COMMUTATORS AND SYSTEMS OF SINGULAR INTEGRAL EQUATIONS. I[1]

BY

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Introduction

In this paper we study certain self-adjoint singular integral operators with matrix coefficients acting on a multi-component Hilbert space H; namely,

$$Lx(\lambda) = A(\lambda)\,x(\lambda) + \frac{1}{\pi i}\mathbf{P}\int_a^b \frac{k^*(\lambda)\,k(\mu)}{\mu-\lambda}\,x(\mu)\,d\mu,$$

where
$$A(\lambda) = \begin{pmatrix} A_{11}(\lambda) & \dots & A_{1n}(\lambda) \\ A_{21}(\lambda) & \dots & A_{2n}(\lambda) \\ \vdots & \vdots & \vdots \\ A_{n1}(\lambda) & \dots & A_{nn}(\lambda) \end{pmatrix},$$

$$k(\lambda) = \begin{pmatrix} k_{11}(\lambda) & \dots & k_{1n}(\lambda) \\ k_{21}(\lambda) & \dots & k_{2n}(\lambda) \\ \vdots & \vdots & \vdots \\ k_{n1}(\lambda) & \dots & k_{nn}(\lambda) \end{pmatrix},$$

where the matrices above have elements which are complex-valued functions of λ, and for almost all λ, $A(\lambda)$ is a bounded Hermitian operator on the Hilbert space H which consists of vectors $x(\lambda) = \{x_1(\lambda), \dots, x_n(\lambda)\}$ with measurable components such that

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$\int_a^b \sum_1^n |x_j(\lambda)|^2 d\lambda < \infty$. $k(\lambda)$ is required to be Hilbert–Schmidt on H for almost all λ. We shall not necessarily take n to be finite in the following, but shall always restrict ourselves in this paper to the case that (a, b) is a finite interval.

It is most remarkable that all the unitary invariants of L can be explicitly obtained under an additional condition on the range of the commutator $[L, \lambda I]$, where I is the identity operator on H.

THEOREM 1.1. *If the (trace class) operator* $C \equiv \int_a^b k^*(\lambda) k(\mu) x(\mu) d\mu$ *has one dimensional range in* H, *then the operator*

$$\frac{I + k(\lambda)(A(\lambda) - \omega)^{-1} k^*(\lambda)}{I - k(\lambda)(A(\lambda) - \omega)^{-1} k^*(\lambda)}$$

for non-real ω *considered as acting on a fixed coefficient Hilbert space,* h, *an* l_2*-space of dimension* n, *has only one eigenvalue different from one and that eigenvalue has the form*

$$\exp\left(\int_{-\infty}^{\infty} g(v, \lambda) \frac{dv}{v - \omega}\right),$$

where $g(v, \lambda)$ *is a measurable function of the pair* (v, λ) *such that* $0 \leqslant g(v, \lambda) \leqslant 1$. *If* n *is finite* $g(v, \lambda)$ *assumes only the values zero and one; furthermore the set in* $R \times R$ *for which* $g(v, \lambda)$ *is the characteristic function is a bounded set with the property that the sets* $\gamma_\lambda = \{v; g(v, \lambda) = 1\}$ *for each fixed* λ *in* (a, b) *consist of exactly* n *disjoint intervals.*

Suppose also, to rule out a trivial degeneration of the matrix $A(\lambda)$, that the smallest closed invariant subspace of L containing the range of C is H, and define $m(\xi) = q$ if $\Gamma_\xi = \{\mu; g(\xi, \mu) = 1\}$ is a union of q disjoint intervals; otherwise, let $m(\xi) = \infty$.

Then the von Neumann spectral multiplicity of L *is* $m(\xi)$ *and the spectral measure of* L *is Lebesgue measure.*

In the above theorem we can show that $g(v, \mu)$ can be calculated from the coefficients $A(\lambda)$ and $k(\lambda)$ by means of the formula

$$g(v, \mu) = \frac{1}{\pi} \arg \det \left[\frac{I + k(\mu)(A(\mu) - v - i0) k^*(\mu)}{I - k(\mu)(A(\mu) - v - i0) k^*(\mu)}\right].$$

When C is not restricted to have one dimensional range it is not yet determined if the *conjecture* that the description of the spectral invariants of L is still given as above is true; nevertheless, it is possible to give a less explicit description of these invariants which has an algorithmic nature and which can be applied in a variety of special cases.

THEOREM 1.2. *There exists a unique analytic operator valued function on h, $E(\omega, z)$, such that*

$$\frac{I-k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}{I+k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}=E^*(\bar{\omega},\lambda-i0)\,E(\omega,\lambda-i0)$$

and

$$\lim_{\omega\to\infty} E(\omega,z)=I.$$

THEOREM 1.3. *There exists a unique positive trace-class on h valued measure $dM_\xi(\cdot)$ such that*

$$E(\xi-i0,x)\,E^*(\xi-i0,\bar{x})=I+\int\frac{dM_\xi(\mu)}{\mu-x}\quad \text{a.a. } \xi.$$

$dM_\xi(\cdot)$ is absolutely continuous with respect to the scalar measure d (Trace $(M_\xi(\cdot)))$. Call the Radon–Nikodym derivative $M'_\xi(\mu)$, and let the jth eigenvalue of $M'_\xi(\mu)$ be denoted by $\lambda_j(\xi,\mu)$, each eigenvalue appearing over again according to its multiplicity, in such a way that

$$0<\ldots\leqslant\lambda_3(\xi,\mu)\leqslant\lambda_2(\xi,\mu)\leqslant\lambda_1(\xi,\mu).$$

Define, for any Borel set of R, the scalar measures

$$M_\xi^{(j)}(\Delta)=\int_\Delta \lambda_j(\xi,\mu)\,d(\text{Trace }(M_\xi(\mu))).$$

Call the L^2 space of complex-valued functions on (a, b) square summable with respect to $dM_\xi^{(j)}(\cdot)\;H_j$.

THEOREM 1.4. *Let $m(\xi)=\sum_{j=1}^{n}\dim(H^j)$, then $m(\xi)$ is the von Neumann spectral multiplicity function for L, and $m(\xi)$ and Lebesgue measure form a complete set of unitary invariants for L.*

These theorems reduce the problem of calculating the unitary invariants of L to the problem of constructing the fundamental solution $E(l, z)$ of the homogeneous Riemann–Hilbert problem

$$\frac{I - k(\lambda)\,(A(\lambda) - \omega)^{-1}k^*(\lambda)}{I + k(\lambda)\,(A(\lambda) - \omega)^{-1}k^*(\lambda)} E^*(\bar{\omega}, \lambda + i0) = E^*(\bar{\omega}, \lambda - i0),$$

$$\lim_{\omega\to\infty} E(\omega, z) = \lim_{z\to\infty} E(\omega, z) = 1.$$

The degree to which our results constitute a solution to the problem of diagonalizing L depends upon how successful we can expect to be in finding a solution to this problem in an explicit and manageable way.

Such Riemann–Hilbert problems have been studied extensively in the literature, and if $A(\lambda)$ and $k(\lambda)$ are sufficiently smooth as functions of λ the problem of calculating $E(l, z)$ is reduced to the problem of solving a Fredholm equation. C.f. N. I. Muschelischwili [15] and I. N. Vekua [18]. When $A(\lambda)$ and $k(\lambda)$ are rational functions of λ it is possible to give a somewhat simpler explicit solution, Vekua [18].

When n is finite and $A(\lambda)$ and $h(\lambda)$ are sufficiently smooth a circle of results beyond the scope of the present paper shows that for fixed $\xi \in \sigma(L)$ the measure d Trace $(M_\xi(\mu))$ is purely atomic, concentrating its mass at only a finite number of points.

In this case the construction by G. F. Mandshewidse [16, 17] of a solution to the Riemann–Hilbert problem by an iterative procedure may be effective for the determination of dim H^j.

We will have, in this case,

$$E(\xi - i0, x)\,E^*(\xi - i0, \bar{x}) = I + \int \frac{dM_\xi(\mu)}{\mu - x} = I + \sum_{j=1}^{n} \frac{M'_\xi[r_j(\xi)]}{r_j(\xi) - x}\,\mu_j(\xi),$$

where $\mu_j(\xi)$ is the positive mass which the measure d Trace $(M_\xi(\cdot))$ concentrates at the real point $r_j(\xi)$.

Thus

$$\frac{1}{2\pi i} \lim_{\eta\downarrow 0} \int_{c_j} (E(\xi - i\eta, x)\,E^*(\xi - i\eta, \bar{x}) - I)\,dx = M'_\xi[r_j(\xi)]\,\mu_j(\xi),$$

where c_j is a sufficiently small circle about $r_j(\xi)$. Approximation of this contour integral by Riemann sums may prove to be numerically possible in certain cases.

In any case, Theorem 1.1 above is deduced from Theorem 1.4.

Our results can be understood abstractly as a means of obtaining the spectral invariants of a bounded self-adjoint operator V from those of another bounded self-adjoint operator U such that $VU - UV = (1/\pi i)C$ where C is a positive operator of trace class.

Thus Theorem 1.1 above provides a complete solution of this abstract problem when C is restricted to be of one dimensional range.

These results are extensions of previous work [1], [2], [3], [4].

1. The determining function

The basic technique of our method consists in the introduction of an operator-valued function of two complex variables, $E(l,z)$, the determining function of the pair $\{U, V\}$, which characterizes the relationship between the two operators U and V such that $VU-UV=(1/\pi i)C$.

We will characterize the class of such determining functions, and show how to construct the direct integral space on which V is diagonal from a determining function.

Let h be the l_2 space of dimension equal to the maximum of the dimension of the range of C and the spectral multiplicity of U. The Schmidt expansion of C has the form $C=\sum\lambda_n^2\varphi_n(\cdot,\varphi_n)$ where $\{\varphi_n\}$ is the complete set of eigenvectors of C, and where the $\{\lambda_n^2\}$ are the corresponding eigenvalues.

If $\{\theta_n\}$ is a complete orthonormal set in h we define a linear transformation k in H whose range is in h by setting $k\varphi_n=\lambda_n\theta_n$ and extending k to all of H by setting $kx=0$ if $Cx=0$. Similarly, we define a transformation $k^*: h\to H$ by setting $k^*\theta_n=\lambda_n\varphi_n$.

Thus we arrive at $C\varphi_n=k^*k\varphi_n=\lambda_n^2\varphi_n$ and so $C=k^*k$.

