A unique continuation theorem for second order parabolic differential operators

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1. Introduction

The purpose of this paper is to show that the unique continuation results in [10] extend naturally to the setting of second order parabolic operators. Previously, we showed that if $P(x, D)$ is a second order elliptic differential operator with a C^∞ real principal part, and if $|P(x, D)u| \leqq |Vu|$ for some $V \in L^{n/2}_{\text{loc}}(\mathbf{R}^n)$, then u must vanish identically if it vanishes in an open subset. In this work (x, t) will denote a generic point in $\mathbf{R}^n \times \mathbf{R}$, with $n \geqq 2$, and we shall assume that $P(x, t, D)$ is a second order elliptic operator acting on $\mathbf{R}^n$ with bounded coefficients whose principal part is real and C^∞. We shall be concerned with parabolic operators of the form

$$L = L(x, t, D, D_t) = \frac{\partial}{\partial t} + P(x, t, D). \tag{1.1}$$

Here we are using the notation $D = D_x = i^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. The natural (non-isotropic) dilations associated to L are given by

$$\delta_\varepsilon(x, t) = (\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n, \varepsilon^2 t), \quad \varepsilon > 0, \tag{1.2}$$

and the homogeneous dimension of $\mathbf{R}^n \times \mathbf{R}$ with this dilation structure is $n+2$. Thus, the natural condition to impose on the potentials $V(x, t)$ in the unique continuation theorem for L is that $V \in L^{(n+2)/2}_{\text{loc}}(\mathbf{R}^n \times \mathbf{R})$.

To state our chief result we first need to recall the definition of the normal set $N(F) \subset T^*(\mathbf{R}^n \times \mathbf{R}) \setminus 0$ associated to a closed subset $F \subset \mathbf{R}^n \times \mathbf{R}$. $N(F)$ will be the set of all $(x_0, t_0, \xi_0, \tau_0)$ where $(x_0, t_0) \in F$, $0 \neq (\xi_0, \tau_0) \in \mathbf{R}^n \times \mathbf{R}$, and

$$\Psi(x, t) \leqq \Psi(x_0, t_0) \quad \text{when} \quad (x_0, t_0) \in F,$$

The author was supported in part by a Sloan Fellowship, a Presidential Young Investigator Award, and NSF grant DMS—8805814.

for some real valued C^∞ function satisfying $d\Psi(x_0, t_0)=\pm(\xi_0, \tau_0)$. If the coefficients of L were analytic, then the classical Holmgren uniqueness theorem (see [5, Chpt. 8]) implies that a solution u of $Lu=0$ must have the property that if $(x_0, t_0, \xi_0, \tau_0)\in N(\operatorname{supp} u)$ then $\xi_0=0$. In other words, there is always unique continuation across non-characteristic hypersurfaces. We shall allow singular potentials, and our main result is:

Theorem 1.1. *Suppose that $Lu\in L^2_{\mathrm{loc}}(X)$ where X is a connected open subset of $\mathbf{R}^n\times\mathbf{R}$. Then if*

$$|Lu(x, t)| \leqq |Vu| \tag{1.3}$$

for some $V\in L^{(n+2)/2}_{\mathrm{loc}}(X)$, it follows that $N(\operatorname{supp} u)$ is non-characteristic for L. Thus, if $(x_0, t_0, \xi_0, \tau_0)\in N(\operatorname{supp} u)$, one must always have $\xi_0=0$.

Corollary 1.2. *Let Ω be a connected open subset of $\mathbf{R}^n$ and $T>0$. Assume that $V\in L^{(n+2)/2}_{\mathrm{loc}}(\Omega\times[-T, T])$, and suppose that (1.3) holds. Then if u vanishes in an open subset $\mathcal{O}\subset\Omega\times[-T, T]$, it follows that u must vanish identically in the horizontal component of $\mathcal{O}$. This means that $u(x, t_0)=0$ if $(x_0, t_0)\in\mathcal{O}$ for some x_0.*

If $V\in L^\infty_{\mathrm{loc}}$ this result is due to Nirenberg [7] in the constant coefficient case and to Saut and Scheurer [8] under the weaker assumptions that the leading coefficients are C^1. Our methods allow one to obtain the essentially optimal results concerning the potential V in the differential inequality; however, they require that one make what are probably non-optimal regularity assumptions on the coefficients involved. Nonetheless, even in the constant coefficient case, it seems that there are no previous unique continuation theorems of this type involving $V\in L^q_{\mathrm{loc}}$ for $q<\infty$; however, Garofalo and Kenig [3] proved certain results for constant coefficient operators under restrictive assumptions regarding the support of u. Similar results for other constant coefficient parabolic differential operators were also proved in Kenig and Sogge [6].

To prove Theorem 1.1 we must show that there is local unique continuation across any non-characteristic hypersurface S. If, locally, such a surface is given by the equation $\Psi(x, t)=0$, there is no loss of generality in assuming that $\Psi(0, 0)=0$ and that $(\partial\Psi/\partial x_n)(0, 0)\neq 0$. This means that S can be locally written as a graph $x_n=\psi(x', t)$, where $x'=(x_1, \dots, x_{n-1})$ denote the first $(n-1)$ coordinates. Furthermore, by making the change of variables $(x', x_n-\psi(x', t), t)$, which preserves the parabolic character of L, we see that we can assume that S is the hyperplane $x_n=0$. Next, if, as in Nirenberg [7], we now use the Holmgren transform

$$(x', x_n, t) \to (x', x_n+t^2+|x'|^2, t),$$

which takes the hyperplane $x_n=0$ to the parabola $x_n=t^2+|x'|^2$, we see that we can assume further that $\operatorname{supp} u\cap\{(x, t)\colon x_n\leqq 0\}=(0, 0)$. Finally, by using the natural geodesic coordinates associated to the principal part of P, we can make one more change of variables so that, in the new coordinates, u has the same support properties, but, modulo a lower order differential operator in x, L is of the form

$$L = \frac{\partial}{\partial t} + D_n^2 + \sum_{j,k<n} g^{jk}(x,t) D_j D_k \tag{1.1'}$$

where g^{jk} is a positive definite matrix.

Putting together all of these straightforward reductions, we conclude that we need only show that u vanishes near $(0, 0)$ when $\operatorname{supp} u$ and L are as above. But standard arguments as in, say, [10] now imply that Theorem 1.1 is a corollary of the following Carleman inequalities.

Theorem 1.3. *Let L be as in* (1.1') *and put*

$$w_\varepsilon = x_n - x_n^2/2\varepsilon. \tag{1.4}$$

If $\varepsilon > 0$ is sufficiently small but fixed and if $\lambda > 0$ is sufficiently large

$$\sum_{|\alpha| \le 1} \lambda^{1+1/(n+1)-|\alpha|} \|e^{-\lambda w_\varepsilon} D_x^\alpha v\|_{L^p} + \|e^{-\lambda w_\varepsilon} v\|_{L^{p'}} \leqq C \|e^{-\lambda w_\varepsilon} L v\|_{L^p} \tag{1.5}$$

whenever $v \in C_0^\infty$ is supported in a sufficiently small neighborhood of $(0, 0)$. *Here $p = 2(n+2)/(n+4)$ and p' is the conjugate exponent $p' = 2(n+2)/n$ which forces $1/p - 1/p' = 2/(n+2)$.*

The proof of Theorem 1.3 is modeled after the corresponding result for second order elliptic operators in [10]. Replacing v by $e^{\lambda w_\varepsilon} v$ one sees that it is enough to obtain the appropriate estimates (see § 3) for the conjugated operators

$$L_\lambda = e^{-\lambda w_\varepsilon} L e^{\lambda w_\varepsilon}.$$

Note that (for fixed (x, t)) the symbol of this operator only vanishes for certain (ξ, τ) with

$$\|(\xi, \tau)\| \approx \lambda$$

where $\|\cdot\|$ always denotes the parabolic norm:

$$\|(\xi, \tau)\| = \sqrt{|\xi|^2 + |\tau|}. \tag{1.6}$$

Inverting L_λ microlocally far enough away from the zero set of the symbol is easy. To handle the other part of the inverse, as in [10], one constructs a parametrix using singular Fourier integrals with complex phase. Choosing the "correct" phase function is the key step, and, as before, it is constructed from an eikonal equation which comes from the factor of the symbol of L_λ which vanishes for large (ξ, τ). This phase function will basically be a sum of two types: the main part will reflect the Euclidean dilations, while the other part will vanish of order two in the space variables along the diagonal, and will reflect the parabolic dilations. To make the necessary estimates for the non-trivial part of the parametrix (and the associated remainder term) we shall use a parabolic version of the equivalence of phase functions and certain variants of the oscillatory integral theorem of Carleson—Sjölin, Hörmander, and Stein (see [11]) that was used in the study of Carleman inequalities for elliptic operators. It is surprising that essentially "elliptic oscillatory integral theorems" alone should be what is needed for proving our Carleman inequalities for parabolic operators, rather than say a variant of the "parabolic restriction theorem" of Strichartz [12]. However, in a different setting, a similar phenomenon was observed in Seeger [9], where estimates for non-isotropic operators were deduced from operators having the standard dilation structure. The experts might also notice that Lemma 2.5 involves both types of operators that were used by Carleson—Sjölin [1] and Fefferman [2] in their different proofs of the disc multiplier theorem.

This paper is organized as follows. In the next section we shall collect the tools which will be needed for our proof of Theorem 1.3. In § 3 we shall complete the proof, and in an appendix we shall prove the "non-isotropic" oscillatory integral lemma that is used. As usual C will denote a generic constant which is not necessarily the same at each occurrence, and $\log\mu$ will always denote the base-2 logarithm of μ. Finally, we are very grateful to Anders Melin for helpful criticisms and comments.

