# Spectral theory for pairs of differential operators

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## 0. Introduction

Recently Pleijel developed a spectral theory generalizing the eigenvalue problem $Su=\lambda Tu$ where $S$ and $T$ are ordinary, formally symmetric differential operators ([3], [13], [14]). His method depends heavily on the fact that the operators are ordinary. To handle the case of partial differential operators the abstract theory of symmetric relations on a Hilbert space may be used. The reason is that a natural setting for the eigenvalue problem seems to be a study of the relation between the functions $u$ and $v$ defined by $Su=Tv$. The abstract theory was first indicated by Arens [1] but he does not discuss the resolvent operators of a symmetric relation and apparently he does not have our application in mind. The theory given in section 1 was outlined in [2].

Section 2 is devoted to the construction of appropriate Hilbert spaces in which to study the relation $Su=Tv$. This involves a certain positivity condition on some linear combination of the operators $S$ and $T$.

In looking for selfadjoint realizations of a formally symmetric differential operator $S$ in $L^2(\Omega)$, $\Omega\subset\mathbf{R}^n$, one usually constructs a maximal operator $S_1$ and a minimal operator $S_0\cdot S_1$ is the (closure of the) operator $u\mapsto Su$ with domain such that $Su$ is defined in some sense and $u$, $Su$ both are in $L^2(\Omega)\cdot S_0$ is the $L^2(\Omega)$-closure of the operator $C_0^\infty(\Omega)\ni u\mapsto Su$. Clearly $S_0\subset S_1$ and one proceeds to prove that $S_0^*=S_1$. The von Neumann extension theory for symmetric operators may now be applied to characterize the possible selfadjoint restrictions of $S_1$. These will be given by conditions on the domain of the operator which in some sense are boundary conditions, since it is essentially on the boundary of $\Omega$ that the domains of $S_0$ and $S_1$ are different.

In this paper we will follow the same course, i.e. we will construct relations associated with $S$ and $T$ which are maximal and minimal in a natural sense and which are each others adjoints. The theory of section 1 may then be used to characterize the selfadjoint realizations of $Su=Tv$ in the particular Hilbert space chosen (there may be several suitable). These discussions are carried out in Section 3.

In Section 4 there is a brief discussion of some classical partial differential eigenvalue problems to which the theory may be applied.

The rest of the paper is devoted to the case when $S$ and $T$ are ordinary differential operators. In Section 5 it is shown that the present theory gives new access to the theory of Pleijel mentioned above. The tool for this is Lemma 5.1, a regularity theorem for weak solutions of $(S-\lambda T)u=Tv$ generalizing the classical one when $T$ is the identity. The proof is based on the du Bois—Reymond lemma of the calculus of variations which gives a more straightforward proof than the one usually employed in the classical situation, see e.g. Lemma 9 of Chapter XIII. 2 of [8]. When $S$ and $T$ have $C^\infty$ coefficients the lemma is a well known fact from the theory of distributions.

The final section contains a brief discussion of the expansion theorems obtained from the spectral theorem and also a simple construction of Green's function for a selfadjoint realization of $Su=Tv$. In certain cases, restricted so as to make consideration of relations as opposed to operators superfluous, similar results were given by Brauer [5]. The method of proof is essentially that of Gårding [9] in dealing with an elliptic operator in $L^2$. It should be remarked that most of the results in Sections 5 and 6 have generalizations if $S$ is assumed to be an elliptic partial differential operator and $T$ a differential operator of lower order. The proofs would be similar apart from the crucial Lemma 5.1.

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## 1.1. Linear relations

Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $|\cdot|_H$. Denote by $H^2$ the Hilbert space $H\oplus H$ and by $\langle \cdot, \cdot\rangle_H$ its inner product. A (closed) linear subset $E\subset H^2$ is called a (closed) linear relation on $H$. Define the boundary operator $\mathcal{U}$: $H^2\to H^2$ by

$$\mathcal{U}(u, v) = (-iv, iu) \quad \text{for} \quad i = \sqrt{-1}, \quad (u, v)\in H^2.$$

Clearly $\mathcal{U}$ is defined everywhere and is isometric, selfadjoint and involutary. Denoting orthogonal complement by $\ominus$ the adjoint $E^*$ of a linear relation $E$ is defined by

$$E^* = H^2\ominus \mathcal{U}E = \mathcal{U}(H^2\ominus E).$$

The adjoint is evidently a closed linear relation on $H$ and is the conjugate set of $E$ in $H^2$ with respect to the Hermitean boundary form

$$B(U,V) = \langle U, \mathcal{U}V\rangle_H \quad \text{for } U \text{ and } V \text{ in } H^2.$$

A third way of stating the same thing is that $(u^*, v^*)\in E^*$ exactly if

$$(u^*, v)_H = (v^*, u)_H \quad \text{for all} \quad (u, v)\in E.$$

An immediate consequence of the definition is

**Proposition 1.1.** *Let $E$ and $F$ be linear relations on $H$. Then $E\subset F$ implies $F^*\subset E^*$. The closure $\bar{E}$ of $E$ is $\bar{E}=E^{**}$ and $(\bar{E})^*=E^*$.*

A relation $E$ will be called symmetric if $E\subset E^*$ and selfadjoint if $E=E^*$. Note that $E$ is symmetric precisely if $B(E, E)=0$.

### 1.2. Extension of symmetric relations

If $F$ is a symmetric extension of $E$ Prop. 1.1 implies $F\subset F^*\subset E^*$ so that a symmetric extension of $E$ is a restriction of $E^*$. It also follows that if $E$ is symmetric, then so is its closure $\bar{E}$ and they have the same adjoint. Hence, in looking for symmetric extensions of $E$ one may as well assume that $E$ is closed which will be done henceforth. Now define

$$D_i = \{U\in E^* | \mathcal{U}U = U\}$$
$$D_{-i} = \{U\in E^* | \mathcal{U}U = -U\}$$

For a closed symmetric relation $E$ one then has the basic

**Theorem 1.2.** $E^*=E\oplus D_i\oplus D_{-i}$ (1.1)

*Proof.* The facts that $D_i$ and $D_{-i}$ are eigenspaces of $\mathcal{U}$ for different eigenvalues and $B(E, E^*)=0$ imply that $E$, $D_i$ and $D_{-i}$ are orthogonal. It remains to show that $D_i\oplus D_{-i}$ contains $E^*\ominus E$. However $U\in E^*\ominus E$ implies $U\in H^2\ominus E$ and thus $\mathcal{U}U\in E^*$. Denoting the identity on $H^2$ by $I$ and using $\mathcal{U}^2=I$ one obtains

$$U_+ = \frac{1}{2}(I+\mathcal{U})U\in D_i \quad \text{and} \quad U_- = \frac{1}{2}(I-\mathcal{U})U\in D_{-i}.$$

Clearly $U=U_+ + U_-$ which proves the theorem.
Define the deficiency indices $n_+$ and $n_-$ of $E$ by

$$n_+ = \dim D_i \quad \text{and} \quad n_- = \dim D_{-i}.$$

The formula (1.1) will now be generalized. Note that $D_i$ and $D_{-i}$ consist of the elements of $E^*$ of the form $(u, iu)$ and $(u, -iu)$ respectively. Without ambiguity one may thus define

$$D_\lambda = \{(u, \lambda u) \in E^*\}$$
$$\tilde{D}_\lambda = \{u \in H | (u, \lambda u) \in E^*\}$$
$$S_\lambda = \{u \in H | (v, \lambda v + u) \in E \text{ for some } v \in H\}$$

for any complex number $\lambda$. The space $\tilde{D}_\lambda$, and by abuse of language also $D_\lambda$, is called the deficiency space and $S_\lambda$ the solvability space of $E$ at $\lambda$. It is easily checked that $B(D_\lambda, D_{\bar\lambda}) = 0$ so that for $\lambda$ real $D_\lambda$ is a nullspace for $B$ whereas if $\operatorname{Im} \lambda > 0$ the form $B$ is positive definite on $D_\lambda$ and negative definite on $D_{\bar\lambda}$. From $B(E, E) = 0$ and $V = (v, \lambda v + u) \in E$ follows

$$0 = B(V, V) = 2 \cdot \operatorname{Im} \lambda \cdot (v, v)_H - 2 \cdot \operatorname{Im} (v, u)_H$$

so that for $\operatorname{Im} \lambda \neq 0$ the Cauchy—Schwarz inequality implies

$$|v|_H \leqq |\operatorname{Im} \lambda|^{-1} |u|_H. \tag{1.2}$$

In particular $v$ is uniquely determined by $u$.

**Lemma 1.3.** *i.* $\tilde{D}_{\bar\lambda} = H \ominus S_\lambda$
*ii. If* $\operatorname{Im} \lambda \neq 0$, *then* $S_\lambda$ *is closed and* $H = S_\lambda \oplus \tilde{D}_{\bar\lambda}$.

*Proof. i.* Every element of $E$ may be written in the form $(v, \lambda v + u)$ where $u \in S_\lambda$. Since

$$B\big((v, \lambda v + u),\ (w, \bar\lambda w)\big) = -i(u, w)_H \tag{1.3}$$

it follows that $w \in \tilde{D}_{\bar\lambda}$ precisely if $w \in H \ominus S_\lambda$.

*ii.* Because of (1.2) the equation $(v, \lambda v + u) \in E$ defines a bounded operator $R_\lambda$: $u \mapsto v$ which is closed because $E$ is. The domain of $R_\lambda$ is $S_\lambda$ so that this is closed. Thus *i.* implies $H = S_\lambda \oplus \tilde{D}_{\bar\lambda}$.

Now put

$$E_\lambda = \{(v, \lambda v + u) \in E^* | u \in \tilde{D}_{\bar\lambda}\}$$

Note that since $E^*$ and $\tilde{D}_{\bar\lambda}$ are closed, so is $E_\lambda$. By taking $w = u$ in (1.3) it follows that $E_\lambda \cap E = D_\lambda \cap E \subset D_\lambda$. It is easily seen that for $\lambda$ non-real $E_\lambda = D_\lambda \dotplus D_{\bar\lambda}$ as a direct topological sum, whereas for $\lambda$ real only $D_\lambda \subset E_\lambda$ holds. Thus $E_\lambda = E_{\bar\lambda}$ and since $D_\lambda \cap D_{\bar\lambda} = \{0\}$ for $\lambda$ non-real, it follows that $E_\lambda \cap E = \{0\}$ for $\lambda$ non-real. The generalization of (1.1) now reads

**Theorem 1.4.** *Let $E$ be a closed, symmetric linear relation on $H$ and $\lambda$ non-real. Then, as a topological direct sum,* $E^* = E \dotplus E_\lambda$.

*Proof.* It has been shown that $E \cap E_\lambda = \{0\}$ and $E_\lambda$ is closed. By Banach's theorem it thus suffices to show that $E^* = E \dotplus E_\lambda$ algebraically. Now, since $H = S_\lambda \oplus$

$\oplus \tilde{D}_{\bar{\lambda}}$ there is for any $(u, v) \in E^*$ a $w \in S_\lambda$ so that $v' = v - \lambda u - w \in \tilde{D}_{\bar{\lambda}}$. According to the definition of $S_\lambda$ there is a $w' \in H$ such that $(w', \lambda w' + w) \in E$. Thus

$$(u, v) - (w', \lambda w' + w) = \big(u - w', \lambda(u - w') + v'\big)$$

belongs to $E^*$ and hence to $E_\lambda$ which proves the theorem.

In the vocabulary of [13], [14] we have proved that $E_\lambda$ is a maximal regular subspace of $E^*$ with respect to $B$ whenever $\lambda$ is non-real.

**Corollary 1.5.** *$D_\lambda$ is a maximal positive (negative) definite subspace of $E^*$ for* $\operatorname{Im} \lambda > 0$ ($\operatorname{Im} \lambda < 0$) *with respect to $B$. Thus*

$$\dim D_\lambda = n_+ \quad \textit{for} \quad \operatorname{Im} \lambda > 0$$

$$\dim D_\lambda = n_- \quad \textit{for} \quad \operatorname{Im} \lambda < 0.$$

One may now characterize the symmetric extensions of $E$.

**Theorem 1.6.** *If $F$ is a closed symmetric extension of the closed symmetric relation $E$ and* $\operatorname{Im} \lambda \neq 0$, *then $F = E \dotplus D$ as a direct topological sum, where $D$ is a subspace of $E_\lambda$ such that*

$$D = \{U + JU \mid U \in \mathcal{D}_J \subset D_\lambda\}$$

*for some linear isometry $J$ of a closed subspace $\mathcal{D}_J$ of $D_\lambda$ onto part of $D_{\bar{\lambda}}$. Conversely, every such space $D$ gives rise to a closed symmetric extension $F = E \dotplus D$ of $E$.*

The proof is obvious after noting that if $U_+, V_+ \in D_\lambda$ and $U_-, V_- \in D_{\bar{\lambda}}$, then $\langle U_+, V_+ \rangle_H = \langle U_-, V_- \rangle_H$ precisely if $B(U_+ + U_-, V_+ + V_-) = 0$. Some immediate consequences of Theorem 1.6 are given below.

