TRACES OF COMMUTATORS OF INTEGRAL OPERATORS

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Introduction

This paper concerns a rather concrete phenomenon in abstract operator algebras. The main examples of the algebras we study are algebras of singular integral operators (pseudo-differential operators of order zero). As everyone knows, the Fredholm index of a pseudo-differential operator depends only on its symbol and the Atiyah-Singer Index Theorem gives an explicit formula for computing the dependence. What we are doing might be though of analogously. The observation behind this paper is that traces of commutators or appropriate higher commutators depend only on "symbols"; then we compute the dependence in one and two dimensions. It emerges that these considerations are closely related to index theory.

The simplest type of operator system which we study is an almost commuting (a.c.) pair of operators on Hilbert space H; that is, a pair X, Y of bounded selfadjoint operators on H with trace class commutator $[X, Y]=XY-YX$. The two main examples of almost commuting pairs are Toeplitz (or Wiener-Hopf) operators with smooth symbol and singular integral operators on the line. (In fact, the singular integral operators and multiplications provide generic examples [14], [17], [19].) In [8] the authors considered an algebra $\mathfrak{A}$ of operators generated by an almost commuting pair and gave a quite satisfying formula for the trace of any commutator $[A, B]$ from $\mathfrak{A}$ in terms of the "symbol" of A

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and of B. The formula which arises easily yields an index formula for operators in $\mathfrak{A}$. The object of this paper is to formulate the trace theory in general, namely, for algebras with more than two generators and for abstract singular integral operators in arbitrary dimensions. We remark that this paper is reasonably self contained, and highly algebraic, and that no knowledge of singular integral or pseudo-differential operators is required for reading § 1–6.

A. Abstract singular integral operators

A selfadjoint algebra of bounded operators which commutes modulo the trace class will be called *almost commuting*. Whereas the pseudo-differential operators of order $\leqslant 0$ on the circle are an almost commuting algebra, such operators on a higher dimensional manifold are not. Thus we introduce a broader class of algebras and with this in mind recall some standard notions.

Given a ring $\mathfrak{A}$ let $\mathfrak{A}_1$ denote the commutator ideal of $\mathfrak{A}$, let $\mathfrak{A}_2$ denote the smallest ideal in $\mathfrak{A}_1$ containing all commutators of elements in $\mathfrak{A}$ with elements in $\mathfrak{A}_1$, and in general let $\mathfrak{A}_n$ be the smallest ideal in $\mathfrak{A}$ containing commutators of elements in $\mathfrak{A}_k$ with elements in $\mathfrak{A}_{n-k-1}$. The sequence of ideals $\mathfrak{A}_j$ is called the *commutator filtration* for $\mathfrak{A}$ and it satisfies $[\mathfrak{A}_j, \mathfrak{A}_k] \subset \mathfrak{A}_{j+k+1}$. The *complete antisymmetric* sum $[A_1, ..., A_m]$ for elements $A_1, ..., A_m$ in $\mathfrak{A}$ is

$$\sum_{\tau \in S_m} \varepsilon(\tau) A_{\tau(1)} A_{\tau(2)} \cdots A_{\tau(m)}$$

where S_m is the symmetric group on $[1, 2, ..., m]$ and ε is the signum character on S_m.

The generalizations of "commutative ring" in most common use (cf. Ch. 8 § 6 [12]) are Lie-nilpotent rings and rings with all large enough antisymmetric sums vanishing. Let us adapt these ideas to our purpose by saying that a selfadjoint operator algebra $\mathfrak{A}$ is *almost nilpotent* (*in n-steps*) if the nth term, $\mathfrak{A}_n$, of the *commutator filtration* for $\mathfrak{A}$ consists of trace class operators; and by saying that $\mathfrak{A}$ is *almost finitely commutative* (*of degree n*) if the antisymmetric sum of any n operators in $\mathfrak{A}$ is a trace class operator. An algebraic identity (Proposition 1.1) reveals that if $\mathfrak{A}$ is almost nilpotent in k steps it is almost finitely commutative of degree $2k$. We now point out that something stronger holds for our chief example. In the algebra $PS(M)$ of pseudo-differential operators of order $\leqslant 0$ on an n-dimensional Riemannian manifold M, the term $\mathfrak{A}_j$ of the *commutator filtration* equals the pseudo-differential operators of order $\leqslant -j$, and computations show that $\mathfrak{A}_{n+1}$ is contained in the trace class (see § 7). Moreover, the Kohn-Nirenberg class of operators is almost finitely commutative of degree $2n$. In addition to pseudo-differential operators, Λ_n, Venugopalkrishna's Toeplitz operators [18] on the $(2n-1)$-sphere satisfy these conditions and so we feel that the correct operator theoretic abstraction of an algebra of singular integral operators is:

Definition 0.1. The selfadjoint algebra $\mathfrak{A}$ is a *cryptointegral algebra* of *dimension* n for $n>1$ provided that $\mathfrak{A}$ is almost nilpotent in $n+1$ steps and that $\mathfrak{A}$ is almost finitely commutative of degree $2n$. For the case $n=1$ the algebra $\mathfrak{A}$ must be almost commuting.

All work in this paper will be set in a crypto-integral algebra. Frequently we shall refer to an operator or a family of operators which lie inside some particular crypto-integral algebra as crypto-integral operators. As can be easily seen (§ 1) every crypto-integral algebra commutes modulo the compact operators. Finite commutativity is a more subtle property and though it is central to our discussion the almost nilpotent algebras might be an abstraction of the singular integral operators suitable for many purposes. Note that in the dimension 1 case, finite commutativity implies nilpotency, while in higher dimensions this will not be true. (There is a parallel anomaly in K-theory distinguishing line bundles from higher dimensional bundles.) One of the basic properties of cryptointegral algebras is that they possess a nice functional calculus (in the operator theoretic sense) which provides an analog of the "full symbol" of a pseudo-differential operator and is useful for certain purposes (see Theorem 3.3).

B. A trace invariant

We shall loosely refer to an "invariant" for a family of operators or an operator algebra as something which is unchanged by unitary equivalence and by trace class perturbations. The closure of a crypto-integral algebra $\mathfrak{A}$ is a C^*-algebra, denoted $C^*\mathfrak{A}$, which commutes modulo the ideal of compact operators $\mathfrak{L}C(H)$ on H. Thus the basic invariants classically used in this situation, to wit, the essential spectrum and the index invariant are basic for the enveloped crypto-integral algebra $\mathfrak{A}$. Let us recall what these invariants are and set our notation. Via the Gelfand map the commutative C^*-algebra $C^*\mathfrak{A}/(\mathfrak{L}C(H)\cap C^*\mathfrak{A})$ is isometrically isomorphic to $C(e\sigma(C^*\mathfrak{A}))$, the continuous functions on some compact Hausdorf space $e\sigma(C^*\mathfrak{A})$. The continuous function associated with the operator is called the symbol of the operator. An operator A in $C^*\mathfrak{A}$ with invertible symbol is called Fredholm and its Fredholm index, defined by dim kernel A $-$dim cokernel A, depends only on the symbol of A. If $X_1, \ldots, X_k$ are selfadjoint generators for $C^*\mathfrak{A}$, then they induce an embedding of $e\sigma(C^*\mathfrak{A})$ into R^k as a compact subset $e\sigma(X_1, \ldots, X_k)$ called the *(joint) essential spectrum* of $X_1, \ldots, X_k$.

This article concerns a new invariant which we now describe. Consider the multilinear functional

$$\operatorname{tr}[A_1, \ldots, A_m]$$

on a crypto-integral algebra $\mathfrak{A}$ of dimension n. In examples this is only well defined if

$m \geqslant 2n$ and we show (Proposition 1.4) that it is identically zero when $m > 2n$. Thus the only trace form of interest is the $2n$-linear form and we call this the *fundamental* (*trace*) *form* of $\mathfrak{A}$. We will denote it by $\mathfrak{T}$.

Proposition 1.2 says that the fundamental form of an algebra vanishes when applied to any operator in $\mathfrak{A}_1$, the commutator ideal of $\mathfrak{A}$. This we call the inducing property because it implies that the fundamental form induces a $2n$-linear functional on the commutative algebra $\mathfrak{A}/\mathfrak{A}_1$. The commutative algebra $\mathfrak{A}/\mathfrak{A}_1$ can be thought of as similar to the symbol algebras mentioned earlier. Thus we might think of the fundamental form $\mathfrak{T}$ as a multilinear function $\mathfrak{T}$ on the symbol algebra for $\mathfrak{A}$.

In the case of the pseudo-differential operators we can compute $\mathfrak{T}$ explicitly and we do so at the end of the paper as an illustration of the abstract theory which the paper develops. For $PS(M)$ the essential spectrum is always identified with the cotangent sphere bundle, $S^*(M)$, of the manifold M and the algebra of symbols is $C^\infty(S^*(M))$. If the manifold has dimension n, then the formula (Theorem 7.1) for the fundamental trace form is

$$tr[A_1, A_2, \ldots, A_{2n}] = \gamma \int_{S^*(M)} a_1 \, da_2 \wedge da_3 \wedge \ldots \wedge da_{2n}.$$

Here the A_j are any operators in $PS(M)$ and the a_j are their symbols; d denotes the familiar exterior differentiation and $\wedge$ denotes the exterior product on differential forms; and γ is a constant. The situation for Toeplitz operators on odd spheres is similar and it is discussed in § 7.

At this point we mention somewhat tangentially that in the process of studying the higher dimensional Toeplitz operators as an example of a crypto-integral algebra we establish a close relationship between them and $PS(T^n)$, the pseudo-differential operators on the n-torus. In fact it turns out that the C^*-algebra generated by Λ_n is the same as that gotten from a natural subalgebra $PS(T^n)$. As a biproduct we obtain that the Toeplitz operator index theorem due to Venugopalkrishna [18] follows from the Atiyah-Singer index theorem.

Let us return to the abstract situation and consider how to compute the fundamental form in an explicit way. If $X_1, \ldots, X_k$ are selfadjoint generators of a crypto-integral algebra $\mathfrak{A}$ of dimension n, then all polynomials

$$p(X_1, \ldots, X_k) = \sum a_{j_1 \ldots j_k} X_1^{j_1} X_2^{j_2} \ldots X_k^{j_k} \tag{1.1}$$

in the X_j are operators which lie in $\mathfrak{A}$ and so may be substituted into the fundamental trace form. The resulting multilinear functional on polynomials can be extended (see § 3) continuously to $C^\infty(R^k)$ giving rise to a multilinear distribution $\mathfrak{T}$ on R^k with compact support. Which order the X_j appear in (I.1) does not effect $\mathfrak{T}$ because of the inducing property. We summarize this construction, the main consequence of § 1, 2, and 3 of the paper, as

THEOREM I. *If $X_1, ..., X_k$ are selfadjoint generators of a crypto-integral algebra of dimension n then there exists a unique continuous $2n$-linear functional $\mathfrak{T}$ on $C^\infty(R^k)$, supported on the joint essential spectrum of $X_1, ..., X_k$, with the property that for any polynomials $p_1, ..., p_{2n}$*

$$\mathfrak{T}(p_1, ..., p_{2n}) = \operatorname{tr}[p_1(X_1, ..., X_k), ..., p_{2n}(X_1, ..., X_k)],$$

where $p_j(X_1, ..., X_k)$ is any operator gotten by formally substituting the operators X_i into the polynomial p_j (in any order whatsoever).

The functional $\mathfrak{T}$ is clearly an invariant for the family of operators $X_1, ..., X_k$ and it is the precise generalization of the invariant classified in [8]. We now show how to represent $\mathfrak{T}$ in dimensions one and two in a manner similar to the representation just described for pseudo-differential operators. The authors suspect for formal reasons that in general the higher dimensional representation will be altered considerably. Indeed, the higher dimensional theory seems like an intriguing open area.

The representations are given in terms of smooth differential forms and we now introduce the necessary notation. Suppose that Ω is an open set in R^k. We will be using C^∞-exterior forms with compact support in Ω. We write such a form ω as $\omega = \sum f_{i_1 \dots i_j} dx_{i_1} \wedge \dots \wedge dx_{i_j}$ with $i_1 < i_2 < \dots < i_j$. The $f_{i_1 \dots i_j}$ are then called the coefficients of ω. An exterior form is said to have compact support in Ω provided that its coefficients have compact support in Ω, that is, they belong to $C_0^\infty(\Omega)$. The set $\Lambda^j(\Omega)$ of k-forms with compact support is a locally convex linear topological space in the topology acquired from $\mathfrak{D} = C_0^\infty(\Omega)$, see Chapter 1, § 2 [10]. Let $C\Lambda^j(\Omega)$ denote the closed forms in $\Lambda^j(\Omega)$ and $E\Lambda^j(\Omega)$ denote the image under exterior differentiation of $\Lambda^{j-1}(\Omega)$. A multilinear functional $l(\, ,\,)$ on $C\Lambda^j(R^k)$ will be said to have compact support if there exists a compact set M in R^k so that $l(w_1, ..., w_n) = 0$ if the support of any w_q does not intersect M. The representation theorem for $\mathfrak{T}$ is

THEOREM II. *If $X_1, ..., X_k$ are selfadjoint generators of a crypto-integral algebra of dimension δ with $\delta = 1$ or 2 and with essential spectrum E, then there is a continuous linear functional l on $C\Lambda^{2\delta}(R^k)$ which vanishes on $E\Lambda^{2\delta}(R^k - E)$ so that the trace form satisfies*

$$\mathfrak{T}(f, g) = l(df \wedge dg)$$

in the one dimensional case ($\delta = 1$) or in the two dimensional case ($\delta = 2$) it satisfies

$$\mathfrak{T}(f, g, h, j) = l(df \wedge dg \wedge dh \wedge dj).$$

The distinctive behavior of the representing functional l suggests that there is a relative homology class λ in $H_{2\delta}(R^k, E)$ associated with l. Indeed this is true and a precise description of the homology group $H_{2\delta}(R^k, E)$ is given in § 6. One would expect that the homology class, since it is canonically arrived at, has some nice properties. In fact, for the one dimensional case, the homology class can be used to compute the index of operators in the algebra generated by $X_1, ..., X_k$. Thus one obtains quite a strong index theorem which we now state although for an absolutely precise formulation of the terminology used in the statement one must see § 6. In connection with the index theorem we shall use matrices of operators and corresponding matrix symbols. Our conventions are as follows: If $\mathfrak{A}$ is an operator algebra, let $M_m(\mathfrak{A})$ denote the $m \times m$ matrices with entries from $\mathfrak{A}$. The elements of $M_m(\mathfrak{A})$ are considered to be operators on $H \oplus H \oplus ... \oplus H$. Let $\Sigma_m(R^k)$ denote the $m \times m$ matrix functions on R^k with entries in $C^\infty(R^k)$, and if G is a topological space, let $CM_m(G)$ denote the continuous $m \times m$ matrix values functions on G. The symbol of an operator U in $M_m(\mathfrak{A})$ is the matrix function in $CM_m(G)$ whose entries are the symbols of the entries of W.

