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On the vector valued Hausdorff–Young inequality

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Abstract. This paper studies Banach space valued Hausdorff–Young inequalities. The largest part considers ways of changing the underlying group. In particular the possibility to deduce the inequality for open subgroups as well as for quotient groups arising from compact subgroups is secured. A large body of results concerns the classical groups $\mathbf{T}^n$, $\mathbf{R}^n$ and $\mathbf{Z}_k$. Notions of Fourier type are introduced and they are shown to be equivalent to properties expressed by finite groups alone.

Introduction

This paper presents a study of the vector valued Fourier transform and deals with a possible analog of the Hausdorff–Young inequality. The realm will be that of locally compact Abelian (LCA) groups together with their Haar measures and general Banach spaces.

Recall that the Hausdorff–Young inequality for complex valued functions on a group simply states that

$$\|\mathcal{F}_G f\|_{L^{p'}(\widehat{G})} \le C\|f\|_{L^p(G)}$$

for suitable normalizations of the group G and its dual $\widehat{G}$. Here p' denotes the conjugate exponent and $\mathcal{F}_G$ is the Fourier transform on G. In the sequel, when there is no doubt about which group is intended, the Fourier transformed function will often be written $\hat{f}$. For infinite groups the inequality can hold only in the range $1\le p\le 2$ and we will unless otherwise stated only consider such values for p. To be precise, this inequality is first proved for $L^1(G)\cap L^p(G)$ and then the operator is extended to the whole space $L^p(G)$. The proof of the inequality for these functions with strong integrability, viz. $L^1\cap L^p$, was one of the very first applications of and indeed a motive for the development of interpolation theory. The tool is the Riesz–Thorin interpolation theorem and this is the basis of the so called complex interpolation functors. We will presently see that the vector valued version will aid in determining the outcome of different interpolation methods.

The obvious generalization is to deal with Banach space valued functions instead of just complex valued ones. The question is now to decide whether the Fourier transform can be extended to a bounded operator

$$\mathcal{F}_G: L^p(G,E) \longrightarrow L^{p'}(\widehat{G},E)$$

where we deal with the Lebesgue–Bochner spaces. We assume throughout that E is a non-zero complex Banach space. We will also agree always to choose the dual measure on $\widehat{G}$, as determined by the group G. Another way to phrase this, is that we require Parseval's identity for complex valued functions to read $\|\hat{f}\|_2 = \|f\|_2$.

Definition.

(1) $M(G,E,p)$ denotes the operator norm of $\mathcal{F}_G$ for its action from $L^p(G,E)$ into $L^{p'}(\widehat{G},E)$.

(2) $B_p = \sqrt{p^{1/p}/p'^{1/p'}}$ denotes the Babenko–Beckner constant.

Note that the quantity $M(G,E,p)$ does not depend on the choice of Haar measure on G, due to the definition of the Fourier transform. By interpolation between Parseval's formula and the inequality $\|\hat{f}\|_\infty \le \|f\|_1$, we find $M(G,\mathbf{C},p) \le 1$ for every group G. The basic examples to be remembered are $M(\mathbf{T},\mathbf{C},p) = 1$ and $M(\mathbf{R}^n,\mathbf{C},p) = B_p^n$. The second is a highly nontrivial improvement first obtained by Babenko [Ba] for those exponents that make p' into an even integer and then by Beckner, in the very important paper [Be], in general. Even though the Babenko–Beckner constant is derived on the real line it plays a vital role also for the circle group. The importance will show in this paper and is also illuminated from another aspect in the second part of the author's paper [A2]. As general examples with $M(G,\mathbf{C},p) = 1$ we have all compact, as well as all discrete groups. This is due to the behavior of constant functions and point masses respectively. A qualitative answer has been obtained by Russo [Ru1]–[Ru5] in case $1<p<2$: $M(G,\mathbf{C},p)<1$ if and only if G does not have an open and compact subgroup. In all remaining cases $M(G,\mathbf{C},p)=1$ obtains. Let us in passing decide to denote the canonical finite cyclic groups by $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$.