**Corollary 1.7.** *The closed symmetric relation $E$ is maximal symmetric precisely if one of $n_+$ and $n_-$ equals zero and selfadjoint precisely if $n_+ = n_- = 0$.*

**Corollary 1.8.** *If* $\operatorname{Im} \lambda \neq 0$ *and $F$ is the symmetric extension of the symmetric relation $E$ given by the isometry $J$ with domain $\mathcal{D}_J \subset D_\lambda$ and range $\mathcal{R}_J \subset D_{\bar{\lambda}}$, then the deficiency spaces of $F$ at $\lambda$ and $\bar{\lambda}$ are given by $D_\lambda \ominus \mathcal{D}_J$ and $D_{\bar{\lambda}} \ominus \mathcal{R}_J$ respectively.*

**Corollary 1.9.** *Every symmetric relation has a maximal symmetric extension. If one of $n_+$ and $n_-$ is finite, then all or none of the maximal symmetric extensions are selfadjoint depending on whether $n_+ = n_-$ or not. If $n_+ = n_- = \infty$ (and $H$ is separable) however, some maximal symmetric extensions are selfadjoint and some are not.*

### 1.3. Resolvents and spectral theory

In Section 1.2 it was seen that if $E$ is a closed symmetric relation, $\operatorname{Im}\lambda\neq 0$ and $u\in H$ given, then the equation $(v, \lambda v+u)\in E$ has a solution $v=R_\lambda u\in H$ precisely if $u$ belongs to a certain subspace $S_\lambda$ of $H$. This solution is unique and $|v|_H\leqq|\operatorname{Im}\lambda|^{-1}|u|_H$, i.e. $\|R_\lambda\|\leqq|\operatorname{Im}\lambda|^{-1}$. Since $H=S_\lambda\oplus\tilde{D}_{\bar\lambda}$ the equation is uniquely solvable for any $u$ precisely if $D_{\bar\lambda}=\{0\}$. If this is not the case one may look for $v$ such that at least $(v, \lambda v+u)\in E^*$.

**Lemma 1.10.** *If* $\operatorname{Im}\lambda\neq 0$, *then for any* $u\in H$ *there is a unique* $v\in S_{\bar\lambda}$ *such that* $(v, \lambda v+u)\in E^*$. *For this* $v$ *holds* $|v|_H\leqq|\operatorname{Im}\lambda|^{-1}|u|_H$.

*Proof.* Any element of $E$ may be written $(R_{\bar\lambda}w, \bar\lambda R_{\bar\lambda}w+w)$ for some $w\in S_{\bar\lambda}$. Thus $(v, \lambda v+u)\in E^*$ precisely if

$$0 = B\big((v, \lambda v+u),\ (R_{\bar\lambda}w, \bar\lambda R_{\bar\lambda}w+w)\big) = -i\{(u, R_{\bar\lambda}w)_H-(v, w)_H\}$$

for all $w\in S_{\bar\lambda}$. Considering $R_{\bar\lambda}$ as an operator from the Hilbert space $S_{\bar\lambda}$ to $H$ we thus have $v=R_{\bar\lambda}^*u$ as the unique solution in $S_{\bar\lambda}$ of the equation $(v, \lambda v+u)\in E^*$. Since $\|R_{\bar\lambda}^*\|=\|R_{\bar\lambda}\|$ the lemma follows.

Writing $v=R_\lambda' u$ for the solution of Lemma 1.10 one obtains $\|R_\lambda'\|\leqq|\operatorname{Im}\lambda|^{-1}$. The operators $R_\lambda$: $S_\lambda\to H$ and $R_\lambda'$: $H\to S_{\bar\lambda}$ are called the resolvent operators at $\lambda$ of $E$ and $E^*$ respectively. It is clear that requesting $(v, \lambda v+u)\in E^*$ for a given $u$ only determines $v$ modulo $\tilde{D}_\lambda$. On the other hand, if $\operatorname{Im}\lambda\neq 0$ and $D_{\bar\lambda}=\{0\}$ it follows that the equations $(v, \lambda v+u)\in E$ and $(v, \bar\lambda v+u)\in E^*$ are uniquely solvable by $v=R_\lambda u$ and $v=R_{\bar\lambda}' u$ respectively for any $u\in H$. Thus maximal symmetric, and à fortiori selfadjoint, relations are particulary wellbehaved.

**Theorem 1.11. (Resolvent relation).** *For* $\lambda$ *and* $\mu$ *non-real one has*

$$R_\lambda-R_\mu = (\lambda-\mu)R_\lambda R_\mu.$$

*If* $D_\lambda=\{0\}$ *one also has*

$$R_\lambda'-R_\mu' = (\lambda-\mu)R_\lambda' R_\mu'.$$

*Proof.* Let $V=(v, \lambda v+u)$ and $W=(w, \mu w+u)$. If $W\in E$, then $V-W\in E$ precisely if $V\in E$ and since $V-W=(v-w, \lambda(v-w)+(\lambda-\mu)w)$ the first formula follows. Similarly, in view of the comments above, the second formula follows.

**Theorem 1.12.** *Assume* $D_\mu=\{0\}$. *Then for* $\operatorname{Im}\lambda\cdot\operatorname{Im}\mu<0$ *the domain of the operator* $R_\lambda$ *is* $H$ *and* $R_\lambda$ *is an analytic function (in the uniform operator topology) of* $\lambda$. *For* $\operatorname{Im}\lambda\cdot\operatorname{Im}\mu>0$ *the same statement is true for* $R_\lambda'$. *If also* $D_{\bar\mu}=\{0\}$, *then* $R_\lambda=R_\lambda'$ *is a normal operator for* $\operatorname{Im}\lambda\neq 0$.

*Proof.* The statement about domains follows from Lemmas 1.3 and 1.10. The rest is a standard consequence of the resolvent relation.

**Theorem 1.13.** *The deficiency space* $D_\lambda$ *of the symmetric relation* $E$ *is an analytic function of* $\lambda$, *regular at least in* $\mathbf{C}\setminus\mathbf{R}$.

*Proof.* Let $F$ be an arbitrary but fixed maximal symmetric extension of $E$ and $R_\lambda$ and $R'_\lambda$ be the resolvent operators at $\lambda$ of $F$ and $F^*$ respectively. Put $A_\lambda=R'_\lambda$ or $A_\lambda=R_\lambda$ depending on whether the deficiency space of $F$ at $\lambda$ is trivial or not. Thus, for $\operatorname{Im}\lambda\neq 0$ the domain of $A_\lambda$ is $H$, $A_\lambda-A_\mu=(\lambda-\mu)A_\lambda A_\mu$ for $\operatorname{Im}\lambda\cdot\operatorname{Im}\mu>0$ and $A_\lambda$ is analytic in $\mathbf{C}\setminus\mathbf{R}$. Note that $F^*\subset E^*$ and define for non-real $\lambda$, $\mu$ an operator $P_{\lambda\mu}: H^2\to H^2$ by

$$P_{\lambda\mu}(u,v)=(u,v)+(\lambda-\mu)(A_\lambda u,\lambda A_\lambda u+u)=(w,\lambda w+v-\mu u)$$

where $w=u+(\lambda-\mu)A_\lambda u$. Clearly $P_{\lambda\mu}$ maps $E^*$ into $E^*$ and $P_{\lambda\mu}D_\mu\subset D_\lambda$. For $\operatorname{Im}\lambda\cdot\operatorname{Im}\mu>0$ the resolvent relation implies $P_{\nu\lambda}P_{\lambda\mu}=P_{\nu\mu}$ and since $P_{\mu\mu}$ is the identity on $H^2$ so is $P_{\mu\lambda}P_{\lambda\mu}$. Hence $P_{\lambda\mu}$ is bijective and $P_{\lambda\mu}D_\mu=D_\lambda$ since $D_\lambda= =P_{\lambda\mu}P_{\mu\lambda}D_\lambda\subset P_{\lambda\mu}D_\mu\subset D_\lambda$. The analyticity of $P_{\lambda\mu}$ as a function of $\mu$ is obvious and that as a function of $\lambda$ follows e.g. from the analyticity of $A_\lambda$. Thus there is a bijection on $H^2$ which is holomorphic in $\lambda$, $\mu$ for $\operatorname{Im}\lambda\cdot\operatorname{Im}\mu>0$ the restriction of which to $D_\mu$ is a bijection $D_\mu\to D_\lambda$ which proves the theorem.

The resolvent operator of a selfadjoint relation has been shown to have all the properties of the resolvent of a selfadjoint operator apart from not being injective. One could now prove a spectral theorem for selfadjoint relations by a modification of the proof for operators. More convenient is to reduce the theorem for relations to that for operators by orthogonalizing away the kernel of the resolvent which will now be done. Assume that $E$ is maximal symmetric and put

$$H_\infty=\{u\in H\,|\,(0,u)\in E\}.$$

Thus $H_\infty$ is the common kernel of all $R_\lambda$. Put

$$\tilde{E}=\{u\in H\,|\,\exists v\in H \text{ with } (u,v)\in E\}=\text{domain of } E,$$

$$H_0=\text{closure of } \tilde{E} \text{ in } H,$$

$$\tilde{E}^*=\{u\in H\,|\,\exists v\in H \text{ with } (u,v)\in E^*\}=\text{domain of } E^*.$$

**Lemma 1.14.** $H=H_0\oplus H_\infty$. *The common kernel of all* $R'_\lambda$ *is* $H_\infty$ *and the closure of* $\tilde{E}^*$ *in* $H$ *is* $H_0$.

*Proof.* Let $(u,v)\in E$. Then $B((u,v),(0,w))=i(u,w)_H$ so that $(0,w)$ is in $E^*$ precisely when $w\in H\ominus\tilde{E}$ which is thus the kernel of $R'_\lambda$. Since $E$ is maximal there is a non-real $\lambda$ with $S_\lambda=H$ (Corollary 1.7 and Theorem 1.12). If $(0,w)\in E^*$ there therefore exists $w'\in H$ with $W=(w',\lambda w'+w)\in E$. Thus, since $(w,w')_H=0$,

$$0=B(W,W)=2\operatorname{Im}\lambda(w',w')_H.$$

Hence $w'=0$ so that $w\in H_\infty$ and $H=H_0\oplus H_\infty$. Repeating the first calculation but assuming that $(u, v)\in E^*$ shows that $H_\infty=H\ominus\tilde{E}^*$ so that the lemma is proved.

Now put

$$E_\infty = \{0\}\times H_\infty \quad \text{and} \quad E_0 = E\cap H_0^2.$$

Then one has

**Theorem 1.15. (Spectral theorem for relations).** *$E_0$ is the graph of a densely defined and maximal symmetric operator on $H_0$ which is selfadjoint if and only if $E$ is. The restrictions of $R_\lambda$ and $R'_\lambda$ to $H_0$ are the resolvent operators of $E_0$ and its adjoint in $H_0$ respectively.*

*Proof.* It has been proved that $H=H_0\oplus H_\infty$ and clearly this implies $E=E_0\oplus E_\infty$. It is also clear that $E_0$ is the graph of a densely defined symmetric operator on $H_0$. The graph of its adjoint is $E_0^*\cap H_0^2$. But $E_0=E\ominus E_\infty$ so that $E_0^*=E^*\oplus\mathcal{U}E_\infty$ which implies that $E_0^*\cap H_0^2=E^*\cap H_0^2$ and since $\tilde{D}_\lambda\subset H_0$ clearly $E$ and $E_0$ have the same deficiency spaces. Since the ranges of $R_\lambda$ and $R'_\lambda$ are $\tilde{E}$ and $\tilde{E}^*$ respectively, their restrictions operate in $H_0$ which finishes the proof.

One may consider $H_\infty$ as an eigenspace belonging to the eigenvalue $\infty$. Hence, in case $E$ is selfadjoint, $H_\infty$ together with the resolution of the identity for $E_0$ gives a resolution of the identity for $E$.

