hat{}(f)=\nu\hat{}(g)$ for every $\nu\in\mathfrak{M}$. Since each complex homomorphism of $\mathfrak{M}$ has the form $\nu\to\nu\hat{}(f\circ\beta)$ for some $f\in\hat{S}$, each $g\in\hat{S}_0$ will be equivalent mod ($\sim$) to some $f\circ\beta$ with $f\in\hat{S}$.

Our hypothesis that $\mu\hat{}(f)\leqslant k<1$ for all $f\in\hat{S}^+$ with $f\neq 1$ becomes, after embedding $\mathfrak{M}$ in $M(S_0)$, $\mu\hat{}(f)\leqslant k<1$ for all $f\in\hat{S}_0^+$ with $f\nsim 1$.

Since $\beta|_G$ is the natural embedding of G in its Bohr compactification $\bar{G}$ and since G is the kernel of S_0 and $\bar{G}$ is the kernel of S, the map $f\to f\circ\beta$ is an isomorphism of $\{f\in\hat{S}: |f|=1\}$ onto $\{g\in\hat{S}_0: |g|=1\}$. We shall, henceforth, call the latter group H. Recall that it is the dual group of G. We shall denote by H_0 the set $\{f\in\hat{S}_0: |f|\sim 1\}=\{f\in\hat{S}_0: f\sim g$ for some $g\in H\}$. Since each element of H has the form $f\circ\beta$ for $f\in\hat{S}$ with $|f|=1$, it follows that $\{\nu\hat{}: \nu\in\mathfrak{M}\}$ separates points in H.

PROPOSITION 5.4. *If p is the idempotent of S_0 contained in G, then $\{\delta_p*\nu: \nu\in\mathfrak{M}\}$ is weak-* dense in $M(G)$.*

Proof. Clearly $\delta_p*\mathfrak{M}=\{\delta_p*\nu: \nu\in\mathfrak{M}\}$ is an L-subalgebra of $M(G)$. It will be weak-* dense if and only if its support T in G is all of G, i.e., if and only if no open set in G is a set of measure zero for all of $\delta_p*\mathfrak{M}$. Now, since $\delta_p*\mathfrak{M}$ is an L-subalgebra of $M(G)$, T is a closed subsemigroup of G.

If $f\in H=\{h\in\hat{S}_0: |h|=1\}=\hat{G}$, then $(\delta_p*\nu)\hat{}(f)=f(p)\nu\hat{}(f)=\nu\hat{}(f)$. Since $\{\nu\hat{}: \nu\in\mathfrak{M}\}$

separates points in H, it follows that T cannot be contained in a proper closed subgroup of G. If $T \neq G$ then by Lemma 2 of [24] there is a continuous homomorphism γ: $G \to R$ such that $(0) \neq T \subset R^+$. However, this implies that for each $x > 0$ the map F_x, $F_x(\nu) = \int e^{-x\gamma(ps)} d\nu(s)$, is a complex homomorphism of $\mathfrak{M}$. If $f_x \in \hat{S}_0$ is the semicharacter such that $F_x(\nu) = \int f_x d\nu$, then clearly $f_x \in \hat{S}_0^+$ and $f_x \neq 1$. However, $\lim_{x \to 0^+} \mu^\wedge(f_x) = \lim_{x \to 0^+} \int e^{-x\gamma(ps)} d\mu(s) = 1$. This contradicts our assumption that $\sup \{\mu^\wedge(f): f \in \hat{S}_0^+, f \neq 1\} < 1$. Hence, $T = G$ and the proof is complete.

If we can prove that some nonzero measure $\nu \in \mathfrak{M}$ is concentrated on G, then the proof of Lemma 5.1 will be complete. In fact, it will then be the case that when $\mathfrak{M}$ is represented as an algebra of measures on S, the kernel $\bar{G}$ of S is not a set of measure zero for ν; by Proposition 2.6 this implies that $\bar{G}$ is the support of a group algebra in $\mathfrak{M}$.