2. Main tools

The easy part of the parametrix for L_λ will involve pseudo-differential operators whose symbols respect the parabolic structure of $\mathbf{R}^n\times\mathbf{R}$ outlined above. We shall say that a is a parabolic symbol of order m, written as $a\in S^m_{\text{par}}$, if a is C^∞ and satisfies

$$|D_\xi^{\alpha_1}D_\tau^{\alpha_2}D_{x,t}^{\beta}a(x,t,\xi,\tau)|\leqq C_{\alpha\beta}\|(\xi,\tau)\|^{m-|\alpha_1|-2\alpha_2} \tag{2.1}$$

for all multi-indices α_j, β.

As in the usual case, the kernels of parabolic pseudo-differential operators are C^∞ away from the diagonal. Furthermore, it is easy to estimate their size, and one has the following result.

Lemma 2.1. *Let a be in S^m_{par}. Then the kernel associated to a,*

$$K(x,t,y,s)=(2\pi)^{-(n+1)}\iint e^{i[\langle x-y,\xi\rangle+(t-s)\tau]}a(x,t,\xi,\tau)\,d\xi\,d\tau,$$

is C^∞ away from the diagonal $(x,t)=(y,s)$ and

$$|K(x,t,y,s)|\leqq C\|(x-y,t-s)\|^{-(n+2)-m}.$$

For fixed m, the constant depends only on finitely many of the constants in (2.1).

Recall that the non-isotropic dimension of $\mathbf{R}^n\times\mathbf{R}$ that is associated to parabolic pseudo-differential operators as above is *not* $n+1$, but $n+2$. Thus, the estimates in Lemma 2.1 are the natural analogue of those for the usual type of pseudo-differential operators.

Next, to estimate the boundedness of the pseudo-differential operators which will appear in our constructions, we shall require the parabolic analogue of the Hardy—Littlewood—Sobolev inequality.

Lemma 2.2. *Suppose that* $1<p<q<\infty$ *and that* $1/p-1/q=\alpha/(n+2)$. *Let*

$$I_\alpha f(x,t)=\iint f(x-y,t-s)\|(y,s)\|^{-(n+2)+\alpha}\,dy\,ds.$$

Then

$$\|I_\alpha f\|_{L^q(\mathbf{R}^n\times\mathbf{R})}\leqq C_{\alpha pq}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R})}.$$

We shall only require the estimates for the special case where $\alpha=2$.

To prove Lemma 2.2, one first notices that, by Minkowski's integral inequality, if we fix t, it is possible to control the norm over $\mathbf{R}^n$ as follows:

$$\|I_\alpha f\|_{L^q(\mathbf{R}^n,dx)}\leqq\int_{-\infty}^{\infty}\Big\|\int f(x-y,t-s)\big(\sqrt{|x-y|^2+|s-t|}\big)^{-(n+2)+\alpha}\,dy\Big\|_{L^q(\mathbf{R}^n,dx)}ds.$$

But, Young's inequality for convolution in $\mathbf{R}^n$, and the relationship between p and q give

$$\Big\|\int f(x-y,t-s)\big(\sqrt{|x-y|^2+|t-s|}\big)^{-(n+2)+\alpha}\,dy\Big\|_{L^q(\mathbf{R}^n,dx)}$$
$$\leqq C|t-s|^{-1+(1/p-1/q)}\Big(\int|f(x,t-s)|^p\,dx\Big)^{1/p}.$$

Thus,

$$\|I_\alpha f\|_{L^q(\mathbf{R}^n\times\mathbf{R})}\leqq C\Big(\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\|f(\cdot,t-s)\|_{L^p(\mathbf{R}^n,dx)}|t-s|^{-1+(1/p-1/q)}\,ds\Big]^q\,dt\Big)^{1/q},$$

and so the desired estimate follows from applying the usual fractional integration theorem for $\mathbf{R}$.

Similar arguments, which reduce the proof of the appropriate boundedness of operators acting on functions of $\mathbf{R}^n\times\mathbf{R}$ to estimating an operator acting on functions of one less variable, will be used throughout.

Another tool we shall require is a parabolic version of the equivalence of phase functions for pseudo-differential operators. This will be useful in constructing the non-trivial part of the parametrix for L_λ.

Suppose that P_φ is a linear operator of the form

$$(P_\varphi u)(x,t)=(2\pi)^{-(n+1)}\iint e^{i\varphi(x,t,y,s,\xi,\tau)}P(x,t,y,s,\xi,\tau)u(y,s)\,d\xi\,d\tau\,dy\,ds,$$

where the symbol $P\in S^m_{\mathrm{par}}$ is assumed to have fixed compact support in (x,t,y,s). We shall assume that φ is C^∞, $\operatorname{Im}\varphi\geqq 0$, and that

$$|\nabla_{\xi,\tau}\varphi|\geqq c|(x-y,t-s)|, \tag{2.2}$$

for some $c>0$. Note that the last assumption involves the Euclidean norm. On the other hand, if we let $\varphi_1=\langle x-y, \xi\rangle+(t-s)\tau$ be the usual phase function for pseudo-differential operators, then we shall also assume that

$$\varphi-\varphi_1 = O(\|(x-y, t-s)\|^2 \|(\xi, \tau)\|), \tag{2.3}$$

and so, in particular,

$$\nabla_\xi(\varphi-\varphi_1) = O(\|(x-y, t-s)\|^2) \quad \text{and} \quad \frac{\partial}{\partial\tau}(\varphi-\varphi_1) = O(\|(x-y, t-s)\|^2 \|(\xi, \tau)\|^{-1}).$$

Under these hypotheses, we have the following result which is the parabolic analogue of the usual equivalence of phase function theorem for pseudo-differential operators.

Lemma 2.3. *Let φ be as above. Then if $P(x, t, y, s, \xi, \tau)\in S^m_{\text{par}}$, P_φ is a parabolic pseudo-differential operator of order m. Moreover, modulo an operator of order $(m-1)$, P_φ equals*

$$(2\pi)^{-(n+1)}\iint e^{i[\langle x-y, \xi\rangle+(t-s)\tau]}P(x, t, x, t, \xi, \tau)\,u(y, s)\,d\xi\,d\tau\,dy\,ds.$$

Proof. To establish this result we need only make some straightforward modifications of proof of the usual result in Hörmander [4]. We first let $\varphi_0(x, t, y, s, \xi, \tau)=\varphi(x, t, y, s, \xi, \tau)$, and, as above, $\varphi_1(x, t, y, s, \xi, \tau)=\langle x-y, \xi\rangle+(t-s)\tau$. Then for $0<\varepsilon<1$, we put $\varphi_\varepsilon(x, t, y, s, \xi, \tau)=(1-\varepsilon)\varphi_0+\varepsilon\varphi_1$, and

$$(P_\varepsilon u)(x, t) = (2\pi)^{-(n+1)}\iint e^{i\varphi_\varepsilon}P(x, t, y, s, \xi, \tau)u(y, s)\,d\xi\,d\tau\,dy\,ds.$$

Since $\varphi_0=\varphi$ satisfies (2.2) and (2.3), we can assume that for all $0\leqq\varepsilon\leqq 1$,

$$|\nabla_{\xi,\tau}\varphi_\varepsilon| \geqq c|(x-y, t-s)| \quad \text{if} \quad P\neq 0 \quad \text{and} \quad \|(\xi, \tau)\| \quad \text{large}. \tag{2.4}$$

We may have to decrease the support of the symbol P near the diagonal; however, since (2.4) holds for $\varepsilon=0$, P_0-P_φ would have to be an integral operator with smooth kernel.

Next, notice that

$$\left(\frac{\partial}{\partial\varepsilon}\right)^j P_\varepsilon u = (2\pi)^{-(n+1)}\iint [i(\varphi_1-\varphi_0)]^j e^{i\varphi_\varepsilon}P(x, t, y, s, \xi, \tau)u(y, s)\,d\xi\,d\tau\,dy\,ds. \tag{2.5}$$

The symbol here

$$[i(\varphi_1-\varphi_0)]^j P(x, t, y, s, \xi, \tau)$$

is *a priori* only in S^{m+j}_{par}, but (2.3) implies that it vanishes like $\|(x-y, t-s)\|^{2j}$ near the diagonal. To exploit this let

$$H = \partial/\partial\tau+\Delta_\xi, \quad \text{and} \quad a_\varepsilon = e^{-i\varphi_\varepsilon}He^{i\varphi_\varepsilon}.$$

Clearly, (2.3) and (2.4) imply that there is a constant $c>0$ so that

$$|a_\varepsilon| > c\|(x-y, t-s)\|^2 \quad \text{for large} \quad \|(\xi, \tau)\|.$$

To use this, notice that if we let A be the adjoint of the operator $(1/a_\varepsilon)H$, then the kernel of the operator in (2.5) equals $(2\pi)^{-(n+1)}$ times

$$\int e^{i\varphi_\varepsilon}A^j\{[i(\varphi_1-\varphi_0)]^j P(x,t,y,s,\xi,\tau)\}\,d\xi\,d\tau.$$

Since one can now check that the symbol in this oscillatory integral is actually in S^{m-j}_{par}, it follows that this kernel becomes arbitrarily smooth as $j\to+\infty$, since (2.4) holds.

