Global properties of differential operators of constant strength

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1. Introduction

For differential operators $P(D)$ with constant coefficients there is a rather complete theory on existence and regularity of solutions of the equation $P(D)u=f$. There exists a solution in every relatively compact open subset of $\mathbf{R}^n$ for an arbitrary right hand side $f\in\mathscr{D}'(\mathbf{R}^n)$ (semi-global existence theorem). In an open set $\Omega\subset\mathbf{R}^n$ the equation can be solved with $u\in C^\infty(\Omega)$ for every $f\in C^\infty(\Omega)$ if Ω is P-convex and with $u\in\mathscr{D}'(\Omega)$ for every $f\in\mathscr{D}'(\Omega)$ if Ω is strongly P-convex. These results are exposed in Hörmander [4, Ch.III].

The class of differential operators of constant strength with variable coefficients (Definition 2.2 below) is closely related to operators with constant coefficients. An operator $P(x, D)$ of constant strength defined in an open set $\Omega\subset\mathbf{R}^n$ can be considered as a bounded perturbation of the operator $P_{x_0}(D)$ with constant coefficients obtained by freezing the coefficients of P at a fixed point $x_0\in\Omega$. Peetre [9] proved that the equation

$$P(x,D)u=f \tag{1.1}$$

can be solved locally for any f (c.f. Hörmander [4, Ch. VII]). Also theorems on differentiability of solutions can be extended to differential operators of constant strength. The operator $P(x, D)$ is hypoelliptic in Ω if it has constant strength and for every $x\in\Omega$ the operator $P_x(D)$ is hypoelliptic (Hörmander [4, Theorem 7.4.1]). M. Taylor [11] has proved that conversely if P is hypoelliptic and of constant strength then $P_x(D)$ is hypoelliptic for every x.

However a semiglobal existence theorem is not valid for all operators of constant strength. In fact by Pliš [10] there is an elliptic operator P_0 of order 4 in $\mathbf{R}^3$ such that there is a function $\varphi\in C_0^\infty$ with

$$^tP_0\varphi=0.$$

A necessary condition for solvability of the equation $P_0 u=f$ in a neighborhood of $\operatorname{supp}\varphi$ is then that $\langle f, \varphi\rangle=0$. There are operators P of constant strength such that no finite number of linear conditions on f are sufficient for solvability of the equation (1.1) in a relatively compact open subset Ω' of the set Ω where P is defined. One example of such an operator is the operator P_0 above considered as an operator in $\mathbf{R}^4$ independent of the last variable. The adjoint of this operator has infinitely many linearly independent solutions with support in a fixed compact set. On the other hand if P is hypoelliptic of constant strength then

$$N = \{\varphi\in\mathscr{E}';\ {}^tP\varphi = 0\}\subset C_0^\infty.$$

Standard compactness arguments give that $N\cap\mathscr{E}'(K)$ is finite dimensional for every compact set K in Ω and one can show that the equation (1.1) can be solved in a neighborhood of K if

$$\langle f, \varphi\rangle = 0, \quad \varphi\in N\cap\mathscr{E}'(K).$$

In Section 3 below we shall give a condition which is sufficient for solvability of the equation (1.1) in an open set $\Omega'\subset\subset\Omega$ when P has constant strength and the right hand side satisfies a finite number of linear conditions (Theorem 3.1). The condition involves so called localizations of P at infinity. If P has constant coefficients then a localization of P at infinity is a differential operator $Q(D)\neq 0$ which is a limit of

$$a_j P(D+\xi_j)$$

when $\xi_j\to\infty$ in $\mathbf{R}^n$ and $a_j\in\mathbf{R}^+$. Localizations at infinity can be defined even for operators of constant strength (Definition 2.3). The condition of Theorem 3.1 is that for no localization Q of P at infinity there should exist $w\in\mathscr{E}'(\Omega)$, $w\neq 0$, such that ${}^tQw=0$. This is also necessary for existence with finite codimension in open relatively compact sets if the solution is required to have the same regularity as in the constant coefficient case (Theorem 3.7). After Theorem 3.7 we give a result which clarifies somewhat the meaning of the condition of Theorem 3.1 (Theorem 3.9).

From Theorem 3.1 it is easy to deduce that if Ω is P-convex then there exists a solution $u\in C^\infty(\Omega)$ of the equation (1.1) for any f in a space of finite codimension in $C^\infty(\Omega)$ (Theorem 3.6). Then it follows from Theorem 1.2.4 in Hörmander [7] that the same is true with $C^\infty(\Omega)$ replaced by $\mathscr{D}'(\Omega)$ if Ω is strongly P-convex. But to be able to decide if a domain is strongly P-convex one needs theorems on singularities of solutions.

In Section 4 we prove some results on existence of singular solutions, which imply certain necessary conditions for Ω to be strongly P-convex. These are generalizations of the following theorem of Hörmander [6].

Theorem 1.1. *Let $P(D)$ be a differential operator with constant coefficients and let $Q(D)$ be a localization of P at infinity such that $\Lambda'(Q)$, the orthogonal space of*

$$\Lambda(Q) = \{\eta\in\mathbf{R}^n;\ Q(\xi+t\eta) = Q(\xi),\ all\ \xi\in\mathbf{R}^n,\ t\in\mathbf{R}\},$$

is different from $\{0\}$. Then there exists a solution $u\in\mathscr{D}'(\mathbf{R}^n)$ of the equation $P(D)u=0$ such that sing supp $u=\Lambda'(Q)$.

The definition of the space $\Lambda(Q)$ can be generalized to operators of constant strength (Definition 2.5). When $P(x, D)$ is of constant strength in an open set $\Omega\subset\mathbf{R}^n$ it is natural to replace $\Lambda'(Q)$ by a component Σ_0 of $\Sigma\cap\Omega$ where Σ is an affine subspace parallel to $\Lambda'(Q)$. With a method of proof similar to the one used in the constant coefficient case one can obtain a result which shows that the statement of Theorem 1.1 with $\Lambda'(Q)$ replaced by Σ_0 is valid for an operator P of constant strength if Ω is small (Theorem 4.2). This gives a new proof of the result of Taylor [11] mentioned above. A global version of Theorem 1.1 is true for an operator of constant strength if some additional conditions hold (Theorem 4.4). We do not know if these are satisfied in general but if P has analytic coefficients they are fulfilled.

I would take the opportunity to thank my teacher, Professor Lars Hörmander, who suggested these problems to me and has given much valuable advice during the work.

2. Definitions and notations

First we recall the definition of an operator of constant strength. If $P(D)$, $D=-i\partial/\partial x$, is a differential operator with constant coefficients the function $\tilde{P}$ is defined by

$$\tilde{P}(\xi) = (\textstyle\sum_\alpha |P^{(\alpha)}(\xi)^2|)^{1/2}.$$

$\tilde{P}$ belongs to the class $\mathscr{K}$ of positive functions k such that

$$k(\xi+\eta) \leqq (1+C|\xi|)^N k(\eta),\quad \text{all}\quad \xi,\eta\in\mathbf{R}^n \tag{2.1}$$

for some positive constants C and N. The functions

$$h_s(\xi) = (1+|\xi|^2)^{s/2} \tag{2.2}$$

belong to $\mathscr{K}$ and are much used.

Definition 2.1. Let P_1 and P_2 be differential operators with constant coefficients. Then $P_1\prec P_2$, i.e., P_1 is weaker than P_2, if there is a constant C such that $\tilde{P}_1(\xi)/\tilde{P}_2(\xi)\leqq C$ for all $\xi\in\mathbf{R}^n$, and $P_1\sim P_2$, i.e., P_1 and P_2 are equally strong, if $P_1\prec P_2$ and $P_2\prec P_1$.

If $P=P(x, D)$ is a differential operator with variable coefficients defined in an open set $\Omega \subset \mathbf{R}^n$ then for each fixed $x \in \Omega$ one can consider the operator P_x with constant coefficients obtained by freezing the coefficients at x.

Definition 2.2. A differential operator P defined in an open set $\Omega \subset \mathbf{R}^n$ is of constant strength if $P_x \sim P_{x'}$ for arbitrary $x, x' \in \Omega$.

In this paper all differential operators will be assumed to have C^∞ coefficients. The letters P and Q will always denote differential operators of constant strength assumed to be defined in an open set $\Omega \subset \mathbf{R}^n$ although that is not stated each time.

A localization at infinity of an operator P of constant strength should as in the constant coefficient case be defined as the limit of

$$a_j P(x, D+\eta_j)$$

when the sequence $\eta_j \to \infty$ in $\mathbf{R}^n$ and $a_j \in \mathbf{R}^+$ are normalizing constants. In view of Definition 2.2 it is natural to take a fixed $x_0 \in \Omega$, define $\tilde{P} = \tilde{P}_{x_0}$, set $a_j = 1/\tilde{P}(\eta_j)$ and thus consider

$$P(x, D+\eta_j)/\tilde{P}(\eta_j). \tag{2.3}$$

There is a subsequence of the sequence η_j such that the limit of (2.3) actually exists. For if R has constant coefficients and is weaker than P_{x_0} then the order of R is at most equal to the order of P_{x_0} so $\{R; R \prec P_{x_0}\}$ is finite dimensional. Let $P_1, \dots, P_N$ be a basis of this vector space. We can write

$$P(x, D) = \sum_{\nu=1}^{N} c_\nu(x) P_\nu(D)$$

where $c_\nu \in C^\infty$. Since $P_\nu \prec P_0$ the coefficient of D^α in $P_\nu(D+\eta_j)/\tilde{P}(\eta_j)$ is a bounded function of η_j for $\nu = 1, \dots, N$ and all multiindices α. Thus there is a subsequence η_{j_k}, which we for simplicity assume is identical with the sequence η_j, such that the coefficient of D^α in

$$P_\nu(D+\eta_j)/\tilde{P}(\eta_j)$$

has a limit for $\nu = 1, \dots, N$ and all α. Then it is clear that there is a differential operator $Q(x, D)$ with C^∞ coefficients such that for all α the coefficient of D^α in (2.3) tends to the corresponding coefficient in $Q(x, D)$ in the $C^\infty(\Omega)$ topology. If another point x_1 is chosen to define $\tilde{P}$ then Q will just be replaced by a constant times Q. Now we can state

Definition 2.3. If P is a differential operator of constant strength let $L(P) = \{Q(x, D);\ Q(x, D) = \lim P(x, D+\eta_j)/\tilde{P}(\eta_j)$ for some sequence $\eta_j \in \mathbf{R}^n$, $\eta_j \to \infty\}$. The elements of $L(P)$ are called localizations of P at infinity.

An operator $Q \in L(P)$ has constant strength for

$$\tilde{Q}_x(\xi) = \lim \tilde{P}_x(\xi+\eta_j)/\tilde{P}(\eta_j).$$

A localization R of Q at ∞ is a localization of P at ∞ for if θ_j is the sequence defining R then it is easily seen that there are subsequences η_{j_ν} and θ_{j_ν} such that $\xi_\nu=\eta_{j_\nu}+\theta_{j_\nu}\to\infty$ and

$$R(x,\xi)=\lim P(x,\xi+\xi_\nu)/\tilde{P}(\xi_\nu).$$

The adjoint ${}^tP$ of P has constant strength if P has and $({}^tP)_x(D)\sim P_x(-D)$ for every x (Hörmander [4, Lemma 7.1.2]). Clearly we have

$${}^tQ(x,D)=\lim {}^tP(x,D-\eta_j)/\tilde{P}(\eta_j)$$

so the adjoint of a localization of P at ∞ is after multiplication by a positive constant a localization of ${}^tP$ at ∞.

The following proposition shows that one need not consider all sequences η_j in order to obtain all localizations of P.

Proposition 2.4. *Let $Q\in L(P)$. Then there is a polynomial in t*

$$\eta(t)=\sum_{j=0}^{J}\theta_j t^j,\quad \theta_j\in\mathbf{R}^n,\quad \eta(t)\to\infty \quad as \quad t\to\infty, \tag{2.4}$$

a number $a>0$ and an integer $\sigma\geqq 0$ such that

$$Q(x,\xi)=\lim_{t\to\infty}P(x,\xi+\eta(t))/at^\sigma.$$

Proof. Let Q be defined by a sequence η_j. After possibly passing to a subsequence we may assume that $P_\nu(\xi+\eta_j)/\tilde{P}(\eta_j)$ has a limit Q_ν for all P_ν in a basis of $\{R;\,R\prec P_{x_0}\}$. It is sufficient to prove that there exist $\eta(t),a,\sigma$ such that $Q_\nu(\xi)=$ $=\lim P_\nu(\xi+\eta(t))/at^\sigma$ for all ν. But that is just Proposition 2.2 in Hörmander [6] applied to the vector valued function $\xi\to(P_1(\xi),\dots,P_N(\xi))$. The proof of that proposition is valid with obvious modifications for a vector valued function.

