AUTOMORPHIC QUASIMEROMORPHIC MAPPINGS IN R^n

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Contents

1. Introduction

1.1. Let G be a discrete group of Möbius transformations in $\bar{R}^n = R^n \cup \{\infty\}$ (i.e., sense preserving conformal automorphisms of R^n) acting on a domain $D \subset \bar{R}^n$ and f: $D \to \bar{R}^n$ a continuous mapping. We say that f is *automorphic* with respect to G if f is discrete (i.e., $f^{-1}(y)$ is discrete in D for any $y \in fD$), open, and $f \circ A = f$ for all $A \in G$. f is said to be *quasiregular* (qr) in D if $D \subset R^n$, $fD \subset R^n$, f is ACL^n (absolutely continuous on lines with partial derivatives locally in L^n) and

$$|f'(x)|^n \leqslant KJ(x, f) \quad \text{a.e. in } D \tag{1}$$

for some $K \in [1, \infty)$. Here $f'(x)$ is the formal derivative of f at $x \in D$, $|f'(x)|$ denotes the supremum norm of the operator $f'(x)$ and $J(x, f) = \det f'(x)$. If $\infty \in D$ or $\infty \in fD$, (1) and the ACL^n property can be checked at a neighborhood of ∞ or at neighborhoods of points of $f^{-1}(\infty)$ by means of auxiliary Möbius transformations which map ∞ to a finite point. If

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(1) and the ACL^n property hold in D, $D\subset \bar{R}^n$ and $fD\subset \bar{R}^n$, for some $K\in[1, \infty)$, we say that f is *quasimeromorphic* (qm). f is said to be *quasiconformal* (qc) in D if f is qm and injective in D. qc, qr, and qm mappings seem to be very reasonable generalizations of conformal, analytic, and meromorphic functions, respectively.

The main purpose of this paper is to study automorphic and in particular qm automorphic mappings in $\bar{R}^n$, $n\geq 2$. One class of automorphic mappings, namely, the family of periodic mappings, has been studied in [11]. Here we are concerned mostly with automorphic mappings for Möbius groups which act on $B^n=\{x\in R^n\colon |x|<1\}$ or on $H^n=\{x\in R^n\colon x_n>0\}$. In chapter 4 we present two examples of qm automorphic mappings for Möbius groups which act on H^3. One of the examples is analogous in some respects to the elliptic modular function.

Given a discrete Möbious group G acting on B^n, there is a standard way of constructing a canonical fundamental set $\tilde{P}$ in B^n which is bounded by $(n-1)$-spheres or $(n-1)$-planes normal to ∂B^n. If B^n/G is of a finite volume then, see [15] or [3], $P=\operatorname{int}\tilde{P}$ has a finite number of faces and either $\bar{P}\cap\partial B^n=\emptyset$ (when B^n/G is compact) or has a finite number of points. The points of $\bar{P}\cap\partial B^n$ will be called *boundary vertices*. It is not hard to see that if f is automorphic with respect to such G, then f has no limit at any boundary point $b\in\partial B^n$, and if in addition B^n/G is compact then f has no radial limit at any point $b\in\partial B^n$. We show (in chapter 6) that in the latter case f assumes almost every value the same (finite) number of times in $\tilde{P}$. This result which is known for meromorphic automorphic mappings in R^2, follows from a more general theorem (derived in chapter 2) on open and discrete mappings from n-dimensional pseudomanifolds into $\bar{R}^n$. With this theorem we prove also the following results, part of which seems to be new even in R^2.

Suppose that G is a Möbius group acting on B^n, with a non-compact orbit space B^n/G of finite volume, and that $f\colon B^n\to\bar{R}^n$ is qm automorphic with respect to G. Let $\tilde{P}$ be a canonical fundamental set for G in B^n, and $N=\sup\operatorname{card}(f^{-1}(y)\cap\tilde{P})$ over all $y\in\bar{R}^n$. Then $N<\infty$ if and only f has a radial limit at every boundary vertex of P. Furthermore, $N<\infty$ implies that f has a radial limit at every parabolic fixed point of G and thus on a dense set on ∂B^n and that $f(U\cap B^n)=\bar{R}^n\setminus A$ for any neighborhood U of any boundary point $b\in\partial B^n$ where A is some fixed set of finite cardinality in $\bar{R}^n$. If in addition $n>2$, then f has a non-empty branch set. These results are not true for non-qm automorphic mappings and the last result is not true for qm mappings in R^2. The elliptic modular function is a counter example.

We also study the growth of qm automorphic mappings $f\colon B^n\to\bar{R}^n$ near ∂B^n. The main tools for this aim are serveral modulus and capacity inequalities which we develop in chapter 5.

Some properties of Möbius groups, which we need for the study of automorphic mappings, are presented in chapter 3.

1.2. *Notation and terminology* will usually be as in [8], [9], [10] and [11]. For $x \in R^n$ we write $x=(x_1, \ldots, x_n)=\sum_{i=1}^n x_i e_i$ where $e_1, \ldots, e_n$ is an orthonormal basis in R^n. For $a \in R^n$ and $r>0$ we denote $B^n(a,r)=\{x \in R^n: |x-a|<r\}$, $B^n=B^n(0,1)$, $S^{n-1}(a,r)=\partial B^n(a,r)$, $S^{n-1}=\partial B^n$, $H^n=\{x \in R^n: x_n>0\}$, and $H(h)=\{x \in R^n: x_n>h\}$.

The hyperbolic distance in B^n or in H^n is denoted by $d(a,b)$. The hyperbolic measure of a set A in B^n or in H^n or in B^n/G or in H^n/G is denoted by $V(A)$. Here G is a discrete Möbius group acting either on B^n or on H^n. The euclidean distance between two sets A and B in R^n is denoted by dist (A, B). The closure $\bar{A}$, the boundary ∂A, and the complement $\complement A$ of sets $A \subset \bar{R}^n$ will always be in $\bar{R}^n$.

2. Open discrete mappings on pseudomanifolds

2.1. Let X and Y be *Hausdorff, connected, locally connected,* and *locally compact* spaces with *countable bases of open sets,* and let $f: X \to Y$ be a *continuous, open,* and *discrete* map. Discreteness of f means that $f^{-1}(y)$ is a discrete set in X, whenever $y \in fX$.

A domain (open connected set) D in X is called a *normal domain* for f if $\bar{D}$ is compact in X and $\partial fD = f\partial D$. Note that if D is a conditionally compact domain in X, then D is a normal domain if and only if $f|D$ defines a closed map $D \to fD$. D is called a *normal neighborhood* of $x \in X$ if D is a normal domain containing x and $\bar{D} \cap f^{-1}f(x)=\{x\}$.

The *branch set* B_f of f is the set of points in X where f fails to define a local homeomorphism. For $y \in Y$ and $A \subset X$, we denote $N(y,f,A)=\operatorname{card}(f^{-1}(y) \cap A)$, $N(f,A)=\sup N(y,f,A)$ over all $y \in Y$, $N(y,f)=N(y,f,X)$, and $N(f)=N(f,X)$.

2.2. LEMMA. *Let f, X, and Y be as in* 2.1.

(i) *If D is a domain in Y and U is a conditionally compact connected component of $f^{-1}D$, then U is a normal domain and $fU=D$.*

(ii) *Every $x \in X$ has arbitrarily small normal neighborhoods.*

Proof. For (i) see Whyburn [21, 5 p. 5]. (ii) follows from (i), Väisälä [18, 5.1], and the fact that for conditionally compact domains U in X, $\partial fU = f\partial U$ is equivalent to the closedness of $f|U$.

2.3. LEMMA. *Let X and Y be as in* 2.1. *Suppose that $f: X \to Y$ is continuous, open, discrete, and closed.*

If $\operatorname{int} fB_f=\emptyset$ *and* $Y\setminus fB_f$ *is connected, then* $N(y, f)$ *is lower-semicontinuous for all* $y\in Y$ *and* $N(y, f)\equiv N(f)<\infty$ *for all* $y\in Y\setminus fB_f$.

Proof. Some of the arguments are due to Väisälä [18, 5.5]. We first show that $N(y, f)<\infty$ for all $y\in Y$. Suppose that $f^{-1}(y)=\{x_1, x_2, \ldots\}$ is infinite. Choose metrics d in X and d' in Y which are compatible with their topologies and a sequence $\{z_1, z_2, \ldots\}$ in X such that $d(z_k, x_k)<1/k$ and $0<d'(f(z_k), y)<1/k$. Then $\{z_1, z_2, \ldots\}$ is closed in X and its image is not closed in Y. Thus $N(y, f)<\infty$ for all $y\in Y$.

Let $y\in Y$ and $f^{-1}(y)=\{x_1, \ldots, x_k\}$. Choose disjoint conditionally compact connected neighborhoods V_i of x_i, $i=1, \ldots, k$ and let U be the y-component of the open set $\bigcap fV_i\setminus f(X\setminus\bigcup V_i)$. Clearly $\{x_1, \ldots, x_k\}\subset f^{-1}U\subset\bigcup V_i$. Hence $N(y, f)\leqslant N(y', f)$ for all $y'\in U$. If $y\in Y\setminus fB_f$, the sets V_i can be chosen so that $f|V_i$ $i=1, \ldots, k$ are homeomorphisms, and in this case $N(y, f)=N(y', f)$ for all $y'\in U$. Thus $N(y, f)$ is continuous in $Y\setminus fB_f$ and lower-semicontinuous in Y.

Now, $\operatorname{int} fB_f=\emptyset$, $N(y, f)$ is semicontinuous and $Y\setminus fB_f$ is connected; hence $N(y, f)\equiv N(f)<\infty$ for all $y\in Y\setminus fB_f$.

2.4. Now, let Y be a connected oriented n-manifold with countable base of open sets and X be a space of a finite oriented pseudo n-manifold K without boundary, i.e., X is homeomorphic to a geometric realization of a finite homogeneously n-dimensional simplical complex K with, see [16],

(i) every $(n-1)$-simplex of K is the face of exactly two n-simplexes of K;

(ii) if s and s' are n-simplexes of K, there is a finite sequence $s=s_1, \ldots, s_m=s'$ of n-simplexes of K such that s_i and s_{i+1} have an $(n-1)$-face in common $i=1, \ldots, m-1$;

(iii) the integral homology group $H_n(K)$ is infinite cyclic; in other words, K is oriented.

In the sequel, *pseudomanifold* will mean either the complex K or its space X. Finally, let D be a domain in X and let $f\colon D\to Y$ be *continuous, open, discrete* and *sense-preserving.*

2.5. THEOREM. *Let* $f\colon D\to Y$ *be as in* 2.4. *Then* $\dim B_f\leqslant n-2$ *and* $\dim fB_f\leqslant n-2$.

Proof. Let Γ be the set in X which corresponds to the geometric realization of the $(n-2)$-skeleton in K. Then $X_1=D\setminus\Gamma$ in an oriented n-manifoid and $f_1=f|X_1\colon X_1\to Y$ is continuous, open and discrete, thus, see [18], $\dim B_{f_1}\leqslant n-2$.