In contrast to the complex valued case it turns out that even for the simple spaces $E = L^r(X)$ it is impossible to derive a Hausdorff–Young inequality for the whole range $1 \le p \le 2$. It is well known that the full range appears for Hilbert spaces and no other. This suggests a notion.

Definition. A Banach space E is said to be of *G-Fourier type p* in case the norm $M(G,E,p)$ is finite.

Remark. This corresponds to the notion of *weak* G-Fourier type as defined in [M2]. The present name will be elaborated further in Section 3. For the classical groups the notion is used in [P1], [P2], [M1], [CS].

The main motivation for this paper is observations made in interpolation theory by Lions–Peetre [LP], Peetre [P1], Cwikel–Sagher [CS] and Milman [M1], [M2]. It is well known that complex interpolation unfortunately produces spaces that are hard to identify. That is in contrast to the real method which often gives identifiable spaces. The observations referred to imply that knowledge on Fourier type makes it possible to equate the outcome of complex and real interpolation. Some examples are helpful.

Theorem. ([P1], [CS]) *Suppose that* $\bar{A}=(A_0,A_1)$ *is a Banach couple such that* A_j *has* $\mathbf{R}$*-Fourier type* p_j*. Then* $(A_0,A_1)_{\theta,p}$ *and* $[A_0,A_1]_\theta$ *both have* $\mathbf{R}$*-Fourier type* p_θ*, where*

$$\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Theorem. ([P1]) *Suppose that* $\bar{A}=(A_0,A_1)$ *is a Banach couple such that* A_j *has* $\mathbf{R}$*-Fourier type* p_j*. One obtains*

$$(A_0,A_1)_{\theta,p}\subseteq[A_0,A_1]_\theta\subseteq(A_0,A_1)_{\theta,p'},\quad \frac{1}{p}=\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

The first result is useful to establish that type p implies every other type r in the interval $r\in[1,p]$ for the same group. This is so since every Banach space trivially is of type 1. The second theorem makes it possible to equate the resulting interpolating spaces in many concrete cases.

Judicious use of the theorems by the above mentioned authors has led to results for interpolation of Besov and Sobolev spaces, trace and Schatten classes as well as some martingale analogues. A basic question is whether $\mathbf{R}$-, $\mathbf{Z}$- and $\mathbf{T}$-Fourier type is one and only one concept. In the papers cited all three notions are used. The first to give thought to this was Bourgain [B2]. An affirmative answer can be extracted from [A2].

Another link is towards type theory within the borders of the geometric theory of Banach spaces. The reason for a connection to exist is the presence of an interaction between the Rademacher functions and the Walsh functions. The latter ones collectively constitute the group dual to the Cantor group $\mathbf{Z}_2^\infty=\{1,-1\}^{\mathbf{N}}$. Recall that Rademacher type p means that for some C and all n

$$\left\|\sum_{j=1}^n x_j r_j\right\|_{L^2([0,1],E)}\leq C\left(\sum_{j=1}^n \|x_j\|_E^p\right)^{1/p}$$

for all $x_j \in E$ and where the r_j are the Rademacher functions. After Kahane it is well known that 2 may be replaced by any $1 \leq r < \infty$. Hence $\mathbf{Z}_2^\infty$-Fourier type implies the same Rademacher type. In [M2] this is noted and commented, whereas [P2] remarks on the more general probabilistic connection.

In the other direction Bourgain [B1], [B2] has results that each Rademacher type result forces a weaker Fourier type. Explicitly, in case $p > 1$, Rademacher type p implies $\mathbf{T}$-Fourier type p^* for some p^* in $]1, p[$ only depending on p. Unfortunately it is not yet possible to fully characterize Fourier type for any single infinite group.