If P has constant coefficients we define $\Lambda(P)$ as in the introduction. If $P_1\prec P_2$ then $\Lambda(P_2)\subset\Lambda(P_1)$. For let $\eta\in\Lambda(P_2)$. Then

$$|P_1(\xi+t\eta)|\leqq\tilde{P}_1(\xi+t\eta)\leqq C\tilde{P}_2(\xi+t\eta)=C\tilde{P}_2(\xi).$$

Hence $P_1(\xi+t\eta)$ must be independent of t. Thus the following definition is independent of the point x_0 chosen.

Definition 2.5. If P has constant strength let $\Lambda(P)=\Lambda(P_{x_0})$. The orthogonal space of $\Lambda(P)$ is denoted by $\Lambda'(P)$.

The class of operators of constant strength is invariant under linear changes of coordinates. Therefore we can choose the coordinate system so that

$$\Lambda'(P)=\{x;\,x_k=0,\ k=j+1,\dots,n\}.$$

Then obviously

$$P(x,D)=\sum_\alpha a^\alpha(x',x'')D_{x'}^\alpha,$$

where $x'=(x_1,\dots,x_j)$, $x''=(x_{j+1},\dots,x_n)$ and $D_{x'}^\alpha$ is a partial derivative which does not contain $\partial/\partial x_k$ for $k=j+1,\dots,n$. Consider a fixed x'' and let

$\Sigma=\{(x', x''); x'\in\mathbf{R}^j\}$. The restriction of P to Σ defines an operator of constant strength in the open set $\Sigma\cap\Omega$ in $\mathbf{R}^j$. We write P_Σ for that operator.

It is immediate from the definitions that $\Lambda'(Q)\subset\Lambda'(P)$ if $Q\in L(P)$. Moreover $\dim \Lambda'(Q)<\dim \Lambda'(P)$ if the sequence η_j defining Q tends to ∞ modulo $\Lambda(P)$. For if θ_J is the coefficient of the largest power of t in (2.4) and $\eta(t)\to\infty$ mod $\Lambda(P)$ then we may assume that $\theta_J\notin\Lambda(P)$. But $\theta_J\in\Lambda(Q)$ since

$$\eta(t+st^{1-J}/J) = \eta(t)+s\theta_J+O(1/t)$$

so that for every real s

$$Q(x, \xi) = \lim_{t\to\infty} P(x, \xi+\eta(t))/at^\sigma = Q(x, \xi+s\theta_J).$$

If the sequence η_j is bounded mod $\Lambda(P)$ then Q is clearly of the form $P(x, \xi+\xi_0)/\tilde{P}(\xi_0)$ for some $\xi_0\in\mathbf{R}^n$. For all $Q\in L(P)$ we have $\dim\Lambda'(Q)<n$ for either $\eta_j\to\infty$ modulo $\Lambda(P)$ and then $\dim\Lambda'(Q)<\dim\Lambda'(P)\leqq n$ or the sequence η_j is bounded modulo $\Lambda(P)$ and then $\dim\Lambda'(Q)\leqq\dim\Lambda'(P)<n$.

These remarks show that if Q is a localization of P at infinity mod $\Lambda(P)$ then Q is somewhat simpler than P. If we take a localization R of Q at infinity mod $\Lambda(Q)$ we get a still simpler localization of P, and so on. When proving an extension of Theorem 1.1 one should first look at the simplest localizations of order $\neq 0$. Therefore we state

Definition 2.6. A differential operator Q of constant strength is of local type if $\Lambda'(Q)\neq\{0\}$ and all localizations of Q which are defined by a sequence η_j which tends to infinity modulo $\Lambda(Q)$ are of order 0.

Let Q be of local type and choose the coordinate system so that $\Lambda'(Q)=$ $=\{(x', x''); x''=0\}$. The definition implies that Q_x is then a hypoelliptic polynomial in the ξ' variables for all x, that is

$$Q^{(\alpha)}(x', x'', \xi')/Q(x', x'', \xi') \to 0$$

when $\xi'\to\infty$ if $\alpha\neq 0$.

The following proposition will imply that in order to prove an extension of Theorem 1.1 it is sufficient to consider localizations of local type.

Proposition 2.7. *For every $Q\in L(P)$ there is an operator $Q'\in L(P)$ of local type such that $\Lambda'(Q')\subset\Lambda'(Q)$.*

Proof. If Q is of local type there is nothing to prove. Otherwise one can find Q_1 of positive order which is a localization of Q at infinity modulo $\Lambda'(Q)$. Then $\Lambda'(Q_1)\subset\Lambda'(Q)$ and $\dim\Lambda'(Q_1)<\dim\Lambda'(Q)$. If Q_1 is of local type the proof is finished, otherwise there is a non constant Q_2 which is a localization of Q_1 at infinity modulo $\Lambda'(Q_1)$, and so on. We get a Q_N of local type after a finite number of steps, for the dimensions of the spaces $\Lambda'(Q)$, $\Lambda'(Q_1)$, ... are strictly decreasing. The operator Q_N belongs to $L(P)$ so the proof is complete.

3. Existence theorems

As before let P be a differential operator of constant strength in an open set $\Omega\subset\mathbf{R}^n$. In this section we prove some existence theorems for the equation $Pu=f$ on compact subsets of Ω.

First we introduce suitable Banach spaces. Let $k\in\mathscr{K}$ and $1\leqq p\leqq\infty$. The space $\mathscr{B}_{p,k}$ is the set of temperate distributions u such that $\hat{u}$ is a function and $\hat{u}k\in L^p$. It is a Banach space with the norm

$$\|u\|_{p,k}=\left((2\pi)^{-n}\int|\hat{u}(\xi)k(\xi)|^p\,d\xi\right)^{1/p}$$

and if $p\neq\infty$ its dual space is $\mathscr{B}_{p',k'}$, where $1/p+1/p'=1$ and $k'(\xi)=1/k(-\xi)$. If $p\neq\infty$ then C_0^∞ is dense in $\mathscr{B}_{p,k}$. If k, $k_1\in\mathscr{K}$ and $k_1(\xi)/k(\xi)\to 0$ when $\xi\to\infty$ then a sequence which is bounded in $\mathscr{B}_{p,k}$ and has supports in a fixed compact set has a subsequence which converges in $\mathscr{B}_{p,k_1}$. For the proofs of these facts see Hörmander [4, section 2.2]. Let Ω' be open and relatively compact. In the study of the equation $Pu=f$ in Ω' we use the quotient spaces

$$\mathscr{B}_{p,k}(\bar{\Omega}')=\mathscr{B}_{p,k}/N_{p,k}(\Omega')$$

where

$$N_{p,k}(\Omega')=\{u\in\mathscr{B}_{p,k};\ u=0\ \text{ in }\ \Omega'\}.$$

If $p\neq\infty$ the dual space of $\mathscr{B}_{p,k}(\bar{\Omega}')$ is $V_{p',k'}(\bar{\Omega}')$, the annihilator of $N_{p,k}(\Omega')$ in $\mathscr{B}_{p',k'}$. It is obvious that $C_0^\infty(\bar{\Omega}')\subset V_{p',k'}(\bar{\Omega}')\subset\mathscr{E}'(\bar{\Omega}')$.

A differential operator P of constant strength in $\Omega\supset\supset\Omega'$ induces a continuous linear map

$$P\colon \mathscr{B}_{p,k\tilde{P}}(\bar{\Omega}')\to\mathscr{B}_{p,k}(\bar{\Omega}').$$

The space $\mathscr{B}_{p,k\tilde{P}}(\bar{\Omega}')$ is clearly independent of the point x_0 chosen to define $\tilde{P}$. The following theorem gives a sufficient condition for the image of P to have finite codimension.

Theorem 3.1. *Let Ω be an open set in $\mathbf{R}^n$ and let P be a differential operator of constant strength in Ω. Assume that*

$$Q\in L(P),\quad w\in\mathscr{E}'(\Omega),\quad {}^tQw=0\Rightarrow w=0. \tag{3.1}$$

Then

$$N=\{w\in\mathscr{E}'(\Omega);\ {}^tPw=0\}\subset C_0^\infty(\Omega).$$

Let Ω' be open, $\Omega'\subset\subset\Omega$. Then $N'=N\cap\mathscr{E}'(\bar{\Omega}')$ is finite dimensional. If $f\in\mathscr{B}_{p,k}(\bar{\Omega}')$ and $\langle f,\varphi\rangle=0$ for all $\varphi\in N'$ there exists some $u\in\mathscr{B}_{p,k\tilde{P}}(\bar{\Omega}')$ such that $Pu=f$ in Ω'.

Theorem 3.1 applies to hypoelliptic operators of constant strength, for if $P_x(D)$ is hypoelliptic for all x then every $Q\in L(P)$ is a nowhere vanishing function.

In this case the result is of course very well known. Operators P with analytic coefficients also satisfy the condition (3.1). In fact every localization Q of P has analytic coefficients. Let $w\in\mathscr{E}'(\Omega)$ and ${}^tQw=0$. Denote the principal part of Q by q. One can find $v\in\mathbf{R}^n$ such that $q(x_0, v)\neq 0$ and then $q(x, v)\neq 0$ for all $x\in\Omega$ since Q has constant strength. Holmgren's uniqueness theorem now implies that $w=0$ in a neighborhood of an affine hyperplane parallel to $\{x;\ \langle x, v\rangle=0\}$ if this is true on one side. Hence w is identically 0. This also shows that the space N is $\{0\}$ when P has analytic coefficients. More generally if each $Q\in L(P)$ has the unique continuation property over all hyperplanes parallel to $\{x;\ \langle x, v\rangle=0\}$ for some v then (3.1) holds. We have that situation for example if $L(P)$ only contains operators of order 1, for a first order operator of constant strength has constant coefficients in the principal part after multiplication by a C^∞ function and a suitable local change of coordinates. For a proof of this see for example Duistermaat—Hörmander [3].

Proof of Theorem 3.1. First we show that $N\subset C_0^\infty$. So let $w\in\mathscr{E}'(\Omega)\diagdown C_0^\infty$ and ${}^tPw=0$. In order to make use of (3.1) we take a point $\xi\in\mathbf{R}^n$ and observe that

$$0=\exp(-i\langle\cdot,\xi\rangle)\,{}^tP(\cdot,D)w/\tilde{P}(-\xi)={}^tP(\cdot,D+\xi)(\exp(-i\langle\cdot,\xi\rangle)w)/\tilde{P}(-\xi). \tag{3.2}$$

Suppose that there exists a sequence $\xi_j\in\mathbf{R}^n$ and constants $t_j\in\mathbf{C}$ such that $\xi_j\to\infty$ and $t_j\exp(-i\langle\cdot,\xi_j\rangle)\,w$ converges in $\mathscr{E}'$ to a distribution $w_0\neq 0$ when $j\to\infty$. We may assume that

$${}^tP(\cdot,D+\xi_j)/\tilde{P}(-\xi_j)\to{}^tQ(\cdot,D)$$

for some $Q\in L(P)$. Then by multiplying (3.2) with t_j and letting j tend to infinity it follows that ${}^tQw_0=0$. But that contradicts (3.1) so the following lemma will complete the proof that $N\subset C_0^\infty$.

Lemma 3.2. *Let $w\in\mathscr{E}'\diagdown C_0^\infty$. Then there is a sequence $\xi_j\to\infty$ in $\mathbf{R}^n$, constants $t_j\in\mathbf{C}$ and a distribution $w_0\in\mathscr{E}'$ not equal to 0 such that $t_j\exp(-i\langle\cdot,\xi_j\rangle)\,w\to w_0$ in $\mathscr{E}'$.*

Proof. It is sufficient to show that there are constants C and N and a sequence $\xi_j\to\infty$ such that

$$|\hat{w}(\xi+\xi_j)/\hat{w}(\xi_j)|\leqq C(1+|\xi|)^N,\quad\forall\,\xi\in\mathbf{R}^n. \tag{3.3}$$

In fact this means that the sequence $\exp(-i\langle\cdot,\xi_j\rangle)\,w/\hat{w}(\xi_j)$ is bounded in $\mathscr{B}_{\infty,h_{-N}}$ where h_{-N} is defined by (2.2). Then there is a subsequence ξ_{j_k} of ξ_j such that

$$\exp(-i\langle\cdot,\xi_{j_k}\rangle)\,w/\hat{w}(\xi_{j_k})$$

has a limit w_0 in $\mathscr{B}_{\infty,h_{-N-1}}$. It is clear that $w_0\neq 0$ for $\hat{w}_0(0)=1$. To prove (3.3) note that since $w\in\mathscr{E}'\diagdown C_0^\infty$ there is a number M such that

$$|\hat{w}(\xi)|\leqq C_1(1+|\xi|)^{M+1/2}$$

for some $C_1>0$ and

$$f(R)=\sup_{|\xi|=R}|\hat{w}(\xi)|(1+R)^{-M}$$

is not a bounded function of R. Then $f(R)\leqq C_1(1+R)^{1/2}$ so

$$S_j=\sup_{R\geqq 0}\big(f(R)-R/j\big)<\infty$$

and is attained at a point R_j. The numbers S_j tend to infinity for if they were bounded then f would be bounded. Then R_j must also tend to infinity. For $R\geqq -R_j$ we have

$$(3.4)\quad f(R+R_j)/f(R_j)\leqq\big(S_j+(R+R_j)/j\big)/f(R_j)=\big(f(R_j)+R/j\big)/f(R_j)\leqq 1+|R|$$

if j is large enough. Let ξ_j be a point where

$$|\xi_j|=R_j,\quad f(R_j)=|\hat{w}(\xi_j)|(1+R_j)^{-M}.$$

For given $\xi\in\mathbf{R}^n$ put $R=|\xi+\xi_j|-R_j$. Then $-R_j\leqq R$ and $|R|\leqq|\xi|$. The definition of f gives that

$$|\hat{w}(\xi+\xi_j)|\leqq f(|\xi+\xi_j|)(1+|\xi+\xi_j|)^M=f(R_j+R)(1+R_j+R)^M.$$

Thus

$$|\hat{w}(\xi+\xi_j)/\hat{w}(\xi_j)|\leqq f(R_j+R)(1+R_j+R)^M/\big(f(R_j)(1+R_j)^M\big)\leqq(1+|R|)(1+|R|)^{|M|}.$$

The last inequality follows from (3.4) and the fact that

$$(1+R_j+R)^M/(1+R_j)^M\leqq(1+|R|)^{|M|}.$$

Hence (3.3) is valid and the lemma is proved.