Γ is compact in X, $B_f\subset\Gamma\cup B_{f_1}$ and $\dim\Gamma=n-2$. On the other hand B_{f_1} is a F_δ set in X, thus, see [5], $\dim B_f\leqslant\dim(B_{f_1}\cup\Gamma)=n-2$, and by [1, 2.1] $\dim fB_f\leqslant n-2$.

2.6. *The local topological index of open discrete mappings on oriented pseudomanifolds.* Let f, D, and Y be as in 2.4, and let $x \in X$. Then the local topological index $i(x, f)$ of f at x may be defined by

$$i(x, f) = N(f, U)$$

where U is any normal neighborhood of x. The definition of $i(x, f)$ is independent of the choice of the normal neighborhood U. Indeed, given two distinct normal neighborhoods U_1 and U_2 of x, choose a normal neighborhood U_3 of x with $U_3 \subset U_1 \cap U_2$. The mappings $f_i = f | U_i$, $i = 1, 2, 3$, are open, discrete, and closed; and since $\dim fB_f \leqslant n-2$ it follows by 2.3 that $N(f_1) = N(f_3) = N(f_2)$.

By definition $i(x, f) \geqslant 1$ for all $x \in D$ and $i(x, f) = 1$ for all $x \in D \setminus B_f$. If f is continuous, open, discrete and sense-reversing we set $i(x, f) = -N(f, U)$, where U is any normal neighborhood of x.

2.7. *Remark.* At points $x \in D \setminus \Gamma$ (see proof of 2.5) where D is locally euclidean the definition of the local topological index here is equivalent to the classical one, see [12, p. 125].

2.8. THEOREM. *Let Y be an oriented n-manifold with countable base of open sets, D a domain in the space X of an oriented pseudo n-manifold without boundary and $f\colon D \to Y$ continuous, open, sense-preserving, discrete, and closed. Then*

$$\sum_{x \in f^{-1}(y)} i(x, f) = N(f, D) < \infty$$

for all $y \in Y$.

Proof. By 2.5 $\dim fB_f \leqslant n-2$ and so by 2.3 $N(y, f, D) = N(f, D) < \infty$ for all $y \in fD \setminus fB_f$.

Given $y \in fD$ with $f^{-1}(y) \cap D = \{x_1, ..., x_k\}$, choose disjoint normal neighborhoods V_j of x_j and let U be the y-component of the open set $\bigcap fV_j \setminus f(D \setminus \cup V_j)$. Then $\{x_1, ..., x_k\} \subset f^{-1}U \subset \cup V_j$ and since $\operatorname{int} fB_f = \emptyset$, by 2.3, $U \setminus fB_f$ has a point y' for which

$$\sum_{x \in f^{-1}(y)} i(x, f) = \sum_{j=1}^{k} i(x_j, f) = \sum_{j=1}^{k} N(f, V_j) = \sum_{j=1}^{k} N(y', f, V_j) = N(f, D).$$

2.9. COROLLARY. *If $f\colon D \to Y$ is as in 2.4 and $D = X$, then Y is compact and*

$$\sum_{x \in f^{-1}(y)} i(x, f) = N(f) < \infty$$

for all $y \in Y$.

2.10. THEOREM. *Let Y be an oriented n-manifold with a countable base of open sets, $\hat{X}$ a space of finite oriented pseudo n-manifold K, $n \geq 2$, $P = \{p_1, ..., p_k\} \subset \hat{X}$, $X = \hat{X} \setminus P$, $\hat{f}: \hat{X} \to Y$ continuous, $f = \hat{f} | X: X \to Y$ open, discrete, and sense-preserving.*

Then $\hat{f}$ is open and discrete and $N(\hat{f}) = N(f) < \infty$.

Proof. Denote $X_1 = \hat{X} \setminus \hat{f}^{-1}\hat{f}P$ and $f_1 = \hat{f} | X_1$. $\hat{f}$ is closed and hence $f_1: X_1 \to f_1 X_1 = Y \setminus \hat{f}P$ is a closed mapping and therefore by 2.8, $N(f_1) < \infty$. Thus $N(y, \hat{f}) \leq N(f_1) + k < \infty$. Consequently $\hat{f}$ is discrete.

Since f is open, it suffices to show that each $p \in P$ has arbitrarily small neighborhoods V such that $\hat{f}(p) \in \operatorname{int} \hat{f}V$. Given a neighborhood U of p, choose a neighborhood $V \subset U$ of p such that $\bar{V} \cap P = \{p\} = \bar{V} \cap \hat{f}^{-1}\hat{f}(p)$. f is open, hence

$$\partial \hat{f}(V \setminus \{p\}) = \partial f(V \setminus \{p\}) \subset \hat{f}\partial V \cup \{\hat{f}(p)\}.$$

Since $\hat{f}\partial V$ and $\hat{f}(p)$ are disjoint and compact in Y, and Y is an n-manifold with $n \geq 2$, $\hat{f}(p)$ has a neighborhood W, homeomorphic to B^n, with $W \setminus \hat{f}(p) \subset \hat{f}(V \setminus \{p\})$. Thus $\hat{f}(p) \in \operatorname{int} \hat{f}V$ and consequently $\hat{f}$ is open.

As $\hat{f}$ is closed and $\dim \hat{f}B_{\hat{f}} \leq n-2$, we may apply 2.3 and conclude that $N(f) = N(f_1) = N(\hat{f})$.

3. Möbius group

3.1. In this chapter we introduce notation, terminology, and some facts about Möbius groups which are needed in the following sections.

3.2. *Möbius transformations* T in $\bar{R}^n$ are defined here as compositions of even number of reflections in $(n-1)$-spheres or $(n-1)$-planes in $\bar{R}^n$. Note that the group $GM(n)$ of all Möbius transformations in $\bar{R}^n$ consists of all sense-preserving conformal automorphisms of $\bar{R}^n$. The subgroups of $GM(n)$ will be called *Möbius groups*. The identity in $GM(n)$ is denoted by I. Let G be a Möbius group, $T \in G$ and $x \in \bar{R}^n$, then Fix T denotes the set of all fixed points of T, $\operatorname{Fix} G = \bigcup_{A \in G \setminus \{I\}} \operatorname{Fix} A$ and $G_x = \{A \in G: A(x) = x\}$ is the *stabilizer* of G at x.

A Möbius transformation $T \in GM(n)$ is called *parabolic* if T has a unique fixed point, called a parabolic point, in $\bar{R}^n$. T is called *loxodromic* if T has exactly two distinct fixed points $a, b \in \bar{R}^n$ and for some $x \in \bar{R}^n \setminus \{a, b\}$ the limit

$$\lim_{k \to \infty} T^k(x) = \lim_{k \to \infty} (\underbrace{T \circ T \circ \ldots \circ T}_{k \text{ times}})(x)$$

is either a or b. a and b are then called loxodromic points. All other elements of $GM(n)$ are called *elliptic*.

All parabolic and loxodromic transformations are of infinite order while elliptic ones may be either of finite or infinite order. The type (parabolic, loxodromic, or elliptic) of Möbius transformations is invariant under conjugation in $GM(n)$. Every parabolic transformation $T \in GM(n)$ is conjugate to a parabolic transformation P of the form $P(x) = Ax + h$, $x \in R^n$, where $A \in O(n) \cap GM(n)$ and $h \in R^n \setminus \{0\}$. T is said to be *strictly parabolic* if $A = I$.

3.3. *A Möbius group* G, is said to be *discrete* if no sequence of distinct elements of G converges (pointwise) to I, or, equivalently, to any $F \in GM(n)$. A point $x \in \bar{R}^n$ is called a *limit point* with respect to G if $T_k(a) \to x$ for some $a \in \bar{R}^n$ and some infinite sequence of distinct elements T_k in G. Other points in $\bar{R}^n$ are called *ordinary points* or *points of discontinuity* of G. The *limit set*, i.e., the set of all limit points will be denoted by $L = L(G)$ and its complement in $\bar{R}^n$, i.e., the *ordinary set* or the *set of discontinuity* of G, by O. O is open and L is either finite or perfect, see [17]. If $O \neq \emptyset$ and $a \in O$, then every point of L is in the cluster set of Ga, see [17]. G is said to be *discontinuous* if $O \neq \emptyset$. Discontinuity implies discreteness. The converse is not true in general; however, see [17, 3.3], if G is discrete and $GD = D$ for some domain $D \subset \bar{R}^n$ with $\operatorname{card} \partial D > 1$, then $L \subset \bar{R}^n \setminus D$. Thus discreteness and discontinuity are the same for Möbius groups which act on such domains D. We shall mostly consider the cases where D is either B^n or H^n.

Let G be discrete Möbius group acting on a domain $D \subset \bar{R}^n$ where D is either B^n or H^n. We say that two points x, $y \in \bar{D}$ or two sets A, $B \subset \bar{D}$ are *G-equivalent* if $y = T(x)$, or $B = T(A)$ respectively, for some $T \in G$. The *canonical projection* of D onto the orbit space D/G will be denoted by π. D/G is connected and its local structure is quite simple. More precisely, every point $z \in D/G$ has a neighborhood which is homeomorphic to B^n/Γ where Γ is a finite subgroup of $O(n) \cap GM(n)$. In fact, Γ is conjugate in $GM(n)$ to the stabilizer G_x where $x \in D \cap \pi^{-1}(z)$. In particular, if $n = 2$ or 3 or if $B^n \cap \operatorname{Fix} G = \emptyset$, then B^n/Γ is always homeomorphic to B^n and so D/G is an oriented manifold. In general D/G is not a manifold.

3.4. *Simple fundamental polyhedra.* Let G be a discrete Möbius group acting on D, where D is either B^n or H^n; and let $x_0 \in D \setminus \operatorname{Fix} G$. The *normal fundamental polyhedron* centered at x_0 is defined by

$$P = \{x \in D: d(x, x_0) < d(x, T(x_0)) \quad \text{for all } T \in G \setminus \{I\}\}.$$

P is a hyperbolic convex polyhedron. ∂P may have finite or infinite number of $(n-1)$-faces. Here we disregard the faces which lie on $\partial P \cap \partial D$. Each $(n-1)$-face is contained in a hyperbolic $(n-1)$-plane

$$H(T, x_0) = \{x \in D: d(x, x_0) = d(x, T(x_0))\} \tag{3.4.1}$$

for some $T \in G \setminus \{I\}$. The faces of P are pairwise G-equivalent by transformations T_1, $T_2, \ldots, \in G$, and G is generated by $T_1, T_2, \ldots$. The union $\tilde{P}$ of P with part of ∂P is a *fundamental set* for G in D. P with its G-equivalent faces being identified is homeomorphic to D/G.

