ISOTROPIC INFINITELY DIVISIBLE MEASURES ON SYMMETRIC SPACES

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§ 1. Introduction

It has been clear for some time that a natural domain for questions of harmonic analysis is the class of symmetric spaces G/K where G is an appropriate Lie group and K an appropriate subgroup [10], [14]. (2)

Now, functions, measures etc. on G/K may be viewed as corresponding objects on G which are invariant under the right action of K, and a convolution operation may be defined for them via the group structure of G.

In this paper we study the representation of probability measures on G/K which are isotropic in the sense that they are, as measures on G, also invariant under the left action of K, and which are infinitely divisible in the sense of the convolution mentioned above. The representation is carried out via the abstract Fourier–Stieltjes transform [13], and the main result is Theorem 6.2 which is analogous to the celebrated Lévy–Khinchine formula for the characteristic function of an infinitely divisible probability measure on the real line.

Certain other results of probabilistic significance are also obtained. The principal one is Theorem 7.2 which is the analogue of a classical theorem of Khinchine [18].

The organization of this paper is as follows. § 2, § 3 are devoted to terminology and recapitulation of known results which will be used often in the sequel. § 4–7 are devoted to the proofs of our theorems in the case when G/K is a symmetric space of the non-compact type. § 8 then indicates briefly the modifications to be made when G/K is of the compact type.

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(1) The partial support of this work by NSF grant no. G-21205 is gratefully acknowledged.

(2) Numbers in square brackets refer to the Bibliography at the end of this paper.

Our work subsumes the work of Bochner [1], Tutubalin [23], Getoor [8] where special cases of our Theorem 6.2 and Corollary 6.3 may be found.

The results of this paper were announced in [6].

§ 2. Preliminaries

Throughout this paper except in § 8, G will be a non-compact connected semi-simple Lie group with a finite center and K vill be a maximal compact subgroup of G. $\mathbf{C}(G)$, $\mathbf{C}_c(G)$, $\mathbf{C}^\infty(G)$ will, as usual, stand respectively for the spaces of continuous functions, continuous functions with compact support and the infinitely differentiable functions on G. We set $\mathbf{C}_c^\infty(G)=\mathbf{C}^\infty(G)\cap\mathbf{C}_c(G)$. Fix a Haar measure dx on G once and for all, and let $\mathbf{L}_1(G)$ be the algebra, with convolution as product, of Borel measurable functions absolutely summable with respect to dx. We denote by $\mathbf{S}(G)$ the set of finite regular non-negative Borel measures on G. $\mathbf{S}(G)$ is a semigroup with convolution as product. For $x\in G$, denote by L_x or $L(x)$ the left translation $y\to xy$ of G and by R_x or $R(x)$ the right translation $y\to yx$. $L(x), R(x)$ induce transformations on $\mathbf{C}(G)$, $\mathbf{C}_c(G)$, $\mathbf{C}^\infty(G)$. Specifically, if $f\in\mathbf{C}(G)$ say, then $f^{L(x)}$ is the function $y\to f(x^{-1}y)$. Similarly $f^{R(x)}=f\circ R(x^{-1})$. Similar but tacit conventions will be made regarding the action of $L(x), R(x)$ on $\mathbf{S}(G), \mathbf{L}_1(G)$.

Write
$$\mathbf{C}(G/K)=\{f\mid f\in\mathbf{C}(G),\ f^{R(k)}=f,\ k\in K\}$$
and
$$\mathbf{C}(K\backslash G/K)=\{f\mid f\in\mathbf{C}(G),\ f^{R(k)}=f^{L(k)}=f,\ k\in K\}.$$

The spaces $\mathbf{C}_c(G/K)$, $\mathbf{C}_c(K\backslash G/K)$, $\mathbf{C}^\infty(G/K)$, $\mathbf{C}^\infty(K\backslash G/K)$ etc. are defined analogously. It is clear that for example $\mathbf{C}(G/K)$ may also be thought of as the space of continuous functions on the symmetric space G/K so that our notation has a built-in consistency.

We shall be particularly interested in

$$\mathbf{S}(K\backslash G/K)=\{\mu\mid\mu\in\mathbf{S}(G);\ \mu^{L(k)}=\mu^{R(k)}=\mu\quad\text{for all}\quad k\in K\}.$$

It is easy to check that this is a semigroup under the convolution product, but, what is more crucial for us is the fact that under our hypotheses, it is a *commutative* semigroup. See for example [10, p. 2]. $\mathbf{S}_0(K\backslash G/K)$ will stand for the subset of $\mathbf{S}(K\backslash G/K)$ consisting of probability measures.

Denote by $\mathbf{D}(G)$ the algebra of differential operators on $\mathbf{C}^\infty(G)$ which commute with the left action of G on $\mathbf{C}^\infty(G)$, and by $\mathbf{D}_0(G)$ the subalgebra of $\mathbf{D}(G)$ consisting of those operators which also commute with the right action of K. Each $D\in\mathbf{D}_0(G)$

clearly leaves $\mathbf{C}^\infty(G/K)$ invariant and it can be shown that the algebra of restrictions of operators $D \in \mathbf{D}_0(G)$ is isomorphic with the algebra $\mathbf{D}(G/K)$ of differential operators on $\mathbf{C}^\infty(G/K)$ which commute with left translation by elements of G. We refer to [15, Chapter X) for elaboration of these definitions.

Let $\mathfrak{g}_0, \mathfrak{k}_0$ be respectively the Lie algebras of G, K. Ad denotes the adjoint representation of G on $\mathfrak{g}_0$. Then there exists a subspace $\mathfrak{p}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, [15, p. 157]. $\mathfrak{p}_0$ can be identified in a natural way with the tangent space to the coset space G/K at $\pi(e) \in G/K$, where $\pi: G \to G/K$ is the natural projection map. $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under the Cartan-Killing form $\mathbf{B}$ of $\mathfrak{g}_0$, and $\mathbf{B}$ is negative definite on $\mathfrak{k}_0 \times \mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$.

Let $\mathfrak{h}_{\mathfrak{p}_0}$ be a maximal abelian subspace of $\mathfrak{p}_0$ and $\mathfrak{h}_0$ a maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{h}_{\mathfrak{p}_0}$. Then $\mathfrak{h}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0 + \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{p}_0$. We write $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$. Denote by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$ and by $\mathfrak{h}$, $\mathfrak{h}_{\mathfrak{p}}$, $\mathfrak{h}_{\mathfrak{k}}$, $\mathfrak{k}$, $\mathfrak{p}$, etc. the subspaces of $\mathfrak{g}$ generated by $\mathfrak{h}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{h}_{\mathfrak{k}_0}$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ respectively. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We fix once and for all a compatible ordering on the duals of the vector spaces $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{h}^* = \mathfrak{h}_{\mathfrak{p}_0} + i\mathfrak{h}_{\mathfrak{k}_0}$, [15, p. 222]. Each root $\alpha \in \Delta$ is real valued on $\mathfrak{h}^*$ so we get in this way an ordering of Δ. Let Δ^+ denote the set of positive roots. We write

$$\Delta^+ = P_+ \cup P_- = \{\alpha \in \Delta^+,\ \alpha \not\equiv 0 \text{ on } \mathfrak{h}_{\mathfrak{p}_0}\} \cup \{\alpha \mid \alpha \in \Delta^+,\ \alpha \equiv 0 \text{ on } \mathfrak{h}_{\mathfrak{p}_0}\}.$$

If $\mathfrak{g}^\alpha$ is the root-space corresponding to $\alpha \in \Delta$, $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}^\alpha$, and if $\mathfrak{n}_0 = \mathfrak{g}_0 \cap \mathfrak{n}$ then a celebrated theorem of Iwasawa says that $\mathfrak{n}_0$ is a nilpotent Lie algebra and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{n}_0$. Further, if $A_\mathfrak{p}$, N are the analytic subgroups of G with Lie algebras $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{n}_0$, respectively, then G admits the Iwasawa decomposition $G = KA_\mathfrak{p}N$ [15, p. 234], and $A_\mathfrak{p}$, N are simply connected. (Of course, $A_\mathfrak{p}$ is a vector group R^l.)

If M' is the normalizer and M the centralizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in K, then it can be shown [15, p. 244] that M is a normal subgroup of M' and M'/M is a *finite* group. This is the Weyl group of the pair (G, K). The Weyl group may be faithfully represented in the natural way as a finite group of linear transformations on $\mathfrak{h}_{\mathfrak{p}_0}$. It preserves the Cartan-Killing form.

Let $\mathbf{S}(\mathfrak{p}_0)$, $\mathbf{S}(\mathfrak{h}_{\mathfrak{p}_0})$ be the symmetric algebras over $\mathfrak{p}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$ respectively [15, p. 391]. The group $\mathrm{Ad}(K)$ operates on $\mathbf{S}(\mathfrak{p}_0)$ and W operates on $\mathbf{S}(\mathfrak{h}_{\mathfrak{p}_0})$ by extension of the action of these groups on $\mathfrak{p}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$ respectively. Let $\mathbf{I}(\mathfrak{p}_0)$, $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ be the corresponding sets of invariants. Since the Cartan-Killing form is non-degenerate on $\mathfrak{p}_0$, we may identify $\mathbf{S}(\hat{\mathfrak{p}_0})$, with $\mathbf{S}(\mathfrak{p}_0)$, $\mathbf{S}(\hat{\mathfrak{h}_{\mathfrak{p}_0}})$ with $\mathbf{S}(\mathfrak{h}_{\mathfrak{p}_0})$, where $V^\wedge$ stands for the dual of the vector space V. In particular $\mathbf{I}(\mathfrak{p}_0)$, $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ can be considered as polynomial functions on $\mathfrak{p}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$ respectively which are invariant under $\mathrm{Ad}(K)$ and W respectively.

It will be quite crucial to us in § 6 that $\mathbf{D}(G/K)$ and $\mathbf{I}(\mathfrak{p}_0)$ are isomorphic as vector spaces. We refer to [15, Theorem 2.7, p. 395] for the proof of this fact. Denote this mapping of $\mathbf{I}(\mathfrak{p}_0)$ onto $\mathbf{D}(G/K)$ by $P \to D_P$, $P \in \mathbf{I}(\mathfrak{p}_0)$.

Finally, we shall have to use, rather crucially, the fact due to Harish-Chandra, that the algebras $\mathbf{D}(G/K)$ and $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ are isomorphic, [15, Theorem 6.15, p. 432], under a mapping Γ of $\mathbf{D}(G/K)$ onto $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$.

The notation we have used is that of [15] which indeed will serve as a blanket reference for the terminology and background material on Lie groups which we use in this paper.

§ 3. Spherical functions

A function ϕ on G is said to be K-spherical (or simply spherical) if $\phi \circ R(k) = \phi \circ L(k) = \phi$ for each $k \in K$. It is said to be an *elementary* or *minimal* spherical function if it satisfies in addition,

$$\int_K \phi(xky)\,dk = \phi(x)\,\phi(y) \tag{3.1}$$

and $\phi(e) = 1$. Here dk stands for the normalized Haar measure of K. The elementary spherical functions can be quite easily shown to be analytic and indeed can be characterized by the following properties: (i) $\phi(e) = 1$; (ii) $\phi \in \mathbf{C}^\infty(K\backslash G/K)$; (iii) ϕ is an eigenfunction of each $D \in \mathbf{D}(G/K)$. Cf. [15, Chapter X, § 3].

A fundamental result, due to Harish-Chandra [13], says that the elementary spherical functions are in one-to-one correspondence with the quotient E_C/W of the space E_C of complex valued linear functionals on $\mathfrak{h}_{\mathfrak{p}_0}$ modulo the action of the Weyl group W on it. More precisely, let $H(x)$ denote the unique element $\in \mathfrak{h}_{\mathfrak{p}_0}$ such that $x = k \cdot \exp H(x) \cdot n$, $k \in K$, $n \in N$, and let $\nu \in E_C$. Then

$$\phi_\nu(x) = \int_K \exp\,(i\nu - \varrho)\;(H(xk))\,dk \tag{3.2}$$

is an elementary spherical function on G, $\phi_\nu \equiv \phi_{\nu'}$ if, and only if, $\nu = s\nu'$ for some $s \in W$, and further *each* elementary spherical function arises in this way from some $\nu \in E_C$. In this formula, $\varrho = \frac{1}{2}\sum_{\alpha \in P_+} \bar{\alpha}$ where $\bar{\alpha}$ is the restriction of α to $\mathfrak{h}_{\mathfrak{p}_0}$.

We shall be especially concerned with a certain subclass of the class of elementary spherical functions; namely, those which arise in the above fashion from *real-valued* linear functionals on $\mathfrak{h}_{\mathfrak{p}_0}$. If we denote by E_R the space of real valued linear functionals on $\mathfrak{h}_{\mathfrak{p}_0}$ (so that $E_C = E_R + iE_R$), and if $\lambda \in E_R$, it is known [13, p. 241] that $\phi_\lambda(x)$

is an elementary spherical function of positive-definite type (though it is not true that all the positive-definite elementary spherical functions arise in this way). We note as a consequence that

$$\phi_\lambda(e)=1, \quad \phi_\lambda(x^{-1})=\overline{\phi_\lambda(x)}, \quad |\phi_\lambda(x)|\leqslant 1 \tag{3.3}$$

for $x\in G$, $\lambda\in E_R$. These facts will be used frequently.