Now it is easy to obtain that the space N' in Theorem 3.1 is finite dimensional. For N' is a closed subspace of L^2. The injection $N'\to\mathscr{H}_{(1)}$ is everywhere defined so by the closed graph theorem

$$\|\varphi\|_{(1)}\leqq C\|\varphi\|_{(0)},\quad \varphi\in N'.$$

A sequence in $\mathscr{E}'(\bar{\Omega}')$ which is bounded in $\mathscr{H}_{(1)}$ has a subsequence which converges in L^2 so this inequality implies that N' is locally compact. A Banach space which is locally compact is finite dimensional so it follows that N' is finite dimensional.

We shall complete the proof of Theorem 3.1 by applying the Hahn—Banach theorem. For that we need the estimate in the following lemma.

Lemma 3.3. *Let the hypothesis of Theorem* 3.1 *be fulfilled. For all* $k\in\mathscr{K}$ *and* $p\in[1,\infty]$ *there is a constant* B *such that if* $h=k\tilde{P}'$ *then*

$$(3.5)\qquad \|v\|_{p,k}\leqq B\|{}^tPv\|_{p,h}\quad\textit{if}\quad v\in V_{p,k}(\bar{\Omega}')\quad\textit{and}\quad\langle v,\varphi\rangle=0,\ \forall\varphi\in N'.$$

The same B *can be used for all* k *satisfying* (2.1) *with fixed* C *and* N.

Proof. Assume that the statement of the lemma is not true. Then for all positive integers j there exists $v_j \in V_{p,k}(\bar{\Omega}')$ orthogonal to N' and k_i satisfying (2.1) such that

(3.6) $$1 = \|v_j\|_{p,k_j} > j\,\|{}^tPv_j\|_{p,h_j},$$

where $h_j = k_j\tilde{P}'$. It is easiest to get a contradiction from (3.6) if $p=\infty$ so we consider that case first. One can then find ξ_j such that

(3.7) $$\hat{v}_j(\xi_j)k_j(\xi_j) \to 1.$$

Define w_j by

$$\hat{w}_j(\xi) = k_j(\xi_j)\hat{v}_j(\xi+\xi_j).$$

By the equality in (3.6) and (2.1) we have

$$|\hat{w}_j(\xi)| \leqq (1+C|\xi|)^N$$

so there is a subsequence which we also denote by w_j which has a limit w in $\mathscr{E}'$. Obviously $w \neq 0$ for (3.7) means that $\hat{w}_j(0) \to 1$. By the remarks before Definition 2.3 any sequence ξ_j in $\mathbf{R}^n$ has a subsequence which defines a localization of P. If ξ_j does not tend to ∞ then the localization is just a constant times a translation of P. Thus after possibly passing to a subsequence

$$Q_j(\cdot,\xi) = {}^tP(\cdot,\xi+\xi_j)/\tilde{P}(-\xi_j) \to {}^tQ(\cdot,\xi).$$

Note that

$$\widehat{Q_j w_j} = \widehat{{}^tPv_j}(\cdot+\xi_j)k_j(\xi_j)/\tilde{P}(-\xi_j).$$

From the inequality in (3.6) it follows that Q_jw_j tends to 0 in the space $\mathscr{B}_{\infty,h_{-s}}$ if s is the constant occurring instead of N in the estimate (2.1) for h and h_{-s} is defined by (2.2). Hence ${}^tQw=0$. Then the sequence ξ_j cannot tend to infinity because (3.1) is valid, so we may assume that ξ_j has a limit $\xi_0 \in \mathbf{R}^n$. Then

$${}^tQ(\cdot,\xi) = {}^tP(\cdot,\xi+\xi_0)/\tilde{P}(-\xi_0), \quad w = Av_0\exp(-i\langle\cdot,\xi_0\rangle)$$

where v_0 is a limit of v_j in $\mathscr{E}'$ and A is a limit of $k_j(\xi_j)$. Clearly v_0 is orthogonal to N' and not equal to 0. But the fact that ${}^tQw=0$ implies now that ${}^tPv_0=0$, that is, $v_0 \in N'$. This is a contradiction so Lemma 3.3 is proved in the case $p=\infty$.

To be able to use the same idea of proof if $p \neq \infty$ one needs a lemma.

Lemma 3.4. *Let $k \in \mathscr{K}$ and $p<\infty$. For $\eta \in \mathbf{R}^n$ define a function $k_\eta \in \mathscr{K}$ by*

(3.8) $$k_\eta(\xi) = (1+|\xi-\eta|^2)^{-M/2}k(\xi), \quad \xi \in \mathbf{R}^n.$$

Let K be a compact set in $\mathbf{R}^n$. If M is sufficiently large the function $\eta \to \|u\|_{\infty,k_\eta}$ belongs to L^p and the norm $\|u\|_{p,k}$ is equivalent to the norm

$$|||u|||_{p,k} = \left(\int \|u\|^p_{\infty,k_\eta}\,d\eta\right)^{1/p}$$

for $u \in \mathscr{E}'(K) \cap \mathscr{B}_{p,k}$.

Proof. It is clear that $\|u\|_{\infty,k_\eta} \geqq |\hat{u}(\eta)k(\eta)|$ so

$$\|u\|_{p,k}^p \leqq \int \|u\|_{\infty,k_\eta}^p \, d\eta.$$

To prove the opposite estimate choose $\chi \in C_0^\infty$ such that $\chi=1$ in a neighborhood of K. Then $\hat{u}=(2\pi)^{-n}\hat{u}*\hat{\chi}$ if $u\in\mathscr{E}'(K)$. If M is large there is a positive constant C_1 such that

$$k(\xi) \leqq C_1(1+|\xi-\theta|)^M k(\theta) \quad \text{for all} \quad \xi, \theta \in \mathbf{R}^n$$

and for $M>0$ we have

$$(1+|\xi-\eta|)^{-M} \leqq (1+|\theta-\eta'|)^{-M}(1+|\xi-\theta|)^M \quad \text{for all} \quad \xi, \eta, \theta \in \mathbf{R}^n.$$

From these estimates, the fact that $\hat{u}=(2\pi)^{-n}\hat{u}*\hat{\chi}$ and Hölder's inequality it follows that

$$|k(\xi)(1+|\xi-\eta|)^{-M}\hat{u}(\xi)| \leqq C_1\|\hat{\chi}(1+|\cdot|)^{2M}\|_{p'}\Big(\int |\hat{u}(\theta)k(\theta)(1+|\theta-\eta|)^{-M}|^p \, d\theta\Big)^{1/p}.$$

If $Mp>n$ we obtain by integrating with respect to η that there is a constant C independent of $u\in\mathscr{E}'(K)\cap\mathscr{B}_{p,k}$ such that

$$\int \|u\|_{\infty,k_\eta}^p \, d\eta \leqq C\|u\|_{p,k}^p.$$

The proof is complete.

End of the proof of Lemma 3.3. Recall that $h=k\tilde{P}'$. Define k_η by (3.8) and h_η in the same way. By Lemma 3.4 one can choose M so large that $\|u\|_{p,k}$ and $|||u|||_{p,k}$ as well as $\|u\|_{p,h}$ and $|||u|||_{p,h}$ are equivalent for $u\in\mathscr{E}'(\bar{\Omega}')$. The functions k_η satisfy the estimate (2.1) with the same constants for all η. Thus it follows from the first part of the proof that

$$\|v\|_{\infty,k_\eta} \leqq B\|{}^tPv\|_{\infty,h_\eta} \quad \text{if} \quad v\in V_{p,k}(\bar{\Omega}') \quad \text{and} \quad v\perp N'.$$

Now (3.5) follows by integrating with respect to η. This completes the proof of Lemma 3.3.

End of the proof of Theorem 3.1. Let $f\in\mathscr{B}_{p,k}(\bar{\Omega}')$ and $v\in C_0^\infty(\bar{\Omega}')$. The estimate (3.5) gives

$$|\langle f, v\rangle| \leqq B\|f\|_{p,k}\|{}^tPv\|_{p',(k\tilde{P})'}$$

if $v\perp N'$. If $f\perp N'$ this inequality is in fact valid for all $v\in C_0^\infty(\bar{\Omega}')$ for v can be written $v=v_1+v_2$ with $v_1\perp N'$, $v_2\in N'$ and when $v_2\in N'$ both sides are 0. The linear form

$${}^tPv \to \langle f, v\rangle$$

is thus continuous on a subspace of $V_{p',(k\tilde{P})'}(\bar{\Omega}')$. By the Hahn—Banach theorem it can be extended to a continuous linear form u on $V_{p',(k\tilde{P})'}(\bar{\Omega}')$ such that $\langle u, {}^tPv\rangle = \langle f, v\rangle$ for all $v\in C_0^\infty(\bar{\Omega}')$. That means $Pu=f$ in Ω' and $u\in\mathscr{B}_{p,k\tilde{P}}(\bar{\Omega}')$ if $p'\neq\infty$. This completes the proof of Theorem 3.1 in case $p\neq 1$.

From the estimate (3.5) one can obtain the following result which contains an existence theorem for the C^∞ case and the statement of Theorem 3.1 for $p=1$. The method of proof is well known.

Theorem 3.5. *Let Ω be an open subset of $\mathbf{R}^n$ and let P be an operator of constant strength in Ω. Assume that* (3.1) *is fulfilled. Let*

$$F = \bigcap_{j=1}^{\infty} \mathscr{B}_{p_j, k_j}, \quad F_1 = \bigcap_{j=1}^{\infty} \mathscr{B}_{p_j, k_j \tilde{P}},$$

where $1 \leqq p_j < \infty$ and $k_j \in \mathscr{K}$. If $\Omega' \subset \subset \Omega$, $f \in F$ and $\langle f, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\bar{\Omega}')$ such that ${}^tP\varphi = 0$ then there exists $u \in F_1$ such that $Pu = f$ in Ω'.

Proof. F and F_1 are Fréchet spaces and so is

$$F_0 = \{f \in F;\ f = 0 \text{ in } \Omega'\}$$

and the quotient space $F_q = F/F_0$. The dual space of F_q is

$$F'_q = \{w;\ w \in \mathscr{B}_{p'_j, k'_j} \text{ for some } j, \langle w, f \rangle = 0 \text{ if } f \in F_0\}.$$

This is of course a subspace of $\mathscr{E}'(\bar{\Omega}')$ containing $C_0^\infty(\bar{\Omega}')$. We have to show that the image of P: $F_1 \to F_q$ is the annihilator of

$$N' = \{\varphi \in F'_q;\ {}^tP\varphi = 0\}.$$

That follows if the range of ${}^tP$ is weakly closed in F'_1 (see e.g. Dieudonné—Schwartz [2, Th. 7]). By a theorem of Banach (see Bourbaki [1, Ch. III, Th. 5]) this means that the intersection of the range of ${}^tP$ and the unit ball in $\mathscr{B}_{p'_j, (k_j \tilde{P})'}$ shall be weakly closed for every j. The weak topology is metrizable on the unit ball since it is equivalent to the weak topology on the unit ball of a dual of a separable Banach space. Let us therefore consider a sequence

$$v_\nu \in F'_q, \quad {}^tPv_\nu = w_\nu, \quad \|w_\nu\|_{p'_j, (\tilde{P}k_j)'} \leqq 1$$

and suppose that w_ν tends to a limit w weakly in F'_1. We may assume that $v_\nu \perp N$ so the estimate (3.5) gives that

$$\|v_\nu\|_{p'_j, k'_j} \leqq B.$$

Then there is a subsequence of v_ν which has a weak limit v in F'. Clearly $v \in F'_q$ and ${}^tPv = w$ so the proof is complete.