We now define *the determining function of the pair* $\{V, U\}$ by setting

$$E(l,z)=1+\frac{1}{\pi i}k(V-l)^{-1}(U-z)^{-1}k^* \quad l\notin\sigma(V),\, z\notin\sigma(U),$$

where 1 denotes the identity operator in h.

$E(l,z)$ is an operator which maps h into h. In fact, $E(l,z)$ maps the subspace of h spanned by those θ_n corresponding to $\lambda_n\neq 0$, onto itself. Let us call this subspace $\mathbf{H}$.

An alternate definition might have been made in terms of the identity

$$(U-x)^{-1}(V-y)^{-1}k^*=(V-y)^{-1}(U-x)^{-1}k^*E(x,y).$$

That is, $E(x,y)$ is a mapping on the domain space h so devised as to compensate for the change in order in which the resolvents are applied.

Proof.

$$(U-x)^{-1}(V-y)^{-1}k^*\left[1+\frac{1}{\pi i}k(V-y)^{-1}(U-x)^{-1}k^*\right]$$

$$=(U-x)^{-1}(V-y)^{-1}k^*+\frac{1}{\pi i}(U-x)^{-1}(V-y)^{-1}C(V-y)^{-1}(U-x)^{-1}k^*.$$

But

$$\frac{1}{\pi i}(U-x)^{-1}(V-y)^{-1}C(V-y)^{-1}(U-x)^{-1}=(V-y)^{-1}(U-x)^{-1}-(U-x)^{-1}(V-y)^{-1},$$

since $VU-UV=(1/\pi i)C$.

Hence $$(U-x)^{-1}(V-y)^{-1}k^*=(V-y)^{-1}(U-x)^{-1}k^*E(x,y),$$

In a similar way we see immediately that

$$E^*(\bar{x},\bar{y})=E^{-1}(x,y).$$

2. Systems of singular integral equations

THEOREM 2.1. *Let U and V be bounded symmetric operators on a separable Hilbert space $\mathcal{H}$. Let C be a positive operator of trace class. Assume that $VU-UV=(1/\pi i)C$. Then V restricted to the smallest closed subspace of $\mathcal{H}$, Γ, which reduces both U and V and which contains the range of C is unitarily equivalent to the singular integral operator L, acting on a certain direct sum of Hilbert spaces, H, in which $U|\Gamma$ is diagonal, defined by setting*

$$Lx(\lambda)=A(\lambda)x(\lambda)+\frac{1}{\pi i}\mathbf{P}\int_{\sigma(U)}\frac{k^*(\lambda)k(\mu)}{\mu-\lambda}x(\mu)\,d\mu$$

for $x(\cdot)\in H$, where $A(\lambda)$ is a bounded symmetric operator on H and where $k(\lambda)$ is bounded on H, and is Hilbert–Schmidt, a.a. $\lambda\in\sigma(U)$. Both of these operators are weakly measurable essentially bounded functions of λ.

Proof. A theorem of C. Putnam [7] asserts that the smallest subspace of $\mathcal{H}$ reducing both U and V and containing the range of C, Γ, is contained in $H_a(U)$, where $H_a(U)$ is the set of elements in $\mathcal{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ; E_λ being the spectral resolution of $U=\int\lambda dE_\lambda$. Furthermore,

$$\|C\|\leqslant\|V\|\cdot(\text{measure}\,[\sigma(U)]).$$

Let H be a minimal direct sum decomposition of Γ into invariant subspaces of $U, \mathcal{H}_{k_i}$, each generated by a cyclic vector k_i. Choose an isometric transformation S; $\Gamma \to H$ such that, for $f \in \Gamma$

$$Sf = \{g_1(\lambda), \ldots, g_n(\lambda)\},$$

and

$$SUf = \{\lambda g_1(\lambda), \ldots, \lambda g_n(\lambda)\} = SUS^{-1}g(\lambda)$$

Set $S_i f = g_i(\lambda)$, and let $\{\varphi_l\}$ be an orthonormal set of eigenvectors of C corresponding to eigenvalues $\{\lambda_l^2\}$. Then $C = \sum_l \lambda_l^2 \varphi_l(\cdot, \varphi_l)$ and

$$SCf = \{\sum_{i,l} S_1\varphi_l \lambda_l^2 (g_i, S_i\varphi_l)_H, \quad \ldots, \quad \sum_{i,l} S_n\varphi_l \lambda_l^2 (g_i, S_i\varphi_l)_H\}.$$

If we define, for each λ, the matrix k^* with the element in the ith row and jth column $(k^*)_{ij} = \lambda_j S_i \varphi_j$, we will have

$$SCf = SCS^{-1} = \int_{\sigma(U)} k^*(\lambda)\, k(\mu)\, g(\mu)\, d\mu,$$

where $k(\mu)$ is the adjoint operator (on H) to $k^*(\mu)$ with matrix elements given by $(k(\mu))_{ij} = \lambda_i \overline{S_j\varphi_i}$. $k(\mu)$ is compact for almost every μ, because

$$\sum_{l,j} \int |S_j(\lambda_l\varphi_l)|^2(\xi)\, d\xi = \sum_l |\lambda_l|^2 < \infty.$$

But this implies that

$$\sum_{l,j} |S_j(\lambda_l\varphi_l)|^2 < \infty \quad \text{a.a. } \xi$$

which, in turn, implies that $k(\mu)$ is Hilbert–Schmidt as an operator on l_2.

The proof that $k(\cdot)$ is a bounded operator on H is slightly more involved. We first note that $\|C\| = \int \|k(t)\|_{l_2}^2 dt$. This follows because

$$\left\| \int k(\tau) x(\tau)\, d\tau \right\|_{l_2} \leqslant \int \|k(\tau)\|_{l_2} \|x(\tau)\|_{l_2} d\tau \leqslant \left(\int \|k(\tau)\|_{l_2}^2 d\tau \int \|x(\tau)\|_{l_2}^2 d\tau \right)^{\frac{1}{2}}.$$

Thus

$$\left\| k^*(t) \int k(\tau) x(\tau)\, d\tau \right\|^2 \leqslant \int \|k^*(t)\|_{l_2}^2 dt \int \|k(\tau)\|_{l_2}^2 d\tau \int \|x(\tau)\|_{l_2}^2 d\tau,$$

and

$$\|Cx\| \leqslant \int \|k(\tau)\|_{l_2}^2 d\tau\, \|x\|.$$

But it is a standard argument to show that the equality may be achieved.

If $VU - UV = (1/\pi i)C \geqslant 0$, where U and V are bounded and symmetric, then C. Putnam has shown [7] that the Schwarz inequality implies that $\| C \| \leqslant \| V \| \cdot$ (measure $[\sigma(U)]$).

In the spectral representation of U, we may write this inequality in the form

$$\int \| k(t) \|_{l_2}^2 dt \leqslant \| V \| \cdot (\text{measure } [\sigma(M)]),$$

where $M = SUS^{-1}$, and if we let $\chi(\Delta)$ be the operator in H which acts by multiplying each component of the vector in H by the characteristic function of an interval, Δ, we will get

$$[\chi(\Delta) M \chi(\Delta)] [\chi(\Delta) L \chi(\Delta)] - [\chi(\Delta) L \chi(\Delta)] [\chi(\Delta) M \chi(\Delta)]$$
$$= \frac{1}{\pi i} \int_{\sigma(U) \cap \Delta} k^*(t) k(\tau) d\tau, \quad t \in \sigma(U) \cap \Delta.$$

Thus
$$\int_{\sigma(U) \cap \Delta} \| k(t) \|_{l_2}^2 dt \leqslant \| V \| \cdot \text{measure } (\sigma(U) \cap \Delta).$$

Now take $\Delta_m = [\alpha_m, \beta_m]$, then the fundamental theorem of the calculus implies that, for almost every t_0,

$$\lim_{m \to \infty} \frac{1}{\beta_m - \alpha_m} \int_{\alpha_m}^{\beta_m} \| k(t) \|_{l_2}^2 dt = \| k(t_0) \|_{l_2}^2$$

provided that $\alpha_m < t_0 < \beta_m$ and $\lim_{m \to \infty} \alpha_m = \lim_{m \to \infty} \alpha_m = t_0$. Hence $\| k(t) \|_{l_2}^2$ is essentially bounded.

Now define the *bounded* operator T on H by

$$Tx(\lambda) = \frac{1}{\pi i} \mathbf{P} \int_{\sigma(U)} \frac{k^*(\lambda) k(\mu)}{\mu - \lambda} x(\mu) d\mu.$$

T satisfies
$$[TM - MT]x = \frac{1}{\pi i} \int_{\sigma(U)} k^*(\lambda) k(\mu) x(\mu) d\mu.$$

If L' is is another bounded operator satisfying this commutator relation, then $A \equiv L' - T$ will commute with M. But the weakly closed ring which is generated by M is the ring of the given decomposition of our space Γ into a direct integral; hence, by a theorem of von Neumann [8] A must be a bounded Borel function of M(1), q.e.d.

Let us compute $E(\omega, z)$ in this representation.

(1) Compare with Xa-Dao-Xeng, On non-normal operators. Chinese Math. 3 (1963), 232–246.

The vectors $\{0, 0, \ldots, 1, 0, \ldots\} = \theta_n$ form a complete orthonormal set in h, while the vectors $\{S_1\varphi_n, S_2\varphi_n, \ldots\} = \theta_n$ form a complete orthonormal set in H.

But $\int k(\lambda)\omega_n(\lambda)d\lambda = \lambda_n\theta_n$, since

$$\int \lambda_i \sum_j \overline{S_j\varphi_i(\mu)}\, S_j\varphi_n(\mu)\, d\mu = \lambda_i(\varphi_i, \varphi_n).$$

Similarly, we deduce that

$$(k^*(\lambda)\theta_n)_i = \sum_j k^*(\lambda)_{ij}\delta_{jn} = k^*(\lambda)_{in} = \lambda_n(S_i\varphi_n)(\lambda).$$

Thus the determining function of $\{U, V\}$ in the spectral representation of U takes the form

$$E(\omega, z) = 1 + \frac{1}{\pi i}\int_{\sigma(U)} k(\lambda)(L-\omega)^{-1}(M-z)^{-1}k^*(\lambda)\,d\lambda.$$

We will study the boundary behaviour of this operator-valued function as $z \to \sigma(U)$ and $\omega \to \sigma(V)$.

Before we do this, however, we wish to describe the strategy which we will pursue in order to achieve a diagonalization of L.