Finally, we need to use the fact that if $D\in\mathbf{D}(G/K)$ then $D\phi_\lambda=\Gamma(D)\,(i\lambda)\phi_\lambda$ where Γ is the isomorphism of $\mathbf{D}(G/K)$ onto $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ mentioned above, and $\Gamma(D)\,(i\lambda)$ stands for the value of the polynomial function $\Gamma(D)$ at the point $i\lambda\in E_C$. (In view of the identification of $\hat{\mathfrak{h}_{\mathfrak{p}_0}}$ with $\mathfrak{h}_{\mathfrak{p}_0}$ via the Cartan-Killing form, this makes sense.) In particular, and this is a remark which we shall use crucially in § 6, if we choose a basis $\Lambda_1, \ldots, \Lambda_l$ for E_R and let $\lambda=\sum_{j=1}^{l}\lambda_j\Lambda_j$, then the eigenvalue corresponding to ϕ_λ of any operator $D\in\mathbf{D}(G/K)$ is a *polynomial* in $\lambda_1, \ldots, \lambda_l$. It is also clear from (3.2) that $\phi_{-i\varrho}\equiv 1$ so that a $D\in\mathbf{D}(G/K)$ annihilates constant functions if, and only if, $\Gamma(D)(\varrho)=0$. It is easy to conclude from these facts and from the fact that $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ contains no *linear* polynomials, that second degree polynomials in $\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ which correspond to second order operators in $\mathbf{D}(G/K)$ which annihilate constants are of the form

$$\sum_{u=1}^{l}\sum_{v=1}^{l} Q_{uv}\,H_u\,H_v-\sum_{u=1}^{l}\sum_{v=1}^{l} Q_{uv}\,\varrho_u\,\varrho_v,$$

where $\varrho_u=\varrho(H_u), u=1, \ldots, l$, $H_u, u=1, \ldots, l$ is an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_0}$, and $\{Q_{uv}\}$ is invariant under W. Further, such an element corresponds to an *elliptic* operator if, and only if, $\{Q_{uv}\}$ is a non-negative definite matrix. We shall have to use these facts in § 6.

§ 4. Fourier–Stieltjes transforms of spherical measures

Let $\mathbf{S}_0(K\backslash G/K)=\mathbf{S}(K\backslash G/K)\cap\{\mu\,|\,\mu(G)=1\}$. For $f\in\mathbf{C}_c(G), \mu\in\mathbf{S}(G)$ we write $\mu(f)=\int_G f(x)\,d\mu(x)$. It is clear that since μ is regular, μ is determined by the values $\mu(f)$, $f\in\mathbf{C}_c(G)$. If, moreover, $\mu\in\mathbf{S}(K\backslash G/K)$ then it would suffice to know $\mu(f)$ for $f\in\mathbf{C}_c(K\backslash G/K)$ in order to determine μ.

DEFINITION 4.1. A sequence $\mu_n\in\mathbf{S}(K\backslash G/K)$ is said to *converge weakly* to $\mu\in\mathbf{S}(K\backslash G/K)$ if $\mu_n(f)\to\mu(f)$ for each $f\in\mathbf{C}_c(K\backslash G/K)$. We then write $\mu_n\to\mu$. The sequence μ_n is said to be Bernoulli convergent to μ if $\mu_n(f)\to\mu(f)$ for each *bounded* $f\in\mathbf{C}(K\backslash G/K)$. We write this as $\mu_n\Rightarrow\mu$.

DEFINITION 4.2. For $\mu \in \mathrm{S}(K\backslash G/K)$ we define the Fourier-Stieltjes transform $\hat{\mu}$ by

$$\hat{\mu}(\lambda) = \int_G \phi_\lambda(x)\,d\mu(x), \quad \lambda \in E_R, \tag{4.1}$$

where the ϕ_λ are as in § 3, (3.2).

It is clear that for $\mu \in \mathrm{S}(K\backslash G/K)$, $\hat{\mu}(\lambda)$ is a bounded continuous complex-valued function on E_R and is invariant under W.

Further $\hat{\mu}(\lambda)$ is definable literally by (4.1) for all those $\lambda \in E_c$ for which the integral (4.1) makes sense. In particular, since $\phi_{-i\varrho}(x) = 1$ for all $x \in G$ by (3.2), it is clear that

$$\hat{\mu}(-i\varrho) = \mu(G). \tag{4.2}$$

We now turn to some properties of this transform.

LEMMA 4.1. *If $\mu, \nu \in \mathrm{S}(K\backslash G/K)$ then the measure $\mu \ast \nu$ defined by*

$$(\mu \ast \nu)(B) = \int_G \mu(By^{-1})\,d\nu(y)$$

is again in $\mathrm{S}(K\backslash G/K)$. Further, $\widehat{(\mu \ast \nu)}(\lambda) = \hat{\mu}(\lambda)\hat{\nu}(\lambda)$, $\lambda \in E_R$.

The proof of the first assertion is easy. The second follows because of the sphericity of μ and ν and the functional equation (3.1) which is satisfied by ϕ_λ. We omit it. Cf. [15, p. 409].

THEOREM 4.1. *Suppose $\mu_1, \mu_2 \in \mathrm{S}(K\backslash G/K)$ and $\hat{\mu}_1 = \hat{\mu}_2$, then $\mu_1 = \mu_2$.*

Proof. Let $\mu = \mu_1 - \mu_2$, then μ is a signed measure of finite total variation and $\hat{\mu} = 0$; we shall prove that $\mu = 0$.

We first assume that μ has a continuous density f with respect to Haar measure on G. The general case will then be settled by approximation.

Let then $d\mu = f dx$ where $f \in \mathbf{L}_1(K\backslash G/K) \cap \mathbf{C}(K\backslash G/K)$. Let $\nu \in E_c$ be such that $\phi_\nu(x) = \int_K \exp\,(i\nu - \varrho)(H(xk))\,dk$ is a *bounded* function on G. It is known that the Haar measures dx, da, dn of G, $A_{\mathfrak{p}}$, N respectively may be so normalized that $dx = \exp 2\varrho(\log a)\,dk\,da\,dn$. [11, Lemma 35]. Therefore we may deduce

$$\int_G \phi_\nu(x)\,d\mu(x) = \int_G \int_K \exp\,(i\nu - \varrho)(H(xk))\,dk \cdot f(x)\,dx$$

$$= \int_G \exp\,(i\nu - \varrho)(H(x)) \cdot f(x)\,dx$$

$$= \int_K \int_{A_\mathfrak{p}} \int_N \exp\,(i\nu + \varrho)\,(\log a) \cdot f(kan)\, dk\, da\, dn$$

$$= \int_{A_\mathfrak{p}} \exp\, i\nu(\log a) \cdot F_f(a)\, da, \tag{4.3}$$

where $F_f(a) = \exp\,\varrho\,(\log a) \cdot \int_N f(an)\,dn$.

Now
$$\int_{A_\mathfrak{p}} |F_f(a)|\, da = \int_{A_\mathfrak{p}} \left| \exp\ \varrho(\log a) \int_N f(an)\, dn \right| da$$

$$\leqslant \int_{A_\mathfrak{p}} \exp\ \varrho(\log a) \int_N |f(an)|\, dn\, da$$

$$= \int_{A_\mathfrak{p}} \int_N \exp\ \varrho(\log a) \cdot |f(an)|\, da\, dn$$

$$= \int_G \phi_0(x)\, |f(x)|\, dx < \infty, \tag{4.4}$$

since $0 \leqslant \phi_0(x) \leqslant 1$ and $f \in \mathbf{L}_1(K\backslash G/K)$. Hence $F_f(a) \in \mathbf{L}_1(da)$. Further, specializing (4.3) when $\nu = \lambda \in E_R$ we get

$$\hat{\mu}(\lambda) = \int \phi_\lambda(x)\, d\mu(x) = \int_{A_\mathfrak{p}} \exp\, i\lambda(\log a) \cdot F_f(a)\, da. \tag{4.5}$$

Thus $\hat{\mu}(\lambda)$ is just the classical Fourier transform of F_f. Since $\hat{\mu}(\lambda) = 0$ and since $F_f \in \mathbf{L}_1(da)$, we conclude that $F_f = 0$. But this means by (4.3) that $\int_G \phi_\nu(x) f(x)\, dx = 0$ for each *bounded* elementary spherical function ϕ_ν. But the Banach algebra $\mathbf{L}_1(K\backslash G/K)$ is known to be semi-simple (see [15, p. 453] where a proof is sketched), that is, if $g \in \mathbf{L}_1(K\backslash G/K)$, $g \neq 0$, then there exists a continuous homomorphism $\chi: \mathbf{L}_1(K\backslash G/K) \to$ complex numbers such that $\chi(g) \neq 0$. On the other hand, it is known also that the continuous homomorphisms of $\mathbf{L}_1(K\backslash G/K)$ are precisely $g \to \int_G \phi_\nu(x) g(x)\, dx$ where ϕ_ν is a *bounded* elementary spherical function. (Cf. Theorem 4.3 of [15, p. 410]. There the theorem proved is for the algebra $\mathbf{C}_c(K\backslash G/K)$, but the same proof applies for our purposes.) Using these facts we conclude that $f = 0$ and hence $\mu = 0$, given that $d\mu = f\, dx$ and that $\hat{\mu} = 0$.

Turning to the general case let $f_j, j = 1, 2, \dots$ be a sequence of functions in $\mathbf{C}_c(K\backslash G/K)$ such that

(i) $f_j \geqslant 0$, $j = 1, 2, \dots$;
(ii) $\int_G f_j(x)\, dx = 1$, $j = 1, 2, \dots$;
(iii) $\int_{G-A} f_j(x)\, dx \to 0$ as $j \to \infty$ for each compact subset A containing e of G.

Such sequences exist. Let σ_j be the measure in $\mathbf{S}(K\backslash G/K)$ whose density with respect to dx is f_j. Then it is easy to check that $\mu * \sigma_j$ has a continuous density w.r.t. dx, viz.

$$(\mu * f_j)(x) = \int_G f_j(y^{-1}x)\,d\mu(y).$$

By Lemma 4.1, $\widehat{\mu * \sigma_j} = \hat{\mu}\,\hat{\sigma}_j$. Hence if $\hat{\mu} = 0$ then $\widehat{\mu * \sigma_j} = 0$. The above discussion then shows that $\mu * \sigma_j = 0$.

But by the choice of the functions f_j, it follows that $f_j * g$ converges to g (as $j \to \infty$) *uniformly* on compact subsets of G for each $g \in \mathbf{C}_c(K\backslash G/K)$. From this it follows easily that $(\mu * \sigma_j)(g) \to \mu(g)$ as $j \to \infty$ for $g \in \mathbf{C}_c(K\backslash G/K)$. Since, however, $\mu * \sigma_j = 0$, it follows that $\mu = 0$. Q.E.D.

The following continuity theorem now follows rather easily.

THEOREM 4.2. *Let* $\mu_j \in \mathbf{S}(K\backslash G/K)$, $j = 1, 2, \dots$. *Then*

(i) *If* $\mu_j \to \mu \in \mathbf{S}(K\backslash G/K)$ *and* $\sup_j \mu_j(G) < \infty$, *then* $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$ *for* $\lambda \in E_R$.

(ii) *If* $\mu_j \Rightarrow \mu \in \mathbf{S}(K\backslash G/K)$, *then* $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$ *and* $\hat{\mu}_j(-i\varrho) \to \hat{\mu}(-i\varrho)$.

(iii) *If* $\hat{\mu}_j(\lambda) \to \beta(\lambda)$ *and* $\sup_j \hat{\mu}_j(-i\varrho) < \infty$, *then there exists* $\mu \in \mathbf{S}(K\backslash G/K)$ *such that* $\mu_j \to \mu$ *and* $\hat{\mu}(\lambda) = \beta(\lambda)$, $\lambda \in E_R$.

(iv) *Suppose* $\hat{\mu}_j(\lambda) \to \beta(\lambda)$, *for* $\lambda \in E_R$ *and* $\lim_{j\to\infty} \hat{\mu}_j(-i\varrho)$ *exists (so that by* (iii) $\beta = \hat{\mu}$ *for some* $\mu \in \mathbf{S}(K\backslash G/K)$). *If* $\lim_{j\to\infty} \hat{\mu}_j(-i\varrho) = \hat{\mu}(-i\varrho)$, *then* $\mu_j \Rightarrow \mu$.

Proof. (i) For each $\lambda \in E_R$, $\phi_\lambda(x)$ is continuous in x and it is known that as a function on G, ϕ_λ vanishes at infinity. (See Theorem 2 of [14, p. 585].) Since

$$\hat{\mu}_j(\lambda) = \int_G \phi_\lambda(x)\,d\mu_j(x)$$

and $\sup_j \mu_j(G) < \infty$, (i) follows rather easily.