The following global existence theorem in a P-convex open set is proved in the same way as Theorem 3.5. As usual $\mathscr{B}^{\mathrm{loc}}_{p,k}(\Omega)$ is the space of $u \in \mathscr{D}'(\Omega)$ such that $\varphi u \in \mathscr{B}_{p,k}$ for all $\varphi \in C_0^\infty(\Omega)$.

Theorem 3.6. *Let Ω be an open set and let P be a differential operator of constant strength in Ω. Assume that* (3.1) *is fulfilled and that Ω is P-convex, that is, for each compact subset K of Ω there exists a compact subset K' of Ω such that*

$$\operatorname{supp}{}^tPw\subset K,\quad w\in\mathscr{E}'(\Omega)\Rightarrow \operatorname{supp} w\subset K'.$$

Let

$$\mathscr{F}=\bigcap_{j=1}^{\infty}\mathscr{B}^{\mathrm{loc}}_{p_j,k_j}(\Omega),\quad \mathscr{F}_1=\bigcap_{j=1}^{\infty}\mathscr{B}^{\mathrm{loc}}_{p_j,k_j\tilde{P}}(\Omega),$$

where $k_j\in\mathscr{K}$ and $1\leqq p_j<\infty$ for all j. For every $f\in\mathscr{F}$ which is orthogonal to the finite dimensional space

$$N=\{\varphi\in C_0^\infty(\Omega);\ {}^tP\varphi=0\}$$

one can then find u such that $u\in\mathscr{F}_1$ and $Pu=f$.

Proof. That N is finite dimensional follows from the P-convexity and Theorem 3.1. $\mathscr{F}$ and $\mathscr{F}_1$ are Fréchet spaces. The dual space of $\mathscr{F}$ is

$$\mathscr{F}'=\{w\in\mathscr{E}'(\Omega);\ w\in\mathscr{B}_{p_j',k_j'},\ \text{some } j\}$$

and the dual space $\mathscr{F}_1'$ of $\mathscr{F}_1$ is defined in the same way except with $(k_j\tilde{P})'$ instead of k_j'. We have to prove that the range of P in $\mathscr{F}$ is the annihilator of N. As in the proof of Theorem 3.5 it follows that this means that the intersection of the range of ${}^tP$ in $\mathscr{F}_1'$ and the unit ball in $\mathscr{E}'(K)\cap\mathscr{B}_{p_j',(k_j\tilde{P})'}$ is weakly closed in $\mathscr{F}_1'$ for every j and every $K\subset\subset\Omega$. Let $\mathscr{W}$ be a filter in this intersection. Thus

$$w\in\mathscr{E}'(K),\quad w={}^tPv\quad\text{for some}\quad v\in\mathscr{F}',\quad \|w\|_{p_j',(k_j\tilde{P})'}\leqq 1$$

for every element w of a set in $\mathscr{W}$. We may assume that v is orthogonal to N. The P-convexity condition and Lemma 3.3 give then that $v\in\mathscr{E}'(K')$ for some compact set K' in Ω and

$$\|v\|_{p_j',k_j'}\leqq B.$$

The ball of radius B in $\mathscr{B}_{p_j',k_j'}$ is weakly compact in $\mathscr{F}'$. The inverse image by ${}^tP$ of $\mathscr{W}$ therefore has a cluster point v_0 weakly in $\mathscr{F}'$. Then ${}^tPv_0$ is a cluster point of $\mathscr{W}$ so the proof is complete.

We shall now prove a converse of Theorem 3.1 by first showing that estimates of the type (3.5) must be valid and then deducing such estimates for ${}^tQ$ when $Q\in L(P)$.

Theorem 3.7. *Let Ω and Ω' be open sets such that $\Omega'\subset\subset\Omega\subset\mathbf{R}^n$ and let P be a differential operator of constant strength in Ω. Let $p\neq\infty$ and assume that $P(\mathscr{B}_{p,k\tilde{P}}(\bar{\Omega}'))$ has finite codimension in $\mathscr{B}_{p,k}(\bar{\Omega}')$ for all $k\in\mathscr{K}$. Then for all $k\in\mathscr{K}$*

and all $Q \in L(P)$ *there is a constant* C *such that*

$$\|v\|_{p',k^tQ} \leqq C\|{}^tQv\|_{p',k}, \quad v \in C_0^\infty(\Omega'). \tag{3.9}$$

If $w \in \mathscr{E}'(\Omega')$, $Q \in L(P)$ *and* ${}^tQw=0$ *then* $w=0$.

One example of an operator such that the condition (3.1) is not fulfilled was given in the introduction. Another can be constructed in the following way. Let $P_0(x, D)$ be the operator of Theorem 2 in Pliš [10] considered as an operator in $\mathbf{R}^4$. Denote the last variable in $\mathbf{R}^4$ by y. If

$$P(x, y, D) = D_y^2 P_0(x, D_x) + D_y Q_1(x, D_x) + Q_2(x, D_x)$$

where Q_1 and Q_2 are of order $\leqq 3$ then P has constant strength. Since $P_0(x, D_x) \in L(P)$ the condition (3.1) is not satisfied. Thus Theorem 3.7 shows that the conclusion of Theorem 3.1 cannot hold for this P.

Proof of Theorem 3.7. Let k_0 be any function in $\mathscr{K}$ and set $h_0 = k_0/\tilde{P}$. Let $\varphi_1, \dots, \varphi_N$ be representatives for a basis in $\mathscr{B}_{p,h_0}(\bar{\Omega}')/P(\mathscr{B}_{p,k_0}(\bar{\Omega}'))$. The space $C^\infty(\bar{\Omega}')$ is dense in $\mathscr{B}_{p,h_0}(\bar{\Omega}')$ since $p \neq \infty$, and $P(\mathscr{B}_{p,k_0}(\bar{\Omega}'))$ is closed as it has finite codimension and P is continuous. Therefore $\varphi_1, \dots, \varphi_N$ can be chosen in $C^\infty(\bar{\Omega}')$. Define a continuous linear operator T from $\mathscr{B}_{p,k_0}(\bar{\Omega}') \oplus \mathbf{C}^N$ to $\mathscr{B}_{p,h_0}(\bar{\Omega}')$ by

$$T(v, a_1, \dots, a_N) = Pv + \sum_{\nu=1}^N a_\nu \varphi_\nu$$

The adjoint ${}^tT$ of T is a continuous linear operator from $V_{p',h_0'}(\bar{\Omega}')$ to $V_{p',k_0'}(\bar{\Omega}') \oplus \mathbf{C}^N$ and

$${}^tT(v) = ({}^tPv, \langle \varphi_1, v\rangle, \dots, \langle \varphi_N, v\rangle).$$

Since T is surjective ${}^tT$ is injective and its image is closed. Then by the closed graph theorem it has a continuous inverse so

$$\|v\|_{p',h_0'} \leqq C\big(\|{}^tPv\|_{p',k_0'} + \sum_{\nu=1}^N |\langle v, \varphi_\nu\rangle|\big), \quad v \in V_{p',h_0'}(\bar{\Omega}'). \tag{3.10}$$

From this inequality we are going to obtain (3.9) for given $Q \in L(P)$ and $k \in \mathscr{K}$. For some sequence $\xi_j \to \infty$ we have

$${}^tQ(\cdot, \xi) = \lim {}^tP(\cdot, \xi + \xi_j)/\tilde{P}(-\xi_j).$$

To get ${}^tQ$ instead of ${}^tP$ in (3.10) it is natural to replace v by $\exp(i\langle x, \xi_j\rangle)v$ and divide both sides by $\tilde{P}(-\xi_j)$. Indeed, $\|{}^tPv\|_{p',k_0'}$ is then replaced by

$$\|{}^tP(\cdot, D+\xi_j)v/\tilde{P}(-\xi_j)\|_{p',k_j} \tag{3.11}$$

where

$$k_j(\xi) = 1/k_0(-\xi-\xi_j).$$

The term $\|v\|_{p',h'_0}$ is replaced by

$$\|v\|_{p',\tilde{P}_j k_j} \tag{3.12}$$

with the notation

$$\tilde{P}_j(\xi) = \tilde{P}(-\xi-\xi_j)/\tilde{P}(-\xi_j).$$

The sum is replaced by

$$\sum_{\nu=1}^{N} |\widehat{\varphi_\nu v}(-\xi_j)/\tilde{P}(-\xi_j)|.$$

Now let j tend to infinity. Then the sum obviously tends to 0 since $v\varphi_\nu \in C_0^\infty$ if $v \in C_0^\infty(\Omega')$. The functions

$${}^tP(\cdot, D+\xi_j)v/\tilde{P}(-\xi_j)$$

tend to ${}^tQv$ in $\mathcal{S}$. Clearly $\tilde{P}_j \to {}^t\tilde{Q}$ uniformly on every compact set and $\tilde{P}_j$ is uniformly bounded by a power of $(1+|\xi|)$. Assume that $k_j(\xi)$ tends to $k(\xi)$ uniformly on every compact set and is uniformly bounded by a power of $(1+|\xi|)$. Then if $p' \neq \infty$ it follows by dominated convergence that the limits of (3.11) and (3.12) as $j \to \infty$ are $\|{}^tQv\|_{p',k}$ and $\|v\|_{p',k^t\tilde{Q}}$ respectively. If $p'=\infty$ it is also clear that we obtain (3.9) when $j\to\infty$. Thus the following lemma applied to k' will complete the proof of (3.9).

Lemma 3.8. *Let $\xi_j \in \mathbf{R}^n$, $\xi_j \to \infty$ and $k \in \mathcal{K}$. Then there is a subsequence ξ_{j_k} of ξ_j and a function $k_0 \in \mathcal{K}$ such that $k_0(\xi+\xi_{j_k}) \to k(\xi)$ uniformly on every compact set.*

Proof. Take a sequence $r_k \in \mathbf{R}$, $r_k \to \infty$, and a subsequence ξ_{j_k} of ξ_j such that the sets $M_k = \{\xi;\ |\xi-\xi_{j_k}| \leqq 2r_k\}$ are all disjoint. To shorten notations we assume that $\xi_{j_k} = \xi_k$. Let $A_k = \{\xi;\, r_k \leqq |\xi-\xi_k| \leqq 2r_k\}$ and $m_k = \{\xi;\ |\xi-\xi_k| \leqq r_k\}$. For $\xi \in A_k$ define $\tilde{\xi}$ by

$$\frac{1}{2}(\xi-\xi_k+\tilde{\xi}-\xi_k) = r_k(\xi-\xi_k)/|\xi-\xi_k|.$$

The point $\tilde{\xi}$ is the reflection of ξ in the tangent plane of ∂m_k where the line through ξ_k and ξ cuts ∂m_k. Geometrically, or by writing down the lengths of $\xi-\eta$ and $\tilde{\xi}-\tilde{\eta}$ by the cosine theorem one easily sees that $|\tilde{\xi}-\tilde{\eta}| \leqq |\xi-\eta|$ if $\xi, \eta \in A_k$. Define a positive function k_0 by

$$\begin{aligned} k_0(\xi) &= k(\xi-\xi_k) \quad &&\text{when} \quad \xi \in m_k \\ k_0(\xi) &= k(\tilde{\xi}-\xi_k) \quad &&\text{when} \quad \xi \in A_k \\ k_0(\xi) &= k(0) \quad &&\text{when} \quad \xi \in \mathbf{R}^n \setminus \cup M_k. \end{aligned}$$

If C and N are the constants occurring in the estimate (2.1) for k it is clear that

$$k_0(\xi) \leqq (1+C|\xi-\eta|)^N k_0(\eta) \tag{3.13}$$

if both ξ and η belong to the same m_k or the same A_k. If $\xi\in m_k$ and $\eta\in A_k$ take a point $\xi_0\in\partial m_k$ such that $|\xi-\xi_0|\leqq|\xi-\eta|$ and $|\eta-\xi_0|\leqq|\xi-\eta|$. Then by applying (3.13) first to ξ and ξ_0 then to ξ_0 and η we get

$$k_0(\xi)\leqq(1+C|\xi-\eta|)^{2N}k_0(\eta).$$

If $\xi\in M_{k_1}$ and $\eta\in M_{k_2}$ we obtain in the same way by taking a point ξ_0 on ∂M_{k_1} and applying this estimate twice that

$$k_0(\xi)\leqq(1+C|\xi-\eta|)^{4N}k_0(\eta).$$

Thus this last inequality is valid for all $\xi,\eta\in\mathbf{R}^n$ and the proof of the lemma is complete.

End of the proof of Theorem 3.7. To obtain the last statement of the theorem we just have to note that $C_0^\infty(\Omega')$ is dense in $\mathscr{E}'(\Omega')\cap\mathscr{B}_{p',k^t\tilde{Q}}$ if $p\neq1$. Hence (3.9) can be extended by continuity to $v\in\mathscr{E}'(\Omega')\cap\mathscr{B}_{p',k^t\tilde{Q}}$. Since any $v\in\mathscr{E}'(\Omega')$ belongs to some $\mathscr{B}_{p,k}$ space with $p\neq1,\infty$ it follows that the equation ${}^tQw=0$ cannot have any nontrivial solution in $\mathscr{E}'(\Omega')$. This completes the proof of Theorem 3.7.