To use this set

$$Q_j = \frac{(-1)^j}{j!}\left(\frac{d}{d\varepsilon}\right)^j P_\varepsilon\big|_{\varepsilon=1}.$$

Then, Taylor's formula gives

$$P_0 = \sum_0^{k-1} Q_j + (-1)^k/k!\int_0^1 \varepsilon^{k-1}(d/d\varepsilon)^k P_\varepsilon\,d\varepsilon.$$

Thus, if we let Q be defined by the formal series $\sum_0^\infty Q_j$, it follows that Q is a pseudo-differential operator, and moreover P_0-Q has a smoothing kernel. Finally since, modulo a pseudo-differential operator in S^{m-1}_{par}, Q_0 equals the pseudo-differential operator with symbol $P(x,t,x,t,\xi,\tau)$ we are done. ∎

The last ingredient we shall require is a variant of the standard oscillatory integral theorems of Carleson—Sjölin, Hörmander, and Stein in [11]. These will allow us to estimate the mapping properties of the main terms in the parametrix for L_λ. Given the non-isotropic nature of the dilations (1.2) associated to L, one should not expect to be able to use the results in [11] alone; however, it is fortunate that in the arguments to follow, one can make a change of variables that allows one to apply easy variants of the usual oscillatory integral theorems.

The operators we shall need to control are of the form

$$R_{\mu,j}f(y) = \int_{\mathbf{R}^{n-1}} 2^{(n-1)j/2} e^{i2^j\mu\psi(y,z)} a_j(y,z) f(z)\,dz, \tag{2.6}$$

for $0\leqq j\leqq \log\mu$, and send functions defined in $\mathbf{R}^{n-1}$ to functions in $\mathbf{R}^n$. We shall assume that both ψ and a_j are in $C^\infty(\mathbf{R}^n\times\mathbf{R}^{n-1})$, with the a_j being supported in a fixed small neighborhood of $\mathcal{N}$ of $(0,0)$ and ψ having non-negative imaginary part. In addition we shall require several technical assumptions.

First, as in Stein [11], we shall assume that ψ is non-degenerate in the sense that its mixed Hessian has maximal rank. Specifically, we require

$$\det\left(\frac{\partial^2\psi}{\partial y_j\partial z_k}\right)_{1\leqq j,\,k\leqq n-1} \neq 0 \quad \text{on } \mathcal{N}. \tag{2.7}$$

In addition, we shall impose the following Carleson—Sjölin condition:

$$\frac{\partial^2\psi}{\partial z_j\partial y_n}(0,0) = 0 \quad \forall j, \quad \text{but} \quad \det\left(\frac{\partial^2}{\partial z_j\partial z_k}[\partial\psi/\partial y_n]\right) \neq 0 \quad \text{on } \mathcal{N}. \tag{2.8}$$

As we shall see, the non-degeneracy condition (2.8) insures that oscillatory integral operators with phase function ψ have better bounds as operators from L^p into itself, rather than form L^p into the dual space $L^{p'}$, for certain exponents p.

Remark 2.4. The model case occurs when

$$\psi(y, z) = |(y_1 - z_1, \dots, y_{n-1} - z_{n-1}, y_n - 1)|$$

and, as above, both y and z are assumed to be close the origin. The phase functions which will occur in the proof of the Carleman inequalities will be related to this one.

Now let us describe the hypotheses for the amplitudes a_j above. In addition to assuming that they have fixed compact support, we shall assume that

$$a_j(y, z) = 0 \quad \text{if} \quad |(y_1 - z_1, \dots, y_{n-1} - z_{n-1})| \geqq 2^{-j}. \tag{2.9}$$

Also, we shall assume that the derivatives of a_j satisfy the natural bounds associated to this support property:

$$|D^{\alpha}_{y',z} D^{\beta}_{y_n} a_j(y, z)| \leqq C_{\alpha\beta} 2^{j|\alpha|}, \quad y' = (y_1, \dots, y_{n-1}), \tag{2.10}$$

for constants $C_{\alpha\beta}$ independent of j.

Having stated the various technical assumptions, the desired result is the following.

Lemma 2.5. *Let ψ and a_j be as above, and let $R_{\mu,j}$ be as in* (2.6), *where $\mu > 1$, and $0 \leqq j \leqq \log \mu$. Then, if $\mathcal{N}$ above is small enough,*

(2.11)

$$\|R_{\mu,j} f\|_{L^{p'}(\mathbf{R}^n)} \leqq C 2^{(n-1)j(1/p - 1/p')/2} \mu^{-(n-1)/p'} \|f\|_{L^p(\mathbf{R}^{n-1})}, \quad 1 \leqq p \leqq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Also, if $1 \leqq p \leqq 2$, and $r = \dfrac{n+1}{n-1} p'$, one has the estimates

$$\|R_{\mu,j} f\|_{L^r(\mathbf{R}^n)} \leqq C 2^{(n-1)j(1/p - 1/r)/2} \mu^{-n/p'} \|f\|_{L^p(\mathbf{R}^{n-1})}. \tag{2.12}$$

The constant C in these inequalities depends only on finitely many of the constants in (2.10) *and remains bounded when ψ belongs to a bounded subset of C^∞ functions satisfying* (2.7) *and* (2.8).

In the case where $j = 0$, this result is the usual oscillatory integral theorem in [11]. On the other hand, in the other extreme case where $j = \log \mu$, the reader can check that (2.11) and (2.12) are a trivial consequence of (2.9), and thus oscillation in (2.6) is irrelevant. The proof of Lemma 2.5 for the other cases is a straightforward modification of the arguments in [10]; however, for the sake of completeness it will be given in an appendix.

For the proof of the Carleman inequalities, we shall actually need a slight variant the inequalities (2.12). By an easy argument which uses Hölder's inequality, the reader can check that (2.9) and (2.12) give

$$\|R_{\mu,j}f\|_{L^r(\mathbf{R}^n)} \leqq C2^{-(n-1)j(1/p-1/r)/2}\mu^{-n/p'}\|f\|_{L^r(\mathbf{R}^{n-1})} \quad \text{if} \quad \frac{2(n+1)}{n-1} \leqq r \leqq \infty, \tag{2.13}$$

and if p and r are related as in (2.12).

As we indicated in Remark 2.4, we shall only be interested in a special case of Lemma 2.5. This will allow us to prove estimates for certain oscillatory integral operators sending functions of n variables now to functions of n variables. These new operators will have phase functions φ which are close to the model function $|x-y|$ in the sense that

$$\left|D^\alpha[\varphi(x,y)-|x-y|]\right| \leqq \varepsilon_0, \quad 0 \leqq |\alpha| \leqq N, \tag{2.14}$$

and will be of the form

$$S_{\mu,j}f(x) = \int_{\mathbf{R}^n} 2^{(n-1)j/2} e^{i2^j\mu\varphi(x,y)} a_j(x,y) f(y)\, dy, \quad x \in \mathbf{R}^n.$$

The analogues of (2.9) and (2.10) for the new function $a_j \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ will be that

$$a_j(x,y) = 0 \quad \text{if} \quad |x-y| \notin [1/2, 1] \quad \text{or} \quad |x'-y'| \geqq 2^{-j} \tag{2.9'}$$

$$|D^\alpha_{x',y'} D^\beta_{x_n,y_n} a_j(x,y)| \leqq C_{\alpha\beta} 2^{j|\alpha|}. \tag{2.10'}$$

If one keeps in mind Remark 2.4, then the usual Carleson—Sjölin arguments (see [1], [11] and the arguments in § 4) together with (2.11) and (2.13) imply that the operators $S_{\mu,j}$ satisfy the same bounds as $R_{\mu,j}$. In particular, we know the $L^p \to L^{p'}$ norm of $S_{\mu,j}$ when p is as in the Carleman inequality. Also, by (2.11) and (2.13), one knows bounds for its $L^p \to L^p$ norm when p is equal to 2 or $2(n+1)/(n-1)$. But, by interpolating between these two estimates and using duality we can also estimate its $L^q \to L^q$ norm when q is one of the exponents in the Carleman inequality. In fact, we can conclude that, for these exponents, the norm must satisfy better bounds[1] than the estimates in (2.13) for $r=2(n+1)/(n-1)$. We collect these facts which will be useful later on in the following.

Corollary 2.6. *Let a_j and φ be as above and assume that* $\operatorname{Im}\varphi \geqq 0$. *Then if, in addition, ε_0 is sufficiently small and N is sufficiently large in* (2.14),

$$\|S_{\mu,j}f\|_{L^{p'}(\mathbf{R}^n)} \leqq C2^{(n-1/n+2)j}\mu^{-(n-1/n+2)(n/2)}\|f\|_{L^p(\mathbf{R}^n)}, \quad p = \frac{2(n+2)}{n+4}. \tag{2.15}$$

[1] For the exponents q which occur in the Carleman inequality, the sharp mapping properties of the $S_{\mu,j}$ from $L^q(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ are not known (even when $j=0$). Fortunately, the bounds corresponding to $q=2(n+1)/(n-1)$ are good enough for our applications.