The hyperbolic measure of Lebesgue measurable sets $A \subset B^n$ is defined by

$$V(A) = \int_A \frac{2^n dm(x)}{(1-|x|^2)^n},$$

and of sets $A \subset H^n$ by

$$V(A) = \int_A \frac{dm(x)}{x_n^n}.$$

The hyperbolic measure of sets $A \subset D/G$, $D = B^n$ or $D = H^n$ is defined by $V(P \cap \pi^{-1}(A))$ where $P \subset D$ is any normal fundamental polyhedron for G. With this normalization $V(D/G) = V(P)$.

If $V(D/G) < \infty$, then every normal fundamental polyhedron P in D has finitely many $(n-1)$-faces and $\bar{P} \cap \partial D$ is either empty or consists of finitely many points, depending on whether D/G is compact or non-compact, see [15] and [3]. In the latter case the points of $\bar{P} \cap \partial D$ are called *boundary vertices*.

Suppose that D/G is non-compact and $V(D/G) < \infty$. A normal fundamental polyhedron P in D will be called *simple* if no two boundary vertices of P are G-equivalent. Other properties of simple fundamental polyhedra are described in the following lemma.

3.5. LEMMA. *Let G be a discrete Möbius group acting on D, $D = B^n$ or H^n. Suppose that D/G is non-compact and $V(D/G) < \infty$. Then*

(i) *every boundary vertex of a normal fundamental polyhedron P is a fixed point for a strictly parabolic element of G;*

(ii) *there is a set $\Sigma \subset D$ with $m_n(\Sigma) = 0$ such that every normal fundamental polyhedron centered in $D \setminus \Sigma$ is simple;*

(iii) *if P is a simple fundamental polyhedron, then every parabolic fixed point of G is G-equivalent to exactly one boundary vertex of P;*

(iv) *every parabolic fixed point is a boundary vertex of a simple fundamental polyhedron;*

(v) *if P and Q are simple fundamental polyhedra with boundary vertices $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_m\}$, respectively, then $k = m$ and each p_i is G-equivalent to one q_j.*

Proof. For (i) see Garland and Raghunathan [3] or Wielenberg [22]. For a proof of (ii) in R^3, see A. Marden [6, 4.2]. With the notion of isometric spheres this proof can be extended to R^n, $n \geq 2$ as follows. Let Π denote the set of all parabolic fixed points of G. For each $p \in \Pi$ and $T \in G \setminus G_p$, let $\Sigma_{p,T}$ denote the set of all points x_0 in D with the property that $p \in \bar{H}(T, x_0)$, see (3.4.1). Note that if $D = H^n$ and $p = \infty$, then $\Sigma_{p,T} = H^n \cap I(T)$, where $I(T) = \{x \in R^n: |T'(x)| = 1\}$ is the isometric sphere of T, otherwise $\Sigma_{p,T} = A^{-1}(\Sigma_{\infty, ATA^{-1}})$, where A is a Möbius transformation with $A(D) = H^n$ and $A(p) = \infty$. In any case $\Sigma_{p,T}$ is part of an $(n-1)$-sphere or an $(n-1)$-plane and $m_n(\Sigma_{p,T}) = 0$. G and Π are countable and $m_n(\text{Fix } T) = 0$ for all $T \in G \setminus \{I\}$, hence

$$\Sigma = (\text{Fix } G) \cup [\bigcup_{p \in \Pi} (\bigcup_{T \in G \setminus G_p} \Sigma_{p,T})]$$

is of measure zero. Moreover, by (i) each boundary vertex of a normal fundamental polyhedron is a parabolic fixed point, hence no $(n-1)$-face $H(T, x_0)$ of a normal fundamental polyhedron P centered at $x_0 \in D \setminus \Sigma$ passes through a boundary vertex p of P unless $T(p) = p$. This proves that P is simple.

(iii) The following argument is due to Leon Greenberg. Let $A \in G$ be a parabolic transformation with a fixed point $p \in \partial D$. Since P has finitely many faces, there exists $\delta > 0$ such that Fix $T \subset \partial P$ whenever $x \in P$ and $T \in G$ are such that T is parabolic and $d(x, T(x)) < \delta$. Indeed, we can first find $\delta_1 > 0$ such that $d(x, T(x)) < \delta_1$, $x \in P$, $T \in G$, implies that either T fixes a boundary vertex of P or else T maps one $(n-1)$-face of P onto another face of P. If such a transformation T is parabolic and does not fix a boundary vertex of P, then $\delta_T = \inf_{x \in P} d(x, T(x)) > 0$. Finally δ can be chosen to be the smallest of these finitely many δ's. Now choose a point $x \in D$ and $g \in G$ such that $d(x, A(x)) < \delta$ and $g(x) \in P$. Then $g \circ A \circ g^{-1}$ is parabolic and $d(g(x), g \circ A \circ g^{-1} g(x)) < \delta$, hence $g(p) = \text{Fix } (g \circ A \circ g^{-1})$ is a boundary vertex of P. Finally, note that P is simple and so p cannot be G-equivalent to any other boundary vertex of P.

(iv) Let $p \in \partial D$ be a parabolic fixed point for G. By (ii) G has at least one simple fundamental polyhedron, say P, and by (iii) $g(p) \in \bar{P}$ for some $g \in G$, thus $g^{-1}(P)$ is a simple fundamental polyhedron with a boundary vertex at p.

(v) By (i) $p_1, \ldots, p_k$ and $q_1, \ldots, q_m$ are parabolic; hence, by (iii) each p_i is G-equivalent to some q_j and vice versa. Since P and Q are simple, no two boundary vertices of P (resp. Q) are G-equivalent, and so (v) follows.

3.6. *The action of a Möbius group near a parabolic point.* Let G be a discrete Möbius group acting on H^n with $V(H^n/G) < \infty$ and suppose that ∞ is fixed for a parabolic element of G. By Lemma 3.5 (iv), G has a simple fundamental polyhedron P with a vertex at ∞, and since $V(H^n/G) < \infty$, P has finitely many faces. Consequently, P has no vertices in the half space $H(h_0)$ for some $h_0 > 0$. The stabilizer G_∞ is a discrete group of euclidean isometries mapping each plane $\partial H(h)$ onto itself in the same manner. $P \cap \partial H(h)$ is a compact $(n-1)$-dimensional euclidean polyhedron of finite diameter which may serve as a fundamental polyhedron for the action of G_∞ on $H(h)$. It thus follows that for points $a \in \partial H(h)$, $h \geq h_0$,

$$\lim_{r\to\infty} \operatorname{card} \{Ga \cap \partial H(h) \cap B^n(a, r)\} = \infty \tag{3.6.1}$$

and that for every $r \in (0, \infty)$ there exists an integer k, depending on r and on the polyhedron P, such that each ball of radius r in $H(h_0)$ can be covered by k G_∞-images of $\bar{P}$. Note, also, that $T(H(h_0)) \cap H(h_0) = \emptyset$ for $T \in G \setminus G_\infty$.

3.7. LEMMA. *Let G be a discrete Möbius group acting on D, $D = B^n$ or H^n with $V(D/G) < \infty$, P a simple fundamental polyhedron, and b a point in ∂D. Then for every neighborhood V of b there is $g \in G$ such that $g(P) \subset V$.*

Proof. If D/G is compact, then P has a finite euclidean diameter; and since the G-images of P cluster at b, the result follows.

If D/G is non-compact, then P has a boundary vertex p which is fixed under a parabolic transformation $A \in G$. Since every point $b \in \partial D$ is a limit point with respect to G and since $G_p \neq G$, it follows, see [17, 3.7], that the G-images of p cluster at every point of ∂D, and in view of 3.6 it is clear that $g \circ A^k(P) \subset V$ for some $g \in G$ and some integer k.

3.8. *The cusp compactification of D/G.* Let G be a discrete Möbius group acting on D, $D = B^n$ or H^n with $V(D/G) < \infty$. $M = D/G$ is either compact or has a natural compactification $\hat{M}$, called the *cusp compactification* and defined as follows. Choose a simple fundamental polyhedron P. Let $Q = \{p_1, ..., p_k\}$ denote the set of all boundary vertices of P. Then $\hat{M} = M \cup Q$, where bases for open neighborhoods at the points $p \in Q$ are given by the families $\{\pi(U(m, p)): m = 1, 2, ...\}$, where $U(m, p) = \{x \in \tilde{P}: |x-p| < 1/m\}$ for $p \in Q \setminus \{\infty\}$ and $U(m, p) = \{x \in \tilde{P}: |x| > m\}$ if $p \in Q \cap \{\infty\}$.

Let Π denote the set of all parabolic fixed points of G. Then, by Lemma 3.5, $\hat{M} = (D \cup \Pi)/G$ holds as a set equivalence, and by the same lemma the space $\hat{M}$ is independent of the choice of P.

In general $\hat{M}$ or even M are not manifolds. The topological nature of $\hat{M}$ is described in the following lemma.

3.9. LEMMA. *$\hat{M}$ is a space of a compact oriented pseudomanifold without boundary, see* 3.8 *and* 2.4.

Proof. We may assume that $D = B^n$ and, by using auxiliary Möbius transformations, that G has a simple fundamental polyhedron P centered at 0. Then $P = B^n \setminus \bigcup_{i=1}^k \bar{B}_i$

where each B_i is a ball orthogonal to S^{n-1}. If B_i, B_j, $i \neq j$, are two such balls, then $\partial B_i \cap \partial B_j$ is contained in a $(n-1)$-plane through 0. Thus we can triangulate the faces of P by using a finite number of planes through 0 so that G-equivalent faces of P have G-equivalent triangulation. Now, this triangulation and 0 as a common vertex define a triangulation of P so that each n-cell has n $(n-1)$-faces lying on planes through 0 and one $(n-1)$-face on some ∂B_i. This triangulation clearly defines a triangulation of $\hat{M}$. Now 2.4 (i) and (ii) are automatically satisfied and 2.4 (iii) holds since the elements of G are sense-preserving.

4. Automorphic mappings: Examples and general properties

4.1. A continuous, open, discrete, and sense-preserving mapping $f\colon D \to \bar{R}^n$ is said to be *automorphic* for a Möbius group G if $f \circ A = f$ for all $A \in G$. It is not hard to see that if $f\colon D \to \bar{R}^n$ is automorphic for G, then G is discontinuous, $GD = D$ and D is a subset of an invariant connected component Σ of the ordinary set O. Consequently G is a function group. Furthermore, since G a clusters at all points of L, whenever $a \in O$, see [17], it follows that $L \subset \partial D$ and that each point of L is an *essential singularity* of f, i.e., there exists no limit of $f(x)$ as $x \in D$ tends to $b \in L$.

The simplest non-trivial automorphic mappings are the periodic mappings. These mappings are invariant under discrete groups of similarities in R^n with a unique limit point at ∞. See [11] for more details.