Digression: Barrier related spectral problems

Let $\tilde{L} = \int_{\sigma(L)} \xi\, dE_\xi$ be a self-adjoint operator on a separable Hilbert space, $\mathcal{H}$, with an absolutely continuous spectral measure. Let $\mathcal{H} = \sum \oplus \mathcal{H}_{k_i}$ be a direct sum decomposition of $\mathcal{H}$ into pairwise orthogonal invariant subspaces of L, minimal in the sense of Hellinger–Hahn, each generated by a cyclic vector k_i. Let $\beta_i(\xi) = (\partial/\partial\xi)\,\|E_\xi k_i\|^2$, and let

$$S_i[f](\xi) = \frac{1}{\beta_i(\xi)}\frac{\partial}{\partial\xi}(f, E_\xi k_i).$$

The following theorem was established in a previous paper [2].

THEOREM

$$P_\xi[f, g] \equiv \frac{\partial}{\partial\xi}(f, E_\xi g) = \frac{1}{2\pi i}\lim_{\eta\downarrow 0}(f, (\tilde{L}-\xi+i\eta)^{-1}g - (\tilde{L}-\xi-i\eta)^{-1}g)$$
$$= \sum_1^{m(\xi)} S_i[f](\xi)\,\overline{S_i[g](\xi)}\,\beta_i(\xi),$$

ere $m(\xi)$ is the von Neumann spectral multiplicity function of $\tilde{L}$. Similarly, any decomposition of $\mathcal{H}$ into a direct sum of reducing subspaces, leads to a bilinear expres-

sion for $P_\xi[f, g]$ in terms of the partial isometries that diagonalize $\tilde{L}$ on the subspaces of the decomposition. The number of terms in such an arbitrary decomposition need not, of course, be equal to the spectral multiplicity. In such a case linear relations will exist between the generalized eigenfunctions that correspond to the partial isometries.

Now we will turn our attention to a way of representing the direct integral Hilbert space on which L becomes diagonal in terms of analytic functions defined ffo the spectrum of M whose boundary values will correspond to generalized eigenfunctions of L.

If $f(\cdot, z)$ is an H-valued analytic function for $z \notin R$, with the property that finite linear combinations of the form $\sum_i a_i f(\cdot, z)$ are dense in the domain of $\tilde{L}$, we define $F_i(\xi, z) \equiv (1/\beta_i(\xi)) S_i(f(\cdot, z))$ to be the indicatrix function of $\tilde{L}$ relative to the analytic generating family $f(\cdot, z)$ and the invariant subspace $\mathcal{H}_{k_i}$. (It follows by an easy argument that it is possible to choose a version of $F_i(\xi, z)$ which is analytic for almost all $\xi \in \sigma(\tilde{L})$ when $z \notin C$.)

Let $\mathcal{H}^*$ be the Hilbert space whose elements are generated from (the equivalence classes of) those functions $g(\xi, z)$ that can be represented as finite linear combinations of the form

$$g(\xi, z) = \sum \alpha_j(\xi) G_j(\xi, z), \quad G_j(\xi, z) \equiv [\beta_j(\xi)]^{-\frac{1}{2}} F_j(\xi, z),$$

where each $\alpha_j(\xi)$ is measurable with $\int_\sigma \sum |\alpha_j(\xi)|^2 d\xi < \infty$ by imposing the scalar product

$$(g, g')_{\mathcal{H}^*} = \int \sum \alpha_j(\xi) \overline{\alpha_j'(\xi)} \, d\xi.$$

Let $\mathcal{H}_\xi$ be the Hilbert space formed from finite linear combinations of the form $a(\xi, x) = \sum \alpha_k P_\xi(x, y_k)$ where α_k and $y_k \notin C$ are arbitrary, by imposing the scalar product $(a, a')\mathcal{H}_\xi = \sum_{i,k} \alpha_i \overline{\alpha_k'} P_\xi(x, y_k)$ when $P_\xi(x, y) = P_\xi[f(\cdot, x), f(\cdot, y)]$.

The author proved the following simple theorems in (2).

THEOREM. *Let $f \in \mathcal{H}$, define $f(\xi, z) \in \mathcal{H}^*$ by setting $f(\xi, z) = \sum_j S_j[f](\xi) F_j(\xi, z)$. Then the correspondence $f(\xi, z) \leftrightarrow f$ furnishes a spectral representation for $\tilde{L}$ in the sense that $f(\xi, z) \leftrightarrow f$ implies $\xi f(\xi, z) \leftrightarrow \tilde{L} f$, and $\mathcal{H}^*$ is the direct integral of the spaces $\mathcal{H}_\xi$ with respect to Lebesgue measure so that the spectral multiplicity $m(\xi)$ of L is equal to the dimension of $\mathcal{H}_\xi$.*

THEOREM. *A necessary and sufficient condition that L have an absolutely continuous spectrum is*

$$\int_\sigma P_\xi[f,g]\,d\xi=(f,g)_{\mathcal{H}} \quad \forall f,g\in\mathcal{H}.$$

These elementary theorems provide the basis for our method.

The idea is that for certain operators $\tilde{L}$ it is possible to find *explicitly* an analytic generating family $f(\cdot\,,z)$ relative to which the indicatrix functions for $\tilde{L}$ all satisfy

(1) $F_i(\xi,z)$ is sectionally holomorphic, $z\notin R$.

(2) $F_i(\xi,\lambda^\pm)$, the boundary values of $F_i(\xi,z)$ as $z\to\lambda\in R$, are finite almost everywhere, and are, in a precise sense, distributions.

(3) There exists a positive purely singular measure of finite total mass $d\mathcal{R}_\lambda(\cdot)$ defined on the Borel sets of the real line such that $\int_\Delta d\mathcal{R}_\lambda(\nu)$ is an integrable function of λ for each Borel set Δ, and for almost all λ

$$F_i(\xi,\lambda^+)=\left(1+\int_{\sigma(\tilde{L})}\frac{d\mathcal{R}_\lambda(\nu)}{\nu-\xi-i0}\right)F_i(\xi,\lambda^-).$$

When these conditions are satisfied we will say that the operator is *barrier related.*

Thus the problem of calculating the unitary invariants of $\tilde{L}$ is transformed into an analysis of the measure $dM_\lambda(\cdot)$, as explained in [3], and such an analysis can be explicitly carried out because it is possible to characterize the solutions of the barrier problem.

In the present work we will need a generalization of the method outlined above; namely, we will find operator-valued indicatrix functions $F_i(\xi,x)$ corresponding to operator valued analytic generating functions $f(\cdot\,,x)$ all acting from h to H such that for $\beta\in h$ and basis vectors $\{\theta_k\}\in h$

(1) $(\beta,F_i(\xi,x)\theta_k)_h=S_{ik}[f(\cdot\,,x)\beta]$;

(2) $F_i(\xi,\lambda+i0)=(1+\int(d\mathcal{R}_\lambda(\nu)/(\nu-\xi-i0))F_i(\xi,\lambda-i0)$, a.a. λ,

where $d\mathcal{R}_\lambda(\cdot)$ is now a positive operator valued singular measure mapping h into h.

The direct integral space $\mathcal{H}^*$ which diagonalizes $\tilde{L}$ will be constructed by forming a Hilbert space from the finite linear combinations $g(x)=\sum\mathcal{D}_\xi(x,y_i)\alpha_i$ where $\alpha_i\in h$, and $\mathcal{D}_\xi(x,y)$ is the operator mapping h into h defined by setting

$$(\beta,\mathcal{D}_\xi(x,y)\alpha)_h=\frac{\partial}{\partial\xi}(f(\cdot\,,x)\beta,E_\xi f(\cdot\,,y)\alpha)$$

$$=\frac{1}{2\pi i}\lim_{\eta\downarrow0}(f(\cdot\,,x)\beta,[(L-\xi-i\eta)^{-1}-(L-\xi+i\eta)^{-1}]f(\cdot\,,y)\alpha)$$

for $\alpha, \beta \in h$. The space $\mathcal{H}_\xi$ is formed by taking as the scalar product

$$(g, g')_{\mathcal{H}_\xi} = \sum_{i,j} (\mathcal{D}_\xi(x_j, x_i)\alpha_i, \alpha_i')$$

for the indicated finite linear combinations, and then taking the completion.

The kernel $\mathcal{D}_\xi(x, y)$ will be a reproducing kernel for the space in the sense that

$$(g(x), \alpha)_{\mathcal{H}_\xi} = (g(y), \mathcal{D}_\xi(x, y)\alpha)_{\mathcal{H}_\xi}.$$

In this paper we will take as our operator-valued analytic generating family $k^*(\lambda)/(\lambda - z)$. The main result (Theorem (3.3)) of the next section is that

$$\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int \frac{k(\mu)}{\mu - x}[(L-\xi+i\eta)^{-1} - (L-\xi-i\eta)^{-1}]\frac{k^*(\mu)}{\mu-\bar{y}}d\mu$$

$$= \frac{1}{2}\frac{E^*(\xi - i0, \bar{x})E(\xi + i0, \bar{y}) - E^*(\xi + i0, \bar{x})E(\xi - i0, \bar{y})}{x - \bar{y}}$$

$$= \mathcal{D}_\xi(x, y)$$

has the properties outlined above.

We will show that $(\alpha, \mathcal{D}_\xi(x, y)\beta)$ permits a bilinear expansion in the form

$$(\alpha, \mathcal{D}_\xi(x, y)\beta) = \sum_{i,j} S_{ij}\left[\frac{k^*(\mu)\, a}{\mu - x}\right](\xi)\, S_{ij}^*\left[\frac{k^*(\mu)\,\beta}{\mu - y}\right](\xi)$$

for certain partial isometries S_{ij}.

At this point it becomes necessary to comment upon another difficulty. If $k(\cdot)$ has a non-trivial null space, then the finite linear combinations $\sum_{m,p}(k^*(\lambda)/(\lambda - z_m))\alpha_p$, $\alpha_p \in h$ will not be dense in H, and may not even form an invariant subspace of L.

Thus, in this case, the partial isometries S_{ij} obtained as outlined above from the bilinear form $\mathcal{D}_\xi(x, y)$ will not be densely defined.

It might happen that the reducing subspaces of L to which the S_{ij} correspond do not have the whole space as direct sum.

We will show now, however, that we

(a) are able to extend the partial isometries to the smallest invariant subspace of both U and V which contains the range of their commutator, and

(b) the extended partial isometries constitute a complete set.