By the structure theorem for l.c.a. groups (cf. [16], Chapter 2), G has an open subgroup G_1 such that $G_1 \approx R^n \times K$ for some $n \geq 0$ and some compact group K. Since G_1 is open and $\delta_p * \mathfrak{M}$ is weak-* dense in G, the open subsemigroup $S_1 = \{s \in S_0: ps \in G_1\}$ is not a set of measure zero for $\mathfrak{M}$. If $\mathfrak{N}$ is the L-subalgebra of $\mathfrak{M}$ consisting of measures concentrated on S_1, then we shall prove that $\mathfrak{N}$ and S_1 satisfy the same conditions we have concerning $\mathfrak{M}$ and S_0. Note that G_1 is the kernel of S_1.

PROPOSITION 5.5. (a) *Each complex homomorphism of $\mathfrak{N}$ has the form $\nu \to \nu^\wedge(f) = \int f d\nu$ for some $f \in \hat{S}_1$;*

(b) *there exists $\mu_1 \in \mathfrak{N}^+$ such that* $\sup \{\hat{\mu_1}(f): f \in \hat{S}_1^+, f \neq 1\} < 1 = \hat{\mu_1}(1)$*; and*

(c) *$\delta_p * \mathfrak{N}$ is weak-* dense in $M(G_1)$.*

Proof. (a) We shall prove that a complex homomorphism of $\mathfrak{N}$ has the form $\nu \to \int_{S_1} f d\nu$ for some $f \in \hat{S}_0$. Then $f|_{S_1}$ will be the required element of $\hat{S}_1$. Now the set of complex homomorphisms of $\mathfrak{N}$ of the form $\nu \to \int_{S_1} f d\nu$ $(f \in \hat{S}_0)$ is a compact subset of the spectrum of $\mathfrak{N}$, since it is the image under restriction of the spectrum of $\mathfrak{M}$. To show that this set is the entire spectrum of $\mathfrak{N}$ we must show that if $\nu_1, ..., \nu_n \in \mathfrak{N}$ and $\hat{\nu_1}(f), ..., \hat{\nu_n}(f)$ do not all vanish for any $f \in \hat{S}_0$, then the equation

$$\nu_1 * \omega_1 + ... + \nu_n * \omega_n = \delta \tag{*}$$

has a solution for $\omega_1, ..., \omega_n \in \mathfrak{N}$. However, since $\hat{S}_0$ determines the spectrum of $\mathfrak{M}$, (*) does have a solution for $\omega_1, ..., \omega_n \in \mathfrak{M}$. If $\omega_i' = \omega_i|_{S_1}$ then $\omega_i' * \nu_i \in \mathfrak{N}$ and $(\omega_i - \omega_i') * \nu_i$ is concentrated on $S_0 \setminus S_1$ for each i. Note that $S_1 \cdot (S_0 \setminus S_1) \subset S_0 \setminus S_1$ and $S_1 \cdot S_1 \subset S_1$. Since S_1 contains the support of δ (which is the identity of S_0), we conclude that $\nu_1 * \omega_1' + ... + \nu_n * \omega_n' = \delta$ as well, and (*) has a solution in $\mathfrak{N}$. This completes part (a).

(b) Suppose that $\sup \{\hat{\mu_1}(f): f \in \hat{S}_+^1, f \neq 1\} = \hat{\mu_1}(1)$ for all $\mu_1 \in \mathfrak{N}^+$. Since $\nu \to \nu^\wedge(f)$ is a complex homomorphism of $\mathfrak{N}$ for $f \in \hat{S}_1$, we have $\nu^\wedge(f)=\nu^\wedge(g)$ for all $\nu \in \mathfrak{N}$ for some $g \in \hat{S}_0$. It follows that $\sup\{\hat{\mu_1}(f)\colon f \in \hat{S}_0^+,\ f \neq 1\}=\hat{\mu_1}(1)$ for all $\mu_1 \in \mathfrak{N}^+$. This implies there is a net $\{f_\alpha\} \subset \hat{S}_0^+$ with $f_\alpha \neq 1$ for each α and $\hat{\mu_1}(f_\alpha) \to \hat{\mu_1}(1)$ for all $\mu_1 \in \mathfrak{N}^+$. Without loss of generality we may assume that $\{f_\alpha\}$ converges to $f \in \hat{S}_0^+$ in the weak topology determined by $\mathfrak{M}$ on $\hat{S}_0$. It follows that $f=1$ on the support of $\mathfrak{N}$ in S_0.