(ii) If $\mu_j \Rightarrow \mu$ then $\mu_j \to \mu$ so by (i) $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$ for $\lambda \in E_R$. But moreover $\hat{\mu}_j(1) \to \hat{\mu}(1)$ (where 1 stands for the function identically equal to 1 on G) and this is precisely the statement $\hat{\mu}_j(-i\varrho) \to \hat{\mu}(-i\varrho)$ since $\phi_{-i\varrho}(x) \equiv 1$.

(iii) The condition $\sup_j \hat{\mu}_j(-i\varrho) = \sup_j \mu_j(G) < \infty$ guarantees by the Helly–Alaoglu theorem that for a subsequence $\{\mu_{j_\alpha}\}$, $\alpha = 1, 2, \dots$, we have $\mu_{j_\alpha} \to \mu \in \mathbf{S}(K\backslash G/K)$. By (i) we have $\hat{\mu}_{j_\alpha}(\lambda) \to \hat{\mu}(\lambda)$, $\lambda \in E_R$. But since $\hat{\mu}_j(\lambda) \to \beta(\lambda)$, it follows that $\hat{\mu}(\lambda) = \beta(\lambda)$.

If $\mu_{j'_\alpha}$ is another weakly convergent subsequence of $\{\mu_j\}$ with limit μ' then we shall have similarly $\hat{\mu}(\lambda)=\beta(\lambda)=\hat{\mu}'(\lambda)$ and so by Theorem 4.1, $\mu'\equiv\mu$. Hence each convergent subsequence of $\{\mu_j\}$ has the same limit μ proving that $\mu_j\to\mu$ and $\beta=\hat{\mu}$.

(iv) Indeed the supplementary condition $\hat{\mu}_j(-i\varrho)\to\hat{\mu}(-i\varrho)$ is to say that $\mu_j(G)\to\mu(G)$ as $j\to\infty$. This strengthens $\mu_j\to\mu$ to $\mu_j\Rightarrow\mu$. Q.E.D.

Let $\hat{\mathrm{S}}, \hat{\mathrm{S}}_0$ be the classes of Fourier-Stieltjes transforms of elements of $\mathrm{S}(K\backslash G/K)$, $\mathrm{S}_0(K\backslash G/K)$ respectively. We have observed that (in Lemma 4.1) $\hat{\mathrm{S}}$ is closed under pointwise multiplication.

Since S_0 is closed under convolution, $\hat{\mathrm{S}}_0$ is also closed under pointwise multiplication. The following lemma shows up another interesting closure property.

Lemma 4.2. *If* $\hat{\mu}\in\hat{\mathrm{S}}$ *then the function* $\beta(\lambda)=\exp t(\hat{\mu}(\lambda)-\hat{\mu}(-i\varrho))\in\hat{\mathrm{S}}_0$ *for each* $t\geqslant 0$,

Proof. It is easy to see that $\hat{\mathrm{S}}$ is closed under linear combinations with non-negative coefficients. Hence for each m the function

$$e^{-\alpha t}\sum_{j=1}^{m}\frac{t^j}{j!}[\hat{\mu}(\lambda)]^j$$

is in $\hat{\mathrm{S}}$ for any $t\geqslant 0$, $\alpha\geqslant 0$. Let us set $\alpha=\hat{\mu}(-i\varrho)\geqslant 0$ in this and call the resulting function $\hat{\mu}_m$. Then it is clear that $\hat{\mu}_m(\lambda)\to\beta(\lambda)$, $\lambda\in E_R$ and further that $\hat{\mu}_m(-i\varrho)\leqslant 1$, $m=1,2,\dots$. Hence by Theorem 4.2 (iii) $\mu_m\to$ some $\sigma\in\mathrm{S}(K\backslash G/K)$ and $\beta(\lambda)=\hat{\sigma}(\lambda)$. Thus $\beta\in\hat{\mathrm{S}}$. But $\beta(-i\varrho)=1$. Hence $\beta\in\hat{\mathrm{S}}_0$. Q.E.D.

§ 5. Poisson measures; infinitely divisible measures

Given $\mu,\nu\in\mathrm{S}(K\backslash G/K)$ we shall write $\mu\nu$ for the convolution of μ and ν; the order being immaterial in view of the commutativity of $\mathrm{S}(K\backslash G/K)$. μ^j will stand for the j-fold convolution of μ with itself. It is clear that $\widehat{\mu\nu}=\hat{\mu}\hat{\nu}$, the product on the right being pointwise.

For $x\in G$ consider the set $\{k_1xk_2\,|\,k_1,k_2\in K\}=\tilde{x}$. It is clear that $\tilde{x}$ is homeomorphic to $K\times K$ in a natural way. Let μ_x be the measure induced on $\tilde{x}$ by $dk\times dk$ on $K\times K$ where dk is, as usual, the normalized Haar measure of K. μ_x may clearly be regarded as a measure on G by setting it 0 outside $\tilde{x}$. We call this extended measure μ_x also. It is clear that $\mu_x\in\mathrm{S}_0(K\backslash G/K)$.

Definition 5.1. The measure

$$\pi_{x,c} = \sum_{j=0}^{\infty} \exp(-c) \cdot c^j \mu_x^j / j!$$

will be called the Poisson measure with jump size x and jump rate c. Here $x \in G$, c is a real number ≥ 0.

It is clear that $\pi_{x,c} \in S_0(K\backslash G/K)$ for each $x \in G$, $c \geq 0$, $\pi_{x,0}$ being μ_e identically for all $x \in G$.

It is an easy computation to verify that $\hat{\mu}_x(\lambda) = \phi_\lambda(x)$, $\lambda \in E_R$ and that $\hat{\pi}_{x,c}(\lambda) = \exp\{c(\phi_\lambda(x) - 1\}$. It is thus clear that $\pi_{x,c} \cdot \pi_{x,d} = \pi_{x,c+d}$.

Definition 5.2. A measure $\mu \in S_0(K\backslash G/K)$ is said to be infinitely divisible if for each positive integer j, there exists a measure $\nu \in S_0(K\backslash G/K)$ such that $\nu^j = \mu$.

It is clear that a product of two infinitely divisible measures $\in S_0(K\backslash G/K)$ is again infinitely divisible.

Lemma 5.1. *Suppose $\mu \in S_0(K\backslash G/K)$ is infinitely divisible. Then $\hat{\mu}(\lambda) \neq 0$ for $\lambda \in E_R$.*

Proof. For any $\mu \in S_0(K\backslash G/K)$ define its adjoint μ^* by $\mu^*(B) = \mu(B^{-1})$. Clearly $\mu^* \in S_0(K\backslash G/K)$ and $(\mu^*)^* = \mu$. Further $(\mu\mu^*)^* = \mu^*\mu = \mu\mu^*$. Hence $\mu\mu^*$ is self-adjoint. Now if μ is infinitely divisible, then there is for each j a measure $\nu_j \in S_0(K\backslash G/K)$ such that $\mu = (\nu_j)^j$. Then clearly $\mu\mu^* = (\nu_j \nu_j^*)^j$ so that $\mu\mu^*$ is infinitely divisible. Also, it is easy to check that $\widehat{\mu\mu^*}(\lambda) = |\hat{\mu}(\lambda)|^2$ so that $\hat{\mu}(\lambda) = 0$ if and only if $\widehat{\mu\mu^*}(\lambda) = 0$. Hence we may assume to begin with that μ is self-adjoint, and for each j there exists a *self-adjoint* measure $\nu_j \in S_0(K\backslash G/K)$ such that $\mu = (\nu_j)^j$. Note that $\hat{\mu}$, $\hat{\nu}_j$ are now real valued for $\lambda \in E_R$. Therefore, since $\hat{\mu}(\lambda) = \{\hat{\nu}_j(\lambda)\}^j$, and $|\hat{\mu}(\lambda)| \leq 1$, we have $\{\hat{\nu}_j(\lambda)\}^2 = \{(\hat{\mu}(\lambda))^2\}^{1/j} \to \beta(\lambda)$ where $\beta(\lambda) = 0$ or 1 according as $\hat{\mu}(\lambda) = 0$ or $\hat{\mu}(\lambda) \neq 0$. Now $\hat{\mu}(0) = \int_G \exp -\{\varrho(H(x))\}\, d\mu(x)$ and so $\hat{\mu}(0) > 0$; therefore $\beta(0) = 1$. But $\hat{\nu}_j^2 \in \hat{S}_0$, hence $\hat{\nu}_j^2(-i\varrho) = 1$ so, by Theorem 4.2 (iii), it follows that $\beta \in \hat{S}$, i.e., $\beta = \hat{\mu}$ for some $\mu \in S(K\backslash G/K)$. In particular, β is continuous on E_R. Since E_R is connected and β only takes the values 0 or 1 as observed above, and finally since $\beta(0) = 1$, it follows that $\beta(\lambda) \equiv 1$, $\lambda \in E_R$. Hence $\hat{\mu}(\lambda) \neq 0$, $\lambda \in E_R$. Q.E.D.

Lemma 5.2. *Suppose $\mu_j \in S_0(K\backslash G/K)$, $j = 1, 2, \ldots$; $\mu_j \Rightarrow \mu$ as $j \to \infty$ and μ_j is infinitely divisible for each j. Then μ is infinitely divisible.*

Proof. Cf. [9].

By hypothesis, there exist measures $\nu_{jm} \in \mathrm{S}_0(K\backslash G/K)$, $j, m = 1, 2, \dots$; such that $\mu_j = (\nu_{jm})^m$ for each j, m. Hence $\mu_j = (\nu_{jm})^m$. Since $\hat{\mu}_j(\lambda) \neq 0$, $\lambda \in E_R$, we have for fixed m, $\hat{\nu}_{jm}(\lambda) \to \{\hat{\mu}(\lambda)\}^{1/m}$ as $j \to \infty$. On the other hand, $\nu_{jm}(-i\varrho) = 1$ for all j, m so Theorem 4.2 (iii) implies $[\hat{\mu}(\lambda)]^{1/m} \in \hat{\mathrm{S}}$, but since $\hat{\mu} \in \hat{\mathrm{S}}_0$, it is clear that $\{\hat{\mu}\}^{1/m} \in \hat{\mathrm{S}}_0$ as well. Hence there exists a $\nu_m \in \mathrm{S}_0(K\backslash G/K)$ such that $\hat{\nu}_m(\lambda) = [\hat{\mu}(\lambda)]^{1/m}$ so that $\mu = (\nu_m)^m$. Since this is true for each m, we are done. Q.E.D.

LEMMA 5.3. *A Poisson measure $\pi_{x,c}$ is infinitely divisible.*

Proof. Indeed $(\pi_{x,c/m})^m = \pi_{x,c}$.

LEMMA 5.4. *A measure $\mu \in \mathrm{S}_0(K\backslash G/K)$ is infinitely divisible if and only if there exists a sequence $\{\mu_j\} \in \mathrm{S}_0(K\backslash G/K)$ such that each μ_j is a convolution of a finite number of Poisson measures, and $\mu_j \Rightarrow \mu$.*

Proof. Necessity: If μ is infinitely divisible then for each m, there is a ν_m such that $(\nu_m)^m = \mu$. Then $\hat{\mu}(\lambda) = \{\hat{\nu}_m(\lambda)\}^m$. Also $\hat{\mu}(\lambda) \neq 0$ for any $\lambda \in E_R$ by Lemma 5.1. Hence $m(\{\hat{\mu}(\lambda)\}^{1/m} - 1) = m(\hat{\nu}_m(\lambda) - 1) \to \log \hat{\mu}(\lambda)$ as $m \to \infty$. Hence

$$\hat{\mu}(\lambda) = \lim_{m \to \infty} \exp m(\hat{\nu}_m(\lambda) - 1).$$

Since
$$\hat{\nu}_m(\lambda) = \int_G \phi_\lambda(x)\, d\nu_m(x),$$

we have
$$\hat{\mu}(\lambda) = \lim_{m \to \infty} \exp m \int_G (\phi_\lambda(x) - 1)\, d\nu_m(x).$$

Our assertion now follows by writing the integral as a limit of suitable Riemann–Stieltjes sums and recalling that $\hat{\pi}_{x,c}(\lambda) = \exp c(\phi_\lambda(x) - 1)$. See [9, p. 74].

Sufficiency: If μ_j is the convolution of a finite number of Poisson measures then μ_j is clearly infinitely divisible by Lemma 5.3 and the remark preceding Lemma 5.1. If $\mu_j \Rightarrow \mu$ then by Lemma 5.2, μ is infinitely divisible. Q.E.D.

COROLLARY 5.4. *$\mu \in \mathrm{S}_0(K\backslash G/K)$ is infinitely divisible if and only if*

$$\hat{\mu}(\lambda) = \lim_{j \to \infty} \exp - \psi_j(\lambda),$$

where $\psi_j(\lambda) = \int_G [1 - \phi_\lambda(x)]\, dL_j(x)$ with $L_j \in \mathrm{S}(K\backslash G/K)$.