Assume for this purpose that partial isometries S_{ij} have been defined on a domain which consists at least of all vectors of the form $\sum_{m,n}(k^*(\lambda)/(\lambda - x_m))\alpha_n$, and that they satisfy the following relation for all $x, y \notin \sigma(M)$ and all $\alpha, \beta \in h$

$$\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_{\sigma(U)}\left(\frac{k(\mu)\beta}{\mu - x}, [(L-\xi+i\eta)^{-1}-(L-\xi-i\eta)^{-1}]\frac{k^*(\mu)\alpha}{\mu-y}\right)d\mu$$

$$= \sum_{i,j} S_{ij}\left[\frac{k^*(\mu)}{\mu - x}\beta\right](\xi)\,\overline{S_{ij}\left[\frac{k^*(\mu)}{\mu-y}\alpha\right]}(\xi) = \frac{\partial}{\partial\xi}\left(E_\xi\frac{k^*(\lambda)}{\lambda-x}\beta, \frac{k^*(\lambda)}{\lambda-y}\alpha\right),$$

where E_ξ is the spectral resolution of L.

Consider the closure $\mathbf{A}$ of the set of finite linear combinations of vectors having the form

$$\sum a_{i,k}(M-x_i)^{-1}(L-y_i)^{-1}k^*\alpha_k,$$

where x_i, y_j are complex numbers and α_k is some vector in h.

Lemma 2.1. *$\mathbf{A}$ is an invariant manifold for both L and M.*

Proof. Note that $(L-y)^{-1}(M-x)^{-1}k^*E(x,y) = (M-x)^{-1}(L-y)^{-1}k^*$ as operators on h. Thus

$$(L-\omega)^{-1}(L-y)^{-1}(M-x)^{-1}k^*\alpha = \frac{1}{\omega-y}(L-\omega)^{-1}(M-x)^{-1}k^*\alpha - \frac{1}{\omega-y}(L-y)^{-1}(M-x)^{-1}k^*\alpha$$

$$= \frac{1}{\omega-y}(M-x)^{-1}(L-\omega)^{-1}k^*E^*(\bar{x},\bar{y})\alpha - \frac{1}{\omega-y}(M-x)^{-1}(L-y)^{-1}k^*E^*(\bar{x},\bar{y})\alpha.$$

Hence, the resolvents of L applied to the finite linear combinations whose closure generates $\mathbf{A}$ have images of the same form. Clearly $\mathbf{A}$ is invariant under the action of the resolvents of M.

Since $\quad (M-x)^{-1}(L-y)^{-1}k^* = (L-y)^{-1}(M-x)^{-1}k^*E(x,y) \quad$ on $\mathbf{H}$

We may set

$$S_{ij}\{\sum_{m,n,o} a_{mno}(M-x_n)^{-1}(L-y_n)^{-1}k^*\alpha_0\}(\xi)$$

$$= S_{ij}\{\sum_{m,n,o} a_{mno}(L-y_n)^{-1}(M-x_m)^{-1}k^*E(x_m,y_n)\alpha_0\}$$

$$= \sum a_{mno}(\xi-y_n)^{-1}S_{ij}[(M-x_m)^{-1}k^*E(x_m,y_n)\alpha_0]$$

and since $E(y_n, x_m)\alpha_0 \in h$, $S_{ij}[(M-x_m)^{-1}k^*E(x_m,y_n)\alpha_0]$ will be determined once we have defined the operators S_{ij} on vectors of the form $k^*(M-x)^{-1}\alpha$.

Since the transformations S_{ij} are bounded, their extensions to **A** are uniquely determined.

LEMMA 2.2. *The partial isometries S_{ij} are complete.*

Proof.

$$\sum_{i,j} S_{ij}[(M-x)^{-1}(L-p)^{-1}k^*(\lambda)\beta](\xi)\overline{S_{ij}[(M-y)^{-1}(L-q)^{-1}k^*(\lambda)\alpha](\xi)}$$

$$=\sum_{i,j} S_{ij}[(L-p)^{-1}(M-x)^{-1}k^*(\lambda)E(p,x)\beta](\xi)\,S_{ij}[(L-q)^{-1}(M-y)^{-1}k^*(\lambda)E(q,y)\alpha](\xi)$$

$$=\frac{1}{\xi-p}\frac{1}{\xi-q}\sum_{i,j} S_{ij}[(M-x)^{-1}k^*(\lambda)E(p,x)\beta](\xi)\overline{S_{ij}[(M-y)^{-1}k^*(\lambda)E(q,y)\alpha](\xi)}$$

$$=\frac{1}{\xi-p}\frac{1}{\xi-q}\frac{\partial}{\partial\xi}(E_\xi(M-x)^{-1}k^*((\lambda)E(p,x)\beta,\,(M-y)^{-1}k^*(\lambda)E(q,y)\alpha)$$

$$=\frac{\partial}{\partial\xi}(E_\xi(L-p)^{-1}(M-x)^{-1}k^*(\lambda)E(p,x)\beta,\,(L-q)^{-1}(M-y)^{-1}k^*(\lambda)E(q,y)\alpha)$$

$$=\frac{\partial}{\partial\xi}(E_\xi(M-x)^{-1}(L-p)^{-1}k^*(\lambda)\beta,\,(M-y)^{-1}(L-q)^{-1}k^*(\lambda)\alpha)$$

and, if we integrate these last equations with respect to $d\xi$, we obtain

$$((M-x)^{-1}(L-p)^{-1}k^*(\lambda)\beta,\,(M-y)^{-1}(L-q)^{-1}k^*(\lambda)\alpha)$$
$$=\sum_{i,j}\int_{\sigma(V)} S_{ij}[(M-x)^{-1}(L-p)^{-1}k^*(\lambda)\beta](\xi)\overline{S_{ij}[(M-y)^{-1}(L-q)^{-1}k^*(\lambda)\alpha](\xi)}\,d\xi.$$

This in turn, implies that

$$\sum_{i,j}\int_{\sigma(V)} S_{ij}[f](\xi)\overline{S_{ij}[g](\xi)}\,d\xi=(f,g)_{\mathcal{H}}$$

for any vectors $f, g \in \mathbf{A}$. This is completeness.

The set **A** defined above is the smallest invariant manifold of both L and M containing the range of the commutator.

3. The Riemann–Hilbert problem for $E(l, z)$ corresponding to the spectral variable of U

Fix λ. Let us denote by $N_\lambda(k)$ the nullspace of $k(\lambda)$ in h, and by $N_\lambda(k)^\perp$ its orthogonal complement. If x and x' are elements of $N_\lambda(k)^\perp$ such that $kx=kx'$, then

$(x-x')\in N_\lambda(k)\cap N_\lambda(k)^\perp=\{0\}$, so that $x=x'$. Thus, the restriction $k_r=k\,|\,N_\lambda(k)^\perp$ k of to $N_\lambda(k)^\perp$ is a one to one linear transformation of $N_\lambda(k)^\perp$ onto $R_\lambda(k)$, the range of $k(\lambda)$. Thus $k(\lambda)$ has linear inverse $j=j(\lambda)$ which is defined on $R_\lambda(k)$.

Let us extend j from $R_\lambda(k)$ to all of h in the following way: for every $x\in h$ there exists a unique $v\in R_\lambda(k)$ and $\omega\in R_\lambda(k)^\perp$ such that $x=v+\omega$. The projection of h on $R_\lambda(k)$ along $R_\lambda(k)^\perp, P_\lambda$, is defined by $P_\lambda x=v$. The transformation $\mathfrak{J}_\lambda=\mathfrak{J}=j_\lambda P_\lambda$ is identical with j_λ on $R_\lambda(k)$ and is defined everywhere in h.

It is clear that

(a) $k\mathfrak{J}k=k$,

(b) $(k\mathfrak{J})\,|_{R(k)}=I\,|_{R(k)}$,

(c) $(\mathfrak{J}k)\,|_{R(\mathfrak{J}k)}=I\,|_{R(\mathfrak{J}k)}$.

Let $h(t)$ be an arbitrary differentiable vector which vanishes outside $(a+\varepsilon, b-\varepsilon)$ for some $\varepsilon>0$ and set

$$\left.\begin{aligned} f(\lambda)&=(L-\omega)^{-1}(M-z)^{-1}h(\lambda),\\ g(\lambda)&=(M-z)^{-1}(L-\omega)^{-1}h(\lambda).\end{aligned}\right\}\quad \operatorname{Im}\omega, \operatorname{Im} z\neq 0,$$

Then
$$[A(\lambda)-\omega]f(\lambda)+\frac{1}{\pi i}\mathbf{P}\int_{\sigma(U)}\frac{k^*(\lambda)k(\mu)}{\mu-\lambda}f(\mu)\,d\mu=(M-z)^{-1}h(\lambda),$$

$$[A(\lambda)-\omega](M-z)g(\lambda)+\frac{1}{\pi i}\mathbf{P}\int_{\sigma(U)}\frac{k^*(\lambda)k(\mu)}{\mu-\lambda}(M-z)g(\mu)\,d\mu=h(\lambda).$$

For $\operatorname{Im}\tau\neq 0$, define

$$F(\tau)=\frac{1}{2\pi i}\int_{\sigma(U)}\frac{k(\tau)f(t)}{t-\tau}\,dt,$$

$$G(\tau)=\frac{1}{2\pi i}\int_{\sigma(U)}\frac{k(t)(M-z)}{t-\tau}\,g(t)\,dt.$$

Then by the Plemelj-Privalow relations, extended for vector-valued integrands (where the subscripts $\pm$ refer to limits in τ taken from above and below the real axis) we have, almost everywhere,

$$f(\lambda)=\mathfrak{J}(F^+-F^-)+f_1,\quad g(\lambda)=(M-z)^{-1}\mathfrak{J}(G^+-G^-)+g_1,$$

where $f_1, g_1\in N(k)$.