Also, one has the estimate

(2.16)
$$\|S_{\mu,j}f\|_{L^q(\mathbf{R}^n)} \leqq C2^{-(n-1/n+1)(j/2)}\mu^{-(n-1/n+1)(n/2)}\|f\|_{L^q(\mathbf{R}^n)}, \quad q \leqq \frac{2(n+2)}{n+4}, \quad \frac{2(n+2)}{n}.$$

3. Proof of Theorem 1.3

Recall that $w_\varepsilon = x_n - x_n^2/2\varepsilon$. If $\varepsilon > 0$ is sufficiently small but fixed, and

$$L_\lambda = e^{-\lambda w_\varepsilon} L e^{\lambda w_\varepsilon},$$

we must show that if $\lambda > 0$ is large

$$\sum_{|\alpha|\leqq 1} \lambda^{1+1/(n+1)-|\alpha|}\|D_x^\alpha v\|_{L^p(\mathbf{R}^n\times\mathbf{R})} + \|v\|_{L^{p'}(\mathbf{R}^n\times\mathbf{R})} \leqq C\|L_\lambda v\|_{L^p(\mathbf{R}^n\times\mathbf{R})}, \tag{3.1}$$

when the function v and the exponents p and p' are as in Theorem 1.3.

If we recall (1.1′), then we see that the symbol of L_λ is

$$L_\lambda(x, t, \xi, \tau) = i\tau + \sum_{j,k<n} g^{jk}(x, t)\xi_j\xi_k + \xi_n^2 - 2i\lambda w_\varepsilon' \xi_n - (\lambda w_\varepsilon')^2 - \lambda w_\varepsilon''.$$

Consequently, if we let A_λ be the differential operator whose adjoint A_λ^* has symbol

$$A_\lambda^*(x, t, \xi, \tau) = -i\tau + \sum_{j,k<n} g^{jk}(x, t)\xi_j\xi_k + \xi_n^2 + 2i\lambda w_\varepsilon'\xi_n - (\lambda w_\varepsilon')^2 \tag{3.2}$$

$$= (w_\varepsilon')^2\left[-i\left(\frac{1}{w_\varepsilon'}\right)^2\tau + \left(\frac{1}{w_\varepsilon'}\right)^2 \sum_{j,k<n} g^{jk}(x, t)\xi_j\xi_k + \left(\frac{1}{w_\varepsilon'}\right)^2\xi_n^2 + 2i\lambda\frac{1}{w_\varepsilon'}\xi_n - \lambda^2\right]$$

then (3.1) follows from

$$\sum_{|\alpha|\leqq 1} \lambda^{1+1/(n+1)-|\alpha|}\|D_x^\alpha v\|_{L^p} + \|v\|_{L^{p'}} \leqq C\|A_\lambda v\|_{L^p}. \tag{3.1′}$$

By adapting the calculus in [10], we claim that we can choose a phase function Φ satisfying the hypotheses of Lemma 2.3 so that the integral operator $T^* = T_\lambda^*$ with kernel

$$(2\pi)^{-(n+1)}\int \frac{e^{i\Phi(x,t,y,s,\xi,\tau)}}{A_\lambda^*(y, s, \xi, \tau)}\, d\xi\, d\tau \tag{3.3}$$

is a suitable right parametrix for A_λ^*. To be more specific, we claim that if v is supported in a small ball B around $(0, 0)\in\mathbf{R}^n\times\mathbf{R}$, then Φ can be chosen so that if T is the adjoint of T^*,

$$T(A_\lambda v) = v + Rv, \tag{3.4}$$

where

$$\|T\|_{(L^p(B),\, L^{p'}(B))} \leqq C \quad\text{and}\quad \|D_x^\alpha T\|_{(L^p(B),\, L^p(B))} \leqq C\lambda^{-1-1/(n+1)+|\alpha|}, \tag{3.5}$$

while the remainder term $R = R_\lambda$ only satisfies

$$\|D_x^\alpha R\|_{(L^q(B),\, L^q(B))} \leqq C\lambda^{-1/(n+1)+|\alpha|}, \quad q = p, p', \tag{3.6}$$

for $|\alpha|\leqq 1$. Here we are of course using the notation that $\|\cdot\|_{(L^r(B),L^s(B))}$ denotes the operator norm from $L^r(B)$ to $L^s(B)$. Since it is easy to check that (3.5) and (3.6) would imply (3.1), we are left with constructing Φ and proving these estimates.

Next, notice that (3.2) implies that

$$A_\lambda^*(x,t,\xi,\tau)=0 \Leftrightarrow (\xi,\tau)\in\delta_\lambda(\Sigma),$$

where δ_λ denotes the non-isotropic dilations defined in (1.2), and

(3.7) $\Sigma=\Sigma_{x,t}=\{(\xi,\tau)\colon\ \xi_n^2+\sum_{j,k<n} g^{jk}(x,t)\xi_j\xi_k=(w_\varepsilon')^2 \text{ and } \tau=2w_\varepsilon'\xi_n\}.$

This set is shown in the following figure.

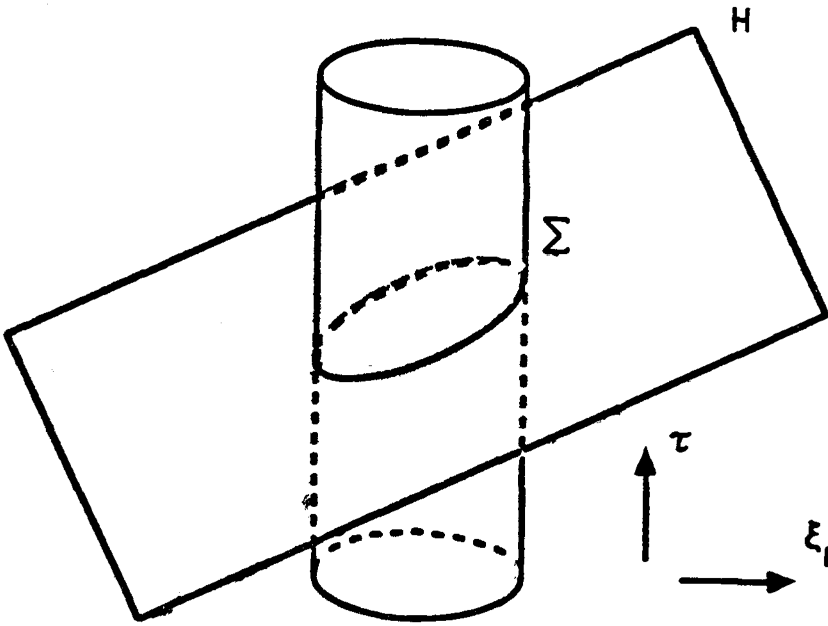

Figure 1

The first step in constructing T^* is to notice that one can use standard arguments to microlocally invert A_λ^* away from $\delta_\lambda(\Sigma)$. Specifically, let $\beta\in C_0^\infty(\mathbf{R})$ equal one near the origin, but have small support. We then define $\beta_0=\beta_{0,\lambda}$ by setting

$$\beta_0(\xi,\tau)=1-\beta(1-|\xi|/\lambda)\beta(\tau/\lambda^2-2\xi_n/\lambda).$$

Then if, as we may, we assume that

$$\sum_{j,k<n} g^{jk}(0,0)D_jD_k=-\Delta_{x'},\quad x'=(x_1,\ldots,x_{n-1}),$$

it follows that β_0 vanishes on $\delta_\lambda(\Sigma)$ (if $|(x,t)|$ is small enough). Moreover, one even has the bounds:

$$|A_\lambda^*(x, t, \xi, \tau)| \geqq c(\|(\xi, \tau)\| + \lambda)^2 \quad \text{on} \quad \operatorname{supp} \beta_0$$

for some $c>0$. Thus, if we let T_0^* be the integral operator with kernel

$$(2\pi)^{-(n+1)} \int \frac{e^{i[\langle x-y, \xi\rangle + (t-s)\tau]}}{A_\lambda^*(y, s, \xi, \tau)} \beta_0(\xi, \tau)\, d\xi\, d\tau,$$

easy arguments would show that its adjoint satisfies

$$T_0(A_\lambda v) = (2\pi)^{-(n+1)} \iint e^{i[\langle x-y, \xi\rangle + (t-s)\tau]} \beta_0(\xi, \tau) v(y, s)\, d\xi\, d\tau\, dy\, ds + R_0 v,$$

where R_0 is an integral operator whose kernel is majorized by a fixed constant times

$$\lambda^{-1} \|(x-y, t-s)\|^{-(n+2)+1}.$$

To prove this one would use Lemma 2.1. If one uses this along with Young's inequality, one sees that R_0 actually satisfies better estimates than those in (3.6) when $\alpha=0$, namely

$$\|R_0\|_{(L^q(B), L^q(B))} \leqq C\lambda^{-1}, \quad q = p, p'.$$

Similar considerations show that $D_x^\alpha R_0$ also satisfies better bounds than those in (3.6).

Likewise, the operator T_0 satisfies better bounds than are needed. In fact, note that T_0 belongs to a bounded subset of S_{par}^{-2}, and also $\lambda^{2-|\alpha|} D_x^\alpha T_0$ is in a bounded subset of S_{par}^0 when $|\alpha| \leqq 1$. With this in mind, one can use Lemmas 2.1 and 2.2, together with Young's inequality to see that T_0 satisfies the analogue of the first inequality in (3.5), as well as an improved version of the second estimate where $1/(n+1)$ is replaced by 1 in the exponent.

Next, let

$$\beta_1(\xi, \tau) = 1 - \beta_0(\xi, \tau).$$

We wish to construct integral operators T_1 and R_1 satisfying the analogues of (3.5)—(3.6) whose adjoints have kernels which are similar to the one in (3.3), and

$$T_1(A_\lambda v) = (2\pi)^{-(n+1)} \iint e^{i[\langle x-y, \xi\rangle + (t-s)\tau]} \beta_1(\xi, \tau) v(y, s)\, d\xi\, d\tau\, dy\, ds + R_1 v. \tag{3.8}$$

As we shall see the adjoint T_1^* will have a kernel of the form

$$(2\pi)^{-(n+1)} \int \frac{e^{i\Phi(x, t, y, s, \xi, \tau)}}{A_\lambda^*(y, s, \xi, \tau)} \beta_1(\xi, \tau)\, d\xi\, d\tau \tag{3.9}$$

where Φ will have to be chosen with some care, but will be as above, and R_1 will behave like λT_1. Thus, if we could construct Φ and prove these estimates, then, by adding T_0 and T_1 and applying Lemma 2.3, we would get an integral operator T as above, and the proof would be complete.