THEOREM III. *Suppose that $X_1, ..., X_k$ are selfadjoint operators which commute modulo the trace class and that λ in $H_2(R^k/E)$ is the homology class induced by the fundamental trace form. If $\mathfrak{A}$ is the C^* algebra generated by $X_1, ..., X_k$ then the index of an operator F in $M_m(\mathfrak{A})$ with unitary symbol having an extension to a matrix function $f = (f_{ij})^m$ in $\Sigma_m(R^k)$ is*

$$\text{index } F = \sum_{i,j=1}^{m} \lambda(df_{ij} \wedge d\bar{f}_{ij}).$$

Connections between this and other work are described extensively in [8] but we review them briefly. The closest antecedent of the work on almost commutating pairs is J. D. Pincus [14], [15]; subsequently he and R. Carey fit the results of [8] into their theory [6], [16]. More recently they considered almost commuting pairs in a type II_∞ factor. We remark that the results (besides § 7) of this paper along with proofs hold in a type II_∞ factor after simple transliteration of terminology. The only modification required is a straightforward one in the proof of Lemma 1.3. Further related work on almost-commuting pairs has been done by C. Berger and Shaw, by K. Clancy and by C. R. Putnam. However, this paper goes in a direction rather different from all the above workers. Perhaps the most closely related work is that of Brown, Douglas and Fillmore [3], [4]. Theorem II associated with one- or two-dimensional cryptointegral algebras, a first or third homology class. The theory of Brown, Douglas, and Fillmore, applies also in this situation and yields an odd homology class. Theorem III says precisely that for dimension one these are the same class and the authors conjecture that for dimension two the homology class coming from Theorem II is the third component of the Brown, Douglas, Fillmore homology class. This is true in the available examples. Although a full exposition would overburden this paper, we remark in passing that $(1/(2\pi i))\, d\bar{f}_{jk} \wedge df_{jk}$ is the 2-homogeneous component of the relativised Chern character of f, considered as an element of $K_1(X)$. It is this fact which in Theorem III permits identification of the absolute class in $H_1(X)$ corresponding to λ with $(1/(2\pi i))$ times the index class, in the sense of Brown, Douglas, and Fillmore [3], [4], for $C^*\mathfrak{A}$.

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§ 1. Crypto-integral operators and the fundamental trace form

In this section we describe some of the basic, easily proven properties of the trace form. These properties all depend on simple algebraic facts.

If $\mathfrak{A}$ is an algebra and $\mathfrak{A} \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \ldots$ is its commutator filtration described in the Introduction, then the sequence of ideals $\mathfrak{A}_i$ not only has the property

$$[A_i, A_j] \in \mathfrak{A}_{i+j+1},$$

for any $A_i \in \mathfrak{A}_i$ or $A_j \in \mathfrak{A}_j$ it also satisfies

$$A_i A_j \in \mathfrak{A}_{i+j},$$

as we now demonstrate. The proof proceeds by induction and begins with an argument which establishes $\mathfrak{A}_1 \mathfrak{A}_j \subset \mathfrak{A}_{j+1}$. Since $\mathfrak{A}_1$ is the ideal generated by commutators of elements in $\mathfrak{A}$ and since $\mathfrak{A}_j$ is an ideal in $\mathfrak{A}$, it suffices to show that a term of the form $[A_0, B_0]A_j$ is in $\mathfrak{A}_{j+1}$, where A_0, $B_0 \in \mathfrak{A}$ and $A_j \in \mathfrak{A}_j$. The identity $[A_0, B_0 A_j] - B_0[A_0, A_j] = [A_0, B_0]A_j$ implies this. Suppose $\mathfrak{A}_{i-1}\mathfrak{A}_j \subset \mathfrak{A}_{j+i-1}$ for all j. Since $\mathfrak{A}_i$ is generated by elements of the form $[A_{i-1-k}, A_k]$ for $k=0, \ldots, i-1$ it suffices to show that $[A_{i-1-k}, A_k]A_j$ is in $\mathfrak{A}_{i+j}$ where again we use the convention $A_j \in \mathfrak{A}_j$. The identity used above applies again to give the result.

A consequence of this is that the commutator ideal of any almost nilpotent algebra is contained in $\mathfrak{L}C(H)$. For if A, B are selfadjoint operators in $\mathfrak{A}$, then $[A, B]$ is skew-selfadjoint and large powers of it are trace class. Thus $[A, B]$ is a compact operator.

Since we shall be working a great deal with antisymmetrizations, we list some properties of them.

PROPOSITION 1.1. *Let $A_1, \ldots, A_l$ be operators in $\mathfrak{A}$. Then*

(a) $[A_1, \ldots, A_l]$

$$= A_1[A_2, \ldots, A_l] - A_2[A_1, A_3, \ldots, A_l] + \ldots + (-1)^{l-1}A_l[A_1, A_2, \ldots, A_{l-1}]$$
$$= (-1)^{l+1}([A_2, \ldots, A_l]A_1 + \ldots + (-1)^{l-1}[A_1, \ldots, A_{l-1}]A_l).$$

(b) *If l is even*

$$[A_1, \ldots, A_l] = 1/2([A_1, [A_2, \ldots, A_l]] - [A_2, [A_1, \ldots, A_l]] - \ldots - [A_l, [A_1, \ldots, A_{l-1}]]).$$

(c) *If l is even, then $[A_1, \ldots, A_l]$ is a sum of $l/2$-fold products of simple commutators $[A_i, A_j]$.*

Proof.

(a) follows by inspection after some slight thought about signs.

(b) is immediate from (a).

(c) Let S_l denote the symmetric group on $\{1, 2, \ldots, l\}$ and let N be the subgroup generated by interchanges of the pairs $\{2j-1, 2j\}$. Given any τ in S_l it is not hard to see that the sum

$$\sum_{\sigma \in N} \varepsilon(\sigma\tau) A_{\sigma\tau(1)} \cdots A_{\sigma\tau(l)}$$

is equal to the product

$$\varepsilon(\tau)[A_{\tau(1)}, A_{\tau(2)}] \cdots [A_{\tau(l-1)}, A_{\tau(l)}].$$

Let τ run over a sequence of coset representations for N in S_l and part (c) of the proposition follows.

The radical of an antisymmetric multilinear functional $\langle , \rangle$ is the set

$$\{A: \langle A, A_1, \ldots, A_n \rangle = 0 \text{ for all } A_1, \ldots, A_n \text{ in } \mathfrak{A}\}.$$

We would like for a trace form

$$\langle A_1, \ldots, A_m \rangle_m \triangleq \operatorname{tr} [A_1, \ldots, A_m]$$

to induce a form on $\mathfrak{A}/\mathfrak{A}_1$. This is obtained by

PROPOSITION 1.2. *If $\mathfrak{A}$ is almost nilpotent in $k+1$ steps and if $\mathfrak{A}$ is almost finitely commutative of degree $2l$ with $l \geq k$, then the commutator ideal $\mathfrak{A}_1$ of $\mathfrak{A}$ is in the radical of $\langle , , \rangle_{2l}$.*

Proof. Proposition 1.1, part (b) says that $[A_1, \ldots, A_{2l}]$ is a sum, a typical term of which is $\Omega = [A_1, [A_2, \ldots, A_{2l}]]$. Then parts (a) and (c) of Proposition 1.1 imply that $[A_2, \ldots, A_{2l}]$ is in $\mathfrak{A}_{k-1}$. Thus Ω is always in $\mathfrak{A}_k$. Now if A_1 is in $\mathfrak{A}_1$, then Ω is in $\mathfrak{A}_{k+1}$ and is consequently trace class or if another A_j is in $\mathfrak{A}_1$, then $[A_2, \ldots, A_{2l}]$ is in $\mathfrak{A}_k$ and so Ω is trace class. The proposition is at this point an immediate consequence of the following lemma and the fact that $\mathfrak{A}$ is a selfadjoint algebra.

LEMMA 1.3. *If X is selfadjoint, if C is compact and if $[X, C]$ is trace class, then* $\operatorname{tr}[X, C]=0$.

Proof. Observe that the commutator $[X, C+C^*]$ in the basis which diagonalizes $C+C^*$ is an infinite matrix with only zeros on its diagonal. Thus $\operatorname{tr}[X, C+C^*]=0$. Likewise $\operatorname{tr}[X, C^*-C]=0$ and the lemma is proved.

The most glaring oddity about Proposition 1.2 is that it is false for $\langle , \rangle_w$ unless w is even. This phenomenon is fundamental and not merely technical. It is a manifestation of a finite dimensional fact. Suppose that H is finite dimensional; then $\langle , \rangle_w$ is identically zero when w is even and not identically zero when w is odd. This proves to be reasonable from a topologist's point of view since one comes to suspect that only the even relative homology classes for E are important anyway.

Next we find that most of the even forms $\langle \ \rangle_{2l}$ are identically zero.

PROPOSITION 1.4. *Suppose that $\mathfrak{A}$ is a *-closed algebra which is almost nilpotent in $k+1$ steps. Then if $l>k$ and $l>1$, the form $\langle , \rangle_{2l}$ is identically zero.*

Proof. If $l>k$, then the operator $[A_2, \ldots, A_{2l}]$ which appears in the argument for Proposition 1.1, is in $\mathfrak{A}_k$. Thus $\Omega=[A_1, [A_2, \ldots, A_{2l}]]$ is always trace class and Lemma 1.3 implies that $\operatorname{tr}\Omega=0$.

Henceforth we shall work only with the $2n$-trace form on an n-dimensional crypto-integral algebra $\mathfrak{A}$; recall it is denoted by $\mathfrak{T}$ and called the *fundamental trace form* for $\mathfrak{A}$. The fundamental trace form is the only interesting trace form on $\mathfrak{A}$; the two propositions show that all higher ones are zero, in examples (§ 7) the lower ones are not well defined, and the odd ones do not depend solely on the "symbols" $\mathfrak{A}/\mathfrak{A}_1$.

The bulk of the next two sections is devoted to developing analytical properties of crypto-integral algebras; this is then applied to trace forms in § 3*b*.

§ 2. Commutator identities

The purpose of this section and the next is to provide a decent functional calculus for crypto-integral algebras. This goal depends to some extent on certain straightforward but somewhat involved algebraic identities concerning commutators in an associative algebra. In this section we formulate the relevant identities.

The main point is to give a more precise description of the commutator filtration. To fix ideas and give the flavor of the arguments, we will begin with the basic case.

Thus let $\mathfrak{A}$ be an associative algebra. Then $\mathfrak{A}$ is implicitly defined over some ring of scalars, which the reader may take to, but in fact need not, be $\mathbf{C}$. Let $\mathfrak{A}_i$, $i=0, 1, 2, \ldots$ be the commutator filtration for $\mathfrak{A}$. Suppose $\{A_\alpha, \alpha \in I\}$ is a set of generators for $\mathfrak{A}$. Then we want to write down, in terms of the A_α, a convenient set of generators for $\mathfrak{A}_i$.

Let us define some objects with which we will be working. A product

$$M = A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_r} = \prod_{i=1}^{r} A_{\alpha_i}$$

is called a *monomial* in the A_α's. We call a commutator $[A_\alpha, A_\beta]$ a basic commutator in the A_α's of weight 1, and inductively, if X is a basic commutator of weight k in the A_α's, then $[A_\beta, X]$ is a basic commutator of weight $k+1$ in the A_α's. If $X_1, \ldots, X_s$ are basic commutators in the A_α's, and the weight of X_i is $w_i \geq 1$ then $\prod_{i=1}^{s} X_i$ is a *basic commutator product* in the A_α's of weight $\sum_{i=1}^{s} w_i$.

The first result we will establish, undoubtedly far from the strongest of its nature, is

PROPOSITION 2.1. *The ideal $\mathfrak{A}_i$ is generated by basic commutator products in the A_α's, of weights s, with $i \leq s \leq 2i-1$.*

This proposition is not at all difficult, but is best approached in gradual steps. We will state the steps as lemmas, most of them self-evident in context. In the following discussion we abbreviate "basic commutator (product)" by b.c(p), and we often suppress the phrase "in the A_α's."

