Endomorphisms of finetely generated projective modules over a commutative ring

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Introduction

The origin of this paper is a misprint (?) in Bourbaki ([4], p. 156, Exercise 13 d). There it is stated that if f is a 2×2-matrix with entries in a commutative ring and $f^2 = 0$ then $(\operatorname{Tr} f)^4 = 0$ and 4 is the smallest integer with this property. Using the Cayley-Hamilton theorem we get $f^2 - af + b1 = 0$ where $a = \operatorname{Tr} f$ and $b = \det f$. Noting that $f^2 = 0$ and taking traces we get $a \cdot \operatorname{Tr} f = a^2 = 2b$. Multiplying the first equation by f gives $bf = 0$ which implies $b \cdot \operatorname{Tr} f = ba = 0$. Hence $a^3 = 2ab = 0$ so 3 and not 4 is the smallest integer above. Experimenting with small m and n one soon makes the conjecture: If f is an $n\times n$-matrix with $f^{m+1} = 0$ then $(\operatorname{Tr} f)^{mn+1} = 0$. This is proved in a somewhat more general setting in 1.7 using exterior algebra.

In Section 1 the characteristic polynomial $\lambda_t(f)$ is defined for an endomorphism $f: P \to P$ where P is a finitely generated projective A-module (A is a commutative ring with 1). If P is free then $\lambda_t(f) = \det(1 + tf)$. The exponential trace formula (in case A contains $\mathbf{Q}$)

$$\lambda_t(f) = \exp\left(-\sum_1^\infty \frac{\operatorname{Tr}(f^i)}{i}(-t)^i\right)$$

connects $\lambda_t(f)$ with the traces of the powers of f.

Various computations of $\lambda_t(f)$ are made in Section 2. By the isomorphism $\operatorname{End}_A(P) \to P^* \otimes_A P$ where $P^* = \operatorname{Hom}_A(P, A)$ every $f: P \to P$ corresponds to a tensor $\sum_i x_i^* \otimes x_i$ with $x_i^* \in P^*$, $x_i \in P$. Let $M(f)$ be the matrix with entries $a_{ij} = \langle x_i^*, x_j\rangle$. Then $\lambda_t(f) = \det(1 + tM(f))$. Even the computation of $\lambda_t(1_P)$

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where 1_P is the identity map is not quite trivial. The result is $\lambda_t(1_P) = \sum_0^n e_i(1 + t)^i$ where the e_i:s are the idempotents given by $\operatorname{Ann} \Lambda^i P = (e_0 + e_1 + \ldots + e_{i-1})A$.

In Section 3 the behaviour of $\lambda_t(f)$ under change of rings and taking duals is studied. Some attempts are made to connect the polynomials $\lambda_t(f)$, $\lambda_t(g)$ and $\lambda_t(f \otimes g)$. In the multiplicative group $\tilde{A} = \{1 + a_1 t + a_2 t^2 + \ldots; a_i \in A\}$ of formal power series with constant term 1 one can define a $*$-multiplication such that $\lambda_t(f \otimes g) = \lambda_t(f) * \lambda_t(g)$. Then $\tilde{A}$ becomes a ring (with ordinary multiplication as addition).

A formula for computing $\lambda_t(f)$ in terms of the minimal polynomial of f and some of the $\operatorname{Tr}(f^i)$:s is given in Section 4.

In Section 5 the definition of $\lambda_t(f)$ is extended to $f: M \to M$ where M is an A-module having a finite resolution of finitely generated projective modules. Some of the results in Section 1 can be generalized to this case. Furthermore $\lambda_t(f)$ is defined for $f =$ chain map of complexes (or map of graded A-modules).

Section 6 contains an attempt to classify all endomorphism of finitely generated projective A-modules, i.e. to compute the K-group $K_0(\operatorname{End} \mathcal{P}(A))$. The characteristic polynomial $\lambda_t(f)$ is sometimes a good enough invariant. This is the case if A is a PID or $A = K[X, Y]$ where K is a field or A is a regular local ring of dimension at most two. Then $K_0(\operatorname{End} \mathcal{P}(A))$ is isomorphic (as a ring) with the direct product of $K_0(A) = \mathbf{Z}$ and the ring of all »rational functions»

$$\frac{1 + a_1 t + \ldots + a_m t^m}{1 + b_1 t + \ldots + b_n t^n}$$

(under multiplication and $*$-multiplication). This generalizes a result by Kelley-Spanier ([8] p. 327) for $A =$ field. The ring of »rational functions» is also isomorphic with a subring of the Witt ring $W(A)$ of A. Finally »trace sequences», $(\operatorname{Tr}(f^i))_1^\infty$ are studied.

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1. The characteristic polynomial

First we fix some notation. A will always denote a commutative ring with unity element 1. $\operatorname{Spec} A$ is the set of all prime ideals $\mathfrak{p}$ of A. If $x \in M$ where M is an A-module we denote by $x_\mathfrak{p}$ the image of x under the localization map $M \to M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

The category of all finitely generated projective A-modules will be denoted by $\mathcal{P}(A)$. If $P \in \mathcal{P}(A)$ then $P_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of finite rank $= \operatorname{rk}_\mathfrak{p} P$. We define $\operatorname{rk} P = \max_\mathfrak{p} \operatorname{rk}_\mathfrak{p} P$. This integer is equal to the minimal number of generators of P. If $\operatorname{rk}_\mathfrak{p} P = \operatorname{rk} P$ for all $\mathfrak{p} \in \operatorname{Spec} A$ we say that P has *constant rank*. Let

$P^* = \mathrm{Hom}_A(P, A)$ be the dual of P. Then for $P \in \mathcal{P}(A)$ there are natural isomorphisms of A-modules

$$\mathrm{End}_A(P^*) \to \mathrm{Hom}_A(P^* \otimes_A P, A) \to \mathrm{Hom}_A(\mathrm{End}_A P, A) \qquad (*)$$

Let Tr be the image of 1_{P^*} under the composed map. We call $\mathrm{Tr}(f)$ the *trace* of $f: P \to P$. This coincides with Bourbakis definition ([3] p. 112).

Definition 1.1:

$$\lambda_t(f) = \sum_{i=0}^{n} \mathrm{Tr}(\Lambda^i f) t^i$$

Here t is an indeterminate, $f: P \to P$ an endomorphism with $P \in \mathcal{P}(A)$, $\Lambda^i f: \Lambda^i P \to \Lambda^i P$ the induced endomorphism of the i:th exterior power of P and $n = \mathrm{rk}P$. Observe that $\Lambda^i P \in \mathcal{P}(A)$ ([4], p. 142).

Remark 1.2. If P is free then $\mathrm{Tr}(f)$ is the usual trace of f and $\lambda_t(f) = \det(1 + tf)$ where 1 = identity of the free $A[t]$-module $P \otimes_A A[t]$. This is a well known formula ([9] p. 436).

PROPOSITION 1.3. *Let $f, g: P \to P$ with $P \in \mathcal{P}(A)$ and $\mathfrak{p} \in \mathrm{Spec}\, A$ be given. Then*

(i) $(\mathrm{Tr} f)_\mathfrak{p} = \mathrm{Tr} f_\mathfrak{p}$
(ii) $(\lambda_t(f))_\mathfrak{p} = \lambda_t(f_\mathfrak{p})$, *i.e. if* $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ *then* $\lambda_t(f) = 1 + a_{1\mathfrak{p}} t + \ldots + a_{n\mathfrak{p}} t^n$
(iii) $\lambda_t(f \circ g) = \lambda_t(g \circ f)$
(iv) $\lambda_t(h \circ f \circ h^{-1}) = \lambda_t(f)$ *if* $h: P \to Q$ *is an isomorphism.*

Proof.

(i) Localization commutes with everything in $(*)$ since all modules involved (P^*, $\mathrm{End}_A(P^*)$ etc.) are in $\mathcal{P}(A)$ ([4], p. 98).
(ii) Localization commutes with exterior powers, $(\Lambda^i f)_\mathfrak{p} = \Lambda^i f_\mathfrak{p}$, so (ii) follows from (i).
(iii) We have $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$ ([3], p. 112) and $\Lambda^i(f \circ g) = \Lambda^i f \circ \Lambda^i g$.
(iv) By (ii) it is sufficient to prove (iv) for P free (and hence Q is free), in which case it is well known.

CAYLEY-HAMILTON THEOREM 1.4. *Let* $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ *and define* $q_f(t) = t^n - a_1 t^{n-1} + \ldots + (-1)^n a_n$. *Then* $q_f(f) = 0$.

Proof. It suffices to show

$$(q_f(f))_\mathfrak{p} = f_\mathfrak{p}^n - a_{1\mathfrak{p}} f_\mathfrak{p}^{n-1} + \ldots + (-1)^n a_{n\mathfrak{p}} \cdot 1_{P_\mathfrak{p}} = 0$$

for all $\mathfrak{p} \in \mathrm{Spec}\, A$. But this follows from the ordinary Cayley-Hamilton theorem for $f_\mathfrak{p}: P_\mathfrak{p} \to P_\mathfrak{p}$ with $P_\mathfrak{p}$ free since

$$t^n - a_{1\mathfrak{p}}t^{n-1} + \ldots + (-1)^n a_{n\mathfrak{p}} = t^{n-rk(P_{\mathfrak{p}})} q_{f_{\mathfrak{p}}}(t).$$

PROPOSITION 1.5. *Let*

$$\begin{array}{ccccccccc} 0 \to & P_d & \to \ldots \to & P_1 & \to & P_0 & \to 0 \\ & \downarrow f_d & & \downarrow f_1 & & \downarrow f_0 & \\ 0 \to & P_d & \to \ldots \to & P_1 & \to & P_0 & \to 0 \end{array}$$

be a commutative diagram with exact row and all $P_i \in \mathcal{P}(A)$. *Then*

$$\sum_0^d (-1)^i \operatorname{Tr} f_i = 0 \quad \textit{and} \quad \prod_0^d \lambda_t(f_i)^{(-1)^i} = 1$$

Proof. Since localization is an exact functor it is (using 1.3 (i), (ii)) sufficient to prove the proposition when all P_i are free. But then it is well known at least for $d = 2$ (see [9], p. 402) and the general case follows by splitting up the long exact sequence into short ones.

COROLLARY 1.6.

$$\operatorname{Tr}(f \oplus g) = \operatorname{Tr} f + \operatorname{Tr} g \quad \textit{and} \quad \lambda_t(f \oplus g) = \lambda_t(f) \cdot \lambda_t(g).$$

THEOREM 1.7. *Let* $f\colon P \to P$ *be given with*

$$P \in \mathcal{P}(A), \quad \operatorname{rk} P = n \quad \textit{and} \quad \lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n.$$

(i) *Assume that* f *is nilpotent with* $f^{m+1} = 0$. *Then* $a_1^{\nu_1} a_2^{\nu_2} \ldots a_n^{\nu_n} = 0$ *if the weight* $\nu_1 + 2\nu_2 + \ldots + n\nu_n > mn$. *The constant* mn *is best possible.*

(ii) *Conversely assume that* $a_1^{\nu_1} a_2^{\nu_2} \ldots a_n^{\nu_n} = 0$ *when* $\nu_1 + 2\nu_2 + \ldots + n\nu_n > k$. *Then* $f^{n+k} = 0$. *The integer* $n + k$ *is best possible.*

Proof. (i) After localizing and using 1.3 (ii) we may assume that P is free of rank n (it is sufficient to consider the case of maximal rank). Let P have basis $e_1, e_2, \ldots, e_n$. Then $\Lambda^n P$ is free with basis $e_1 \wedge e_2 \wedge \ldots \wedge e_n$. Now we claim that

$$a_r e_1 \wedge e_2 \wedge \ldots \wedge e = \sum_{i_1 < i_2 < \ldots < i_r} e_1 \wedge \ldots \wedge f e_{i_1} \wedge \ldots \wedge f e_{i_2} \wedge \ldots \wedge f e_{i_r} \wedge \ldots \wedge e_n \qquad (**)$$

By definition we have $a_r = \operatorname{Tr}(\Lambda^r f)$. Let $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}$ be a fixed basis element of $\Lambda^r P$ (with $i_1 < i_2 < \ldots < i_r$). Then

$$\Lambda^r f(e_{i_1} \wedge \ldots \wedge e_{i_r}) = f e_{i_1} \wedge \ldots \wedge f e_{i_r} = C_{i_1 i_2 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r} + \text{other terms}.$$

Hence

$$a_r = \operatorname{Tr}(\Lambda^r f) = \sum_{i_1 < i_2 < \ldots < i_r} C_{i_1 i_2 \ldots i_r}$$

Expanding the right hand side in (**) one easily gets

$$\left(\sum_{i_1<i_2<\ldots<i_r} C_{i_1i_2\ldots i_r}\right) e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

and the claim is proved.

Using (**) several times we get

$$a_1^{\nu_1}a_2^{\nu_2}\ldots a_n^{\nu_n}(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = \sum f^{s_1}e_1 \wedge f^{s_2}e_2 \wedge \ldots \wedge f^{s_n}e_n$$

where the sum is taken over all $s_1, s_2, \ldots, s_n$ such that $s_1 + s_2 + \ldots + s_n = \nu_1 + 2\nu_2 + \ldots + n\nu_n$ which by assumption is larger than mn. Hence each term contains an $s_i > m$ and $f^{s_i} = 0$. Therefore the right hand side is zero and the first part of (i) is proved.

To see that mn is best possible let A be the commutative ring generated by 1, $\alpha_1, \ldots, \alpha_n$ with the only relations $\alpha_1^{m+1} = \alpha_2^{m+1} = \ldots = \alpha_n^{m+1} = 0$. Let f be the map given by the diagonal matrix

$$f = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & 0 & \\ & 0 & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

Then $f^{m+1} = 0$ and $a_1^{\nu_1}\ldots a_n^{\nu_n} = \sum \alpha_1^{s_1}\alpha_2^{s_2}\ldots\alpha_n^{s_n}$ where the sum runs over all $s_1, s_2, \ldots, s_n$ with $s_1 + s_2 + \ldots + s_n = \nu_1 + 2\nu_2 + \ldots + n\nu_n$. If $\nu_1 + 2\nu_2 + \ldots + n\nu_n \leq mn$ then there is a term $\alpha_1^{s_1}\ldots\alpha_n^{s_n}$ with all $s_i \leq m$ and hence $a_1^{\nu_1}\ldots a_n^{\nu_n} \neq 0$.

(ii) Assume that $a_1^{\nu_1}\ldots a_n^{\nu_n} = 0$ if $\nu_1 + 2\nu_2 + \ldots + n\nu_n > k$. By the Cayley-Hamilton theorem we have

$$f^n = a_1 f^{n-1} - a_2 f^{n-2} + \ldots \pm a_n 1,$$

Multiplying by f and using Cayley-Hamilton again we get

$$f^{n+1} = a_1^2 f^{n-1} + \ldots \pm a_1 a_n 1.$$

Repeating the procedure several times we get

$$f^r = q_{r-n+1}f^{n-1} + q_{r-n+2}f^{n-2} + \ldots + q_r \cdot 1$$

where q_i is a polynomial in $a_1, \ldots, a_n$ of weight i. If $r = k + n$ then $q_r = q_{r-1} = q_{r-n+1} = 0$ and we get $f^r = 0$.