The following theorem shows that in order to verify that the condition (3.1) is fulfilled it is sufficient to consider C_0^∞ densities in certain subspaces.

Theorem 3.9. *Let Ω be an open set in $\mathbf{R}^n$ and let P be a differential operator of constant strength in Ω. Assume that for all $Q\in L(P)$ and all affine subspaces Σ parallel to $\Lambda'(Q)$*

(3.14) $$\varphi\in C_0^\infty(\Sigma\cap\Omega),\quad {}^tQ_\Sigma\varphi=0\Rightarrow\varphi=0.$$

Then for all $Q\in L(P)$ we have

(3.15) $${}^tQw=0,\quad w\in\mathscr{E}'(\Omega)\Rightarrow w=0.$$

Recall that Q_Σ is the operator Q considered as a differential operator in the open set $\Sigma\cap\Omega$ of Σ.

Proof of Theorem 3.9. The theorem is proved by induction over the dimension n.

1. When $n=1$ all localizations of P at infinity are nowhere vanishing functions so (3.15) is trivially valid.

2. Assume that the theorem is true for all differential operators in open sets of $\mathbf{R}^j$ when $j<n$ and let P be an operator in $\Omega\subset\mathbf{R}^n$ satisfying the hypothesis of the theorem. Let $Q\in L(P)$. Then $\dim\Lambda'(Q)<n$. Let Σ be parallel to $\Lambda'(Q)$ and consider the operator Q_Σ in the open set $\Sigma\cap\Omega\subset\mathbf{R}^j$. If $R\in L(Q)$ then $R\in L(P)$.

Hence all $R\in L(Q_\Sigma)$ satisfy (3.14) and then the induction hypothesis gives that the conclusion of the theorem is valid for Q_Σ, that is

$$R\in L(Q_\Sigma),\quad w\in\mathscr{E}'(\Omega\cap\Sigma),\quad {}^tRw=0\Rightarrow w=0.$$

Here $\mathscr{E}'(\Omega\cap\Sigma)$ denotes distributions of compact support in the open set $\Omega\cap\Sigma\subset\mathbf{R}^j$ and not a space of distributions in $\mathbf{R}^n$. Thus Q_Σ satisfies the condition (3.1) and therefore Theorem 3.1 and (3.14) imply that ${}^tQ_\Sigma$ has no non trivial solution in $\mathscr{E}'$. Then Lemma 3.3 gives that if $\Omega'\subset\subset\Omega$ and $k\in\mathscr{K}$ there is a constant C such that

$$\|\varphi\|_{2,k^t\tilde{Q}}\leqq C\|{}^tQ_\Sigma\varphi\|_{2,k}\quad\text{for all}\quad\varphi\in C_0^\infty(\Omega'\cap\Sigma). \tag{3.16}$$

We will prove such an estimate for functions $\Phi\in C_0^\infty(\Omega)$ with ${}^tQ$ instead of ${}^tQ_\Sigma$. Then (3.15) will follow easily. Choose coordinates (x',x'') so that

$$\Lambda'(Q)=\{(x',x'');\ x''=0\},\quad\Sigma=\{(x',x'');\ x''=x_0''\}.$$

Now ${}^t\tilde{Q}$ depends only on the ξ' variables. Let Ω' be relatively compact in Ω and choose $\Phi\in C_0^\infty(\Omega)$ so that $(x',x_0'')\in\Sigma\cap\Omega'$ if $(x',x'')\in\operatorname{supp}\Phi$. Then the function $\hat{\Phi}''(\cdot,\xi'')$ given by

$$\hat{\Phi}''(x',\xi'')=\int\exp(-i\langle x'',\xi''\rangle)\,\Phi(x',x'')\,dx''$$

belongs to $C_0^\infty(\Omega'\cap\Sigma)$. Take k_1 and $k_2\in\mathscr{K}$ such that k_1 only depends on the ξ' variables and k_2 only depends on the ξ'' variables. If we apply (3.16) to $\hat{\Phi}''(x',\xi'')$, multiply by $k_2(\xi'')$ and integrate it follows that

$$\|\Phi\|_{2,k^t\tilde{Q}}\leqq C\|{}^tQ(x',x_0'',D_{x'})\Phi\|_{2,k}, \tag{3.17}$$

if $k=k_1k_2$. To obtain ${}^tQ(x',x'',D_{x'})$ in the right hand side note that since Q has constant strength we have

$${}^tQ(x',x'',D_{x'})-{}^tQ(x',x_0'',D_{x'})=\textstyle\sum_j c_j(x',x'')Q_j(D_{x'})$$

with some $Q_j\prec{}^tQ_{x_0}$, $c_j\in C^\infty$ such that $c_j(x',x_0'')=0$. If the support of Φ is sufficiently near Σ this shows that

$$\|{}^tQ(x',x_0'',D_{x'})\Phi\|_{2,k}\leqq\|{}^tQ\Phi\|_{2,k}+1/(2C)\|\Phi\|_{2,k^t\tilde{Q}}. \tag{3.18}$$

The estimates (3.17) and (3.18) imply that

$$\|\Phi\|_{2,k^t\tilde{Q}}\leqq 2C\|{}^tQ\Phi\|_{2,k} \tag{3.19}$$

if the support of Φ belongs to

$$\{(x',x'')\in\Omega;\ |x''-x_0''|<\varepsilon\text{ and }(x',x_0'')\in\Sigma\cap\Omega'\} \tag{3.20}$$

and ε is sufficiently small. The estimate (3.19) can be extended by continuity to $w\in\mathscr{B}_{2,k^t\tilde{Q}}$ with support in the set (3.20). Now let $w\in\mathscr{E}'(\Omega)$ and ${}^tQw=0$. There is a partition of unity in Ω consisting of functions χ_ν depending only on the x''

variables such that an estimate of the form (3.19) is valid for each $\chi_\nu w$. Since Q contains only derivatives in the x' variables we have ${}^tQ(\chi_\nu w)=0$. It follows that $w=0$ so the proof is complete.

We shall end this section by considering operators P of constant strength defined in an open set $\Omega\subset\mathbf{R}^n$ such that $\Lambda'(P)\neq\mathbf{R}^n$. Then P is a localization of itself at infinity. Thus (3.1) cannot hold if the adjoint of some P_Σ with Σ parallel to $\Lambda'(P)$ has a non trivial solution in $\mathscr{E}'$. Clearly a necessary condition for solvability of the equation $Pu=f$ is in general that the restriction of f to each Σ parallel to $\Lambda'(P)$ satisfies a number of linear conditions. Examples of operators with $\Lambda'\neq\mathbf{R}^n$ are the non-hypoelliptic operators of local type. For these operators we shall prove an existence theorem which will be used in the next section.

First we introduce some convenient notations. For an operator Q of local type there are coordinates

$$(x', x'') = (x'_1, \ldots, x'_{n'}, x''_1, \ldots, x''_{n''})$$

such that

$$\Lambda'(Q) = \{(x', x'');\ x'' = 0\}.$$

It is natural to assume that Q is defined in a product domain

$$\Omega_c = \Omega\times\{x'';\ |x''| < c\}$$

for some Ω open in $\mathbf{R}^{n'}$ and $c>0$. For $|x''|<c$ we denote the operator $Q(x', x'', D_{x'})$ in Ω by $Q_{x''}$.

Theorem 3.10. *Let Q be of local type defined in a product domain Ω_c as above and let ω be relatively compact in Ω. If ε is small enough there is for each x'' with $|x''|<\varepsilon$ defined a linear operator $E_{x''}$ from $\mathscr{D}'(\Omega)$ to $\mathscr{D}'(\Omega)$ such that if $f\in C^\infty(\Omega_\varepsilon)$ and*

$$u(\cdot, x'') = E_{x''}(f(\cdot, x'')) \tag{3.21}$$

then $u\in C^\infty(\Omega_\varepsilon)$. In addition $Qu=f$ near $\bar\omega\times\{x'';\ |x''|<\varepsilon\}$ if for certain finitely many functions $a_1, \ldots, a_M\in C^\infty(\Omega_\varepsilon)$ such that $a_j(\cdot, x'')\in C_0^\infty(\Omega)$ for all x'', j we have

$$\langle a_j(\cdot, x''), f(\cdot, x'')\rangle = 0, \quad j = 1, \ldots, M, \quad |x''| < \varepsilon.$$

If there is a neighborhood of $\bar\omega$ where the equation $Q_0U=F$ can be solved for all F then $E_{x''}$ can be chosen so that all a_j vanish.

Proof. By Theorem 4.2 in Hörmander [5] there is for all x'' a properly supported pseudo-differential operator $A_{x''}$ in Ω which is a parametrix of $Q_{x''}$. The construction of $A_{x''}$ shows that its symbol is a C^∞ function of (x', x'', ξ'). Thus

$$Q_{x''}A_{x''}G = G+T_{x''}G, \quad G\in\mathscr{D}'(\Omega) \tag{3.22}$$

where $T_{x''}$ is a properly supported integral operator in Ω with a kernel which is a C^∞ function of x'' with values in $C^\infty(\Omega\times\Omega)$. If $T_{x''}$ is replaced by $K_{x''}=\chi T_{x''}$, where $\chi\in C_0^\infty(\Omega)$, $\chi=1$ near $\bar{\omega}$, then (3.22) is still valid near $\bar{\omega}$. Since $T_{x''}$ is properly supported the kernel of $K_{x''}$ has support in a fixed compact set in $\Omega\times\Omega$ for all x'' near 0. The equation $G+K_{x''}G=F$ can be solved by classical Fredholm theory. For the sake of completeness we give a proof.

Lemma 3.11. *If ε is sufficiently small then there exists for all x'' with $|x''|<\varepsilon$ a properly supported integral operator $R_{x''}$ with a kernel which is a C^∞ function of x'' with values in $C_0^\infty(\Omega\times\Omega)$, such that*

$$(I+K_{x''})(I+R_{x''})=I-H_{x''}$$

where

$$H_{x''}F=\sum_{j=1}^M \langle a_j(\cdot,x''),F\rangle\varphi_j,$$

$\varphi_j\in C_0^\infty(\Omega)$ and a_j are as in the statement of Theorem 3.10.

Proof. K_0 is a compact operator from $\mathscr{H}_{(s)}$ to $\mathscr{H}_{(s)}$ for all s so it follows that $I+K_0$ is a Fredholm operator in all the spaces $\mathscr{H}_{(s)}$. Note that $F\in\mathscr{H}_{(s)}$ if and only if $F+K_0F\in\mathscr{H}_{(s)}$. Since $\psi+{}^tK_0\psi=0$ implies that $\psi\in C_0^\infty(\Omega)$ there are finitely many linearly independent functions $\psi_1,\dots,\psi_M\in C_0^\infty(\Omega)$ such that

$$F\in\operatorname{Im}(I+K_0)\Leftrightarrow\langle F,\psi_j\rangle=0,\quad j=1,\dots,M.$$

The operator $I+K_0$ is bijective from the orthogonal complement of its null space in L^2 to its range in L^2. By the closed graph theorem it has a continuous inverse Y between these spaces. Denote the orthogonal projection in L^2 on the null space of $I+{}^t\bar{K}_0$ by H_0 and the orthogonal projection in L^2 on the null space of $I+K_0$ by P_0. Note that

$$H_0F=\sum_{j=1}^M\langle F,\psi_j\rangle\bar{\psi}_j$$

if $\psi_1,\dots,\psi_M$ are chosen orthonormal in L^2. If $I+R_0=Y(I-H_0)$ then

$$(I+R_0)(I+K_0)F=F-P_0F,\quad (I+K_0)(I+R_0)F=F-H_0F$$

for all $F\in L^2$. After multiplication of the first identity by K_0 from the left we obtain

$$K_0R_0+K_0^2+K_0R_0K_0+K_0P_0=R_0+K_0+K_0R_0+H_0.$$

It follows then that

$$R_0=-H_0-K_0+K_0P_0+K_0^2+K_0R_0K_0.$$

This shows that R_0 is an operator with C^∞ kernel of compact support in $\Omega\times\Omega$ for K_0, H_0 and P_0 have this property.

Put $(K_{x''}-K_0)(I+R_0)=V_{x''}$. Since $V_0=0$ there exists $\varepsilon>0$ such that the operator $I+V_{x''}$ has an inverse $I+S_{x''}$ in L^2, say, when $|x''|<\varepsilon$. This inverse is a C^∞ function of x'' with values in $L(L^2, L^2)$. A computation similar to the one carried out for R_0 above gives that

$$S_{x''} = -V_{x''} + V_{x''}^2 + V_{x''} S_{x''} V_{x''},$$

so it follows that $S_{x''}$ is in fact an operator with a kernel which is a C^∞ function of x'' with values in $C_0^\infty(\Omega\times\Omega)$. Thus the statement of the lemma holds for $I+R_{x''}=(I+R_0)(I+S_{x''})$ and $H_{x''}F=\Sigma\langle(I+S_{x''})F, \psi_j\rangle\bar{\psi}_j$, so the proof is complete.