To finish the construction, recall that in the elliptic case studied in [10], the phase function was constructed from an eikonal equation arising from the non-elliptic factor of the symbol. A similar thing happens here. Recall from (3.2) that the imaginary part of A_λ^* vanishes when $\xi_n/w'_\varepsilon-\tau/[2\lambda(w'_\varepsilon)^2]=0$. On account of this, it is natural to factor the symbol with respect to this variable. By using the quadratic rule and (3.2) one finds that

$$A_\lambda^*(x,t,\xi,\tau) = G_\lambda(x,t,\xi,\tau)\big((\xi_n/w'_\varepsilon-\tau/[2\lambda(w'_\varepsilon)^2])-i[b_\lambda(x,t,\xi,\tau)-\lambda]\big),$$

where G_λ is in S^1_{par} on $\operatorname{supp}\beta_1$ and is elliptic with respect to the parabolic norm. In fact, it satisfies

$$|G_\lambda| \geqq c\big(\|(\xi,\tau)\|+\lambda\big) \quad \text{some} \quad c>0.$$

The other factor, of course, does not enjoy this property since it vanishes on $\delta_\lambda(\Sigma)$. The function b_λ appearing is real (on the support of β_1) and is given by the formula

$$b_\lambda(x,t,\xi,\tau) = \frac{1}{w'_\varepsilon}\sqrt{\sum\nolimits_{j,k<n} g^{jk}(x,t)\xi_j\xi_k-\frac{\xi_n\tau}{\lambda w'_\varepsilon}+\frac{\tau^2}{4\lambda^2(w'_\varepsilon)^2}}\,.$$

The important thing, though, is that if $\varepsilon>0$ is small, then on the support of β_1

$$\text{(3.10)}\qquad \begin{gathered} |b_\lambda(x,t,\xi,\tau)| \leqq C\|(\xi,\tau)\| \\ \frac{\partial}{\partial x_1}b_\lambda \geqq c\varepsilon^{-1}\|(\xi,\tau)\| \quad \text{some} \quad c>0. \end{gathered}$$

In the last expression, we have to assume, as we may, that both $|x_1|$ and the support of β above are small, and use the fact that $(\partial/\partial x_1)w'_\varepsilon\approx\varepsilon^{-1}$.

Since G_λ is "elliptic", we are led to construct the phase function Φ in (3.9) from the other factor of A_λ^*. Specifically, if we let

$$B_\lambda(x,t,\xi,\tau) = \xi_n/w'_\varepsilon-\tau/[2\lambda(w'_\varepsilon)^2]-ib_\lambda(x,t,\xi,\tau),$$

then, following [10], we would like Φ to satisfy the eikonal equation

$$\text{(3.11)}\qquad B_\lambda(x,t,\Phi_x,\tau) = B_\lambda(y,s,\xi,\tau), \quad (\xi,\tau)\in\operatorname{supp}\beta_1$$

and, in addition, have the properties that

$$\text{(3.12)}\qquad \Phi = \langle x-y,\xi\rangle+(t-s)\tau+O\big(\|(x-y,t-s)\|^2\|(\xi,\tau)\|\big)$$

$$\text{(3.13)}\qquad \operatorname{Im}\Phi \geqq 0.$$

Here we are using the notation that $\Phi_x=\nabla_x\Phi$. We should emphasize the fact that (3.11) does *not* involve t derivatives of Φ; this is because the parabolic nature of the problem requires us to treat the space and time directions differently. On the other hand, since (3.12) implies that $\partial\Phi/\partial t-\tau=O\big(\|(\xi,\tau)\|\big)$, we shall be able to handle the error term which arises from the fact that (3.11) involves only x derivatives. It is mainly for this reason that it is crucial that the quadratic term in (3.12) involve the parabolic norm and *not* the Euclidean norm of (ξ, τ).

Since B_λ is complex valued and in general only C^∞, a solution to (3.11)—(3.12) need not exist. However, let us now argue that if certain natural conditions are placed on Φ which would guarantee (3.12), then an essentially unique approximate solution will exist which will satisfy (3.13) if the number $\varepsilon>0$ occurring in the definition of w_ε is small enough. We shall then see that this approximate solution to the eikonal equation will serve our purposes.

To construct Φ let us first "freeze" $x_n=y_n$ in the coefficients. Then, we must consider the *real* boundary value problem:

$$b_\lambda(x', y_n, t, \varphi_x, \tau) = b_\lambda(y, s, \xi, \tau)$$

$$\varphi = 0 \quad \text{when} \quad \langle x-y, \xi\rangle = 0 \quad \text{and} \quad \varphi_x = \xi \quad \text{when} \quad (x, t) = (y, s).$$

This equation has a unique solution $\varphi(x, t, y, s, \xi, \tau)$ for small (x, t) is close to (y, s). Further, one can check that φ must be real and be of the form

$$\varphi = \langle x-y, \xi\rangle + O(\|(x-y, s-t)\|^2 \|(\xi, \tau)\|).$$

Next, we try to solve the following complex boundary value problem involving an unknown function $\psi(x, t, y, s, \xi, \tau)$:

$$B_\lambda(x, t, \varphi_x+\psi_x, \tau) = B_\lambda(y, s, \xi, \tau)$$

$$\psi = 0 \quad \text{when} \quad x_n = y_n \quad \text{and} \quad \psi_x = 0 \quad \text{when} \quad (x, t) = (y, s).$$

As in (3.11), we need only worry about the case where the parameters (ξ, τ) belong to the support of β_1. Since $(\partial/\partial\xi_n)B_\lambda(0, 0, \varphi_x+\xi, \tau)\neq 0$ this is an elliptic non-linear boundary value problem; however, as we pointed out before, an exact solution need not exist since B_λ is complex valued. Nonetheless, results in Treves [13, Chpt. 10], [14] imply that an (essentially unique) approximate solution ψ must always exist when (x, t) and (y, s) are close and small. This function will be an approximate solution in the sense that, for every N,

$$B_\lambda(x, t, \varphi_x+\psi_x, \tau) - B_\lambda(y, s, \xi, \tau) = O(|\mathrm{Im}\,\psi|^N \|(\xi, \tau)\|^{1-N}), \tag{3.14}$$

and because of the choice of φ, it must be of the form

$$\psi = O(|x_n-y_n|^2 \|(\xi, \tau)\|). \tag{3.15}$$

Furthermore, by using (3.10) and Taylor's formula, one can argue that if the $\varepsilon>0$ appearing in the definition of w_ε is small enough and $\beta_1(\xi, \tau)\neq 0$,

$$\mathrm{Im}\,\psi \geqq c|x_n-y_n|^2 \|(\xi, \tau)\|, \tag{3.16}$$

where $c>0$. For similar arguments see [10]. As usual, (3.14) will allow us to argue essentially as if ψ were an exact solution in what follows, while (3.16) will be crucial for the L^p estimates of the parametrix.

We can now finally say what the phase function Φ in our parametrix (3.9) is. If φ and ψ are as above, we shall take

$$\Phi(x, t, y, s, \xi, \tau) = \varphi(x, t, y, s, \xi, \tau)+\psi(x, t, y, s, \xi, \tau)+(t-s)\tau.$$

Clearly, then Φ satisfies (3.12), and

$$\operatorname{Im}\Phi \geqq c|x_n-y_n|^2\|(\xi, \tau)\|. \tag{3.13'}$$

Next, set

$$a = iA_\lambda^*(x, t, D_x, D_t)(\varphi+\psi)\in S^1_{\text{par}}.$$

Then,

$$A_\lambda^*(T_\lambda^* v) = (2\pi)^{-(n+1)}\iint e^{i\Phi}\beta_1(\xi, \tau)v(y, s)\, d\xi\, d\tau\, dy\, ds+R_{1,0}^* v, \tag{3.8'}$$

where $R_{1,0}^*$ is an integral operator whose kernel equals $(2\pi)^{-(n+1)}$ times

$$\int \frac{\beta_1 a e^{i\Phi}}{A_\lambda^*(y, s, \xi, \tau)}\, d\xi\, d\tau+\int e^{i\Phi}\frac{G_\lambda(x, t, \varphi_x+\psi_x, \tau)}{G_\lambda(y, s, \xi, \tau)}\beta_1\, d\xi\, d\tau+O(\lambda^{-N}). \tag{3.17}$$

The last term comes from (3.14) and (3.13′). Equation (3.8′) resembles (3.8), and, in fact, if we let R_1^* equal $R_{1,0}^*$ plus the operator

(3.18)

$$(2\pi)^{-(n+1)}\iint e^{i[\langle x-y, \xi\rangle+(t-s)\tau]}\beta_1 v\, d\xi\, d\tau\, dy\, ds-(2\pi)^{-(n+1)}\iint e^{i\Phi}\beta_1 v\, d\xi\, d\tau\, dy\, ds,$$

then we get (3.8) by taking adjoints. But the equivalence of phase function lemma, Lemma 2.3, implies that the adjoint of the operator in (3.18) satisfies the same bounds as the operator R_0 above. Also, since $\varphi_x+\psi_x=\xi$ when $(x, t)=(y, s)$, if one recalls that G_λ is bounded below in the parabolic norm, then it is not hard to argue that the integral operator corresponding to the second summand in (3.17) has an adjoint satisfying the same bounds.