Thus

$$(A-\omega)\mathcal{J}(F^+-F^-)+(A-\omega)f_1+k^*(F^++F^-)=(M-z)^{-1}h,$$

$$(A-\omega)\mathcal{J}(G^+-G^-)+(A-\omega)g_1+k^*(G^++G^-)=h.$$

Let $\tilde{g}_1=(M-z)^{-1}g_1$. Then

$$(A-\omega)\mathcal{J}[(M-z)^{-1}G^+-(M-z)^{-1}G^-]+(A-\omega)\tilde{g}_1+k^*[(M-z)^{-1}G^++(M-z)^{-1}G^-]$$
$$=(M-z)^{-1}h(\lambda),$$

and thus

$$(A-\omega)\mathcal{J}[F^+-(M-z)^{-1}G^+]+k^*[F^+-(M-z)^{-1}G^+]-[(A-\omega)\mathcal{J}[F^+-(M-z)^{-1}G^+]$$
$$+k^*[F^--(M-z)^{-1}G^-]]=(A-\omega)[f_1-g_1]$$

or

$$[(A-\omega)\mathcal{J}+k^*][F^+-(M-z)^{-1}G^+]-[(A-\omega)\mathcal{J}-k^*][F^--(M-z)^{-1}G^-]=(A-\omega)[f_1-g_1].$$

Thus

$$[\mathcal{J}+(A-\omega)^{-1}k^*][F^+-(M-z)^{-1}G^+]-[\mathcal{J}-(A-\omega)^{-1}k^*][F^--[M-z]^{-1}G^-]=f_1-g_1.$$

Thus

$$\mathcal{J}[(F^+-(M-z)^{-1}G^+)-(F^--(M-z)^{-1}G^-)]+(A-\omega)^{-1}k^*[(F^+-(M-z)^{-1})G^+)$$
$$+(F^--(M-z)^{-1}G^-)]=f_1-g_1$$

but $(F^+-(M-z)^{-1}G^+)-[F^--(M-z)^{-1}G^-]\in R(k)$ and $k\mathcal{J}|_{R(k)}=I_{R(k)}$. Thus, if we multiply this last equation by k we will get

$$[1+k(A-\omega)^{-1}k^*][F^+-(M-z)^{-1}G^+]-[1-k(A-\omega)^{-1}k^*][F^--(M-z)^{-1}G^-]$$
$$=k(f_1-g_1)=0.$$

At this point we will make use of some results due to I. C. Gohberg and M. G. Krein [8], which generalize results of Muschelischwili and Vekua, on the factorization of finite dimensional matrices.

Let $\mathcal{R}$ be the ring of functions $\mathcal{F}(\lambda)$ of the form $\mathcal{F}(\lambda)=C+\int_{-\infty}^{\infty}f(t)e^{it\lambda}dt$, $-\infty\leqslant\lambda\leqslant\infty, f\in L$, C constant.

By $\mathcal{R}^+$ denote the subring of $\mathcal{R}$ of functions $\mathcal{F}(\lambda)$ of the form $\mathcal{F}(\lambda)=C+\int_0^\infty f(t)e^{it\lambda}dt$, and by $\mathcal{R}^-$ the functions of the form $C+\int_{-\infty}^0 f(t)e^{it\lambda}dt$. Every function in $\mathcal{R}^+$ is defined by means of a function which is holomorphic inside the upper half plane

$\prod_+$ and which is continuous up to the boundary. Similarly, functions in $\mathcal{R}^-$ are defined by means of a function holomorphic in the lower half plane $\prod_-$ and continuous up to the boundary.

Let $\mathcal{R}_{n\times n}$ denote the ring of $n\times n$ matrices whose entries are all elements of $\mathcal{R}$, and let $\mathcal{R}^+_{n\times n}$ and $\mathcal{R}^-_{n\times n}$ denote the corresponding ring of matrices whose entries are respectively in $\mathcal{R}^+$ and in $\mathcal{R}^-$.

THEOREM (Gohberg and Krein). *In order that the non-singular matrix function $T(\lambda)\in\mathcal{R}_{n\times n}$ possess a representation of the form $T(\lambda)=\varphi_+(\lambda)\varphi_+^*(\lambda)$ in which $\varphi_+(\lambda)\in\mathcal{R}^+_{n\times n}$ and* determinant $(\varphi_+(\lambda))\neq 0$, $\lambda\in\prod_+$, *it is necessary and sufficient that $T(\lambda)$ be positive definite.*

In addition, it follows from results of those authors that $\varphi_+^*(\lambda)\in\mathcal{R}^-_{n\times n}$.

Suppose now that $A(\lambda)$ and $k(\lambda)$ are n-dimensional operators and that they belong to the ring $\mathcal{R}_{n\times n}$. (We will carry out a series of calculations on this assumption, and then by passing to a limit in the final step, do away with these restrictions.)

We have already shown that

$$[1+k(A-\omega)^{-1}k^*][F^+-(M-z)^{-1}G^+]=[1-k(A-\omega)^{-1}k^*][F^--(M-z)^{-1}G^-],$$

and with A and k restricted by the smoothness and dimensionality restrictions imposed just above we can now assert the existence of matrix functions $\varphi_+(\lambda)$ and $\varphi_-(\lambda)\in\mathcal{R}^+_{n\times n}$ and $\mathcal{R}^-_{n\times n}$, respectively such that

$$\varphi_+(\lambda)=\frac{1-k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}{1+k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}\,\varphi_-(\lambda)$$

for ω real and not in $\sigma(L)$.(1)

But this equation taken in conjunction with the immediately preceding equation implies that

$$\varphi_+^{-1}(\lambda)[(M-z)F^+(\lambda)-G^+(\lambda)]=\varphi_-^{-1}(\lambda)[(M-z)F^-(\lambda)-G^-(\lambda)].$$

Since the left-hand side of this equation belongs to the ring $\mathcal{R}^-_{n\times n}$ and the right-hand side belongs to the ring $\mathcal{R}^+_{n\times n}$(2) it follows, by a simple extension of Liouville's theorem, that each side equals a constant matrix.

(1) Since Theorem 5.1 shows that we are factoring a positive operator.

(2) Using a well-known theorem of N. Wiener. See [8] p. 249.

Now, we can evaluate the constant by analytically continuing λ to τ (in either half-plane). Then we will have

$$\varphi(\tau)^{-1}[(\tau-z)F(\tau)-G(\tau)]=-\varphi(z)^{-1}G(z)$$

by setting $\tau=z$.

But

$$G(z)=\frac{1}{2\pi i}\int k(t)g(t)\,dt,$$

thus
$$G^{+}(\lambda)-(M-z)F^{+}(\lambda)=\varphi(\lambda+i0)\,\varphi(z)^{-1}G(z),$$

$$G^{-}(\lambda)-(M-z)F^{-}(\lambda)=\varphi(\lambda-i0)\,\varphi(z)^{-1}G(z),$$

so
$$(G^{+}(\lambda)-G^{-}(\lambda))-(M-z)(F^{+}(\lambda)-F^{-}(\lambda))=(\varphi(\lambda+i0)-\varphi(\lambda-i0))\,\varphi(z)^{-1}G(z).$$

Thus
$$k(\lambda)(M-z)\,g(\lambda)-(M-z)\,k(\lambda)\,f(\lambda)=(\varphi(\lambda+i0)-\varphi(\lambda-i0))\,\varphi(z)^{-1}G(z)$$

and
$$k(\lambda)(M-z)\,[(L-\omega)^{-1}(M-z)^{-1}h(\lambda)-(M-z)^{-1}(L-\omega)^{-1}h(\lambda)]$$
$$=-[\varphi(\lambda+i0)-\varphi(\lambda-i0)]\,\varphi(z)^{-1}G(z).$$

But
$$(L-\omega)^{-1}(M-z)^{-1}-(M-z)^{-1}(L-\omega)^{-1}$$
$$=(L-\omega)^{-1}(M-z)^{-1}\left(\frac{LM-ML}{\pi i}\right)(M-z)^{-1}(L-\omega)^{-1}.$$

Thus, by substituting this relation, and evaluating $G(z)$, we obtain

$$-k(\lambda)(M-z)(L-\omega)^{-1}(M-z)^{-1}k^{*}(\lambda)\int k(t)(M-z)^{-1}(L-\omega)^{-1}h(t)\,dt$$

$$=[\varphi(\lambda+i0)-\varphi(\lambda-i0)]\frac{\varphi(z)^{-1}}{2}\int k(t)(M-z)^{-1}(L-\omega)^{-1}h(t)\,dt.$$

Since $h(t)$ is arbitrary the integral in this expression ranges through the range of the operator $\int k(t)\,dt$, which is $\mathbf{H}$.

In $\mathbf{H}$ we, accordingly, have the operator identity

$$-k(\lambda)(L-\omega)^{-1}(M-z)^{-1}k^{*}(\lambda)=\frac{(M-z)^{-1}}{2}[\varphi(\lambda+i0)-\varphi(\lambda-i0)]\,\varphi(z)^{-1}.$$

Now multiply both sides by z and let $z\to\infty$, to get

$$-k(\lambda)(L-\omega)^{-1}k^{*}(\lambda)=\tfrac{1}{2}[\varphi(\lambda+i0)-\varphi(\lambda-i0)].$$

For a contour C enclosing $\sigma(U)$ oriented in the clockwise way, we can use simple estimates and Cauchy's theorem to show that

$$\int_{\sigma(U)} \frac{\varphi(\lambda+i0)-\varphi(\lambda-i0)}{\lambda-z}\,d\lambda=\int_C \frac{\varphi(\theta)}{\theta-z}\,d\theta$$

but, for z outside C, the Cauchy residue theorem tells us that

$$\int_C \frac{\varphi(\theta)}{\theta-z}\,d\theta=2\pi i[\varphi(z)-1].$$

Thus, by analytic continuation throughout the domain of analyticity

$$\varphi(z)=1-\frac{1}{\pi i}\int_{\sigma(U)} k(\lambda)\,(M-z)^{-1}(L-\omega)^{-1}k^*(\lambda)\,d\lambda.$$

Since $E^*(\bar\omega,\bar z)=E^{-1}(\omega,z)$ we have proved

THEOREM 3.1a.

$$E(\omega,z)=\varphi(z)^{-1} \textit{ on } \mathbf{H}.$$

We have assumed in the foregoing paragraphs that $B(\lambda)$ and $k(\lambda)$ are finite dimensional matrices continuous in λ. The general case where $A(\lambda)$ and $k(\lambda)$ are weakly measurable essentially bounded functions of λ can now be obtained by an approximation argument(1), and we finally obtain

THEOREM 3.2.

$$\frac{I-k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}{I+k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}=E^*(\bar\omega,\lambda-i0)\,E(\omega,\lambda-i0).$$

THEOREM 3.3.

The following relation is an operator identity on h

$$\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_a^b \frac{k(\lambda)}{\lambda-x}\left((L-\xi+i0)^{-1}-(L-\xi-i0)^{-1}\right)\frac{k^*(\lambda)}{\lambda-\bar y}\,d\lambda$$

$$=\frac{1}{2}\,\frac{E^*(\xi-i0,\bar x)\,E(\xi+i0,\bar y)-E^*(\xi+i0,\bar x)\,E(\xi-i0,\bar y)}{x-\bar y}.$$

(1) See also R. G. Douglas: On factoring positive operator functions. J. Math. Mech. 16 (1966), 119–126, Theorem 4.