As the original research was completed and printed in [A1] I was informed of the concurrent but independent research of García-Cuerva, Kazarian and Torrea [GKT], which was kindly communicated to me in manuscript form. There is of course some overlap but the two presentations follow different aspects of a beginning theory. The present work is above all concentrated on the algebraic dependence on the group G. In addition, the goal has been to establish the results in a way to decide how the bound $M(G, E, p)$ depends on the group. The paper [GKT] on the other hand focuses on the geometric restrictions on E imposed by Fourier type. There a thorough investigation concerning the exact exponent p yielding Fourier type is presented. Most importantly they treat Orlicz spaces.

During the revision of this work Fernando Cobos pointed out a further closely related paper by König [K].

The purpose of Section 1 is to present the basic results that connect the different bounds $M(G, E, p)$ as the group changes. A short selection of concrete calculations is included. The proofs of the factorization theorems are partitioned into Sections 2 and 3. The first of these relies heavily on measure theory. Finally, Section 4 deals with questions on how Fourier type may be established and to what extent a smaller number of test groups would suffice.

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1. Specific bounds

In this section some known bounds on the quantity $M(G, E, p)$ are collected for natural groups and simple Banach spaces. In addition, we will prove some general bounds on $M(G, E, p)$ and recall a few relations between different bounds from [A2].

To begin with let us record an immediate observation.

Proposition. *For each subspace $F \subseteq E$ the bound $M(G, F, p) \leq M(G, E, p)$ holds. In particular $M(G, E, p) \geq M(G, \mathbf{C}, p)$.*

Proof. The supremum involved is performed over a smaller space when determining $M(G,F,p)$.

In order to efficiently study the Hausdorff–Young inequality we will need to deduce its validity when the group is enlarged. Specifically we want to start out with small groups and extend them. As in all areas of group theory there are factorization theorems of different kinds. For the present purpose we need to relate the Hausdorff–Young inequality on the whole group to the inequalities on a subgroup and its factor group. To be specific:

Theorem 1.1. *For each closed subgroup H of an LCA group G*

$$M(G,E,p) \le M(G/H,E,p)M(H,E,p).$$

The path for establishing the theorem is somewhat cumbersome and is therefore deferred to the next section. The model of proof is the same that Weil followed in his demonstration of the Plancherel theorem on LCA groups: a property on H and G/H is often transportable to G. We will use a relativized Fourier transform (cf. [Re]).

Ideally, one would also like to bound the quantity $M(G,E,p)$ from below in terms of the subgroups. This turns out to be tricky and apparently calls for a number of cases. For the time being we will content ourselves with two results, each of which will be referred to when mentioning factorization in their respective contexts.

Proposition 1.2. *In case $G \simeq G_1 \oplus G_2$ there are bounds*

$$M(G_1,E,p)M(G_2,\mathbf{C},p) \le M(G,E,p) \le M(G_1,E,p)M(G_2,E,p).$$

Proof. The rightmost inequality follows from the theorem above. A much simpler proof is readily obtained through the factorization of the characters on $G_1 \oplus G_2$. A careful proof is written out in [GKT]. For the leftmost inequality we make a simple construction. Given $g \in C_c(G_1,E)$ the complex valued function h in $C_c(G_2)$ is chosen arbitrarily. Our test function will be $f\colon G \to E$, $f(t_1,t_2)=g(t_1)h(t_2)$. Then one easily finds $\mathcal{F}_G f(\varrho_1,\varrho_2)=\mathcal{F}_{G_1}g(\varrho_1)\mathcal{F}_{G_2}h(\varrho_2)$. Consequently

$$\|f\|_p = \|g\|_p\|h\|_p \quad \text{and} \quad \|\mathcal{F}_G f\|_{p'} = \|\mathcal{F}_{G_1}g\|_{p'}\|\mathcal{F}_{G_2}h\|_{p'},$$

from which

$$\frac{\|\mathcal{F}_{G_1}g\|_{p'}}{\|g\|_p} = \frac{\|\mathcal{F}_G f\|_{p'}}{\|f\|_p} \Bigg/ \frac{\|\mathcal{F}_{G_2}h\|_{p'}}{\|h\|_p}$$

follows. The freedom in the choice of h allows us to deduce the leftmost inequality.