### 1.4. Semi-bounded relations

One may define the essential spectrum of a closed symmetric relation $E$ as

$$e\sigma(E) = \{\lambda\in\mathbf{C}\,|\,S_\lambda \text{ is not closed}\}.$$

The stability under finite-dimensional extensions of $E$ follows immediately. However, in general $e\sigma(E)$ is not closed. By inspection of the proof of Theorem 1.4 it is seen that if $\varrho\in\mathbf{R}\setminus e\sigma(E)$, then $E^*$ is the linear hull of $E$ and $E_\varrho$. In fact, if $E'_\varrho=E_\varrho\ominus(D_\varrho\cap E)$ then as a direct topological sum $E^*=E\dotplus E'_\varrho$. That $D_\varrho\cap E\neq\{0\}$ means of course that $\varrho$ is in the pointspectrum $p\sigma(E)$. Thus the conclusion of Theorem 1.4 holds for $\varrho\notin e\sigma(E)\cup p\sigma(E)=\sigma(E)$. For $\varrho$ real $D_\varrho$ is always a null-space for $B$ so that $E\dotplus D'_\varrho$, where $D'_\varrho=D_\varrho\ominus(D_\varrho\cap E)$, is a symmetric extension of $E$. This ext ension is in fact selfadjoint if $\varrho\notin e\sigma(E)$ which is easily seen using the fact that $E^*=E\dotplus E'_\varrho$. Thus unless $e\sigma(E)=\mathbf{R}$ the symmetric relation $E$ always has a selfadjoint extension, and $n_+=n_-=\dim D'_\varrho$. Obviously $\varrho$ is an eigenvalue of $E\dotplus D'_\varrho$ of multiplicity $\dim D_\varrho$. A slight elaboration shows that when $\varrho\notin e\sigma(E)$, then one can find a selfadjoint extension of $E$ for which the eigenspace at $\varrho$ is any space $D$ with $D'_\varrho\subset D\subset D_\varrho$.

In practise a case of interest is when $E$ is bounded from below, i.e.

$$d = \inf(u, v)_H, \quad \text{inf taken over} \quad (u, v)\in E \quad \text{with} \quad |u|_H = 1,$$

is finite. Assume $\operatorname{Re}\lambda < d$ and $(v, \lambda v+u)\in E$. Then it follows easily from $d(v, v)_H \leqq (v, \lambda v+u)_H$ that

$$|v|_H \leqq |d-\lambda|^{-1}|u|_H.$$

Thus $S_\lambda$ is closed and $R_\lambda$ defined and analytic for any $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda<d$. If $\varrho<d$ then $E^*=E\dotplus E_\varrho$ and $E\dotplus D_\varrho$ is a selfadjoint relation which is easily seen to have the lower bound $\varrho$. By a reasoning entirely similar to the one employed by Friedrichs for operators one can show that there is a selfadjoint relation extending $E$ and with no decrease in the lower bound $d$.

## 2. A Hilbert space

Let $\Omega$ be an open, connected part of $\mathbf{R}^n$ and $a_{\alpha\beta}$ a finite number of locally integrable functions defined in $\Omega$. Here $\alpha$, $\beta$ are multi-indices in $\mathbf{R}^n$, i.e. $\alpha=(\alpha_1, \ldots, \alpha_n)$ with non-negative integers $\alpha_j$ and the length of $\alpha$ is $|\alpha|=\sum\alpha_j$. Consider the form

$$(u, v)_P = \int_\Omega \sum a_{\alpha\beta} D^\alpha u \cdot \overline{D^\beta v}.$$

If $m$ is an upper bound for $|\alpha|$, $|\beta|$ this is defined at least for $u$ and $v$ in $C_0^m(\Omega)$. As usual $D^\alpha=(-i)^{|\alpha|}\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}$ where $\partial_j=\dfrac{\partial}{\partial x_j}$. We will also consider the integral forms

$$(u, v)_{k,M} = \int_M \sum_{|\alpha|=k} D^\alpha u \cdot \overline{D^\alpha v}, \quad (u, v)_k = (u, v)_{k,\Omega}$$

and the corresponding semi-norms $|u|_{k,M}=\sqrt{(u, u)_{k,M}}$. Here $M$ is any (measurable) subset of $\Omega$. If the partial differential operator $P$ may be written $P=\sum D^\beta a_{\alpha\beta} D^\alpha$ we say that $(\cdot, \cdot)_P$ is a Dirichlet integral belonging to $P$ on $\Omega$. This is motivated by the fact that $(u, v)_P=(Pu, v)_0$ if $u$ and $v$ are in $C_0^\infty(\Omega)$ which follows on integration by parts. Note that a given operator $P$ has many different Dirichlet integrals. We assume that the matrix $(a_{\alpha\beta})$ is hermitean. Thus $P$ is formally symmetric. Let $C_*^m$ be a linear subset of $C^m(\Omega)$ containing $C_0^m(\Omega)$ such that $\sum a_{\alpha\beta} D^\alpha u \cdot \overline{D^\beta u}\in L^1(\Omega)$ for $u\in C_*^m$. More generally, $L^1(\Omega)$ may be replaced by the functions $w\in L^1_{\mathrm{loc}}(\Omega)$ such that $\lim_j \int_{K_j} w$ exists, where $\{K_j\}_1^\infty$ is a fixed increasing sequence of compacts with $\cup K_j=\Omega$. In view of the applications sketched in Section 4 we make the following

*Definition 2.1.* $(\cdot,\cdot)_P$ is called $r$-positive over $C_*^m$ if for every compact $M\subset\subset\Omega$ there exists $C_M>0$ such that

$$(u,u)_{r,M}\leqq C_M(u,u)_P \quad \text{for every} \quad u\in C_*^m. \tag{2.1}$$

Thus $\|u\|=\sqrt{(u,u)_P}$ is a semi-norm on $C_*^m$. Later we shall prove

**Theorem 2.2.** *Suppose for each $x\in\Omega$ there exists $C_x>0$, a natural number $n_x$ and a neighbourhood $\mathcal{O}_x\subset\Omega$ of $x$ such that*

$$(u,u)_{n_x,\mathcal{O}_x}\leqq C_x(u,u)_P \quad \text{for every} \quad u\in C_*^m.$$

*Then $(\cdot,\cdot)_P$ is $r$-positive over $C_*^m$ with $r=\min_{x\in\Omega} n_x$.*

If for each complex vector $\{\zeta_\alpha\}_{|\alpha|\leqq m}$ one has

$$\sum_{|\alpha|=n_x}|\zeta_\alpha|^2\leqq C_x\sum a_{\alpha\beta}(y)\zeta_\alpha\bar{\zeta}_\beta \quad \text{for every} \quad y\in\mathcal{O}_x \tag{2.2}$$

it follows that $(\cdot,\cdot)_P$ is $r$-positive over

$$\mathcal{L}=\left\{u\in C^m(\Omega)\,\middle|\,\sum a_{\alpha\beta}D^\alpha u\cdot\overline{D^\beta v}\in L^1(\Omega)\right\}.$$

The linearity of $\mathcal{L}$ follows from the positivity of the matrix $A=(a_{\alpha\beta})$. More generally, let $B$ be the positive root of $A^2$, which is easily seen to have elements in $L^1_{\text{loc}}(\Omega)$. Let $A(u,v)=\sum a_{\alpha\beta}D^\alpha u\cdot\overline{D^\beta v}$ and $B(u,v)$ be similarly defined. If $\sqrt{\int_\Omega B(u,u)}$ is equivalent to $\|u\|$ over $\mathcal{L}$ then $\mathcal{L}$ is linear because

$$\int_\Omega|A(u,v)|\leqq\int\sqrt{B(u,u)}\sqrt{B(v,v)}\leqq\sqrt{\int_\Omega B(u,u)}\sqrt{\int_\Omega B(v,v)}.$$

To see that equivalence between the metrics given by $A$ and $B$ does not require $A$ to be pointwise positive, consider

$$(u,v)_P=\int_I(pu'\bar{v}'+qu\bar{v})$$

where $I$ is a real interval and $p>0$, $q$ are in $L^1_{\text{loc}}(I)$. For $u\in C^1$ $u(x)=u(y)+\int_y^x u'$ so that by Schwarz' inequality

$$|u(x)|\leqq|u(y)|+\left|\int_y^x\frac{1}{p}\int_y^x p|u'|^2\right|^{1/2}.$$

Put $q_+=1/2(|q|+q)$ and $q_-=q_+-q$ and assume that $q_+$ does not vanish identically on $I$. Squaring, multiplying by $q_+(y)q_-(x)$ and integrating over $I\times I$ one obtains by easy estimates that

$$\int_I q_-|u|^2\leqq\int_I q_-\left\{\left(\int_I q_+\right)^{-1/2}+\left(\int_I\frac{1}{p}\right)^{1/2}\right\}^2\int_I\{p|u'|^2+q_+|u|^2\}.$$

It follows that $\int_I \{p|u'|^2+|q|\cdot|u|^2\}$ is equivalent to $(\cdot,\cdot)_P$ if

$$\int_I q_-\Big\{\Big(\int_I q_+\Big)^{-1/2}+\Big(\int_I \frac{1}{p}\Big)^{1/2}\Big\}^2<1 \tag{2.3}$$

in which case $(\cdot,\cdot)_P$ is also 0-positive over those $u\in C^1(I)$ for which $(u,u)_P<+\infty$. To see this, replace $q$ in (2.3) by $q-\varepsilon\chi_M$ where $\varepsilon>0$ is small and $\chi_M$ the characteristic function of $M$. Clearly $p$ and $q$ can be chosen e.g. so that (2.3) holds but $q$ is strictly negative outside a compact subinterval of $I$.

If $(\cdot,\cdot)_P$ is $r$-positive over $C_*^m$ Theorem 2.2 implies that for any compact part $M$ of $\Omega$ with non-empty interior $(\cdot,\cdot)_+=(\cdot,\cdot)_P+(\cdot,\cdot)_{0,M}$ is 0-positive over $C_*^m$ (in fact, by Theorem 2.5 $M\subset\subset\Omega$ needs only have positive measure). Hence $(\cdot,\cdot)_+$ is a scalar product on $C_*^m$ which may thus be completed to a Hilbert space $H_+$ with norm $\|\cdot\|_+$. Theorem 2.2 implies that all such norms with different $M$ are equivalent. Thus, in the sequel, let $M$ be a fixed, non-degenerate and compact interval in $\Omega$. That $H_+$ does not depend on the value of $m$ in $C_*^m$ is shown by

**Lemma 2.3.** *Put $C_*^\infty=H_+\cap C^\infty(I)$. Then $C_*^\infty$ is dense in $H_+$.*

*Proof.* We must show that for $u\in C_*^m$ and $\varepsilon>0$ there exists $u_\varepsilon\in C_*^\infty$ with $\|u-u_\varepsilon\|_+<\varepsilon$. Let $\{K_j\}$ be an increasing sequence of compacts with $\cup K_j=\Omega$ and let $\{\varphi_j\}$ be a partition of unity subordinate to $\{K_{j+1}\setminus K_{j-1}\}$ where $K_0=\emptyset$. If $u\in C_*^m$ then $\varphi_j u\in C_0^m(\Omega)\subset C_*^m$. Put $u_j=\psi_j*\varphi_j u$ where $0\leqq\psi_j\in C_0^\infty(\Omega)$ so that $u_j\in C_0^\infty(\Omega)$. If $\int\psi_j=1$ and $\operatorname{supp}\psi_j$ is sufficiently close to 0, then $\sum_{|\alpha|\leqq m}\sup|D^\alpha(u_j-\varphi_j u)|$ is arbitrarily small so we may choose $\psi_j$ so that $\operatorname{supp}u_j\subset K_{j+1}\setminus K_{j-1}$ and $\|u_j-\varphi_j u\|_+\leqq\varepsilon 2^{-(j+2)/2}$. Since $(u_j-\varphi_j u, u_k-\varphi_k u)_+=0$ for $|j-k|>1$ and $u_j-\varphi_j u\in C_0^m(\Omega)\subset C_*^m$ one obtains $\sum(u_j-\varphi_j u)\in H_+$ and $\|\sum(u_j-\varphi_j u)\|_+\leqq\varepsilon$. Hence $u_\varepsilon=\sum u_j\in H_+\cap C^\infty(\Omega)$ and $\|u_\varepsilon-u\|_+<\varepsilon$. Note that $\sum u_j$ may not converge in $H_+$ and that we may require that $\sum a_{\alpha\beta}D^\alpha u_\varepsilon D^\beta u_\varepsilon\in L^1$ if $\sum a_{\alpha\beta}D^\alpha u\overline{D^\beta u}\in L^1$.