Proof. Obvious.

§ 6. The Lévy–Khinchine formula

We begin with two lemmas which will be crucial. Recall that the Cartan–Killing form $\mathbf{B}$, being positive definite on $\mathfrak{p}_0 \times \mathfrak{p}_0$, induces an inner product on $\mathfrak{p}_0$ and hence on $\mathfrak{h}_{\mathfrak{p}_0}$ as well. We denote by $\|X\|$ the norm of $X \in \mathfrak{p}_0$ according to this inner product. Let $H_1, \ldots, H_l$ be an orthonormal basis of $\mathfrak{h}_{\mathfrak{p}_0}$, and let $\Lambda_1, \ldots, \Lambda_l$ be the dual basis of E_R. We shall denote by $d\lambda$ the Euclidean measure induced on E_R by this basis. $\|\cdot\|$ will also denote the norm in E_R. Finally, since $\mathfrak{p}_0$ is essentially the tangent space at $\pi(e)$ to the symmetric space G/K (where π is the natural projection $G \to G/K$), $\mathbf{B}$ induces a metric on G/K. For $x \in G$ we write $|x|$ for the distance of $\pi(x)$ from $\pi(e)$ according to this metric. It is clear that $|x|$ is a spherical function of x since $\mathbf{B}$ is invariant under Ad (k), $k \in K$.

LEMMA 6.1. *Let $d > 0$ be a positive number and let V_d be the set $\{\lambda \mid \|\lambda\| \leqslant d\}$ in E_R. For $x \in G$, let*

$$q_d(x) = \int_{V_d} [1 - \operatorname{Re} \phi_\lambda(x)]\, d\lambda.$$

Then $q_d(x)$ is a spherical function $\geqslant 0$, $q_d(x)$ tends to a positive *limit $q_d(\infty)$ as $x \to \infty$ on G, and $q_d(x) = 0$ if and only if $x \in K$.*

Proof. The sphericity of $q_d(x)$ is obvious, as also is the fact that $q_d(x) \geqslant 0$ (cf. (3.3)). Further, since $\phi_\lambda(x) \to 0$ as $x \to \infty$ [14], it is clear that $q_d(x) \to \text{Volume}\ (V_d) = q_d(\infty) > 0$ as $x \to \infty$. If $x \in K$, then $q_d(x) = q_d(e) = 0$ since $\operatorname{Re} \phi_\lambda(e) = \phi_\lambda(e) = 1$ for all $\lambda \in E_R$. Thus it only remains to show that *if* $q_d(x) = 0$ *then* $x \in K$.

Assume then that

$$q_d(x) = \int_{V_d} [1 - \operatorname{Re} \phi_\lambda(x)]\, d\lambda = 0.$$

Then, since by (3.3), $|\operatorname{Re} \phi_\lambda(x)| \leqslant 1$ and since $\phi_\lambda(x)$ is a continuous function of λ, it follows that $\operatorname{Re} \phi_\lambda(x) = 1$ for all $\lambda \in V_d$. But using (3.2), this means that

$$1 = \int_K \cos \lambda(H(xk)) \cdot \exp - \varrho(H(xk))\, dk, \quad \lambda \in V_d. \tag{6.1}$$

If $H(x) = \sum_{j=1}^{l} a_j(x) H_j$, $\lambda = \sum_{j=1}^{l} \lambda_j \Lambda_j$, it is easy to see that $\operatorname{Re} \phi_\lambda(x)$ depends differentiably on $(\lambda_1, \ldots, \lambda_l)$ and so we have from (6.1)

$$0=\sum_{j=1}^{l}\frac{\partial^2}{\partial\lambda_j^2}[\mathrm{Re}\ \phi_\lambda(x)]\cdot|_{\lambda_1=\dots=\lambda_l=0}=-\int_K\sum_{j=1}^{l}[a_j(xk)]^2\cdot\exp\ -\varrho(H(xk))\,dk. \tag{6.2}$$

It follows from this that $\sum_{j=1}^{l}[a_j(xk)]^2=0$, i.e., that $H(xk)=0$ for all $k\in K$.

Since $H(ky)=H(y)$ for any $y\in G$, $k\in K$ by the very definition of $H(y)$, it follows that

$$H(kxk')=0 \text{ for } k,k'\in K. \tag{6.3}$$

Now $x=\exp X\cdot k_0$, $x\in\mathfrak{p}_0$, $k_0\in K$, and further it is known that $\mathfrak{p}_0=\bigcup_{k\in K}\mathrm{Ad}(k)\mathfrak{h}_{\mathfrak{p}_0}$, see [12, p. 616]. From these facts it is easy to conclude that there exist $k_1,k_2\in K$ such that $z=k_1xk_2\in A_{\mathfrak{p}}$, and $H(z)=0$ because of (6.3). But since $h\to H(h)$ is one–one for $h\in A_{\mathfrak{p}}$, this means that $k_1xk_2=z=e$. So that $x\in K$, concluding the proof. Q.E.D.

LEMMA 6.2. *With the same notation as above we claim*

(i) $0\leqslant q_d(x)\leqslant 2$ Volume (V_d).

(ii) *There exist constants* I_d, J_d *such that*

$$0<I_d\leqslant\frac{1+|x|^2}{|x|^2}q_d(x)\leqslant J_d,\quad \textit{for all}\quad x\in G\quad \textit{such that}\ |x|>0. \tag{6.4}$$

Proof. (i) is trivial. As for (ii), $q_d(x)\to$ Volume (V_d) as $x\to\infty$ on G implies that $(1+|x|^2)\,|x|^{-2}q_d(x)\to$ Volume (V_d) as $x\to\infty$ on G. Since $(1+|x|^2)\,|x|^{-2}q_d(x)$ is continuous on $\{x|\,|x|>0\}$, we shall be done as soon as we prove that $|x|^{-2}q_d(x)$ is bounded away from 0 and ∞ as $|x|\to 0$.

To this end, let $x=\exp X\cdot k$, $X\in\mathfrak{p}_0$, $k\in K$, let $r=\|X\|$, $\tilde{X}=X/\|X\|$. Then $x=\exp r\tilde{X}\cdot k_x$, $\tilde{X}\in\mathfrak{p}_0$, $\|\tilde{X}\|=1$, $k_x\in K$, and $x^{-1}=k_x^{-1}\exp -r\tilde{X}$. Now since ϕ_λ is spherical, we have $\phi_\lambda(x)=\phi_\lambda(\exp r\tilde{X})=\phi_\lambda(\exp \mathrm{Ad}(k)r\tilde{X})$, $k\in K$. Therefore,

$$\begin{aligned}\phi_\lambda(x)&=\phi_\lambda(\exp r\tilde{X})=\int_K\phi_\lambda(\exp r\ \mathrm{Ad}(k)\tilde{X})\,dk\\&=\sum_{j=0}^{\infty}\frac{r^j}{j!}\int_K[(\mathrm{Ad}(k)\tilde{X})^j\phi_\lambda](e)\,dk=\sum_{j=0}^{\infty}r^j(D_j^x\phi_\lambda)(e),\end{aligned} \tag{6.5}$$

where we have set $D_j^x=$ the differential operator $\int_K(\mathrm{Ad}(k)\tilde{X})^j\,dk/j!$ and where for the last step but one we used the fact that ϕ_λ is analytic (cf. § 3).

Similarly

$$\phi_\lambda(x^{-1})=\sum_{j=0}^{\infty}(-)^j r^j(D_j^x\phi_\lambda)(e) \tag{6.6}$$

so that
$$\operatorname{Re} \phi_\lambda(x) = \tfrac{1}{2}\,[\phi_\lambda(x) + \phi_\lambda(x^{-1})] = \sum_{j=0}^{\infty} r^{2j}\,(D_{2j}^x\,\phi_\lambda)\,(e). \tag{6.7}$$

Note that by its very construction $D_{2j}^x \in \mathbf{D}_0(G)$ for each j. We examine D_2^x more closely. Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathfrak{p}_0$ such that $X_1, \ldots, X_l$ is an orthonormal basis for $\mathfrak{h}_{\mathfrak{p}_0}$. Let $\tilde{X} = \sum_{i=1}^n \xi_i X_i$ so that $\sum_{i=1}^n \xi_i^2 = 1$. Then $\operatorname{Ad}(k)\,\tilde{X} = \sum_{i=1}^n \xi_i \operatorname{Ad}(k)\,X_i$ so

$$(D_2^x\,\phi_\lambda)\,(e) = \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i\,\xi_j \left(\left[\int_K \operatorname{Ad}(k)\,X_i \operatorname{Ad}(k)\,X_j\,dk\right]\phi_\lambda\right)(e)$$

from which it is clear that $(D_2^x\,\phi_\lambda)\,(e)$ depends continuously on $(\xi_1, \ldots, \xi_n)$. Now from (6.7) we have
$$1 - \operatorname{Re} \phi_\lambda(x) = -r^2\,(D_2^x\,\phi_\lambda)\,(e) + o\,(r^2), \quad r \to 0 \tag{6.8}$$

so that $-(D_2^x\,\phi_\lambda)\,(e) \geqslant 0$ since $1 - \operatorname{Re} \phi_\lambda(x) \geqslant 0$. Now $D_2^x \in \mathbf{D}_0(G)$, hence D_2^x corresponds to an operator $\tilde{D}_2^x \in \mathbf{D}\,(G/K)$. But ϕ_λ is an eigenfunction of each operator in $\mathbf{D}\,(G/K)$ (cf. § 3). Hence $(D_2^x\,\phi_\lambda)\,(e) = (\tilde{D}_2^x\,\phi_\lambda)\,(e) = P_2^x(\lambda)\,\phi_\lambda(e) = P_2^x(\lambda)$ where $P_2^x(\lambda)$ is a second degree polynomial in $\lambda_1, \ldots, \lambda_l$. (6.8) implies that $-P_2^x(\lambda) \geqslant 0$. Being a polynomial in $\lambda_1, \ldots, \lambda_l$, $P_2^x(\lambda)$ is $=0$ only on a null set in V_d and it follows that

$$-\int_{V_d} P_2^x(\lambda)\,d\lambda > 0.$$

Now elementary considerations show that the $o\,(r^2)$ in (6.8) is uniform in $\lambda \in V_d$ so that integrating (6.8) with respect to λ one has

$$q_d(x) = r^2 \cdot \int_{V_d} - P_2^x(\lambda)\,d\lambda + o\,(r^2). \tag{6.9}$$

As observed above, $(D_2^x\,\phi_\lambda)\,(e) = P_2^x(\lambda)$ depends continuously on $(\xi_1, \ldots, \xi_n)$. Hence so does $\int_{V_d} - P_2^x(\lambda)\,d\lambda$, We have seen above that $\int_{V_d} - P_2^x(\lambda)\,d\lambda > 0$. Hence as $(\xi_1, \ldots, \xi_n)$ vary on $\sum_{i=1}^n \xi_i^2 = 1$, there exist constants I_d', J_d' such that

$$0 < I_d' \leqslant -\int_{V_d} P_2^x(\lambda)\,d\lambda \leqslant J_d'.$$

Using this, and (6.9) and remembering that $\|X\|^2/|x|^2 \to 1$ as $\|X\| \to 0$, we see that $|x|^{-2}\,q_d(x)$ is bounded away from 0 and ∞ as $|x| \to 0$, proving the lemma. Q.E.D.

COROLLARY 6.2. *For a spherical measure L, the integrals*

$$\int q_d(x)\,dL(x) \quad \textit{and} \quad \int \frac{|x|^2}{1+|x|^2}\,dL(x)$$

are either both finite or both infinite (d being any fixed positive real number). [1]

We have seen in Corollary 5.4 that $\mu \in \mathrm{S}_0(K\backslash G/K)$ is infinitely divisible if and only if $\hat{\mu}(\lambda) = \exp - \psi(\lambda)$, where $\psi(\lambda) = \lim_{j\to\infty} \psi_j(\lambda)$ with $\psi_j(\lambda) = \int [1-\phi_\lambda(x)]\,dL_j(x)$, $L_j \in \mathrm{S}(K\backslash G/K)$, $j = 1, 2, \ldots$. We now turn to the representation of functions $\psi(\lambda)$ which can arise in this way.

THEOREM 6.1. *Let*

$$\psi_j(\lambda) = \int [1-\phi_\lambda(x)]\,dL_j(x), \quad L_j \in \mathrm{S}(K\backslash G/K), \quad j = 1, 2, \ldots.$$

Suppose that $\lim_{j\to\infty} \psi_j(\lambda) = \psi(\lambda)$. Then there exists a constant c, a spherical measure L and a second order elliptic differential operator $D \in \mathbf{D}(G/K)$ which annihilates constants, such that

$$\psi(\lambda) = c - P_D(\lambda) + \int_{|x|>0} [1-\phi_\lambda(x)]\,dL(x), \tag{6.10}$$

where $P_D(\lambda)$ is the eigenvalue of D corresponding to the eigenfunction ϕ_λ, i.e., $D\phi_\lambda = P_D(\lambda)\phi_\lambda$. Further,

$$\int \frac{|x|^2}{1+|x|^2}\,dL(x) < \infty. \tag{6.11}$$

For such a $\psi(\lambda)$, $\exp - \psi(\lambda) \in \hat{\mathrm{S}}_0$ if and only if $c = 0$.