Proof. We have already established the fundamental identity

$$k(\lambda)(L-\omega)^{-1}(M-y)^{-1}k^*(\lambda)=\frac{1}{2}(M-y)^{-1}H^*(\bar{\omega},\lambda)E(\omega,y),$$

where
$$H(\omega,\lambda)\equiv E(\omega,\lambda+i0)-E(\omega,\lambda-i0).$$

Thus
$$\frac{k(\lambda)}{\lambda-x}(L-\omega)^{-1}\frac{k^*(\lambda)}{\lambda-y}=\frac{1}{2}\frac{H^*(\bar{\omega},\lambda)}{(\lambda-x)(\lambda-y)}E(\omega,y),$$

but
$$\frac{1}{\lambda-x}\frac{1}{\lambda-y}=\frac{1}{x-y}\left(\frac{1}{\lambda-x}-\frac{1}{\lambda-y}\right)$$

and
$$\frac{1}{2\pi i}\int_a^b\frac{E^{-1}(\omega,\lambda-i0)-E^{-1}(\omega,\lambda+i0)}{\lambda-z}d\lambda=1-E^{-1}(\omega,z).$$

Hence, since

$$\frac{1}{2}\frac{1}{x-y}\{(1-E^{-1}(\omega,y))E(\omega,y)-(1-E^{-1}(\omega,x))E(\omega,y)\}=\frac{1}{2}\frac{1}{x-y}\{E^*(\bar{\omega},\bar{x})E(\omega,y)-1\},$$

we find that

$$\mathcal{P}_\xi(x,y)=\frac{1}{2}\frac{1}{x-\bar{y}}\{E^*(\xi-i0,\bar{x})E(\xi+i0,y)-E^*(\xi+i0,\bar{x})E(\xi-i0,y)\}.$$

4. Absolute continuity of spectral measure

THEOREM 4.1. *Under the hypothesis of Theorem I, V restricted to the smallest subspace of H, Γ, reducing both U and V and containing the range of C has an absolutely continuous spectral measure.*

Proof. If $VU-UV=(1/\pi i)C$ then the theorem of Putnam to which we have referred before [7] affords the quickest proof. Putnam's theorem asserts that $\Gamma\subset H_a(U)$. Set $W=-U$. Then $WV-VW=(1/\pi i)C$. Thus $\Gamma\subset H_a(V)$, q.e.d.

Another proof of this result can also be given. We can evaluate $\int_\sigma P_\xi(x,y)\,d\xi$ by residues to get $\int P_\xi(x,y)\,d\xi=(f,g)_{\mathcal{H}}$. The characterization of absolutely continuous spectral measures given in the digression then implies the result of the theorem.

5. The dual Hilbert problem

THEOREM 5.1. *There exists a positive one parameter operator-valued family of measures $d\mathcal{R}_\lambda(\cdot)$ such that*

$$\frac{1+k(\lambda)(A(\lambda)-l)^{-1}k^*(\lambda)}{1-k(\lambda)(A(\lambda)-l)^{-1}k^*(\lambda)}=1+\int\frac{d\mathcal{R}_\lambda(\nu)}{\nu-l}.$$

Proof. For any invertible operator Q, $\operatorname{Im} Q = -Q(\operatorname{Im} Q^{-1})Q^*$. But if $a(l) = k(\lambda)(A(\lambda)-l)^{-1}k^*(\lambda)$ then $a(l)$ has imaginary part positive in the upper half plane, and we can find a positive measure $dS_\lambda(\cdot)$ such that

$$a(l) = \int \frac{dS_\lambda(\nu)}{\nu - l}.$$

Now

$$\operatorname{Im}\left[\frac{1}{1-\int \frac{dS_\lambda(\nu)}{\nu-l}}\right] = -\left[1-\int\frac{dS_\lambda(\nu)}{\nu-l}\right] \operatorname{Im}\left[1-\int\frac{dS_\lambda(\nu)}{\nu-l}\right]\left[\int 1-\int\frac{dS_\lambda(\nu)}{\nu-l}\right]^*$$

$$= \left[1-\int\frac{dS_\lambda(\nu)}{\nu-l}\right] \operatorname{Im}\int\frac{dS_\lambda(\nu)}{\nu-l}\left[1-\int\frac{dS_\lambda(\nu)}{\nu-l}\right]^* \geqslant 0,$$

hence, using the fact that $1-a(l)\to 1$ when l becomes infinite, we can conclude that there exists a positive measure $dR_\lambda(\cdot)$ such that

$$\frac{1}{1-\int\frac{dS_\lambda(\nu)}{\nu-l}} = 1+\int\frac{dR_\lambda(\nu)}{\nu-l}.$$

Let
$$E_2(l) = \frac{1}{1-\int\frac{dS_\lambda(\nu)}{\nu-l}} = 1+\int\frac{dR_\lambda(\nu)}{\nu-l}$$

and
$$E_1(l) = 1+\int\frac{dS_\lambda(\nu)}{\nu-l}.$$

Then
$$1-E_2^{-1}(l) = \int\frac{dS_\lambda(\nu)}{\nu-l} = E_1(l)-1$$

so that
$$E_1(l)\,E_2(l) = [2-E_2^{-1}]E_2 = 2E_2-1 = 1+\int\frac{2dR_\lambda(\nu)}{\nu-l}, \quad \text{q.e.d.}$$

It is clear that the measure $dR_\lambda(\cdot)$ is of trace class, since the operator $k(\lambda)$ is Hilbert–Schmidt.

We have encountered the operators $E(l,z)$ as the solution of the Riemann–Hilbert problem

$$\frac{1-k(\lambda)(A(\lambda)-l)^{-1}k^*(\lambda)}{1+k(\lambda)(A(\lambda)-l)^{-1}k^*(\lambda)}E^*(\bar{l},\lambda+i0) = E^*(\bar{l},\lambda-i0).$$

We now wish to show that the duality between U and V extends to a duality between their respective spectral variables z and l. That is, we will establish

THEOREM 5.2. *There exists a positive one parameter family of h-valued operator measures $dM(\cdot)$ such that*

$$E^*(\xi - i0, z) = E^*(\xi + i0, z)\left(I + \int \frac{dM_\xi(\mu)}{\mu - \bar{z}}\right).$$

Proof. $$\mathcal{D}_\xi^*(x, y) = \frac{1}{2}\frac{E^*(\xi + i0, \bar{y})\, E(\xi - i0, \bar{x}) - E^*(\xi - i0, \bar{y})\, E(\xi + i0, \bar{x})}{\bar{x} - y}.$$

Thus $$E(\xi - i0, y)\, \mathcal{D}_\xi^*(x, y)\, E^*(\xi - i0, x) =$$

$$\frac{1}{2}\frac{1}{\bar{x} - y}[E(\xi - i0, \bar{x})\, E^*(\xi - i0, x) - E(\xi - i0, y)\, E^*(\xi - i0, \bar{y})].$$

Now set $y = x$, then

$$0 \leqslant \frac{1}{\operatorname{Im} x} \operatorname{Im} E(\xi - i0, \bar{x})\ E^*(\xi - i0, x),$$

since $\mathcal{D}_\xi(x, x)$ is a positive operator. Hence, when $\operatorname{Im} x > 0$, $\operatorname{Im} E(\xi - i0, \bar{x})\, E^*(\xi - i0, x) \geqslant 0$. Thus, by the operator generalization of the familiar representation theorem for functions analytic in the upper half plane with positive imaginary part, we can conclude that

$$E(\xi - i0, \bar{x})\, E^*(\xi - i0, x) = I + \int \frac{dM_\xi(\mu)}{\mu - \bar{x}}$$

(since $E(\omega, z) \to I$ as $z \to \infty$).

THEOREM 5.3. *$E(\xi - i0, z)\, E^*(\xi - i0, \bar{z}) - I$ is of trace class.*

Proof.

$$\begin{aligned} &E(l, z)\, E^*(l, \bar{z}) \\ &= \left(I + \frac{1}{\pi i}\int k(\lambda)\,(L - l)^{-1}(M - z)^{-1}k^*(\lambda)\, d\lambda\right)\left(I - \frac{1}{\pi i}\int k(\lambda)\,(M - z)^{-1}(L - \bar{l})k^*(\lambda)\, d\lambda\right) \\ &= I + \frac{1}{\pi i}\int k(\lambda)\,(L - l)^{-1}(M - z)^{-1}k^*(\lambda)\, d\lambda - \frac{1}{\pi i}\int k(\lambda)\,(M - z)^{-1}(L - \bar{l})k^*(\lambda)\, d\lambda \\ &\quad + \frac{1}{\pi^2}\int k(\lambda)\,(L - l)^{-1}(M - z)^{-1}k^*(\lambda)\, d\lambda \int k(\mu)\,(M - z)^{-1}(L - \bar{l})k^*(\mu)\, d\mu. \end{aligned}$$

But $\quad (L-l)^{-1}(M-z)^{-1}-(M-z)^{-1}(L-l)^{-1}=(M-z)^{-1}(L-l)^{-1}\frac{C}{\pi}(L-l)^{-1}(M-z)^{-1}.$

Thus

$$E(l,z)\,E^*(l,\bar{z})=I+\frac{1}{\pi i}\int k(\lambda)\,(M-z)^{-1}(L-l)^{-1}\frac{C}{\pi i}(L-l)^{-1}(M-z)^{-1}k^*(\lambda)\,d\lambda$$

$$+\frac{1}{\pi i}\int k(\lambda)\,(M-z)^{-1}[(L-l)^{-1}-(L-\bar{l})^{-1}]\,k^*(\lambda)\,d\lambda$$

$$+\frac{1}{\pi^2}\int k(\lambda)\,(L-l)^{-1}(M-z)^{-1}\,k^*(\lambda)\,d\lambda\int k(\mu)\,(M-z)^{-1}(L-\bar{l})^{-1}\,k^*(\mu)\,d\mu$$

$$=I-\frac{1}{\pi^2}\int k(\lambda)\,(M-z)^{-1}(L-l)^{-1}\,k^*(\lambda)\,d\lambda\int k(\mu)\,(L-l)^{-1}(M-z)^{-1}k^*(\mu)\,d\mu$$

$$+\frac{1}{\pi i}\int k(\lambda)\,(M-z)^{-1}[(L-l)^{-1}-(L-\bar{l})^{-1}]k^*(\lambda)\,d\lambda$$

$$+\frac{1}{\pi^2}\,k(\lambda)\,(L-l)^{-1}(M-z)^{-1}k^*(\lambda)\,d\lambda\int k(\mu)\,(M-z)^{-1}(L-\bar{l})^{-1}k^*(\mu)\,d\mu.$$