We first present two examples of qr automorphic mappings $f\colon H^3 \to R^3$ for groups G acting on H^3. In the first example G has exactly two limit points and $B_f = \emptyset$. In the second example every point on ∂H^3 is a limit point for G and thus ∂H^3 is the natural boundary of f.

4.2. *Example.* Let G be the cyclic Möbius group generated by the stretching $x \to 2x$, $x \in R^3$. $G(H^3) = H^3$ and $P = \{x \in H^3\colon 1 < |x| < 2\}$ is a fundamental domain for the restriction of G to H^3. Denote $S_k = \{x = H^3\colon |x| = 2^k\}$, $k = 0, \pm 1, \pm 2, \ldots$. $\bar{P}$ is homeomorphic to $\bar{B}^3$, and there is a continuous mapping $f\colon \bar{P} \to R^3$ such that $f|P$ is qc, $f(\bar{P})$ is a closed solid torus and $f(x) = f(2x) \in \partial H^3$ for $x \in S_0$. In particular $f(S_0) = f(S_1) \subset \partial H^3$. Now extend f to the G images of $\bar{P}$ by means of inversions in S_k and reflections in $f(S_k) \subset \partial H^3$. The extensions are denoted again by f. Note that a reflection in S_k maps S_{k-1} onto S_{k+1}. We thus end with a qr mapping $f\colon H^3 \to R^3$, which is automorphic for G. Note that $f(H^3)$ is bounded, $N(f, H^3) = \infty$, f has exactly two essential singularities 0 and ∞ and $B_f = \emptyset$, i.e., f is a local homeomorphism.

4.3. *Example.* Let P_1 be the semi-infinite prism in H^3 which is bounded by the planes ∂H^3, $\pi_i = \{x \in R^3\colon x_i = 0\}$, $i = 1, 2$, and $\pi_3 = \{x \in R^3\colon x_1 + x_2 = 1\}$; P_2 be the prism in H^3 bounded by ∂H^3, π_1, π_2 and the plane $\pi_4=\{x\in R^3: x_1-x_2=1\}$. Denote $D_i=P_i\setminus\bar{B}^3$, $i=1, 2$, $D=\operatorname{int}(\bar{D}_1\cup\bar{D}_2)$, $\tilde{D}=\bar{D}_1\cup D_2$. The vertices $(1, 0, 0)$, $(0, 1, 0)$, ∞ and $(0, 0, 1)$ of D_1 are denoted by p_i, $i=1, ..., 4$, respectively.

Let G be the Möbius group which is generated by the following transformations: T_1=rotation of $90°$ about the line $l_1=\pi_1\cap\pi_3=\{x\in R^3: x_1=0, x_2=1\}$, T_2=rotation of $180°$ about the circle $l_2=\partial B^3\cap\pi_1=\{x\in R^3: |x|=1, x_1=0\}$ (T_2 can be obtained by composing an inversion in ∂B^3 and a reflection in π_1), T_3=rotation of $180°$ about the line $l_3=\pi_1\cap\pi_2=\{x\in R^3: x_1=x_2=0\}$. T_1, T_2, T_3 are elliptic of orders 4, 2, and 2 with the fixed sets l_1, l_2, and l_3, respectively.

The faces of D are pairwise congruent by T_k, $k=1, 2, 3$ (we consider $\partial B^3\cap\partial D_i$ and $\pi_2\cap D_i$, $i=1, 2$ as two pairs of distinct faces of D). Each T_k maps H^3 onto itself. The angle between any two adjacent faces of D is either $180°$, $90°$, or $45°$. Moreover, the three edges of D_1 which meet at $p_4=(0, 0, 1)$ are mutually orthogonal at p_4. It thus follows that G is a discrete group acting on H^3, D is a fundamental domain for G, $V(H^3/G)=V(D)<\infty$, and $L=\partial H^3$.

We now construct a qr mapping $f: H^3\to R^3$ which is automorphic with respect to G. Noting that the neighborhoods of p_i, $i=1, 2$, rel. D_1 are conformally equivalent to a neighborhood of $p_3=\infty$ rel. D_1, it is not hard to apply [4, sec 8] to conclude that there is a continuous and injective mapping $f: \bar{D}_1\to\bar{H}^3$ such that $f|D_1$ is qc, $f\partial D_1=\partial H^3$ and $f(\infty)=\infty$. Denote $f(p_i)=a_i$ $i=1, ..., 4$. Now extend f to $\tilde{D}$ by means of reflections in π_2 and ∂H^3, respectively, and then continue and extend f infinitely many times by means of all possible reflections and inversions with respect to ∂H^3 and to the faces of TD_1, $T\in G$, respectively. We end with a qm mapping, denoted again by f, which is well defined everywhere in H^3, $fH^3=\bar{R}^3\setminus\{a_1, a_2, a_3\}=R^3\setminus\{a_1, a_2\}$. f is automorphic with respect to G, and since $L=\partial H^3$, it follows that every point in ∂H^3 is an essential singularity of f; and so ∂H^3 is the natural boundary of f. $B_f=\bigcup_{T\in G}\bigcup_{i=1}^{6}T(s_i)$ where s_i, $i=1, ..., 6$, are the edges of D_1. fB_f is rather simple. It consists of a topological triangle with vertices at the points a_1, a_2, and a_3 and three curves which extend from a_i, $i=1, 2, 3$, to ∞.

In some respect, f is analogous to the elliptic modular function in R^2.

4.4. *Remark.* The last two examples prove that the class of qm automorphic mappings in R^3 is not empty. In a forthcoming paper we shall prove that every discrete Möbius group which acts on D, $D=B^n$ or H^n, with $V(D/G)<\infty$, $n\geq 2$, has qm automorphic mappings. We do not know whether every discrete Möbius group acting on B^n has a qm automorphic mapping.

4.5. Let G be a discrete Möbius group acting on D, $D=B^n$ or H^n, with $V(D/G)<\infty$,

$P \subset D$ a simple fundamental polyhedron and suppose that $\bar{P} \cap \partial D \neq \emptyset$. Let $Q = \{p_1, ..., p_k\}$ be the set of all boundary vertices of P. Finally, let $f: D \to \bar{R}^n$ be an automorphic mapping with respect to G.

We say that f *has the limits* $a_1, ..., a_k$ *in* P if $\lim f(x) = a_i$ as $x \to p_i$ in $\tilde{P}$, $i = 1, ..., k$. The existence of the limits a_i, $i = 1, ..., k$ in P implies that the induced mapping $\mathring{f}: M \to \bar{R}^n$, $M = D/G$, which satisfies $f = \mathring{f} \circ \pi$, has a continuous extension $\hat{f}: \hat{M} \to \bar{R}^n$ with $\hat{f}(p_i) = a_i$, $i = 1, ..., k$.

Since π is continuous, G is discontinuous in D, and f is open and discrete, it follows that $\mathring{f}$ is open and discrete, and thus Theorem 2.10 yields that $\hat{f}$ is open and discrete and that

$$N(\hat{f}) = N(\mathring{f}) = N(f, \tilde{P}) < \infty. \tag{4.5.1}$$

Furthermore, we may use 2.6 to define the local topological index $i(x, \hat{f})$ of $\hat{f}$ at points $x \in \hat{M}$.

It is not hard to see that

$$i(x, f) = N(x, G)\, i((i \circ \pi)(x), \hat{f}) \tag{4.5.2}$$

for $x \in B^n$, where $N(x, G) = \operatorname{card} G_x$ and $i: M \to M$. Moreover by Corollary 2.9 and (4.5.1) it follows that

$$\sum_{x \in \hat{f}^{-1}(y)} i(x, \hat{f}) = N(f, \tilde{P}) < \infty \tag{4.5.3}$$

for all $y \in \bar{R}^n$.

5. Modulus and capacity inequalities in B^n/G

In this chapter we prove several modulus inequalities in $M = D/G$ where D is either B^n or H^n and G is a discrete Möbius group acting in D with $V(D/G) < \infty$. These inequalities will be used in chapter 6.

5.1. *Modulus of path families in* B^n/G. Let G be a discrete Möbius group acting on B^n with $V(B^n/G) < \infty$. Let Γ be a path family in $M = B^n/G$, i.e., each $\gamma \in \Gamma$ is a non-constant continuous mapping $\gamma: \Delta \to M$ where $\Delta \subset R^1$ is an interval. The modulus $M_G(\Gamma)$ of Γ is defined by

$$M_G(\Gamma) = \inf_{\varrho \in F_G(\Gamma)} \int_P \varrho^n dm. \tag{5.1.1}$$

where P is a simple fundamental polyhedron for G, see 3.4, and $F_G(\Gamma)$ is the set of all non-negative Borel-functions $\varrho: B^n \to \dot{R}^1$ such that

$$\varrho(x) = \varrho(T(x))\,|T'(x)| \tag{5.1.2}$$

for all $T \in G$ and

$$\int_{\gamma^*} \varrho\, ds \geq 1 \tag{5.1.3}$$

where $\gamma^*: \Delta \to B^n$ is any locally rectifiable (maximal) lift of γ under the canonical projection $\pi: B^n \to B^n/G$, i.e., $\pi \circ \gamma^* = \gamma$. For more details on the line integral (5.1.3) see [19, sec. 4]. Note that $M_G(\Gamma)$ is independent of the choice of the fundamental polyhedron P. In fact

$$\int_D \varrho^n dm = \int_P \varrho^n dm \tag{5.1.4}$$

where D is any measurable fundamental set for G. To show this let $G = \{T_1, T_2, \ldots\}$, $D_i = D \cap T_i(\tilde{P})$, and $P_i = T_i^{-1}(D_i)$, $i = 1, 2, \ldots$. Now $D_i \cap D_j = \emptyset$ for $i \neq j$ and $D = \cup D_i$ since $\tilde{P}$ is a fundamental set. The same reason yields $P_i \cap P_j = \emptyset$ for $i \neq j$ and $\tilde{P} = \cup P_i$. Now (5.1.2) implies

$$\begin{aligned}\int_D \varrho^n dm &= \sum_i \int_{D_i} \varrho^n dm = \sum_i \int_{P_i} \varrho(T_i(x))^n J(x, T_i)\, dm(x) = \sum_i \int_{P_i} \varrho(T_i(x))^n |T_i'(x)|^n dm(x) \\ &= \sum_i \int_{P_i} \varrho^n dm = \int_{\tilde{P}} \varrho^n m = \int_P \varrho^n dm.\end{aligned}$$

Note also that if (5.1.3) holds for some lift γ^* then it holds for all lifts. Indeed, if $\gamma_1^*: \Delta \to B^n$ is another lift of γ, then the subarcs $\gamma^*(\Delta) \cap T_i(\tilde{P})$, $T_i \in G$, are in one to one correspondence with the subarcs $\gamma_1^*(\Delta) \cap T_j(\tilde{P})$, $T_j \in G$. Since corresponding subarcs are G-equivalent, the result follows from (5.1.2) and the transformation formula for line integrals, see [19, sec. 4]. Observe that in general $\pi: B^n \to B^n/G$ need not be a covering mapping, hence there may not exist $T \in G$ with $T \circ \gamma_1^* = \gamma^*$.