Definition 2.1 states, for $r=0$, that there is a continuous mapping $\mathbf{i}: H_+\to L^2_{\text{loc}}(\Omega)$ which is the identity on $C_*^m$. Suppose that

$$D^\gamma a_{\alpha\beta}\in L^2_{\text{loc}}(\Omega)\quad\text{for}\quad \gamma\leqq\beta. \tag{2.4}$$

Then $(u,\varphi)_P=(u,P\varphi)_0$ if $u\in C_*^m$ and $\varphi\in C_0^\infty(\Omega)$. By taking limits it follows that $\operatorname{Ker}\mathbf{i}\subset H_+\ominus C_0^\infty(\Omega)$ so that $\mathbf{i}$ is injective if $C_0^\infty(\Omega)$ is dense in $H_+$. More generally one has

**Lemma 2.4.** *If* (2.4) *holds and the mapping*

$$C_*^m\ni u\mapsto\sum a_{\alpha\beta}D^\alpha u\cdot\overline{D^\beta v}\in L^1(\Omega) \tag{2.5}$$

*is continuous for every $v\in C_*^\infty$, then $\mathbf{i}$ is injective.*

*Proof.* If $C_*^m \ni u_j \to u \in H_+$ and $C_0^\infty(\Omega) \ni \varphi_k \nearrow 1$, then (2.5) implies

$$(u, v)_+ = \lim_k \lim_j \int_\Omega \varphi_k \sum a_{\alpha\beta} D^\alpha u_j \cdot \overline{D^\beta v} \quad \text{for} \quad v \in C_*^\infty.$$

Putting $P_k = \sum D^\beta \varphi_k a_{\alpha\beta} D^\alpha$ integration by parts gives

$$\int_\Omega \varphi_k \sum a_{\alpha\beta} D^\alpha u_j \cdot \overline{D^\beta v} = (u_j, P_k v)_0.$$

If $u_j \to 0$ in $L^2_{\text{loc}}(\Omega)$ this implies $u=0$ since $P_k v \in L^2(\Omega)$ has compact support and $C_*^\infty$ is dense in $H_+$.

It is easily verified that $(u, v) \mapsto \sum a_{\alpha\beta} D^\alpha u \cdot \overline{D^\beta v}$ is a bounded Hermitean mapping from $C_*^m \times C_*^m$ to $L^1(\Omega)$ if $\|\cdot\|$ is equivalent to the metric associated with the matrix $B$ introduced just after (2.2) so that (2.5) is certainly satisfied in this case. To slightly simplify the statements it will be assumed that $H_+$ is continuously embedded in $L^2_{\text{loc}}(\Omega)$, i.e. that $\mathbf{i}$ is injective, in the following sections. This can be achieved by simply orthogonalizing away $\operatorname{Ker} \mathbf{i}$ which would not change the minimal relation (see Section 3). Put

$$K = \{u \in H_+ \mid \|u\| = 0\}.$$

Then, by the Cauchy—Schwarz' inequality, one has $(K, H_+)_P = 0$ and similarly $(\operatorname{Ker} \mathbf{i}, H_+)_{0,M} = 0$ so that $(\operatorname{Ker} \mathbf{i}, K)_+ = 0$. Thus $\mathbf{i}|_K$ is injective.

**Theorem 2.5.** $\mathbf{i}K$ *is a set of polynomials of degree* $<r$ *and hence* $\dim K < \infty$. *If* $H$ *is a subspace of* $H_+$ *such that* $H \cap K = \{0\}$ *then* $\|\cdot\|$ *and* $\|\cdot\|_+$ *are equivalent norms on* $H$.

To prove Theorems 2.2 and 2.5 we need some lemmas.

**Lemma 2.6.** *(Poincaré inequality). For any non-degenerate bounded interval* $J \subset \mathbf{R}^n$ *there is a constant* $C$ *such that for* $u \in C^1(\bar{J})$

$$|u|_{0,J} \leqq C\left(|u|_{1,J} + \left|\int_J u\right|\right).$$

A proof for $n=2$ which immediately extends to the general case can be found in [6, Chapter VII, Section 3.1.].

**Lemma 2.7.** *Let* $B$ *be a Banach space with norm* $\|\cdot\|$ *and let* $|\cdot|$ *be a bounded semi-norm on* $B$ *such that on some subspace* $B_0$ *with* codim $B_0$ *finite* $\|\cdot\|$ *and* $|\cdot|$ *are equivalent norms. Then they are equivalent norms on any subspace* $B_1$ *on which* $|\cdot|$ *is a norm.*

*Proof.* On $B_2 = B_0 \cap B_1$ the norms are equivalent so $|\tilde{u}| = \inf_{u \in \tilde{u}} |u|$ defines a norm on $B_1/B_2$. However, $\dim B_1/B_2 \leqq \dim B/B_0 < \infty$ so that $B_1/B_2$ is complete under $|\cdot|$. Thus $B_1$ is complete under $|\cdot|$ and hence $\|\cdot\|$ and $|\cdot|$ are equivalent on $B_1$ by Banach's theorem.

**Lemma 2.8.** *Let* $J \subset I$ *be bounded intervals in* $\mathbf{R}^n$ *with non-empty interior and* $p \leqq q < r$ *natural numbers. Then there is a constant* $K$ *such that for* $u \in C^r(\bar{I})$

$$|u|_{q,I} \leqq K(|u|_{r,I} + |u|_{p,J}).$$

*Proof.* Put $\|u\| = \sum_{k=p}^{r} |u|_{k,I}$. Let $P_j$ denote the set of polynomials in $\mathbf{R}^n$ of degree at most $j$. By repeated application of the Poincaré inequality one obtains, for $u \in C^r(\bar{I})$

$$\|u\| \leqq A_0 |u|_{r,I} + \sum_{p \leqq |\alpha| \leqq r} A_\alpha \left| \int_I D^\alpha u \right|. \tag{2.6}$$

This inequality has an obvious sense also on $C^r(\bar{I})/P_{p-1}$ on which $\|\cdot\|$ is a norm so that we may complete to the Banach space $B$. All terms in (2.6) are bounded by $\|\cdot\|$ so that (2.6) extends to all of $B$. $|\cdot|_{r,I}$ vanishes on $K = P_{r-1}/P_{p-1}$ and on no other elements of $B$. If it did, then $|\cdot|_{r,I}$ would vanish on so large a subspace of $B$ that one could find a non-zero vector for which the right hand side of (2.6) vanishes. Now $|\cdot|_{p,J}$ is a bounded semi-norm on $B$ which does not vanish on $P_{r-1}/P_{p-1}$. Thus Lemma 2.7 implies that $|\cdot|_{r,I} + |\cdot|_{p,J}$ and $\|\cdot\|$ are equivalent norms on $B$. The lemma evidently follows.

*Proof of Theorem 2.2.* According to assumption, $M$ may be covered with open intervals $\mathcal{O}_x$, hence by a finite number $\mathcal{O}_1, \ldots, \mathcal{O}_q$ of intervals, and since $\Omega$ is connected we may assume that $\cup \mathcal{O}_j$ is connected. Recall that there are constants $C_j$ and $n_j$ such that $|u|_{n_j, \mathcal{O}_j} \leqq C_j \|u\|$. We may assume that $n_1 = r$. Since $\cup \mathcal{O}_j$ is connected there is for any $j$, $1 \leqq j \leqq q$, a sequence $j_1, \ldots, j_s$ such that $j_1 = j$, $j_s = 1$ and $\mathcal{O}_{j_k} \cap \mathcal{O}_{j_{k+1}} = w_k$ is a non-empty open interval. It follows that

$$|u|_{r, \mathcal{O}_{j_k}} \leqq K(|u|_{n_{j_k}, \mathcal{O}_{j_k}} + |u|_{r, w_k}) \leqq K(C_{j_k} \|u\| + |u|_{r, \mathcal{O}_{j_{k+1}}})$$

where we have used Lemma 2.8 and the assumption. Putting these inequalities together one obtains, for some constants $A_j$, $B_j$ and $K_j$

$$|u|_{r, \mathcal{O}_j} \leqq A_j \|u\| + B_j |u|_{r, \mathcal{O}_1} \leqq K_j \|u\|.$$

The theorem follows with $C = \sum K_j$ since $|u|_{r,M} \leqq |u|_{r, \cup \mathcal{O}_j} \leqq \sum_j |u|_{r, \mathcal{O}_j}$.

*Proof of Theorem 2.5.* Let $J$ be an arbitrary non-degenerate compact interval in $\Omega$. Repeated use of the Poincaré inequality gives

$$|u|_{0,J} \leqq A |u|_{r,J} + \sum_{|\alpha| \leqq r} A_\alpha \left| \int_J D^\alpha u \right| \quad \text{for} \quad u \in C_*^m. \tag{2.7}$$

For $|\alpha| \leqq r$ Lemma 2.8 shows that $\int_J D^\alpha u$ is bounded by $|u|_{r,J} + |u|_{0,J}$ and thus, $\|\cdot\|_+$ being both $r$-positive and 0-positive, by $\|u\|_+$. Hence (2.7) extends to all of $H_+$. As in the proof of Lemma 2.8 one sees that if $u \in H_+$, then $|u|_{r,J}$ can only vanish if the restriction to $J$ of $\mathbf{i}u$ is a polynomial of degree at most $r-1$. Since $\|\cdot\|$ is $r$-positive this is true for any $u \in H_+$ for which $\|u\|$ vanishes. The first statement of the theorem follows since $J$ is arbitrary. For $J=I$ Theorem 2.2 and (2.7) give

$$\|u\|_+ \leqq A_0\|u\|+\sum_{|\alpha|\leqq r} A_\alpha \left|\int_I D^\alpha u\right|.$$

Hence a direct application of Lemma 2.7 gives the desired result.

## 3. Maximal and minimal relations

Let $S$ and $T$ be formally symmetric partial differential operators in an open connected domain $\Omega\subset\mathbf{R}^n$. Their coefficients are assumed to be so regular that $Su$ and $Tu$ may be formed and are locally square integrable for any sufficiently differentiable function $u$. We now make the basic

**Assumption 3.1.** *There is a Dirichlet integral $(\cdot,\cdot)_P$ belonging to some $P=aS+$ $+bT$ with $a$ and $b\in\mathbf{R}$, a space $C_*^m$ and an integer $r$ such that $(\cdot,\cdot)_P$ is $r$-positive over $C_*^m$.*

The discussion in Section 2 implies that we can choose a Hilbert space $H$ which is continuously embedded in $L^2_{\mathrm{loc}}(\Omega)$ and for which $(\cdot,\cdot)_P$ is the scalar product. Now define

$$E_{\max}=\{(u,v)\in H^2|(u,S\varphi)_0=(v,T\varphi)_0 \text{ for all } \varphi\in C_0^\infty(\Omega)\}$$

where $(\cdot,\cdot)_0$ is the scalar product of $L^2(\Omega)$. Hence $E_{\max}$ consists of the pairs $(u,v)\in H^2$ for which $Su=Tv$ in a weak sense. Thus $E_{\max}$ is in a natural sense the maximal relation associated with $S$ and $T$ in $H$. The continuity of the embedding $H\subset L^2_{\mathrm{loc}}(\Omega)$ implies that $E_{\max}$ is a closed linear relation on $H$. Let $L_0^2$ denote the functions in $L^2(\Omega)$ with (essentially) compact support in $\Omega$. Then the mapping

$$H\ni u\mapsto (u,f)_0,\quad f\in L_0^2.$$

is a continuous linear form on $H$ because of the continuity of the embedding. Thus there is a unique linear operator $G: L_0^2\to H$ such that

$$(u,f)_0=(u,Gf)_P \quad\text{for}\quad u\in H,\quad f\in L_0^2$$

(In fact $G=\mathbf{i}^*$.) The minimal relation is now defined by

$$E_{\min}=\{(GT\varphi, GS\varphi)\in H^2|\varphi\in C_0^\infty(\Omega)\}.$$

Clearly $E_{\min}^*=E_{\max}$ because for $(u,v)\in H^2$ one has

$$(u,S\varphi)_0-(v,T\varphi)_0=(u,GS\varphi)_P-(v,GT\varphi)_P.$$

If $E_{\min}\subset E_{\max}$, i.e. $E_{\min}$ is symmetric, we are ready to apply the abstract theory of Section 1. Unfortunately it seems difficult to give exact conditions for the choice of $H$ so that $E_{\min}$ becomes symmetric. It is not even clear if it is always possible to make such a choice. However, some reasonably general sufficient conditions for symmetry will be given below.

Put $Q=bS-aT$ so that $Q$ is formally symmetric. Assuming $a, b$ normalized so that $a^2+b^2=1$ and $\varphi$ and $\psi$ in $C_0^\infty(\Omega)$ one obtains

$$\begin{gathered}(GT\varphi, GS\psi)_P-(GS\varphi, GT\psi)_P = (GP\varphi, GQ\psi)_P-(GQ\varphi, GP\psi)_P = \\ (P\varphi, GQ\psi)_0-(GQ\varphi, P\psi)_0 = (\varphi, GQ\psi)_P-(GQ\varphi, \psi)_P.\end{gathered} \tag{3.1}$$

This must vanish in order that $E_{\min}$ be symmetric.

**Theorem 3.2.** *Let $K'$ be a subspace of $K=\{u\in H_+ \mid \|u\|=0\}$ such that $QK'=0$. If $C_0^\infty(\Omega)\subset H\dotplus K'$, then $E_{\min}$ is symmetric on $H$.*

*Proof.* By assumption we can write $\varphi\in C_0^\infty(\Omega)$ as $\varphi=\varphi_0+\varphi_1$, where $\varphi_0\in H$ and $\varphi_1\in K'$. By Cauchy—Schwarz' inequality we have $(K', H)_P=0$ so that (3.1) becomes

$$(\varphi_0, GQ\psi)_P-(GQ\varphi, \psi_0)_P = (\varphi_0, Q\psi)_0-(Q\varphi, \psi_0)_0 = (\varphi, Q\psi)_0-(Q\varphi, \psi)_0 = 0$$

which proves the theorem.