To show that $n + k$ is best possible let $A = Z[X_1, X_2 \ldots, X_n]/I$ where $X_1, \ldots, X_n$ are indeterminates and I is the ideal generated by all monomials in $X_1, \ldots, X_n$ of weight $k + 1$. Put

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & (-1)^{n-1} & a_n \\ 1 & 0 & 0 & . & - & \\ 0 & 1 & 0 & . & - & \\ - & - & - & 0 & -a_2 & \\ 0 & 0 & 0 & 1 & a_1 & \end{pmatrix}$$

where a_i is the residue of X_i. Then a calculation shows that

$$\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$$

and that $f^{n+k-1} \neq 0$.

COROLLARY 1.8. *f is nilpotent if and only if all coefficients $a_i (i \geq 1)$ of $\lambda_t(f)$ are nilpotent.*

PROPOSITION 1.9. *Given $f: P \to P$ with $P \in \mathcal{P}(A)$. If $f^{\otimes \nu} = f \otimes f \otimes \ldots \otimes f = 0$* then $f^\nu = 0$.

Proof. Localizing we may assume that P is free of rank n. Let (a_{ij}) be the matrix of f in some basis and I the ideal in A generated by the coefficients (a_{ij}). The entries of the matrix of $f^{\otimes \nu}$ are just all possible products of ν of the a_{ij}:s. Since $f^{\otimes \nu} = 0$ we get $I^\nu = 0$. The entries (c_{ij}) of the matrix of f^ν are certain sums of products of ν of the a_{ij}:s. Hence $c_{ij} \in I^\nu$ and $c_{ij} = 0$ for all i, j and $f^\nu = 0$.

THEOREM 1.10 (exponential trace formula). *Let $f: P \to P$ be A-linear with $P \in \mathcal{P}(A)$. Then*

$$-t\lambda_t(f)^{-1} \frac{d}{dt} \lambda_t(f) = \sum_1^\infty \operatorname{Tr}(f^i)(-t)^i$$

Proof. Setting $b_i = \operatorname{Tr}(-f)^i$ and $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ we must prove

$$-(a_1 t + 2a_2 t^2 + \ldots + n a_n t^n) = (1 + a_1 t + \ldots + a_n t^n) \sum_1^\infty b_i t^i$$

Comparing the coefficients of t^i on both sides one finds $b_i = Q_i(a_1, \ldots, a_n)$ where the Q_i:s are certain polynomials with integer coefficients. Localizing at $\mathfrak{p} \in \operatorname{Spec} A$ we have to show $b_{i\mathfrak{p}} = Q_i(a_{1\mathfrak{p}}, \ldots, a_{n\mathfrak{p}})$. Hence it is sufficient to show the formula when P is free and f is a matrix. Then $b_i = Q_i(a_1, \ldots, a_n)$ becomes a polynomial identity (over $\mathbf{Z}$) in the coefficients of the matrix f. Therefore it is enough to consider the case $A = \mathbf{Z}[X_{11}, \ldots, X_{nn}]$ which is a domain of characteristic zero. Let K be the quotient field of K and $\bar{K}$ the algebraic closure of K. Over $\bar{K}$ the formula is easy to prove. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of f we have

$\lambda_t(f) = \prod_{\nu=1}^n (1 + \lambda_\nu t)$. Taking the logarithmic derivative, expanding $\lambda_\nu(1 + \lambda_\nu t)^{-1}$ into power series and using $\mathrm{Tr}\,(f^i) = \sum_{\nu=1}^n \lambda_\nu^i$ we get the desired formula.

Remark 1.11. In the theory of differential equations there is a »continuous» analogue of the formula above: Let $U(t)$ and $B(t)$ be $n \times n$-matrices with real entries, depending on a parameter t, satisfying

$$\frac{d}{dt}\, U(t) = B(t)U(t) \quad \text{and} \quad U(0) = 1.$$

Then

$$\det U(t) = \exp \int_0^t \mathrm{Tr}\, B(s)ds.$$

It is well known that if $\mathrm{Tr}\,(f^i) = 0$ for $i = 1, 2, \ldots, n$ where f is an $n \times n$-matrix over a field of characteristic zero then f is nilpotent. Our next result is a generalization of this.

We will call the ring A *torsion-free* if it is torsion-free as an abelian group, i.e. $na = 0$ with $n \in Z$ and $a \in A$ implies $n = 0$ or $a = 0$.

PROPOSITION 1.12. *Assume that A is torsion-free. Let $f\colon P \to P$ be A-linear where $P \in \mathcal{P}(A)$ has rank n. If* $\mathrm{Tr}\,(f^i) = 0$ *for n consecutive i:s then f is nilpotent.*

Proof. Assume that $\mathrm{Tr}\,(f^r) = \mathrm{Tr}\,(f^{1+r}) = \ldots = \mathrm{Tr}\,(f^{r+n-1}) = 0$. Multiplying Cayley-Hamilton by f^r we get

$$f^{n+r} = a_1 f^{n+r-1} - a_2 f^{n+r-2} + \ldots \pm a_n f^r$$

Taking traces on both sides we get $\mathrm{Tr}\,(f^{n+r}) = 0$. Repeating the procedure we get $\mathrm{Tr}\,(f^\nu) = 0$ for all $\nu \geq r$. Put $g = f^r$. Then $\mathrm{Tr}\,(g^\nu) = 0$ for $\nu = 1, 2, \ldots$. Using the exponential trace formula for g we find $\dfrac{d}{dt}\, \lambda_t(g) = 0$ which implies $\lambda_t(g) = 1$ since A has no torsion. Cayley-Hamilton applied to g gives $g^n = 0$, i.e. $f^{nr} = 0$.

Remark 1.13. The proposition is true if A has no s-torsion for $s \leq \mathrm{rk}P$.

Remark 1.14. If A is a field of characteristic 2 then $\mathrm{Tr}\, 1_P^\nu = 0$ for P free of rank 2.

Remark 1.15. If we assume that A is torsion-free we can give another proof of the fact that $f^{\otimes \nu} = 0 \Rightarrow f$ nilpotent (compare 1.9). Put $b_i = \mathrm{Tr}\,(f^i)$. Then $(f^i)^{\otimes \nu} = (f^{\otimes \nu})^i = 0$ implies $\mathrm{Tr}\,((f^i)^{\otimes \nu}) = (\mathrm{Tr}\,(f^i))^\nu = b_i^\nu = 0$ for $i = 1, 2, \ldots$. Comparing coefficients in the exponential trace formula we get $a_1 = b_1$, $2a_2 = b_1^2 - b_2, \ldots$. Since A has no torsion all a_i:s are nilpotent. Then f is also nilpotent by 1.8.

2. Some computations

First a generalization 1.3 (iii):

PROPOSITION 2.1. *Given* $f\colon P \to Q$ *and* $g\colon Q \to P$ *with* $P, Q \in \mathcal{P}(A)$. *Then*

$$\operatorname{Tr}(f \circ g) = \operatorname{Tr}(g \circ f) \quad \textit{and} \quad \lambda_t(f \circ g) = \lambda_t(g \circ f)$$

Proof. After localization we may assume that P and Q are free. The formula for the trace is then easily proved and

$$\operatorname{Tr} \Lambda^i(f \circ g) = \operatorname{Tr}(\Lambda^i f \circ \Lambda^i g) = \operatorname{Tr}(\Lambda^i g \circ \Lambda^i f) = \operatorname{Tr} \Lambda^i(g \circ f)$$

finishes the proof.

We continue with describing a method for computing $\lambda_t(f)$. P denotes always a module in $\mathcal{P}(A)$ of rank n.

THEOREM 2.2. *We have* $\operatorname{End}_A(P) \cong P^* \otimes_A P$. *Let* $f\colon P \to P$ *correspond to* $\sum_{i=1}^m x_i^* \otimes x_i$ *in* $P^* \otimes_A P$. *Let* $M(f)$ *be the* $m \times m$*-matrix with entries* $\langle x_i^*, x_j \rangle$ *at place* (i, j). *Then*

$$\lambda_t(f) = \det(1 + tM(f))$$

In particular the right hand side is independent of the choice of representatives for the tensor. The x_i*:s can be chosen as a minimal generator set of* P.

Proof. First we reduce to the case when P is free. Let $\mathfrak{p}$ be a prime ideal in A. Localizing at $\mathfrak{p}$ we get a commutative diagram

$$\begin{tikzcd} P^* \otimes_A P \arrow[r, "u", "\cong"'] \arrow[d] & \operatorname{End}_A P \arrow[d] \\ P^*_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} P_{\mathfrak{p}} \arrow[r, "\cong"'] & \operatorname{End}_{A_{\mathfrak{p}}} P_{\mathfrak{p}} \end{tikzcd}$$

where the star in the south west corner means $\operatorname{Hom}_{A_{\mathfrak{p}}}(\cdot, A_{\mathfrak{p}})$. Hence if $f\colon P \to P$ corresponds to $\sum_1^m x_i^* \otimes x_i$ then $f_{\mathfrak{p}}\colon P_{\mathfrak{p}} \to P_{\mathfrak{p}}$ corresponds to $\sum_1^m (x_i^*)_{\mathfrak{p}} \otimes x_{i\mathfrak{p}}$ and by using 1.3 (ii) we may assume that P is free. Let now $y_1, \ldots, y_n$ be a basis for P and $h_1, \ldots, h_n$ a dual basis for P^*, i.e. $\langle h_i, y_j \rangle = \delta_{ij}$. Given $f\colon P \to P$ let it correspond to

$$\sum_{i,j} a_{ji}(h_i \otimes y_j) = \sum_{j=1} \Big(\sum_{i=1} a_{ji} h_i \Big) \otimes y_j = \sum_{j=1} y_j^* \otimes y_j \quad \text{in} \quad P^* \otimes_A P, \quad \text{i.e.} \quad y_j^* = \sum_{i=1}^n a_{ji} h_i.$$

Hence the (j, k):th entry in the matrix is

$$\langle y_j^*, y_k \rangle = \sum_{i=1}^n a_{ji} \langle h_i, y_k \rangle = a_{jk}.$$

Now $u: P^* \otimes_A P \to \mathrm{End}_A P$ is given by $x^* \otimes x \mapsto (y \mapsto \langle x^*, y\rangle x)$ so $f = u(\sum_{i,j} a_{ji}h_i \otimes y_j)$ means $f(x_k) = \sum_{i,j} a_{ji}\langle h_i, x_k\rangle y_j = \sum_j a_{jk}y_j$.

It follows that f has the matrix (a_{jk}) in the basis $y_1, \ldots, y_n$. Thus the formula is true if the x_i:s form a basis for P.

Let now $\sum_1^m x_i^* \otimes x_i$ be another representation of f. Assume that

$$x_i = \sum_{j=1}^n c_{ji}y_j \quad \text{and} \quad x_i^* = \sum_{k=1}^n d_{ik}h_k$$

Then

$$\sum_{i=1}^m x_i^* \otimes x_i = \sum_{i=1}^m \sum_{j,k} c_{ji}d_{ik}h_k \otimes y_j = \sum_{j,k} \left(\sum_{i=1}^m c_{ji}d_{ik}\right) h_k \otimes y_j = \sum_{j,k} a_{jk}h_k \otimes y_j$$

where

$$(a_{jk}) = CD \quad \text{with} \quad C = (c_{ji}) \quad \text{and} \quad D = (d_{ik})$$

(here C and D are $n \times m$- and $m \times n$-matrices respectively). The (i, k):th entry of the matrix in the formula is

$$\langle x_i^*, x_k\rangle = \sum_{\nu,j} d_{i\nu}c_{jk}\langle h_\nu, y_j\rangle = \sum_{j=1}^n d_{ij}c_{jk}.$$

Thus this matrix is DC and we are done since $\lambda_t(f) = \det(1 + tCD))$ by the first part of the proof and $\det(1 + tCD) = \det(1 + tDC)$ by 2.1.

Next we compute $\lambda_t(1_P)$ where 1_P is the identity map of $P \in \mathcal{P}(A)$.

THEOREM 2.2 (Goldman). (i) $\mathrm{Tr}(1_P) = \sum_0^n ie_i$ *and* $\lambda_t(1_P) = \sum_1^n e_i(1 + t)^i$ *where* $e_0, e_1, \ldots, e_n$ *are orthogonal idempotents with* $e_0 + e_1 + \ldots + e_n = 1$.

(ii) $\mathrm{Ann}(\wedge^i P) = (e_0 + e_1 + \ldots + e_{i-1})A$. *Furthermore the* e_i*:s are uniquely determined by* P.

Remark. Some of the e_i:s might be zero, e.g. if P is constant rank n, then $e_0 = e_1 = \ldots = e_{n-1} = 0$.

Proof. (i) Let $\mathbf{Z}$ have the discrete topology. Then rk: Spec $A \to \mathbf{Z}$ given by $\mathfrak{p} \to \mathrm{rk}_\mathfrak{p} P$ is a continuous function. Hence $X_i = \{\mathfrak{p} \in \mathrm{Spec}\, A_0;\ \mathrm{rk}_\mathfrak{p} P = i\}$ is both open and closed. It follows that Spec $A = X_0 \cup X_1 \cup \ldots \cup X_n$ where the union is disjoint. But to this covering of Spec A corresponds a unique »partition of unity»

$$1 = e_0 + e_1 + \ldots + e_n \text{ where } e_i(x) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases} \text{ i.e. } 1 - e_i \in \mathfrak{p} \text{ for all } \mathfrak{p} \in X_i$$

and $e_i \in \mathfrak{p}$ for all $\mathfrak{p} \notin X_i$: (see Swan [12] p. 140). This means that the e_i:s are orthogonal idempotents.

Now we claim that $\lambda_t(1_P) = \sum_0^n e_i(1 + t)^i$.

Fix a prime $\mathfrak{p} \in X_i$. Then the localization at $\mathfrak{p}$ of the left hand side is $(\lambda_t(1_P))_{\mathfrak{p}} = \lambda_t(1_{P_{\mathfrak{p}}}) = (1+t)^i$ since $P_{\mathfrak{p}}$ is free of rank i. To compute the localization of the right hand side we need $e_{k\mathfrak{p}}$. But $e_i e_j = 0$ with $e_i \notin \mathfrak{p}$ implies $e_{j\mathfrak{p}} = 0$ in $A_{\mathfrak{p}}$ for $j \neq i$. Furthermore $e_i(1-e_i) = 0$ with $e_i \notin \mathfrak{p}$ implies $e_{i\mathfrak{p}} = 1$ in $A_{\mathfrak{p}}$. Thus $(\sum_0^n e_j(1+t)^j)_{\mathfrak{p}} = (1+t)^i = (\lambda_t(1_P))_{\mathfrak{p}}$ and we are done since $\mathfrak{p} \in \operatorname{Spec} A$ was arbitrary.

(ii) $\Lambda^i P$ is in $\mathcal{P}(A)$ and thus $\operatorname{Ann}(\Lambda^i P) = eA$ where e is a uniquely determined idempotent (Goldman [6] p. 33). Now $(\Lambda^i P)_{\mathfrak{p}} = 0$ if and only if $\operatorname{rk}_{\mathfrak{p}} P < i$ if and only if $\mathfrak{p} \in X_0 \cup X_1 \cup \ldots \cup X_{i-1}$. This is the case if and only if $eA = \operatorname{Ann}(\Lambda^i P) \not\subset \mathfrak{p}$ if and only if $e \notin \mathfrak{p}$. Thus $e(x) = 0$ if and only if $x \in X_i \cup \ldots \cup X_n$ (and hence $e(x) = 1$ otherwise). But $e_0 + e_1 + \ldots + e_{i-1}$ is a candidate satisfying these conditions. By uniqueness we get

$$e = e_0 + e_1 + \ldots + e_{i-1}.$$

Putting $i = 1$ we get e_0 uniquely. Since $e_0 + e_1$ is unique e_1 is unique etc.