LEMMA 2.2. *The product of two b.c.p's of weights s and t is a b.c.p of weight $s+t$.*

LEMMA 2.3. *The commutator of two b.c.'s of weights s and t is a sum of b.c.'s of weights $s+t+1$.*

Proof. The proof will be a repeated application of the Jacobi identity. Let X and Y be basic commutators in the A_α's, and suppose $X=[A_\beta, Z]$. Then $[X, Y]=[A_\beta, [Y, Z]]+[Z, [A_\beta, Y]]$. If we write $Z=[A_\gamma, W]$, and so forth, the lemma follows.

The next two statements are consequences of the basic identity for commutators in an associative algebra:

$$[AB, C] = A[B, C]+[A, C]B. \tag{2.1}$$

LEMMA 2.5. *The commutator of two b.c.p.'s of weights s and t is a sum of b.c.p.'s of weight $s+t+1$.*

LEMMA 2.6. *The commutator of a b.c.p. of weight s and a monomial is a sum of terms of the form $M_1 X M_2$, where M_1 and M_2 are monomials, and X is a b.c.p. of weight $s+1$.*

Now return to consideration of Proposition 2.1. We proceed by induction on i. We make some preliminary reductions. Observe that if X is a b.c. of weight greater than i, then X may be regarded as a sum of terms, each of the form $M_1 Y M_2$, where Y is a b.c. of weight exactly i, and the M_j are monomials. Also if X is a b.c.p. of weight greater than $2(i-1)$, then either some basic commutator among the factors of X has weight at least i, or all have weight at most $(i-1)$. In either case X may be expressed as a sum of terms of the form $M_1 Y M_2$, where Y is a b.c.p. of weight j with $i \leqslant j \leqslant 2(i-1)$, and the M_k are monomials. From these observations and the discussion at the beginning of § 1, we see that to prove Proposition 2.1 we have only to verify the following fact: Consider terms $L_1 X L_2$ and $M_1 Y M_2$ where the L_k and M_k are monomials, while X and Y are b.c.p.'s the sum of whose weights is at least $i-1$. Then the commutator of these two terms is a sum of other terms of the form

$$N_1 Z N_2 \text{ where the } N_k \text{ are monomials and } Z \text{ is a b.c.p. of weight at least } i. \tag{2.2}$$

We first compute

$$\begin{aligned}
[L_1XL_2,\, M_1YM_2] &= L_1X[L_2,\, M_1YM_2] + L_1[X,\, M_1YM_2]L_2 + [L_1,\, M_1YM_2]XL_2 \\
&= L_1XM_1Y[L_2,\, M_2] + L_1M_1Y[X,\, M_2]L_2 + M_1Y[L_1,\, M_2]XL_2 \\
&\quad + L_1XM_1[L_2,\, Y]M_2 + L_1M_1[X,\, Y]M_2L_2 + M_1[L_1,\, Y]M_2XL_2 \\
&\quad + L_1X[L_2,\, M_1]YM_2 + L_1[X,\, M_1]YM_2L_2 + [L_1,\, M_1]YM_2XL_2.
\end{aligned}$$

Then from the lemmas above, we see this is a sum of terms of the form $N_1Z_1N_2Z_2N_3Z_3N_4$, where the N_j are monomials, and the Z_k are b.c.p.'s the sum of whose weights is at least i. Thus what we need to show is how to bring all of the Z_k's together. Consider for example the product $Z_1N_2Z_2$. We may write

$$Z_1N_2Z_2 = N_2Z_1Z_2 + [Z_1,\, N_2]Z_2. \tag{2.3}$$

The first term on the right-hand side is what we want, while the second term is, by Lemma 2.5, a sum of the form $N_1'Z_1'N_2'Z_2$, where Z_1' has weight one more than Z_1. Thus successive applications of identity (2.3) puts (2.3) into the form (2.2) thereby establishing Proposition 2.1.

Next we refine Proposition 2.1 slightly to obtain a result suitable for our applications. We still have our algebra $\mathfrak{A}$ generated by a set $\{A_\alpha\colon \alpha \in I\}$. As hinted earlier, we have been suitably vague about the ring Λ over which $\mathfrak{A}$ was an algebra. We will now go so far as to assume Λ contains certain elements $\{\lambda_{\alpha\beta};\, \alpha \in I,\, \beta \in J\}$, where J is another indexing set. The $\lambda_{\alpha\beta}$ may be thought of as indeterminates. We proceed by considering a new ring $\mathfrak{A}'$, which has as generators the A_α's, and also elements $(A_\alpha - \lambda_{\alpha\beta})^{-1}$. (To be precise, we adjoin to $\mathfrak{A}$ elements $B_{\alpha\beta}$ and divide by relations $B_{\alpha\beta}(A_\alpha - \lambda_{\alpha\beta}) = 1$.) Then, as an extension of Proposition 2.1, we have

PROPOSITION 2.7. *The ideals $\mathfrak{A}'_i$ are generated by basic commutator products in the A_α's of weights s, with $i \leqslant s \leqslant 2(i-1)$.*

We will not give the details of the proof which proceeds very much as above. We do, however, give the essential additional identity needed. If A is invertible, then

$$[A^{-1}, B] = -A^{-1}[A, B]A^{-1}. \tag{2.4}$$

Hence also $[(A-\lambda_{\alpha\beta})^{-1}, B] = -(A-\lambda_{\alpha\beta})^{-1}[A, B](A-\lambda_{\alpha\beta})^{-1}$.

COROLLARY 2.8. *Suppose the number of α's is finite; order them, and set $R_\beta = \Pi_\alpha\,(A_\alpha - \lambda_{\alpha\beta})^{-1}$. Then any basic commutator product of weight i in the R_β may be expressed as a sum of terms M_1XM_2, where X is a basic commutator product of weight $j \geqslant i$ in the A_α and M_1 and M_2 are monomials in the A_α and the $(A_\alpha - \lambda_{\alpha\beta})^{-1}$.*

We want to know a similar fact about the complete antisymmetrization $[R_{\beta_1}, \ldots, R_{\beta_{2n}}]$. Using Proposition 1.1 part 3, and successive applications of (2.4) we obtain the following result.

LEMMA 2.9. *The complete antisymmetrization $[R_{\beta_1}, \ldots, R_{\beta_{2n}}]$ may be written in the form $(R_{\beta_1} \ldots R_{\beta_{2n}})^2 U + V$ where U is the complete antisymmetrization of the R_β^{-1} and V belongs to $\mathfrak{A}'_{n+1}$.*

Note that U may be expressed as a sum of complete antisymmetrizations in monomials in the A_α's, with monomials in the $\lambda_{\alpha\beta}$'s as coefficients. Further manipulation of the sort that produced Lemma 2.9 shows that the complete antisymmetrization of monomials in the A_α's may be written modulo $\mathfrak{A}_{n+1}$ as a sum of terms of the form MY, where M is a monomial in the A_α's and Y is a complete antisymmetrization of the A_α's.

§ 3. Functional calculus

In this section we will investigate the extent to which one can take functions of crypto-integral operators. The map (functional calculus) (1.1) on polynomials $p \xrightarrow{e} p(X_1, \ldots, X_k)$ given in the Introduction can be extended by standard methods to functions p whose Fourier transform is in $L^1(\mathbf{R}_k)$. However, the range of this extended mapping will in all probability not lie in any one crypto-integral algebra. So one asks to what extent is it reasonable that a crypto-integral algebra should be closed under forming functions of its members? The answer is, one may form differentiable functions. By way of motivation observe that if $X_1, \ldots, X_{2n}$ are nice generators of $PS(U)$, where U is open in $\mathbf{R}^n$, then the range of the map e above is dense in $PS(U)$ because the X_j satisfy strong commutation relations. Thus, the functional calculus gives a parametrization of $PS(U)$ by C^∞ functions which is alternative to the usual one given by the "full symbols."

A major biproduct of the functional calculus is that the trace form $\mathfrak{T}$ extends from polynomials to a continuous multilinear functional on $C^\infty(\mathbf{R}^k)$. This is discussed in Part B of the section.

A. Functional calculus

According to our definitions, and Proposition 2.1, and the remarks following Lemma 2.9, we have the following criterion for crypto-integrality.

Criterion 3.1. In order for a selfadjoint family Y of operators to generate a crypto-integral algebra of dimension n it is necessary and sufficient that:

(1) the commutator of any two elements of Y be compact;
(2) any b.c.p. of weight m, with $n+1 \leqslant m \leqslant 2n$ be trace class; and
(3) the complete antisymmetrization of any $2n$ elements of Y be trace class.

Of course (1) is redundant, but it seems harmless to emphasize it. The condition (2) guarantees almost nilpotency in n steps by Proposition 2.1, and (3) then insures almost finite commutativity by the remarks following Lemma 2.9.

Looking at this criterion, we see that it imposes conditions on only a finite number of elements of Y at a time. Therefore the subsets of $\mathfrak{L}(H)$ satisfying Criterion 3.1 form an inductive family in the sense of Zorn's lemma and so maximal such sets exist. These maximal sets will clearly be algebras, for a set of operators is contained in the algebra it generates.

Thus let $\mathfrak{A}$ be a maximal selfadjoint set (algebra) satisfying Criterion 3.1. We will show $\mathfrak{A}$ is closed under C^∞ operations on its elements. We will work with the C^∞ completion of the standard (Dunford) functional calculus. Let us recall the general procedure. Let $T \in \mathfrak{L}(H)$ be any operator, and let $\sigma(T) \subseteq \mathbf{C}$ be the spectrum of T, so $\sigma(T)$ is a non-empty compact set in $\mathbf{C}$. Let $R_\lambda(T) = (T-\lambda)^{-1} (\lambda \notin \sigma(T))$ be the resolvent of T. Then R_λ is an holomorphic $\mathfrak{L}(H)$-valued function on $\mathbf{C} - \sigma(T)$. Now let f be a complex-valued function which is holomorphic on some neighborhood U of $\sigma(T)$, and let γ be a smooth curve in U, such that the winding number of γ with respect to every element of $\sigma(T)$ is 1. Then one defines

$$f(T) = \frac{i}{2\pi} \int_\gamma f(\lambda) R_\lambda(T) \, d\lambda.$$

It can be verified that $e_T: f \to f(T)$ is an homomorphism from the algebra of (germs of) holomorphic functions on $\sigma(T)$ to $\mathfrak{L}(H)$.

In our applications, T will be selfadjoint, so that $\sigma(T) \subseteq R$, and e_T may be extended considerably. Now recall some details of this business. We consider a slightly generalized situation. We fix some interval $[a, b] \subseteq \mathbf{R}$, and we assume we have some function $S(\lambda)$, which is holomorphic on $\mathbf{C} - [a, b]$ and takes values in, for example, some Banach space E. We will moreover assume that $S(\lambda)$ has "polynomial growth near $[a, b]$," i.e., that $\|S(\lambda)\| \leqslant \min \{|\lambda - c| : c \in [a, b]\}^{-k}$ for some $k > 0$, where $\| \ \|$ is the norm in E. Now let γ_t for $t > 0$ be a family of curves going around $[a, b]$, such that, as $t \to 0$, the curves squeeze down very smoothly on $[a, b]$. (For example, the γ_t could consist of two parallel lines with circular caps at distance t from $[a, b]$.) Then if f is a complex-valued function holomorphic on a neighborhood of $[a, b]$, the integral

$$e_S(f) = \int_{\gamma_t} f(\lambda) S(\lambda) d\lambda$$

is defined and independent of t, as long as t is small enough.

A major point about e_S is that it may be written as an E-valued distribution acting on R^1 test functions and having support in $[a, b]$. To check this the first step is to select some point, say a, in $[a, b]$, and write

$$S(\lambda) = \left(\frac{d}{d\lambda}\right)^{k+1} Q(\lambda) + \sum_{j=1}^{k+1} e_j (\lambda - a)^{-j},$$

where each e_j is in E and $Q(\lambda)$ is E-valued and uniformly bounded. To obtain this representation begin naively by integrating $S(\lambda)$, then to make the indefinite integral a single-valued function subtract the residue $-e_1(\lambda - a)^{-1}$; having done this integrate again and so on k more times. Now take the expression just obtained for S and substitute it into the integral which defines e_S. After integrating by parts $k+1$ times and shrinking the γ_t's down to $[a, b]$ one obtains a distribution E_S on R^1 with the property $e_S(f) = E_S(f_{|R^1})$ for an f which is holomorphic near $[a, b]$. Clearly e_S can be identified with E_S thereby insuring that it has a unique continuous extension to $C^\infty(\mathbf{R})$ as an E-valued distribution supported on $[a, b]$. We may recover S from E_S by the rule $S(\lambda) = E_S(r(\lambda))$ where $r(\lambda)$ is the function $r(\lambda)(\xi) = (\xi - \lambda)^{-1}$ in $C^\infty(\mathbf{R})$.

These remarks have rather straightforward extensions to several operators. Thus let $T_1, \ldots, T_k$ be selfadjoint operators (denote the family by T), each of whose spectra is contained in the interval $[a, b]$, and let γ_t be as above. Then for any f on $\mathbf{C}^k$, holomorphic in a neighborhood of $[a, b]^k$, we may define

$$e_T(f) = \left(\frac{i}{2\pi}\right)^k \int_{(\gamma_t)^k} f(\lambda_1, \ldots, \lambda_k) R_{\lambda_1}(T_1) \ldots R_{\lambda_k}(T_k) d\lambda_1 \ldots d\lambda_k. \tag{3.2}$$

for any t small enough.

We will call this the normal-ordered Dunford functional calculus for the T_i. It extends in the manner suggested above from holomorphic functions to $C^\infty(\mathbf{R}^k)$. We call the extension the *normal-ordered C^∞ functional calculus*, and denote it again by e_T.