The same definitions hold when B^n is replaced by H^n.

5.2. A path $\gamma: \Delta \to M$ is said to be rectifiable or absolutely continuous, if it has a lift $\gamma^*: \Delta \to B^n$ which is rectifiable or absolutely continuous, respectively. Clearly these properties do not depend on the choice of the lift. In the same way a set $A \subset M$ is called measurable, a set of measure zero, or a Borel set if $\pi^{-1}(A)$ has a corresponding property in B^n with respect to the Lebesgue measure and the usual topology of B^n. Recall that $\hat{M}$ denotes the cusp compactification of M and $i: M \to \hat{M}$ the natural inclusion.

5.3. LEMMA. *Let Γ be a family of paths $\gamma: [a, b) \to M$ such that either γ is nonrectifiable or $i \circ \gamma(t)$ tends to a limit in $\hat{M} \setminus iM$ as $t \to b$. Then $M_G(\Gamma) = 0$.*

Proof. Let Γ' be the family of all the paths in Γ which satisfy the second condition stated in the lemma. Then $M_G(\Gamma)=M_G(\Gamma')$, for if $\gamma\in\Gamma\setminus\Gamma'$, then the function $\varrho(y)=\varepsilon|T'(y)|$, $T(y)\in\tilde{P}$, $T\in G$, belongs to $F_G(\Gamma\setminus\Gamma')$. Hence

$$M_G(\Gamma\setminus\Gamma')\leqslant\int_P\varepsilon^n dm\leqslant\varepsilon^n m(P)$$

which shows $M_G(\Gamma\setminus\Gamma')=0$. To prove that $M_G(\Gamma')=0$, let the boundary vertices of a simple fundamental polyhedron P be $\{p_1,\ldots,p_k\}\subset\partial B^n$. Pick $A_i\in GM(n)$ such that $A_i(B^n)=H^n$ and $A_i(p_i)=\infty$. Let $h_0>0$ be so large that for all i, $H(h_0)$ meets only those faces of $A_i(P)$ which terminate at ∞, see 3.6. Clearly, it is enough to show that for large h, $M(\Gamma_h)=0$ where $\Gamma_h=\{\gamma\in\Gamma': A_i(\gamma^*(a))\in\complement H(h)\text{ for some }i\}$. Fix $h\geqslant h_0$ so large that the sets $\tilde{P}\cap A_i^{-1}(H(h))$ are disjoint. Let $\varepsilon>0$. Denote $\tilde{P}_i=\tilde{P}\cap A_i^{-1}(H(h)\setminus H(h+1/\varepsilon))$. Define $\varrho_\varepsilon:\tilde{P}\to R^1$ as $\varrho_\varepsilon(x)=\varepsilon|A_i'(x)|$, $x\in\tilde{P}_i$, and $\varrho_\varepsilon(x)=0$, $x\in\tilde{P}\setminus\cup\tilde{P}_i$. Extend ϱ_ε to $\varrho_\varepsilon^*: B^n\to R^1$ by setting $\varrho_\varepsilon^*(y)=\varrho_\varepsilon(x)|T'(y)|$ where $T\in G$ is such that $T(y)=x$, $x\in\tilde{P}$. Now $\varrho_\varepsilon^*\in F_G(\Gamma_h)$, since if γ^* is any rectifiable lift of $\gamma\in\Gamma_h$ terminating at p_i, then

$$\int_{\gamma^*}\varrho_\varepsilon^* ds=\int_{A_i^{-1}\circ\gamma_i^*}\varrho_\varepsilon^* ds=\int_{\gamma_i^*}\varrho_\varepsilon^*(A_i^{-1}(x))|A_i^{-1\prime}(x)||dx|\geqslant\varepsilon\int_h^{h+1/\varepsilon}|dx|=1$$

where $A_i\circ\gamma^*=\gamma_i^*$. On the other hand

$$\int_P\varrho_\varepsilon^{*n}dm=\varepsilon^n\sum_{i=1}^k\int_{\tilde{P}_i}|A_i'(x)|^n dm(x)=\varepsilon^n\sum_{i=1}^k\int_{\tilde{P}_i}J(x,A_i)\,dm(x)=\varepsilon^n\sum_{i=1}^k m(A_i(\tilde{P}_i))\leqslant\varepsilon^{n-1}C$$

where, by 3.6, C is independent of ε. The lemma follows.

5.4. *Capacity of condensers in B^n/G or in B^n/G.* Let G and $M=B^n/G$ be as in 5.1. A pair $E=(A,C)$ is called a *condenser* in M if A is open in M and C is a non-empty compact subset of A. The *capacity* $\operatorname{cap}_G E$ of E is defined as follows: If $C=M$ we let $\operatorname{cap}_G E=0$, otherwise $\operatorname{cap}_G E=M_G(\Gamma_E)$ where Γ_E is the family of all paths $\gamma:[a,b)\to A$ in M such that $\gamma(a)\in\partial C$ and $i\circ\gamma(t)\to\partial i(A)$ as $t\to b$. Here i is the natural inclusion of M into its cusp compactification $\hat{M}$.

A pair $E=(A,C)$ is called a condenser in $\hat{M}$ if A is open in $\hat{M}$ and $C\neq\emptyset$ is compact in A. Furthermore, for the sake of simplicity we shall always assume that ∂C is compact in $i(M)$. Such a compact set C will be called *admissible*. The capacity of E is defined by $\operatorname{cap}_G E=\operatorname{cap}_G(i^{-1}A, i^{-1}\partial C)$.

In the same way we define condensers and their capacities in H^n/G and in $H^{n^\wedge}/G$ if G acts on H^n.

5.5. *Some special condensers in B^n/G or in B^{n^*}/G.* Lemma 5.3 implies that $\operatorname{cap}_G(M, C) = 0 = \operatorname{cap}_G(\hat{M}, C)$ for all admissible compact sets C in M or in $\hat{M}$, respectively. The same lemma also implies that $\operatorname{cap}_G(A, C) = \operatorname{cap}_G(i^{-1}A, i^{-1}C)$ for any condenser (A, C) in $\hat{M}$ such that $i^{-1}C$ is compact in $i^{-1}A$.

5.6. Suppose now that G acts on H^n, that P is a simple fundamental polyhedron with a boundary vertex at ∞, and that $E = (A, C)$ is a condenser in $M = H^{n^*}/G$. Note that ∞ is a point in $\hat{M}$ as well as in $\bar{R}^n$.

LEMMA. *There exists $h_0 > 0$ such that if*

(i) *C and $\complement A$ are connected,*

(ii) *$\infty \in C$ and $\partial H(h_0) \subset (i \circ \pi)^{-1} \complement A$,*

(iii) *$(i\circ\pi)^{-1}C$ and $(i\circ\pi)^{-1}\complement A$ both meet $\partial H(h)$ for some $h \geq h_0$,*

then $\operatorname{cap}_G E \geq \delta$ for some $\delta > 0$ which is independent of h.

Proof. Let $h' > 0$ be such that $H(h')$ only meets those faces of ∂P which terminate at ∞ and let $d = \operatorname{diam}(P \cap \partial H(h')) = \operatorname{diam}(P \cap \partial H(h_1))$, $h_1 \geq h'$, see 3.6. Define $h_0 = h' + 2d$ and let $h \geq h_0$.

Fix $x_0 \in \partial H(h) \cap P$. By 3.6 there exists an integer k depending only on G such that

$$B^n(x_0, 2d) \subset \operatorname{int} \sum_{i=1}^{k} T_i(\tilde{P}) = S,$$

where $T_i \in G$ and $T_i(\infty) = \infty$.

Let $E' = (i^{-1}A, i^{-1}\partial C)$ and let $\Gamma = \Gamma_{E'}$. Fix $\varrho \in F_G(\Gamma)$. Then by (5.1.2)

$$\int_P \varrho^n dm = \frac{1}{k} \int_S \varrho^n dm.$$

The proof will now be completed if we find a lower bound for $\int_S \varrho^n dm$.

Let $d < r < 2d$. Then (i)–(iii) imply that $S^{n-1}(x_0, r)$ meets both $(i\circ\pi)^{-1}\partial C$ and $(i\circ\pi)^{-1}A$. Hence [19, sec. 10] yields

$$\int_{S^{n-1}(x_0, r)} \varrho^n ds \geq d_n/r,$$

where $d_n > 0$ depends only on n. Integrating from d to $2d$ gives

$$\int_S \varrho^n dm \geq d_n \log 2.$$

Since $\varrho \in F_G(\Gamma)$ was arbitrary, this gives the required estimate.

5.7. *Capacity of condensers in $\bar{R}^n$.* A pair $E=(A, C)$ is called a condenser in $\bar{R}^n$ if A is open in $\bar{R}^n$ and C is a non-empty compact subset of A, cf. [8, 5.2], [9, 2.7]. We define the capacity cap E of E as follows: If $A=C$ we let cap $E=0$, otherwise, we let cap $E=M(\Gamma_E)$ where Γ_E is the family of all paths which join C and ∂A in A, i.e., paths γ: $[a, b)\to A$ such that $\gamma(a)\in C$ and $\gamma(t)\to\partial A$ as $t\to b$. Here $M(\Gamma_E)$ denotes the usual modulus of Γ_E, see [19]. This definition is equivalent to the potential theoretic definition in [8, 5.4].

5.8. *Condensers and automorphic mappings.* Let G and $M=B^n/G$ be as in 5.2. Suppose that f: $B^n\to\bar{R}^n$ is a *qm* automorphic mapping with respect to G. The induced mapping $\tilde{f}$: $M\to\bar{R}^n$ is then continuous and open, thus, if $E=(A, C)$ is a condenser in M, then $\tilde{f}E=(\tilde{f}A, \tilde{f}C)$ is a condenser in $\bar{R}^n$. If, in addition, $\tilde{f}$ has a continuous extension $\hat{f}$: $\hat{M}\to\bar{R}^n$, then by 2.10 $\hat{f}$ is open. It thus follows that for any condenser $E=(A, C)$ in $\hat{M}$, $\hat{f}E=(\hat{f}A, \hat{f}C)$ is a condenser in $\bar{R}^n$.

A condenser $E=(A, C)$ in M (resp. in $\hat{M}$) is called *normal* if A is a normal domain of $\tilde{f}$ (resp. $\hat{f}$). In this case $\tilde{f}$ (resp. $\hat{f}$) defines a closed mapping $A\to\tilde{f}A$ (resp. $A\to\hat{f}A$).