Conversely, given $D \in \mathbf{D}(G/K)$ and a spherical measure L satisfying the above conditions, the function

$$-P_D(\lambda) + \int_{|x|>0} [1-\phi_\lambda(x)]\,dL(x)$$

is the limit as $j \to \infty$ of functions $\psi_j(\lambda)$ which arise from measures $L_j \in \mathrm{S}(K\backslash G/K)$ according to the recipe $\psi_j(\lambda) = \int [1-\phi_\lambda(x)]\,dL_j(x)$.

Our proof is in broad conformity with classical lines, see for example [2, Chapter 3]. For reasons of space we refrain from giving an extensive treatment like the one there.

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(1) When integration is performed on all of G, we shall frequently omit G from the symbol, unless risk of confusion is present.

Proof. Suppose that

$$\psi_j(\lambda) = \int [1 - \phi_\lambda(x)]\,dL_j(x), \quad L_j \in \mathbf{S}(K\backslash G/K)$$

and $\psi_j(\lambda) \to \psi(\lambda)$ as $j \to \infty$ for $\lambda \in E_R$. Then $\exp -\psi_j(\lambda) \to \exp -\psi(\lambda)$. On the other hand, by Lemma 4.2, $\exp -\psi_j(\lambda) \in \hat{\mathbf{S}}_0$ so that by Theorem 4.2 (iii), $\exp -\psi(\lambda) \in \hat{\mathbf{S}}$. In particular, $\exp -\psi(\lambda)$ and hence $\psi(\lambda)$ is continuous on E_R.

Let $q_1(x)$ be the function of Lemmas 6.1, 6.2 with $d=1$. We have

$$\operatorname{Re}\psi_j(\lambda) = \int [1 - \operatorname{Re}\phi_\lambda(x)]\,dL_j(x), \quad j = 1, 2, \ldots. \tag{6.12}$$

Since the integrand is non-negative we integrate w.r.t. λ and using Fubini's theorem, we get

$$\int_{V_1} \operatorname{Re}\psi_j(\lambda)\,d\lambda = \int q_1(x)\,dL_j(x), \quad j = 1, 2, \ldots. \tag{6.13}$$

However, $\operatorname{Re}\psi_j(\lambda) \to \operatorname{Re}\psi(\lambda)$ as $j \to \infty$ and since these functions are non-negative and continuous,

$$\int_{V_1} \operatorname{Re}\psi_j(\lambda)\,d\lambda \to \int_{V_1} \operatorname{Re}\psi(\lambda)\,d\lambda, \quad j \to \infty; \tag{6.14}$$

in particular, $M = \sup_j \int_{V_1} \operatorname{Re}\psi_j(\lambda)\,d\lambda < \infty$. Thus

$$\int q_1(x)\,dL_j(x) \leqslant M < \infty. \tag{6.15}$$

We now break up $\int q_1(x)\,dL_j(x)$ into

$$\int_{|x|\leqslant 1} q_1(x)\,dL_j(x) + \int_{|x|>1} q_1(x)\,dL_j(x),$$

and since each of these is non-negative, we have

$$\int_{|x|>1} q_1(x)\,dL_j(x) \leqslant M, \tag{6.16}$$

$$\int_{|x|\leqslant 1} q_1(x)\,dL_j(x) \leqslant M, \quad j = 1, 2, \ldots. \tag{6.17}$$

But because of Lemma 6.1, $q_1(x) \geqslant \delta > 0$ for $|x| > 1$, hence by (6.16)

$$\int_{|x|>1} dL_j(x) \leqslant M/\delta. \tag{6.18}$$

(6.17), (6.18) together imply easily that there exists a spherical measure L on $\{x \mid |x|>0\}$ such that

$$\int_{0<|x|\leqslant 1} q_1(x)\,dL(x) + \int_{|x|>1} dL(x) < \infty \tag{6.19}$$

and such that for each $f\in C(K\backslash G/K)$ which vanishes at e and at ∞, we have

$$\int f(x)\,dL_{j_\alpha}(x) \to \int f(x)\,dL(x), \quad j_\alpha \to \infty \tag{6.20}$$

for a suitably chosen subsequence $\{L_{j_\alpha}\}$ of $\{L_j\}$. We rename L_{j_α} as L_j.(1) Note that since $q_1(x)\geqslant \delta>0$ on $|x|>1$ and since $q_1(x)$ is continuous and tends to a positive limit as $x\to\infty$, (6.19) is equivalent to

$$\int_{|x|>0} q_1(x)\,dL(x) < \infty. \tag{6.21}$$

Now consider $\int_{|x|>1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)$. Because of (6.18) the numbers

$$\gamma_j = \int_{|x|>1} dL_j(x)$$

are bounded, so that a subsequence of them will converge to γ_∞ say, while, since $\operatorname{Re}\phi_\lambda$ vanishes at ∞, (6.20) implies

$$\int_{|x|>1} \operatorname{Re}\phi_\lambda(x)\,dL_j(x) \to \int_{|x|>1} \operatorname{Re}\phi_\lambda(x)\,dL(x)$$

so that, setting $c=\gamma_\infty - \int_{|x|>1} dL(x)$, and passing to an appropriate subsequence of $\{L_j\}$, we have

$$\int_{|x|>1} [1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x) \to c + \int_{|x|>1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL(x). \tag{6.22}$$

Now we turn to $\int_{|x|\leqslant 1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)$. Each such term is non-negative and therefore,

$$0 \leqslant \int_{|x|\leqslant 1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x) \leqslant \int [1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x) = \operatorname{Re}\psi_j(\lambda). \tag{6.23}$$

(1) In the following proof we shall pass repeatedly to subsequences of $\{L_j\}$, and call these subsequences $\{L_j\}$ again. We may clearly do so since we are concerned with the representation of $\lim_{j\to\infty}\psi_j(\lambda)$.

Since Re $\psi_j(\lambda) \to$ Re $\psi(\lambda)$, if follows that for a subsequence, $\int_{|x|\leqslant 1}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x)$ converges.

Let $\varepsilon > 0$ be a fixed number. We have

$$\int_{|x|\leqslant 1}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x) = \int_{|x|\leqslant\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x) + \int_{|x|>\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x). \tag{6.24}$$

As to the second term, we may write it as

$$\int_{|x|>\varepsilon}\frac{1-\mathrm{Re}\,\phi_\lambda(x)}{q_1(x)}\cdot q_1(x)\,dL_j(x).$$

Since in this expression, the integrand is bounded and continuous (cf. Lemma 6.1), since the range of integration excludes $\{x\,|\,|x|=0\}$ and since $q_1(x)\,dL_j \to q_1(x)\,dL$ on $\{x\,|\,|x|>0\}$, we see that

$$\int_{|x|>\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x) \to \int_{|x|>\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL(x).$$

(Provided, of course, that $r=\varepsilon$ is a point of continuity of the monotone function of r given by $\int_{r\leqslant x\leqslant 1} dL(x)$; this function has only countably many points of discontinuity and we exclude these once and for all.) Thus in (6.24), the left side and the 2nd term on the right side both converge as $j\to\infty$ and so must the remaining term, to yield

$$\begin{aligned}\lim_{j\to\infty}&\int_{|x|\leqslant 1}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x)\\ &= \lim_{j\to\infty}\int_{|x|\leqslant\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x) + \int_{\varepsilon<|x|\leqslant 1}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL(x).\end{aligned} \tag{6.25}$$

Letting $\varepsilon\to 0$ through permissible values, the 2nd term on the right, being monotone in ε and bounded by the left side (which is independent of ε), is seen to converge to

$$\int_{0<|x|\leqslant 1}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL(x),$$

so that because of (6.25),

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\int_{|x|\leqslant\varepsilon}[1-\mathrm{Re}\,\phi_\lambda(x)]\,dL_j(x)$$

is seen to exist; we have

$$\lim_{j\to\infty}\int_{|x|\leq 1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)$$
$$=\lim_{\varepsilon\to 0}\lim_{j\to\infty}\int_{|x|\leq\varepsilon}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)+\int_{0<|x|\leq 1}[1-\operatorname{Re}\phi_\lambda(x)]\,dL(x). \tag{6.26}$$

Let us now examine the first term more closely. Recall the device and notation used in proving Lemma 6.2. We have then

$$1-\operatorname{Re}\phi_\lambda(x)=-r^2P_2^x(\lambda)+o(r^3). \tag{6.27}$$

Integrating (6.27) and recalling that $\int q_1(x)\,dL_j(x)<\infty$ and $q_1(x)\sim|x|^2\sim r^2$ as $r\to 0$ because of Lemma 6.2 (iii), it follows that

$$\int_{|x|\leq\varepsilon}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)=\int_{|x|\leq\varepsilon}r^2P_2^x(\lambda)\,dL_j(x)+\eta_j, \tag{6.28}$$

where $\sup_j\eta_j\to 0$ as $\varepsilon\to 0$, so that

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\int_{|x|\leq\varepsilon}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)=\lim_{\varepsilon\to 0}\lim_{j\to\infty}-\int_{|x|\leq\varepsilon}r^2P_2^x(\lambda)\cdot dL_j(x). \tag{6.29}$$

The existence of the limit on the right being clear from (6.28). However, for fixed x, $P_2^x(\lambda)$ is a quadratic polynomial in $\lambda_1,\dots,\lambda_l$ being the eigenvalue of $D_2^x\in\mathbf{D}(G/K)$ corresponding to the eigenfunction ϕ_λ. By our remarks at the end of § 3,

$$-P_2^x(\lambda)=\sum_{u,v=1}^{l}Q_{u,v}(x)\cdot\lambda_u\cdot\lambda_v+\sum_{u,v=1}^{l}Q_{u,v}(x)\,\varrho_u\cdot\varrho_v.$$

It follows easily from this that the limit on the right of (6.29) must necessarily be of the form

$$\sum_{u,v=1}^{l}Q_{uv}\lambda_u\lambda_v+\sum_{u,v=1}^{l}Q_{uv}\varrho_u\varrho_v=-P_D(\lambda)$$

say. Further, since for each x, $P_2^x(s\lambda)=P_2^x(\lambda)$ for $s\in W$, it is clear that $P_D(s\lambda)=P_D(\lambda)$, so that $P_D\in\mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$. Hence (cf. § 3) there exists a second order operator $D\in\mathbf{D}(G/K)$ whose eigenvalue for ϕ_λ is $P_D(\lambda)$. The fact that $P_D(-i\varrho)=0$ means D annihilates constants, while the fact that $\{Q_{uv}\}$ is a non-negative definite matrix is precisely the ellipticity assertion, cf. § 3. Thus

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\int_{|x|\leq\varepsilon}[1-\operatorname{Re}\phi_\lambda(x)]\,dL_j(x)=-P_D(\lambda), \tag{6.30}$$

where $D\phi_\lambda=P_D(\lambda)\phi_\lambda$. Collecting the results (6.22), (6.26), (6.30) we have

$$\text{Re } \psi(\lambda) = \lim_{j\to\infty} \text{ Re } \psi_j(\lambda) = c - P_D(\lambda) + \int_{|x|>0} [1 - \text{Re } \phi_\lambda(x)] dL(x). \tag{6.31}$$

Let us now look at the imaginary part of $\psi(\lambda)$. We have

$$\text{Im } \psi_j(\lambda) = \int \text{Im } \phi_\lambda(x) dL_j(x) = \int_{|x|\leqslant 1} \text{Im } \phi_\lambda(x) dL_j(x) + \int_{|x|>1} \text{Im } \phi_\lambda(x) dL_j(x). \tag{6.32}$$

As to the second term, Im $\phi_\lambda(x)$ vanishes at ∞ and hence, by (6.20),

$$\int_{|x|>1} \text{Im } \phi_\lambda(x) dL_j(x) \to \int_{|x|>1} \text{Im } \phi_\lambda(x) dL(x)$$

for a subsequence (to which we may pass). With regard to the first term, we may use the device of Lemma 6.2 to get

$$\text{Im } \phi_\lambda(x) = \frac{1}{2i} [\phi_\lambda(x) - \phi_\lambda(x^{-1})] = \frac{1}{i} \sum_{j=0}^{\infty} r^{2j+1} (D^x_{2j+1} \phi_\lambda)(e). \tag{6.33}$$

Now $D^x_1 = \int_K \text{Ad}(k) \tilde{X} dk$. Since $\tilde{X} \in \mathfrak{p}_0$ and $\text{Ad}(K)\mathfrak{p}_0 \subset \mathfrak{p}_0$, it follows that $D_1 \in \mathfrak{p}_0$ and is invariant under $\text{Ad}(k')$, $k' \in K$. But this implies that $D_1 = 0$, [5], [15]. Hence by (6.33),

$$\text{Im } \phi_\lambda(x) = o(r^2) \quad \text{as} \quad r \to 0. \tag{6.34}$$