Definition 2.3: We define the *determinant of* f by $\det f = \lambda_1(f - 1_P)$ for $f: P \to P$ with $P \in \mathcal{P}(A)$.

First we note that $\det 1_P = \lambda_1(0) = 1$. If P is free then $\det(f)$ coincides with the usual determinant of a matrix for f. If $\operatorname{rk} P = n$ then there exists Q such that $P \oplus Q = F$ where F is free of rank n. Clearly $Q \in \mathcal{P}(A)$. Localizing at $\mathfrak{p} \in \operatorname{Spec} A$ we get $P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}} = F_{\mathfrak{p}}$ where $P_{\mathfrak{p}}, Q_{\mathfrak{p}}, F_{\mathfrak{p}}$ are free $A_{\mathfrak{p}}$-modules of rank $r = rk_{\mathfrak{p}} P$, $n - r$ and n, respectively. We get $(\det(f \oplus 1_Q))_{\mathfrak{p}} = \det(f_{\mathfrak{p}} \oplus 1_{Q_{\mathfrak{p}}}) = \det f_{\mathfrak{p}} \cdot \det 1_{Q_{\mathfrak{p}}} = \det f_{\mathfrak{p}} = (\det f)_{\mathfrak{p}}$. Hence we could also have defined $\det f$ as $\det(f \oplus 1_Q)$ where the last det is the ordinary determinant of a matrix for $f \oplus 1_Q$. Thus $\det f$ is the same as Goldman's determinant ([6] p. 29). We state some properties of $\det(f)$.

PROPOSITION 2.4. (i) $\det(f \circ g) = \det f \det g$.

(ii) f *is an ismorphism if and only if* $\det f$ *is invertible in* A.

We now collect some formulas for $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ where $f: P \to P$ with $P \in \mathcal{P}(A)$ and $rkP = n$.

PROPOSITION 2.5. (i) $\lambda_t(\Lambda^k f) = 1 + a_k t + \ldots + a_n^{\binom{n-1}{k-1}} t^{\binom{n}{k}}$. *In particular*

(ii) $\lambda_t(\Lambda^n f) = 1 + a_n t$.

(iii) $\lambda_t(\Lambda^{n-1} f) = 1 + a_{n-1} t + a_{n-2} a_n t^2 + a_{n-3} a_n^2 t^3 + \ldots + a_1 a_n^{n-2} t^{n-1} + a_n^{n-1} t^n$.

(iv) $\lambda_t(f^2) = 1 + (a_1^2 - 2a_2)t + (2a_4 - 2a_1 a_3 + a_2^2)t^2 + \ldots + a_n^2 t^n$.

Proof. Since λ_t and Λ^k commute with localization we may assume that P is free. Using the technique employed in proving the exponential trace formula 1.10 we may even assume that A is an algebraically closed field. If

$$\lambda_t(f) = \prod_1^n (1 + \lambda_i t) = 1 + a_1 t + \ldots + a_n t^n$$

we have

$$\lambda_t(\Lambda^k f) = \prod_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (1 + \lambda_{i_1}\lambda_{i_2} \ldots \lambda_{i_k} t).$$

The first two formulas now follow easily.

(iii) We may assume that $a_n = \prod_1^n \lambda_i \neq 0$. Then we have

$$\lambda_t(\Lambda^{n-1}f) = \prod_1^n \left(1 + \frac{a_n}{\lambda_i} t\right) = a_n^n t^n \prod_1^n \left(1 + \frac{\lambda_i}{a_n t}\right) \cdot \prod_1^n \frac{1}{\lambda_i} =$$

$$= a_n^{n-1} t^n \left(1 + a_1 \cdot \frac{1}{a_n t} + a_2 \frac{1}{a_n^2 t^2} + \ldots + \frac{a_{n-1}}{a_n^{n-1} t^{n-1}} + \frac{a_n}{a_n^n t^n}\right) =$$

$$= 1 + a_{n-1} t + a_{n-2} a_n t^2 + \ldots + a_1 a_n^{n-2} t + a_n^{n-1} t^n.$$

(iv) Set $t = -s^2$. Then

$$\lambda_t(f^2) = \det(1 - s^2 f^2) = \det(1 - sf) \cdot \det(1 + sf) = \lambda_{-s}(f)\lambda_s(f) =$$

$$= (1 - a_1 s + a_2 s^2 - + \ldots + (-1)^n a_n s^n)(1 + a_1 s + a_2 s^2 + \ldots + a_n s^n) =$$

$$= 1 + (2a_2 - a_1^2)s^2 + (2a_4 - 2a_1 a_3 + a_2^2)s^4 + \ldots + a_n^2(-s^2)^n$$

We keep the notation from above and furthermore $e_0, e_1, \ldots, e_n$ are the idempotents in theorem 2.2.

PROPOSITION 2.6. (i) $\det f = \sum_0^n a_i e_i$ *where* $a_0 = 1$.

(ii) *If* f *is invertible then* $\det f$ *is a unit in* A *and* $\lambda_t(f^{-1}) = \sum_0^n d_k t^k$ *where* $d_k = \sum_{i=k}^n c_{i-k} e_i$ *with* c_i *given by* $(\det f)^{-1} = (\sum_0^n a_i e_i)^{-1} = \sum_{i=0}^n c_i e_i$ (i.e. if $e_i \neq 0$ then $c_i e_i$ is the inverse of $a_i e_i$ in the subring Ae_i).

Proof. (i) Localization at $\mathfrak{p} \in X_i$ (for the notation see the proof of 2.2) gives

$$(\sum_0^n a_j e_j)_\mathfrak{p} = \sum_0^n a_{j\mathfrak{p}} e_{j\mathfrak{p}} = a_{j\mathfrak{p}} \quad \text{since} \quad e_{j\mathfrak{p}} = \delta_{ij}.$$

But $(\det f)_\mathfrak{p} = (\lambda_1(f - 1_p)_\mathfrak{p} = \lambda_1(f_\mathfrak{p} - 1_{P_\mathfrak{p}}) = \det(f_\mathfrak{p})$ since $P_\mathfrak{p}$ is free. Furthermore $P_\mathfrak{p}$ has rank i (since $\mathfrak{p} \in X_i$) and hence $(\lambda_t(f))_\mathfrak{p} = \lambda_t(f_\mathfrak{p}) = 1 + \ldots + \det f_\mathfrak{p} \cdot t^i$ and $a_{i\mathfrak{p}} = \det f_\mathfrak{p}$. This proves (i).

(ii) It is sufficient to show the formula locally. Fix a $\mathfrak{p} \in X_\nu$. Then $P_\mathfrak{p}$ is free of rank ν and we get

$$(\lambda_t(f^{-1})_{\mathfrak{p}} = \lambda_t(f_{\mathfrak{p}}^{-1}) = \det(1 + tf_{\mathfrak{p}}^{-1}) = (\det f_{\mathfrak{p}})^{-1} \det(t \cdot 1_{P_{\mathfrak{p}}}) \det(1 + t^{-1}f_{\mathfrak{p}}) =$$

$$= (\sum_0^n c_{j\mathfrak{p}} e_{j\mathfrak{p}}) t^\nu \sum_{j=0}^{\nu} a_{j\mathfrak{p}} t^{-j} = c_{\nu\mathfrak{p}} \sum_{j=0}^{\nu} a_{j\mathfrak{p}} t^{\nu-j} \quad \text{since} \quad e_{j\mathfrak{p}} = \delta_{j\nu}.$$

On the other hand

$$(\sum_0^n d_k t^k)_{\mathfrak{p}} = \sum_0^n d_{k\mathfrak{p}} t^k = \sum_{k=0}^n (\sum_{i=k}^n c_{i\mathfrak{p}} a_{(i-k)\mathfrak{p}}) t^k = \sum_{k=0}^{\nu} c_{\nu\mathfrak{p}} a_{(\nu-k)\mathfrak{p}} t^k = c_{\nu\mathfrak{p}} \sum_{j=0}^{\nu} a_{j\mathfrak{p}} t^{\nu-j} \text{ with } j = \nu - k.$$

Hence the localizations of both sides agree.

3. The behaviour of λ_t under change of rings, taking duals and forming of tensor products

PROPOSITION 3.1. *Let $\phi\colon A \to B$ be a ringhomomorphism (with $\phi(1) = 1$) and $f\colon P \to P$ an A-linear map with $P \in \mathcal{P}(A)$. Then $P \otimes_A B$ is in $\mathcal{P}(B)$ and*

$$\lambda_t^B(f \otimes 1_B) = \phi(\lambda_t^A(f)).$$

Proof. The first statement is well known. Since $\Lambda_B^i(P \otimes_A B)$ is naturally isomorphic as B-module to $(\Lambda_A^i P) \otimes_A B$ it is sufficient to prove $\mathrm{Tr}_B(f \otimes 1_B) = \phi(\mathrm{Tr}_A(f))$ which is well known.

PROPOSITION 3.2. *Every $f\colon P \to P$ with P in $\mathcal{P}(A)$ induces $f^*\colon P^* \to P^*$ where $P^* = \mathrm{Hom}_A(P, A)$ is in $\mathcal{P}(A)$. Furthermore*

$$\mathrm{Tr}\, f^* = \mathrm{Tr}\, f \quad \text{and} \quad \lambda_t(f^*) = \lambda_t(f).$$

Proof. For every $\mathfrak{p} \in \mathrm{Spec}(A)$ we get a natural $A_{\mathfrak{p}}$-isomorphism

$$(P^*)_{\mathfrak{p}} = (\mathrm{Hom}_A(P, A))_{\mathfrak{p}} \xrightarrow[\cong]{h} \mathrm{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, A_{\mathfrak{p}}) = (P_{\mathfrak{p}})^*$$

and we have a commutative diagram

$$\begin{array}{ccc} (P^*)_{\mathfrak{p}} & \xrightarrow[\cong]{h} & (P_{\mathfrak{p}})^* \\ \downarrow{\scriptstyle (f^*)_{\mathfrak{p}}} & & \downarrow{\scriptstyle (f_{\mathfrak{p}})^*} \\ (P^*)_{\mathfrak{p}} & \xrightarrow[\cong]{h} & (P_{\mathfrak{p}})^* \end{array}$$

Hence $(f^*)_{\mathfrak{p}} = h^{-1} \circ (f_{\mathfrak{p}})^* \circ h$. It follows

$$(\lambda_t(f^*))_{\mathfrak{p}} = \lambda_t((f^*)_{\mathfrak{p}}) = \lambda_t(h^{-1} \circ (f_{\mathfrak{p}})^* \circ h) = \lambda_t((f_{\mathfrak{p}})^*)$$

by 1.3 (iv). But $(P_{\mathfrak{p}})^*$ is free and

$$\lambda_t((f_\mathfrak{p})^*) = \det(1 + (f_\mathfrak{p})^*) = \det(1 + f_\mathfrak{p}) = \lambda_t(f_\mathfrak{p}) = (\lambda_t(f))_\mathfrak{p}.$$

This proves the formula for λ_t and taking the coefficient of t we get the formula for the trace.

Next we turn to the tensor product of two A-linear maps $f: P \to P$ and $g: Q \to Q$ with P, Q in $\mathcal{P}(A)$. For completeness we quote

PROPOSITION 3.3. $\operatorname{Tr}(f \otimes g) = \operatorname{Tr} f \cdot \operatorname{Tr} g$.

There is a corresponding formula for λ_t but it is more complicated. It is convenient to introduce some notation:

Let $\tilde{A}$ denote the set of all formal power series $1 + a_1 t + a_2 t^2 + \ldots$ over A with constant term 1. Then $\tilde{A}$ is an abelian group under multiplication. We define »$*$- *multiplication*» in $\tilde{A}$ such that the following formula is valid

$$\lambda_t(f \otimes g) = \lambda_t(f) * \lambda_t(g).$$

This defines $*$ for all polynomials in $\tilde{A}$ since $1 + a_1 t + \ldots + a_n t^n = \lambda_t(f)$ where $f: A^n \to A^n$ is given by the matrix

$$f = \begin{pmatrix} 0 & 0 & \ldots & 0 & \pm a_n \\ 1 & 0 & & & \mp a_{n-1} \\ 0 & 1 & \ldots & & \pm a_{n-2} \\ \ldots & \ldots & \ldots & 0 & -a_2 \\ 0 & & 0 & 1 & a_1 \end{pmatrix}$$

PROPOSITION 3.4. *If* $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ *and*

$$\lambda_t(g) = 1 + b_1 t + \ldots + b_m t^m$$

then

$$\lambda_t(f \otimes g) = (1 + a_1 t + \ldots + a_n t^n) * (1 + b_1 t + \ldots + b_m t^m) = 1 + d_1 t + \ldots + d_{mn} t^{mn}$$

where

$$\begin{aligned}
d_1 &= a_1 b_1 \\
d_2 &= a_1^2 b_2 + a_2 b_1^2 - 2a_2 b_2 \\
d_3 &= a_1^3 b_3 + a_3 b_1^3 + a_1 a_2 b_1 b_2 - 3a_1 a_2 b_3 - 3a_3 b_1 b_2 + 3a_3 b_3 \\
d_4 &= a_1^2 a_2 b_1 b_3 + a_1 a_3 b_1^2 b_2 - a_1 a_3 b_1 b_3 + a_1^4 b_4 + a_4 b_1^4 + 4a_1 a_3 b_4 + 4a_4 b_1 b_3 - 2a_1 a_3 b_2^2 - \\
&\quad - 2a_2^2 b_1 b_3 + 2a_2^2 b_4 + 2a_4 b_2^2 - 4a_4 b_4 - 4a_1^2 a_2 b_4 - 4a_4 b_1^2 b_2 + a_2^2 b_2^2 \\
&\quad \ldots \\
d_{mn-1} &= a_n^{m-1} a_{n-1} b_m^{n-1} b_{m-1} \\
d_{mn} &= a_n^m b_n^m.
\end{aligned}$$

Proof. Just as in the proof of 1.10 we may assume that A is an algebraically closed field of characteristic zero. Then

$$\lambda_t(f) = \prod_1^n (1 + \lambda_i t), \quad \lambda_t(g) = \prod_1^m (1 + \mu_j t)$$

and

$$\lambda_t(f \otimes g) = \prod_{i,j} (1 + \lambda_i \mu_j t)$$

Using formulas for symmetric functions (see [1] p. 258) it is possible to compute $d_1, d_2, d_3, \ldots$ A better way is to use the exponential trace formula 1.10. Put $p_i = \operatorname{Tr} f^i$, $q_i = \operatorname{Tr} g^i$ and $r_i = \operatorname{Tr}(f \otimes g)^i$. Then $r_i = p_i q_i$ since $\operatorname{Tr}(f \otimes g)^i = \operatorname{Tr}(f^i \otimes g^i) = \operatorname{Tr} f^i \operatorname{Tr} g^i$. The exponential trace formula applied to f gives $a_1 t + 2a_2 t^2 + \ldots + n a_n t^n = (1 + a_1 t + \ldots + a_n t^n)(p_1 t - p_2 t^2 + p_3 t^3 - \ldots)$ and hence

$$\begin{aligned} a_1 &= p_1 \\ 2a_2 &= a_1 p_1 - p_2 \\ 3a_3 &= a_2 p_1 - a_1 p_2 + p_3 \\ 4a_4 &= a_3 p_1 - a_2 p_2 + a_1 p_3 - p_4 \\ &\ldots \end{aligned}$$

Solving for the p_i:s we get

$$\begin{aligned} p_1 &= a_1 \\ p_2 &= a_1^2 - 2a_2 \\ p_3 &= a_1^3 - 3a_1 a_2 + 3a_3 \\ p_4 &= a_1^4 - 4a_1^2 a_2 + 4a_1 a_3 + 2a_2^2 - 4a_4 \\ &\ldots \end{aligned}$$

There are similar formulas connecting the b_i:s and q_i:s (d_i:s and r_i:s). The latter give

$$\begin{aligned} d_1 &= r_1 = p_1 q_1 = a_1 b_1 \\ 2d_2 &= d_1 r_1 - r_2 = a_1^2 b_1^2 - p_2 q_2 = a_1^2 b_1^2 - (a_1^2 - 2a_2)(b_1^2 - 2b_2) = 2(a_1^2 b_2 + a_2 b_1^2 - 2a_2 b_2) \\ 3d_3 &= d_2 r_1 - d_1 r_2 + r_3 = d_2 p_1 q_1 - d_1 p_2 q_2 + p_3 q_3 = a_1 b_1 (a_1^2 b_2 + a_2 b_1^2 - 2a_2 b_2) - \\ &\quad - a_1 b_1 (a_1^2 - 2a_2)(b_1^2 - 2b_2) + (a_1^3 - 3a_1 a_2 + 3a_3)(b_1^3 - 3b_1 b_2 + 3b_3) = \\ &\quad = 3(a_1^3 b_3 + a_3 b_1^3 - 3a_1 a_2 b_3 - 3a_3 b_1 b_2 + 3a_3 b_3 + a_1 a_2 b_1 b_2) \end{aligned}$$

We omit the calculation of d_4.