Let T be the support of $\mathfrak{M}$ in S_0 and note that T is a closed subsemigroup of S_0. Furthermore, if $T_x = T \cap \{s \in S_0\colon ps \in xG_1\}$ for $x \in G$, then Proposition 5.4 implies that $T_x \neq \emptyset$ for each $x \in G$ (recall that G_1 and, hence, each xG_1 is open in G). Note that $T_p = T \cap S_1$ is the support of $\mathfrak{N}$ and $T_x \cdot T_y \subset T_{x \cdot y}$ for each $x, y \in G$.

If $s \in T$ then $s \in T_x$ for some x. If we choose $t \in T_{x-1}$, then $st \in T_p$ and, hence, $f(st)=1$. However, $f \in \hat{S}_0^+$ then implies that $f(s)=f(t)=1$. Hence, $f=1$ on the support of $\mathfrak{M}$. This implies that $\lim \mu^\wedge(f_\alpha)=\mu^\wedge(f)=1$ which contradicts our original assumption that $\sup\{\mu^\wedge(g)\colon g \in \hat{S}_0^+,\ g \neq 1\}<1$. This completes part (b).

(c) Since G_1 is an open subgroup of G, part (c) follows immediately from Proposition 5.4.

Now that we have proved Proposition 5.5 we will forget about $\mathfrak{N}$, S_1, and μ_1 and simply assert that, without loss of generality, we may assume that our group G has the form $G = R^n \times K$ for some compact group K and some integer n.

If it were true that $n=0$, i.e., that $G=K$, then the dual group H of G would be a discrete open subset of the spectrum of $\mathfrak{M}$. The Shilov idempotent theorem would then immediately imply that the Haar measure of G was an element of $\mathfrak{M}$, and we would be finished. The fact that n may not be zero forces us to resort to a combinatorial procedure introduced in [22].

We define a map $\varrho\colon S_0 \to R^n$ to be the map $s \to p \cdot s\colon S_0 \to G$ followed by the projection of G onto R^n.

Definiton 5.1. If A is a compact subset of R^n, set $\varphi_A(s)=\sup\{e^{-x\cdot\varrho(s)}\colon x \in A\}$ for each $s \in S_0$, and $\varphi'_A(s)=\varphi^{-1}_{-A}(s)=\inf\{e^{-x\cdot\varrho(s)}\colon x \in A\}$ (cf. Definition 2.2 of [22]).

Several of the remaining propositions will not be proved here. They have proofs which differ in no essential way from the proofs of similar propositions in [22]. In each case, the only difference is that in [22] the underlying space was a group, whereas here it is the semigroup S_0. In particular, the following proposition is proved essentially as in Lemma 2.1 of [22] (it is trivial in any case):

PROPOSITION 5.6. *If A and B are nonempty compact subsets of R^n, $x \in R^n$, and $s, t \in S_0$, then:*

(a) φ_A *is continuous on* S_0*;*

(b) $\varphi_A(s\cdot t)\leqslant\varphi_A(s)\varphi_A(t)$ *and* $\varphi'_A(st)\leqslant\varphi'_A(s)\varphi_A(t)$*;*

(c) $\varphi_A=\varphi_{\langle A\rangle}$, *where* $\langle A\rangle$ *is the convex hull of* A*;*

(d) $B\subset A$ *implies* $\varphi'_A\leqslant\varphi'_B\leqslant\varphi_B\leqslant\varphi_A$*;*

(e) $\varphi_{A\cup B}=\max(\varphi_A,\varphi_B)$*; and*

(f) $\varphi_{A+x}(s)=e^{-x\cdot\varrho(s)}\varphi_A(s)$.

Definition 5.2. (cf. Definition 2.3 of [22]). (1) Let $\mathfrak{M}_{\text{loc}}$ denote the linear space of (possibly infinite) measures ν on the ring of bounded Borel sets of S_0, such that $\nu|_C\in\mathfrak{M}$ for each compact set $C\subset S_0$;

(2) for $\nu\in\mathfrak{M}_{\text{loc}}$ set $\|\nu\|_A=\int\varphi_A(s)d|\nu|(s)$ and $\|\nu\|'_A=\int\varphi'_A(s)d|\nu|(s)$ for each compact set $A\subset R^n$;

(3) for each compact $A\subset R^n$ set $\mathfrak{M}(A)=\{\nu\in\mathfrak{M}_{\text{loc}}\colon\ \|\nu\|_A<\infty\}$ and $\mathfrak{M}'(A)=\{\nu\in\mathfrak{M}_{\text{loc}}\colon\ \|\nu\|'_A<\infty\}$.