If $K=\{0\}$, i.e. $(\cdot, \cdot)_P$ is 0-positive over $C_*^m$, we may take $H=H_+$ which gives a symmetric $E_{\min}$ since $C_0^\infty(\Omega)\subset H_+$. Note that in some sense $C_0^\infty(\Omega)\subset$ $\subset H\dotplus K'$ means that the restriction of $H_+$ to $H\dotplus K'$ is given by boundary conditions. It is clear that Theorem 3.2 can not always be applied if the closure of $C_0^\infty(\Omega)$ in $H_+$ has codimension $<\dim K$. To give an example of this we use the following lemma, which is closely related to Theorem 2.1 in [11].

Let $I$ be a real interval and $(u, v)_+=\int_I(pu'\overline{v'}+qu\bar{v})$ where $p>0$ and $q\geqq 0$ locally are absolutely continuous and integrable respectively. Assume $\int_I q\neq 0$ and let $H_+$ be the completion with respect to $(\cdot, \cdot)_+$ of all functions in $C^1(I)$ giving a finite value to $(u, u)_+$.

**Lemma 3.3.** *$C_0^\infty(I)$ is dense in $H_+$ if and only if $1/p+q$ is not integrable in any neighbourhood of an endpoint of $I$.*

Now put $(u, v)_P=\int_I pu'\overline{v'}$ which is 1-positive over those $u\in C^1(I)$ for which it is finite. The norm-square of the auxiliary space $H_+$ is of the form $\int_I(p|u'|^2+q|u|^2)$ where $q$ is positive and in $L^1(I)$. Lemma 3.3 shows that $1/p$ must be integrable near at least one endpoint of $I$ if there is a non-trivial subspace of $H_+$ which contains $C_0^\infty(I)$. However, if this holds it follows from the proof of the lemma that the function $\equiv 1$ is not in the hull of $C_0^\infty(I)$. Hence $1/p$ integrable in one half of $I$ is a necessary and sufficient condition for the existence of proper subspaces containing $C_0^\infty(I)$ on which $\int_I p|u'|^2$ is a norm equivalent to the norm of $H_+$.

*Proof of Lemma 3.3.* Suppose $1/p\in L^1(J)$ where $J\subset I$ is an interval with $\int_J q>0$. For $x, y\in J$ one obtains easily from $u(x)=u(y)+\int_y^x u'$ that for $u\in C^1(I)$ the inequalities

$$|u(x)-u(y)| \leqq \Big\{\int_y^x \frac{1}{p}\Big\}^{1/2}\|u\|_+ \quad \text{and} \quad |u(x)| \leqq \Big\{\Big(\int_J q\Big)^{-1/2}+\Big(\int_J \frac{1}{p}\Big)^{1/2}\Big\}\|u\|_+$$

hold. Hence $\lim_{x\to a} u$ is a bounded linear form on $H_+$ if $a$ is an endpoint of $I$ near which $1/p$ is integrable. If $u$ is in the closure of $C_0^\infty(I)$ in $H_+$ it follows that $\lim_a u=0$. However, if also $q$ is integrable near $a$ any function which is $\equiv 1$ near $a$, $\equiv 0$ near the other endpoint of $I$ and in $C^\infty(I)$ is in $H_+$ so that $C_0^\infty(I)$ is not dense in $H_+$ in this case. The argument also shows that there is a 2-dimensional space outside the closure of $C_0^\infty(I)$ if $1/p+q\in L^1(I)$.

On the other hand, it is well known, and follows from Lemma 5.1, that $u$ is in $H_+\ominus C_0^\infty(I)$ precisely if $u\in C^1(I)\cap H_+$, $u'$ is locally absolutely continuous and $-(pu')'+qu=0$. Hence, for $J\subset I$ and such a $u$,

$$0 \leqq \int_J (p|u'|^2+q|u|^2) = [p\overline{u}'u]_J.$$

Thus $p\bar{u}'u$ is increasing and has finite monotone limits in both endpoints of $I$. If both limits are $0$ then $(u, u)_+=0$ so $u\equiv 0$. Thus suppose that $\lim_a p\bar{u}'u\neq 0$ where $a$ is an endpoint of $I$. Then, near $a$, $(p\bar{u}'u)^{-1}\in L^\infty$ so $u'/u\in L^1$ near $a$ since $p|u'|^2\in L^1(I)$. Thus $\lim_a \exp\big(\int_c^x u'/u\big)\neq 0$ exists, if $c$ is near $a$. However, $\big(u\cdot\exp\big(-\int_c^x u'/u\big)\big)'=0$ so

$$u(x) = u(c)\exp\Big(\int_c^x u'/u\Big).$$

Thus $\lim_a u\neq 0$ exists and since $\lim_a p\bar{u}'u\neq 0$ also $\lim_a pu'\neq 0$. Since $p|u'|^2$ and $q|u|^2$ both are in $L^1(I)$, multiplication by the bounded functions $(pu')^{-2}$ and $u^{-2}$ respectively shows that $1/p+q$ is integrable near $a$. Thus the lemma is proved and it is easy to see that $\dim H_+\ominus C_0^\infty(I)$ is 2, 1 or 0 depending on whether $1/p+q$ is integrable near both, just one or neither of the endpoints of $I$.

If Theorem 3.2 can not be used to find an appropriate space $H$ there is another method of obtaining a symmetric $E_{\min}$ which was exploited by Pleijel already in [12] for special choices of operators and boundary conditions. For this one must assume that $Q$ has a Dirichlet integral $(\cdot, \cdot)_Q$ which is a bounded hermitean form on $H_+$. Let $\dim K=k$ and define

$$\begin{aligned} K_0 &= \{u\in K|(u, H_+)_Q = 0\}, \quad \dim K_0 = k_0,\\ K_1 &= \{u\in K|(u, K)_Q = 0\}, \quad \dim K_1 = k_1,\\ H_1 &= \{u\in H_+|(u, K)_Q = 0\}. \end{aligned}$$

Then $\operatorname{codim} H_1 = k - k_0$ and $H_1 \cap K = K_1$ so that $H_1$ must be further restricted to obtain a space $H$ for which $H \cap K = \{0\}$ so that $\|\cdot\|$ may serve as norm on $H$.

**Lemma 3.4.** *Let $v \in K_1$. Then there exists $w \in H_+$, determined modulo $K$ and with $w \in K$ if and only if $v \in K_0$, such that for all $u \in H_+$*

$$(u, v)_Q = (u, w)_P.$$

*Proof.* Only the existence needs a proof. Put $\tilde{H} = H_+ \ominus K$. Then $\tilde{H}$ is a Hilbert space with norm $\|\cdot\|$ and $\tilde{H} \ni u \mapsto (u, v)_Q$, $v \in H_+$ is a bounded linear form on $\tilde{H}$ since

$$|(u, v)_Q| \leqq C\|u\|_+ \|v\|_+ \leqq C'\|v\|_+ \|u\|$$

for some constants $C$ and $C'$ by the boundedness of $(\cdot, \cdot)_Q$ on $H_+$ and since $\|\cdot\|_+$ and $\|\cdot\|$ are equivalent norms on $\tilde{H}$. Hence there is a unique linear operator $\tilde{G} : H_+ \to \tilde{H}$ such that

$$(u, v)_Q = (u, \tilde{G}v)_P \tag{3.2}$$

for $u \in \tilde{H}$ and $v \in H_+$. However, when $v \in K_1$ this holds for any $u$ in $H_+$ since both sides of (3.2) are invariant when an element of $K$ is added to $u$. This proves the lemma.

Now put $L = \tilde{G}K_1$. From $\operatorname{Ker} \tilde{G} = K_0$ follows $\dim L = k_1 - k_0$ and from $(H_1, K)_Q = 0$ follows $(H_1, L)_P = 0$. Put

$$H_2 = \{u \in H_1 | (u, L)_Q = 0\}.$$

Now $H_2 \cap L = \{0\}$ because $(H_2, K)_Q = 0$ and if $v \in K_1$, $\tilde{G}v \in H_2$ one has

$$(\tilde{G}v, \tilde{G}v)_P = (\tilde{G}v, v)_Q = 0$$

so $\tilde{G}v = 0$. But $H_2 \cap K \subset H_1 \cap K = K_1$ so assuming $v \in H_2 \cap K$ the same computation gives $H_2 \cap K = K_0$. Now let $H$ be any subspace of $H_2$ such that $H \cap K_0 = 0$ and $\operatorname{codim}_{H_2} H = k_0$. Then one has

**Theorem 3.5.** *With $H$ defined as above $E_{\min}$ is symmetric*

Thus whenever $Q$ has a Dirichlet integral which is bounded as a hermitean form on $H_+$ there is a choice of $H$ with $k \leqq \operatorname{codim} H = k + k_1 - k_0 \leqq 2k$ for which $E_{\min}$ is symmetric. Note that the only freedom of choice for the Hilbert space this method permits is the choice of $H$ as a subspace of $H_2$. The most unfavorable case, i.e. when the largest $\operatorname{codim} H$ is obtained, occurs when $K_0 = \{0\}$ and $K_1 = K$ and permits no freedom at all in the choice of $H$.

*Proof of Theorem 3.5.* For $\varphi \in C_0^\infty(\Omega)$ we may write $\varphi=\varphi_0+\varphi_1+\varphi_2$ with $\varphi_0 \in H$, $\varphi_1 \in K$ and $\varphi_2 \in L$. Then (3.1) becomes

$$(\varphi_0+\varphi_1+\varphi_2, GQ\psi)_P-(GQ\varphi, \psi_0+\psi_1+\psi_2)_P = (\varphi_0, Q\psi)_0-(Q\varphi, \psi_0)_0 =$$
$$(\varphi_0, \psi)_Q-(\varphi, \psi_0)_Q = (\varphi_0, \psi_1+\psi_2)_Q-(\varphi_1+\varphi_2, \psi_0)_Q = 0$$

since $(H_+, K)_P=(H, L)_P=(H, K)_Q=(H, L)_Q=0$ by construction.

## 4. Examples

The following are examples of situations which may be analyzed with the aid of the preceding theory. In all cases the integrand of $(\cdot, \cdot)_P$ is positive so according to the remarks after (2.2) one can choose $C_*^m=\{u \in C^m(\Omega) | (u, u)_P < \infty\}$. This is assumed in this section.

*i.* $\Delta\Delta u=\lambda\Delta u$. One has $S=\Delta\Delta$ and $T=\Delta$ and may take $P=-T$. As Dirichlet integral we choose

$$(u, v)_P = (u, v)_1 = \int_\Omega \operatorname{grad} u \, \overline{\operatorname{grad} v}.$$

Then $(\cdot, \cdot)_P$ is 1-positive, vanishing on constants, and $Q=-\Delta\Delta$ which clearly also vanishes on constants. Hence, according to Theorem 3.2, choosing as Hilbert space any hyperplane in $H_+$ not containing the polynomial 1, the minimal relation will be symmetric. In fact, in this case the minimal relation will be the graph of a symmetric, but not necessarily densely defined or closeable, operator (Holds whenever $P=bT$). In particular, if $C_0^\infty(\Omega) \subset H$, it will be the graph of the operator $\Delta$, defined on $C_0^\infty(\Omega)$. In general, one finds for an element $(G\Delta\varphi, G\Delta\Delta\varphi)$ in $E_{\min}$ that

$$(G\Delta\varphi, G\Delta\Delta\varphi)_P = -(\Delta\varphi, \Delta\varphi)_0 \leqq 0.$$

Hence the minimal relation is bounded from above and one may use the method of Friedrichs to obtain a selfadjoint extension which in a generalized sense corresponds to the Dirichlet boundary condition.

*ii.* $\Delta\Delta u=\lambda\Delta u$. The same relation as in *i.* may be considered in other Hilbert spaces. Choose $P=\Delta\Delta-b\Delta$, let $a$ and $b$ be constants and assume, for simplicity of notation, $\Omega \subset \mathbf{R}^2$. Let

$$(u, v)_{(a)} = \int_\Omega \Delta u \overline{\Delta v}+a(2\partial_1\partial_2 u \overline{\partial_1\partial_2 v}-\partial_1^2 u \overline{\partial_2^2 v}-\partial_2^2 u \overline{\partial_1^2 v})$$

and put $(\cdot, \cdot)_P=(\cdot, \cdot)_{(a)}+b(\cdot, \cdot)_1$. For $0<a<1$ this is easily seen to be a Dirichlet integral for $P$ which is 2-positive for $b=0$ and 1-positive for $b>0$, in which case one may also allow $a=0$. For $b>0$ everything may be handled as in *i.* and the minimal relation is bounded from above by $b$. In case $b=0$, $(\cdot, \cdot)_P$ vanishes on all polynomials of degree 1, but since also $Q=-b\Delta\Delta-\Delta$ does this, any $H$ chosen according to Theorem 3.2 will give a symmetric minimal relation.