Let $E=(A, C)$ be a condenser in M. The *minimal multiplicity* of $\tilde{f}$ on C is defined by

$$M(\tilde{f}, C)=\inf_{y\in\tilde{f}C}\ \sum_{x\in\tilde{f}^{-1}(y)\cap C} i(x, \tilde{f}) \tag{5.8.1}$$

see [7, 3.5]. Clearly $M(\tilde{f}, C)<\infty$ and $M(\tilde{f}, C)\leqslant N(\tilde{f}, A)$. If $\tilde{f}$ has a continuous extension $\hat{f}$ to M, then $M(\hat{f}, C)$ is defined for condensers $E=(A, C)$ in $\hat{M}$ in the same way. Observe that in this case $N(\hat{f}, A)$ is always finite.

5.9. THEOREM. *Suppose that f: $B^n\to R^n$ is a qm automorphic mapping for a discrete group G acting on B^n with $V(M)<\infty$, $M=B^n/G$. Then*

$$\operatorname{cap}_G E\leqslant K_0(f)N(\tilde{f}, A)\operatorname{cap}\tilde{f}E \tag{5.9.1}$$

for all normal condensers $E=(A, C)$ in M, and

$$\operatorname{cap}\tilde{f}E\leqslant\frac{K_I(f)}{M(\tilde{f}, C)}\operatorname{cap}_G E \tag{5.9.2}$$

for all condensers $E=(A, C)$ in M. Furthermore, if $\tilde{f}$ has a continuous extension $\hat{f}$ to $\hat{M}$, then (5.9.1) *and* (5.9.2) *hold with $\tilde{f}$ and M being replaced by $\hat{f}$ and $\hat{M}$, respectively.*

Proof. Since the details of the proof are similar to those in [8, 6.4], [7], and [20, 3.17], we shall only give an outline.

For (5.9.1) suppose that $E=(A, C)$ is a normal condenser in M. If $A=C$ there is nothing to prove. Suppose $A\neq C$ and let $\varrho'\in F(\Gamma_{\tilde{f}E})$. Define ϱ: $B^n\to\dot{R}^1$ by setting $\varrho(x)=$

$\varrho'(f(x))L(x, f)$ for $x \in \pi^{-1}(A)$ and $\varrho(x) = 0$ otherwise. Here $L(x, f) = \overline{\lim}_{h\to 0} |f(x+h) - f(x)|/|h|$. Let Γ_0 be the family of all paths $\gamma \in \Gamma_E$ such that the corresponding lift γ^* in B^n is rectifiable and f is absolutely continuous on γ^*. By Fuglede's theorem [2], see also [19], $M_G(\Gamma_0) = M_G(\Gamma_E)$. The change of variables in a line integral shows that

$$\int_{\gamma^*} \varrho \, ds \geq \int_{\tilde{f} \circ \gamma} \varrho' ds. \tag{5.9.3}$$

A is a normal domain of $\tilde{f}$, hence $\tilde{f} \circ \gamma \in \Gamma_{\tilde{f}E}$ and thus the right-hand side of (5.9.3) is ≥ 1. On the other hand

$$\varrho(x) = \varrho'(f(x))L(x, f) = \varrho'(f(T(x)))L(x, f \circ T) = \varrho'(f(T(x)))L(T(x), f)\,|T'(x)| = \varrho(T(x))\,|T'(x)|$$

for all $x \in \pi^{-1}(A)$ and $T \in G$. Since the last formula holds trivially for $x \in B^n \setminus \pi^{-1}(A)$, it follows that $\varrho \in F_G(\Gamma_0)$. This gives

$$\begin{aligned} \operatorname{cap}_G E = M_G(\Gamma) = M_G(\Gamma_0) &\leq \int_P \varrho^n dm = \int_{P \cap \pi^{-1}(A)} \varrho'(f(x))^n L(x, f)\, dm(x) \\ &\leq K_0(f) \int_{P \cap \pi^{-1}(A)} \varrho'(f(x))^n J(x, f)\, dm(x) \\ &\leq K_0(f) \int_{\tilde{f} A} \varrho'(y)^n N(y, \tilde{f}, A)\, dm(y) \leq K_0(f)\, N(\tilde{f}, A) \int_{R^n} \varrho' dm. \end{aligned}$$

Since this holds for every $\varrho' \in F(\Gamma_{\tilde{f}E})$, (5.9.1) is proved. Clearly the same proof applies when $\tilde{f}$ and M are replaced by $\hat{f}$ and $\hat{M}$.

For (5.9.2) set $m = M(\tilde{f}, C)$ and let $\gamma'\colon [a, b) \to \tilde{f}A$ be a path in $\Gamma_{\tilde{f}E}$. Then $C \cap \tilde{f}^{-1}(\gamma'(a))$ consists of finitely many points $x_1, \ldots, x_k$ with $\sum_{j=1}^{k} i(x_j, \tilde{f}) \geq m$. A result by Rickman [14] on path lifting, see also [20, 3.12], implies the existence of paths $\gamma_j\colon [a, b_j) \to A$ in Γ_E with $\tilde{f} \circ \gamma_j \subset \gamma'$ and such that card $\{j\colon \gamma_j(t) = x\} \leq 1$ whenever $x \in M \setminus B_{\tilde{f}}$ and $t \in \bigcup_j [a, b_j)$.

Let $\varrho \in F_G(\Gamma_E)$. Denote by E the set of all points $x \in B^n$ where f is differentiable and $J(x, f) > 0$. Then E is a Borel set, $m(B^n \setminus E) = 0$, and $B_f \subset B^n \setminus E$ by [8, 2.26 and 8.2]. Define $\varrho'\colon R^n \to \dot{R}^1$ by

$$\varrho'(y) = \begin{cases} m^{-1} \sup\limits_{B} \sum\limits_{x \in B} \varrho(x)/l(f'(x)), & \phi \neq f^{-1}(y) \subset E, \\ \infty \qquad , & \phi \neq f^{-1}(y) \not\subset E, \\ 0 \qquad , & f^{-1}(y) = \phi, \end{cases} \tag{5.9.4}$$

where B runs through all subsets of $\tilde{P} \cap f^{-1}(y)$ such that card $B \leqslant m$. One can follow the proof in [20, p. 6] to conclude that ϱ' is a Borel function.

Let Γ_0 be the family of all paths β in $\bar{R}^n$ such that either β is non-rectifiable or there is a path α in B^n such that $f \circ \alpha \subset \beta$ and f is not absolutely precontinuous on α in the terminology of [20, p. 4]. Then $M(\Gamma_0)=0$ and so for $\Gamma' = \Gamma_{\bar{f}E} \setminus \Gamma_0$, $M(\Gamma') = M(\Gamma_{\bar{f}E})$. Hence it suffices to show

$$M(\Gamma') \leqslant \frac{K_I(f)}{m} M_G(\Gamma_E). \tag{5.9.5}$$

Arguing as in [20, pp. 6–7] it is not hard to see that

$$\int_\gamma \varrho' ds \geqslant 1$$

for every $\gamma \in \Gamma'$, and consequently $\varrho' \in F(\Gamma')$. This in turn implies

$$M(\Gamma') \leqslant \int_{R^n} \varrho'^n dm, \tag{5.9.6}$$

and the estimate

$$\int_{R^n} \varrho'^n dm \leqslant m^{-1} K_I(f) \int_P \varrho^n dm \tag{5.9.7}$$

follows from (5.9.4), Hölder's inequality, a transformation formula for Lebesgue integrals, and from the facts that $m(P \setminus E)=0$ and $f|(P \setminus E)$ is locally quasiconformal, see [20, p. 8]. Now (5.9.6) and (5.9.7) imply the required inequality (5.9.5). The same proof applies to condensers in $\hat{M}$.

5.10. COROLLARY. *If E is a condenser in M, then*

$$\operatorname{cap} \bar{f}E \leqslant K_I(f) \operatorname{cap}_G E. \tag{5.10.1}$$

Here M and $\bar{f}$ may be replaced by $\hat{M}$ and $\hat{f}$, respectively.

5.11. *Remarks.*

(a) It is possible to prove more general modulus inequalities than (5.9.1) and (5.9.2). For instance, using the arguments of 5.9 one can show that

$$M_G(\Gamma) \leqslant K_0(f) N(\bar{f}, A) M(\bar{f}\Gamma),$$

where Γ is any path family in a Borel set $A \subset M$, see [8, sec. 3]. Here $\bar{f}\Gamma = \{\bar{f} \circ \gamma : \gamma \in \Gamma\}$. However, we shall only need the special inequalities (5.9.1) and (5.9.2).

(b) Theorem 5.9 and Corollary 5.10 are true as well if G acts on H^n.

6. Automorphic mappings: Value distribution, boundary behavior and the branch set

In this chapter we study several aspects of qm automorphic mappings in the hyperbolic space D, $D=B^n$ or H^n. We start with theorems about the radial and angular limits of such mappings. We then study the value distribution of qm automorphic mappings. Next, we consider the branch set B_f of an automorphic qm mappings f. It turns out that in this point there are differences between $n=2$ and $n\geqslant 3$. Finally, we describe the growth of qm automorphic mappings near parabolic points. Our main tools are the theorems on open discrete mappings of chapter 2 and the capacity inequalities that we derived in chapter 5.

6.1. *Radial and angular limits.* Let $f\colon D\to\bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on D, $D=B^n$ or H^n, with $V(D/G)<\infty$. Let $P\subset D$ be a simple fundamental polyhedron with respect to G, see 3.4, and let $Q=\bar{P}\cap\partial D$ denote the set of all boundary vertices of P, possibly $Q=\emptyset$. Recall, see 4.5, that f is said to have a limit at $p\in Q$ in $\tilde{P}$ if $\lim f(x)$ exists as $x\to p$ in $\tilde{P}$. Due to the nature of G, the structure of P near p, and the invariance of f under G it follows that if f has a limit at $p\in Q$ in P then f has an *angular limit* at p in D. If in addition $D=B^n$ then f has a *radial limit* at p.

One of the main results in the next four sections is Theorem 6.5 which says that f has limit at all boundary vertices of P in $\tilde{P}$ if and only if $N(f,\tilde{P})<\infty$.

6.2. THEOREM. *Let G, P, Q, and f be as in* 6.1. *If $p\in Q\neq\emptyset$ and $N(f,\tilde{P}\cap U)<\infty$ for some neighborhood U of p, then f has a limit at p in $\tilde{P}$.*

Proof. We may assume that $D=H^n$ and $p=\infty$. Let U be a neighborhood of ∞ with $N(f,\tilde{P}\cap U)=N<\infty$. Choose $y\in\bar{R}^n$ such that $N(y,f,\tilde{P}\cap U)=N$ and let $f^{-1}(y)\cap\tilde{P}\cap U=\{y^1,\dots,y^N\}$. Choose disjoint normal neighborhoods U_i of y^i, $i=1,\dots,N$, see 2.1. Let $U'=\cap\, U_i$ and $V=\cap\, fU_i$. Then $V\subset \complement f((U\cap\tilde{P})\setminus G(U'))$. Pick $h>0$ so large that $H(h)\subset R^n\setminus U'$ and $H(h)$ does not contain any vertices of P. Denote $R=\{x\in R^n\colon |x_j|<1,\ j=1,\dots,n\}$.