As in Lemma 2.2 of [22], the properties of φ_A and φ'_A described in Proposition 5.6(b) can be used to prove that the convolution product $\nu*\omega$ exists and is in $\mathfrak{M}'(A)$ if $\nu\in\mathfrak{M}(A)$ and $\omega\in\mathfrak{M}'(A)$. In fact $\|\nu*\omega\|'_A\leqslant\|\nu\|_A\|\omega\|'_A$. Furthermore, if both ν and ω are in $\mathfrak{M}(A)$ then $\nu*\omega\in\mathfrak{M}(A)$ and $\|\nu*\omega\|_A\leqslant\|\nu\|_A\|\omega\|_A$. It follows that $\mathfrak{M}(A)$ is a Banach algebra and $\mathfrak{M}'(A)$ is a Banach module over $\mathfrak{M}(A)$.

If A and B are compact sets in R^n with $B\subset A$, then Proposition 5.6 (b) implies that $\mathfrak{M}(A)\subset\mathfrak{M}(B)\subset\mathfrak{M}'(A)$. Hence each $\mathfrak{M}(B)$ for $B\subset A$ is a submodule of the $\mathfrak{M}(A)$-module $\mathfrak{M}'(A)$. Also, since $\varphi_A=\varphi_{\langle A\rangle}$we have $\mathfrak{M}(A)=\mathfrak{M}(\langle A\rangle)$ and we may as well restrict attention to convex subsets of R^n.

The next proposition follows from Lemmas 2.6 and 2.7 of [22], which also hold in our present situation.

Proposition 5.7. *If A is a compact, convex subset of R^n, then the correspondence $B\to\mathfrak{M}(B)$ (B compact, convex in A) defines a convex stack of submodules of the $\mathfrak{M}(A)$-module $\mathfrak{M}'(A)$ in the sense of Definition 3.4. of* [22].

If $\mu_1,\ldots,\mu_n\in\mathfrak{M}(A)$ then the equation

$$\mu_1*\nu_1+\ldots+\mu_n*\nu_n=\delta \tag{1}$$

may have a solution $\nu_1,\ldots,\nu_n\in\mathfrak{M}(B)$ for some compact, convex sets $B\subset A$ and not for others. If $x\in A$ we say (1) is locally solvable at x if (1) has a solution $\nu_1,\ldots,\nu_n\in\mathfrak{M}(A\cap V)$ for some compact neighborhood V of x.

Proposition 5.8. *If $x\in A$ and A is compact and convex in R^n, then for $\mu_1,\ldots,\mu_n\in\mathfrak{M}(A)$ the equation (1) is locally solvable at x if and only if for each $f\in\hat{S}_0$, $\int e^{-x\cdot\varrho(s)}f(s)\,d\mu_i(s)\neq0$ for some i.*

Proof. If we set $dT_x\nu = e^{x\cdot\varrho}d\nu$ for $\nu\in\mathfrak{M}$, then T_x is clearly an isomorphism-isometry of $\mathfrak{M}=\mathfrak{M}(\{0\})$ onto $\mathfrak{M}(\{x\})$ (cf. [22], Lemma 2.4). Since each complex homomorphism of $\mathfrak{M}$ has the form $\nu\to\nu^{\wedge}(f)=\int f d\nu$ for some $f\in\hat{S}_0$, it follows that each complex homomorphism of $\mathfrak{M}(\{x\})$ has the form $\nu\to\int e^{-x\cdot\varrho}f d\nu$. Hence, if for each $f\in\hat{S}_0$ there is an i such that $\int e^{-x\cdot\varrho}f d\mu_i\neq 0$, then equation (1) will have a solution in $\mathfrak{M}(\{x\})$. It now follows as in Lemma 2.5 of [22] that (1) has a solution in $\mathfrak{M}(A\cap V)$ for some neighborhood V of x.