We could immediately have seen that the terms $a_1^3b_1^3$, $a_1^3b_1b_2$ would be missing in d_3 since they would occur in $(1 + a_1t) * (1 + b_1t + b_2t_2^2)$ which only has degree $1 \cdot 2 = 2$. Similarly $a_1b_2b_1^3$ will not occur.

To get the last terms one can use

$$(1 + a_1t + \ldots + a_nt^n) * (1 + b_1t + \ldots + b_mt^m) =$$
$$= a_n^m b_m^n t^{mn}\left(1 + \frac{a_{n-1}}{a_n} t^{-1} + \frac{a_{n-2}}{a_n} t^{-2} + \ldots\right) * \left(1 + \frac{b_{m-1}}{b_m} t^{-1} + \frac{b_{m-2}}{b_m} t^{-2} + \ldots\right)$$

In particular the number of monomials occurring in d_{mn-i} is the same as in d_i. Let s_k denote the number of monomials in d_k for large m, n (say $m, n \geq k$). The computation of s_k seems to be quite a problem.

By formally factoring

$$1 + a_1t + a_2t^2 + \ldots + a_nt^n = (1 + \alpha t)(1 + \beta t)(1 + \gamma t)(t + \delta t) \ldots$$

we find that the term containing, say $b_4^2b_1^2$, of

$$(1 + a_1t + \ldots) * (1 + b_1t + b_2t^2 + \ldots) =$$
$$= (1 + b_1\alpha t + b_2\beta^2t^2 + \ldots)(1 + b_1\beta t + b_2\beta^2t^2 + \ldots) \ldots$$

is $-\alpha^4\beta^4\gamma\delta$. Using the large fold-out tables of *Faa de Bruno:* Theorie des formes binaires, Turin 1876, we find the following results $s_1 = 1$, $s_2 = 3$, $s_3 = 6$, $s_4 = 15$, $s_5 = 28$, $s_6 = 64$, $s_7 = 116$, $s_8 = 234$, $s_9 = 373$, $s_{10} = 814$, $s_{11} = 1508$.

The method based on couning zeroes in tables cannot be generalized to k larger than 11.

Now back to defining $*$-multiplication in $\tilde{A}$. By the computations above it is clear that if we cut off the power series in the left hand side of

$$(1 + a_1t + \ldots) * (1 + b_1t + \ldots) = 1 + d_1t + \ldots + d_kt^k + \ldots$$

and take $*$ of the remaining polynomials of degree n and m respectively, then $d_k =$ the coefficient of t^k will not depend on n and m if $n, m \geq k$. Hence we can define d_k in this way. Then $\tilde{A}$ becomes a commutative ring with ordinary multiplication as addition and $*$-multiplication as multiplication. The unity element is $1 + t$. Clearly $\tilde{A}$ is torsionfree (as abelian group). Furthermore $\lambda_t(f) \mapsto \lambda_t(A^kf)$ induces a λ-ring structure on $\tilde{A}$ (it is even a special λ-ring, see [1], p. 257).

We denote by $N(A) = \{a \in A_0;\ a \text{ is nilpotent}\}$ the nilradical of a ring A.

PROPOSITION 3.5. (i) *If A is torsion free then*

$$N(\tilde{A}) \subseteq \widetilde{N(A)} = \{1 + a_1t + a_2t^2 + \ldots;\ a_i \in N(\tilde{A})\}.$$

(ii) *If A is noetherian then* $N(\tilde{A}) \subseteq \widetilde{N(A)}$.

Proof. (i) Assume that $(1 + a_1t + a_2t^2 + \ldots)^{*k} = 1$. The left hand side is $1 + c_1t + c_2t^2 + \ldots$ with $c_1 = a_1^k$ and in general $c_n = m_na_n^k + a$ polynomial of weight nk containing at least one of $a_1, a_2, \ldots, a_{n-1}$. Here m_n is an integer. We proceed by induction over n. We have $a_1^k = 0$ so $a_1 \in N(A)$. Assume now that $a_1, a_2, \ldots, a_{n-1} \in N(A)$. Since $c_n = 0$ we get $m_na_n^k \in N(A)$ and $a_n \in N(A)$ since A is torsion free.

(ii) If A is noetherian then $N(A)$ is nilpotent, say $N(A)^k = 0$. Hence the product of any k elements of $N(A)$ is zero. The computation above shows that all monomials occurring in c_n contain at least k factors among the $a_1, \ldots, a_n \in N(A)$. It follows that $(1 + a_1t + \ldots)^{*k} = 1$.

We will return to the ring $\tilde{A}$ in Section 6.

PROPOSITION 3.6. *Given* $f\colon P \to P$, $g\colon Q \to Q$ *with* $P, Q \in \mathcal{P}(A)$. *Then we have an induced map*

$$\mathrm{Hom}\,(f, g)\colon \mathrm{Hom}_A\,(P, Q) \to \mathrm{Hom}_A\,(P, Q) \quad \textit{where} \quad \mathrm{Hom}_A\,(P, Q) \in \mathcal{P}(A)$$

defined by $u \mapsto g \circ u \circ f$. *Then*

$$\mathrm{Tr}\,\mathrm{Hom}\,(f, g) = \mathrm{Tr}\,f \cdot \mathrm{Tr}\,g \quad \textit{and} \quad \lambda_t(\mathrm{Hom}\,(f, g)) = \lambda_t(f) * \lambda_t(g).$$

Proof. We have a natural isomorphism $Q \cong Q^{**}$ which induces natural isomorphisms

$$\mathrm{Hom}_A\,(P, Q) \cong \mathrm{Hom}_A\,(P, Q^{**}) \cong \mathrm{Hom}_A\,(P \otimes_A Q^*, A) = (P \otimes_A Q^*)^*$$

Hence we get $\mathrm{Tr}\,(\mathrm{Hom}\,(f, g) = \mathrm{Tr}\,(f \otimes g^*)^*$ and $\lambda_t(\mathrm{Hom}\,(f, g)) = \lambda_t(f \otimes g^*)^*)$. Using 3.2 twice and the definition of $*$-multiplication we get the desired formulas.

4. Relations between $\lambda_t(f)$ and minimal polynomials of f

PROPOSITION 4.1. *Let* $f\colon M \to M$ *be* A*-linear with* M *a finitely generated* A*-module. Then there is a* m o n i c *polynomial* $q \in A[t]$ *of minimal degree such that* $q(f) = 0$. (q *will be called a* m i n i m a l p o l y n o m i a l o f f). *The degree of* q *is at most equal to the minimal number of generators of* M.

Proof. Let n be the minimal number of generators of M. Then we have a surjection $A^n \xrightarrow{\pi} M \longrightarrow 0$. Since A^n is free we can find $g\colon A^n \longrightarrow A^n$ such that

$$\begin{tikzcd} A^n \arrow[r, "\pi"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "f"] & 0 \\ A^n \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

commutes. Now g satisfies a monic polynomial q_1 of degree n by the Cayley-Hamilton theorem. Using this in the diagram gives

$$\begin{tikzcd} A^n \arrow[r, "\pi"] \arrow[d, "0 = q_1(g)"'] & M \arrow[r] \arrow[d, "q_1(f)"] & 0 \\ A^n \arrow[r, "\pi"'] & M \arrow[r] & 0 \end{tikzcd}$$

from which it follows that $q_1(f) = 0$.

Remark 4.2. The polynomial q is not unique in general. If $A = Z/(4)$ then $f = \left(\begin{smallmatrix}2&2\\0&2\end{smallmatrix}\right)$ satisfies both $f^2 = 0$ and $f^2 + 2f = 0$.

PROPOSITION 4.3. *Given $f\colon P \to P$ with P in $\mathcal{P}(A)$. Assume that f has minimal polynomial q and put $\tilde{q}(t) = (-t)^\nu q(-t^{-1})$ where $\nu =$ degree of q. Then $\lambda_t(f)$ satisfies the following differential equation in $A[t]$*

$$t\lambda_t(f)^{-1}\frac{d}{dt}\lambda_t(f) = \frac{\tilde{q}\cdot\psi \,(\mathrm{mod}\; t^{\nu+1})}{\tilde{q}}$$

where $\psi(t) = b_1t - b_2t^2 + b_3t^3 \ldots$ with $b_i = \mathrm{Tr}\, f^i$. If $q(0) = 0$ we may take $(\mathrm{mod}\; t^\nu)$ in the formula above.

Proof. Assume that $q(t) = t^\nu + c_1t^{\nu-1} + \ldots + c_kt^{\nu-k}$. Taking the trace of $0 = f^\nu + c_1f^{\nu-1} + \ldots + c_kk^{\nu-k}$ we get $0 = b_\nu + c_1b_{\nu-1} + \ldots + c_kk_{\nu-k}$ where in case $k = \nu$ we put $b_0 = \mathrm{Tr}\, 1_P$. Multiplying by f and taking traces again gives

$b_{\nu+1} + c_1b_\nu + \ldots + c_kb_{\nu-k+1} = 0$ etc. Now $\tilde{q}(t) = 1 - c_1t + c_2t^2 - \ldots \pm c_kt^k$ and

$$\begin{aligned}\tilde{q}(t)\psi(t) &= (1 - c_1t + c_2t^2 - \ldots \pm c_kt^k)(b_1t - b_2t^2 + b_3t^3\ldots) = \\ &= (\text{terms of degree } < \nu) \pm (b_\nu + c_1b_{\nu-1} + \ldots + c_kb_{\nu-k})t \pm \\ &\pm (b_{\nu+1} + c_1b_\nu + \ldots + c_kb_{\nu-k+1})t^{\nu+1} + \ldots\end{aligned}$$

Here all terms of degree higher than ν vanish and the coefficient of t^ν is zero unless $k = \nu$ in which case it is $(-1)^{\nu-1}c_k\,\mathrm{Tr}\, 1_P$. The exponential trace formula gives

$$t\lambda_t(f)^{-1}\frac{d}{dt}\lambda_t(f) = \psi(t)$$

and multiplying by $\tilde{q}(t)$ finishes the proof.

Remark 4.4. If A contains the rational numbers $\mathbf{Q}$ then $\lambda_t(f)$ is determined by a minimal polynomial q of f and $b_1, b_2 \ldots, b_{\nu-1}$ where $\nu =$ degree of q.

Example 4.5. Assume that $A \supseteq Q$. Let $f\colon P \to P$ have minimal polynomial $q(t) = t^2 - t$, i.e., f is a non-trivial idempotent in $\mathrm{End}_A P$. Then $\tilde{q}(t) = 1 + t$ and if we apply 4.3 we get (since $q(0) = 0$)

$$t\lambda_t(f)^{-1}\frac{d}{dt}\lambda_t(f) = \frac{((1+t)(b_1t - b_2t^2 \ldots))(\bmod t^2)}{1+t} = \frac{b_1t}{1+t}$$

which implies $\lambda_t(f) = (1+t)^{b_1} = (1+t)^{\operatorname{Tr} f}$.

If $f^3 = f$, i.e. $q(t) = t^3 - t$ one finds similarly

$$\lambda_t(f) = (1+t)^{\frac{b_2+b_1}{2}} \cdot (1-t)^{\frac{b_2-b_1}{2}}$$

Example 4.6. Let G be a finite group of order n and $A[G]$ the group algebra. Let $f\colon A[G] \to A[G]$ be given by left multiplication with $\sigma \in G$. If σ has order k then the minimal polynomial of f is $q(t) = t^k - 1$ and $\tilde{q}(t) = 1 + (-1)^k t^k$. Using 4.3 and the fact that $b_1 = b_2 = \ldots = b_{k-1} = 0$ and $b_k = n$ we get

$$\lambda_t(f) = (1 - (-1)^k t^k)^{\frac{n}{k}}$$

5. Endomorphisms of modules having finite resolutions of finitely generated projective modules

Let $\mathcal{H}(A)$ denote the category of A-modules M such that M has a finite resolution in $\mathcal{P}(A)$. We want to define $\lambda_t(f)$ for $f\colon M \to M$ when $M \in \mathcal{H}(A)$. For this we need some preparations.

Definition 5.1. Let $\operatorname{End}\mathcal{P}(A)$ denote the category of endomorphisms of modules in $\mathcal{P}(A)$, i.e. the objects are endomorphism $f\colon P \to P$ with $P \in \mathcal{P}(A)$ and a morphism u from f to $g\colon Q \to Q$ (where $Q \in \mathcal{P}(A)$) is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "f"'] & Q \arrow[d, "g"] \\ P \arrow[r, "u"] & Q \end{tikzcd}$$

Then $K_0(\operatorname{End}\mathcal{P}(A))$ is defined as the free abelian group generated by (the isomorphism classes of) the objects in $\operatorname{End}\mathcal{P}(A)$ modulo the subgroup generated by all $[f] - [f'] - [f'']$ where

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d, "f"] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & P' \arrow[r] & P \arrow[r] & P'' \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact row. Similarly we define $\operatorname{End}\mathcal{H}(A)$ and $K_0(\operatorname{End}\mathcal{H}(A))$.

PROPOSITION 5.2. *The embedding* $\operatorname{End}\mathcal{P}(A) \to \operatorname{End}\mathcal{H}(A)$ *induces an isomorphism* $i\colon K_0(\operatorname{End}\mathcal{P}(A)) \xrightarrow[\cong]{} K_0(\operatorname{End}\mathcal{H}(A))$.

Proof. The usual proof does not apply since $f\colon P \to P$ with $P \in \mathcal{P}(A)$ is not a projective object in the abelian category of *all* endomorphisms (which is isomorphic to the category of modules over $A[t]$). Fortunately Swan has formulated a theorem general enough for our purposes (see [12] p. 235. Theorem 16.12). Put $\mathcal{P} = \operatorname{End}\mathcal{P}(A)$ and $\mathcal{M} = \operatorname{End}\mathcal{H}(A)$. Then the assumptions in 16.12 are fulfilled. Indeed,

(1) Clearly $\operatorname{End}\mathcal{P}(A)$ and $\mathcal{H}(A)$ are closed under direct sums

(2) If

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r, "u"] \arrow[d, "f'"] & P \arrow[r, "v"] \arrow[d, "f"] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & P' \arrow[r, "u"] & P \arrow[r, "v"] & P'' \arrow[r] & 0 \end{tikzcd}$$

is exact and commutative then $P, P'' \in \mathcal{P}(A)$ implies $P' \in \mathcal{P}(A)$ and $P, P'' \in \mathcal{H}(A)$ implies $P' \in \mathcal{H}(A)$ (see Bass [2], p. 122, Proposition 6.3).