THEOREM 3.3. *Let $\mathfrak{A}$ be a maximal selfadjoint set in $\mathfrak{L}(H)$ satisfying Criterion* 3.1. *Then:*

(a) *$\mathfrak{A}$ is an algebra;*

(b) *if $\{A_i\}_{i=1}^k$ is any collection of selfadjoint elements in $\mathfrak{A}$, then $\mathfrak{A}$ contains the normal ordered C^∞ functional calculus in the A_i; and*

(c) *if $S \in M_m(\mathfrak{A})$ is any selfadjoint element, then $M_m(\mathfrak{A})$ contains the C^∞-functional calculus in S.*

Remark. It will become clear from the proof that appropriate analogues of (b) hold for other sets of elements of $\mathfrak{A}$. In particular, note that we don't really need the A_i to be selfadjoint for (b). It would suffice that they have real spectra and that their resolvents have polynomial growth.

Proof. Point (a) is already proved. To prove (b) we must show that adding any element of the prospective functional calculus to $\mathfrak{A}$ preserves the conditions of Criterion 3.1. We will ignore condition (i), since, as we remarked, it is redundant.

First, we note we may add the resolvents of the A_i to $\mathfrak{A}$. This is immediate from Proposition 2.7 and Lemma 2.9. Next put $(\lambda_1, \ldots, \lambda_k) = \lambda$ and write $R_A(\lambda) = \Pi_{i=1}^k (A_i - \lambda_i)^{-1}$, for $\lambda_i \notin [a, b]$. Consider any b.c.p. of weight at least $n+1$, in the $R_A(\lambda)$ for various λ, or consider the complete antisymmetrization of $2n$ of the $R_A(\lambda)$. Corollary 2.8 and Lemma 2.9 now imply these expressions are holomorphic trace class valued functions of the λ's away from the singularities of R_A, and that they have polynomial growth near the singularities of R_A. Part (b) of the theorem now follows from properties of the functional calculus already discussed.

Remark. In fact, we see that each b.c.p. of weight at least $n+1$, as well as the complete antisymmetrization of $2n$ operators, define, when composed with e_A, trace-class valued multilinear distributions of compact support on $\mathbf{R}^k$.

Now we consider (c). We need an identity analogous to (2.4). Let S be in $M_m(\mathfrak{A})$, and let S_{ij} be the entries of S. For a given $B \in \mathfrak{A}$, let $\tilde{B}$ be the element of $M_m(\mathfrak{A})$ such that

$\tilde{B}_{ij}=\delta_{ij}B$, where δ_{ij} is Kronecker's δ. Then it is easy to verify that $[S, \tilde{B}]_{ij}=[S_{ij}, B]$. Now let $R_S(\lambda)$ be the resolvent of S. Then $[R_S(\lambda)_{ij}, B]=[R_S(\lambda), \tilde{B}]_{ij}=-(R_S(\lambda)[S, \tilde{B}]R_S(\lambda))_{ij}=-\sum_{k,l} R_S(\lambda)_{ik}[S_{kl}, B]R_S(\lambda)_{lj}$. With this analog of identity (2.4) one may show, for example, that if $\mathfrak{B}$ is the algebra generated by $\mathfrak{A}$ and the $R_S(\lambda)_{kl}$ then the ideal $\mathfrak{B}_i$ equals $\mathfrak{B}\mathfrak{A}_i\mathfrak{B}$, in analogy with Proposition 2.7. Thus one concludes $\mathfrak{B}=\mathfrak{A}$ by maximality of $\mathfrak{A}$. An analog of Corollary 2.8, Lemma 2.9 and basic properties of the functional calculus allow one to finish part (c).

The next result is complementary to Theorem 3.3 and parallel to results in [8] § 3. It's proof involves nothing new, so we omit it. Let N_p denote the Schatten p-class of operators. These are operators with the property that the p-th power of their absolute value is trace class. Note that $\mathfrak{A}_{n+1}\subset N_1$ implies $\mathfrak{A}_1\subset N_{n+1}$.

PROPOSITION 3.4. *The map* e_A: $C^\infty(\mathbf{R}^n)\to\mathfrak{A}$ *defined by* (3.2) *for any set* $A=\{A_1, ..., A_n\}\subseteq\mathfrak{A}$ *of selfadjoint elements has the following properties*

(i) $e_A(f)^*-e_A(\bar{f})\in N_{n+1}\cap\mathfrak{A}_1$. *Here* $\bar{f}$ *is the complex conjugate of* f. *Moreover the map* $e_A(f)^*-e_A(\bar{f})$ *is an* N_{n+1}*-valued distribution of compact support.*

(ii) $e_A(f)e_A(g)-e_A(fg)\in N_{n+1}\cap\mathfrak{A}_1$, *and the corresponding bilinear map is continuous in* f *and* g *to* N_{n+1}.

(iii) $[e_A(f), e_A(g)]\in N_{n+1}\cap\mathfrak{A}_1$, *and again is continuous in* f *and* g.

If $n=1$, then we can write N_1 rather than N_2 in the above.

B. Trace forms

If we compose the fundamental trace form with e_A we get a $2n$-linear functional on $C^\infty(\mathbf{R}^k)$ which we will denote by $\mathfrak{T}_A$. This much is evident from the foregoing:

PROPOSITION 3.5. *The* $2n$*-linear functional on* $C^\infty(\mathbf{R}^k)$ *given by* $\mathfrak{T}_A(f_1, ..., f_{2n})=\operatorname{tr}([e_A(f_1), ..., e_A(f_{2n})])$ *is a well defined continuous and antisymmetric form.*

Now we show that $\mathfrak{T}_A$ has support on $e\sigma(A_1, ..., A_k)$. We begin by observing the behavior of $\mathfrak{T}_A$ under coordinate transformations. Let $g_1, ..., g_m\in C^\infty(\mathbf{R}^k)$ be real-valued functions. We may interpret the g_i as defining a map θ: $\mathbf{R}^k\to\mathbf{R}^m$ by the formula $\theta(x)=(g_1(x), ..., g_m(x))$ for $x\in\mathbf{R}^k$. Put $B_i=\frac{1}{2}(e_A(g_i)+e_A(g_i)^*)$. Then $e_A(g_i)-B_i\in\mathfrak{A}_1$ by Proposition 3.4. Now consider the map e_B: $C^\infty(\mathbf{R}^m)\to\mathfrak{A}$ defined by the B_i. By Propositions 3.4 and 1.2, we see that if $\{p_j\}_{j=1}^{2n}$ are polynomials on $\mathbf{R}^m$, then $\operatorname{tr}[e_B(p_1), ..., e_B(p_{2n})]=\operatorname{tr}[e_A(p_1\circ\theta), ..., e_A(p_{2n}\circ\theta)]$, so by continuity (Proposition 3.5) we deduce

PROPOSITION 3.6. *With notations as above,* $\mathfrak{T}_B=\mathfrak{T}_A\circ\theta$.

Now let $E=e\sigma(A_1, ..., A_k)\subseteq\mathbf{R}^k$.

PROPOSITION 3.7. *The 2n-linear form* $\mathfrak{T}_A$ *on* $C^\infty(\mathbf{R}^k)$ *is supported on* E, *in the sense that* $\mathfrak{T}_A(f_1, \ldots, f_{2n})=0$ *if any* f_i *vanishes in a neighborhood of* E.

Proof. It suffices to prove the result for real-valued $f_i \in C_0^\infty(\mathbf{R}^k)$. By definition of essential spectrum, if f vanishes on E, the operator $e_A(f)$ is compact. Then $\frac{1}{2}(e_A(f)+e_A(f)^*)=B$ is compact selfadjoint, so if $g \in C^\infty(\mathbf{R})$ vanishes in a neighborhood of the origin, the operator $g(B)$ has finite rank. Therefore, by Proposition 3.6, Proposition 1.1 and Lemma 1.3, we conclude that $\mathfrak{T}_A(g \circ f_1, f_2, \ldots, f_{2n})=0$ for any $\{f_i\}_{i=2}^{2n} \subseteq C^\infty(\mathbf{R}^k)$. However, it is not hard to show that any $h \in C_0^\infty(\mathbf{R}^k)$ which vanishes in a neighborhood of E is a linear combination of functions of the form $g \circ f$, where f vanishes on E and $g \in C^\infty(\mathbf{R})$ vanishes near zero. Thus the proposition is proved.

§ 4 Generalized Wallach's lemma

In the preceding sections we have shown that it is natural to associate with a family of crypto-integral operators a multilinear form on $C^\infty(R^n)$ which is supported on the joint essential spectrum of these operators. Now we turn to the problem of classifying the trace forms.

The point behind classification of the two-form is that $\operatorname{tr}[R, S]=0$ if R and S commute. From this we may conclude easily that for polynomials p, q in $P(R^n)$ we have $\mathfrak{T}(p, q)=0$ if p and q are both functions of a third polynomial. This property of $\mathfrak{T}$ is essential. We call it the *collapsing property*. In particular the collapsing property implies $\mathfrak{T}$ is antisymmetric. This was exploited via Wallach's lemma in [8] to get the representation theorem in [8]. Our representation scheme which has been successful in dimensions one and two generalizes this procedure.

The core of this scheme is elegant enough to warrant presenting it for all dimensions even though our application is to dimensions one and two. That is what we do in this section. The considerations in this section are almost purely algebraic and they make sense in a fairly broad algebraic context. We are informed by Larry Brown that the main result can be derived alternately from the theory of Kähler differentials in algebraic geometry. We give here a first principles proof. It is self contained and though its relationships with existing algebraic theories is not explored—surely some exist.

Denote by S_p the p-multilinear map from $C^\infty(R^n)$ to $\Lambda^p(R^n)$ given by the formula $S_p(f_1, f_2, \ldots, f_p)=df_1 \wedge df_2 \wedge \ldots \wedge df_p$. One sees by Poincare's lemma that the associated map of the p-th tensor power of $C^\infty(R^n)$ to $\Lambda^p(R^n)$ is surjective to the closed p-forms. For if ψ is a closed $(p+1)$-form, then $\psi=d\xi$ by Poincaré's lemma, and if we write $\xi=\sum_{i_j<i_{j+1}} g_{i_1 i_2 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$, then we obtain $\psi=\sum S_{p+1}(g_{i_1 \ldots i_p}, x_{i_1}, \ldots, x_{i_p})$. An ele-

mentary computation shows that if f_i and f_j, with $i \neq j$, are both functions of some third function, then $S_p(f_1, \ldots, f_p) = 0$. Thus S_p has what we might call the "p-fold collapsing property". The elegant fact is that S_p is the universal p-linear form with this property.

GENERALIZED WALLACH'S LEMMA: *Let φ be a (continuous) p-linear functional on $C^\infty(R^n)$ having compact support, and suppose φ has the p-fold collapsing property. Then there is a continuous linear functional l with compact support on the closed p-forms on R^n such that $\varphi = l \cdot S_p$.*

Proof. To prove this, we resort to a string of sublemmas.

We first establish an algebraic form of the lemma, then pass to the C^∞ form by continuity. As before $P(R^n)$ is the algebra of polynomials on R^n. We write $P\Lambda^p(R^n)$ for the space of differential forms with polynomial coefficients. It is not hard to show that the Poincaré lemma holds algebraically. In fact, it is rather pretty: the closed p-forms with homogeneous coefficients of degree m and the co-closed $(p-1)$ forms with degree $(m+1)$ coefficients form isomorphic irreducible Gl_n modules. Therefore S_p: $\otimes^p P(R^n) \to P\Lambda^p(R^n)$ is onto the closed forms. Let us now consider a p-linear functional φ on $P(R^n)$ with the p-fold collapsing property.

SUBLEMMA 1. *Let $r(\varphi)$ be the set of $f \in P(R^n)$ such that $\varphi(f, g_2, \ldots, g_p) = 0$ for any choice of g_i, $2 \leqslant i \leqslant p$. Then $r(\varphi)$ is a subalgebra of $P(R^n)$.*

Proof. Indeed, suppose f_1 and f_2 are in $r(\varphi)$. Then letting g_i be arbitrary we have

$$\varphi((\alpha f_1 + \beta f_2 + g_2), (\alpha f_1 + \beta f_2 + g_2)^2, g_3, \ldots, g_p) = 0$$

by the collapsing property. Expanding this out gives

$$\varphi(g_2, \alpha^2 f_1^2 + 2\alpha\beta f_1 f_2 + \beta^2 f_2^2 + 2\alpha f_1 g_2 + 2\beta f_2 g_2 + g_2^2, g_3, \ldots, g_p) = 0.$$

This equation holds identically in α and β. Considering the coefficient of $\alpha\beta$, we see that $f_1 f_2 \in r(\varphi)$, which implies the lemma.

SUBLEMMA 2. *Given $\{f_i\}_{i=1}^p$, with $f_i \in r(\varphi)$ for $i \geqslant 2$, then φ is identically zero on the subalgebra generated by the f_i.*

Proof. By Sublemma 1, it suffices to show f_1 is in the radical of φ restricted to this subalgebra. Consider the equation

$$0 = \varphi\left(f_1 + \sum_{i=2}^p \alpha_i f_i, \left(f_1 + \sum_{i=2}^p \alpha_i f_i\right)^n, g_3, g_4, \ldots, g_p\right),$$

where the g_i are arbitrary. Since $f_i \in r(\varphi)$ for $i \geqslant 2$, this gives

$$\varphi\left(f_1, \left(f_1 + \sum_{i=2}^{p} \alpha_i f_i\right)^n, g_3, \ldots, g_p\right) = 0.$$

By choosing the constants α_i properly and summing, we can achieve in the second place an arbitrary polynomial in the f_i, $1 \leqslant i \leqslant p$. This establishes the sublemma.