We now define a C_1 valued indefinite integral on $(\mathcal{R},\sum_\beta)$. (C_1 denotes trace class, T_r is the trace, $\mathcal{R}$ is the real line.) A C_1-valued measure μ on $(\mathcal{R},\sum_\beta)$ is said to be a C_1-valued indefinite integral on $(\mathcal{R},\sum_\beta)$ if there exists a C_1-valued function $\mu\in L_1(T_r\,\mu, C_1)$ such that

$$\mu(e)=\int_e \mu'(s)\,T_r\,\mu(ds).$$

THEOREM (Kuroda [9]). *Let $E(e)$ be a spectral measure on $(\mathcal{R},\sum_\beta)$ and let $A\in C_2$. Then the set function $A^*E(e)A$, $e\in\sum_\beta$ is a C_1-valued indefinite integral on $(\mathcal{R},\sum_\beta)$.*

THEOREM (de Branges [10]). *Let σ be a scalar measure on $(\mathcal{R},\sum_\beta)$ and let $x(\lambda)$ be a C_1-valued σ-measurable function which satisfies $(1+\lambda^2)^{-1}x(\lambda)\in L_1(\sigma, C_1)$ and which is positive a.a. λ. Then for a.a. λ*

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{1}{\mu-(\lambda\pm i\varepsilon)}x(\mu)\,\sigma(d\mu)\ \textit{exists in}\ C_2(H).$$

These two results enable us to analyze each factor in the products of integrals above. Thus, by Kuroda's theorem we may write

$$\int k(t)\,(L-l)^{-1}k^*(t)\,(M-z)^{-1}\,dt$$

$$=\int\left[\int\left[\frac{\partial k(t)\,E(\nu)\,k^*(t)}{\partial_\nu\,T_r\,k(t)\,E(\nu)\,k^*(t)}\right]\frac{d\nu[T_r\,k(t)\,E(\nu)\,k^*(t)]}{\nu-l}\right]\frac{dt}{t-\tau}$$

with

$$\chi(\nu)\equiv\frac{\partial k(t)\,E(\nu)\,k^*(t)}{\partial_\nu\,T_r[k(t)\,E(\nu)\,k^*(t)]}\in C_1.$$

De Brange's theorem tells us that the inner integral exists as a limit in C_2 as l approaches the real axis.

Since the product of two H.S. operators is in C_1, the theorem will be proved if we can show that the last integral has a limiting value in C_1.

This is however implied by the familiar lemma. Let σ be a scalar measure on $(\mathcal{R}, \sum_\beta)$ and let $\chi(\lambda)$ be a C_1-valued σ-measurable function which satisfies $(1+\lambda^2)^{-1}\chi(\lambda)\in L_1(\sigma, C_1)$. Then

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{\varepsilon}{(\mu-\lambda)^2+\varepsilon^2}\chi(\mu)\,d\sigma(\mu)=\pi\sigma'(\lambda)\,\chi(\lambda)\quad\text{a.e.}$$

Take $\chi(\nu)$ as before, and note that the absolute continuity of dE_ν implies the existence of the Radon–Nikodym-derivative

$$\frac{d_\nu[T_r\,k(t)\,E(\nu)\,k^*(t)]}{d_\nu}.$$

6. The decomposition, definition of S_{ij}

THEOREM 6.1.

$$\mathcal{D}_\xi(x,y)=\frac{1}{2}E^*(\xi+i0,\bar{x})\int\frac{dM_\xi(\mu)}{(\mu-x)\,(\mu-\bar{y})}E(\xi+i0,\bar{y}).$$

Proof.

$$E(\xi-i0,x)=\left(1+\int\frac{dM_\xi(\mu)}{\mu-x}\right)E(\xi+i0,x),$$

$$E^*(\xi+i0,\bar{x})\left(1+\int\frac{dM_\xi(\mu)}{\mu-x}\right)=E^*(\xi-i0,\bar{x}).$$

Thus,

$$\mathcal{D}_\xi(x, y) = \frac{E^*(\xi - i0, \bar{x})\, E(\xi + i0, \bar{y}) - E^*(\xi + i0, \bar{x})\, E(\xi - i0, \bar{y})}{2(x - \bar{y})}$$

$$= \frac{1}{2}\frac{1}{x-\bar{y}}\left[E^*(\xi + i0, \bar{x})\left[1 + \int \frac{dM_\xi(\mu)}{\mu - x} - 1 - \int \frac{dM_\xi(\mu)}{\mu - \bar{y}}\right] E(\xi + i0, \bar{y})\right]$$

$$= \frac{1}{2} E^*(\xi + i0, \bar{x}) \int \frac{dM_\xi(\mu)}{(\mu - x)(\mu - \bar{y})} E(\xi + i0, \bar{y}).$$

We have seen in Theorem 5.3. that $\int dM_\xi(\mu)/(\mu - z)$ is an operator of trace class. From this it follows that the operator $dM_\xi(\cdot)$ is of trace class; furthermore it is absolutely continuous with respect to the scalar measure $dT_r M_\xi(\cdot)$, and therefore there exists a C_1-valued operator $M'_\xi(\cdot)$ such that

$$M_\xi(e) = \int_e M'_\xi(\mu)\, dT_r\, M_\xi(\mu)$$

for any Borel set e.

Lemma (M. Rosenberg [14]).

$$0 \leqslant M'_\xi(\lambda) \leqslant I \quad (\textit{a.e. with respect to } dT_r\, M_\xi(\cdot))$$

Proof. Take a fixed $x \in l_2$, then for each Borel set e,

$$(M_\xi(e)x, x) = \int_e (M'_\xi(\lambda)\, x, x)\, dT_r\, M_\xi(\lambda) \geqslant 0.$$

Hence, $(M'_\xi(\mu)x, x) \geqslant 0$ on $\mathcal{R} - N_x$, where $T_r\, M_\xi(N_x) = 0$.

Since l_2 is separable, there exists a countably dense subset of $l_2, \{x_i\}$, such that $(M'_\xi(\lambda)x_i, x_i) \geqslant 0$ on $\mathcal{R} - N_{x_i}$. But $T_r\, M_\xi(N \equiv \bigcup N_{x_i}) = 0$. Hence $M'_\xi(\lambda) \geqslant 0$ on $\mathcal{R} - N$.

In a similar way we can establish that $M'_\xi(\lambda) \leqslant I$ almost everywhere with respect to the trace measure.

Now we can write

$$\int \frac{dM_\xi(\mu)}{\mu - z} = \int M'_\xi(\mu) \frac{dT_r\, M_\xi(\mu)}{\mu - z}.$$

Thus

$$\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int\left(\frac{k^*(\lambda)}{\lambda-\bar{x}}\alpha,[(L-\xi+i0)^{-1}-(L-\xi-i0)^{-1}]\frac{k^*(\lambda)}{x-\bar{y}}\beta\right)_h d\lambda$$

$$=\frac{1}{2}\left(E(\xi+i0,\bar{x})\alpha,\int\frac{dM_\xi(\mu)}{(\mu-x)(\mu-\bar{y})}E(\xi+i0,\bar{y})\beta\right)_h$$

$$=\frac{1}{2}\int\frac{(E(\xi+i0,\bar{x})\alpha,M'_\xi(\mu)E(\xi+i0,\bar{y})\beta)_h}{(\mu-\bar{x})(\mu-y)}dT_r M_\xi(\mu).$$

Denote by $\{\theta_j(\xi,\mu)\}$ the complete orthonormal set in h consisting of eigenvectors of $M'_\xi(\mu)$. Let the corresponding eigenvalues be called $\{\lambda_j(\xi,\mu)\}$, (we consider these numbers to be ordered so that $0\leqslant\ldots\leqslant\lambda_2(\xi,\mu)\leqslant\lambda_1(\xi,\mu)\leqslant 1$). Then Parseval's identity enables us to conclude that the last expression above is equal to:

$$\tfrac{1}{2}\sum_j\int\frac{(\alpha,E^*(\xi+i0,\bar{x})\theta_j(\xi,\mu))_h}{\mu-\bar{x}}\frac{\overline{(\beta,E^*(\xi+i0,\bar{y})\theta_j(\xi,\mu))_h}}{\mu-y}\lambda_j(\xi,\mu)\,dT_r M_\xi(\mu).$$

Now for fixed ξ consider the L_2 space formed with respect to the measure $\int_\Delta\lambda_j(\xi,\mu)\,dT_r M_\xi(\mu)\equiv M_\xi^{(j)}(\Delta)$. Call this space $L_2(dM_\xi^{(j)}(\cdot))$. Form the direct integral of these spaces with respect to $d\xi$. In the direct integral space select a complete orthonormal set $\{P_{i,j}(\xi,\mu)\}$. This set will have the property that for almost all ξ $\{P_{i,j}(\xi,\mu)\}$ is a complete orthonormal set in $L_2(dM_\xi^{(j)}(\cdot))$; moreover, it will be ordered. Therefore, we can again use Parseval's equality to assert that the above expression is equal to

$$\tfrac{1}{2}\sum_{i,j}\int\frac{P_{i,j}(\xi,\mu)}{\mu-x}(\alpha,E^*(\xi+i0,\bar{x})\theta_j(\xi,\mu))_h\,dM_\xi^{(j)}(\mu)$$

$$\times\overline{\int\frac{P_{i,j}(\xi,\mu)}{\mu-\bar{y}}(\beta,E^*(\xi+i0,y)\theta_j(\xi,\mu))_h\,dM_\xi^{(j)}(\mu)}.$$

Now define

$$S_{ij}\left[\frac{k^*(\lambda)}{\lambda-x}\alpha\right](\xi)=2^{-\frac{1}{2}}\int\frac{P_{i,j}(\xi,\mu)}{\mu-x}(\alpha,E^*(\xi+i0,\bar{x})\theta_j(\xi,\mu))_h\,dM_\xi^{(j)}(\mu).$$

LEMMA. *If* $\sum_{i,j}a_{ij}S_{ij}\left[\frac{k^*(\lambda)}{\lambda-x}\alpha\right](\xi)=0$, *for all* $x\notin\sigma(U)$

for all $\alpha\in h$, *for certain complex valued constants* a_{ij}, *then* $a_{ij}=0$ *all* i,j.