(3) Given any $f\colon M \to M$ with $M \in \mathcal{H}(A)$ there exists a finite resolution in $\operatorname{End}\mathcal{P}(A)$, i.e.

$$\begin{tikzcd} 0 \arrow[r] & P_d \arrow[r] \arrow[d, "f_d"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & P_d \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

is commutative with exact row and all $P_i \in \mathcal{P}(A)$. This is easily proved.

Now the inverse ψ of $i\colon K_0(\operatorname{End}\mathcal{P}(A)) \to K_0(\operatorname{End}\mathcal{H}(A))$ is given by

$$\psi([f]) = \sum_0^d (-1)^i [f_i]$$

and it is shown in [12] that the right hand side is independent of the choice of the resolution $(*)$.

THEOREM 5.3. *Given $f\colon M \to M$ with $M \in \mathcal{H}(A)$. Consider the resolution $(*)$ in $\operatorname{End}\mathcal{P}(A)$ above. Then*

$$\sum_0^d (-1)^i \operatorname{Tr} f_i \quad \textit{and} \quad \prod_0^d \lambda_t(f_i)^{(-1)^i}$$

are independent of the choice of the resolutions and the liftings f_i of f.

Proof. For $f\colon P \to P$ with $P \in \mathcal{P}(A)$, $f \mapsto \lambda_t(f)$ is a map from (isomorphism classes in) $\operatorname{End}\mathcal{P}(A)$ to $\tilde{A}$. If $0 \to (P', f') \to (P, f) \to (P'', f'') \to 0$ is exact in $\operatorname{End}\mathcal{P}(A)$ we have (by (1.5) $\lambda_t(f) = \lambda_t(f')\lambda_t(f'')$.

Hence by the universal property of $K_0(\operatorname{End}\mathcal{P}(A))$ we have a factorization

$$\begin{tikzcd} \operatorname{End} \mathcal{P}(A) \arrow[r, "{[\ ]}"] \arrow[dr, "\lambda_t"'] & K_0(\operatorname{End} \mathcal{P}(A)) \arrow[d] \\ & \tilde{A} \end{tikzcd}$$

Assume now that (M, f) in $\operatorname{End} \mathcal{H}(A)$ has two resolutions

$$0 \to (P_d, f_d) \to \ldots \to (P_0, f_0) \to, (M, f) \to 0$$

and

$$0 \to (P'_{d'}, f'_{d'}) \to \ldots \to (P'_0, f'_0) \to (M, f) \to 0$$

in $\operatorname{End} \mathcal{P}(A)$. By the proof of 5.2 we have

$$\sum_0^d (-1)^j [f_j] = \sum_0^{d'} (-1)^j [f'_j] \quad \text{in} \quad K_0 (\operatorname{End} \mathcal{P}(A)$$

and thus

$$\prod_0^d \lambda_t(f_j)^{(-1)^j} = \prod_0^{d'} \lambda_t(f'_j)^{(-1)^j} \quad \text{in} \quad \tilde{A}.$$

The statement about the trace follows from taking the coefficient of t in the formula for λ_t.

Now we can safely make the

Definition 5.4. For $f: M \to M$ with M in $\mathcal{H}(A)$ we define

$$\chi(f) = \sum^d (-1)^i \operatorname{Tr} f_i \quad \text{and} \quad \lambda_t(f) = \prod^d \lambda_t(f_i)^{(-1)^i}$$

where the f_i:s are given in (*).

PROPOSITION 5.5. *Let*

$$\begin{tikzcd} 0 \arrow[r] & M_k \arrow[r] \arrow[d, "f_k"] & \ldots \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_0 \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & M_k \arrow[r] & \ldots \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact row and all M_i *in* $\mathcal{H}(A)$. *Then*

$$\sum_0^k (-1)^i \chi(f_i) = 0 \quad \textit{and} \quad \prod_0^k \lambda_t(f_i)^{(-1)^i} = 1$$

Proof. Consider the diagram (see the proof of 5.2)

$$\begin{tikzcd} K_0(\operatorname{End} \mathcal{P}(A)) \arrow[r, shift left, "i"] \arrow[d, "\lambda_t"'] & K_0(\operatorname{End} \mathcal{H}(A)) \arrow[l, shift left, "\psi"] \arrow[dl, "\tilde{\lambda}_t"] \\ \tilde{A} & \end{tikzcd}$$

where we denote λ_t by $\tilde{\lambda}_t$ on $\operatorname{End}\mathcal{H}(A)$. The definition of ψ and $\tilde{\lambda}_t$ means exactly that $\tilde{\lambda}_t = \lambda_t \circ \psi$. Now given an exact sequence

$$0 \to (M_k, f_k) \to \dots \to (M_0, f_0) \to 0$$

in $\operatorname{End}\mathcal{H}(A)$ we get $\sum_0^k (-1)^i [f_i] = 0$ in $K_0(\operatorname{End}\mathcal{H}(A))$ and hence

$$\prod_0^k \tilde{\lambda}_t [f_i]^{(-1)^i} = 1$$

Taking the coefficient of t we get the formula for χ.

COROLLARY 5.6. *$\chi(f \oplus g) = \chi(f) + \chi(g)$ and $\lambda_t(f \oplus g) = \lambda_t(f) \cdot \lambda_t(g)$.*

Next we generalize the exponential trace formula

PROPOSITION 5.7. *If $f\colon M \to M$ with $M \in \mathcal{H}(A)$ then*

$$-t\lambda_t(f)^{-1} \frac{d}{dt} \lambda_t(f) = \sum_1^\infty \chi(f^i)(-t)^i \quad \textit{in } \tilde{A}.$$

Proof. Let $0 \to (P_d, f_d) \to \dots \to (P_0, f_0) \to (M, f) \to 0$ be a resolution in $\operatorname{End}\mathcal{P}(A)$. Taking logarithmic derivatives of

$$\lambda_t(f) = \prod_{j=0}^d \lambda_t(f_j)^{(-1)^j}$$

we get (using the exponential trace formula)

$$-t\lambda_t(f)^{-1} \frac{d}{dt} \lambda_t(f) = \sum_{j=0}^d (-1)^j \left(-t\lambda_t(f_j)^{-1} \frac{d}{dt} \lambda_t(f_j) \right) =$$

$$= \sum_{j=0}^d (-1)^j \sum_{i=1}^\infty (-1)^i \operatorname{Tr}(f_j^i) t^i = \sum_{i=1}^\infty (-1)^i \left(\sum_{j=0}^d (-1)^j \operatorname{Tr}(f_j^i) \right) = \sum_{i=1}^\infty (-1)^i \chi(f^i) t^i$$

since

$$0 \to (P_d, f_d^i) \to \dots \to (P_0, f_0^i) \to (M, f^i) \to 0$$

is a resolution of (M, f^i).

THEOREM 5.8. *Let $f\colon M \to M$ with $M \in \mathcal{H}(A)$ be nilpotent, $f^{m+1} = 0$. Then there is a resolution*

$$0 \to (P_d, f_d) \to \dots \to (P_0, f_0) \to (M, f) \to 0$$

in $\operatorname{End}\mathcal{P}(A)$ such that all $f_i^{m+1} = 0$.

Assume that $\mathrm{rk}P_i = n_i$ *and* $\lambda_t(f) = 1 + \sum_1^\infty c_i t^i$. *Then all the* c_i:*s are nilpotent and* $c_1^{\nu_1} c_2^{\nu_2} \dots c_k^{\nu_k} = 0$ *if the weight* $\nu_1 + 2\nu_2 + \dots + k\nu_k > m \sum_0^d n_i$. *It follows that* $\lambda_t(f)$ *is a polynomial of degree*

$$\leq n_0 + mn_1 + n_2 + mn_3 + \dots + \begin{cases} n_d & \text{if } d \text{ is even} \\ mn_d & \text{if } d \text{ is odd.} \end{cases}$$

Proof. The existence of the projective resolution such that $f_i^{m+1} = 0$ is precisely Proposition 6.2, p. 653 in Bass [2]. Now $\lambda_t(f)$ is a product of factors

$$\lambda_t(f_i) = 1 + a_1 t + \dots + a_{n_i} t^{n_i}$$

or their inverses. By 1.7 any monomial in the a_j:s vanishes provided its weight is larger than mn_i. Inverting the polynomial $\lambda_t(f_i) = 1 + a_1 t + \dots + a_{n_i} t^{n_i}$ we find that $\lambda_t(f_i)^{-1}$ is a polynomial of degree at most mn_i and the coefficient of t^ν is a polynomial in the a_j:s where every term has weight ν. Taking the alternating product of the $\lambda_t(f_i)$:s we get $\lambda_t(f) = 1 + c_1 t + c_2 t^2 + \dots$ where c_ν is a sum of terms of the type

$$a_1^{r_1} \dots a_{n_0}^{r_{n_0}} \dots b_1^{s_1} \dots b_{n_d}^{s_{n_d}} \tag{**}$$

if $\lambda_t(f_0) = 1 + a_1 t + \dots + a_{n_0} t^{n_0} \dots, \lambda_t(f_d) = 1 + b_1 t + \dots + b_{n_d} t^{n_d}$.

Furthermore the weight of the monomial (**) is

$$\nu = r_1 + 2r_2 + \dots + n_0 r_{n_0} + \dots + s_1 + 2s_2 + \dots + n_d s_{n_d}.$$

Let now $c = c_1^{\nu_1} c_2^{\nu_2} \dots c_k^{\nu_k}$ be a monomial in the c_i:s of weight

$$\nu_1 + 2\nu_2 + \dots + k\nu_k > m \sum_{i=0}^{d} n_i.$$

Then c is a sum of monomials of type (**) such that their weight

$$r_1 + 2r_2 + \dots + n_0 r_{n_0} + \dots + s_1 + 2s_2 + n_d s_{n_d} = \nu_1 + 2\nu_2 + \dots + k\nu_k > m \sum_0^d n_i.$$

Hence at least one of the factors

$$(a_1^{r_1} \dots a_n^{r_n}), \dots, (b_1^{s_1} b_2^{s_2} \dots b_{n_d}^{s_{n_d}})$$

has weight $> mn_1, \dots, mn_d$ respectively and this factor is zero by 1.7.

The estimate of the degree of $\lambda_t(f)$ is clear from the previous considerations.

COROLLARY 5.9. *Assume that the ring* A *is* r e d u c e d, *i.e. the nilradical* $N(A) = 0$. *Then* $\lambda_t(f) = 1$ *for all nilpotent* $f: M \to M$ *with* $M \in \mathcal{X}(A)$.

We denote the *projective dimension* of an A-module M with $dh_A M$.

PROPOSITION 5.10. *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$, residue field $k = A/\mathfrak{m}$, and M a finitely generated A-module. If $d = dh_A M$ is finite then $M \in \mathcal{X}(A)$ and $\lambda_t^A(1_M) = (1+t)^{\chi^A(1_M)}$ where*

$$\chi^A(1_M) = \sum_{i=0}^{d} (-1)^i \dim_k \operatorname{Tor}_i^A (M, k)$$

Proof. Choose a minimal free resolution

$$0 \to P_d \to \dots \to P_1 \to P_0 \to M \to 0$$

with $n_i = rk_A P_i = \dim_k \operatorname{Tor}_i^A (M, k)$ (see Serre [10] p. IV — 47). Then

$$\lambda_t(1_M) = \prod_0^d \lambda_t(1_{P_i})^{(-1)^i} = \prod_0^d (1+t)^{(-1)^i n_i} = (1+t)^{\sum_0^d (-1)^i n_i}.$$

But

$$\chi(1_M) = \sum_0^d (-1)^i \operatorname{Tr} 1_{P_i} = \sum_0^d (-1)^i n_i.$$

PROPOSITION 5.11. *Let A be a regular local noetherian ring with residue field k. Then $k \in \mathcal{X}(A)$ and $\lambda_t^A(1_k) = 1$.*

Proof. Putting $M = k$ in 5.10 we get

$$\chi^A(1_k) = \sum_0^d (-1)^i \dim_k \operatorname{Tor}_i^A (k, k) = \sum_0^d (-1)^i \binom{d}{i} = (1-1)^d = 0$$

since $\dim_k \operatorname{Tor}_i^A (k, k) = \binom{d}{i}$ where $d =$ global dimension of A if A is a regular local noetherian ring.

PROPOSITION 5.12. *Let $\phi: A \to B$ be a flat ring homomorphism, i.e. B is flat as an A-module. If $f: M \to M$ with $M \in \mathcal{X}(A)$, then $M \otimes_A B \in \mathcal{X}(B)$ and*

$$\lambda_t^B(f \otimes 1_B) = \phi(\lambda_t^A(f)).$$

Proof. Let

$$\begin{array}{ccccccccc} 0 \to & P_d & \to \dots \to & P_0 & \to & M & \to 0 \\ & \downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\ 0 \to & P_d & \to \dots \to & P_0 & \to & M & \to 0 \end{array}$$

be a projective resolution. Then the exactness is preserved after taking $\cdot \otimes_A B$ since B is A-flat. Furthermore each $P_i \otimes_A B$ is B-projective and finitely generated as B-module. Hence $M \otimes_A B \in \mathcal{H}(B)$ and since $\phi(\lambda_t^A(f_i)) = \lambda_\tau^B(f_i \otimes 1_B)$ by 3.1 we finish the proof by taking alternating products.

Corollary 5.13. *Let A be an integral domain and K its quotient field. Then*

$$\lambda_t^A(f) = \lambda_t^K(f \otimes 1_K)$$

Proof. The inclusion $A \to K$ is flat.

Corollary 5.14. *Let A be an integral domain and $f\colon M \to M$ where M is a torsion module in $\mathcal{H}(A)$. Then $\lambda_t(f) = 1$.*

Proof. Since M is torsion we have $M \otimes_A K = 0$ and hence

$$\lambda_t^A(f) = \lambda_t^K(f \otimes 1_K) = \lambda_t^K(0) = 1$$

by 5.13.

Corollary 5.15. *Let A be a Dedekind ring and $f\colon M \to M$ A-linear where M is finitely generated. Then $M = T \oplus P$ where T is a torsion module and P is projective and torsion free.*

Furthermore $f(T) \subseteq T$ and $\lambda_t(f) = \lambda_t(f_P)$ where $f_P\colon P \to P$ is the »torsion free part» of f.

Proof. First we note that $M \in \mathcal{H}(A)$ since A is noetherian and gl. dim $A \leq 1$. Then $M = T \oplus P$ is just Bourbaki [5] p. 79, Corollaire. Now $\mathrm{Hom}_A(T, P) = 0$ so we get the following diagram using matrix representation

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "f_T"] & T \oplus P \arrow[r, "{(0,\,1)}"] \arrow[d, "f = \begin{pmatrix} f_T & h \\ 0 & f_P \end{pmatrix}"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"'] & T \oplus P \arrow[r, "{(0,\,1)}"'] & P \arrow[r] & 0.
\end{tikzcd}$$

From 5.6 and 5.14 it follows that

$$\lambda_t(f) = \lambda_t(f_T) \cdot \lambda_t(f_P) = 1 \cdot \lambda_t(f_P) = \lambda_t(f_P).$$

We now extend the definitions of χ and λ_t to endomorphisms of graded modules and complexes.