There is no difficulty in showing, remembering (6.17) and $q_1(x) \sim r^2$ as $r \to 0$, that

$$\int_{|x|\leqslant 1} \text{Im } \phi_\lambda(x) dL_j(x) \to \int_{|x|\leqslant 1} \text{Im } \phi_\lambda(x) dL(x). \tag{6.35}$$

Putting together (6.32), (6.35), we have

$$\int \text{Im } \phi_\lambda(x) dL_j(x) \to \int \text{Im } \phi_\lambda(x) dL(x). \tag{6.36}$$

It follows now from (6.31), (6.36) that

$$\begin{aligned} \psi(\lambda) = \lim_{j\to\infty} \psi_j(\lambda) &= \lim_{j\to\infty} [\text{Re } \psi_j(\lambda) + i \text{ Im } \psi_j(\lambda)] \\ &= \lim_{j\to\infty} \left[\int [1 - \text{Re } \phi_\lambda(x)] dL_j(x) - i \int \text{Im } \phi_\lambda(x) dL_j(x) \right] \\ &= c - P_D(\lambda) + \int_{|x|>0} [1 - \phi_\lambda(x)] dL(x). \end{aligned} \tag{6.37}$$

Thus $$\exp -\psi(\lambda) = \exp -\left(c - P_D(\lambda) + \int_{|x|>0} [1-\phi_\lambda(x)]\,dL(x)\right)$$

and so $\exp -\psi(\lambda) \in \hat{\mathbf{S}}_0$ if and only if $\exp -\{\psi(-i\varrho)\} = 1$, i.e., if and only if $\psi(-i\varrho) = 0$. But the quadratic polynomial $P_D(\lambda)$ vanishes at $\lambda = -i\varrho$ as seen above, while the integral vanishes at $\lambda = -i\varrho$ since $\phi_{-i\varrho} \equiv 1$. Hence $\psi(-i\varrho) = c$ so $\exp -\psi(\lambda) \in \hat{\mathbf{S}}_0$ if and only if $c = 0$. The proof of the direct half of the theorem is thus finished.

We now turn to the converse part of the theorem. Suppose then that we are given an elliptic second order differential operator $D \in \mathbf{D}(G/K)$ which annihilates constants and a spherical measure L on $\{x \mid |x| > 0\}$ satisfying (6.11). We shall presently prove that there exist $N_j \in \mathbf{S}(K\backslash G/K)$ such that

$$\lim_{j\to\infty} \int [1-\phi_\lambda(x)]\,dN_j(x) = -P_D(\lambda).$$

Granting the existence of such N_j, let M_j be the measure in $\mathbf{S}(K\backslash G/K)$ defined by $M_j(A) = L(A \cap \{x \mid |x| \geqslant 1/j\})$ and let $L_j = N_j + M_j$. Then it is clear that

$$\lim_{j\to\infty} \int [1-\phi_\lambda(x)]\,dL_j(x) = -P_D(\lambda) + \int_{|x|>0} [1-\phi_\lambda(x)]\,dL(x).$$

Thus, to finish the proof of Theorem 6.1, we have to produce $N_j \in \mathbf{S}(K\backslash G/K)$ such that $\int [1-\phi_\lambda(x)]\,dN_j(x) \to -P_D(\lambda)$ as $j \to \infty$. Recall that (cf. § 2) $\mathbf{D}(G/K)$ and $\mathbf{I}(\mathfrak{p}_0)$ are isomorphic as vector spaces under an isomorphism $P \to D_P$ from $\mathbf{I}(\mathfrak{p}_0)$ onto $\mathbf{D}(G/K)$, and further the order of D_P = degree of P. Thus given $D \in \mathbf{D}(G/K)$ of the above description there exists $P \in \mathbf{I}(\mathfrak{p}_0)$ such that $D = D_P$. Since D is second order, P is of degree 2. The ellipticity of D is to say that the quadratic part of P is non-negative definite. P has no constant term since D annihilates constants. Hence,

$$P(X_1, \ldots, X_n) = \sum_{u=1}^{n} \sum_{v=1}^{n} Q_{uv} X_u X_v + \sum_{u=1}^{n} b_u X_u.$$

But since P is invariant under $\operatorname{Ad}(K)$, it is easily seen that each of the two terms on the right must be so invariant. But there are no vectors $\in \mathfrak{p}_0$ invariant under $\operatorname{Ad}(K)$ except the null vector. Hence $\sum b_u X_u = 0$. So

$$P(X_1, \ldots, X_n) = \sum_{u,v=1}^{n} Q_{uv} X_u X_v. \tag{6.38}$$

Since Q_{uv} is symmetric and non-negative definite it is clear that

$$P(X_1, \ldots, X_n) = \sum_{v=1}^{n} c_v Y_v^2, \tag{6.39}$$

where $c_v \geqslant 0$ and the Y_v are certain linear forms in $X_1 \ldots, X_n$; so that $Y_v \in \mathfrak{p}_0$. We may clearly assume that $\|Y_v\| = 1$, $v = 1, \ldots, n$. Now since P is invariant under Ad (K), we have from (6.39)

$$P(X_1, \ldots, X_n) = \sum_{v=1}^{n} C_v \int_K \text{Ad } (k) Y_v^2 dk = \sum_{v=1}^{n} c_v P_v \quad \text{say}, \tag{6.40}$$

where $P_v \in \mathbf{I}(\mathfrak{p}_0)$, $v = 1, \ldots, n$.

Since $P \to D_P$ is linear, it is clear, setting $D_v = D_{P_v}$, that

$$D = \sum_{v=1}^{n} c_v D_v. \tag{6.41}$$

Let $P_v(\lambda)$ be the eigenvalue corresponding to ϕ_λ of D_v, i.e., $D_v \phi_\lambda = P_v(\lambda) \phi_\lambda$. Then $P_D(\lambda) = \sum_{v=1}^{n} c_v P_v(\lambda)$. We shall now construct a sequence of measures

$$N_{j,v} \in \mathbf{S}(K \backslash G/K), \ j = 1, 2, \ldots,$$

such that

$$\int [1 - \phi_\lambda(x)] dN_{j,v}(x) \to -P_v(\lambda), \quad j \to \infty; \quad v = 1, \ldots, n. \tag{6.42}$$

Then if we set $N_j = \sum_{v=1}^{n} c_v N_{j,v}$, we shall clearly have

$$\begin{aligned} \int [1 - \phi_\lambda(x)] dN_j(x) &= \sum_{v=1}^{n} c_v \int [1 - \phi_\lambda(x)] dN_{j,v}(x) \\ &\to \sum_{v=1}^{n} -c_v P_v(\lambda), \ j \to \infty \\ &= -P_D(\lambda). \end{aligned} \tag{6.43}$$

Thus we shall be through as soon as we construct $N_{j,v} \in \mathbf{S}(K \backslash G/K)$ satisfying (6.42).

To do this, let ε_j be a sequence of positive numbers such that $\varepsilon_j \to 0$, $j \to \infty$. Let $N_{j,v}$ be the measure

$$\frac{1}{\varepsilon_j^2} \mu_{\exp \varepsilon_j Y_v}.$$

(Recall here the notation introduced in the second paragraph of § 5.) Then

$$\int [1 - \phi_\lambda(x)] dN_{j,v}(x) = \frac{1}{\varepsilon_j^2} [1 - \phi_\lambda(\exp \varepsilon_j Y_v)]. \tag{6.44}$$

Using (6.5) on this, we get easily

$$\begin{aligned}\int [1-\phi_\lambda(x)]\,dN_{j,v}(x) &= \frac{1}{\varepsilon_j^2}[1-\phi_\lambda(\exp \varepsilon_j Y_v)] \\ &= -\frac{1}{\varepsilon_j^2}\sum_{m=1}^{\infty} \varepsilon_j^m \left(\left[\int_K \mathrm{Ad}\,(k)\, Y_v^m\, dk\right]\phi_\lambda\right)(e) \\ &= -\frac{1}{\varepsilon_j^2}\sum_{m=1}^{\infty} \varepsilon_j^m (D_{v,m}\phi_\lambda)(e), \end{aligned} \tag{6.45}$$

where $D_{v,m} = \int_K \mathrm{Ad}\,(k)\, Y_v^m\, dk$.

Now $D_{v,1}$ being a linear form $\in \mathbf{I}(\mathfrak{p}_0)$, must equal 0 while $D_{v,2} = D_v$ in our previous notation. Hence we have

$$\int [1-\phi_\lambda(x)]\,dN_{j,v}(x) = -(D_v\phi_\lambda)(e) + o(\varepsilon_j). \tag{6.46}$$

Since $(D_v\phi_\lambda)(e) = P_v(\lambda)\cdot\phi_\lambda(e) = P_v(\lambda)$, and since $\varepsilon_j \to 0$ as $j\to\infty$, we have from (6.46)

$$\int [1-\phi_\lambda(x)]\,dN_{j,v}(x) \to -P_v(\lambda) \text{ as } j\to\infty. \tag{6.47}$$

and the proof of Theorem 6.1 is finally finished. Q.E.D.

In view of Theorem 6.1 and Lemma 5.2, we can now state the following.

THEOREM 6.2. *A measure $\mu \in \mathrm{S}_0(K\backslash G/K)$ is infinitely divisible if and only if its Fourier–Stieltjes transform $\hat\mu(\lambda)$, $\lambda\in E_R$ has the representation*

$$\hat\mu(\lambda) = \exp\left\{P_D(\lambda) - \int_{|x|>0}[1-\phi_\lambda(x)]\,dL(x)\right\}, \tag{6.48}$$

where L is a spherical measure satisfying

$$\int \frac{|x|^2}{1+|x|^2}\,dL(x) < \infty$$

and $P_D(\lambda)$ is the eigenvalue corresponding to the eigenfunction ϕ_λ of a second order elliptic differential operator $D \in \mathbf{D}(G/K)$ which annihilates constants.

In view of the remarks on page 265 of [16], the following corollary is not without interest.

COROLLARY 6.3. *Every infinitely divisible measure $\mu \in \mathrm{S}_0(K\backslash G/K)$ can be represented as μ^1 in a continuous one-parameter convolution semi-group $\{\mu^t\}_{t>0}$ with $\mu^t \Rightarrow \mu_e$ as $t\downarrow 0$ (recall the notation of § 5).*

Proof. Indeed let μ^t be defined by $(\mu^t)^\wedge = (\hat{\mu})^t$. Q.E.D.

We now turn to the question of uniqueness.

THEOREM 6.3. *In the representation* (6.48), *the measure L and the operator D are determined uniquely by μ.*

Proof. We shall follow the method used in [23 p. 194]. Suppose then $\mu \in \mathbf{S}_0(K\backslash G/K)$ is such that μ satisfies (6.48). Let ξ_t be a separable Markov process taking values in the symmetric space G/K, whose transition function $P(t, p, B)$ satisfies the following two conditions:

(i) $P(t, p, B) = P(t, \boldsymbol{\pi}(e), x^{-1}B)$ where $p = \boldsymbol{\pi}(x)$, $x \in G$, and B is a Borel subset of G/K, $\boldsymbol{\pi}$ being the natural projection $G \to G/K$.

(ii) If μ^t is the measure in $\mathbf{S}_0(K\backslash G/K)$ which corresponds to the measure $P(t, \boldsymbol{\pi}(e), \cdot)$ on G/K, in the natural way (the existence of μ^t being guaranteed by (i)) then

$$(\mu^t)^\wedge = (\hat{\mu})^t. \tag{6.49}$$

It is not hard to show that the sample functions of such a process will, with probability one, only have discontinuities of the first kind. (Indeed, the sample functions of ξ_t may be obtained as limits with probability one of sequences of functions each of which is the trajectory of a Brownian motion, interlarded with finitely many independent Poisson jumps, the convergence of the sequence being uniform on compact subsets of the parameter set $[0, \infty\rangle$. See our note [7].)

We may prove just as in [4], [17] that the mathematical expectation of the number of jumps of the trajectories of the process ξ_t between time 0 and t, the magnitudes (in the sense of the metric on G/K) of which lie between r_1 and r_2, is precisely

$$t \int_{r_1 < |x| < r_2} dL(x). \tag{6.50}$$

Since this mathematical expectation is determined by the transition function $P(\cdot, \cdot, \cdot)$, if follows that the measure L is determined by μ. It follows that the term $P_D(\lambda)$ in (6.48) is also determined by μ.

Now we have seen that $P_D(\lambda)$ is the eigenvalue, corresponding to ϕ_λ, of an elliptic second order operator in $D(G/K)$. To finish the proof of our theorem it is enough to prove that if $D^1, D^2 \in \mathbf{D}(G/K)$ and if $D^1\phi_\lambda = D^2\phi_\lambda$ for each $\phi_\lambda, \lambda \in E_R$ then $D^1 = D^2$. But if Γ is the isomorphism $\Gamma: \mathbf{D}(G/K) \to \mathbf{I}(\mathfrak{h}_{\mathfrak{p}_0})$ (cf. § 2) then it is known

[15 p. 430–432] that $D\phi_\lambda = \Gamma(D)(i\lambda)\phi_\lambda$ where $\Gamma(D)(i\lambda)$ is the value at $i\lambda \in E_C$ of the polynomial function $\Gamma(D) \in \mathrm{I}(\mathfrak{h}_{\mathfrak{p}_0})$. Thus, if $D^1\phi_\lambda = D^2\phi_\lambda$, it follows that $\Gamma(D^1)(i\lambda) = \Gamma(D^2)(i\lambda)$, $\lambda \in E_R$. Hence $\Gamma(D^1) = \Gamma(D^2)$ and since Γ is an isomorphism, we conclude $D^1 = D^2$. Q.E.D.