End of the proof of Theorem 3.10. If we do not require that all the functions a_j vanish then Lemma 3.11 shows that $E_{x''}=A_{x''}(I+R_{x''})$ has the desired properties. The proof of Lemma 3.11 shows that if K_0 is 0 then H will be 0 so all the functions a_j vanish in that case. If the equation $Q_0U=F$ can be solved for all F in an open set $\omega'\supset\supset\omega$ then $A_{x''}$ can be modified so that K_0 becomes 0. In fact by the closed graph theorem there is a continuous linear operator B from $L^2(\omega')$ to $\mathscr{B}_{2,\tilde{Q}}(\bar{\omega}')$ such that $Q_0BF=F$. Let $\psi\in C_0^\infty(\omega')$, $\psi=1$ near $\bar{\omega}$. The operator $\psi BT_{x''}$ is properly supported and it has C^∞ kernel since Q_0 is hypoelliptic. If $\tilde{A}_{x''}=A_{x''}-\psi BT_{x''}$ then

$$Q_{x''}(\tilde{A}_{x''}G) = G+T_{x''}G-Q_{x''}(\psi BT_{x''}G) = G+\tilde{T}_{x''}G,$$

where the latter equality is a definition. We have $\tilde{T}_0G=0$ in the open set where $\psi=1$. Thus if χ has support in this set and $K_{x''}=\chi\tilde{T}_{x''}$ then $K_0=0$. This completes the proof of the theorem.

4. Solutions with singularities in affine subspaces

In this section we prove extensions of Theorem 1.1. Let P be of constant strength, defined in an open set Ω, $Q\in L(P)$, Σ an affine subspace parallel to $\Lambda'(Q)$ and Σ_0 a component of $\Sigma\cap\Omega$. The first step is to rephrase the negation of the statement of Theorem 1.1 as an inequality. For a positive integer m let

$$\mathscr{F} = \{u\in C^m(\Omega);\ u\in C^\infty(\Omega\setminus\Sigma_0),\ Pu\in C^\infty(\Omega)\}.$$

$\mathscr{F}$ is a Fréchet space with the weakest topology making the maps

$$\mathscr{F}\ni u \to C^m(\Omega), \quad \mathscr{F}\ni u\to C^\infty(\Omega\setminus\Sigma_0), \quad \mathscr{F}\ni u\to Pu\in C^\infty(\Omega)$$

continuous. From the closed graph theorem one easily obtains the following lemma.

Lemma 4.1. *Let V be open, relatively compact in Ω. If*

$$\{u;\ u\in\mathscr{F},\ u\in C^{m+1}(\bar{V})\}$$

is of the second category in $\mathscr{F}$ *then there exist* $\nu\in\mathbf{Z}^+$, $K_1\subset\subset\Omega$ *and* $K_2\subset\subset\Omega\setminus\Sigma_0$ *such that*

$$\sum_{|\alpha|=m+1}\sup_V |D^\alpha u| \leqq C\{\sum_{|\alpha|\leqq m}\sup_{K_1}|D^\alpha u|+\sum_{|\alpha|\leqq\nu}\sup_{K_1}|D^\alpha(Pu)|+\sum_{|\alpha|\leqq\nu}\sup_{K_2}|D^\alpha u|\} \tag{4.1}$$

for all $u\in C^{m+1}(\overline{V})\cap\mathscr{F}$.

If we prove that (4.1) is always false when V is a neighborhood of a point in Σ_0 then there exists a function $u\in\mathscr{F}$ such that u is not in C^{m+1} in a neighborhood of any point in Σ_0. For if $u\in\mathscr{F}$ and $u\in C^{m+1}$ in a neighborhood of some point in Σ_0 then

$$u\in\big(\bigcup_{x_0,r} C^{m+1}(\{x;\ |x-x_0|\leqq r\})\big)\cap\mathscr{F}$$

where the union is taken over a countable dense set of points x_0 in Σ_0 and countably many $r>0$. Since a countable union of sets of the first category in $\mathscr{F}$ is of the first category the assertion follows. This also shows that it is sufficient to consider Q of local type. For by Proposition 2.7 there exists some Q' of local type such that $\Lambda'(Q')\subset\Lambda'(Q)$. If $x_0\in\Sigma_0$ is a given point one can therefore find a component Σ_0' of $\Sigma'\cap\Omega$ for some Σ' parallel to $\Lambda'(Q')$ such that $x_0\in\Sigma_0'\subset\Sigma_0$. If we prove that (4.1) cannot be valid for any $K_2\subset\subset\Omega\setminus\Sigma_0'$ then it cannot be valid for any $K_2\subset\subset\Omega\setminus\Sigma_0$. Hence it is no restriction to assume that Q is of local type.

By Proposition 2.4 we may assume that

$$Q(x,D)=\lim_{t\to\infty}P(x,D+\eta(t))/at^\sigma$$

where $\eta(t)$ is a polynomial in t, $a>0$ and σ is a positive integer. Note that

$$R_t(x,D)=Q(x,D)-P(x,D+\eta(t))/at^\sigma$$

has coefficients which are $O(t^{-1})$. To prove that (4.1) is false one should construct functions u^t such that the derivatives of u^t of order $m+1$ are large compared with those of order $\leqq m$, Pu^t is small and u^t is 0 in K_2. If $u_0\in C^\infty$ the function

$$u^t=\exp\big(i\langle\cdot,\eta(t)\rangle\big)u_0/at^\sigma,$$

or just u for short, satisfies the first requirement if t is large. We have

$$Pu=\exp\big(i\langle\cdot,\eta(t)\rangle\big)P(\cdot,D+\eta(t))u_0/at^\sigma$$

so if $Qu_0=0$ then Pu will be equal to $-\exp\big(i\langle\cdot,\eta(t)\rangle\big)R_tu_0$ and thus the supremum of $|Pu|$ over K_1 is $O(t^{-1})$. To get a still better approximation we try to solve

$$\exp\big(-i\langle\cdot,\eta(t)\rangle\big)P\big(u_1\exp\big(i\langle\cdot,\eta(t)\rangle\big)/at^\sigma\big)=R_tu_0.$$

Since the left hand side is approximately Qu_1 and we only have an existence theorem for Q we replace this equation by

$$Qu_1 = R_t u_0.$$

The coefficients of R_t are $O(t^{-1})$ so u_1 should be $O(t^{-1})$. We have

$$P((u_0+u_1)\exp(i\langle\cdot,\eta(t)\rangle)/at^\sigma) = -\exp(i\langle\cdot,\eta(t)\rangle)R_t u_1.$$

If we could solve $Qu_2=R_t u_1$ so that $u_2=O(t^{-2})$, $Qu_3=R_t u_2$ so that $u_3=O(t^{-3})$, and so on we could define

$$u^t = \sum_{j=0}^N u_j \exp(i\langle\cdot,\eta(t)\rangle)/at^\sigma.$$

Now the supremum of $|Pu|^t$ over K_1 decreases as t^{-N} as $t\to\infty$. By multiplying u_0 with a cutoff function which depends only on the variables of $\Lambda(Q)$ and is 0 in K_2 we could achieve that $u^t=0$ in K_2.

This idea of proof is easiest to carry through if the equation $Qu=f$ can be solved near $\Sigma_0\cap K_1$ for an arbitrary right hand side, so we consider that case first. Then the equations $Qu_1=R_t u_0$, $Qu_2=R_t u_1$, and so on, can be solved successively if there is just one function u_0 such that $Qu_0=0$ to start with.

Theorem 4.2. *Let Ω be open in $\mathbf{R}^n$ and let P be a differential operator of constant strength in Ω. Let $Q\in L(P)$ be of local type, let Σ be an affine subspace parallel to $\Lambda'(Q)$ and Σ_0 a component of $\Sigma\cap\Omega$. Assume that*

$$v\in\mathscr{E}'(\Sigma_0),\quad {}^tQ_\Sigma v = 0 \Rightarrow v = 0. \tag{4.2}$$

Denote by S the set of all $x\in\Sigma_0$ such that for all $\omega\subset\subset\Sigma_0$ and all neighborhoods V of x there exists $U_0\in C^\infty(\Sigma_0)$ such that $Q_\Sigma U_0=0$ in ω and $U_0\not\equiv 0$ in $V\cap\Sigma_0$. Then for all positive integers m there exists $u\in C^m(\Omega)$ such that $Pu\in C^\infty(\Omega)$ and $S\subset\operatorname{sing\,supp} u\subset\Sigma_0$.

Proof. We have to prove that the inequality (4.1) is false for any neighborhood V of a point $x_0\in S$ and given K_1, K_2, v. We shall construct functions u^t as indicated above. Choose an open set Ω' such that $K_1\cup K_2\subset\Omega'\subset\subset\Omega$ and an open set $\omega\subset\subset\Sigma_0$ such that $\Sigma_0\cap\Omega'\subset\subset\omega$. The coordinates $x=(x',x'')$, $x'\in\mathbf{R}^{n'}$, $x''\in\mathbf{R}^{n''}$ can be chosen so that $\Sigma=\{x;\ x''=0\}$. Put

$$\omega_\varepsilon = \{x;\ x'\in\omega,\ |x''|<\varepsilon\}$$

and define $Q_{x''}$ as before Theorem 3.10. If ε is small then $\omega_\varepsilon\subset\subset\Omega$, $K_2\cap\omega_\varepsilon=\emptyset$ and the intersection of Ω' with the boundary of ω_ε is contained in $\{x;\ |x''|=\varepsilon\}$. The condition (4.2) implies that the equation $Q_\Sigma U=F$ can be solved for any F in an open relatively compact subset of Σ_0. Then Theorem 3.10 gives that if ε is sufficiently small one can for any $f\in C^\infty(\Omega)$ find $u\in C^\infty(\Omega)$ such that $Qu=f$ in ω_ε.

We need a function $u_0 \not\equiv 0$ such that $Qu_0=0$ in Ω'. Since $x_0 \in S$ there is some $U_0 \in C^\infty(\Sigma_0)$ such that $Q_\Sigma U_0=0$ near $\bar{\omega}$ and $U_0 \not\equiv 0$ in $V \cap \Sigma_0$. Choose a function $\chi \in C_0^\infty(\mathbf{R}^{n''})$ such that $\chi=1$ near $\{x;\ x''=0\}$, $\chi=0$ when $|x''|>\varepsilon/2$, and a function $\psi \in C_0^\infty(\Sigma_0)$ such that $\psi=1$ near $\bar{\omega}$ and $Q_\Sigma U_0=0$ near $\operatorname{supp}\psi$. Let $u \in C^\infty(\Omega)$ be a solution of

$$Qu = \psi Q_{x''} U_0$$

in ω_ε. Since $u(\cdot, x'')$ is a linear function of $\psi Q_{x''} U_0$ we have $u(\cdot, x'')=0$ when $x''=0$. Thus if $u_0=0$ when $|x''|>\varepsilon$ and $u_0(x', x'')=(\psi(x')U_0(x')-u(x', x''))\chi(x'')$ when $|x''|<\varepsilon$, then $u_0 \in C^\infty(\Omega)$, $Qu_0=0$ in Ω', $u_0=0$ when $|x''|>\varepsilon/2$ and $u_0=U_0$ in $\Sigma_0 \cap \omega$.