Definition 5.16. Let $M = \oplus_0^d M_i$ be a graded A-module with all $M_i \in \mathcal{H}(A)$. If $f\colon M \to M$ is a homomorphism of degree zero, i.e. $f(M_i) \subseteq M_i$, we put $f_i =$ the restriction of f to M_i and define

$$\chi^{gr}(f) = \sum_0^d (-1)^i \chi(f_i) \quad \text{and} \quad \lambda_t^{gr}(f) = \prod_0^d \lambda_t(f_i)^{(-1)^i}.$$

Note that $\chi^{gr}(f)$ and $\lambda_t^{gr}(f)$ in general do not agree with $\chi(f)$ and $\lambda_t(f)$ where M is considered just as an A-module.

Similarly if

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_d & \xrightarrow{\delta_d} & C_{d-1} & \to \dots \to & C_1 & \xrightarrow{\delta_1} & C_0 & \longrightarrow & 0 \\ & & \downarrow f_d & & \downarrow f_{d-1} & & \downarrow f_1 & & \downarrow f_0 & & \\ 0 & \longrightarrow & C_d & \xrightarrow{\delta_d} & C_{d-1} & \to \dots \to & C_1 & \xrightarrow{\delta_1} & C_0 & \longrightarrow & 0 \end{array}$$

for short $f\colon C \to C$ is a chainmap of a finite complex C with all C_i in $\mathcal{H}(A)$, we define

$$\chi(f) = \sum_0^d (-1)^i \operatorname{Tr} f_i \quad \text{and} \quad \lambda_t(f) = \prod_0^d \lambda_t(f_i)^{(-1)^i}.$$

PROPOSITION 5.17. *Let $f\colon C \to C$ be as above. Assume that all homology modules $H_i(C)$ are in $\mathcal{H}(A)$. Then*

$$\chi(f) = \chi^{gr}(H_*(f)) \quad \textit{and} \quad \lambda_t(f) = \lambda_t^{gr}(H_*(f))$$

where $H_*(f)\colon H_*(C) \to H_*(C)$ is the induced endomorphism of the graded homology module $H_*(C) = \oplus_0^d H_i(C)$.

Proof. Put $K_i = \operatorname{Ker} \delta_i$ and $B_i = \operatorname{Im} \delta_{i+1}$. Then we have exact sequences

$$0 \to K_i \to C_i \to B_{i-1} \to 0$$

$$0 \to B_i \to K_i \to H_i(C) \to 0.$$

Now $B_0 = C_0 \in \mathcal{H}(A)$ and $C_1 \in \mathcal{H}(A)$ so $K_1 \in \mathcal{H}(A)$ by Bass [2] p. 122, Proposition 6.3. Since $H_1(C) \in \mathcal{H}(A)$ we also get $B_1 \in \mathcal{H}(A)$. By induction all $B_i, K_i \in \mathcal{H}(A)$. We get induced maps

$$\begin{array}{ccccccccc} 0 \to & K_i & \to & C_i & \to & B_{i-1} & \to 0 \\ & \downarrow g_i & & \downarrow f_i & & \downarrow h_{i-1} & \\ 0 \to & K_i & \to & C_i & \to & B_{i-1} & \to 0, \end{array} \qquad \begin{array}{ccccccccc} 0 \to & B_i & \to & K_i & \to & H_i(C) & \to 0 \\ & \downarrow h_i & & \downarrow g_i & & \downarrow H_i(f) & \\ 0 \to & B_i & \to & K_i & \to & H_i(C) & \to 0. \end{array}$$

Using 5.5 several times and taking alternating products all $\lambda_t(g_i)$ and $\lambda_t(h_i)$ cancel and we get the wanted formula for $\lambda_t(f)$.

Remark 5.18. The condition $H_i(C) \in \mathcal{H}(A)$ is satisfied if A is a regular noetherian ring.

COROLLARY 5.19. *If* $f\colon C \to C$ *and* $g\colon C \to C$ *are chain homotopic maps of complexes then* $\lambda_t(f) = \lambda_t(g)$.

PROPOSITION 5.20. *Let* $f\colon C \to C$ *be a chain map as above. Then*

$$-t\lambda_t(f)^{-1}\frac{d}{dt}\lambda_t(f) = \sum_{j=1}^{\infty} \chi(f^j)(-t)^j$$

Proof. Take the logarithmic derivative of $\lambda_t(f) = \prod_0^d \lambda_t(f_i)^{(-1)^i}$ and use 5.7.

PROPOSITION 5.21. *Given* $f\colon M \to M$ *and* $g\colon N \to N$ *with* $M, N \in \mathcal{H}(A)$. *Assume that* $\operatorname{Tor}_i(M, N) \in \mathcal{H}(A)$ *for all* $i \geq 0$. *Then*

$$\lambda_t(f) * \lambda_t(g) = \lambda_t^{gr}(\operatorname{Tor}_*(f, g))$$

where $\operatorname{Tor}_*(M, N) = \oplus_{i \geq 0} \operatorname{Tor}_i(M, N)$ *and* $\operatorname{Tor}_*(f, g)$ *is the induced graded map.*

Proof. Let

$$0 \to (P_m, f_m) \to \ldots \to (P_0, f_0) \to (M, f) \to 0$$

and

$$0 \to (Q_n, g_n) \to \ldots \to (Q_0, g_0) \to (N, g) \to 0$$

be resolutions in $\operatorname{End}\mathcal{P}(A)$. Then

$$\lambda_t(f) = \prod_0^n \lambda_t(f_i)^{(-1)^i} \quad \text{and} \quad \lambda_t(g) = \prod_0^n \lambda_t(g_j)^{(-1)^j}.$$

Taking the tensor product of the complexes we get a complex $C = (C_k)_{k=0}^{m+n}$ and a chain map $h = (h_k)_0^{m+n}\colon C \to C$ where

$$C_k = \bigoplus_{i+j=k} P_i \otimes Q_j \quad \text{and} \quad h_k = \bigoplus_{i+j=k} (f_i \otimes g_j).$$

Then

$$H_k(C) = \operatorname{Tor}_k(M, N) \quad \text{and} \quad H_k(h) = \operatorname{Tor}_k(f, g).$$

Now

$$\lambda_t(h_k) = \lambda_t\Big(\bigoplus_{i+j=k} (f_i \otimes g_j)\Big) = \prod_{i+j=k} \lambda_t(f_i \otimes g_j) = \prod_{i+j=k} \lambda_t(f_i) * \lambda_t(g_j)$$

and

$$\lambda_t(h) = \prod_{k=0}^{m+n} \lambda_t(h_k)^{(-1)^k} = \prod_{i=0}^{m} \prod_{j=0}^{n} \lambda_t(f_i) * \lambda_t(g_j))^{(-1)^{i+j}} =$$

$$= \prod_{i=0}^{m} \lambda_t(f_i)^{(-1)^i} * \prod_{j=0}^{m} \lambda_t(g_j)^{(-1)^j} = \lambda_t(f) * \lambda_t(g).$$

But $\lambda_t(h) = \lambda_t^{gr}(H_*(h)) = \lambda_t^{gr}(\mathrm{Tor}_*(f, g))$ by 5.17 and we are done.

Remark 5.22. If $M, N \in \mathcal{H}(A)$ implies $M \otimes_A N \in \mathcal{H}(A)$ for all M, N then also $\mathrm{Tor}_i(M, N) \in \mathcal{H}(A)$ for $i \geq 1$. This is the case if A is a regular noetherian ring.

To prove this we use induction on $\mathrm{dh}\, M$. If $\mathrm{dh}\, M = 0$, i.e. M is projective, we have nothing to prove. Assume that $\mathrm{dh}\, M = m \geq 1$. Choose an exact sequence

$$0 \to K \to F \to M \to 0$$

where F is free. Then $\mathrm{dh}\, K = m - 1$ and $K \in \mathcal{H}(A)$ since F and M are in $\mathcal{H}(A)$. The long exact sequence is

$$\ldots \to \underbrace{\mathrm{Tor}_2(F, N)}_{=0} \to \mathrm{Tor}_2(M, N) \to \mathrm{Tor}_1(K, N) \to \underbrace{\mathrm{Tor}_1(F, N)}_{=0} \to \mathrm{Tor}_1(M, N) \to$$

$$\to K \otimes_A N \to F \otimes_A N \to M \otimes_A N \to 0.$$

By assumption $K \otimes N$, $F \otimes N$, $M \otimes N \in \mathcal{H}(A)$ and thus $\mathrm{Tor}_1(M, N) \in \mathcal{H}(A)$ by Bass [2] p. 122. Furthermore by the induction hypothesis $\mathrm{Tor}_1(K, N) \in \mathcal{H}(A)$ and hence $\mathrm{Tor}_2(M, N) \cong \mathrm{Tor}_1(K, N) \in \mathcal{H}(A)$. Similarly $\mathrm{Tor}_i(M, N) \in \mathcal{H}(A)$ for $i \geq 2$.

Example 5.23 (M. Schlessinger). If $M, N \in \mathcal{H}(A)$ then $M \otimes_A N$ may not be in $\mathcal{H}(A)$. Let A be the local ring at the singular point $(0, 0)$ of the curve $x^3 - y^2 = 0$. Then $A/(x)$ and $A/(y)$ have homological dimension one (since $0 \to A \xrightarrow{x} A \to A/(x) \to 0$ is exact) but $A/(x) \otimes A/(y) \cong A/(x, y) = k =$ the residue field which has infinite homological dimension (as A-module) since A is not regular.

COROLLARY 5.24. *If M or N is projective and both are in $H(A)$ then*

$$\lambda_t(f \otimes g) = \lambda_t(f) * \lambda_t(g)$$

(it is not more general to assume M only flat since M flat and $M \in \mathcal{H}(A)$ implies M is projective).

Example 5.25. Let X be a polyhedron (or any topological space such that $H_*(X, \mathbf{Z})$ is finitely generated) and $g: X \to X$ a continuous map. Then there is an induced homomorphism of graded abelian groups

$$H_*(X) = \bigoplus_0^d H_i(X, \mathbf{Z}) \quad \text{with} \quad d = \dim X.$$

Then (since $\mathbf{Q}$ is $\mathbf{Z}$-flat)

$$\lambda_t(g_*) = \lambda_t(g_* \otimes 1_Q) = \prod_{i=0}^{d} \lambda_t(H_i(g_*)^{(-1)^i}$$

is exactly $\tilde{\zeta}_g(-t)$ where $\tilde{\zeta}_g$ is the »false» ζ-function of g (see Smale [11] p. 768). It would be interesting to consider (co-)homology with other coefficients. The Lefschetz number is just $\chi(g_*) =$ the coefficient of t in $\lambda_t(g_*)$.

PROPOSITION 5.26. *Assume that* $A = \prod_s^{i=1} A_i$ *is a direct product of rings. Then* $1 = e_1 + \ldots + e_s$ *where* $e_1, \ldots, e_s$ *are orthogonal idempotents and* $A_i \simeq Ae_i$. *Given an* A*-linear map* $f: M \to M$ *with* M *in* $\mathcal{H}(A)$ *then* $M = \oplus_1^s M_i$ *where* $M_i = e_i M$ *can be considered as an* A_i*-module in* $\mathcal{H}(A_i)$. *Let* $f_i: M_i \to M_i$ *be the restriction of* f *to* M_i. *Then*

$$\pi_i(\lambda_t^A(f)) = \lambda_t^{A_i}(f_i)$$

where $\pi_i: A \to A_i$ *is the canonical projection.*

Proof. Since A_i is a direct summand of A it follows that A_i is a projective (and hence flat) A-module. Then

$$M \otimes_A A_i \in \mathcal{H}(A_i) \quad \text{and} \quad \pi_i(\lambda_t^A(f)) = \lambda_t^{A_i}(f \otimes 1_{A_i})$$

by 5.12. Finally $M \otimes_A A_i \cong e_i M = M_i$ as A_i-modules and $f \otimes 1_{A_i}$ may be identified with $f_i: M_i \to M_i$.

COROLLARY 5.27. *Let* A *be a noetherian regular ring. Then* $A = \prod_1^s A_i$ *where the* A_i*:s are integral domains. Let* M *be a finitely generated* A*-module and* $f: M \to M$ *as in* 5.26. *Then*

$$\pi_i(\lambda_t^A(f)) = \lambda_t^{A_i}(f_i) = \lambda_t^{K_i}(f_i \otimes 1_{K_i})$$

where K_i *is the quotient field of* A_i.

Proof. First M is in $\mathcal{H}(A)$ since A is noetherian and $\text{gl.}\dim A < \infty$. The direct product decomposition of the ring is Kaplansky [7], p. 119, Theorem 168.

6. K-theory of endomorphisms

In this section we make an attempt to classify the endomorphisms of finitely generated projective A-modulus (for notation see 5.1).

We have two ringhomomorphisms

$$K_0\,(\text{End}\,\mathcal{P}(A)) \to K_0(A)$$

defined by

$$(P, f) \mapsto P \quad \text{and} \quad K_0(A) \to K_0\,(\text{End}\,\mathcal{P}(A))$$

defined by $P \mapsto (P, 0)$.

Since the latter map is the right inverse of the first one we get a split exact sequence

$$0 \to K_0(A) \to K_0\,(\mathrm{End}\,\mathcal{P}(A)) \to \tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A)) \to 0$$

(compare Bass [2], p. 652) which defines $\tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A))$. Hence

$$K_0\,(\mathrm{End}\,\mathcal{P}(A)) \simeq K_0(A) \times \tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A))$$

and we can consider λ_t defined on $\tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A))$ since $\lambda_t(0) = 1$.

PROPOSITION 6.1. *Let* $A = \prod_1^s A_i$. *Then* $K_0\,(\mathrm{End}\,\mathcal{P}(A)) \simeq \prod_1^s K_0\,(\mathrm{End}\,\mathcal{P}\,(A_i))$.

Proof. We have $1 = e_1 + \ldots + e_s$ where $e_1, \ldots e_s$ are orthogonal idempotents (see 5.26). Given $f\colon P \to P$ with $P \in \mathcal{P}(A)$ we get $f_i\colon P_i \to P_i$ where $P_i = e_i P \in \mathcal{P}(A_i)$. Define

$$\Psi\colon K_0\,(\mathrm{End}\,\mathcal{P}(A)) \to \prod_{i=1}^{s} K_0\,(\mathrm{End}\,\mathcal{P}(A_i))$$

by

$$[f] \to ([f_i])_{i=1}^s$$

Conversely given $([g_i])_1^s$ in $\prod_{i=1}^s K_0\,(\mathrm{End}\,\mathcal{P}(A_i))$ where $g_i\colon P_i \to P_i$ with $P_i \in \mathcal{P}(A_i)$, define $[g] \in K_0\,(\mathrm{End}\,\mathcal{P}(A))$ by $g(x) = g(\sum_1^s x_i) = \sum_1^s g_i(x_i)$

$$\text{if } x = \sum_1^s x_i \in P = \bigoplus_1^s P_i \text{ with } x_i \in P_i \text{ for } i = 1, 2, \ldots, s.$$

Then $P = \oplus_1^s P_i \in \mathcal{P}(A)$ and $g\colon P \to P$ is A-linear.

The maps Ψ and $([g_i])_1^s \mapsto [g]$ are easily seen to be each others inverses. Furthermore Ψ is a ringhomomorphism since f_i can be identified with $f \otimes 1_{A_i}$ and A_i is A-flat.