SUBLEMMA 3. *Suppose $\varphi(f_1, \ldots, f_p)$ is always zero if $f_1, \ldots, f_k$ is an arbitrary k-tuple chosen from $x_1, x_2, \ldots, x_{n-1}$ (the first $n-1$ standard co-ordinate functions). Then $\varphi(f_1, \ldots, f_p)$ is always zero if $f_1, \ldots, f_{k-1}$ is an arbitrary $(k-1)$-tuple chosen from $x_1, x_2, \ldots, x_{n-1}$.*

Proof. Select $x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}$ and consider the $(p-k+1)$-linear functional ψ on $P(R^n)$ given by

$$\psi(g_1, g_2, \ldots, g_{p-k+1}) = \varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}, g_1, g_2, \ldots, g_{p-k+1}).$$

Then ψ clearly has the $(p-k+1)$-fold collapsing property, and by hypothesis, $r(\psi)$ contains x_i for $1 \leqslant i \leqslant n-1$. By Sublemma 2 the functional ψ is trivial, and this implies the desired result.

Using Sublemma 3 repeatedly, and then appealing to Sublemma 2 again we obtain

SUBLEMMA 4. *If for some $k \geqslant 1$ we have $\varphi(f_1, f_2, \ldots, f_p) = 0$ whenever $f_1, f_2, \ldots, f_k$ are chosen among $x_1, x_2, \ldots, x_{n-1}$, then we have $x_i \in r(\varphi)$ for $i \leqslant n-1$. Hence φ is trivial.*

Now we come to the main step. We will construct inductively a sequence of linear functionals m_j, $0 \leqslant j \leqslant n-p$, on appropriate subspaces of $p\Lambda^p(R^n)$, which will define functionals on the closed p-forms by restriction. Each of the P-linear functionals $\phi_i = \phi - \Sigma_{j=0}^{i} m_j \circ S_p$ will have the p-fold collapsing property. We will show inductively that $\phi_k(f_1, \ldots, f_p)$ is identically zero whenever $f_1, \ldots, f_{p-1}$ is a $(p-1)$-tuple of elements chosen from $x_1, \ldots, x_{p+k-1}$. Then Sublemma 4 will show ϕ_{n-p} is identically 0, and the algebraic form of our lemma will be proved.

Suppose, then, that we have defined m_i for $i \leqslant k$. We define m_{k+1} as follows. First, let m_{k+1} be zero on any $f dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ unless $i_1 < i_2 < \ldots < i_{p-1} = p+k$. (Of course then $i_p > p+k$). On the other hand, suppose $i_1, \ldots, i_{p-1}$ are as just specified. Take $f \in P(\mathbf{R}^n)$ and define

$$\psi(f) = \sum_{i_p > p+k} \left(\frac{\partial f}{\partial x_{i_p}}\right) dx_{i_1} \wedge \ldots \wedge dx_{i_{p-1}} \wedge dx_{i_p}.$$

The map ψ depends also on the i_j, but we suppress this in the notation. The $\psi(f)$ evidently form a certain linear subspace of $P\Lambda^p$. On this subspace define m_{k+1} by

$$m_{k+1}(\psi(f)) = \phi_k(x_{i_1}, x_{i_2}, \ldots, x_{i_{p-1}}, f). \tag{4.1}$$

We should verify this equation is actually a definition. This amounts to showing that if $\psi(f)=0$, then the right side of (4.1) also vanishes. But if $\psi(f)=0$ then clearly, f depends only on $x_1, ..., x_{p+k}$, so by our inductive hypothesis, Sublemma 4 shows that $\phi_k(x_{i_1}, ..., x_{i_{p-1}}, f)=0$. Thus m_{k+1} is well-defined.

Now let us verify $\phi_{k+1}=\phi_k-m_{k+1}\circ S_p$ has the desired properties. Choose $i_1 \leqslant i_2<...<i_{p-1}\leqslant p+k$. If $i_{p-1}<p+k$, then $m_{k+1}\circ S_p(x_{i_1}, ..., x_{i_{p-1}}, f)=0$ by construction, so $\phi_{k+1}(x_{i_1}, ..., x_{i_{p-1}}, f)=\phi_k(x_{i_1}, ..., x_{i_{p-1}}, f)$ and this vanishes by induction. If, however $i_{p-1}=p+k$, then

$$\begin{aligned}\phi_{k+1}(x_{i_1}, \ldots, x_{i_{p-1}}, f) &= \phi_k(x_{i_1}, \ldots, x_{i_{p-1}}, f) - m_{k+1}(dx_{i_1} \ldots \wedge \ldots \wedge dx_{i_{p-1}} \wedge df) \\ &= \phi_k(x_{i_1}, \ldots, x_{i_{p-1}}, f) - \sum_{j=1}^{n} m_{k+1}\left(\frac{\partial f}{\partial x_j} dx_{i_1} \wedge \ldots \wedge dx_j\right).\end{aligned}$$

By our definition of m_{k+1}, the terms in this sum with $j \leqslant p+k$ are zero, and the sum of the terms for $j>p+k$ is just $\phi_k(x_{i_1}, ..., x_{i_{p-1}}, f)$. Thus $\phi_{k+1}(x_{i_1}, ..., x_{i_{p-1}}, f)=0$ if $i_1<i_2<...<i_{p-1}\leqslant p+k$, as desired. This completes our construction

Now we describe how to pass from the algebraic to the C^∞ version of the lemma. On $P(R_n)$ we have

$$\phi = \sum_{j=0}^{n-p} m_j \circ S_p. \tag{4.2}$$

The definition of the m_j and the fact that the map

$$f \to \sum_{p>p+k=i_{p-1}} \left(\frac{\partial f}{\partial x_{i_p}}\right) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

is an open map from C^∞ onto the p-forms $\sum_{j>p+k} g_j dx_{i_1} \wedge ... \wedge dx_{i_{p-1}} \wedge dx_j$ satisfying $\partial g_i/\partial x_j = \partial g_j/\partial x_i$ for $i, j>p+k$ imply that m_j extends to a continuous linear functional of compact support on $C\Lambda^p(R^n)$. Thus (4.2) and the representation theorem hold on $C_0^\infty(R^n)$ or alternatively on $C^\infty(R^n)$ since g and l have compact support. This concludes the proof of generalized Wallach's theorem.

To close out this section, we give a condition that further reduces the collapsing property and which is useful in concrete situations.

PROPOSITION 4.2. *In order for a p-linear functional φ on $C^\infty(R^n)$ to have the p-fold collapsing property, it is sufficient that $\varphi(f_1, ..., f_p)=0$ whenever*

(i) *any f_i is constant; or*

(ii) *$f_i=f_j$ for any $i \neq j$; or*

(iii) *$f_i=f_j^2$ for some i and j.*

Proof. Clearly it is enough to work with a bilinear functional. Then it will be enough to show that if φ is antisymmetric and $1 \in r(\varphi)$, and $\varphi(f, f^2)=0$ for any f, then $\varphi(f^n, f^m)=0$ for any f and any n, m. Consider the formal sum $\Sigma\, \alpha_i f^i$. Then the equation $\varphi(\Sigma\, \alpha_i f^i, (\Sigma\, \alpha_i f^i)^2)=0$ is a formal identity in the α_i. Put $\varphi(f^m, f^n)=c_{m,n}$. Then we compute the coefficient of $\alpha_i\alpha_j\alpha_k$ in the above equation to be

$$\begin{array}{ll} 2c_{i,j+k}+2c_{k,i+j}+2c_{j,k+i} & \text{if } i \neq j \neq k, \\ 2c_{i,i+k}+c_{k,2i} & \text{if } i = j \neq k, \\ c_{i,2i} & \text{if } i = j = k. \end{array}$$

Therefore, we see in all cases we have the equations $c_{i,j+k}+c_{k,i+j}+c_{j,k+i}=0$, with the supplementary equations $c_{0,i}=c_{i,1}=c_{i,2i}=c_{i,j}+c_{j,i}=0$. We consider these equations when $i=1$ and j and k are arbitrary. For $j+k=n$ we get the system of equations

$$\begin{array}{l} 0 = 2c_{1,n}+c_{n-1,1}, \\ 0 = c_{1,n}+c_{2,n-1}+c_{n-2,3}, \\ 0 = c_{1,n}+c_{3,n-2}+c_{n-3,4}, \\ 0 = c_{1,n}+2c_{n/2,(n/2)+1}, \quad \text{or} \quad c_{1,n}+c_{(n-1)/2,(n+3)/2}, \end{array}$$

according as n is odd or even. Note this is a system in $c_{i,j}$ for $i<j$ since $c_{i,j}=-c_{j,i}$ if $i \geqslant j$. The matrices of these systems are

$$\begin{pmatrix} 2 & -1 & & & & \\ 1 & 1 & -1 & & & \\ 1 & 0 & 1 & -1 & & \\ \vdots & & & & & \\ 1 & & & & 1 & -1 \\ 1 & & & & & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & & & & \\ 1 & 1 & -1 & & & \\ 1 & 0 & 1 & -1 & & \\ \vdots & & & & & \\ 1 & & & & 1 & -1 \\ 1 & & & & & 1 \end{pmatrix}.$$

It is easy to establish by elementary row operations beginning at the bottom and working up, that these matrices have full rank. Thus the only solution to the system is $c_{i,j}=0$ for all i, j. This finishes the proposition.

§ 5. Representation of the trace forms

Theorem II can be proved very quickly by putting together the results that have already been proved. This section is devoted to doing this.

Proof of theorem. One dimensional case, $\delta=1$. As we saw in § 4, the fundamental trace form $\mathfrak{T}$ when restricted to polynomials has the collapsing property. By continuity, see Proposition 3.5, the form $\mathfrak{T}$ on $C^\infty(R^k)$ also has the collapsing property. Wallach's lemma implies the existence of the required continuous linear functional. The fact that $\mathfrak{T}$ is supported on E, see Proposition 3.7, is equivalent to the statement on support required by the theorem.

Two dimensional case, $\delta=2$. The collapsing property is less apparent, than it was before. We use Proposition 4.2 to establish it. Conditions (i) and (ii) of this proposition are obviously satisfied. Condition (iii) depends on an algebraic identity

$$\begin{aligned}[A^2, B, C, D] = A[A, B, C, D]+[A, B, C, D]A+[[A, B], [A, [C, D]]] \\ +[[A, C], [A, [D, B]]]+[[A, D][A, [B, C]]],\end{aligned} \tag{5.1}$$

which we verify in the next paragraph. Each of the last three terms in this expression is clearly in $\mathfrak{A}_3$ and consequently each is trace class. Thus $\operatorname{tr}[A^2, A, C, D]$ has the form $\operatorname{tr}[,]+\operatorname{tr}[,]+\operatorname{tr}[,]$ where each commutator is trace class. Lemma 1.3 implies that $\operatorname{tr}[A^2, A, C, D]=0$. This is condition (iii).

Now we verify the crucial identity. First note that

$$\begin{aligned}[A^2, B, C, D] = [A^2, B][C, D]+[C, D][A^2, B]+[A^2, D][B, C]+[B, C][A^2, D] \\ +[A^2, C][D, B]+[D, B][A^2, C].\end{aligned}$$

The first, third, and fifth terms can be written

$$\begin{aligned}A([A, B][C, D]+[A, D][B, C]+[A, C][D, B])+[A, B]A[C, D] \\ +[A, D]A[B, C]+[A, C]A[D, B],\end{aligned}$$

and the first 3 terms of this expression differs from $A[A, B, C, D]$ by $A[(C, D][A, B]+[B, C][A, D]+[D, B][A, C])$. This along with similar reasoning applied to the second, fourth, and sixth terms of the original formula give that

$$\begin{aligned}&[A^2, B, C, D]-A[A, B, C, D]-[A, C, D, B]A \\ &\quad = [A, B]A[C, D]+[A, D]A[B, C]+[A, C]A[D, B]-A[C, D][A, B] \\ &\qquad -A[B, C][A, D]-A[D, B][A, C]+[C, D]A[A, B]+[B, C]A[A, D] \\ &\qquad +[D, B]A[A, C]-[A, B][C, D]A-[A, D][B, C]A-[A, C][D, B]A,\end{aligned}$$

which equals

$$\begin{aligned}[[A, B], A[C, D]]+[[A, D], A[B, C]]+[[A, C], A[D, B]] \\ +[[C, D]A, [A, B]]+[[B, C]A, [A, D]]+[[D, B]A, [A, C]],\end{aligned}$$

and these six terms combine to give the last three terms of (5.1).

§ 6. The trace form homology class and index theory

The preceding section described how to associate with a family $X_1, \ldots, X_k$ of one or two dimensional crypto-integral operators a linear functional on closed differential forms which vanish on boundaries of forms, supported off of the joint essential spectrum of the family. This suggests that we may associate a relative homology class to the family $X_1, \dots X_k$. Indeed one can.

Paragraph A of this section discusses the precise topological setting for this construction. The machinery in paragraph A allows one to obtain the relative homology class immediately from Theorem II and shows that in a reasonable sense this can be identified with an absolute homology class of E.

As was described in the introduction, this homology class for the one dimensional case is closely related to index structure. In paragraph B we prove an index theorem.

A. Relative deRham cohomology on R^n

The appropriate cohomology for our purposes is sheaf cohomology. It would take us too far afield to go through the basic definitions of sheaf theory, so we confine our attention to constructing the sheaves we will use. All the theory we will need can be learned quite quickly from [9], § 2; a fuller account is found in [7].

If U^n is an open subset of R^n, then the assignment $U \to C^\infty(U)$ defines a presheaf on R^n. The associated sheaf is called the sheaf of germs of smooth functions on R^n. It is a sheaf of rings. We will denote it by $\tilde{C}^\infty(R^n)$. Then $C^\infty(R^n)$ is the space of global sections of $\tilde{C}^\infty(R^n)$, and $C^\infty(U)$ is the space of local sections over U of $\tilde{C}^\infty(R^n)$.