*iii.* $\Delta\Delta u = \lambda\partial_1\partial_2 u$. Here one must choose $P = \Delta\Delta$ and may take

$$(\cdot, \cdot)_P = (\cdot, \cdot)_{(a)}, \quad 0 < a < 1.$$

Then $Q = \partial_1\partial_2$ so that $Q$ vanishes on the polynomials on which $(\cdot, \cdot)_P$ vanishes. Hence also in this case we will have no difficulty in choosing a Hilbert space. In this case the minimal relation is not semi-bounded. However, if $\Omega$ is bounded and of decent regularity, $Q$ will have Dirichlet integrals that are bounded in $H_+$ and this fact may be used to construct selfadjoint extensions of the minimal relation.

*iv.* $-\Delta u = \lambda k u$. Here $k$ is a function which is not assumed to have a fixed sign in $\Omega$ and is even allowed to vanish on parts, but not all, of $\Omega$. This is one of the cases Pleijel considers in [12]. We choose $P = -\Delta$ and $(\cdot, \cdot)_P = (\cdot, \cdot)_1$. Then $Q = k$ and it may of course happen that $\int_\Omega k u \bar{v}$ is a bounded form on $H_+$ in which case we may follow Pleijel and apply Theorem 3.5. However, if this fails the possibility that one may use Theorem 3.2 still remains. Since $k \not\equiv 0$ one must choose $H$ such that $C_0^\infty(\Omega) \subset H$. Suppose for example that there is a compact piece $\Gamma$ of $\partial\Omega$ which is sufficiently smooth. Then $Lu = \int_\Gamma u$ is a continuous linear form on $H_+$. Clearly $L(1) \neq 0$ and $L$ vanishes on $C_0^\infty(\Omega)$ so one may put $H = \{u \in H_+ | Lu = 0\}$ and then apply Theorem 3.2. This method seems both fairly general ands atisfactory in that it uses boundary conditions to determine $H$. Hence it may deserve some elaboration. For this, we allow other domains than open ones in the definition of $r$-positiveness.

**Theorem 4.1.** *Let $(\cdot, \cdot)_P$ be $r$-positive over $C_*^m$ on $\Gamma \cup \Omega$ where $\Gamma$ is a compact, connected part of $\partial\Omega$ with strictly positive $(n-1)$ dimensional measure. Assume that there is a bounded neighbourhood $\mathcal{O}$ of $\Gamma$ such that $\mathcal{O} \cap \Omega$ is a properly regular domain in the sense of* [7, p. 21]. *Then any element $u$ of $H_+$ has strong $L^2$-derivatives of all orders $< r$ on $\Gamma$ and the mapping*

$$H_+ \ni u \mapsto \{D^\alpha u\}_{|\alpha| < r} \tag{4.1}$$

*is continuous in the sense that there is a constant $C$ such that*

$$\sum_{|\alpha| < r} \int_\Gamma |D^\alpha u|^2 \leqq C \|u\|_+^2 \quad \text{for} \quad u \in H_+.$$

The proof depends on the fact that Lemma 2.6, the Poincaré inequality, can be proved for properly regular domains. It follows that Theorems 2.2 and 2.5 hold for $\Gamma \cup \Omega$. Now one may apply the result on the continuity of the trace operator (4.1) as proved e.g. in lecture 4 of [7] and the theorem follows. This shows that if $P_\Gamma$ is a differential operator of order at most $r-1$ and with coefficients, the restrictions of which to $\Gamma$ are in $L^2(\Gamma)$, then the linear form $H_+ \ni u \mapsto \int_\Gamma P_\Gamma u$ is continuous. Hence such forms may be used for restricting $H_+$. It is evident that they vanish on $C_0^\infty(\Omega)$, and for a given polynomial $p$ of degree at most $r-1$ one can always find a boundary form of this type that does not vanish on $p$.

## 5. The case of ordinary operators

Consider an ordinary differential operator $L\neq 0$ on the interval $I\subset\mathbf{R}$ of the form

$$L = \sum D^j a_{ij} D^i \tag{5.1}$$

This operator is called $A^0$-proper if $a_{ij}\in A^i\cap A^j$ where $A^j$ denotes the space $W^{j,2}_{\mathrm{loc}}(I)$ of functions $u\in C^{j-1}(I)$ for which $D^{j-1}u$ is locally absolutely continuous and $D^j u$, which exists a.e., is in $L^2_{\mathrm{loc}}(I)$. However, all statements made in the sequel remain true if $A^j$ is instead interpreted as $W^{j+d,2}_{\mathrm{loc}}(I)$ or $C^{j+d}(I)$ for some fixed $d$, $0\leqq d\leqq +\infty$. Only the proof of Lemma 5.1 needs some, mainly simplifying, modifications. Since the formal adjoint of $L$ is $L^+=\sum D^i\overline{a_{ij}}D^j$ also $L^+$ is $A^0$-proper. By carrying out differentiations in (5.1) it is seen that in addition to $a_{ij}\in A^i\cap A^j$ we may also assume that $L$ is on Jacobi form, i.e. $a_{ij}\equiv 0$ if $|i-j|>1$ and $a_{j,j-1}=a_{j-1,j}$. One may also carry out the differentiations in the Jacobi form to obtain

$$L = \sum_{j=0}^{M} l_j D^j, \quad l_M \not\equiv 0,$$

where the functions $l_j\in A^0$. If $M=2m$ or $2m-1$ it is easily seen that $l_{j+M-m}\in A^j$ for $j\geqq 0$. In particular $l_M\in A^m$. If $L$ has fixed order, i.e. $l_M(x)\neq 0$ throughout $I$, we say that $L$ is regular. Suppose that $L$ is regular $A^0$-proper and let $K=\sum_{j=0}^{N} k_j D^j$ be another $A^0$-proper operator such that $M>N$. Then we have the basic

**Lemma 5.1.** *Let $u\in L^2_{\mathrm{loc}}(I)$ and $v\in A^p$ for some $p\geqq 0$ and assume that*

$$(u, L\varphi)_0 = (v, K\varphi)_0 \tag{5.2}$$

*for all $\varphi\in C_0^\infty(I)$. Then $u\in A^s$ where $s=\min(M, M-N+p)\geqq M-N$.*

The proof depends on

**Lemma 5.2.** (du Bois-Reymond). *Assume that $u\in L^1_{\mathrm{loc}}(I)$ and for all $\varphi\in C_0^\infty(I)$ one has $(u, D^m\varphi)_0=0$. Then, after correction on a null-set, $u$ is a polynomial of degree at most $(m-1)$.*

The proof is well known, at least for $m=1$, see e.g. [4, Section 6]. The general case may be handled by induction.

*Proof of Lemma 5.1.* By repeated partial integrations in (5.2) one obtains

$$\Big(\sum_{j=0}^{M} F_{M-j}(l_j u-\bar{k}_j v),\, D^M\varphi\Big)_0 = 0 \quad \text{for all} \quad \varphi\in C_0^\infty(I).$$

Here $F_i(w)$ denotes a function in $A^i$ such that $D^i F_i(w)=w$. It follows from Lemma 5.2 that for some polynomial $q$

$$\sum_{j=0}^{M} F_{M-j}(l_j u-\bar{k}_j v) = q.$$

Since $l_M \neq 0$ everywhere in $I$ and $k_M \equiv 0$ this may be written

$$u = (\bar{l}_M)^{-1}\{q - \sum_{j=0}^{M-1} F_{M-j}(\bar{l}_j u - \bar{k}_j v)\}. \tag{5.3}$$

According to assumption $u \in L^2_{loc}(I) = A^0$. Suppose that, for some $d \geqq 0$, we have $u \in A^j$ for $j = d$ but not for $j > d$. It follows from the known regularity of $l_j$, $k_j$ and $v$ that the right hand side of (5.3) is in $A^s$, where $s = \min(d+1, m, M-N+p)$. Thus the assertion follows unless $M-N+p > m$. In that case we have at least $u \in A^m$. By carrying out differentiation in the Jacobi form of $K$ one obtains

$$K = \sum_{i \leqq p} D^i(\sum_{j < M-m} b_{ji} D^j)$$

with $b_{jp} \in A^t$ where $t = p + \max(0, j-n)$ if $N = 2n$ or $2n-1$. Furthermore $b_{ji} \in \in A^j \cap A^i$ for $i < p$. Integrating by parts in (5.2) it then follows that

$$\int_I \sum \bar{a}_{ji} D^i u \overline{D^j \varphi} = \int_I \sum \bar{b}_{ji} D^i v \overline{D^j \varphi} \quad \text{for all} \quad \varphi \in C_0^\infty(I).$$

Integrating by parts in the other direction one obtains

$$\int_I \sum_{j=0}^m F_{m-j}(\sum_i (\bar{a}_{ji} D^i u - \bar{b}_{ji} D^i v)) \cdot \overline{D^m \varphi} = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(I).$$

As before there is then a polynomial $q$ such that

$$D^m u = (\bar{l}_M)^{-1}\{q - \sum_{i=0}^{m-1} \bar{a}_{mi} D^i u - \sum_{j=0}^{m-1} F_{m-j}(\sum_i (\bar{a}_{ji} D^i u - \bar{b}_{ji} D^i v))\} \tag{5.4'}$$

if $l_M = a_{mm}$. If $a_{mm} = 0$, so that $l_M = 2a_{m,m-1}$, the formula reads

$$D^m u = \tag{5.4''}$$

$$= (\bar{l}_M)^{-1}[D\{q - \sum_{i=0}^{m-2} \bar{a}_{mi} D^i u - \sum_{j=0}^{m-1} F_{m-j}(\sum_i (\bar{a}_{ji} D^i u - \bar{b}_{ji} D^i v))\} - D^{m-1} u D \bar{a}_{m,m-1}],$$

where the term $\bar{a}_{m-1,m} D^m u$ does not appear in the last sum. From these formulas it is easily concluded that if $u \in A^{m+d}$ for some $d \geqq 0$, then the right hand side of (5.4) is in $A^s$ where $s = \min(M-m, d+1, M-m-N+p)$. This proves the lemma.

**Corollary 5.3.** *Under the assumptions of lemma* 5.1 *and if* $p \geqq N$, *then* $u$ *is a classical solution of the equation* $L^+ u = K^+ v$.

*Proof.* Lemma 5.1 implies sufficient differentiability so that integrating by parts in (5.2) one obtains

$$\int_I (L^+ u - K^+ v)\bar{\varphi} = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(I)$$

from which the corollary follows.

Returning to spectral theory, let $S$ and $T$ be formally symmetric ordinary differential operators over $I$ having a symmetric minimal relation $E_{\min}$ in the Hilbert space $H \subset L^2_{loc}(I)$ with scalar product $(\cdot, \cdot)_P$, a Dirichlet integral over $I$ belonging to $P = aS + bT$ for some real numbers $a$, $b$. We assume furthermore

(i) $S$ and $T$ are $A^0$-proper on $I$.
(ii) S is regular of order $M$ and the maximal order $N$ over $I$ of $T$ satisfies $N<M$.

Put $E$=closure of $E_{\min}$ so that $E^*=E_{\max}$. Then we have

**Corollary 5.4.** *For all* $(u, v)\in E^*$ *the function* $u$ *belongs to* $W_{\text{loc}}^{q,2}(I)$ *where* $q=M-N$.

This is an immediate consequence of Lemma 5.1 by the definition of $E^*$ and since $H\subset L_{\text{loc}}^2(I)$.

**Corollary 5.5.** $E_\lambda\subset A^M\times A^M$ *for every* $\lambda\in\mathbf{C}$.

This follows since $\tilde{D}_{\bar{\lambda}}$, according to Lemma 5.1 with $L=S-\lambda T$, $K=T$, is in $A^M$. Repeating the argument with $L=S-\bar{\lambda}T$ proves the corollary. Now define (cf. Pleijel [14])

$$E[I] = \{(u, v)\in E^* | v\in A^N\}$$

$$E^\perp = E[I]\cap(E[I])^*.$$

By Corollary 5.3 the space $E[I]$ consists of all pairs $(u, v)\in H^2\cap A^M\times A^N$ for which $Su=Tv$ classically. By Corollary 5.5 we have $E_\lambda\subset E[I]$.

**Theorem 5.6.** *If* $v\in A^N\cap H$, *then for any non-real number* $\lambda$ *there is an element* $(u, \lambda u+v)\in E[I]$.

**Theorem 5.7.** $E[I]=E^\perp\dot{+}E_\lambda$ *as a direct sum for any non-real* $\lambda$.

In [3], [13] and [14] these theorems are proved by a limiting process directly from the standard existence theorems for differential equations. They then form the basis for the spectral theory.