Let x^i, $i=1,2,\dots$, be a sequence of points in $\tilde{P}$ such that $x^i\to\infty$ as $i\to\infty$. We may assume that $x^i_n>i+h$, otherwise choose a subsequence. For each integer i let $f_i\colon R\to\bar{R}^n$ be the mapping defined by $f_i(x)=f(ix+x^i)$, $x\in R$. Then $K(f_i)\leqslant K(f)$ and each f_i maps R into $\bar{R}^n\setminus V$; hence $\{f_i\}$ is a normal family, see [9, 3.17].

Let $\{f_{i_k}\}$ be a subsequence of $\{f_i\}$ which converges uniformly on compact subsets of R. By [13] the limit function f_0 is qm. Let

$$R'=\{x\in\partial H^n\colon |x_j|\leqslant\tfrac12,\quad j=1,2,\dots,n-1\}.$$

Then, by (3.6.1), $\lim_{k\to\infty} N(f_{i_k}, R')=\infty$, and since R' is compact in R, f_0 cannot be discrete [12, p. 131], see also [11, 8.3]. Hence f_0 is a constant, say α. We may assume $\alpha\neq\infty$. We shall show that $\lim f(x)=\alpha$ as $x\to\infty$ in $\tilde{P}$.

Let $\varepsilon>0$ and let $P'\colon R^n\to R^{n-1}=\partial H^n$ denote the projection $x\mapsto x-x_n e_n$. By 3.6 the euclidean diameter d_0 of $P'(P)$ is finite, therefore there exists an integer k_0 such that $\operatorname{diam}(i_k R')>2d_0$ and $f_{i_k}(R')\subset B^n(\alpha,\varepsilon)$ for all $k\geqslant k_0$. Fix x^i such that $x^i_n>i_{k_0}+h$ and then pick $k>k_0$ so that $i_k+h>x^i_n$. Define

$$S_k=\{x\in\tilde{P}\colon i_{k_0}<x_n-h<i_k\}.$$

Then $f(\partial S_k)\subset f_{i_k}(R')\cup f_{i_{k_0}}(R')\subset B^n(\alpha,\varepsilon)$; and since f is open and omits V, $f(S_k)\subset B^n(\alpha,\varepsilon)$. Since $x^i\in S_k$, this implies $\lim_{i\to\infty} f(x^i)=\alpha$.

6.3. For the sake of simplicity we state the following theorem only for mappings in B^n. The corresponding result in H^n follows at once.

THEOREM. *Let $f\colon B^n\to\bar{R}^n$ be a qm automorphic mapping with respect to a discrete Möbius group G acting on B^n with $V(B^n/G)<\infty$. If B^n/G is non-compact and if $N(f,F)<\infty$ for some fundamental set $F\subset B^n$ with respect to G, then*

(i) *the set of all parabolic fixed points of G is dense in ∂B^n, and*

(ii) *f has a radial limit at every parabolic fixed point of G.*

Proof. For (i) see the proof of Lemma 3.7. For (ii), let $p\in\partial B^n$ be a parabolic point. By 3.5 (iv), G has a simple fundamental polyhedron P with a boundary vertex at p. Hence $N(f, R^n\cap\tilde{P})=N(f,\tilde{P})=N(f,F)<\infty$ and thus (ii) follows by 6.2.

6.4. *Remark.* Let $f\colon B^n\to\bar{R}^n$ a qm automorphic mapping for a discrete Möbius group acting on B^n. If B^n/G is compact then f has no radial limit at any point of ∂B^n.

6.5. THEOREM. *Let $f\colon D\to\bar{R}^n$ be a qm automorphic mapping with respect to a discrete Möbius group G acting on D with $V(D/G)<\infty$, where $D=B^n$ or H^n; and let P be a simple fundamental polyhedron with non-empty set of boundary vertices $Q=\bar{P}\cap\partial D$.*

Then f has a limit at each $p\in Q$ in P if and only if $N(f,\tilde{P})<\infty$; or equivalently: the induced map $\tilde{f}\colon M\to R^n$ has a continuous extension $\hat{f}\colon\hat{M}\to R^n$ if and only if $N(\tilde{f})=N(f,\tilde{P})<\infty$. Here $\hat{M}$ is the cusp compactification of $M=D/G$, see 3.8 and 4.5.

Proof. If f has a limit at each point of Q, then $\tilde{f}$ has a continuous extension $\hat{f}$ on $\hat{M}$ and by 2.10, $N(f,\tilde{P})=N(\tilde{f})=N(\hat{f})<\infty$. Conversely, if $N(f,\tilde{P})=N(\tilde{f})<\infty$, then, by 6.2, f has a limit at each point $p\in Q$ in P and thus $\tilde{f}$ has a continuous extension $\hat{f}$ on $\hat{M}$.

6.6. *Value distribution of qm automorphic mappings.* In the next five sections we extend the results of [9, 4.4 and 4.6] and [11, 8.2] to *qm* automorphic mappings.

6.7. A compact set C in $\bar{R}^n$ is said to be of *zero capacity* if either $C=\emptyset$ or else if cap $(A, C)=0$ whenever $A\subset R^n$ is open and $C\subset A$, see [9, 2.12]. In this case we write cap $C=0$.

6.8. THEOREM. *Let $f\colon D\to\bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on D, $D=B^n$ or H^n, with $V(D/G)<\infty$. Then* cap $\complement f(U\cap D)=0$, *whenever $U\subset\bar{R}^n$ is open and $U\cap\partial D\neq\emptyset$.*

Proof. We may assume that $D=B^n$. Let $P\subset B^n$ be a simple fundamental polyhedron for G. Since $g(\bar{P})\subset U$ for some $g\in G$, see 3.7, and $f(\bar{P})=f(B^n)$, it suffices to show that cap $\complement fB^n=0$. Let C be a non-degenerate continuum in P. Consider the condenser $E=(B^n/G, \pi(C))$ in $M=B^n/G$, and $\tilde{f}E=(fB^n, fC)$ in $\bar{R}^n$. Then (5.10.1) and 5.5 imply

$$\operatorname{cap}(\complement fC, \complement fB^n)=\operatorname{cap}(fB^n, fC)=\operatorname{cap}\tilde{f}E\leqslant K_I(f)\operatorname{cap}_G E=0;$$

and since fC is a non-degenerate continuum in $\bar{R}^n$, it follows by [9, 3.11] that cap $\complement fU=0$.

6.9. COROLLARY. *Let $f\colon D\to\bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on D with $V(D/G)<\infty$, where $D=B^n$ or H^n. Then* cap $\complement fD=0$ *and in particular $\overline{fD}=\bar{R}^n$.*

6.10. THEOREM. *Let $f\colon D\to\bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on D, $D=B^n$ or H^n, with $V(D/G)<\infty$.*

If $N=N(f, F)<\infty$ for some fundamental set $F\subset D$, then card $(\bar{R}^n\setminus fD)<\infty$ *and*

$$\sum_{x\in f^{-1}(y)\cap F}\frac{i(x,f)}{N(x,G)}=N \tag{6.12.1}$$

for all points y in R^n with the possible exception of a finite set of points. Here $N(x, G)=$ card G_x *and $i(x, f)$ denotes the local topological index of f at x, see 2.6.*

Proof. If $D/G=M$ is compact then $fD=\tilde{f}M=\bar{R}^n$. If M is non-compact then $N<\infty$ implies, by 6.5, that $\tilde{f}$ has a continuous extension $\hat{f}$ on $\hat{M}$, see 4.5. By 2.10, $\hat{f}$ is open and so $\hat{f}\hat{M}=\bar{R}^n$ and consequently $fD=\tilde{f}M\supset\hat{f}\hat{M}\setminus\hat{f}Q=\bar{R}^n\setminus\hat{f}Q$. Thus card $(\bar{R}^n\setminus fD)\leqslant$ card $\hat{f}Q\leqslant$ card Q. The rest of the theorem follows from (4.5.2) and (4.5.3).

6.11. *Remark.* The assumption $V(D/G)<\infty$ in 6.8–6.10 is essential as shown by example 4.2.

6.12. *The branch set of qm automorphic mappings.* One of the main differences between plane and space *qm* mappings is the branch set, cf. [10]. This difference also occurs in automorphic mappings and is described in this section.

THEOREM. *Let f: $D \to \bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on D, $D = B^n$ or H^n, with $V(D/G) < \infty$.*

Then $B_f \neq \emptyset$ and $\partial D \subset \bar{B}_f$ whenever at least one of the following conditions holds:

(i) $n \geq 2$ *and D/G is compact.*

(ii) $n \geq 3$ *and $f(D) \subset R^n$.*

(iii) $n \geq 3$ *and $N(f, F) < \infty$ for some fundamental set $F \subset D$ with respect to G.*

Proof. (i) If $B_f = \emptyset$, then f defines a covering map $D \to \bar{R}^n$ and since $\bar{R}^n$ is simply connected it follows that f is a homeomorphism. But D is not homeomorphic to $\bar{R}^n$. Hence $B_f \neq \emptyset$, and 3.7 implies that $\partial D \subset \bar{B}_f$.

(ii) Note that in this case D/G is non-compact since otherwise $fD = \tilde{f}(D/G) = \bar{R}^n$ contrary to the assumption $fD \subset R^n$. Let P be a simple fundamental polyhedron. We may assume that $D = H^n$ and that P has a boundary vertex at ∞. Then G_∞ has a strictly parabolic transformation say $A(x) = x + h$ where $h \neq 0$ and normal to e_n. If $B_f = \emptyset$ and $n \geq 3$ then by [10, 2.3] there is $\alpha \in (0, 1)$ depending only on n and on $K(f)$ such that $f | B^n(a, \alpha R)$ is injective whenever $B^n(a, R) \subset H^n$. Choose $a \in H^n$ and $R > 0$ such that $B^n(a, R) \subset H^n$ and such that $|h| < 2\alpha R$. Then $B^n(a, \alpha R)$ contains at least two points x and $A(x)$ which are G-equivalent and which are thus mapped by f onto the same point, contradicting the injectiveness of $f | B^n(a, \alpha R)$. It thus follows that $B_f \neq \emptyset$, and 3.7 implies that $\partial D \subset \bar{B}_f$.