Again for U open in R^n, let $\Lambda^p(U)$ be the smooth exterior forms of degree p on U. Then $\Lambda^p(U)$ is a module over $C^\infty(U)$. It is a free module of rank $\binom{n}{p}$ with basis $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ for any p-tuple of integers i_j, with $1 \leqslant i_j \leqslant n$ and $i_j < i_{j+1}$. The direct sum $\Lambda^*(U) = \oplus_{p=0}^n \Lambda^p(U)$ is a graded ring over $C^\infty(U)$ with associative multiplication (the standard wedge product) satisfying $f dx_i = dx_i f$ for $f \in C^\infty(U)$ (we agree that $\Lambda^\circ(U) = C^\infty(U)$). As U varies, the $\Lambda^p(U)$ fit together to form a presheaf and we denote the associated sheaf by $\tilde{\Lambda}^p(R^n)$. The $\tilde{\Lambda}^p$ are sheaves of modules over the sheaf $\tilde{C}^\infty$. The sections of $\tilde{\Lambda}^p(R^n)$ over an open $U \subseteq R^n$ is just $\Lambda^p(U)$. The direct sum of the $\tilde{\Lambda}^p$ forms a sheaf $\tilde{\Lambda}^*$ of graded algebras over $\tilde{C}^\infty$.

We have the exterior differentiation mappings d_p: $\Lambda^p(U) \to \Lambda^{p+1}(U)$, which fit together to give $\Lambda^*(U)$ the structure of a graded differential complex, and one has the formula $d_{p+q}(\alpha \wedge \beta) = (d_p \alpha) \wedge \beta + (-1)^p \alpha \wedge d_q \beta$ for $\alpha \in \Lambda^p(U)$ and $\beta \in \Lambda^q(U)$. The classical Poincaré lemma [9] says that the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \Lambda^0(R^n) \xrightarrow{d_0} \Lambda^1(R^n) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Lambda^n(R^N) \to 0 \tag{6.1}$$

is exact. In general, we have $d_p \cdot d_{p-1} = 0$, so that $\operatorname{im} d_{p-1} \subseteq \ker d_p$; the quotient $H^p(U) =$

$\ker d_p / \operatorname{im} d_{p-1}$ is the p-th deRham cohomology group of U. Thus the Poincaré lemma just says $H^p(R^n) = 0$.

It is evident that the d_p's behave well with respect to restriction, so that they define sheaf maps $\tilde{d}_p$: $\tilde{\Lambda}^p \to \tilde{\Lambda}^{p+1}$. On this level, the Poincaré lemma asserts that

$$0 \to \tilde{\mathbf{C}} \to \tilde{\Lambda}^0 \xrightarrow{\tilde{d}_0} \tilde{\Lambda}^1 \xrightarrow{\tilde{d}_1} \cdots \xrightarrow{\tilde{d}_{n-1}} \tilde{\Lambda}^n \to 0 \tag{6.2}$$

is an exact sequence of sheaves. Here $\tilde{C}$ denotes the constant sheaf with stalk $\mathbf{C}$. Now the sheaves $\tilde{\Lambda}^p$ allow partitions of unity so they are what is known as fine or soft sheaves. Thus, the sequence (6.2) is a resolution of the constant sheaf $\tilde{\mathbf{C}}$ by fine sheaves. In particular, the deRham groups $H^p(U)$ are just the sheaf cohomology groups of U with coefficients in the constant sheaf $\tilde{\mathbf{C}}$.

Now consider a compact set $X \subseteq R^n$. We let $\Lambda^p(U/X)$ be the space of smooth p-forms on $U \subseteq R^n$ which vanish in a neighborhood of X, and we let $\tilde{\Lambda}^p(R^n/X)$ be the associated sheaf. It is easy to check that the stalks of $\tilde{\Lambda}^p(R^n/X)$ are the same as the stalks of $\tilde{\Lambda}^p(R^n)$ off of X, but that they are zero on X. Also $\Lambda^p(R^n/X)$ is the space of global sections of $\tilde{\Lambda}^p(R^n/X)$. We see $\tilde{\Lambda}^p(R^n/X)$ is a subsheaf of $\tilde{\Lambda}^p(R^n)$. Let $\tilde{\Lambda}^p(X)$ be the quotient sheaf $\tilde{\Lambda}^p(R^n)/\tilde{\Lambda}^p(R^n/X)$. Then the stalks of $\tilde{\Lambda}^p(X)$ are zero except on X, so $\tilde{\Lambda}^p(X)$ may be considered to be a sheaf on X.

The sheaves $\tilde{\Lambda}^p(R^n/X)$ and $\tilde{\Lambda}^p(X)$ are evidently soft sheaves. Moreover, we clearly have $d_p(\Lambda^p(U/X)) \subseteq \Lambda^{p+1}(U/X)$, so that $\tilde{d}_p$ preserves $\tilde{\Lambda}^*(R^n/X) (= \oplus_{p=0}^{n} \tilde{\Lambda}^p(R^n/X))$. Thus $\tilde{d}_p$ factors to a sheaf map $\tilde{d}_p$: $\tilde{\Lambda}^p(X) \to \tilde{\Lambda}^{p+1}(X)$. The Poincaré lemma again applies and says that

$$0 \to \tilde{C}_{|X} \to \tilde{\Lambda}^0(X) \xrightarrow{\tilde{d}_0} \Lambda^1(X) \xrightarrow{\tilde{d}_1} \cdots \xrightarrow{\tilde{d}_{n-1}} \tilde{\Lambda}^n(X) \to 0 \tag{6.3}$$

is an exact resolution by fine sheaves of the constant sheaf $\tilde{C}$ on X. Thus this resolution may be used to compute the sheaf cohomology groups of X with coefficients in $\tilde{C}$. Of course, for reasonable X, they are the same as any other cohomology groups of X with coefficients in C.

The sheaves $\tilde{\Lambda}^p(R^n/X)$ also give a fine resolution of a certain sheaf Y on R^n; namely, Y is the sheaf which is the constant sheaf $\tilde{C}$ on $R^n - X$ and the zero sheaf on X. We denote the cohomology groups of Y by $H^p(R^n/X)$ and call them the relative (deRham) cohomology groups of X in R^n. Clearly the sequence $0 \to Y \to \tilde{C} \to \tilde{C}_{|X} \to 0$ is an exact sequence of sheaves. By definition, the sequence $0 \to \tilde{\Lambda}^p(R^n/X) \to \tilde{\Lambda}^p(R^n) \to \tilde{\Lambda}^p(X) \to 0$ is also exact for each p. Moreover, all of these exact sequences are compatible with the resolutions of Y, $\tilde{C}$ and $\tilde{C}_X$ constructed by exterior differentiation, so that they fit together to form an exact sequence of exact fine resolutions of an exact sequence of sheaves. Thus, we obtain in the usual manner a long exact sequence of cohomology groups:

$$\to H^p(R^n) \to H^p(X) \xrightarrow{\partial_p^*} H^{p+1}(R^n/X) \to H^{p+1}(R^n) \to \dots . \tag{6.4}$$

Since $H^p(R^n)=0$ for all $p \geqslant 1$, we have the canonical ismorphism $H^p(X) \overset{\partial_p^*}{\cong} H^{p+1}(R^n/X)$. Let us be explicit about this isomorphism. If $\alpha \in H^p(X)$, then we may pick a form $\varphi \in \Lambda^p(R^n)$ which represents α. We write $\alpha=[\varphi]$. By the definition of the differential in $\tilde{\Lambda}^p(X)$, we see that $d_p\varphi$ vanishes in a neighborhood of X. Thus $d\varphi \in \Lambda^{p+1}(R^n/X)$. Moreover, clearly, $d_{p+1}(d_p\varphi)=0$, so $d_p\varphi$ represents a class $[d_p\varphi] \in H^{p+1}(R^n/X)$. The formula $\partial_p^*([\varphi])=[d_p\varphi]$ holds.

It is clear that $H^p(X)$ and $H^p(R^n/X)$ are complex vector spaces. We let $H_p(X)$ and $H_p(R^n/X)$ denote the dual vector spaces. We refer to these as the (deRham) homology groups of X or of R^n relative to X. Again, they are in reasonable cases canonically identifiable with more classical homology groups. Of course, the adjoint of ∂_p^* gives an isomorphism $\partial_{*p}: H_{p+1}(R^n/X) \to H_p(X)$.

B. The index theorem

Theorem II yields a relative homology class in the sense of the preceding subsection as we now see. Suppose that $X_1, \dots, X_k$ are self adjoint crypto-integral operators of dimension δ where δ is one or two. The representation theorem says precisely that there is a linear functional l on $C\Lambda^{2\delta}(R^k)$ which vanishes on $d\Lambda^{2\delta-1}(R^k/E)$. Thus the restriction of l to $\Lambda^{2\delta}(R^k/E)$ may actually be factored to $H^{2\delta}(R^k/E)$. In other words, l may be used in this way to define a relative homology class λ_δ in $H_{2\delta}(R^k/E)$. By the boundary isomorphism, we get a class $\partial_*(\lambda_\delta)$ in $H_{2\delta-1}(E)$.

Now we turn to the index theory for the one dimensional case. Instead of proving Theorem III immediately, we look first at an algebra with two generators since everything in this case is extremely graphic and provides good motivation for the general case. Consider X_1 and X_2 self-adjoint operators with trace class commutators. Set $T=X_1+iX_2$ and consider $\mathbf{C}$ and R^2 to be identified in the natural way. If U is a partial isometry in the C^* algebra generated by X_1 and X_2 whose symbol extends to a smooth function u on R^2, then since index $U=\operatorname{tr}(UU^*-U^*U)$ a purely formal application of the representation theorem yields

$$\text{index } U = l(du \wedge d\bar{u}) = \lambda(du \wedge d\bar{u}). \tag{6.5}$$

In fact, the first equality was proved in [8]. The fact that l vanishes on $d(\Lambda^1(R^2/E))$ says that l is a constant multiple k_j of Lebesgue measure on each component U_j of R^2-E. It is not hard to use the formula above and evaluate the k_j; they are

$$k_j = \frac{-1}{2\pi i} \text{ index } (T - z_j I)$$

where z_j is any point in U_j. Now it is folklore that (integration over) the connected components of $\mathbf{C} - E$ gives an integral basis for $H_2(R^2/E)$. Thus we intuitively associate an assignment of integers λ_j to each component of $\mathbf{C} - E$ with a class λ in $H_2(R^2/E)$. This is precisely what was done above. So, in summary, we conclude that the homology class λ canonically associated with the fundamental trace form corresponds to the index information contained in T. The operator T was important here because it generates $C^*\mathfrak{A}(X_1, X_2)$. When $n > 2$ the C^*-algebra does not have one generator and the relationship described here cannot hold. However, the basic phenomenon carries over in a simple and complete fashion via Theorem III which will now be proved.

Proof of Theorem III: Suppose that $X = \{X_1, ..., X_k\}$ is an almost commuting family of operators, and let $\mathfrak{A}$ be a maximal selfadjoint set containing the X_i such that $[R, S]$ is trace class for any R, S in $\mathfrak{A}$. Suppose $R = \{R_{ij}\}$ and $S = \{S_{ij}\}$ are in $M_m(\mathfrak{A})$. Then $[R, S]$ is not necessarily trace class, or even compact. Nevertheless, the formal trace of $[R, S]$ is $\sum_{i,j} R_{ij}S_{ji} - \sum_{ij} S_{ij}R_{ji} = \sum_{i,j} [R_{ij}, S_{ji}]$, and this is trace class, so we may take its trace. Thus $\langle R, S\rangle = \sum_{i,j} \operatorname{tr}[R_{ij}, S_{ji}]$ is a well-defined bilinear form on $M_m(\mathfrak{A})$. Moreover, if $[R, S]$ happens to be trace class, then $\langle R, S\rangle = \operatorname{tr}[R, S]$. In particular, if $U \in M_m(\mathfrak{A})$ is a Fredholm partial isometry, we have $\langle U^*, U\rangle = -\text{ind } U$.

Consider an operator F in the C^*-algebra generated by $M_m(\mathfrak{A})$ whose symbol f in $CM_m(E)$ is invertible. Write f in polar form $f = u|f|$, where u is a unitary valued function in $CM_m(E)$ and $|f| = \sqrt{f^*f}$. Standard approximation arguments allow us to choose a map $\varphi: R^k \to M_m(\mathbf{C})$ such that the entries φ_{ij} of φ are smooth functions, φ takes on unitary values in a neighborhood of E, and such that $\varphi|_E$ is arbitrarily close to u. Our functional calculus, Theorem 3.3 part c, allows us to associate with φ an operator $e_X(\varphi)$ in $M_m(\mathfrak{A})$. Since φ and f are in the same homotopy class, the index of F equals the index of $e_X(\varphi)$.

The representation theorem yields

$$\langle e_X(\phi), e_X(\phi)^*\rangle = \sum_{i,j=1}^{m} \lambda(d\varphi_{ij} d\bar{\varphi}_{ij})$$

If the operator $e_X(\phi)$ were a partial isometry, then its index would simply equal $\langle e_X(\phi), e_X(\phi)^*\rangle$. Even though it is not a partial isometry, we shall see that it is reasonably close to one. Namely, let UT denote the polar decomposition of $e_X(\phi)$. The operator U is a partial isometry with the same index as $e_X(\phi)$. Moreover, $e_X(\phi) - U$ is a compact operator.