*Proof of Theorem 5.6.* According to Lemma 1.10 there is an element $(u, \lambda u+v)\in E^*$. Applying Lemma 5.1 with $L=S-\bar{\lambda}T$, $K=T$ it follows that $u\in A^M$. Hence $(u, \lambda u+v)\in$ $\in E[I]$.

*Proof of Theorem 5.7.* Since $E_\lambda\subset E[I]$ an intersection of $E^*=E\dot{+}E_\lambda$ by $E[I]$ shows that $E[I]=E\cap E[I]\dot{+}E_\lambda$. From $E_\lambda\subset E[I]\subset E^*$ one obtains $E\subset(E[I])^*\subset E_\lambda^*$ and hence $E\cap E[I]\subset E^\perp\subset E_\lambda^*\cap E[I]$. From the definition of adjoint and $E^*=E\dot{+}E_\lambda$ it is easily seen that $E_\lambda^*\cap E^*=E$. Thus $E^\perp=E\cap E[I]$ and hence $E[I]=E^\perp\dot{+}E_\lambda$.

**Theorem 5.8.** $E[I]$ *is dense in* $E^*$ *and* $E^\perp$ *in* $E$.

*Proof.* The proof of Theorem 5.7 shows that $\overline{E^\perp}$ is a subspace of $E$. It follows that the sum $\overline{E^\perp}\dot{+}E_\lambda$ is topological and hence that

$$\overline{E[I]} = \overline{E^\perp}\dot{+}E_\lambda \quad (\operatorname{Im}\lambda\neq 0).$$

Thus from $\overline{E[I]}=E^*$ follows $\overline{E^\perp}=E$. To prove $\overline{E[I]}=E^*$ write an arbitrary element $U\in E^*$ in the form $U=(u, \lambda u+v)$ where $\operatorname{Im}\lambda\neq 0$. Let $\tilde{v}\in A^N\cap H$ and determine $w\in S_{\bar\lambda}$ so that $W=(w, \lambda w+\tilde{v}-v)\in E^*$. This may be done and $\|w\|\leqq \leqq|\operatorname{Im}\lambda|^{-1}\|\tilde{v}-v\|$ according to lemma 1.10. Now take $\tilde{v}$ so that $\|\tilde{v}-v\|$ is arbitrarily small which then also holds for $\|w\|$. Put $\tilde{u}=u+w$. Then $\tilde{U}=U+W=(\tilde{u}, \lambda\tilde{u}+\tilde{v})$ is an arbitrarily good approximation of $U$, and $\tilde{U}\in E[I]$ since $\tilde{v}\in A^N$ and $\tilde{u}\in A^M$ according to Lemma 5.1 with $L=S-\bar\lambda T$ and $K=T$. This proves the theorem.

Theorem 5.8 implies that $E^\perp$ and $E$ have the same selfadjoint extensions. In this sense our theory agrees with that of Pleijel in [13], [14]. The regularity requirements for the coefficients of $S$ and $T$ made in these papers are roughly the same as here. Certain positivity assumptions related to an increasing sequence $\{J_k\}_0^\infty$ of compact subintervals of $I$ with $\cup J_k=I$ are made with respect to a Dirichlet integral

$$(u, v)_P=\int_I \sum a_{ij}D^i u\overline{D^j v}$$

belonging to $P=T$ in one and $P=S$ in the other paper. Apart from a trivial translation of the spectral parameter this covers the possible choices of $P$ in the present theory. Setting

$$(u, v)_{P,k}=\int_{J_k} \sum a_{ij}D^i u\overline{D^j v}$$

these positivity assumptions read, in slight paraphrasing,

i. *The order $p$ of $P$ is constant in $I$.*
ii. *If $u\in A^p$ does not vanish in $J_0$, then $(u, u)_{P,0}>0$.*
iii. *$(u, u)_{P,k}$ increases with $k$ for any fixed $u\in A^p$.*

Put $C_*^p=\{u\in C^p(I)|\lim_k (u, u)_{P,k}<+\infty\}$, which is easily seen to be linear. The condition iii. is used in [13], [14] to prove Theorems 5.6 and 5.7. However, the example at the beginning of Section 2 shows that it need not be satisfied by a Dirichlet integral which is 0-positive over $C_*^p$. On the other hand, i., ii. and iii. imply that $(\cdot, \cdot)_P$ is 0-positive over $C_*^p$. To sketch a proof, consider the Hilbert space $H$ obtained by completion of $W^{p,2}(J_k)$ in the norm-square $(u, u)_{P,k}+(u, u)_{0,J_k}$. The problem of finding an element of $H$ which minimizes $(u, u)_{P,k}$ under the side condition $(u, u)_{0,J_k}=1$ is equivalent to finding the first eigenfunction of the eigenvalue problem $Pu=\lambda u$ in $J_k$ with free boundary conditions in Courant's sense. Because of i. this eigenvalue problem has a discrete spectrum so that there exists a minimizing function $u_k\in W^{p,2}(J_k)$ such that $(u_k, u_k)_{0,J_k}=1$ and $(u_k, u_k)_{P,k}\leqq(u, u)_{0,J_k}^{-1}(u, u)_{P,k}$ for all $u\in H$. Because of assumptions ii. and iii. we must have $C_k=(u_k, u_k)_{P,k}$ strictly positive and

$$(u, u)_P\geqq(u, u)_{P,k}\geqq C_k(u, u)_{0,J_k} \quad \text{for any} \quad u\in C_*^p.$$

Since $\{J_k\}_0^\infty$ exhausts $I$ it follows that $(\cdot, \cdot)_P$ is 0-positive over $C_*^p$.

## 6. Eigenfunction expansions and Green's function

Let $\sigma$ be a positive measure on $\mathbf{R}$ and $\nu\colon \mathbf{R}\to\{1, 2, \dots, \infty\}$ $\sigma$-measurable. If $f_j\colon \mathbf{R}\to\mathbf{C}$ are $\sigma$-measurable and $f(\xi)=(f_1(\xi), \dots, f_{\nu(\xi)}(\xi))$, then the set of (equivalence classes of) functions $\xi\mapsto f(\xi)$ for which $\sum_{j=1}^{\nu(\xi)}|f_j(\xi)|^2$ is $\sigma$-integrable is a Hilbert space $L^2(\sigma, \nu)$ with scalar product $(\cdot, \cdot)_\sigma$ and norm $\|\cdot\|_\sigma$. The spectral theorem (von Neumann, [10]) states that for every selfadjoint operator $E_0$ on a Hilbert space $H_0$ there is a space $L^2(\sigma, \nu)$, with $\sigma$ determined modulo equivalent measures and $\nu$ determined $\sigma$-a.e., and a unitary mapping $\mathscr{F}\colon H_0\ni u\mapsto\hat{u}\in L^2(\sigma, \nu)$ such that $\widehat{E_0 u}(\xi)=\xi\hat{u}(\xi)$. Finally, $u\in\mathscr{D}_{E_0}$ precisely if $\xi\hat{u}(\xi)\in L^2(\sigma, \nu)$. Let $S$ and $T$ be ordinary differential operators on the interval $I\subset\mathbf{R}$ satisfying the assumptions of Section 5 and assume that there exists a selfadjoint realization $E$ of $Su=Tv$ in the Hilbert space $H$ with norm $\|\cdot\|=\sqrt{(\cdot, \cdot)_P}$. Thus according to Theorem 1.15 and the spectral theorem, there is a space $L^2(\sigma, \nu)$ and a unitary mapping $\mathscr{F}$ from $H_0=H\ominus H_\infty$ to $L^2(\sigma, \nu)$ with $\hat{v}(\xi)=\xi\hat{u}(\xi)$ for $(u, v)\in E_0=E\cap H_0^2$. For convenience, define $\mathscr{F}H_\infty=\{0\}$ so that $\hat{v}(\xi)=\xi\hat{u}(\xi)$ for any $(u, v)$ in $E$ and $(u, w)_P= =(\hat{u}, \hat{w})_\sigma$ if at least one of $u$ and $w$ is in $H_0$. We will need the following lemma, which is similar to theorem 5.8.

**Lemma 6.1.** *The set $E_0\cap A^M\times A^M$, where $M=$order $S$, is dense in $E_0$.*

*Proof.* If $(u, v)\in E_0$ then $R_i(v-iu)=u$ and for $\varepsilon=0$ one can find $\tilde{v}\in A^M\cap H$ so that $\|v-iu-\tilde{v}\|<\varepsilon$. Since $\|R_i\|\leqq 1$ this implies that $\|u-\tilde{u}\|<\varepsilon$, where $\tilde{u}=R_i\tilde{v}$. Hence $(\tilde{u}, i\tilde{u}+\tilde{v})$ approximates $(u, v)$. However, $\tilde{u}\in\tilde{E}\subset H_0$ so that Lemma 5.1 implies $\tilde{u}\in A^M\cap H_0$. It remains to show only that one may take $\tilde{v}\in A^M\cap H_0$. This follows, however, since we have just seen that $A^M\cap\tilde{E}$ is dense in $\tilde{E}$ and thus in $H_0$. Hence we may take $\tilde{v}\in A^M\cap\tilde{E}$ which proves the lemma.

Let $s(\cdot, \cdot)$ be a fundamental solution for $S$, i.e. if $\varphi\in A^M$ with compact support one has $\varphi(x)=(S\varphi, s(x, \cdot))_0$. Such a function is easily constructed, e.g. by solving an appropriate Cauchy problem, and has the property

$$(x, y)\mapsto D_y^j s(x, y) \text{ is locally bounded for } j\leqq M-1 \text{ and}$$
$$\text{continuous except when } j=M-1 \text{ and } x=y. \tag{6.1}$$

Now suppose $(u, v)\in E_0\cap A^M\times A^M$. Let $J\subset\subset I$ and $\varphi_J\in C_0^\infty(I)$ be realvalued and $\equiv 1$ in a neighbourhood of $J$. Then one has

$$u(x)=\varphi_J u(x)=(S\varphi_J u, s(x, \cdot))_0=(Su, \varphi_J s(x, \cdot))_0+(K_J u, s(x, \cdot))_0$$

for $x\in J$. Here $K_J=[S, \Phi_J]$ is the commutator of $S$ and the operator $\Phi_J$ of multiplication by $\varphi_J$. Thus $K_J$ is a skew-symmetric differential operator with coefficients in $L_0^2(I)$ and of order $<M$. Now $Su=Tv$ and order $T<M$ so that $T\varphi_J s(x, \cdot)$ and $K_J s(x, \cdot)$ are in $L_0^2(I)$ and thus in the domain of $G$ (see section 3). Integration by parts gives thus

$$u(x) = (v, T\varphi_J s(x, \cdot))_0-(u, K_J s(x, \cdot))_0 = (v, GT\varphi_J s(x, \cdot))_P-(u, GK_J s(x, \cdot))_P.$$

Setting $a_J(x, \cdot)=\mathscr{F}(GT\varphi_J s(x, \cdot))$, $b_J(x, \cdot)=\mathscr{F}(GK_J s(x, \cdot))$ and $e_J(x, \xi)=\overline{\xi a_J(x, \xi)-b_J(x, \xi)}$ one obtains, since $v\in H_0$,

$$u(x) = (\hat{u}, \overline{e_J(x, \cdot)})_\sigma \quad \text{for} \quad x\in J. \tag{6.2}$$

Clearly $e_J(x, \cdot)\in L^2(\sigma_-, \nu)$ where $d\sigma_-=(1+|\xi|^2)^{-1}d\sigma$. Since $u\in\tilde{E}$ means that $\hat{u}$ and $\xi\hat{u}(\xi)\in L^2(\sigma, \nu)$ one has $\mathscr{F}\tilde{E}=L^2(\sigma_+, \nu)$ where $d\sigma_+=(1+|\xi|^2)d\sigma$. Taking limits it is thus clear from Lemma 6.1 that (6.2) holds for any $u\in\tilde{E}$. It follows also that $e_J(x, \cdot)=e_{J'}(x, \cdot)$ if $x\in J\cap J'$. Thus $e(x, \cdot)=\lim_{J\to I} e_J(x, \cdot)$ is defined as an element of $L^2(\sigma, \nu)$ and

$$u(x) = (\hat{u}, \overline{e(x, \cdot)})_\sigma \quad \text{for} \quad x\in I \quad \text{and} \quad u\in\tilde{E}.$$

If $u\in H_0\setminus\tilde{E}$ put, for any compact $J$, $\hat{u}_J(\xi)=\hat{u}(\xi)$ for $\xi\in J$ and $=0$ otherwise. Clearly $\hat{u}_J\in L^2(\sigma_+, \nu)$ and $\hat{u}_J\to\hat{u}$ in $L^2(\sigma, \nu)$ when $J\to\mathbf{R}$. Thus

$$(\hat{u}_J, \overline{e(x, \cdot)})_\sigma \to u \quad \text{in} \quad H_0 \quad \text{when} \quad J\to\mathbf{R} \tag{6.3}$$

which is the desired formula for the inverse transform $\mathscr{F}^{-1}\colon L^2(\sigma, \nu)\to H_0$. To show that this is an "eigenfunction expansion" first note that as an $L^2(\sigma, \nu)$ valued function $x\mapsto e(x, \cdot)$ is continuous. This follows easily from (6.1) and the fact that $\mathscr{F}G\colon L_0^2(I)\to L^2(\sigma, \nu)$ is continuous. We need the following