If $\nu\in\mathfrak{M}(A)$ then we set $\tilde{\nu}(x, f)=\int e^{-x\cdot\varrho}f d\nu$ for $x\in A$, $f\in\hat{S}_0$. If ν has compact support then $\tilde{\nu}(x,f)$ is defined for all $(x,f)\in R^n\times\hat{S}_0$. Recall, if $H_0=\{f\in\hat{S}_0\colon |f|\sim 1\}$ then $H=H_0/(\sim)$ is the dual group of G. Thus, each element of H_0 is equivalent to a function $s\to e^{-iy\cdot\varrho(s)}\gamma(ps)$ for some $y\in R^n$ and $\gamma\in\hat{K}$. Hence, for $f=e^{-iy\cdot\varrho}\gamma\in H_0$ and $z=x+iy$ we have $\tilde{\nu}(x,f)=\nu^{\vee}(z,\gamma)=\int e^{-z\cdot\varrho(s)}\gamma(p\cdot s)\,d\nu(s)$. If ν has compact support then obviously the function $\nu^{\vee}(z,\gamma)$ is a holomorphic function of $z\in\mathbf{C}^n$ for each $\gamma\in\hat{K}$.

PROPOSITION 5.9. *Each function on $\mathbf{C}^n\times\hat{K}$ which is holomorphic on $\mathbf{C}^n\times\{\gamma\}$ for each $\gamma\in\hat{K}$ can be uniformly approximated on compact sets by functions of the form $\nu^{\vee}$ with $\nu\in\mathfrak{M}$ with compact support.*

Proof. Note that $\nu^{\vee}(z,\gamma)=\int_{R^n\times K}e^{-z\cdot t}\gamma(k)\,d\omega(t,k)$, where $\omega=\delta_p*\nu\in M(G)=M(R^n\times K)$, is just the Laplace transform of ω. As in Lemma 1.3 of [22], the space of all Laplace transforms of compactly supported measures on G is dense in the space of holomorphic functions on $\mathbf{C}^n\times\hat{K}$. Hence, the proposition follows from the fact that $\{\nu*\delta_p\colon \nu\in\mathfrak{M}\}$ is weak-* dense in $M(G)$ (Proposition 5.4) (cf. Lemma 2.8 of [22]).

PROPOSITION 5.10. *There exists an n-simplex $A\subset R^n$, with $0\in A$, and measures $\mu_1, \dots, \mu_n\in\mathfrak{M}(A)$ such that $\tilde{\mu}_1(0,1)=\dots=\tilde{\mu}_n(0,1)=0$, but $\tilde{\mu}_1, \dots, \tilde{\mu}_n$ do not vanish simultaneously at any (x,f) with $x\in\delta A$ or with $f\in\hat{S}_0\setminus H_0$.*

Proof. Recall that $|\int f d\mu|\leqslant\int|f|d\mu\leqslant k<1$ if $f\in\hat{S}_0\setminus H_0$, and $\int 1 d\mu=1$. Since μ is inner regular, we can replace μ with a measure $\mu'\in\mathfrak{M}$ which has compact support and satisfies $|\int f d\mu'|<k'<1$ for $f\in\hat{S}_0\setminus H_0$, and $\int 1 d\mu'=1$. It follows that there is a compact neighborhood U of 0 in R^n such that $|\tilde{\mu}'(x,f)|<k'<1$ for $x\in U$ and $f\in\hat{S}_0\setminus H_0$. We choose $\mu_1=\delta-\mu'$ and note that $|\tilde{\mu}_1(x,f)|>1-k'>0$ if $f\in\hat{S}_0\setminus H_0$ and $x\in U$, but $\tilde{\mu}_1(0,1)=0$. Since $\hat{S}_0$ is compact, it follows that the set $N=\{(x,f)\in U\times\hat{S}_0\colon \tilde{\mu}_1(x,f)=0\}$ is a compact subset of $U\times H_0$.