Now one can find $u_1, u_2, \ldots$ such that $Qu_0=R_t u_0$, $Qu_2=R_t u_1$, and so on. Since

$$R_t(x, D) = \textstyle\sum_{k=1}^K t^{-k} R_k(x, D),$$

where R_k are operators with C^∞ coefficients, we just solve $Qu_{1,k}=R_k u_0$ for each k and set

$$u_1^t = \textstyle\sum_{k=1}^K t^{-k} u_{1,k}.$$

The next right hand side, $R_t u_1^t$, will also be a sum of powers of t with some functions as coefficients. For each coefficient function c_k we take a solution $u_{2,k}$ of the equation $Qu_{2,k}=c_k$ and then define u_2^t as a sum of powers of t with the coefficients $u_{2,k}$ such that $Qu_2^t=R_t u_1^t$. In this way we continue with the following equations. Thus we take $u_{j,k} \in C^\infty(\Omega)$ such that

$$\begin{aligned}
&Qu_{1,k} = R_k u_0 && k = 1, \ldots, K \\
&Qu_{2,k} = \textstyle\sum_{i+\mu=k} R_i u_{1,\mu} && k = 2, \ldots, 2K \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \\
&Qu_{N,k} = \textstyle\sum_{i+\mu=k} R_i u_{N-1,\mu} && k = N, \ldots, NK
\end{aligned} \tag{4.3}$$

in ω_ε. Since $u_0=0$ when $|x''|>\varepsilon/2$ we can choose $u_{j,k}$ such that $u_{j,k}=0$ when $|x''|>\varepsilon/2$ for all j, k. Thus the equations (4.3) are valid in Ω' if we set $u_{j,k}=0$ outside ω_ε. For $j=1, 2, \ldots$ let

$$u_j^t = \textstyle\sum_{k=j}^{jK} t^{-k} u_{j,k}$$

and write $u_0^t=u_0$. Then we have

$$Qu_j^t = R_t u_{j-1}^t$$

for all j. Now put

$$u^t = \textstyle\sum_{j=0}^N \exp\left(i\langle \cdot, \eta(t)\rangle\right) u_j^t / a t^\sigma.$$

The functions u^t belong to $C^\infty(\Omega)$. They are constructed so that

$$Pu^t = -\exp\left(i\langle \cdot, \eta(t)\rangle\right) R_t u_N^t$$

in Ω' and $R_t u_N^t$ is a sum of powers of t where the highest power occurring is t^{-N-1}. Recall that the highest power of t in the expansion of $\eta(t)$ is t^J. Now look at the terms in (4.1) with $u=u^t$. The last term in the right hand side is zero and for some C_1, C_2 and $C>0$ we have

$$\sum_{|\alpha|\leqq m} \sup_{K_1} |D^\alpha u^t| \leqq C_1 t^{Jm-\sigma}$$

$$\sum_{|\alpha|\leqq \nu} \sup_{K_1} |D^\alpha P u^t| \leqq C_2 t^{J\nu-(N+1)}$$

$$\sum_{|\alpha|=m+1} \sup_{V} |D^\alpha u^t| = C \sup_{V} |u_0|\, t^{J(m+1)-\sigma} + O(t^{J(m+1)-\sigma-1}).$$

If N is so large that $J\nu-(N+1)<J(m+1)-\sigma$ we get a contradiction when $t\to\infty$ for $\sup_V |u_0|\neq 0$ since $u_0=U_0$ in Σ_0. This proves the theorem.

In general (4.2) is not fulfilled for an operator of local type. However if the domain Ω is small enough (4.2) holds and the set S in Theorem 4.2 is not empty. That gives a new proof of the following corollary which was proved with other methods by Taylor [11]. It is a converse of Theorem 7.4.1 in Hörmander [4].

Corollary 4.3. *Let P be a differential operator of constant strength in an open set $\Omega\subset\mathbf{R}^n$. Assume that* $\operatorname{sing\,supp} u=\operatorname{sing\,supp} Pu$ *for all $u\in\mathcal{D}'(\Omega)$. Then P_x is a hypoelliptic polynomial for all $x\in\Omega$, that is $P_x^{(\alpha)}(\xi)/P_x(\xi)\to 0$ when $\xi\to\infty$ if $\alpha\neq 0$.*

Proof. If P_x is not a hypoelliptic polynomial for some $x\in\Omega$ there exists some $Q\in L(P)$ of positive order. The proof of Proposition 2.7 shows that Q can be chosen of local type. We have to verify that (4.2) is fulfilled for $\Sigma_0=\Sigma\cap\omega$ when Σ is parallel to $\Lambda'(Q)$ and ω is small enough and that the set S in Theorem 4.2 is non-empty. Theorem 7.3.1 in Hörmander [4] shows that if ω is small there is a linear mapping E: $\mathcal{E}'(\mathbf{R}^{n'})\to\mathcal{E}'(\mathbf{R}^{n'})$ such that $E^t Q_\Sigma v=v$ in Σ_0 if $v\in\mathcal{E}'(\Sigma_0)$. This implies (4.2). That S is non-empty follows from Lemma 4.7 below which states that there are infinitely many linearly independent $U\in\mathcal{D}'(\Sigma_0)$ such that $Q_\Sigma U=0$ if ω is small. Since Q_Σ is hypoelliptic it follows that $U\in C^\infty$. Now Theorem 4.2 shows that there exists $u\in\mathcal{D}'(\omega)$ such that $\operatorname{sing\,supp} u\neq\emptyset$ and $Pu\in C^\infty$. This completes the proof.

Let Q be any operator of local type and as before Σ_0 a component of $\Sigma\cap\Omega$ for some Σ parallel to $\Lambda'(Q)$. We know that (4.2) is in general not valid and we wish to prove that (4.1) cannot be true for any $K_1\subset\subset\Omega$, $K_2\subset\subset\Omega\setminus\Sigma_0$, $\nu\in\mathbf{Z}^+$ if V is a neighborhood of $x_0\in\Sigma_0$, even if the hypothesis (4.2) is omitted. In order to deduce a contradiction from (4.1) as in the proof of Theorem 4.2 one must first have a function u_0 such that $Qu_0=0$ and $\sup_V |u_0|\neq 0$ and then be able to solve the system of equations (4.3). In the following theorem we will show that if we omit (4.2) but assume that the equation $Q_\Sigma U=0$ has infinitely many solutions which are linearly independent in $\Sigma_0 \cap V$, instead of just one, then it is possible to find such a function u_0.

Theorem 4.4. *Let P be a differential operator of constant strength in an open set $\Omega \subset \mathbf{R}^n$. Let $Q \in L(P)$ be of local type and let Σ_0 be a component of $\Sigma \cap \Omega$ where Σ is an affine subspace parallel to $\Lambda'(Q)$. Let S be the set of all $x \in \Sigma_0$ such that for all neighborhoods V_0 of x in Σ_0 and all $\omega \subset\subset \Sigma_0$ the space $\{u|_{V_0};\ u \in C^\infty(\Sigma_0),\ Q_\Sigma u = 0 \text{ in } \omega\}$ has infinite dimension. Here $u|_{V_0}$ denotes the restriction of u to V_0. Then for all positive integers m there exists $u \in C^m(\Omega)$ such that $Pu \in C^\infty(\Omega)$ and $S \subset \operatorname{sing\,supp} u \subset \Sigma_0$.*

Proof. It is sufficient to show that for all $V = \{x;\ |x - x_0| \leqq r\}$, $x_0 \in S$ the inequality (4.1) is not valid for any K_1, K_2, ν. Assume that (4.1) is true for some neighborhood V of $x_0 \in S$ and some K_1, K_2, ν. A contradiction will follow as in the proof of Theorem 4.2 if we show:

(4.4) For all $\omega \subset\subset \Sigma_0$, $N \in \mathbf{Z}^+$ and $\varepsilon > 0$ there exists $u_0 \in C^\infty(\omega_\varepsilon)$ such that $\sup_V |u_0| \neq 0$, $Qu_0 = 0$ in ω_ε, $u_0 = 0$ near $\{(x', x'');\ |x''| = \varepsilon\}$ and there are $u_{j,k} \in C^\infty(\omega_\varepsilon)$ vanishing near $\{(x', x'');\ |x''| = \varepsilon\}$ which are solutions of (4.3) in ω_ε.

Here the coordinates and ω_ε are as in the proof of Theorem 4.2. Thus let ω, N and ε be given. Take an open set ω' such that $\omega \subset\subset \omega' \subset\subset \Sigma_0$. Recall that Theorem 3.10 gives a number $\varepsilon' > 0$, which we may assume is equal to ε, and functions $a_1, \dots, a_M \in C^\infty(\omega'_\varepsilon)$ such that $a_i(\cdot, x'') \in C_0^\infty(\omega')$ for all i and x''. If $f \in C^\infty(\omega'_\varepsilon)$ and satisfies

$$\langle a_i(\cdot, x''), f(\cdot, x'')\rangle = 0 \quad \text{when} \quad |x''| < \varepsilon, \quad i = 1, \dots, M$$

then there is a solution $u \in C^\infty(\Omega')$ given by (3.21) of the equation $Qu = f$ in ω_ε.

We have to find a function u_0 which satisfies the conditions (4.4). Let $U_1, U_2, \dots \in C^\infty(\Sigma_0)$ be solutions of the equation $Q_0 U_j = 0$ in ω' which are linearly independent in $\Sigma_0 \cap V$. Any linear combination

$$u(x', x'') = \sum_{j=1}^J c_j(x'') U_j(x')$$

with $c_j \in C^\infty(\mathbf{R}^{n''})$ is a solution of $Qu = 0$ in $\omega' \cap \Sigma_0$. If we could find a solution of $Qu' = Qu$ in ω_ε which vanishes when $x'' = 0$ we could define $u_0 = u - u'$ and thus obtain a non trivial solution of $Qu_0 = 0$ in ω_ε. This is possible if

$$\langle a_i(\cdot, x''), Qu(\cdot, x'')\rangle = \sum_{j=1}^J c_j(x'') \langle a_i(\cdot, x''), Q_{x''} U_j\rangle = 0 \tag{4.5}$$

when $|x''| < \varepsilon$, $i = 1, \dots, M$. If $c_1, \dots, c_J$ satisfy this condition we can in view of

(3.21) let

$$u_0(\cdot, x'') = \sum_{j=1}^{J} c_j(x'')(U_j - E_{x''}Q_{x''}U_j). \tag{4.6}$$

This function u_0 belongs to $C^\infty(\omega'_\varepsilon)$, $Qu_0=0$ in ω_ε and $u_0=u$ in $\omega\cap\Sigma_0$.

Now we will find what conditions the functions c_j have to satisfy in order that (4.3) can be solved. Consider the first row. Define R_k for $k=1, \dots, K$ as in the proof of Theorem 4.2. We can write

$$R_k = \sum_\alpha R_k^\alpha D_{x''}^\alpha$$

where R_k^α are differential operators not containing $D_{x''}$. In view of (4.6) we have

$$R_k u_0 = \sum_j \sum_\alpha f_{k,\alpha,j}^1(x', x'')(D_{x''}^\alpha c_j) \tag{4.7}$$

for some functions $f_{k,\alpha,j}^1 \in C^\infty(\omega'_\varepsilon)$. Thus the first row of (4.3) can be solved if

$$\langle R_k u_0(\cdot, x''), a_i(\cdot, x'')\rangle = \sum_{j=1}^{J} A_{k,i,j}^1 c_j = 0, \tag{4.8}$$
$$|x''| < \varepsilon, \quad i = 1, \dots, M, \quad k = 1, \dots, K,$$

where $A_{k,i,j}^1$ are differential operators with C^∞ coefficients. The order of $A_{k,i,j}^1$ is less than or equal to the order of P. If (4.8) is fulfilled let

$$u_{1,k}(\cdot, x'') = E_{x''}(R_k u_0(\cdot, x'')), \quad k = 1, \dots, K. \tag{4.9}$$

We have $u_{1,k}\in C^\infty(\omega'_\varepsilon)$ and the $u_{1,k}$ are by (3.21) solutions of the first row of (4.3) in ω_ε. To solve the second row we note that the expression (4.7) for $R_k u_0$ combined with (4.9) shows that

$$\sum_{i+\mu=k} R_i u_{1,\mu} = \sum_j \sum_\alpha f_{k,\alpha,j}^2(x', x'')(D_{x''}^\alpha c_j) \tag{4.10}$$

for some $f_{k,\alpha,j}^2 \in C^\infty(\omega'_\varepsilon)$. In the same way as above we see that the functions (4.10) satisfy the conditions for the existence of solutions $u_{2,k}\in C^\infty(\omega'_\varepsilon)$ of the second row of (4.3) in ω_ε if

$$\sum_{j=1}^{J} A_{k,i,j}^2 c_j = 0 \quad \text{when} \quad |x''| < \varepsilon, \quad i = 1, \dots, M, \quad k = 2, \dots, 2K, \tag{4.11}$$

for certain differential operators $A_{k,i,j}^2$ with C^∞ coefficients. In this way we continue with the following rows in (4.3). Thus if the functions c_j satisfy (4.5), (4.8), (4.11) and the corresponding conditions arising from the later rows, then for u_0 defined by (4.6) there exist $u_{j,k}\in C^\infty(\omega'_\varepsilon)$ which are solutions of (4.3) in ω_ε. We rewrite the conditions on c_j as

$$\sum_{j=1}^{J} A_{ij} c_j = 0 \quad \text{for} \quad i = 1, \dots, I \quad \text{when} \quad |x''| < \varepsilon. \tag{4.12}$$

All A_{ij} are differential operators of order less than a fixed number G which only depends on N and the order of P. The number I depends only on N and K. The following lemma shows that it is possible to solve (4.12) if J is large enough.

Lemma 4.5. *Let $A_{i,j}$, $i=1, \dots, I$, $j=1, 2, 3, \dots, J$ be differential operators with C^∞ coefficients in an open set $\Omega\subset\mathbf{R}^n$ such that order $A_{ij}\leqq G$ for all i and j. If J is larger than a certain number which only depends on I and G, then in every neighborhood of a point $x_0\in\Omega$ one can find a point x_1 and C^∞ functions $b_1, \dots, b_J$ in a neighborhood of x_1 such that $b_1(x_1), \dots, b_J(x_1)$ are not all 0 and*

$$\sum_{j=1}^{J} A_{ij}(b_j\chi) = 0, \quad i = 1, \dots, I, \tag{4.13}$$

if supp χ *belongs to a sufficiently small neighborhood of* x_1.