Replacing $e_X(\phi)$ by its adjoint and taking a trace class perturbation if necessary, we may assume that $e_X(\phi)$, and therefore T, is invertible. Then $T^2 = e_X(\phi)^* e_X(\phi)$ is in $M_m(\mathfrak{A})$ and invertible. Now Theorem 3.3 implies that T, T^{-1} and hence U belongs to $M_m(\mathfrak{A})$. Thus Lemma 1.3 implies $\langle e_X(\phi), e_X(\phi)^* \rangle = \operatorname{tr}[U, U^*]$, completing the proof of Theorem III.

§ 7. Examples

In this section we shall see how the foregoing theory looks when specialized to some concrete examples. The first subsection treats pseudo-differential operators of non-positive order. We prove that they form a crypto-integral algebra of the appropriate dimension, and that the fundamental trace form just gives the fundamental class on the cotangent sphere bundle of the manifold. In the second subsection we state analogous results for Toeplitz operators on the $2n-1$ sphere which follow from forthcoming work of the second author. We also discuss how Toeplitz operators compare to pseudo-differential operators and derive the Toeplitz index theorem from the pseudo-differential operator index theorem.

A. Pseudo-differential operators

We begin with a discussion of pseudo-differential operators. There are quite a few different classes of such operators. The ones appropriate here are the original ones of Kohn-Nirenberg [13] or an extended class intermediate between these and the ones of Hörmander [11]. Now we set conventions. Suppose U is an open set in R^n. For any integer m, denote by $S^m(U)$ the set of all smooth functions $p(x, \xi)$ on $U \times R^n$ such that, for every compact $K \subset U$ and multi-indices α, β, we have

$$\left| \frac{\partial^\beta}{\partial x} \frac{\partial^\alpha}{\partial \xi} p(x, \xi) \right| \leqslant C_{\alpha, \beta, K} (1 + |\xi|)^{m - |\alpha|}. \tag{7.1}$$

The subclass $A(U)$ of $S^0(U)$ which we shall use has the property that for any $S^0(U)$ function p in $A(U)$ there is a smooth function $\sigma^{\sim}(p)(x, \xi)$ which outside the disk $|\xi| \leqslant 1$ is homogeneous of order 0 in the ξ variable, and a positive number ε such that

$$p(x, \xi) - \sigma^{\sim}(p)(x, \xi) \in S^{-\varepsilon}(U).$$

For any p in $A(U) \cap S^m(U)$ the pseudo-differential operator P with total symbol p is the map from $L^2(U)$ functions of compact support to locally L^2 functions defined by extension of the formula

$$Pu = (2\pi)^{-n} \int p(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} d\xi$$

from a dense set of functions. The operator P is said to have order m and the symbol of P, denoted $\sigma(P)(x,\theta)$, is the function on $U\times\{\xi: |\xi|=1\}$ defined by $\sigma(P)(x,\theta)=\sigma^{\sim}(p)(x,\theta)$ for x in U and $|\theta|=1$. The function p will be called the "full symbol" of P.

Let $M^n=M$ be a smooth, compact, n-dimensional manifold. Let w be an everywhere positive smooth density on M. Using w we may define an inner product $(,)_w$ on $C^\infty(M)$ by the formula $(f,g)_w=\int_M f\bar{g}w$. Let $\mathcal{H}_w$ be the corresponding Hilbert space. The pseudodifferential operators behave well under coordinate changes [11] and so one can define a class $PS(M)$ of bounded operators on $\mathcal{H}_w$ whose localizations are pseudo-differential operators having full symbol in $A(U)$ as above.

Let $T^*(M)$ be the cotangent bundle of M. Let $S^*(M)\subseteq T^*(M)$ be the cotangent sphere bundle of M, and $D^*(M)\subseteq T^*(M)$ the associated disk bundle. Then $D(M)$ is a manifold of dimension $2n$, with boundary $\partial D(M)=S^*(M)$. To pin down $S^*(M)$ and $D^*(M)$ as actual physical subsets of $T^*(M)$, the reader may assume a Riemannian metric on M has been specified. Actually $S^*(M)$ is more properly and invariantly thought of as the bundle of rays in $T^*(M)$, forming a natural boundary for $T^*(M)$; but this doesn't matter. It is known that the commutator ideal of $PS(M)$ consists of compact operators, and that the joint essential spectrum of $PS(M)$ may be identified with $S^*(M)$. Note that the symbol, $\sigma(P)$, of an operator P in $PS(M)$ is well defined and is a function on $S^*(M)$.

THEOREM 7.1.

(a) *Given a compact, smooth, n-dimensional manifold M, the associated algebra $PS(M)$ is crypto-integral of dimension n.*

(b) *Given operators $A_1, \ldots, A_{2n}\subseteq PS(M)$, let $f_1,\ldots,f_{2n}\subseteq C^\infty(S^*(M))$ be the symbols of the A_i. Then*

$$\operatorname{tr}[A_1,A_2,\ldots,A_{2n}]=\gamma\int_{S^*(M)} f_1\,df_2\wedge df_3\wedge\ldots\wedge df_{2n} \tag{7.2}$$

where $\gamma=n!(2\pi i)^{-n}$.

Remark. The right hand side of (7.2) is antisymmetric in the f_i because of Stoke's theorem, or integration by parts. A more balanced formula may be obtained by considering extensions $\tilde{f}_i$ of the f_i from $S^*(M)$ to $D^*(M)$. (One may choose such extensions in a systematic manner if one wishes.) The Stokes' theorem again implies

$$\operatorname{tr}[A_1,\ldots,A_{2n}]=\gamma\int_{D(M)} d\tilde{f}_1\wedge\ldots\wedge d\tilde{f}_{2n}.$$

Remark. This theorem along with the Atiyah-Singer index theorem gives one an explicit formula for the index of any Fredholm operator in $M_m(PS(M))$ in terms of the trace of an antisymmetrization of operators in $PS(M)$.

Proof. We will first prove the theorem for pseudo-differential operators having compact support on an open set U. The general case will then follow from the behavior of $PS(M)$ under localization. The Hilbert space on which these operators act is taken to be $L^2(U)$ where one uses Lebesgue measure. If one instead were to use an equivalent measure, this would have the effect of conjugating the operators P involved by an invertible operator A, namely sending P to APA^{-1}, and all considerations involving traces are insensitive to this.

First observe that if T is a trace class pseudo-differential operator with full symbol $t(x, \xi)$ on $U \times R^n$ having compact support in U, then tr T is the trace of the integral operator with kernel

$$q(x,s) = (2\pi)^{-n} \int_{R^n} t(x,\xi)\, e^{-i(s-x)\cdot\xi} d\xi.$$

Thus

$$\mathrm{tr} T = (2\pi)^{-n} \int_{U \times R^n} t(x,\xi) d\xi\, dx.$$

It is straightfoward to check that if $t(x, \xi)$ is dominated by $(1+|\xi|)^{-(n+\varepsilon)}$, then T is trace class. Also recall that the composition of two operators P, Q with full symbol p, q compactly supported in U is a pseudo-differential operator whose full symbol equals

$$\sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial \xi} p(x,\xi) \frac{\partial^\alpha}{\partial x} q(x,\xi).$$

From this it follows that the product of operators of order m and n yield an operator of order at most $n+m$ and that the commutator of two such operators is order $n+m-1$. These facts immediately guarantee that the order zero pseudo-differential operators are almost nilpotent in $n+1$ steps. The fact that the antisymmetrization of $2n$ operators is trace class is more subtle and will emerge from the forthcoming discussion.

The next step in the proof is to show that the total symbol of the operator $[A_1, \ldots, A_{2n}]$ has the form

$$i^n n! J(a_1, a_2, \ldots, a_{2n})(x,\xi) + E(a_1, \ldots, a_{2n})(x,\xi), \tag{7.3}$$

where the a_j are full symbols of the A_j, J is the Jacobian matrix of the $\{a_j\}_{j=1}^{2n}$, and E has order $-(n+1)$. In fact E is a sum of products of derivatives of the a_j where each product contains at least $n+1$ derivatives in the ξ directions.

To verify (7.3), we recall from part 3) of Proposition 1.1 that the antisymmetrization of $A_1, \ldots, A_{2n}$ is a sum of $2^{-n}n!$ terms like

$$[A_1, A_2][A_3, A_4] \ldots [A_{2n-1}, A_{2n}].$$

Clearly the full symbol of this product has order $-n$ and the leading term is

$$i^n\left(\sum_{|k|=1} \frac{\partial^k a_1}{\partial x}\frac{\partial^k a_2}{\partial \xi} - \frac{\partial^k a_2}{\partial x}\frac{\partial^k a_1}{\partial \xi}\right) \cdots \left(\sum_{|w|=1} \frac{\partial^w a_{2n-1}}{\partial x}\frac{\partial^w a_{2n}}{\partial \xi} - \frac{\partial^w a_{2n}}{\partial x}\frac{\partial^w a_{2n-1}}{\partial \xi}\right).$$

Upon expanding this out, one obtains $(2n)^n$ terms, of which n^n have the form

$$i^n \frac{\partial a_1}{\partial x_{i_1}}\frac{\partial a_2}{\partial \xi_{i_1}}\frac{\partial a_3}{\partial x_{i_2}}\frac{\partial a_4}{\partial \xi_{i_2}} \cdots \frac{\partial a_{2n-1}}{\partial x_{i_n}}\frac{\partial a_n}{\partial \xi_{i_n}},$$

where $1 \leqslant i_j \leqslant n$. The other $(2^n-1)n^n$ terms are obtained from this last expression by making interchanges in i of the pairs (a_{2j-1}, a_{2j}) and multiplying by $(-1)^i$. Hence, keeping Proposition 1.1 in mind, we see there are altogether $n!n^n$ terms in the n-th order symbol of $[A_1, \ldots, A_{2n}]$ and these terms are obtained from the standard term above by performing arbitrary permutations in the i_k and inserting the appropriate sign. If we fix $i_1, i_2, \ldots, i_n$ and perform the permutation, we see the result will be identically zero unless $i_j \neq i_k$ for $k \neq j$. If however we do have $i_j \neq i_k$ for $k \neq j$, then the i_j are just $1, 2, \ldots, n$ in some order, and we just obtain 1 copy of $i^n J(a_1, \ldots, a_{2n})$. Summing over all possible permutations $i_1, \ldots, i_n$, we do obtain $i^n n! J(a_1, \ldots, a_{2n})$, as claimed.

The first consequence of (7.3) is that $[A_1, \ldots, A_{2n}]$ is trace class as we shall now demonstrate. Since the full symbol of A_j is in $A(U)$ the function $e_j(x, \xi) = (a_j(x, \xi) - \tilde{\sigma}(a_j))(x, \xi)$ is of order $-\varepsilon_j$ for some positive ε_j. If $g_1, \ldots, g_{2n-1}$ are functions of order less then or equal to zero, then $J(g_1, \ldots, g_{2n-1}, e_j)$ has order $-n-\varepsilon_j$ and so $[A_1, \ldots, A_{2n}]$ will be trace class if $J(\tilde{\sigma}(a_1), \ldots, \tilde{\sigma}(a_{2n}))$ is integrable on $U \times R^n$. However, this function vanishes identically outside of $D(U)$ since it is the Jacobian of a map which outside of $D(U)$ has rank only $2n-1$, by homogeneity of the a_i's. Consequently it is integrable.

The trace of $[A_1, \ldots, A_{2n}]$ is just the integral of $i^n n!(2\pi)^{-n}J$ plus the integral of E. We wish to show E does not contribute to the trace. Define A_j^t to be the pseudo-differential operator with symbol $a_j(x, \xi/t)$ which we denote by a_j^t. The function $a_1^t - a_1$ has order $-\varepsilon$ and so if the corresponding operator D_1^t appears in a commutator $[D_1^t, [A_2, \ldots, A_{2n}]]$ or $[A_2, [D_1^t, A_3, \ldots, A_{2n}]]$ the commutator is trace class and has trace zero. Repetition of this principle and Proposition 1.1, part 2, yield that $\operatorname{tr}[A_1^t, \ldots, A_{2n}^t]$ is independent of t. Of course $\int_{U \times R^n} J(a_1^t, \ldots, a_{2n}^t)$ is also independent of t. However,

$$\left|\int E(a_1^t, \ldots, a_{2n}^t)(x, \xi)\, dx\, d\xi\right| \leqslant \frac{1}{t^{n+1}} \int \left|E(a_1, \ldots, a_{2n})(x, \xi/t)\right| dx\, d\xi$$

for $t>1$, so if we make the change of variables $\eta=\xi/t$ and then take the limit $t\Rightarrow\infty$, we happily find that $\int E=0$. Therefore

$$\operatorname{tr}[A_1, A_2, \ldots, A_{2n}] = i^n \frac{n!}{(2\pi)^n} \int_{U\times R^n} J(a_1, a_2, \ldots, a_{2n}),$$

which by Stokes theorem is

$$\frac{n!}{(2\pi i)^n} \lim_{r\to\infty} \int_U \int_{|\xi|=r} a_1 da_2 \wedge da_3 \wedge \ldots \wedge da_{2n},$$

which in turn is

$$\frac{n!}{(2\pi i)^n} \lim_{r\to\infty} \int_U \int_{|\xi|=r} \sigma(A_1) d\sigma(A_2) \wedge \ldots \wedge d\sigma(A_{2n}).$$

However, this last integral is independent of r for $r>1$ since the integrand is exact. This gives the desired trace formula (7.2) for U.