Putting all of this together, we have shown that, if we now set

$$R_{1,1}^* v = (2\pi)^{-(n+1)}\iint \frac{\beta_1 a e^{i\Phi}}{A_\lambda^*(y, s, \xi, \tau)}v\, d\xi\, d\tau\, dy\, ds, \tag{3.19}$$

then our task would be over if we could show that T_1 satisfies the bounds in (3.5), while $R_{1,1}$ satisfies those in (3.6). For later reference we note that, since $a\in S^1_{\text{par}}$, the support properties of β_1 imply that $R_{1,1}$ behaves like λT_1.

Let us first concentrate on proving the desired estimates for T_1. First, we define the dilated (and transposed) phase functions

$$\Phi_\lambda(x, t, y, s, \xi, \tau) = -\lambda^{-1}\bar{\Phi}(y, s, x, t, \lambda\xi, \lambda^2\tau). \tag{3.20}$$

Then, a change of scale argument and (3.9) implies that the kernel of T_1 equals a constant multiple times

$$(3.9') \qquad K_1(x,t,y,s) = \lambda^n \int \frac{e^{i\lambda\Phi_\lambda(x,t,y,s,\xi,\tau)}}{L(x,t,\xi-i\nabla w_\varepsilon,\tau)} \beta(1-|\xi|)\beta(\tau-2\xi_n)\,d\xi\,d\tau.$$

Here we have used the fact that $\bar{A}_\lambda(x,t,\lambda\xi,\lambda^2\tau)=\lambda^2 L(x,t,\xi-\nabla w_\varepsilon(x),\tau)$, if $\bar{A}_\lambda$ denotes the complex conjugate.

Notice that (3.12) implies that on the support of the integrand

$$\Phi_\lambda = \langle x-y,\xi\rangle+\lambda(t-s)\tau+O(|x-y|^2+|s-t|),$$

which means, that, if λ is large but $|x-y|$ is small, Φ_λ is close to the model phase function $\langle x-y,\xi\rangle+\lambda(t-s)\tau$. The extra factor of λ with the $t-s$ variable complicates things, and we basically have to split the operator T_1 as $T_1=T_1^0+T_1^1$, where T_1^0 comes from the part of the kernel where $|t-s|$ is small compared to λ^{-1}. To handle the first piece, we shall apply the estimates in Corollary 2.6 corresponding to the special case $j=0$. On the other hand, to estimate the norm of T_1^1, we shall need to use all of the estimates in this oscillatory integral lemma.

Let us now be more specific about the splitting of T_1. Let $\eta\in C_0^\infty(\mathbf{R})$ have the property that

$$\eta(s)=1 \quad \text{if} \quad |s|\leqq 1 \quad \text{and} \quad \eta(s)=0 \quad \text{if} \quad |s|\geqq 2.$$

We then let T_1^0 be the integral operator with kernel

$$K_1^0=\eta(\lambda(t-s))K_1,$$

and let T_1^1 be the difference between T_1 and T_1^0. Of course this means that the kernel of T_1^1 vanishes for $|t-s|$ smaller than λ^{-1}. By Minkowski's inequality, we would have the desired bounds for T_1 if we could show that both T_1^0 and T_1^1 satisfy (3.5).

Let us first handle T_1^0. We noted before that in K_1 the s and t variables are weighted more heavily by an extra factor of λ. On account of this, it is more natural to consider a dilated version, $\tilde{T}_1^0$, of T_1^0. This will be the integral operator whose kernel is

$$\tilde{K}_1^0(x,t,y,s)=K_1^0(x,t/\lambda,y,s/\lambda).$$

We would have the right estimates for T_1^0 if we could show that the scaled version satisfies

$$(3.5.0) \qquad \|\tilde{T}_1^0\|_{(L^p,L^{p'})}\leqq C\lambda^{1-(1/p-1/p')} \quad \text{and} \quad \|D_x^\alpha \tilde{T}_1^0\|_{(L^p,L^p)}\leqq C\lambda^{-1/(n+1)+|\alpha|}.$$

To prove these, we shall need to take a closer look at the kernel. By (3.9′),

$$(3.21) \quad \tilde{K}_1^0(x,t,y,s)=\lambda^n\eta(t-s)\int \frac{e^{i\lambda\tilde{\Phi}_\lambda(x,t,y,s,\xi,\tau)}}{L(x,t/\lambda,\xi-i\nabla w_\varepsilon,\tau)}\beta(1-|\xi|)\beta(\tau-2\xi_n)\,d\xi\,d\tau,$$

where

$$\tilde{\Phi}_\lambda=\Phi_\lambda(x,t/\lambda,y,s/\lambda,\xi,\tau)=\langle x-y,\xi\rangle+(t-s)\tau+O(|x-y|^2+|(t-s)/\lambda|).$$

Since we need only prove (3.5.0) when $\tilde{T}_1^0$ acts on functions with small support, we see that $\tilde{\Phi}_\lambda$ is close to the usual Euclidean phase function $\langle x-y, \xi\rangle+(t-s)\tau$ when λ is large. This fact will allow us to compute the oscillatory integral in (3.21) and eventually apply Corollary 2.6.

To rewrite $\tilde{K}_1^0$ in a more useful way, recall that the symbol in the last oscillatory integral is singular on the set Σ defined in Figure 1. Away from Σ, however, it is C^∞. And, in fact, since $L(y, s, \xi-i\nabla w_\varepsilon, \tau)$ vanishes only of first order on this set, we are in a situation that is similar to the one studied in [10].

In fact, let $\Pi=\Pi_{y_n}$: $\mathbf{R}^{n+1}\to\mathbf{R}^{n+1}$ denote the projection onto the n dimensional subspace

$$H = H_{y_n} = \{(x, t)\colon\ t = 2w_\varepsilon'(y_n)x_n\}\subset\mathbf{R}^{n+1}.$$

This is the hyperplane containing the set Σ in Figure 1. Unfortunately, Σ is not a sphere in H, so matters are complicated slightly. Nonetheless, if one repeats the stationary phase arguments in [10], one sees that

$$(3.22)\quad \tilde{K}_1^0 = \sum\nolimits_{\nu=0,1} \lambda^{(n-1)/2} a_\nu(x, t, y, s)\frac{e^{i\lambda\varphi_\nu(x,t,y,s)}}{|\Pi(x-y, t-s)|^{(n-1)/2}|(x-y, t-s)|},$$

where $a_\nu=a_{\nu,\lambda}$ and $\varphi_{\nu,\lambda}$ are C^∞ when $\Pi(x-y, t-s)\neq 0$ and have the following properties. First, there is a fixed non-singular transformation A: $H\to H$, sending the unit sphere into an ellipse so that $(-1)^\nu\varphi_\nu$ is close to $|A\Pi(x-y, t-s)|$. To be more specific, given $\varepsilon_0>0$ and N finite, if (x, t) is close to (y, s) and λ is large,

$$(3.23)\quad \left|D^\alpha[(-1)^\nu\varphi_\nu-|A\Pi(x-y, t-s)|]\right| \leqq \varepsilon_0|\Pi(x-y, t-s)|^{1-|\alpha|},\quad 0\leqq|\alpha|\leqq N,$$

provided that $|\Pi(x-y, t-s)|\geqq\lambda^{-1}$. On the other hand, for each fixed N, the functions a_ν in (3.22) satisfy

$$(3.24)\qquad |D^\alpha a_\nu| \leqq C_\alpha\big(1+\lambda(x_n-y_n)^2\big)^{-N}|\Pi(x-y, t-s)|^{-|\alpha|}.$$

The rapid decay in the x_n-y_n direction of course occurs because of (3.13′).

We are now in a position to prove (3.5.0). In the second inequality there, it is easy to see that the arguments giving the estimates for $\alpha=0$ can be adapted to prove those for $|\alpha|=1$ since $D_x^\alpha\tilde{T}_1^0\approx\lambda\tilde{T}_1^0$ when $|\alpha|=1$. Therefore, for simplicity, we shall only treat the case of $\alpha=0$. To prove these estimates, we shall need to break up the kernel dyadically with respect to the H variables. To this end, choose $\varrho\in C_0^\infty(\mathbf{R})$ satisfying

$$\operatorname{supp}\varrho\subset[1/4, 1],\quad\text{and}\quad \sum\nolimits_{-\infty}^{\infty}\varrho(2^{-k}s) = 1,\quad s>0.$$

We then, for $k=0, 1, 2, \dots$, let $\tilde{T}_{1,k}^0$ be the integral operator with kernel

$$\tilde{K}_{1,k}^0 = \begin{cases}\varrho\big(\lambda 2^{-k}|\Pi(x-y, t-s)|\big)\tilde{K}_1^0, & k>0\\ \big[1-\sum_{j\leqq 0}\varrho\big(\lambda 2^{-j}|\Pi(x-y, t-s)|\big)\big]\tilde{K}_1^0, & k=0.\end{cases}$$

We claim that, if B is a small enough ball around $(0, 0)\in\mathbf{R}^n\times\mathbf{R}$ and if λ is sufficiently large, then Young's inequality and Corollary 2.6 yield

$$\|\tilde{T}^0_{1,k}f\|_{L^{p'}(B)} \leqq C\lambda^{1-(1/p-1/p')}2^{-2k/(n+2)}\|f\|_{L^p(B)} \tag{3.25}$$

$$\|\tilde{T}^0_{1,k}f\|_{L^p(B)} \leqq C\lambda^{-1/(n+1)}(\lambda 2^{-k})^{-(n-1)/2(n+1)}\|f\|_{L^p(B)}, \tag{3.26}$$

whenever $f\in C_0^\infty(B)$. By summing a geometric series, it is plain that these two sets of inequalities imply the desired result. The series converge since $\tilde{T}^0_{1,k}f(x)=0$ when $x\in B$ if $k\geqq\log\lambda$ and B is small.