§ 7. Generalized limits and infinitely divisible measures

The following definition has a well-known classical motivation.

Definition 7.1. A measure $\mu \in \mathrm{S}_0(K\backslash G/K)$ is called a generalized limit if there exist, for each positive integer j, measures $\mu_{jr} \in \mathrm{S}_0(K\backslash G/K)$, $1 \leqslant r \leqslant r_j$ with $r_j \to \infty$ as $j \to \infty$, such that

(i) $\hat{\mu}_j(\lambda) = \prod_{r=1}^{r_j} \hat{\mu}_{jr}(\lambda) \to \hat{\mu}(\lambda)$ as $j \to \infty$, $\lambda \in E_R$;

(ii) $\max\limits_{1 \leqslant r \leqslant r_j} |\hat{\mu}_{jr}(\lambda) - 1| \to 0$ as $j \to \infty$ uniformly for λ in compact subsets of E_R.

Theorem 7.1. *Condition* (ii) *of Definition 7.1 is equivalent to*

$$\max_{1 \leqslant r \leqslant r_j} \int_{A^c} d\mu_{jr}(x) \to 0 \text{ as } j \to \infty \tag{7.1}$$

for each compact subset A containing e of G. Here A^c stands for the complement of A.

The proof, which offers no difficulty, is omitted (cf. [8, p. 1298]).

Lemma 7.1. *If $\mu \in \mathrm{S}_0(K\backslash G/K)$ is infinitely divisible, then μ is a generalized limit.*

Proof. Let $r_j = j$, and let μ_{jr} be defined by means of its Fourier–Stieltjes transform as follows:

$$\hat{\mu}_{jr}(\lambda) = [\hat{\mu}(\lambda)]^{1/j} = \exp\left(\frac{1}{j}\left\{P_D(\lambda) - \int_{|x|>0} [1 - \phi_\lambda(x)]\, dL(x)\right\}\right), \quad 1 \leqslant r \leqslant r_j = j. \tag{7.2}$$

Then it is clear that $\hat{\mu}_j = \prod_{r=1}^{r_j} \hat{\mu}_{jr} = \hat{\mu}$ for each j so $\mu_j = \mu$. Also (7.2) ensures that condition (ii) of Definition 7.1 is fulfilled. Q.E.D.

Lemma 7.2. *Suppose that $\mu \in \mathrm{S}_0(K\backslash G/K)$ is a generalized limit, and let μ_j, μ_{jr} be as in Definition 7.1. Set $\beta_{jr}(\lambda) = 1 - \hat{\mu}_{jr}(\lambda), \lambda \in E_R$. Then there exists a $d > 0$ such that*

$$\sup_{\lambda \in V_d} \sup_j \sum_{r=1}^{r_j} |\operatorname{Re} \beta_{jr}(\lambda)| < \infty, \tag{7.3}$$

where $V_d = \{\lambda \mid \|\lambda\| \leqslant d\}$.

Proof. Since $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$, $\lambda \in E_R$ and $\hat{\mu} \in \hat{\mathrm{S}}_0$, we have $\mu_j \Rightarrow \mu$ by Theorem 4.2 (iv). In particular, the convergence $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$ is uniform in every compact neighborhood of 0 in E_R. Now, since $\mu \in \mathrm{S}_0(K\backslash G/K)$, $\hat{\mu}(0) > 0$; hence there exists a $d > 0$ such that $\hat{\mu}(\lambda) \neq 0$ for $\lambda \in V_d$. Therefore $\log |\hat{\mu}_j(\lambda)|^2 \to \log |\hat{\mu}(\lambda)|^2$ uniformly for $\lambda \in V_d$. Let us write $B(\lambda) = \log |\hat{\mu}(\lambda)|^2$. Then it is clear that if $\varepsilon > 0$ is given, we have, for large j

$$B(\lambda) + \varepsilon \geqslant \log |\hat{\mu}_j(\lambda)|^2 = \sum_{r=1}^{r_j} \log |\hat{\mu}_{jr}(\lambda)|^2. \tag{7.4}$$

Let $\alpha_{jr}(\lambda) = 1 - |\hat{\mu}_{jr}(\lambda)|^2$. Then $|\alpha_{jr}(\lambda)| \to 0$ as $j \to \infty$ uniformly in $1 \leqslant r \leqslant r_j$ and $\lambda \in V_d$. Therefore it easily follows that for large j,

$$\begin{aligned}
\sum_{r=1}^{r_j} \log |\hat{\mu}_{jr}(\lambda)|^2 &= \sum_{r=1}^{r_j} \log 1 - \alpha_{jr}(\lambda) \\
&\geqslant \frac{1}{2} \sum_{r=1}^{r_j} |\alpha_{jr}(\lambda)| \\
&= \frac{1}{2} \sum_{r=1}^{r_j} 1 - |\hat{\mu}_{jr}(\lambda)|^2 \\
&= \frac{1}{2} \sum_{r=1}^{r_j} \left\{ 1 - \left(\int \operatorname{Re} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 - \left(\int \operatorname{Im} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 \right\}.
\end{aligned} \tag{7.5}$$

Since μ_{jr} obey Theorem 7.1, and $\phi_\lambda(e) = 1$, we have, given $\eta > 0$, for large j

$$\int \operatorname{Re} \phi_\lambda(x)\, d\mu_{jr}(x) > 1 - \eta \tag{7.6}$$

so that

$$\begin{aligned}
1 - \left(\int \operatorname{Re} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 &= \left[1 - \int \operatorname{Re} \phi_\lambda(x)\, d\mu_{jr}(x) \right] \left[1 + \int \operatorname{Re} \phi_\lambda(x)\, d\mu_{jr}(x) \right] \\
&\geqslant (2 - \eta) \int [1 - \operatorname{Re} \phi_\lambda(x)]\, d\mu_{jr}(x).
\end{aligned} \tag{7.7}$$

Now consider $(\int \operatorname{Im} \phi_\lambda(x) d\mu_{jr}(x))^2$; we have for any compact neighborhood A of e,

$$\begin{aligned}
\left(\int_A \operatorname{Im} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 &\leqslant 2 \left(\int_A \operatorname{Im} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 + 2 \left(\int_{A^c} \operatorname{Im} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 \\
&= J_1 + J_2 \quad \text{say.}
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
J_1 = 2 \left(\int_A \operatorname{Im} \phi_\lambda(x)\, d\mu_{jr}(x) \right)^2 &\leqslant 2 \int_A d\mu_{jr}(x) \cdot \int_A |\operatorname{Im} \phi_\lambda(x)|^2\, d\mu_{jr}(x) \\
&\leqslant 2 \int_A |\operatorname{Im} \phi_\lambda(x)|\, d\mu_{jr}(x),
\end{aligned} \tag{7.9}$$

since $\mu_{jr} \in S_0(K\backslash G/K)$ and $|\mathrm{Im}\ \phi_\lambda(x)| \leqslant 1$. Now, for a sufficiently small neighborhood A of e, $1-\mathrm{Re}\ \phi_\lambda(x)>0$ except at $x=e$ for $x \in A$ and further, (6.34), (6.27) imply that $\mathrm{Im}\ \phi_\lambda(x)/(1-\mathrm{Re}\ \phi_\lambda(x)) \to 0$ as $x \to e$. Hence, given $\eta>0$ we can choose A so small that $|\mathrm{Im}\ \phi_\lambda(x)| \leqslant \eta(1-\mathrm{Re}\ \phi_\lambda(x))$ for $x \in A$. Making this choice, (7.9) yields

$$J_1 \leqslant 2\eta \int_A [1-\mathrm{Re}\ \phi_\lambda(x)]\, d\mu_{jr}(x). \tag{7.10}$$

As for J_2, we have

$$\begin{aligned} J_2 = 2\left(\int_{A^c} \mathrm{Im}\ \phi_\lambda(x)\, d\mu_{jr}(x)\right)^2 &\leqslant 2\int_{A^c} d\mu_{jr}(x) \cdot \int_{A^c} |\mathrm{Im}\ \phi_\lambda(x)|^2\, d\mu_{jr}(x) \\ &\leqslant 4\int_{A^c} d\mu_{jr}(x) \cdot \int_{A^c} [1-\mathrm{Re}\ \phi_\lambda(x)]\, d\mu_{jr}(x), \end{aligned} \tag{7.11}$$

where we used, for the last step, the following

$$\begin{aligned} (\mathrm{Im}\ \phi_\lambda(x))^2 = |\phi_\lambda(x)|^2 - (\mathrm{Re}\ \phi_\lambda(x))^2 &\leqslant 1-(\mathrm{Re}\ \phi_\lambda(x))^2 \\ &= (1+\mathrm{Re}\ \phi_\lambda(x))\,(1-\mathrm{Re}\ \phi_\lambda(x)) \leqslant 2(1-\mathrm{Re}\ \phi_\lambda(x)). \end{aligned} \tag{7.12}$$

Since μ_{jr} obey Theorem 7.1, for large enough j we have

$$J_2 \leqslant 4\eta \int_{A^c} [1-\mathrm{Re}\ \phi_\lambda(x)]\, d\mu_{jr}(x). \tag{7.13}$$

From (7.8), (7.10), (7.13), we have

$$\left(\int \mathrm{Im}\ \phi_\lambda(x)\, d\mu_{jr}(x)\right)^2 \leqslant 4\eta \cdot \int [1-\mathrm{Re}\ \phi_\lambda(x)]\, d\mu_{jr}(x), \tag{7.14}$$

using (7.4), (7.5), (7.7), (7.14) we have finally

$$B(\lambda)+\varepsilon \geqslant \tfrac{1}{2}(2-5\eta)\sum_{r=1}^{r_j} \int 1-\mathrm{Re}\ \phi_\lambda(x)\, d\mu_{jr}(x) = \tfrac{1}{2}(2-5\eta)\sum_{r=1}^{r_j} \mathrm{Re}\ \beta_{jr}(\lambda). \tag{7.15}$$

Since η may be as small as we please, $\mathrm{Re}\ \beta_{jr}(\lambda) \geqslant 0$, and since $B(\lambda)$ is continuous for $\lambda \in V_d$, Lemma 7.2 clearly follows from (7.15). Q.E.D.

LEMMA 7.3. *Let $\mu \in S_0(K\backslash G/K)$ be a generalized limit. Then with the same notation as in Lemma 7.2, we have, for $\lambda \in E_R$,*

$$\sup_j \sum_{r=1}^{r_j} |\beta_{jr}(\lambda)| < \infty \cdot \tag{7.16}$$

Proof.
$$|\beta_{jr}(x)| = \left|\int [1-\phi_\lambda(x)]\,d\mu_{jr}(x)\right|$$
$$\leqslant \int_{|x|\leqslant\tau} |1-\phi_\lambda(x)|\,d\mu_{jr}(x) + \int_{|x|>\tau} |1-\phi_\lambda(x)|\,d\mu_{jr}(x)$$
$$= J_1 + J_2 \quad \text{say.} \tag{7.17}$$

Now, it is an easy consequence of Lemma 6.2 and (6.27) that for each $d>0$, $|1-\phi_\lambda(x)| \leqslant M_d q_d(x)$ for $|x|\leqslant\tau$, M_d being a constant (depending on τ, which is fixed in this discussion). Hence

$$J_1 \leqslant M_d \int_{|x|\leqslant\tau} q_d(x)\,d\mu_{jr}(x). \tag{7.18}$$

As for J_2, we have by Lemma 6.1 that $q_d(x) \geqslant \delta > 0$ on $|x|>\tau$ (once again, δ may depend on τ), so that

$$J_2 \leqslant 2\int_{|x|>\tau} d\mu_{jr}(x) \leqslant \frac{2}{\delta}\int_{|x|>\tau} q_d(x)\,d\mu_{jr}(x). \tag{7.19}$$

Hence
$$|\beta_{jr}(\lambda)| \leqslant M'_d \int q_d(x)\,d\mu_{jr}(x), \tag{7.20}$$

where M'_d is a constant. Finally

$$\sum_{r=1}^{r_j} |\beta_{jr}(\lambda)| \leqslant M'_d \sum_{r=1}^{r_j} q_d(x)\,d\mu_{jr}(x)$$
$$= M'_d \sum_{r=1}^{r_j} \iint_{V_d} [1-\operatorname{Re}\phi_{\lambda'}(x)]\,d\lambda'\,d\mu_{jr}(x)$$
$$= M'_d \int_{V_d} \sum_{r=1}^{r_j} |\operatorname{Re}\beta_{jr}(\lambda')|\,d\lambda'. \tag{7.21}$$

In view of Lemma 7.2, this last inequality concludes the proof of Lemma 7.3. Q.E.D.