Definition 6.2. We define the subring of »rational functions»

$$\tilde{A}_0 = \left\{ \frac{1 + a_1 t + \ldots + a_m t^m}{1 + b_1 t + \ldots + b_n t^n}\,;\ a_i, b_j \in A \right\}$$

of $\tilde{A}$ (where $\tilde{A}_0$ has the induced operations).

PROPOSITION 6.3. $\lambda_t\colon \tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A)) \to \tilde{A}$ *is a* λ*-ringhomomorphism with image* $\tilde{A}_0$.

Proof. This follows from the definitions made after 3.3.

THEOREM 6.4. $\tilde{A}_0$ *is a direct summand (as an abelian group) of* $\tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A))$.

Proof. We have to construct a right inverse σ of

$$\lambda_t\colon K_0\,(\mathrm{End}\,\mathcal{P}(A)) \to \tilde{A}_0$$

For this purpose it is convenient to view an endomorphism $f\colon P \to P$ as an $A[t]$-module with the action defined by $t \cdot x = f(x)$ for $x \in P$. Maps between endomorphisms correspond exactly to $A[t]$-linear maps. Let S be the multiplicative set of all monic polynomials in $A[t]$. Then $S^{-1}P = 0$, i.e. P is killed by some monic polynomial, which follows from the Cayley-Hamilton theorem. Summing up, put $T_0(A[t], S) = K_0\{P \in \mathrm{Mod}\,A[t];\ P$ is projective as an A-module and $S^{-1}P = 0\}$ then

$$T_0(A[t], S) \cong K_0\,(\mathrm{End}\,\mathcal{P}(A)).$$

Given $g(t) = 1 + a_1t + \ldots + a_nt^n$ in $\tilde{A}_0$ define $\sigma\colon \tilde{A}_0 \to T_0(A[t], S)$

$$\text{by} \quad \sigma(g(t)) = A[t]/\tilde{g}(t) \quad \text{where} \quad \tilde{g}(t) = t^n g^{-1/t}$$

Over in $K_0\,(\mathrm{End}\,\mathcal{P}(A))$ this means

$$\sigma(g(t)) = \begin{pmatrix} 0 & 0 & 0 & & 0 & \pm a_n \\ 1 & 0 & 0 & & 0 & \pm a_{n-1} \\ 0 & 1 & 0 & & 0 & \pm a_{n-2} \\ 0 & 0 & 0 & 1 & 0 & -a_2 \\ 0 & 0 & 0 & 0 & 1 & a_1 \end{pmatrix}$$

and $\sigma(g(t))$ is an endomorphism of a free A-module.

Then σ is additive, i.e. $\sigma(g(t)h(t)) = \sigma(g(t)) + \sigma(h(t))$.

Indeed we have an exact sequence in $\mathrm{Mod}\,A[t]$

$$0 \to A[t]/(\tilde{g}(t)) \to A[t]/(\tilde{g}(t)\tilde{h}(t)) \to A[t]/(\tilde{h}(t)) \to 0$$

since $\tilde{g}(t)$ and $\tilde{h}(t)$ are non-zero-divisors in $A[t]$. Since

$$\lambda_t(\sigma(g(t)) = 1 + a_1t + \ldots + a_nt = g(t)$$

we have $\lambda_t \circ \sigma = id$ as we wanted.

Corollary 6.5. *Let A be a regular noetherian ring. Then $\tilde{A}_0$ is a direct summand (as abelian group) of $K_0\,(\mathrm{End}\,\mathcal{P}(A)) = K_0\,(\mathrm{End}\,\mathcal{M}(A))$ (here $\mathcal{M}(A)$ is the category of finitely generated A-modules).*

Proof. If A is regular noetherian then every module has finite homological dimension and $\mathcal{M}(A) = \mathcal{H}(A)$. By 5.27 $A = \prod_1^s A_i$ where the A_i:s are integral domains. The rest follows from $\tilde{A}_0 \cong \prod_1^s \tilde{A}_{i_0}$, 5.27, 6.1 and 6.4.

Theorem 6.6. *The map $\lambda_t\colon \tilde{K}_0\,(\mathrm{End}\,\mathcal{P}(A)) \to \tilde{A}_0$ is a ring isomorphism in the following cases*

(i) *A is a PID.*
(ii) *$A = B[X]$ where B is a PID, e.g. $A = K[X, Y]$ where K is a field.*
(iii) *A is a noetherian regular local ring of dimension ≤ 2.*

Proof. Using the notation in the proof of 6.4 and Bass [2] p. 492 we have

$$K_0 (\operatorname{End} \mathcal{P}(A) \cong K_0 (\operatorname{End} \mathcal{H}(A)) = K_0 (\operatorname{End} \mathcal{M}(A)) \cong G_0(A[t], S) =$$

K_0 of the category of $A[t]$-modules killed by some monic polynomial.
Now $A[t]$ is noetherian so given any M as above we have a filtration in $\operatorname{Mod} A[t]$

$$M = M_0 \supset M_1 \supset \ldots \supset M_k = 0$$

such that

$$M_i/M_{i+1} \cong A[t]/\widetilde{\mathfrak{p}_i}$$

where the $\widetilde{\mathfrak{p}_i}$:s are prime ideals in $A[t]$. Since M is killed by a monic polynomial so is M_i and $A[t]/\widetilde{\mathfrak{p}_i}$ which means that $\widetilde{\mathfrak{p}_i}$ contains a monic polynomial. Let $\mathfrak{p}_i = \widetilde{\mathfrak{p}_i} \cap A$ and put $\mathfrak{p}_i' = (\mathfrak{p}_i, f_i)$ where f_i is a monic polynomial in $\widetilde{\mathfrak{p}_i}$ of minimal degree. Now we claim that $\mathfrak{p}_i'$ is a prime ideal in $A[t]$.

We have

$$A[t]/\mathfrak{p}_i' = A[t]/(\mathfrak{p}_i, f_i) \cong (A/\mathfrak{p}_i)[t]/(\overline{f_i})$$

where $\overline{f_i}$ is the residue of f_i in $A/\mathfrak{p}_i[t]$. Furthermore $\overline{f_i}$ is irreducible in $A/\mathfrak{p}_i[t]$ since $\overline{f_i} = \overline{g_i h_i}$ implies $f_i = g_i h_i + q_i$ with $q_i \in \mathfrak{p}_i A[t]$. We can choose g_i and h_i monic and $g_i h_i \in \widetilde{\mathfrak{p}_i}$ since f_i and q_i are in $\widetilde{\mathfrak{p}_i}$. Hence g_i or h_i is in $\widetilde{\mathfrak{p}_i}$ since $\widetilde{\mathfrak{p}_i}$ is prime. But f_i has minimal degree so $g_i = 1$ or $h_i = 1$ and we have shown that $\mathfrak{p}_i'$ is prime in $A[t]$. Evidently $\mathfrak{p}_i' \subseteq \widetilde{\mathfrak{p}_i}$ and $\mathfrak{p}_i' \cap A = \widetilde{\mathfrak{p}_i} \cap A$ so $\mathfrak{p}_i' = \widetilde{\mathfrak{p}_i}$ by Serre [10] p. III. 17, Lemma 3.

Hence $G_0(A[t], S)$ is generated by all $A[t]/(\mathfrak{p}, f)$ where $\mathfrak{p} \in \operatorname{Spec} A$ and f is a monic polynomial such that $\bar{f}$ is irreducible in $A/\mathfrak{p}[t]$. We will show that only the case $\mathfrak{p} = 0$ is interesting. We treat the three cases separately.

(i) Assume that A is a PID and $0 \neq \mathfrak{p} = pA$. Then there is an exact sequence

$$0 \to A[t]/(f) \xrightarrow{p} A[t]/(f) \to A[t]/(\mathfrak{p}, f) \to 0$$

This shows that $[A[t]/(\mathfrak{p}, f)] = 0$ if $\mathfrak{p} \neq 0$.

(ii) If $A = B[X]$ where B is a PID then a prime ideal $\mathfrak{p} \neq 0$ in A is either principal or of the form $\mathfrak{p} = (p, g)$ where $p \in B$ is a prime element in B and $g \in B[X]$ is such that $\bar{g} \in B/pB[X]$ is irreducible.

The case $\mathfrak{p}$ principal is treated as in (i) and in the second case

$$0 \to A[t]/(p, f) \xrightarrow{\bar{g}\cdot} A[t]/(p, f) \to A[t]/(p, g, f) \to 0$$

is exact.

Hence $[A[t]/(\mathfrak{p}, f)] = 0$.

(iii) Let now A be a noetherian regular local ring of dimension ≤ 2. If $\dim A = 0$ or 1 then A is a field or a PID. Assume therefore $\dim A = 2$. Let $\mathfrak{p} \neq 0$ be a prime ideal in A. If ht $\mathfrak{p} = 1$ then $\mathfrak{p}$ is principal since A is a UFD (Bourbaki [5], p. 33) and we are back in case (i). If ht $\mathfrak{p} = 2$ then $\mathfrak{p}$ is the maximal ideal in A and $\mathfrak{p} = (x_1, x_2)$ where x_1, x_2 is an A-sequence. Hence the map

$$A/(x_1) \xrightarrow{\bar{x}_2 \cdot} A/(x_1)$$

is injective. Then

$$0 \to A[t]/(x_1, f) \xrightarrow{\bar{x}_2 \cdot} A[t]/(x_1, f) \to A[t]/(x_1, x_2, f) \to 0$$

is exact and

$$[A[t]/(\mathfrak{p}, f)] = 0.$$

Hence in all three cases $G_0(A[t], S)$ is generated by all $A[t]/(f)$ where f is an irreducible monic polynomial. Recall the maps in the proof of 6.4

$$G_0(A[t], S) \underset{\sigma}{\overset{\lambda_t}{\rightleftarrows}} \tilde{A}_0$$

where we saw $\lambda_t \circ \sigma = id$. The subgroup $K_0(A) \simeq \mathbf{Z}$ of $K_0(\operatorname{End} \mathcal{P}(A)) = G_0(A[t], S)$ has the generator $A[t]/(t)$. It follows that $\sigma \circ \lambda_t = id$ on the rest of the generators $A[t]/(f)$ and hence $\tilde{K}_0(\operatorname{End} \mathcal{P}(A)) \simeq \tilde{A}_0$ which ends the proof.

We now turn to the study of the K_0-groups of some full subcategories of $\operatorname{End} \mathcal{P}(A)$. The first one is (see Bass [2] p. 652)

$$\mathcal{N}il\, \mathcal{P}(A) = \{f \in \operatorname{End} \mathcal{P}(A);\ f \text{ is nilpotent}\}$$

Definition 6.7. Let $\widetilde{N(A)}_0$ denote the subring of $\tilde{A}_0$ consisting of all »rational functions»

$$\frac{1 + a_1 t + \ldots + a_m t^m}{1 + b_1 t + \ldots + b_n t^n}$$

where all a_i, b_j are nilpotent. Since $(1 + b_1 t + \ldots + b_n t^n)^{-1}$ in this case is a polynomial we have

$$\widetilde{N(A)}_0 = \{1 + c_1 t + \ldots + c_k t^k;\ c_i \in N(A)\}.$$

PROPOSITION 6.8. *λ_t: $K_0(\mathcal{N}il\, \mathcal{P}(A)) \to \widetilde{N(A)}_0$ is a surjective ringhomomorphism. Furthermore $\widetilde{N(A)}_0$ is a direct summand (as abelian group) of $K_0(\mathcal{N}il\, \mathcal{P}(A))$.*

Proof. We only have to check that all the a_i:s in $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ are nilpotent if f is nilpotent. This was done in 1.7 and 1.8. The last part follows from 6.4.

Remark 6.9. The subcategory of $\mathcal{N}il\mathcal{P}(A)$ consisting of all zero maps $0: P \to P$ can be identified with $\mathcal{P}(A)$. It follows that $K_0(\mathcal{N}il\mathcal{P}(A))$ contains $K_0(\mathcal{P}(A)) = K_0(A)$ as a direct summand (see Bass [2] p. 652)

$$K_0(\mathcal{N}il\mathcal{P}(A)) = K_0(A) \oplus \mathrm{Nil}(A).$$

Since $\lambda_t(0) = 1$ we have $K_0(A) \subseteq \operatorname{Ker} \lambda_t$ so the proposition shows that Nil (A) contains $\widetilde{N(A)}_0$ as a direct summand.

Proposition 6.10. *The map*

$$\Psi: K_0(A) \to \{\sum_{i=1}^{s} e_i(1+t)^{n_i};\ n_i \in \mathbf{Z} \text{ and } e_1, \ldots, e_s \text{ are orthogonal idempotents with sum } 1\}$$

defined by $[P] \mapsto \lambda_t(1_P)$ *is a split surjective ring homomorphism. The right hand side considered as a subring of* $\tilde{A}$ *is isomorphic to the ring of all continuous functions from* Spec A *to* $\mathbf{Z}$ *(where* $\mathbf{Z}$ *has the discrete topology). The kernel of* Ψ *is equal to the Jacobson radical of* $K_0(A)$, *which is also equal to* $N(K_0(A))$.

Proof. Given $P \in \mathcal{P}(A)$ with $rkP = n$ let

$$X_j = \{\mathfrak{p} \in \operatorname{Spec} A;\ rkP_{\mathfrak{p}} = j\} \quad \text{(compare the proof of 2.2.)}$$

Let $e_0, e_1, \ldots, e_n$ be the corresponding indempotents in A. Then

$$\lambda_t(1_P) = \sum_{i=0}^{n} e_i(1+t)^i \ \text{ defines } \ \Psi.$$

To construct a right inverse Θ of Ψ consider the map

$$\sum_{i=1}^{k} e_i(1+t)^{n_i} \overset{\Theta}{\mapsto} \left[\bigoplus_{n_i \geq 0} A_i^{n_i}\right] - \left[\bigoplus_{n_j < 0} A_j^{-n_j}\right] = [P] - [Q]$$

where $e_1, \ldots, e_k$ are orthogonal idempotents with sum one, $n_i \in \mathbf{Z}$, and $A_i = Ae_i \in \mathcal{P}(A)$. One verifies that Θ is a ring homomorphism. We want $\lambda_t \circ \Theta = id$.

First

$$(Ae_i)_{\mathfrak{p}} = A_{\mathfrak{p}} e_{i\mathfrak{p}} = \begin{cases} A_{\mathfrak{p}} & \text{if } \mathfrak{p} \in X_i \\ 0 & \text{otherwise,} \end{cases}$$

where X_i is the closed and open subset of Spec A corresponding to e_i. Hence $rk_{\mathfrak{p}} P = n_i$ and $(\lambda_t(1_P))_{\mathfrak{p}} = (1+t)^{n_i}$ for $\mathfrak{p} \in X_i$.

But

$$(\sum_{i=1}^{k} e_i(1+t)^{n_i})_{\mathfrak{p}} = (1+t)^{n_i} \quad \text{for} \quad \mathfrak{p} \in X_i.$$

Furthermore

$$(\sum_{n_j<0} e_j(1+t)^{-n_j})^{-1} = \sum_{n_j<0} e_j(1+t)^{n_j} .$$

and we have shown that $\lambda_t \circ \Theta = id$.

The map

$$\sum_{1}^{k} e_i(1+t)_0^{n_i} \overset{\xi}{\mapsto} f$$

where $f(x) = n_i$ if $x \in X_i$, gives the isomorphism between the ring on the right hand side above and the ring of all continuous functions f: Spec $A \to \mathbf{Z}$.