Let us start with (3.25). One can check that, when $k=0$, the L^r norm of the kernel is $O(\lambda^{1-(1/p-1/p')})$ if r satisfies $1/r=1-(1/p-1/p')=n/(n+2)$. Consequently, the estimate in (3.25) for $k=0$ follows from Young's inequality.

To prove the bounds for $k>0$, we shall first estimate the mapping properties of an operator acting on functions of one less variable. Recall that a similar argument was used in the proof of Lemma 2.2. This time, if x_n and y_n are fixed, and if $x'=(x_1, \dots, x_{n-1})$, the new operator will be given by

$$(\tilde{T}^0_{1,k})'g(x', t) = \int_{\mathbf{R}^n}\tilde{K}^0_{1,k}(x, t, y, s)g(y', s)\,dy'\,ds. \tag{3.27}$$

Of course the $L^p(\mathbf{R}^n)\to L^{p'}(\mathbf{R}^n)$ norm of this operator equals $(2^k/\lambda)^{n[1-(1/p-1/p')]}$ times the norm of the *dilated* operator

$$\int\tilde{K}^0_{1,k}(\alpha x', x_n, \alpha t, \alpha y', y_n, \alpha s)g(y', s)\,dy'\,ds, \quad \alpha = 2^k/\lambda. \tag{3.28}$$

However, if we use Corollary 2.6 (where the parameters there are $j=0$ and $\mu=2^k$), we see that the operator in (3.28) must have an $L^p\to L^{p'}$ norm which is majorized by

$$\lambda^{(n-1)/2}(2^k/\lambda)^{-(n-1)/2}2^{-k(n-1)/p'}\big((x_n-y_n)^2+(2^k/\lambda)^2\big)^{-1/2}.$$

But, then a little arithmetic shows that if $r=(n+2)/n$, as above, then the $L^p\to L^{p'}$ norm of the operator in (3.27) is controlled by a constant multiple of

$$\lambda^{1-(1/p-1/p')}2^{-2k/(n+2)}\times\big[(\lambda/2^k)^{-1+1/r}\big((x_n-y_n)^2+(2^k/\lambda)^2\big)^{-1/2}\big].$$

Finally since the L^r norm of the term in the brackets is uniformly bounded, we get the desired result from Young's inequality for $\mathbf{R}$, if we repeat the argument in the proof of Lemma 2.2.

Notice that so far we have not used the rapid decay of the kernel in the x_n-y_n direction, i.e. (3.24). However, to prove the $L^p\to L^p$ inequalities (3.26) this will be crucial. In fact, if one argues as above, except uses the $L^p\to L^p$ estimates in Corollary 2.6, then one finds that the $L^p\to L^p$ norm of the restricted operator (3.27) is dominated by

$$\lambda^{-1/(n+1)}(\lambda 2^{-j})^{-(n-1)/2(n+1)}\times\big[\lambda^{1/2}\big(1+\lambda(x_n-y_n)^2\big)^{-N}\big]$$

for every N. However, because the term in the brackets is uniformly in L^1 when $N>1$, it is clear that we also get (3.28). This finishes the proof of the estimates for $\tilde{T}_1^0$.

To finish the estimates for the main part of the parametrix, we must prove that T_1^1 satisfies

(3.5.1) $\quad \|T_1^1\|_{(L^p(B), L^{p'}(B))} \leqq C \quad \text{and} \quad \|D_x^\alpha T_1^1\|_{(L^p(B), L^p(B))} \leqq C\lambda^{-1-1/(n+1)+|\alpha|}.$

Since the estimates for $D_x^\alpha T_1^1$, $|\alpha|=1$, follow from the same arguments, we shall only consider the case where $\alpha=0$ in the second inequality.

Recall that the kernel of T_1^1,

$$K_1^1 = [1-\eta(\lambda(s-t))]K_1$$

vanishes when $\lambda^{-1}\leqq|s-t|\leqq c_0$ where c_0 can be assumed to be as small as we wish. Thus, if we let ϱ be as above and set

$$K_{1,j}^1 = \varrho(\lambda 2^{-j}|s-t|)K_1^1,$$

then the desired estimates would follow if we could show that the associated integra operator satisfies

(3.29) $$\|T_{1,j}^1\|_{(L^p, L^{p'})} \leqq C\lambda^{-1/(n+2)}2^{-((n+1)j)/n(n+2)}$$
and
$$\|T_{1,j}^1\|_{(L^p, L^p)} \leqq C\lambda^{-1-1/(n+1)}2^{-((n-1)j)/2n(n+1)}$$
when $0\leqq j\leqq\log\lambda$.

As before, it is more natural to consider a dilated version of the operator. But in this case the dilated operator, $\tilde{T}_{1,j}^1$, will have a kernel of the form

$$\tilde{K}_{1,j}^1 = K_{1,j}^1(2^j x', x_n, \lambda 2^j t, 2^j y', y_n, \lambda 2^j s).$$

Note that $\tilde{K}_{1,j}^1$ vanishes when $|x'-y'|$ is larger than a constant times 2^{-j} or $|t-s|\notin[1/4, 1]$. Furthermore, the stationary phase arguments that give (3.22) imply that

$$\tilde{K}_{1,j}^1 = \sum\nolimits_{\nu=0,1} \lambda^{(n-1)/2} a_{\nu,j}(x, t, y, s)2^{-j((n+1)/2)}e^{i2^j\lambda\varphi_{\nu,j}(x,t,y,s)},$$

where now (if the support of B is small) $\varphi_{\nu,j}$ can be assumed to satisfy

$$\left|D^\alpha\left[(-1)^\nu\varphi_{\nu,j}-\left|A\Pi(x'-y', 2^{-j}(x_n-y_n), t-s)\right|\right]\right| \leqq \varepsilon_0, \quad 0\leqq|\alpha|\leqq N,$$
while
$$|D_{x',y'}^\alpha D_{t,s}^\beta a_{\nu,j}| \leqq C_\alpha(1+\lambda(x_n-y_n)^2)^{-N},$$
for every fixed N.

If one keeps in mind the support properties of $a_{\nu,j}$ and repeats the above arguments, then the estimates in Corollary 2.6 for $\mu=\lambda$ and $0\leqq j\leqq\log\lambda$

give that

$$\|\tilde{T}^1_{1,j}\|_{(L^p, L^{p'})} \leqq C[\lambda^{1-(1/p-1/p')} 2^{nj(1/p-1/p')}] \times \lambda^{-1/(n+2)} 2^{-((n+1)j/n(n+2))}$$

$$\|\tilde{T}^1_{1,j}\|_{(L^p, L^p)} \leqq C[\lambda^{-1/(n+1)} 2^{nj}] \times 2^{-((n-1)j/2n(n+1))}.$$

Since these estimates are equivalent to (3.29), this finishes the proof of (3.5).

To finish matters, we have to complete the proof of (3.6). That is, we need to show that the remainder oqerator $R_{1,1}$ in (3.19) satisfies the bounds in (3.6). However, since $R_{1,1} \approx \lambda T_1$, it is not hard to see that the arguments for T_1 will also give the desired estimates for $R_{1,1}$. This completes the proof of the Carleman inequalities for the operator L.

4. Appendix: non-isotropic Carleson—Sjölin estimates

In this section we shall sketch a proof of Lemma 2.5. As we pointed out before, this is just a modification of arguments in Stein [11]. One of the main ingredients in the proof of our oscillatory integral lemma is the following L^2 oscillatory integral theorem.

Lemma 4.1. *Suppose that $a_j(x, y)$ is a function of $x, y \in \mathbf{R}^d$ which vanishes when $|x|$ or $|y|$ is $\geqq 1$ or $|x-y| \geqq 2^{-j}$. In addition assume that*

$$|D_x^\alpha a_j(x, y)| \leqq C_\alpha 2^{j|\alpha|} \quad \text{for all} \quad \alpha, \tag{4.1}$$

where the constants are independent of $j \geqq 0$. Let $\varphi(x, y) \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ be real and satisfy the non-degeneracy condition

$$\det(\partial^2\varphi/\partial x_j \partial y_k) \neq 0 \quad \text{for} \quad |x|, |y| \leqq 1. \tag{4.2}$$

Then if

$$(I_{\mu,j} f)(x) = \int_{\mathbf{R}^d} 2^{jd/2} e^{i2^j\mu\varphi(x,y)} a_j(x, y) f(y)\, dy,$$

it follows that there is a constant C independent of j so that

$$\|I_{\mu,j} f\|_{L^2(\mathbf{R}^d)} \leqq C\mu^{-d/2} \|f\|_{L^2(\mathbf{R}^d)}. \tag{4.3}$$

Proof. The desired inequality holds if and only if

$$\|I^*_{\mu,j} I_{\mu,j} f\|_2 \leqq C\mu^{-d} \|f\|_2. \tag{4.3'}$$

But the kernel of the operator in (4.3′) is

$$2^{jd} \int_{\mathbf{R}^d} e^{i2^j\mu[\varphi(x,y)-\varphi(x,z)]} a_j(x, y) \overline{a_j(x, z)}\, dx. \tag{4.4}$$