**Lemma 6.2.** *Let $I\ni x\mapsto\tilde{e}(x, \cdot)$ be a continuous $L^2(\sigma_-, \nu)$ valued function. Then one can for each $x$ choose an element $e(x, \xi)$ of the equivalence class $\tilde{e}(x, \cdot)$ so that $e(x, \xi)$ is measurable with respect to the product measure $dxd\sigma$.*

*Proof.* Let $\{x_j\}$ be a dense sequence in $I$ and choose for each $x_j$ a representative $e(x_j, \xi)$ for $\tilde{e}(x_j, \cdot)$. Put $e_n(x, \xi)=e(x_j, \xi)$ where $x_j$ is the closest to $x$ of $x_1, \ldots, x_n$ (if there are two equally close, take the left one). It is clear that $e_n$, being piecewise independent of $x$, is measurable. Furthermore, since $\tilde{e}(x, \cdot)$ is locally uniformly continuous it is clear that $e_n(x, \cdot)\to\tilde{e}(x, \cdot)$ locally uniformly in $L^2(\sigma_-, \nu)$. By a standard device we may thus for every compact $J\subset\subset I$ choose a subsequence $\{e_{n_j}(x, \xi)\}$ which for each $x\in J$ converges to $\tilde{e}(x, \cdot)$ except on a $\sigma$-nullset. But the sequence $\{e_{n_j}\}$ of $dxd\sigma$-measurable functions converges except on a $dxd\sigma$-measurable set $M$, whose trace for fixed $x\in J$ is a $\sigma$-nullset. Hence $M\cap J\times\mathbf{R}$ is a $dxd\sigma$-nullset and one may take $e(x, \xi)=\lim_j e_{n_j}(x, \xi)$ for $x\in J$. The lemma follows.

We will assume that a choice of representatives for $e(x, \cdot)$ making $e(x, \xi)$ measurable has been made, and then it immediately follows from the continuity of $e(x, \cdot)$ that $e$ is $L^2(dx d\sigma_-, v)$ on $J \times \mathbf{R}$ for any $J \subset\subset I$. For $f \in L_0^2(I)$ and $u \in \tilde{E}$ one then obtains

$$(\widehat{Gf}, \hat{u})_\sigma = (Gf, u)_P = (f, u)_0 = (f, (\hat{u}, \overline{e(x, \cdot)})_\sigma)_0 = ((f, e(\cdot, \xi))_0, \hat{u})_\sigma$$

the change of order of integration being legitimate by the absolute convergence of the double integral. Since $\tilde{E}$ is dense in $H_0$ one obtains

$$\widehat{Gf}(\xi) = (f, e(\cdot, \xi))_0 \quad \sigma\text{-almost everywhere.} \tag{6.4}$$

It follows also that $e(\cdot, \xi) \in L^2_{\mathrm{loc}}(I)$ for $\sigma$-almost all $\xi$, and setting $e(\cdot, \xi) \equiv 0$ in the exceptional set this holds everywhere. From $(GT\varphi, GS\varphi) \in E_{\min} \subset E$ for $\varphi \in C_0^\infty(I)$ it follows that $\widehat{GS\varphi}(\xi) = \xi \widehat{GT\varphi}(\xi)$ $\sigma$-a.e. After setting $e(\cdot, \xi) \equiv 0$ on a certain $\sigma$-nullset, which may depend on $\varphi$, one obtains by setting $f = T\varphi$ and $f = S\varphi$ respectively in (6.4) that

$$((S - \xi T)\varphi, e(\cdot, \xi))_0 = 0 \quad \text{for all} \quad \xi. \tag{6.5}$$

Repeating the same procedure for a denumerable set $\Phi$ of functions $\varphi$ we have still only changed $e(\cdot, \xi)$ on a $\sigma$-nullset. The set $\Phi$ can be taken dense in $C_0^\infty(I)$ with the usual topology, and then (6.5) will hold for all $\varphi \in C_0^\infty(I)$ by continuity since $e(\cdot, \xi) \in L^2_{\mathrm{loc}}(I)$ for all $\xi$. According to Lemma 5.1 this implies that $e(\cdot, \xi) \in A^M$ for each $\xi$ and satisfies $(S - \xi T)e(\cdot, \xi) = 0$. It is easy, but somewhat tedious, to see that there is an element $\hat{\lambda} \in L^2(\sigma_+, v)$ such that $\sum_{j=1}^{v(\xi)} \hat{\lambda}_j(\xi) e_j(\cdot, \xi) = 0$ but $\hat{\lambda}(\xi) = 0$ only for those $\xi$ for which the components $e_j(\cdot, \xi)$ of $e(\cdot, \xi)$ are linearly independent. This implies

$$\lambda(x) = (\mathscr{F}^{-1}\hat{\lambda})(x) = (\hat{\lambda}, \overline{e(x, \cdot)})_\sigma = 0$$

so that $\hat{\lambda} = 0$. Hence $e(\cdot, \xi)$ has linearly independent components for $\sigma$-almost all $\xi$. Since $e_j(\cdot, \xi)$ is a solution of the regular $M^{\text{th}}$ order equation $(S - \xi T)u = 0$ this implies that $v(\xi) \leqq M$ $\sigma$-a.e. It is also easy to see that $\mu$ is a proper eigenvalue of $E$ if and only if $\sigma$ has a jump at $\mu$, i.e. $\sigma(\{\mu\}) > 0$. In this case all components of $e(\cdot, \mu)$ are in $\tilde{E}$, being inverse transforms of functions with support $\{\mu\}$, and they span the eigenspace at $\mu$.

Since the range of $G$ is dense in $H$, a consequence of $H \subset L^2_{\mathrm{loc}}(I)$, a formula for computing $\hat{u}$ is given by (6.4). However, it is not clear in general how to find $f \in L_0^2(I)$ so that $Gf$ approximates a given $u \in H$. A more satisfactory formula is obtained by making the following

**Assumption 6.3.** *For* $\varphi \in L^\infty(I)$ *there exists* $C_\varphi = C_\varphi(u)$ *such that*

i. $|(u, v)_{P_\varphi}| \leqq C_\varphi \|v\|$ *for* $v \in H$ *and any fixed* $u \in H$.
ii. $C_\varphi$ *depends continuously on* $u$ *for fixed* $\varphi$.
iii. *If* $\varphi_j \searrow 0$ *when* $j \to \infty$, *then* $C_{\varphi_j} \to 0$ *for any fixed* $u$.

Here we have defined, first for $u, v \in C_*^\infty$ and then by continuity,

$$(u, v)_{P_\varphi} = \int_I \varphi \sum a_{ij} D^i u \overline{D^j v}$$

so that $(\cdot, \cdot)_{P_\varphi} = (\cdot, \cdot)_P$ if $\varphi \equiv 1$, and if $\varphi \in C_0^\infty(I)$ we define

$$P_\varphi = \sum D^j \varphi a_{ij} D^i.$$

The assumption is easily verified if $\|\cdot\|$ is equivalent to the norm $\sqrt{\int B(\cdot, \cdot)}$ introduced just after (2.2). For $\varphi \in C_0^\infty(I)$ and $u \in A^M \cap H$ one obtains from Assumption 6.3

$$(u - GP_\varphi u, u - GP_\varphi u)_P = (u, u - GP_\varphi u)_{P_{1-\varphi}} \leqq C_{1-\varphi} \|u - GP_\varphi u\|$$

so that $GP_\varphi u \to u$ in $H$ when $\varphi \nearrow 1$. Thus also $\widehat{GP_\varphi u} \to \hat{u}$ in $L^2(\sigma, v)$ and from (6.4) one obtains

$$\widehat{GP_\varphi u}(\xi) = \big(u, e(\cdot, \xi)\big)_{P_\varphi}$$

so that

$$\big(u, e(\cdot, \xi)\big)_{P_\varphi} \to \hat{u} \quad \text{in} \quad L^2(\sigma, v) \quad \text{when} \quad \varphi \nearrow 1. \tag{6.5}$$

This has been proved for $u \in A^M \cap H$ but follows easily from Assumption 6.3 in general. In fact it follows that $\varphi$ may be taken as the characteristic function of a compact subinterval of $I$ so that (6.5) together with (6.3) gives a perfect analogy to the classical theory of the Fourier transform.

We give now a brief discussion of the Green's function of $E$. For $\operatorname{Im} \lambda \neq 0$ we have $(R_\lambda u, \lambda R_\lambda u + u) \in E$ so it is clear that $(\xi - \lambda)\widehat{R_\lambda u}(\xi) = \hat{u}(\xi)$, hence that $R_\lambda u(x) = \big((\xi - \lambda)^{-1} \hat{u}, \overline{e(x, \cdot)}\big)_\sigma = \big(\hat{u}, \overline{(\xi - \lambda)^{-1} e(x, \cdot)}\big)_\sigma$. However, since $e(x, \cdot) \in L^2(\sigma_-, v)$ clearly $(\xi - \lambda)^{-1} e(x, \cdot) \in L^2(\sigma, v)$ and is thus the transform of $g_\lambda(x, \cdot) \in H$. Therefore

$$R_\lambda u(x) = \big(u, \overline{g_\lambda(x, \cdot)}\big)_P$$

valid pointwise for any $u \in H$. Thus $g_\lambda(\cdot, \cdot)$ is the kernel of the resolvent $R_\lambda$, i.e. the Green's function of $E$ at $\lambda$. Using $R_\lambda^* = R_{\bar\lambda}$ and the resolvent relation (Theorem 1.11) one obtains the basic properties

$$g_\lambda(x, y) = \overline{g_{\bar\lambda}(y, x)}$$

$$g_\lambda(x, y) - g_\mu(x, y) = (\lambda - \mu)\big(g_\lambda(\cdot, y), \overline{g_\mu(x, \cdot)}\big)_P.$$

If $P=T$ one also obtains, using $E_{\min}\subset E$, that if $C_0^\infty(I)\subset H$, then

$$(S-\lambda T)_x g_\lambda(x, y) = 0 \quad \text{for} \quad y \neq x.$$

However, if $P=aS+bT$ with $a\neq 0$ one obtains instead

$$g_\lambda(x, y) = \tilde{g}_\lambda(x, y)-a(a\lambda+b)^{-1}p(x, y)$$

where $(S-\lambda T)_x\tilde{g}_\lambda(x, y)=0$ for all $x\neq y$ and $p$ is the kernel of the operator $G$ on $L_0^2(I)$, or equivalently, of the evaluation operator on $H$

$$Gf(x) = (f, \overline{p(x, \cdot)})_0, \quad u(x) = (u, \overline{p(x, \cdot)})_P.$$

One may construct $p$ in a way similar to the construction of $e(\cdot, \cdot)$ and it is easily seen that $P_x p(x, y)=0$ for $x\neq y$ and that $p(x, y)=\overline{p(y, x)}$. In an appropriate sense one may also show that $g_\lambda(x, \cdot)$ satisfies the (abstract) boundary condition which determines $E$ as a restriction of $E_{\max}$, but we will not give any details here.

Finally we mention that since the dependence on $\xi$ of $e(x, \xi)$ is rather arbitrary it is convenient to redefine $L^2(\sigma, v)$ somewhat as is usually done in dealing with ordinary differential operators. One chooses a basis $E(x, \xi)=(E_1(x, \xi), \ldots, E_M(x, \xi))$ for the solutions of $(S-\xi T)u=0$ which is analytic in $\xi$, e.g. by solving Cauchy problems with data analytic in $\xi$. Then $e(x, \xi)=A(\xi)E(x, \xi)$ where $A(\xi)$ is a $v(\xi)\times M$ matrix. Setting $B=A^*A$ the matrix $B$ is non-negative with rank $v(\xi)$. Let $d\varrho=Bd\sigma$, the spectral matrix, and $L^2(\varrho)$ be the Hilbert space of equivalence classes of functions $U=(U_1, \ldots, U_M)$ for which

$$(U, U)_\varrho = \int U^*BU\,d\sigma < \infty.$$

Then $U(\xi)=\lim_{\varphi\nearrow 1}(u, \overline{E(\cdot, \xi)})_{P_\varphi}$ defines an element of $L^2(\varrho)$ (under assumption 6.3) and $\hat{u}=AU$ so that

$$u(x) = (U, E(x, \cdot))_\varrho$$

and the mapping $H\ni u\to U\in L^2(\varrho)$ diagonalizes $E$, vanishes on $H_\infty$ and is unitary on $H_0$. Clearly $d\varrho$ is uniquely determined once $E(x, \cdot)$ has been fixed.

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