The composite $\xi \circ \Psi$ is precisely the rank map rk. It follows that

$$\text{Ker}\ \Psi = \text{Ker}\ (rk) = \text{the Jacobson radical of } K_0(A)$$

(for the last statements see Swan [12] p. 169).

COROLLARY 6.11. *Let A be noetherian. Then A has a finite number, say k, of irreducible idempotents and $K_0(A)$ contains $\mathbf{Z}^k$ as a direct summand.*

By the previous results the study of the structure of $\tilde{A}_0$ seems interesting. In case A contains the rational numbers $\tilde{A}_0$ is related to sequences of traces of the powers of a matrix (see 6.13).

Definition 6.12. A sequence $(b_1, b_2, \ldots)$ of elements in A is called a *trace sequence* if there is some $f: P \to P$ with $P \in \mathcal{P}(A)$ such that $b_i = \text{Tr}\,(f^i)$ for all $i \geq 1$.

One may of course assume that P is free.

PROPOSITION 6.13. *Assume that $A \supseteq \mathbf{Q}$.*

(i) *Then there is a ringisomorphism $\phi: \tilde{A} \to \prod_1^\infty A$ where the latter ring can be identified with all sequences under componentwise addition and multiplication.*

(ii) *$\tilde{A}_0$ is isomorphic to the ring of all sequences which are differences of trace sequences.*

Proof. (i) Define ϕ as the composition

$$1 + a_1t + \ldots \mapsto \frac{a_1t + 2a_2t^2 + \ldots}{1 + a_1t + a_2t^2 + \ldots} = b_1t - b_3t^3 \ldots \mapsto (b_1, b_2, b_3, \ldots)$$

The inverse is given by

$$(b_1, b_2 \ldots) \mapsto \exp \int_0^t (b_1 - b_2 s + b_3 s^3 \ldots)$$

where $\int_0^t$ is A-linear and $\int_0^t s^k = \dfrac{t^{k+1}}{k+1}$.

Clearly ϕ is additive (essentially it is the logarithmic derivative). To see that ϕ is multiplicative one uses the same technique as in the proof of 3.4, the key fact being $\operatorname{Tr}(f \otimes g)^i = \operatorname{Tr}(f^i)\operatorname{Tr}(g^i)$.

(ii) The restriction of ϕ to $\tilde{A}_0$ will do. By the exponential trace formula

$$\phi\left(\frac{\lambda_t(f)}{\lambda_t(g)}\right) = (b_i)_1^\infty - (c_i)_1^\infty \quad \text{where} \quad b_i = \operatorname{Tr} f^i \quad \text{and} \quad c_i = \operatorname{Tr} g^i.$$

Remark 6.14. If A is a finite field with q elements then ϕ in (i) is neither injective nor surjective. Indeed $\lambda_t(f^{q^\nu}) = \lambda_t(f)$ for $\nu = 1, 2, \ldots$ In particular $b_{q^\nu} = b_1$ and hence every $(b_i)_1^\infty$ in the image of ϕ must have this property.

Definition 6.15. The *Witt ring* $W(A)$ of A consists of all sequences $(x_i)_1^\infty$ where $x_i \in A$ (Witt vectors) with addition and multiplication defined such that for every $n \geq 1$

$$(x_i)_1^\infty \mapsto \sum_{d|n} d x_d^{n/d}$$

is a ring homomorphism $W(A) \to A$. The right hand side $b_n = \sum_{d|n} d x_d^{n/d}$ is called the n:th ghost component of $(x_i)_1^\infty$. We have a ring isomorphism $W(A) \to \tilde{A}$ defined by

$$(x_i)_1^\infty \mapsto \prod_{i=1}^{\infty} (1 - x_i(-t)^i).$$

Many of the previous results can be formulated in the Witt ring instead of $\tilde{A}$. E.g. 6.6. becomes

PROPOSITION 6.16. *If A is a PID ($A = B[X]$ where B is a PID)* or A is a regular local ring of dimension ≤ 2 then $K_0(\operatorname{End} \mathcal{P}(A))$ *is isomorphic with the subring $W_0(A)$ of $W(A)$ consisting of all Witt vectors having differences of trace sequences as ghost components.*

Thus we have four rings: $K_0(\operatorname{End} \mathcal{P}(A))$, $\tilde{A}_0$, the ring of differences of trace sequences and $W_0(A)$. They are all isomorphic if A is a field of characteristic zero. In case A is also algebraically closed they are also isomorphic to the group ring $\mathbf{Z}[A^*]$ where A^* is the multiplicative group of non-zero elements in A. The isomorphism $\tilde{A}_0 \to \mathbf{Z}[A^*]$ is given by

$$\prod_i (1 + \lambda_i t)^{\nu_i} \mapsto \sum_i \nu_i \lambda_i$$

and is actually defined for any algebraically closed field.

Assume now that $f: P \to P$ is nilpotent, say $f^{m+1} = 0$ and $rkP = n$. Consider the image $(x_i)_1^\infty$ in $W(A)$ of $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$. Since x_k is a polynomial of weight k in $a_1, a_2, \ldots, a_k$ we find (using 1.7) that all x_i are nilpotent and $x_k = 0$ if $k > mn$. We can now reformulate 6.8 as follows.

PROPOSITION 6.17. *There is a surjective ring homomorphism from* $K_0(\mathcal{N}il\,\mathcal{P}(A))$ *onto the ring of Witt vectors* $(x_i)_1^\infty$ *where almost all* $x_i = 0$ *and all* x_i *are nilpotent. The latter is a direct summand (as abelian group) of* Nil (A).

PROPOSITION 6.18. *The following are equivalent for a sequence* $(b_1, b_2, \ldots)$ *in* A

(i) $(b_1, b_2, \ldots)$ *is a trace sequence,*

(ii) *there exist* $a_1, a_2, \ldots, a_n$ *in* A *such that*

$b_1 = a_1$

$b_2 = a_1 b_1 - 2a_2$

$b_3 = a_1 b_2 - a_2 b_1 + 3a_3$ *(Newton's formulas)*

$\ldots$

$b_n = a_1 b_{n-1} - a_2 b_{n-2} + \ldots + (-1)^n a_{n-1} b_1 + (-1)^{n+1} n a_n$

and

$b_{n+i} - a_1 b_{n+i-1} + \ldots + (-1)^n a_n b_i = 0$ *for all* $i \geq 1$,

(iii) *there exists an integral extension* $A' \supseteq A$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n \in A'$, *zeroes of a monic polynomial in* $A[t]$ *of degree* n, *such that*

$$b_i = \sum_{\nu=1}^{n} \lambda_\nu^i \quad \text{for all} \quad i \geq 1,$$

(iv) *(if* $A \supseteq \mathbf{Q}$*)*

$$\exp\left(-\sum_1^\infty \frac{b_i}{i}(-t)^i\right)$$

is a polynomial.

Proof. (i) $\Rightarrow$ (ii): Assume that $b_i = \operatorname{Tr}(f^i)$ where $f: P \to P$ with $P \in \mathcal{P}(A)$ and $rkP = n$. Assume that $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$. Comparing the coefficients on both sides in the exponential trace formula we get Newton's formulas.

(ii) $\Rightarrow$ (i): Assume that $(b_1, b_2, \ldots)$ satisfies the condition (ii). Let $f: A^n \to A^n$ be such that $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$. The exponential trace formula then gives $b_i = \operatorname{Tr}(f^i)$.

(i) $\Rightarrow$ (iii): Assume that $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ and $b_i = \operatorname{Tr}(f^i)$. Since $t^n \lambda_{1/t}(f)$ is a monic polynomial there exists an integral extension A' of A such that $t^n \lambda_{1/t}(f)$ splits into linear factors in $A'[t]$ (Bass [2], p. 118, Lemma 5.10). It follows that

$$\lambda_t(f) = \prod_{\nu=1}^{n} (1 + \lambda_\nu t) \quad \text{with} \quad \lambda_\nu \in A'.$$

Taking logarithmic derivatives on both sides and comparing with the exponential trace formula gives $b_i = \sum_{\nu=1}^n \lambda_\nu^i$.

(iii) $\Rightarrow$ (ii): Assume that $\lambda_1, \ldots, \lambda_n$ are zeroes of $t^n - a_1 t^{n-1} + \ldots + (-1)^n a_n$ with $a_1, \ldots, a_n$ in A. Then $b_i = \sum_{\nu=1}^n \lambda_\nu^i$ and $a_1, \ldots, a_n$ satisfy Newton's formulas in (ii). In particular we have $b_i \in A$.

(i) $\Rightarrow$ (iv): see 1.10.

(iv) $\Rightarrow$ (ii): Taking logarithmic derivatives of

$$\exp\left(-\sum_1^\infty \frac{b_i}{i}(-t)^i\right) = 1 + a_1 t + \ldots + a_n t^n$$

and comparing coefficients we get (ii).

Example 6.19. The Fibonacci sequence $(1, 3, 4, 7, 11, 18, \ldots)$ is a trace sequence in $\mathbf{Z}$. We have $b_{i+2} - b_{i+1} - b_i = 0$, so $a_1 = 1$ and $a_2 = -1$. The initial conditions $b_1 = a_1 = 1$ and $b_2 = a_1 b_1 - 2a_2 = 3$ are satisfied. We get $\lambda_t(f) = 1 + t - t^2$ and the corresponding matrix

$$f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

PROPOSITION 6.20. *If A is a finite ring then a trace sequence is periodic. If the trace sequence comes from $f\colon P \to P$ with $\mathrm{rk}P = n$ then the period is at most $k^n - 1$ where k is the number of elements in A.*

Proof. Assume that $b_i = \mathrm{Tr}(f^i)$ with $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$. Then $b_{n+i} = a_1 b_{n+i-1} - a_2 b_{n+i-2} + \ldots \pm a_n b_i$ for $i \geq 1$ by 6.14 (ii).

Hence an element in the trace sequence is completely determined by the n preceding elements. There are only k^n choices of these preceding n elements. Thus among $k^n + n$ consecutive b_i:s there must be two identical sets of n consecutive b_i:s. Thus the period is at most $k^n - 1$.

Remark 6.21. The maximal period $k^n - 1$ may occur as the Fibonacci sequence (mod 2) shows $(1, 1, 0, 1, 1, 0, \ldots)$ with $k = 2$ and $n = 2$. (See 6.19.)

Remark 6.22. The sequence of maps $f, f^2, f^3, \ldots$ is also periodic if A is finite. If A has k elements and f is represented by an $n \times n$-matrix then two maps in the sequence $f, f^2, \ldots, f^{k^{n^2}+1}$ must coincide since there are at most k^{n^2} distinct $n \times n$-matrices.

PROPOSITION 6.23. *Let A be a finite field with q elements. Assume that $b_i = \mathrm{Tr}(f^i)$ with $\lambda_t(f) = 1 + a_1 t + \ldots + a_n t^n$ irreducible in $A[t]$. Then the period of the trace sequence $(b_1, b_2, \ldots)$ divides $q^n - 1$.*

Proof. Let $\lambda_t(f) = \prod_{\nu=1}^n (1 + \lambda_\nu t)$ be the factorization of $\lambda_t(f)$ with $\lambda_\nu \in K$ where K is the splitting field of $\lambda_t(f)$ over A.

Then $b_i = \sum_{\nu=1}^{n} \lambda_\nu^i$. Now $A[\lambda_\nu]$ is a field with q^n elements and $\lambda_\nu^{q^n-1} = 1$ in $A[\lambda_\nu]$ and hence in K. It follows that $b_{i+q^n-1} = b_i$ for all $i \geq 1$. Thus the period of $(b_1, b_2, \ldots)$ divides $q^n - 1$.

COROLLARY 6.24. *If $\lambda_t(f)$ is a product of irreducible polynomials of degrees $n_1, n_2, \ldots, n_s$ respectively then the period of the trace sequence $(\mathrm{Tr}\,(f^i))_1^\infty$ divides the l.c.m. of $q^{n_1} - 1, q^{n_2} - 1, \ldots, q^{n_s} - 1$.*

Remark 6.25. It seems to be quite hard to predict the period from the characteristic polynomial $\lambda_t(f)$. The following results are not too useful for practical computations.

PROPOSITION 6.26. *Given $b_i = \mathrm{Tr}\,(f^i)$.*
(i) *Let $q \in A[t]$ be any polynomial such that $q(f) = 0$ (e.g. $q = t^n\lambda_{-1/t}(f)$ or $q =$ a minimal polynomial of f). If $q \mid t^r - 1$ then $(b_i)_1^\infty$ is periodic and the period s divides r.*
(ii) *Conversely assume that $(b_i)_1^\infty$ is periodic with period s. Assume further that A is a UFD and $\lambda_t(f)$ is irreducible of degree ≥ 1. Then $t^n\lambda_{-1/t}(f) \mid t^s - 1$.*

Proof. (i) We have $t^r - 1 = q(t)h(t)$ for some h in $A[t]$. Since $q(f) = 0$ we get $f^r = 1$ so $f^{r+\nu} = f^\nu$ for all $\nu \geq 1$. It follows $b_{\nu+r} = b_\nu$ and $s \mid r$.

(ii) The exponential trace formula gives

$$\frac{d}{dt}\lambda_t(f) = \lambda(f)(b_1 - b_2t + bt^2 \ldots) = \lambda_t(f)(b_1 - b_2t + \ldots - (-1)^s b_s t^{s-1})(1 - (-t)^s)^{-1}$$

since $b_{i+s} = b_i$.

Hence $\lambda_t(f) \mid (1 - (-t)^s) \cdot \dfrac{d}{dt}\lambda_t(f)$ and $\lambda_t(f) \mid (1 - (-t)^s)$ which implies $t^n\lambda_{-1/t}(f) \mid (t^s - 1)$.

COROLLARY 6.27. *Assume that A is a UFD and that $\lambda_t(f)$ is irreducible. Then $(b_i)_1^\infty$ is periodic if and only if*

$$t^n\lambda_{-1/t}(f) \mid t^r - 1$$

for some $r \geq 1$ and the period s is the smallest r with this property.

Remark 6.28. If $\lambda_t(f)$ is not irreducible but the product $\lambda_t(f) = h_1h_2 \ldots h_k$ where $h_1, \ldots, h_k$ are irreducible of degrees $n_1, \ldots, n_k$ respectively, then the period is the l.c.m. of $s_1, s_2, \ldots, s_k$ where s_i is the smallest integer > 0 such that

$$t^{n_i}h_i(-1/t) \mid t^{s_i} - 1.$$

Example 6.29. Let $A = Z/(13)$ and

$$f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then $\lambda_t(f) = 1 + t + t^3 = (1 - 2t)(1 + 3t - 6t^2)$ where $1 + 3t - 6t^2$ is irreducible. We get

$$t^3 \lambda_{-1/t}(f) = (t + 2)(t^2 - 3t + 6)$$

Now $t + 2 | t^6 - 1$ and $t^2 - 3t + 6 | t^{168} - 1$ since the splitting field of $t^2 - 3t + 6$ has $13^2 = 169$ elements.

Thus $6 | s$ and $s | 168$ where s is the period of $\mathrm{Tr}\,(f^i) = (1, 1, 4, 5, 6, 10, \ldots)$. By actually computing the period one finds $s = 168$ and hence 168 is the smallest integer $r \geq 0$ such that $t^2 - 3t + 6 | t^r - 1$.

Added in proof: In a paper »The Grothendieck ring of the category of endomorphisms», to appear in J. Algebra, the author proves Theorem 6.6 for any commutative ring.

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