THEOREM 7.2. *$\mu \in S_0(K\backslash G/K)$ is a generalized limit if and only if μ is infinitely divisible.*

Proof. In view of Lemma 7.1, it is enough to prove that if μ is a generalized limit, then μ is infinitely divisible.

Le μ_j, μ_{jr}, $\beta_{jr}(\lambda)$ etc. be as above. Let $d>0$ be any real number. Condition (ii) of Definition 7.1 implies that for sufficiently large j (how large depending only on d),

$\hat{\mu}_{jr}(\lambda) \neq 0$ for $\lambda \in V_d$, $1 \leqslant r \leqslant r_j$. Hence $\hat{\mu}_j(\lambda) \neq 0$, $\lambda \in V_d$ so that $\log \hat{\mu}_j(\lambda)$ may be defined. For such j we have

$$\begin{aligned} -\log \hat{\mu}_j(\lambda) &= -\sum_{r=1}^{r_j} \log \hat{\mu}_{jr}(\lambda) \\ &= -\sum_{r=1}^{r_j} \log (1-\beta_{jr}(\lambda)) = \sum_{r=1}^{r_j} \sum_{m=1}^{\infty} (m)^{-1} (\beta_{jr}(\lambda))^m \\ &= \sum_{r=1}^{r_j} \beta_{jr}(\lambda) + \sum_{r=1}^{r_j} \sum_{m=2}^{\infty} (m)^{-1} (\beta_{jr}(\lambda))^m = J_1 + J_2 \quad \text{say.} \end{aligned} \tag{7.22}$$

Where, for the expansions, we used the fact, guaranteed by Definition 7.1, that $\max_{1 \leqslant r \leqslant r_j} |\beta_{jr}(\lambda)| \to 0$ as $j \to \infty$ uniformly for $\lambda \in V_d$.

Now
$$\begin{aligned} |J_2| &\leqslant \sum_{r=1}^{r_j} \sum_{m=2}^{\infty} (m^{-1}) \, |\beta_{jr}(\lambda)|^m \\ &\leqslant \frac{1}{2} \sum_{r=1}^{r_j} \sum_{m=2}^{\infty} |\beta_{jr}(\lambda)|^m = \frac{1}{2} \sum_{r=1}^{r_j} \frac{|\beta_{jr}(\lambda)|}{1-|\beta_{jr}(\lambda)|} \\ &\leqslant \max_{1 \leqslant r \leqslant r_j} |\beta_{jr}(\lambda)| \sum_{r=1}^{r_j} |\beta_{jr}(\lambda)| \to 0 \quad \text{as} \quad j \to \infty \end{aligned} \tag{7.23}$$

uniformly for $\lambda \in V_d$ in view of Lemma 7.3. Hence, by (7.22) we have

$$\hat{\mu}_j(\lambda) - \exp - \left\{ \sum_{r=1}^{r_j} \beta_{jr}(\lambda) \right\} \to 0 \tag{7.24}$$

as $j \to \infty$, uniformly for $\lambda \in V_d$. But since $\hat{\mu}_j(\lambda) \to \hat{\mu}(\lambda)$ by hypothesis, it follows that

$$\lim_{j \to \infty} \exp - \left\{ \sum_{r=1}^{r_j} \beta_{jr}(\lambda) \right\} = \hat{\mu}(\lambda), \quad \lambda \in E_R. \tag{7.25}$$

Now, $\exp - \{\sum_{r=1}^{r_j} \beta_{jr}(\lambda)\} = \exp - \{\int [1-\phi_\lambda(x)]\, dL_j(x)\}$ where $L_j = \sum_{r=1}^{r_j} \mu_{jr} \in S(K \backslash G/K)$. Therefore, by Theorem 6.2, $\exp - \{\sum_{r=1}^{r_j} \beta_{jr}(\lambda)\} = \hat{\nu}_j(\lambda)$ where $\nu_j \in S_0(K \backslash G/K)$ and is infinitely divisible. Since $\hat{\nu}_j(\lambda) \to \hat{\mu}(\lambda)$, $\lambda \in E_R$ and since $\mu \in S_0(K \backslash G/K)$, Theorem 4.2 (iv) implies that $\nu_j \Rightarrow \mu$. Therefore, by Lemma 5.2, μ is infinitely divisible. Q.E.D.

The following theorem is implicit in our work above and we mention it without formal proof.

THEOREM 7.3. *Let* $\hat{\mu}_{jr} \in \hat{S}_0$, $j = 1, 2, \ldots, 1 \leqslant r \leqslant r_j$, $r_j \to \infty$ *as* $j \to \infty$, *and suppose that*

$$\lim_{j \to \infty} \max_{1 \leqslant r \leqslant r_j} |1 - \hat{\mu}_{jr}(\lambda)| = 0$$

uniformly for λ in compact subsets of E_R. Let

$$\hat{\mu}_j(\lambda) = \prod_{r=1}^{r_j} \hat{\mu}_{jr}(\lambda).$$

Then $\hat{\mu}_j \to \hat{\mu} \in \hat{\mathbf{S}}_0$ if and only if the measures Ω_j defined by

$$\Omega_j(B) = \sum_{r=1}^{r_j} \int_B \frac{|x|^2}{1+|x|^2}\, d\mu_{jr}(x)$$

are Bernoulli convergent to a measure $\Omega \in \mathbf{S}(K\backslash G/K)$.

If this condition is satisfied, then there exists a uniquely determined second order elliptic differential operator $D \in \mathbf{D}(G/K)$ which annihilates constants such that μ has the representation (6.48), *the measure L of that formula being related to Ω by*

$$L(B) = \int_B \frac{1+|x|^2}{|x|^2}\, d\Omega(x).$$

We end this section with some comments. Our Theorem 6.2. which was the instigation for this paper, can be obtained from Hunt's results [16] *if* we assume the truth of Corollary 6.3. Thus a part of the justification for our results is that in our situation Corollary 6.3 *is a consequence and not an assumption.* As Hunt remarks, Corollary 6.3 is not true in his more general set-up. Indeed, there are reasons to believe that such a result is true only when some commutativity is present as in our set-up, where $\mathbf{S}(K\backslash G/K)$ is commutative.

Our work shows that the function exp $P_D(\lambda)$ is the Fourier–Stieltjes transform of a measure in $\mathbf{S}_0(K\backslash G/K)$, for each $D \in \mathbf{D}(G/K)$ of the description in Theorem 6.2. It is natural to call these measures Gaussian, for obvious reasons. One can at this point formulate and prove many theorems analogous to those of classical central limit theory, e.g., the theorems of Lindeberg–Feller on convergence to Gaussian distributions. Another class of problems of interest is the discussion of Stable laws; cf. § 10 of [8]. We do not occupy ourselves with these questions in the present paper.

When the symmetric space G/K has rank 1 so that it is two-point homogeneous, our results become rather more explicit. In this case, there is, up to positive multiples, only one elliptic second order differential operator in $\mathbf{D}(G/K)$, viz. the Laplace–Beltrami operator of G/K. The space of double cosets $K\backslash G/K$ is one dimensional in this case and our theory is a theory of infinitely divisible elements in certain convolution semigroups of measures on the half-line $[0, \infty\rangle$, the convolution being, of course, different from the usual one. In this form, our work subsumes [8], [23], as rather special cases.

§ 8. The case of a compact symmetric space

The case where G/K is a *compact* symmetric space can be dealt with by substantially the same methods as we have followed above. We do not wish to give complete details but will merely outline our results and point out one or two places where the argument of the previous sections has to be modified.

It is well known that the elementary K-spherical functions of positive-definite type are in one-to-one correspondence with equivalence classes of irreducible unitary representations of G which are of class 1 with respect to K, i.e., representations T such that the trivial representation of K occurs in the reduction of the restriction of T to K. If G/K is compact, then it also is easy to show that there are only countably many distinct elementary K-spherical functions on G and these are *all* of positive-definite type. Indeed if χ is the character of a finite-dimensional unitary irreducible representation of G which is of class 1 with respect to K then the function

$$\phi(x) = \int_K \chi(x^{-1}k)dk, \quad x \in G \tag{8.1}$$

is an elementary spherical function on G and all the elementary spherical functions arise in this way.

Letting $\phi_0, \phi_1, \dots;$ $(\phi_0 \equiv 1)$ be an enumeration of these elementary spherical functions one may now define for $\mu \in S(K\backslash G/K)$ the sequence of Fourier–Stieltjes coefficients by

$$\hat{\mu}(n) = \int \phi_n(x) d\mu(x). \tag{8.2}$$

By utilizing Peter–Weyl theory, it is possible to show that any complex-valued continuous *spherical* function on G may be approximated *uniformly* by finite complex linear combinations of the elementary spherical functions and this can be seen easily to lead to the fact that if $\hat{\mu}(n) = 0$ for all $n = 0, 1, 2 \dots$ then $\mu = 0$. The elementary spherical functions ϕ_n still satisfy (3.1), and this has the implication that $\widehat{\mu\nu} = \hat{\mu}\hat{\nu}$, for $\mu, \nu \in S(K\backslash G/K)$. The analogue of the continuity theorem (Theorem 4.2) is trivial.

The notion of infinite divisibility may be introduced just as before, but due to the disconnectedness of the domain of the Fourier transform, it is no longer true (as examples easily show) that for an infinitely divisible measure $\mu \in S_0(K\backslash G/K)$, $\hat{\mu}(n) \neq 0$ for any n. Instead one has to make this assumption *ad hoc*. With this assumption, the theorems of § 5 are true with only minor changes which will be obvious to the reader.

The function $q_1(x)$ of Lemma 6.1 is now to be replaced by the function $1-\operatorname{Re}\phi_1(x)$ (the arbitrariness of the choice of ϕ_1 may be, for example, decided by choosing ϕ_1 to be that elementary non-constant spherical function which has the smallest eigenvalue with respect to the Laplace–Beltrami operator of G/K). Lemma 6.1 which was used in the non-compact case to get the estimate (6.18) is now superfluous in view of the compactness of G and only the left half of the inequalities in (ii) of Lemma 6.2 has content. The method of proof of this inequality has, of course, to be slightly modified.

The Lie algebra $\mathfrak{g}_0$ of G may be decomposed as $\mathfrak{k}_0+\mathfrak{p}_*$ and $\mathfrak{p}_*$ is identifiable with the tangent space to G/K at $\pi(e)$. It is still true that $\mathbf{I}(\mathfrak{p}_*)$ and $\mathbf{D}(G/K)$ are isomorphic as vector spaces, and this remark enables us to retain the essential idea in the proof of the above inequality.

Theorem 6.2 finds the following replacement in

THEOREM 8.1. *Suppose $\mu\in\mathbf{S}_0(K\backslash G/K)$, and that $\hat{\mu}(n)\neq 0$ for any n. Then μ is infinitely divisible if and only if*

$$\hat{\mu}(n)=\exp\left(P_D(n)-\int_{|x|>0}1-\phi_n(x)\,dL(x)\right), \tag{8.3}$$

where $P_D(n)$ is the eigenvalue corresponding to ϕ_n of an elliptic second order differential operator $D\in\mathbf{D}(G/K)$ and L is a spherical measure on $\{x\,|\,|x|>0\}$ such that

$$\int[1-\operatorname{Re}\phi_1(x)]\,dL(x)<\infty.$$

We remark that this last condition on L can be shown to be equivalent to $\int|x|^2dL(x)<\infty$.

Corollary 6.3 and the uniqueness Theorem 6.3 can be proved in the same fashion, and indeed, the counterpart of the Theorem 7.2 also holds exactly as written down in [6], the method of proof being as above. The reader is invited to fill in on this sketchy outline.

Theorem 8.1 is the general version of the result of Bochner in [1], where

$$G=\mathbf{SO}(n),\ K=\mathbf{SO}(n-1);\ n\geqslant 3.$$

To be sure, Bochner considers a slightly larger class of heat equations obtainable from the radial part of the Laplace–Beltrami operator of $\mathbf{SO}(n)/\mathbf{SO}(n-1)$ by continuation of a parameter (depending on n) and his full results may be thought of as giving results for "spheres" of fractional dimension. We do not know whether similar results could be obtained by suitable analogous continuation of the radial part of the Laplace–Beltrami operators of compact symmetric spaces G/K which are two-point homogeneous.

A specialization of our results is the following: Taking $G = \mathbf{SU}(n+1)$, $K = \mathbf{U}(n)\, n \geqslant 2$, one gets G/K = complex projective space. The spherical functions ϕ_k can be shown to be the Jacobi polynomials $P_k^{(\frac{1}{2}(n-1),0)}$ in the notation of [22]. This result may be regarded with a certain amount of curiosity since we are not aware of a *classical* proof of the positivity property [1, p. 24] of the Jacobi polynomials which is brought out by (3.1), on which is based Bochner's proof of his result in [1].

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Received August 12, 1963