The non-degeneracy hypothesis (4.2) on the phase function means that

$$|\nabla_x[\varphi(x, y) - \varphi(x, z)]| \geqq c|y-z| \quad \text{some} \quad c > 0 \tag{4.5}$$

if x and z are sufficiently close. After possibly contracting the support of a_j in the definition of $I_{\mu,j}$ we can always assume that this is the case in the above oscillatory integral. But then, (4.5) and a straightforward integration by parts argument which takes into account (4.1) and the support properties of $x\to a_j(x, y)$ (when y is fixed) gives that (4.4) is $O((1+\mu|y-z|)^{-N})$ for any N. This clearly yields (4.3′). ∎

We now claim that this gives the $L^p(\mathbf{R}^{n-1})\to L^{p'}(\mathbf{R}^n)$ estimates in Lemma 2.5 for the operators

$$(R_{\mu,j}f)(y) = \int_{\mathbf{R}^{n-1}} 2^{(n-1)j/2} e^{i2^j\mu\psi(y,z)} a_j(y, z) f(y)\, dy.$$

In fact, if we recall (2.7), then Lemma 4.1 with $d=(n-1)$ implies that whenever y_n is fixed

$$\|(R_{\mu,j}f)(\cdot, y_n)\|_{L^2(\mathbf{R}^{n-1})} \leqq C\mu^{-(n-1)/2}\|f\|_{L^2(\mathbf{R}^{n-1})}.$$

Since we are assuming that a_j has compact support, this clearly gives (2.11) for $p=2$. Inequality (2.11) is trivial for $p=1$, and, hence, by the M. Riesz interpolation theorem, the $L^p\to L^{p'}$ estimates in Lemma 2.5 must hold for all $1\leqq p\leqq 2$.

To finish the proof of Lemma 2.5, we need to prove (2.12). However, by interpolating with the trivial inequality for $p=1$, one sees it is enough to prove the estimate for the other endpoint, which by duality, would follow from

$$(4.6)\quad \|R^*_{\mu,j}f\|_{L^2(\mathbf{R}^{n-1})} \leqq C2^{((n-1)/2(n+1))j}\mu^{-(n(n-1)/2(n+1))}\|f\|_{L^p(\mathbf{R}^n)},\quad p = \frac{2(n+1)}{n+3}.$$

But, if $T_{\mu,j}=R_{\mu,j}R^*_{\mu,j}$, then

$$\|R^*_{\mu,j}f\|_2^2 = \int_{\mathbf{R}^n} T_{\mu,j}f\bar{f}\, dx \leqq \|T_{\mu,j}f\|_{L^{p'}(\mathbf{R}^n)}\|f\|_{L^p(\mathbf{R}^n)}.$$

Consequently, (4.6) would be true if

$$(4.6')\quad \|T_{\mu,j}f\|_{L^{p'}(\mathbf{R}^n)} \leqq C2^{((n-1)/(n+1))j}\mu^{-(n(n-1)/n+1)}\|f\|_{L^p(\mathbf{R}^n)},\quad p = \frac{2(n+1)}{n+3}.$$

Note that, unlike $R_{\mu,j}$, the operators $T_{\mu,j}$ send functions of n variables to functions of n variables. Also, notice that the kernel of $T_{\mu,j}$ is

$$(4.7)\qquad K_{\mu,j}(x, z) = \int_{\mathbf{R}^{n-1}} 2^{(n-1)j} e^{i2^j\mu[\psi(x,y)-\psi(z,y)]} a_j(x, y)\overline{a_j(z, y)}\, dy.$$

The notation may be a bit confusing since now $y=(y_1, \ldots, y_{n-1})$ denotes a vector in $\mathbf{R}^{n-1}$. Keeping this in mind, let us define an analytic family of kernels as follows. Fix a real $\eta\in C_0^\infty(\mathbf{R})$ satisfying $\eta(s)=1$ for s near 0. Then, like in [11], we define for $\zeta\in\mathbf{C}$, the analytic family of distributions

$$K^\zeta_{\mu,j}(x, z) = \frac{e^{\zeta^2}}{\Gamma(\zeta)}\int_{\mathbf{R}^n} 2^{(n-1)j} e^{i2^j\mu[(\psi(x,y)-\psi(z,y))+(x_n-z_n)y_n]} a_j(x, y)\overline{a_j(z, y)}\eta^2(y_n)(y_n)_+^{-1+\zeta}\, dy\, dy_n.$$

Then, if $T^{\zeta}_{\mu,j}$ is the associated integral operator, we have

$$T^{\zeta}_{\mu,j} = T_{\mu,j}, \quad \text{if} \quad \zeta = 0.$$

Consequently, by invoking Stein's analytic interpolation theorem as in [11], we see that (4.6′) follows from:

$$\|T^{\zeta}_{\mu,j} f\|_2 \leqq C2^{-j}\mu^{-n}\|f\|_2, \quad \mathrm{Re}(\zeta) = 1 \tag{4.8}$$

$$\|T^{\zeta}_{\mu,j} f\|_\infty \leqq C2^{(n-1)j}\|f\|_1, \quad \mathrm{Re}(\zeta) = -\frac{n-1}{2}. \tag{4.9}$$

Let us start with the first inequality. To begin, note that (4.8) holds if and only if the oscillatory integral operators

$$2^{((n-1)/2)j}\int_{\mathbf{R}^n} e^{i2^j\mu[\psi(x,y)+x_ny_n]}a_j(x, y)\eta(y_n)(y_n)_+^{-1+\zeta}f(y, y_n)\, dy\, dy_n$$

send $L^2 \to L^2$ with norm $O(2^{-j/2}\mu^{-n/2})$ when $\mathrm{Re}(\zeta)=1$. However, for such ζ, $(y_n)_+^{-1+\zeta}$ is a uniformly bounded function, and so, by taking adjoints, we see that we would be done if we could show that the operators

$$2^{((n-1)/2)j}\int_{\mathbf{R}^n} e^{-i2^j\mu[\psi(z,y)+z_ny_n]}a_j(z, y)\eta(y_n)g(z)\, dz$$

enjoy the same mapping properties. This last statement is true if and only if the integral operators with kernel

$$2^{(n-1)j}\int_{\mathbf{R}^n} e^{i2^j\mu[(\psi(x,y)-\psi(z,y))+(x_n-z_n)y_n]}a_j(x, y)\overline{a_j(z, y)}\eta^2(y_n)\, dy\, dy_n \tag{4.10}$$

send $L^2 \to L^2$ with norm $O(2^{-j}\mu^{-n})$. But, by (2.7) and (2.8), we can assume that the $n\times n$ matrix

$$\left(\frac{\partial^2}{\partial x_j \partial y_k}[\psi(x, y)+x_ny_n]\right)$$

is non-singular. Therefore, by repeating the proof of Lemma 4.1, one can check that the kernel (4.10) is dominated by

$$(1+\mu|x-z|)^{-N}(1+2^j\mu|x_n-z_n|)^{-N}$$

for any N, if x and z are sufficiently close, as we may always assume. Finally, since this means that the L^1 norm of the kernel (4.10) is $O(2^{-j}\mu^{-n})$, we get (4.8).

To prove the other inequality, (4.9), first recall that

$$\frac{e^{\zeta^2}}{\Gamma(\zeta)}\int_{-\infty}^{\infty} e^{i2^j\mu(x_n-z_n)y_n}\eta^2(y_n)(y_n)_+^{-1+\zeta}\, dy_n$$
$$= O\big((1+2^j\mu|x_n-y_n|)^{(n-1)/2}\big) \quad \text{if} \quad \mathrm{Re}(\zeta) = -\frac{n-1}{2}.$$

Consequently, since (4.9) holds if and only if $K^{\zeta}_{\mu,j}(x, z)=O(2^{(n-1)j})$, we would be done if we could show that the original kernel in (4.7) satisfies the bounds

$$|K_{\mu,j}(x, z)| \leqq C(1+2^{-j}\mu|x-z|)^{-((n-1)/2)}. \tag{4.9'}$$

To prove this we of course have to use our non-degeneracy assumptions (2.7) and (2.8) for the phase function ψ. First of all, if $(x-z)/|x-z|$ is in a small fixed neighborhood of $\pm(0, ..., 0, 1)\in S^{n-1}$, the second half of (2.8) implies that

$$\left|\det\left(\frac{\partial^2}{\partial y_j \partial y_k}[\psi(x, y)-\psi(z, y)]\right)\right| \geqq c|x-z| \quad \text{some} \quad c>0.$$

This together with the usual stationary phase estimates (see e.g. [5], [11]) gives (4.9′) in this case. On the other hand, we claim that (2.7) and the first half of (2.8) imply that, if $(x-z)/|x-z|$ is outside of a fixed neighborhood of $\pm(0, ..., 0, 1)$, then $K_{\mu,j}=O((1+2^{-j}\mu|x-z|)^{-N})$ for any N, provided that the amplitudes a_j vanish outside of a sufficiently small neighborhood $\mathcal{N}$ of the origin in $\mathbf{R}^n\times\mathbf{R}^{n-1}$. Under these assumptions, we have that in $\mathcal{N}$,

$$|\nabla_y[\psi(x, y)-\psi(z, y)]| \geqq c|x-z| \quad \text{some} \quad c>0,$$

which clearly yields the claim by integration by parts. Thus, if the neighborhood $\mathcal{N}$ in Lemma 2.5 is sufficiently small we have the desired result, and this completes the proof.

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Received January 11, 1989

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