Proof. The hypothesis immediately implies that

$$\int \sum_{i,j} a_{ij} \frac{P_{i,j}(\xi,\mu)}{\mu - x} \theta_j(\xi,\mu)\, dM_\xi^{(j)}(\mu)$$

is the null vector of h. Since we can take $\alpha = E(\xi + i0, x)\tau$ where τ ranges throughout h. But functions of the form $\sum \alpha_\varrho/(\mu - x_\varrho)$ are dense in $L_2(\sigma(U), d\mu)$; thus, the vanishing of

$$\int \sum_{i,j} a_{ij} \frac{P_{i,j}(\xi,\mu)}{\mu - x} \theta_j(\xi,\mu)\, \lambda_j(\xi,\mu)\, dT_r M_\xi(\mu)$$

for an infinite unbounded sequence of non-real values of x implies that

$$\sum_{i,j} a_{ij} P_{i,j}(\xi,\mu)\, \theta_j(\xi,\mu)\, \lambda_j(\xi,\mu) = 0$$

on the support of $dT_r M_\xi(\cdot)$. Now

$$\sum_{i,j} a_{i,j} P_{i,j}(\theta_j(\xi,\mu), \theta_k(\xi,\mu))_h \lambda_j(\xi,\mu) = \sum_i a_{i,k} P_{i,k}(\xi,\mu)\, \lambda_k(\xi,\mu) = 0.$$

And we can multiply the last expression by $\overline{P_{r,k}(\xi,\mu)}$ and integrate with respect to $dT_r M_\xi(\mu)$ to get

$$\sum_i a_{i,k} \int P_{i,k}(\xi,\mu)\, \overline{P_{r,k}(\xi,\mu)}\, dM_\xi^{(k)}(\mu) = 0$$

But $\{P_{i,k}(\xi,\mu)\}$ is a complete orthonormal set in $L_2(dM_\xi^{(k)}(\mu))$. Hence

$$\textstyle\sum_i a_{i,k}\delta_{i,r} = a_{rk} = 0, \quad \text{q.e.d.}$$

We have seen that

$$\frac{\partial}{\partial \xi}\left(E_\xi \frac{k^*(\lambda)}{\lambda - x}\alpha, \frac{k^*(\lambda)}{\lambda - y}\beta\right) = \sum_{i,j} S_{ij}\left[\frac{k^*(\lambda)}{\xi - x}\alpha\right](\xi)\overline{S_{ij}\frac{k^*(\lambda)}{\lambda - y}\beta(\xi)}.$$

This equation, in turn, implies that

$$\frac{\partial}{\partial \xi}\left(E_\xi (L-p)^{-1}\frac{k^*(\lambda)}{\lambda - x}\alpha, \frac{k^*(\lambda)}{\lambda - y}\beta\right) = \frac{1}{\xi - p}\sum_{i,j} S_{ij}\left[\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi)\overline{S_{ij}\left[\frac{k^*(\lambda)}{\lambda - y}\beta\right](\xi)}.$$

Now consider the S_{ij} extended to $\mathbf{A}$, the smallest invariant manifold of both L and M containing the range of the commutator (as in the sequel to Lemma 2.2). Then

$$S_{ij}\left[(L-p)^{-1}\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi)$$

is defined, and

$$\frac{\partial}{\partial \xi}(E_\xi f, g) = \sum_{i,j} S_{ij}[f](\xi)\,\overline{S_{ij}[g](\xi)}, \qquad f, g \in \mathbf{A}.$$

Thus, we can conclude that

$$\sum_{i,j} S_{ij}\left[(L-p)^{-1}\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi)\,\overline{S_{ij}\left[\frac{k^*(\lambda)}{\lambda - y}\beta\right](\xi)}$$

$$= \sum_{i,j} \frac{1}{\xi - p} S_{ij}\left[\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi)\,\overline{S_{ij}\left[\frac{k^*(\lambda)}{\lambda - y}\beta\right](\xi)}$$

all $\alpha, \beta \in h$, $x, y \notin \sigma(M)$. And the previous lemma makes it possible for us to conclude that

$$S_{ij}\left[(L-p)^{-1}\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi) = \frac{1}{\xi - p} S_{ij}\left[\frac{k^*(\lambda)}{\lambda - x}\alpha\right](\xi).$$

7. The commutator with one dimensional range

In this section we will specialize our results to the case where C has one dimensional range.

THEOREM 7.1. *There exists a measurable function $g(\nu, \mu)$, $\nu \in \sigma(V)$, $\mu \in \sigma(U)$, such that $0 \leqslant g(\nu, \mu) \leqslant 1$, and*

$$E(l, z) = \begin{bmatrix} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\sigma(U)}\int_{\sigma(V)} g(\nu, \mu)\frac{d\nu}{\nu - l}\frac{d\mu}{\mu - z}\right\} & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & \cdot & \cdot & \dots \end{bmatrix}.$$

If either U or V has finite spectral multiplicity, then $g(\nu, \mu)$ takes on only the values zero and one, i.e. it is the characteristic function of some set in $\sigma(U) \times \sigma(V)$.

Proof. In Theorem 5.1 we have shown that we may write

$$\frac{1 + k(\lambda)(A(\lambda) - \omega)^{-1}k^*(\lambda)}{1 - k(\lambda)(A(\lambda) - \omega)^{-1}k^*(\lambda)} = 1 + \int \frac{d\mathcal{R}_\lambda(v)}{v - \omega}.$$

Since C has one dimensional range the matrix $\mathcal{R}_\lambda(\cdot)$ essentially reduces to a scalar and the above operator on h takes the form

$$\begin{bmatrix} 1+\displaystyle\int\frac{dR_\lambda(v)}{v-\omega} & 0 & 0 & \cdots\cdots \\ 0 & 1 & 0 & \cdots\cdots \\ 0 & 0 & 1 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{bmatrix}$$

By a theorem of Verblunsky [11] we may set

$$1+\int\frac{dR_\lambda(v)}{v-\omega}=\exp\left\{\int g(v,\lambda)\frac{dv}{v-\omega}\right\}$$

for a $g(v,\lambda)$ with $0\leqslant g(v,\lambda)\leqslant 1$, and, by a theorem of Aronszajn and Donoghue [12], $g(v,\lambda)$ is a characteristic function if and only if $dR_\lambda(\cdot)$ is a singular measure.

If U has finite spectral multiplicity, then $A(\lambda)$ is a finite dimensional symmetric matrix and hence has only a finite number of eigenvalues. Thus the singularity of the measure $dG_\lambda(\cdot)$ follows in a simple way.

Now form

$$\log\det\left[\frac{1+k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}{1-k(\lambda)\,(A(\lambda)-\omega)^{-1}k^*(\lambda)}\right]=\log\det E(\omega,\lambda+i0)-\log\det E(\omega,\lambda-i0).$$

But
$$\log\det\left(1+\int\frac{dR_\lambda(v)}{v-\omega}\right)=\int g(\nu,\lambda)\frac{d\nu}{\nu-\omega}.$$

Hence, since $E(\omega,z)\to 1$ as $\omega\to\infty$,

$$\det\ E(\omega,z)=\exp\left\{\frac{1}{2\pi i}\iint g(\nu,\mu)\frac{d\nu}{\nu-\omega}\frac{d\mu}{\mu-z}\right\}.$$

But since $E(\omega,z)$ is necessarily diagonal with all its eigenvalues equal to unity except for the first, this proves the indicated representation. (It now follows exactly as in references [1, 2] that the spectral multiplicity of V can be calculated as follows: Let $\Lambda_\nu=\{\mu;\,g(\nu,\mu)=1\}$ then if Λ_ν is the union of p disjoint intervals, the spectral multiplicity function, $m(\nu)=p$; otherwise it is infinite.)

To complete the proof of Theorem 1.1 we wish to examine the spectral multiplicity of the operator U.

To do this we simply note that $(-U)V-V(-U)=(1/\pi i)k^*k$. Thus the pair $\{V,-U\}$ satisfies our requirements and we may calculate the determining function, $\varepsilon(y,x)$, corresponding to this pair.

The two determining functions $E(x, y)$ and $\varepsilon(y, x)$ satisfy

$$(-U-x)^{-1}(V-y)^{-1}k^* = (V-y)^{-1}(-U-x)^{-1}k^*\varepsilon(y, x),$$

or
$$(U-x)^{-1}(V-y)^{-1}k^* = (V-y)^{-1}(U-x)^{-1}k^*\,\varepsilon(y, -x).$$

But
$$(U-x)^{-1}(V-y)^{-1}k^* = (V-y)^{-1}(U-x)^{-1}k^*E(x, y),$$

hence $k^*\varepsilon(y, -x) = k^*E(x, y)$. However, since the range of both E and ε is contained in the range of k, we can conclude that the corresponding principal eigenvalues (which we denote by a superscript $\sim$) satisfy

$$\tilde{E}(x, y) = \tilde{\varepsilon}(y, -x),$$

that is
$$\tilde{\varepsilon}(z, l) = \exp \frac{1}{2\pi i}\int_{\sigma(V)}\int_{\sigma(U)} g(\nu, -\mu)\frac{d\nu}{\nu - l}\frac{d\mu}{\mu - z}.$$

From this formula we see that the spectral multiplicity of U is also computed from knowledge of $g(\nu, \mu)$ by the same rule as was followed for V, but in the other variable.

Thus when the spectral multiplicity of U is taken to be n, for each point in the spectrum of U, we may conclude that the set $\gamma_j = \{v; g(v, -\lambda) = 1\}$ consists of exactly n disjoint intervals for each $-\lambda \in \sigma(-U)$, i.e., each $\lambda \in (a, b)$.

Remark. We have defined $E(y, x)$ so that

$$(U-x)^{-1}(V-y)^{-1}k^* = (V-y)^{-1}(U-x)^{-1}k^*E(y, x).$$

Suppose that S^{-1} is an isometry from $\mathcal{H}$ onto another Hilbert space, say the space in which U has the spectral representation M.

In this space we may write $SMS^{-1} = U$, $SLS^{-1} = V$, and

$$(M-x)^{-1}(L-y)^{-1}\hat{k}^* = (L-y)^{-1}(M-x)^{-1}\hat{k}^*\hat{E}(y, x).$$

What is the relation between $\hat{E}(y, x)$ and $E(y, x)$?

If $\mathfrak{J}^*$ denotes the pseudo-inverse of $\hat{k}^*$ defined as $\mathfrak{J}$ was before, then the above formulae give

$$(\hat{\mathfrak{J}}^* S^{-1}k^*)\,E(y, x)\,(\mathfrak{J}^* S\hat{k}^*) = \hat{E}(y, x).$$

When C has finite dimensional range $\mathfrak{J}^*$ and $\hat{\mathfrak{J}}^*$ are bounded, and $E(y, x)$ and $\hat{E}(y, x)$

are similar. However since both of these two operators are unitary for values of y, x real and outside $\sigma(V), \sigma(U)$ and similar normal operators are clearly unitarily equivalent, we can conclude that $E(y, x)$ and $\hat{E}(y, x)$ are unitarily equivalent as operators on l_2.

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