Let $W=\{(z,\gamma)\in\mathbf{C}^n\times\hat{K}\colon \operatorname{Re} z\in\operatorname{int} U\}$ and $V=\{(z,\gamma)\in W\colon \mu_1^{\vee}(z,\gamma)=0\}$. Then V is an $(n-1)$-dimensional subvariety of W (cf. [7], III.c.), $(0,1)\in V$, and V has compact closure in $\mathbf{C}^n\times\hat{K}$. It is a simple matter to choose functions $f_2, \dots, f_n$, holomorphic on $\mathbf{C}^n\times\hat{K}$, such that $V\cap\{(z,\gamma)\in W\colon f_2(z,\gamma)=\dots=f_n(z,\gamma)=0\}$ is a finite set containing $(0,1)$. By Proposition

5.9, we can find $\mu_2, \ldots, \mu_n \in \mathfrak{M}$, with compact support, such that each $\mu_i^{\check{}}$ approximates f_i sufficiently close on $\bar{V}$ that $(0, 1) \in V \cap \{(z, \gamma) \in W: \check{\mu}_2(z, \gamma) = \ldots = \check{\mu}_n(z, \gamma) = 0\}$ is compact in V. It follows from III.B.17 of [7] that this set will also be finite. It is now a trivial matter to choose an n-simplex A with $0 \in \operatorname{int} A \subset A \subset \operatorname{int} U$ such that no common zero (z, γ) of $\check{\mu}_1, \ldots, \check{\mu}_n$ has $\operatorname{Re} z \in \delta A$. Hence $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$ do not have a common zero $(x, f) \in \delta A \times \hat{S}_0$. We already have that $\tilde{\mu}_1$ does not vanish on $A \times (S_0 \setminus H_0)$. This completes the proof.

Proof of Lemma 5.1. Let A and $\mu_1, \ldots, \mu_n$ be as in the above proposition. Since $\tilde{\mu}_1(x, f), \ldots, \tilde{\mu}_n(x, f)$ do not vanish simultaneously for $x \in \delta A$, Proposition 5.8 implies that equation (1) is locally solvable at each $x \in \delta A$. However, since $\tilde{\mu}_1(0, 1) = \ldots = \tilde{\mu}_n(0, 1) = 0$, equation (1) is not locally solvable at 0 and, hence, is not solvable in $\mathfrak{M}(A)$. By Theorem 4.2 of [22], there is a measure $\lambda \in \mathfrak{M}'(A)$ such that for $\omega \in \mathfrak{M}(A)$ the equation

$$\mu_1 * \nu_1 + \ldots + \mu_n * \nu_n = \omega \tag{2}$$

can be solved in $\mathfrak{M}(A)$ if and only if $\omega * \lambda = 0$. It follows that $\lambda \neq 0$ but $\mu_i * \lambda = 0$ for each i.

If $f \in \hat{S}_0$ set $d\mu_i^f = f d\mu_i$ and $d\lambda^f = f d\lambda$. The fact that f is a semicharacter implies that $\mu_i^f * \lambda^f = (\mu_i * \lambda)^f = 0$ for each i. However, if $f \in \hat{S}_0 \setminus H_0$ then $|gh| \neq 1$ for $g \in \hat{S}_0$ and $(\mu_i^f)^{\check{}}(x, g) = \check{\mu}_i(x, fg)$. It follows that $(\mu_1^f)^{\check{}}, \ldots, (\mu_n^f)^{\check{}}$ do not vanish simultaneously on $A \times \hat{S}_0$. Hence, by Theorem 4.1 of [22], the equation $\mu_1^f * \nu_1 + \ldots + \mu_n^f * \nu_n = \delta$ can be solved in $\mathfrak{M}(A)$. This implies that $\lambda^f = \nu_1 * (\mu_1^f * \lambda^f) + \ldots + \nu_n * (\mu_n^f * \lambda^f) = 0$. We conclude that the support of λ in S_0 is a subset of $\{s \in S_0: f(s) = 0 \text{ if } f \sim g \in \hat{S}_0 \setminus H_0\}$. Since $\hat{S}_0$ separates points in S_0 and G is the kernel of S_0, it follows that $G = \{s \in S_0: f(s) = 0 \text{ if } f \sim g \in \hat{S}_0 \setminus H_0\}$. We conclude that λ is concentrated on G. Since $\lambda|_C \in \mathfrak{M}$ for each compact set C, it follows that there is a nonzero element of $\mathfrak{M}$ concentrated on G. We noted earlier that this suffices to prove that $\bar{G} \subset S$ is the support of a group algebra in $\mathfrak{M}$.

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