(iii) Let $Q = \hat{M} \setminus iM$, where i: $M \to \hat{M}$ denotes the inclusion map from $M = D/G$ into its cusp compactification $\hat{M}$. Note that $N(\tilde{f}) = N(f, F)$, hence the assumption $N(f, F) < \infty$ implies, by Theorem 6.5, that $\tilde{f}$: $M \to \bar{R}^n$ has a continuous extension $\hat{f}$: $\hat{M} \to \bar{R}^n$.

Suppose that $B_f = \emptyset$. Then f defines a covering map $D \setminus f^{-1}\hat{f}Q \to \bar{R}^n \setminus \hat{f}Q$. Indeed, let P be a simple fundamental polyhedron for G. Fix $y \in \bar{R}^n \setminus \hat{f}Q$ and let $\{x_1, \ldots, x_m\} = [\tilde{P} \cap f^{-1}(y)] \setminus f^{-1}\hat{f}Q$. Since f is a local homeomorphism, there exist neighborhoods U_i of x_i such that $f | U_i$ are homeomorphisms, $fU_i = V$ for some neighborhood V of y, $i = 1, \ldots, m$ and such that $\pi(U_i) \neq \pi(U_j)$ for $i \neq j$. Then $\{gU_i$: $1 \leq i \leq m$, $g \in G\}$ is a set of disjoint components of $f^{-1}V$ required in the definition of a covering map. The assumption $n \geq 3$ implies that $\bar{R}^n \setminus \hat{f}Q$ is simply connected and thus $f | D \setminus f^{-1}\hat{f}Q$ is a homeomorphism contradicting the fact that f is automorphic. It thus follows that $B_f \neq \emptyset$, and 3.7 implies that $\partial D \subset \bar{B}_f$.

6.13. *Remarks.* (a) The assumption $V(D/G) < \infty$ in Theorem 6.12 is essential as shown by example 4.2.

(b) The assumption $n \geqslant 3$ is essential in 6.12 (ii) and (iii). The elliptic modular function in R^2 is a counterexample.

(c) We do now know whether the conditions $f(D) \subset R^n$ and $N(f, F) < \infty$ in 6.12 (ii) and (iii), respectively, are essential. Our guess is that they are not.

6.14. *Behavior of qm automorphic mappings at their natural boundary.* In the following sections we consider *qm* automorphic mappings $f: D \to \bar{R}^n$ for discrete Möbius groups G acting on D, $D = B^n$ or H^n, with $V(B^n/G) < \infty$ and study their growth near ∂D. In view of Remark 6.4 and Theorem 6.3 we shall consider only the case where D/G is non-compact and $N(f, F) < \infty$ for some fundamental set $F \subset D$ with respect to G. We show here that the rate of growth of f near parabolic points is similar to the growth of $(n-1)$-periodic *qm* mappings near ∞, see Theorem 8.7 and Corollary 8.11 of [11]. The main tools are the capacity inequalities of section 5.

6.15. Let $f: H^n \to \bar{R}^n$ be a *qm* automorphic mapping for a discrete Möbius group G acting on H^n with $V(H^n/G) < \infty$, and suppose that ∞ is fixed for a parabolic element of G. Then, by Lemma 3.5 (iv), G has a simple fundamental polyhedron P with a boundary vertex at ∞. Suppose that $N(f, \tilde{P}) < \infty$. Then by Theorem 6.3 f has a limit at ∞ in $\tilde{P}$. Suppose that the limit is ∞. In this case the induced mapping $\tilde{f}: M \to \bar{R}^n$ has a continuous extension $\hat{f}: \hat{M} \to \bar{R}^n$ with $\hat{f}(\infty) = \infty$. Here $\hat{M}$ denotes the cusp compactification of $M = H^n/G$, see 3.8 and 4.5.

For $h > 0$ let

$$M(h) = \sup\{|f(x)| : x \in H^n, x_n = h\},$$

$$m(h) = \inf\{|f(x)| : x \in H^n, x_n = h\}.$$

6.16. THEOREM. *Under the assumptions of* 6.15

(i) $A_1 e^{\alpha h} \leqslant M(h) \leqslant A_2 e^{\beta h}$,

(ii) $A_3 e^{\alpha h} \leqslant m(h) \leqslant A_4 e^{\beta h}$,

for all sufficiently large h, where A_i $i = 1, \ldots, 4$ are positive constants depending on f,

$$\alpha = \left[\frac{\omega i(\infty, \hat{f})}{K_I(f) A}\right]^{1/(n-1)}, \quad \beta = \left[\frac{\omega K_0(f)\, i(\infty, \hat{f})}{A}\right]^{1/(n-1)},$$

where A and ω are the $(n-1)$-measures of $\{x \in P: x_n = h\}$ and S^{n-1}, respectively, and $i(\infty, \hat{f})$ denotes the local topological index of $\hat{f}$ at ∞, see 2.6.

Proof. Pick $A \in GM(n)$ with $|A(x)| = 1/|x|$ and let $g = A \circ f$, $\tilde{g} = A \circ \tilde{f}$ and $\hat{g} = A \circ \hat{f}$, then g is *qm* automorphic mapping with respect to G, g has the dilatations of f, $N(f, \tilde{P}) = N(g, \tilde{P})$ and $\hat{g}(\infty) = 0$.

For $r>0$ let $U(r)$ denote the ∞-component of $\hat{g}^{-1}B^n(r)$. Choose $r_0>0$ so small that for $0<r\leqslant r_0$, $U(r)$ is a normal neighborhood of ∞ and $\complement U(r)$ is connected, see 2.2. Since $\hat{g}\,|\,U(r)$ is a closed map, Theorem 2.8 implies that for all $r\in(0,r_0]$

$$N(\hat{g},U(r))=i(\infty,\hat{g})=M(g,\overline{U(r)});$$

where $M(\hat{g},\overline{U(r)})$ denotes the minimal multiplicity of $\hat{g}$ on $\overline{U(r)}$, see (5.8.1).

Let $C=\{x_n\colon x\in(i\circ\pi)^{-1}(\partial U(r_0))\}$, $b_1=\inf C$, and $b_2=\sup C$. For $h>0$ let $\hat{H}(h)$ be the closure of the set $(i\circ\pi)H(h)$ in $\hat{M}$. Choose $h_0>0$ such that the requirements in 5.6 are satisfied and such that the compact set $\hat{g}\hat{H}(h)\subset B^n(r_0)$ for all $h\geqslant h_0$. For $h>0$, let

$$M'(h)=\sup\{|g(x)|\colon x_n=h\},$$
$$m'(h)=\inf\{|g(x)|\colon x_n=h\}.$$

Then $M(h)=1/m'(h)$ and $m(h)=1/M'(h)$.

For $h>h_0$, $E=(U(r_0),\,\bar{U}(m'(h))$ is a condenser in $\hat{M}$ and $\hat{g}E=(B^n(r_0),\,\bar{B}^n(m'(h)))$ is a condenser in $\bar{R}^n$. Since $\bar{U}(m'(h))\subset\hat{H}(h)$, Theorem 5.9 implies

$$\omega\left(\log\frac{r_0}{m'(h)}\right)^{1-n}=\operatorname{cap}\hat{g}E\leqslant\frac{K_I(f)}{i(\infty,\hat{f})}\operatorname{cap}_G E\leqslant\frac{K_I(f)\,A}{i(\infty,\hat{f})\,(h-b_2)^{n-1}}.$$

This yields $A_1e^{\alpha h}\leqslant 1/m'(h)$ and the left side of (i) follows.

For the left side of (ii) it suffices to show that

$$M'(h)\leqslant cm'(h) \tag{6.16.1}$$

for all $h>h_1$ for some $c>0$. Fix $h>h_0$ and suppose that $M'(h)>m'(h)$. Then $E=(U(M'(h)),\,\bar{U}(m'(h)))$ is a normal condenser in $\hat{M}$ and $\hat{g}E=(B^n(M'(h)),\,\bar{B}^n(m'(h)))$ and by Theorem 5.9

$$\operatorname{cap}_G E\leqslant K_0(f)\,i(\infty,\hat{f})\operatorname{cap}\hat{g}E=K_0(f)\,i(\infty,\hat{f})\,\omega\left(\log\frac{M'(h)}{m'(h)}\right)^{1-n}. \tag{6.16.2}$$

On the other hand the condenser E satisfies the assumptions of Lemma 5.6, thus cap $E\geqslant\delta>0$ where δ depends on G. This combined with (6.16.2) yields (6.16.1).

To prove the right side of (ii), consider the normal condenser $E=(U(r_0),\,\bar{U}(M'(h)))$ in $\hat{M}$, $h>h_0$, and its image $\hat{g}E=(B^n(r_0),\,\bar{B}^n(M'(h)))$ in $\bar{R}^n$; then Theorem 5.9 implies

$$A(h-b_1)^{1-n}\leqslant\operatorname{cap}_G E=K_0(f)\,i(\infty,\hat{f})\operatorname{cap}\hat{g}E=K_0(f)\,i(\infty,\hat{f})\,\omega\left(\log\frac{r_0}{M'(h)}\right)^{1-n}.$$

This gives $m(h)=1/M'(h)\leqslant A_4e^{\beta h}$, and by (6.16.1) the right side of (i) follows, too.

6.17. COROLLARY. *Let $f: B^n \to \bar{R}^n$ be a qm automorphic mapping for a discrete Möbius group G acting on B^n with $V(B^n/G) < \infty$ and with a parabolic fixed point at $p \in \partial B^n$. For $r \in (0, 1/2)$, let $S(r) = \{x \in B^n: |x-(1-r)p| = r\}$.*

If $N(f, F) < \infty$ for some fundamental set $F \subset B^n$, then:

(i) *The radial limit $a = \lim_{t\to 1-0} f(tp)$ exists.*

(ii) *For all sufficiently small $r > 0$,*

$$A_1 e^{\gamma r} \leqslant m(r) \leqslant M(r) \leqslant A_2 e^{\delta r},$$

where

$$M(r) = \sup_{x\in S(r)} |f(x)| \text{ and } m(r) = \inf_{x \in S(r)} |f(x)| \text{ when } a = \infty$$

and

$$M(r) = \sup_{x\in S(r)} |f(x)-a| \text{ and } m(r) = \inf_{x \in S(r)} |f(x)-a| \text{ otherwise.}$$

Here A_1 and A_2 are positive constants which depend on f, γ and δ are positive constants when $a = \infty$ and negative constants otherwise. In any case γ and δ depend only on G, $N(f, F)$ and the maximal dilatation of f.

Proof. (i) is contained in Theorem 6.3 (ii). For (ii), first replace f by $B \circ f \circ A$ where A is a Möbius transformation with $A(H^n) = B^n$ and $A(\infty) = p$ and $B \in GM(n)$ with $B(a) = \infty$. This brings us to the situation in 6.15, where the punctured spheres $S(r)$ correspond to the $(n-1)$-planes $\partial H(h)$, $h > 0$. The result, then, follows by 6.16.

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