Proof. If

$$\sum_{j=1}^{J} b_j{}^t A_{ij} = 0 \quad \text{in } \Omega, \quad i = 1, \dots, I,$$

then (4.13) is valid for all $\chi\in\mathscr{D}'(\Omega)$. For some C^∞ functions $d_{\alpha,i,j}$ we have

$$\sum_{j=1}^{J} b_j{}^t A_{ij} = \sum_{|\alpha|\leqq G} \left(\sum_{j=1}^{J} b_j d_{\alpha,i,j}\right) D^\alpha.$$

Label (α, i) for $i=1, \dots, I$, $|\alpha|\leqq G$ as a sequence with indices $\sigma=1, \dots, S$. It is thus sufficient to find b_j such that

$$\sum_{j=1}^{J} d_{\sigma,j} b_j = 0 \quad \text{for} \quad \sigma = 1, \dots, S \tag{4.14}$$

where S is a number which depends only on I and G. The rank of the matrix $(d_{\sigma,j}(x))$ is $\leqq S$ in Ω. In a given neighborhood Ω_1 of x_0 there is a point x_1 such that

$$\operatorname{rank}\left(d_{\sigma,j}(x_1)\right) = \max_{x\in\Omega_1} \operatorname{rank}\left(d_{\sigma,j}(x)\right) = r.$$

Then we can assume that

$$D(x) = \det\left(d_{\sigma,j}(x)\right)_{\sigma,j=1}^{r} \neq 0$$

in a neighborhood of x_1. Assume that $J>S$. Let $b_{r+1}=1, b_{r+2}=\dots=b_J=0$ and define $b_1, \dots, b_r$ so that the equations (4.14) are satisfied for $\sigma=1, \dots, r$. Then all b_j are C^∞ in a neighborhood of x_1. For all x near x_1 the later equations are linear combinations of the first r equations since the rank was maximal at x_1. Hence (4.14) is valid also for $\sigma=r+1, \dots, S$ in a neighborhood of x_1. The proof is complete.

End of the proof of Theorem 4.4. We have to find solutions of (4.12). If J is larger than a certain number we can by Lemma 4.5 choose a sequence $x''_\nu\to 0$ and functions $b_1^1, \dots, b_J^1, \dots, b_1^\nu, \dots, b_J^\nu, \dots\in C^\infty$ such that

$$\sum_{j=1}^{J} A_{ij}(b_j^\nu\chi) = 0, \quad i = 1, \dots, I,$$

if supp χ is contained in a sufficiently small neighborhood of x''_ν. For all ν some $b_j^\nu(x''_\nu)$ is equal to 1. Let $c_j^\nu=b_j^\nu\chi_\nu$ where $\chi_\nu\in C_0^\infty(\mathbf{R}^{n''})$, $\chi_\nu(x''_\nu)=1$ and supp χ_ν is contained in the permissible neighborhood of x''_ν. Then for all ν the equations (4.12) are satisfied for $c_j=c_j^\nu$.

We claim that the function

$$u_0(\cdot, x'') = \sum_{j=1}^{J} c_j^\nu(x'')(U_j - E_{x''} Q_{x''} U_j)$$

satisfies (4.4) if ν is chosen large. Since the functions c_j^ν satisfy the conditions (4.12) we have $Qu_0=0$ and there exist $u_{j,k}\in C^\infty(\omega_\varepsilon')$ which are solutions of (4.3) in ω_ε. We may choose χ_ν such that $c_j^\nu=0$ when $|x|''>\varepsilon/2$ and then clearly u_0 and $u_{j,k}$ vanish near $\{(x', x''); |x''|=\varepsilon\}$. To prove that $\sup_V |u_0|\neq 0$ if ν is chosen large note that

$$E_{x_\nu''} Q_{x_\nu''} U_j \to 0 \quad \text{in} \quad L^2(\omega) \quad \text{when} \quad \nu \to \infty.$$

In fact $Q_0 U_j=0$ and the norm of the operator $E_{x_\nu''}$ is bounded independently of x_ν''. We may assume that $|b_\nu^j(x_\nu'')|\leq 1$ for $j=1, \dots, J$ and all ν and therefore take a subsequence which we also denote by x_ν'' such that $b_j^\nu(x_\nu'')\to B_j$ when $\nu\to\infty$. One B_j must be different from 0. Hence

$$\sum_{j=1}^{J} c_\nu^j(x_\nu'')(U_j - E_{x_\nu''} Q_{x_\nu''} U_j) \to \sum_{j=1}^{J} B_j U_j$$

in $L^2(\omega)$ when $\nu\to\infty$. The limit is not identically 0 in $V\cap\Sigma_0$ since the functions U_j were linearly independent in $V\cap\Sigma_0$. It follows that u_0 is not identically 0 in V if we choose ν large. The proof of Theorem 4.4 is thus complete.

What remains in order to extend Theorem 1.1 to all operators of constant strength is to show that the set S in Theorem 4.4 is equal to Σ_0. This is easy to prove if Q has analytic coefficients. However in that case we always have (4.2) so we would only need to verify that for each $x\in\Sigma_0$ and ω such that $x\in\omega\subset\subset\Sigma_0$ there exists $u_0\in C^\infty(\Sigma_0)$ so that $Q_\Sigma u_0=0$ in ω and $x\in\operatorname{supp} u_0$. But since it follows by practically the same proof that S is equal to Σ_0 we will prove that.

Theorem 4.6. *Let Q be a differential operator of constant strength with analytic coefficients. If Ω and Ω_1 are open sets such that Q is defined in a neighborhood of $\bar{\Omega}$ and $\Omega_1\subset\subset\Omega\subset\subset\mathbf{R}^n$, $n>1$, then the space $\{u|_{\Omega_1};\ u\in\mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}),\ Qu=0$ in $\Omega\}$ has infinite dimension.*

If Q had constant coefficients the theorem would be trivial for then Q has infinitely many different exponential solutions and these are linearly independent in any open set. In the following lemma we prove by means of a perturbation argument used in Hörmander [4, Ch. VII] that there are infinitely many linearly independent solutions in Ω_1 if Ω_1 is small. Lemma 4.7 also completes the proof of Corollary 4.3.

Lemma 4.7. *Let Q be a differential operator of constant strength defined in a neighborhood of a point x_0. If Ω_1 is a sufficiently small neighborhood of x_0 then the space $\{u\in\mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1);\ Qu=0\}$ has infinite dimension.*

Proof. We can write

$$Q(x, D) = Q_{x_0}(D) + \sum_j c_j(x) Q_j(D)$$

with some $Q_j \prec Q_{x_0}$, $c_j \in C^\infty$ and $c_j(x_0)=0$. Let $E \in \mathscr{B}^{\mathrm{loc}}_{2,\tilde{Q}}$ be a fundamental solution of Q_{x_0}. If $u \in L^2(\Omega_1)$ let u_0 be the function which is equal to u in Ω_1 and vanishes elsewhere. Denote by E_0 the linear operator

$$L_2(\Omega_1) \ni u \to \text{restriction of } (E * u_0) \text{ to } \Omega_1 \in \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1).$$

As in the proof of Theorem 7.2.1 in Hörmander [4] we find that if Ω_1 is sufficiently small there is for all $f \in L^2(\Omega_1)$ a unique $g \in L^2(\Omega_1)$ such that

$$g + \sum_j c_j(x) Q_j E_0 g = f. \tag{4.15}$$

The operator Q_{x_0} has infinitely many different exponential solutions $v_1, v_2, \ldots$. Let g_k be the solution of (4.15) for

$$f = -Qv_k = -\sum_j c_j(x) Q_j(D) v_k.$$

The functions

$$u_k = v_k + E_0 g_k \in \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1)$$

satisfy $Qu_k = 0$ in Ω_1. They are linearly independent since

$$v_k = u_k - E_0 Q_{x_0} u_k$$

and the functions v_k are linearly independent. This completes the proof of the lemma.

Proof of Theorem 4.6. We will prove that there are infinitely many linearly independent solutions in Ω_1 which can be extended to solutions in Ω. The method of proof is well known (see Malgrange [8, Ch. 3, Théorème 1]). Note that Ω_1 may be replaced by a smaller subset. Let

$$N = \{u \in \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega});\ Qu = 0\}, \quad N_1 = \{u \in \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1);\ Qu = 0\}$$

and let R be the restriction operator $\mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}) \to \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1)$. If we prove that the annihilator of $R(N)$ is equal to the annihilator of N_1 then the Hahn—Banach theorem implies that $\overline{R(N)} = N_1$. The space N_1 is infinite dimensional by Lemma 4.7 so then it will follow that $R(N)$ is infinite dimensional. The image of the map

$$Q : \mathscr{B}_{2,\tilde{Q}}(\bar{\Omega}_1) \to \mathscr{B}_{2,1}(\bar{\Omega}_1) = L^2(\Omega_1)$$

is equal to $\mathscr{B}_{2,1}(\bar{\Omega}_1)$ by Theorem 7.3.1 in Hörmander [4]. Hence the annihilator of N_1 is the image of ${}^tQ$, that is

$$N_1^0 = \{{}^tQv \in V_{2,\tilde{Q}'}(\bar{\Omega}_1);\ v \in V_{2,1}(\bar{\Omega}_1)\}.$$

Here $V_{2,1}(\bar{\Omega}_1)$ is the space of functions in $L^2(\mathbf{R}^n)$ vanishing almost everywhere outside Ω_1. The annihilator of N is given in the same way with Ω_1 replaced by Ω. The annihilator of $R(N)$ consists of those elements in $V_{2,\tilde{Q}'}(\bar{\Omega}_1)$ which annihilate N when they are considered as elements of $V_{2,\tilde{Q}'}(\bar{\Omega}_1)$ so we have

$$R(N)^0 = \{{}^tQv \in V_{2,\tilde{Q}'}(\bar{\Omega}_1);\ v \in V_{2,1}(\bar{\Omega})\}.$$

Let $w \in R(N)^0$. Then $w = {}^tQv$ for some $v \in V_{2,1}(\bar{\Omega}) \subset \mathscr{E}'(\bar{\Omega})$ and $\operatorname{supp} w \subset \bar{\Omega}_1$. Holmgren's uniqueness theorem now implies, if Ω_1 is chosen convex, that $\operatorname{supp} v \subset \bar{\Omega}_1$, for a hyperplane which is non-characteristic at one point in Ω is non-characteristic everywhere since Q has constant strength. Now it follows that $w \in N_1^0$, for $v \in V_{2,1}(\bar{\Omega})$ and $\operatorname{supp} v \subset \bar{\Omega}_1$ means that $v \in V_{2,1}(\bar{\Omega}_1)$.

From Theorem 4.6 and Theorem 4.4 (or Theorem 4.2) we now obtain an extension of Theorem 1.1 to operators of constant strength with analytic coefficients.

Theorem 4.8. *Let P be a differential operator of constant strength with analytic coefficients in an open set $\Omega \subset \mathbf{R}^n$. Let $Q \in L(P)$ be of positive order, Σ parallel to $\Lambda'(Q)$ and Σ_0 a component of $\Sigma \cap \Omega$. Then there exists $u \in \mathscr{D}'(\Omega)$ such that $Pu \in C^\infty(\Omega)$ and* $\operatorname{sing\,supp} u = \Sigma_0$.

References

1. Bourbaki, N., *Espaces vectoriels topologiques*, Paris (1953—55).
2. Dieudonné, J., Schwartz, L., La dualité dans les espaces $(\mathscr{F})$ et $(\mathscr{LF})$, *Ann. Inst. Fourier (Grenoble)*, **1** (1949), 61—101.
3. Duistermaat, J. J., Hörmander, L., Fourier integral operators II, *Acta Math.* **128** (1972), 183—269.
4. Hörmander, L., *Linear partial differential operators*, Springer-Verlag, Berlin—Heidelberg—New York (1969).
5. Hörmander, L., Pseudo-differential operators and hypoelliptic equations, *American Mathematical Society Proceedings of Symphosia in Pure Mathematics, Vol. X., Singular Integrals*, Providence (1967), 138—183.
6. Hörmander, L., On the singularities of solutions of partial differential equations, *Comm. Pure Appl. Math.* **23** (1970), 329—358.
7. Hörmander, L., On the existence and the regularity of solutions of linear pseudo-differential equations, *Enseignement Math. T. XVII*, fasc. 2 (1971), 99—163.
8. Malgrange, B., Existence et approximation de solutions des équations aux derivées partielles et des équations de convolution, *Ann. Inst. Fourier (Grenoble)* **6** (1955—56), 271—355.
9. Peetre, J., *Théorèmes de regularité pour quelques classes d'operateurs differentiels*, Thesis, Lund (1959).
10. Pliš, A., A smooth linear elliptic equation without any solution in a sphere, *Comm. Pure Appl. Math.* **14** (1961), 599—617.
11. Taylor, M., Gelfand theory of pseudo differential operators and hypoelliptic operators, *Trans. Amer. Math. Soc.* **153** (1971), 495—510.

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