Now we pass to the general case. The point is that $PS(M)$ is pseudo-local in the following sense. Throughout this section M_h will denote the operation of multiplication by the function h. The function space on which it acts will vary but in a way which will always be clear from context; here this space is $C^\infty(M)$. Suppose f and g are two smooth functions on M with disjoint supports. Then, for any $T\in PS(M)$, the operator $M_f T M_g$ is a smooth integral operator, and in particular is trace class. Also, if $U\subseteq M$ is any open set, then the algebra $\mathfrak{A}(U)=\{M_f T M_g: f, g\in C_0^\infty(U), T\in PS(M)\}$ depends only on the diffeomorphism type of U, not on the embedding of U in M. This is the excision property for pseudo-differential operators.

If M is any compact manifold let $\{U_i\}$ be a finite covering of M by coordinate disks. For convenience we will assume that given U_i, then the union of those U_j for which $U_i\cap U_j\neq\phi$ is contained in some suitable coordinate disk. It is not hard to see that such covers exist. Let $\{h_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$. Then given any $T\in PS(M)$, we may write $T=\sum_{i,j} M_{h_i} T M_{h_j}$. In this way, we may replace any basic commutator product in elements of $PS(M)$ by a sum of basic commutator products which either contain only terms of the form $AM_{h_i}BM_{h_j}C$ where h_i and h_j have disjoint support, or are basic commutator products in operators all belonging to $\mathfrak{A}(V)$ for some suitable coordinate disk V. The former type are clearly trace class. Whether or not a given type of expression involving commutators in elements of $PS(M)$ is always trace class is reduced to the same question asked only of elements of $\mathfrak{A}(V)$ for some standard disk V. In other words, such a question is a local question. From our investigations on U, we may conclude that $PS(M)$ is crypto-integral of dimension $n=\dim M$ for any manifold M.

Now we may turn to the fundamental form on $PS(M)$. It is clear that both the right and left hand sides of (7.2) have the following properties:

(i) They are equal if all A_i, $1 \leqslant i \leqslant 2n$ are in $\mathfrak{A}(V)$ for some disk $V \subseteq M$. This of course follows from the computation on R^n.

(ii) Suppose for some i and j, $A_i = M_{g_i} B_i M_{h_i}$ and $A_j = M_{g_j} B_j M_{h_j}$ where some pair from g_i, g_j, h_i and h_j have disjoint supports. The right hand side of (6.1) is zero because f_i and f_j, the symbols of A_i and A_j, have disjoint supports. Secondly, the product $A_1 A_2 \dots A_{2n}$ and any permutation of it is trace class. Consequently the antisymmetrization of $A_1, \dots, A_{2n}$ can be written as the sum of trace class commutators and so the left hand side of (6.1) is zero.

(iii) Each side is determined by its values on $\mathfrak{A}(V)$ for disks V and by property (ii). This follows from the partition of unity argument mentioned above.

From (i), (ii) and (iii) above, statement (b) of Theorem 7.1 follows immediately for a general manifold M, so the proof of that theorem is complete.

Remark. The localization property used here to reduce the computation on a general manifold to the computation on R^n is in fact a general phenomenon in crypto-integral algebras. Precisely, suppose $\mathfrak{A}$ is a crypto-integral algebra closed under C^∞ functional calculus. Suppose $\mathfrak{B} \subseteq \mathfrak{A}$ is a commutative subalgebra. Suppose T_1 and $T_2 \in \mathfrak{B}$ and T_1 and T_2 have disjoint supports in the maximal ideal space of $C^*\mathfrak{B}$. Then $T_1 A T_2$ is trace class for any $A \in \mathfrak{A}$ for it is not hard to show that there is an S in $\mathfrak{A}$ commuting with T_1 and T_2 such that $ST_1 = T_1$, and $ST_2 = 0$. Then $[S, T_1 A T_2] = T_1 A T_2$, so $T_1 A T_2 \in \bigcap_{i>0} \mathfrak{A}_i$ (so $T_1 A T_2$ is of order $-\infty$) and in particular is trace class.

B. Venugopalkrishna's Toeplitz operators

Let us briefly recall the definition of these operators [18]. Let $D \subset C^n$ be the unit disk in complex n-space, and let $S^{2n-1} = \partial D$ be the unit sphere. Let $H^2(D)$ be the subspace of $L^2(D)$ consisting of holomorphic functions in D. Let P be the projection of L^2 onto H^2. For any continuous function f on D, define the Toeplitz operators τ_f on $H^2(D)$ by $\tau_f = PM_f$.

The algebra of Toeplitz operators is denoted Λ_n and the symbol of τ_f is $f_{|S^{2n-1}}$. One can perform the same construction for a pseudo-convex domain with smooth boundary. However, in the case of D, one has a pleasant orthogonal basis for H^2, namely

$$b_k = \pi^{-n/2} \left[\frac{(n+|k|)!}{k!} \right]^{1/2} z^k, \tag{7.4}$$

where k is the conventional multi-index with entries in $(Z^+)^n$. This basis allows one to identify $H^2(D)$ with $l^2((Z^+)^n)$ (see [5] for details). If $a \in Z^n$, then we define translation by a

to be the operator on $l^2(Z^n)$ whose action on $f\in l^2(Z^n)$ is $[T_a f](b)=f(b-a)$. One can compute that τ_{z_i} when viewed on $l^2((Z^+)^n)$ has the form $M_{f_i}T_{e_i}$ where e_i is the standard ith basis vector in $(Z^+)^n$ and the function f_i is

$$f_i(x)=\left(\frac{x_i}{x_1+\ldots+x_n+n}\right)^{1/2}. \tag{7.5}$$

Toeplitz operators and pseudo-differential operators are very closely related. For, if M is a manifold and $U\subset S^*(M)$ is a coordinate ball, then the C^*-algebra generated by pseudo-differential operators whose symbols are constant outside U is isomorphic to the C^*-algebra generated by Λ_n. One can see this on general grounds by combining the Atiyah-Singer Index Theorem [1], the theory of Brown-Douglas-Fillmore [3], and Venugopalkrishna's Index Theorem [18]. We take a more concrete point of view instead and write down explicitly such a correspondence between Λ_n and a natural subalgebra of $PS(T^n)$, the pseudo-differential operators on the n-torus. A benefit of this approach is a demonstration that Venugopalkrishna's index theorem follows from the Atiyah-Singer index theorem.

By means of the Fourier transform the pseudo-differential operator $PS(T^n)$ can be conveniently regarded as being an algebra of operators on $l_2(Z^n)$ generated by the following two types of operators:

(i) translation operators: if $a, b\in Z^n$, and $f\in l_2(Z^n)$, then $T_a(f)(b)=f(b-a)$ is translation by a;

(ii) multiplication operators: M_g with g the restriction to Z^n of a smooth function on R^n which satisfies

$$\left|\frac{\partial^\alpha}{\partial\xi}g(\xi)\right|\leqslant C_\alpha(1+|\xi|)^{-|\alpha|} \tag{7.6}$$

as well as the obvious analog of the asymptotic condition which characterizes functions in $A(U)$.

This approach to pseudo-differential operators is workable and the original proof of Theorem 7.1 was given in this framework. Also treating these operators as weighted shifts is gratifying to operator theorists who have had a long standing affection for weighted shifts. Now we turn to Toeplitz operators.

The function f_i of (7.5) defined on $(R^+)^n$ has radial asymptotic limits and its derivatives $\partial^k/\partial x^k$ satisfy the appropriate growth condition (7.6) provided that $|x|/x_i$ is bounded. Thus the generators $\tau_{t_i}=M_{f_i}T_{e_i}$ for the Toeplitz algebra seem something like $PS(T^n)$ operators but they have a "rough symbol" along the hyperplane $x_i=0$. We shall identify certain Toeplitz operators with operators in $PS(T^n)$ whose interesting action is just that of the Toeplitz operators. Let $\mathcal{G}$ denote all polynomials in $\tau_{z_i}, \tau^*_{z_j}$. Put $w=\prod_{i=1}^n f_i^2$ where f_i is as in (7.5). Note w vanishes on the coordinate planes $x_i=0$. Also the operator M_w can be written $\prod_{i=1}^{n}(\tau_{z_i}\tau_{z_i}^*)$ and consequently is in $\mathcal{G}$. Furthermore, any continuous function β of M_w is in the C^*-algebra generated by $\mathcal{G}$. Pick β to be a one-one C^∞ function from R^+ to R^+ which along with all of its derivatives vanishes at the origin. The salient feature of β is that $\beta(w)\,f_i$ is smooth and satisfies (7.6) thereby insuring that the operator $\beta(M_w)\tau_{z_i} = M_{\beta(w)f_i}T_{e_i}$ in the Toeplitz algebra is also in $PS(T^n)$.

Let A be the selfadjoint polynomial algebra generated by the constants and the operators $\beta(M_w)\tau_{z_i}$. Then if $R \in M_m(A)$ is Fredholm, it is clear that the index of R is the same whether R is considered as operating on $l^2((Z^+)^n)$ or on $l^2(Z^n)$. In the latter situation of course the index of R is computed by the Atiyah-Singer Index Theorem. On the other hand, A is contained in Λ_n and it is clear that the symbols represented by A are uniformly dense in the functions constant on the zeros of $w_{|S^{2n-1}}$. Thus it is clear from standard arguments that each homotopy class of symbols for $M_m(\Lambda_n)$ contains a symbol from $M_m(A)$. (One shrinks the zero-locus of $w_{|S^{2n-1}}$ to a point.) Thus Atiyah and Singer compute the index homomorphism for Λ_n and this computation is just the content of Venugopalkrishna's result.

We also remark that one can construct a subalgebra of $PS(T^n)$ which is reduced by $l^2((Z^+)^n)$, which restricts faithfully to $l^2((Z^+)^n$, and which modulo the compacts are dense (uniform norm) in Λ_n. There are several constructions one can use to obtain such subalgebras, all of which are straightforward and follow after a few moments of thought.

Now we forsake the C^*-algebra generated by Λ_n and discuss finer structure. To show that Λ_n is a cryptointegral algebra it suffices to show that the τ_{z_i} and their adjoints satisfy Criterion 3.1. This can be done by getting growth estimates analogous to (7.6) for the f_i with differences $f_i(x+e_j)-f_i(x)$ replacing derivatives $\partial/\partial x$. Although this is the method which the authors used to verify that Λ_n is crypto-integral, we shall not present it because it has been outmoded by recent work of the second author which will appear elsewhere. The new approach springs from a very concrete study of Fourier analysis on the Heisenberg group, in which pseudo-differential and Toeplitz operators fit into a common framework. The Kohn-Nirenberg formalism has a natural interpretation in this setup and standard procedures for pseudo-differential operators apply also to Toeplitz operators. One can use this to prove the analog of (7.1) for Λ_n:

THEOREM 7.2. (a) *Λ_n is a crypto-integra-algebra of dimension n,* (b) *If $g_1, \ldots, g_{2n}$ are smooth functions on the $2n$-dimensional unit disk D in C^n, then*

$$\operatorname{tr}[\tau_{g_1}, \ldots, \tau_{g_{2n}}] = \frac{1}{\operatorname{vol}(\mathrm{D})}\int_D dg_1 \wedge dg_2 \wedge \ldots \wedge dg_{2n}.$$

Remark. vol (D) equals $(2\pi i)^n/n!$ with respect to the volume form $d\bar{z}_1 \wedge dz_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_n$.

References

[1]. ATIYAH, M. & SINGER, I. M., Index of elliptic operators I, II, III. *Ann. of Math.*, 87 (1968).

[2]. BROWN, L., The determinant invariant for operators with trace class self-commutators. *Proceedings of a Conference on Operator Theory*, Springer Lecture Notes 345 (1973), 58–128.

[3]. BROWN, L., DOUGLAS, R. G. & FILLMORE, P., Extensions of C*-algebras operators with compact self-commutators and K-homology. *Bull. A.M.S.* (1973).

[4]. —— Unitary equivalence modulo the compact operators, *Proceedings of a Conference on Operator Theory*, Springer Lecture Notes 345 (1973).

[5]. COBURN, L., Toeplitz operators on odd spheres. *Proceedings of a Conference on Operator Theory*, Springer Lecture Notes 345 (1973).

[6]. CAREY, R. & PINCUS, J. D., On an invariant for certain operator algebras. To appear.

[7]. GODEMENT, R., *Theorie des faisceaux*, Hermann, Paris 1958.

[8]. HELTON, J. W. & HOWE, R. E., Integral operators, traces, index and homology. *Proceedings of Conference on Operator Theory*, Springer Lecture Notes 345 (1973).

[9]. HIRZEBRUCH, F., Topological methods in algebraic geometry Springer, 1966.

[10]. HÖRMANDER, L., *Linear Partial Differential Operators*. Springer, 1963.

[11]. —— Pseudodifferential operators and hypoelliptic equations. *Amer. Math. Symposia in Pure Math.*, 10 (1967), 138–183.

[12]. JACOBSON, N., *Structure of Rings*, A.M.S. Coll. Vol. 37 (1956).

[13]. KOHN, J. J. & NIRENBERG, L., An algebra of pseudodifferential operators. *Comm. Pure and Appl. Math.*, 18 (1965), 269–305.

[14]. PINCUS, J. D., Commutators and systems of singular integral equations I. *Acta Math.*, 121 (1968), 219–250.

[15]. —— The determining function method in the treatment of commutator systems. *Proc. International Conference on Operator Theory, Tihany* (Hungary) (1970).

[16]. —— On the trace of commutators in the algebra of operators generated by an operator with trace class self-commutators. To appear.

[17]. ROSENBLOOM, M., A spectral theory for self-adjoint singular integral operators. *Amer. J. Math.*, 88 (1966), 314–328.

[18]. VENUGOPALKRISHNA, M., Fredholm operators associated with strongly pseudo-convex domains in C^n. *J. Functional Analysis*, 9 (1972), 349–373.

[19]. XA-DAO-XENG. On non-normal operators. *Chinese Math.*, 3 (1963), 232–246.

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