Let us recall a version of the most important direct sum for topological groups.

Decomposition of LCA groups. *Let G be an LCA group. There exist an integer $n\geq 0$ and an LCA group G_1 having an open and compact subgroup such that $G\simeq \mathbf{R}^n\oplus G_1$.*

This is a standard result in the theory and may be found in e.g. [Re]. The most important use of the lower estimate above stems from the known numbers $M(\mathbf{R}^n,\mathbf{C},p)=B_p^n$ and $M(G,\mathbf{C},p)=1$ for each group G with a compact and open subgroup (cf. the introduction).

Proposition 1.3. *When H is an open subgroup of an LCA group G the following bound obtains*:

$$M(H,E,p)\leq M(G,E,p).$$

Proposition 1.3′. *For compact subgroups H the following holds*:

$$M(G/H,E,p)\leq M(G,E,p).$$

As the best proof of these uses an integral representation developed for the proof of Theorem 1.1 we will return to them in due time. Observe that Proposition 1.3 is false for general closed subgroups: $M(\mathbf{Z},\mathbf{C},p)=1$ and $M(\mathbf{R}/\mathbf{Z},\mathbf{C},p)=1$ exceed $M(\mathbf{R},\mathbf{C},p)=B_p$.

A very useful concept throughout functional analysis is duality. It turns out in the present context that the duality of Pontryagin and the one in Banach space theory fit together perfectly.

Proposition 1.4. *The bound in the Hausdorff–Young inequality is preserved under duality*:

$$M(G,E,p)=M(\widehat{G},E',p).$$

Proof. We begin by proving that $M(G,E',p)\leq M(\widehat{G},E,p)$. To that end let $f\colon G\to E'$ be simple: $f=\sum_{k=1}^{n} f_k\chi_{A_k}$, $\sum|A_k|<\infty$. It follows that

$$\begin{aligned}\|\hat f\|_{L^{p'}(\widehat{G},E')}&=\sup\left\{\int_{\widehat{G}} a(\sigma)\|\hat f(\sigma)\|\,d\sigma \mid a\in C_c(\widehat{G}),\ \|a\|_p=1\right\}\\&=\sup_a\int_{\widehat{G}}\Big[\sup_{h(\sigma)}\operatorname{Re}\langle \hat f(\sigma),h(\sigma)\rangle\Big]\,d\sigma,\end{aligned}$$

where $h(\sigma)\in E$, with $\|h(\sigma)\|\leq a(\sigma)$. Fortunately

$$\langle \hat f(\sigma),h(\sigma)\rangle=\sum_k\int_{A_k}\langle f_k,h(\sigma)\rangle\overline{\sigma(x)}\,dx=\sum_k\langle f_k,h(\sigma)\rangle\int_{A_k}\overline{\sigma(x)}\,dx,$$

whence the measurability of $\sigma \mapsto \int_{A_k} \overline{\sigma(x)}\, dx$ guarantees

$$\|\hat{f}\|_{L^{p'}(\widehat{G},E')} = \sup\left\{ \mathrm{Re} \int_{\widehat{G}} \langle \hat{f}(\sigma), h(\sigma)\rangle \, d\sigma \mid h \in C_c(\widehat{G}, E),\ \|h\|_{p,E} = 1 \right\}.$$

However

$$\int_{\widehat{G}} \langle \hat{f}(\sigma), h(\sigma)\rangle \, d\sigma = \int_G \langle f(x), \hat{h}(-x)\rangle \, dx,$$

whence

$$\begin{aligned} \|\hat{f}\|_{L^{p'}(\widehat{G},E')} &\le \sup_h \int_G \|f(x)\|_{E'} \|\hat{h}(-x)\|_E \, dx \le \sup_h \|f\|_{p,E'} \|\hat{h}\|_{p',E} \\ &\le M(\widehat{G}, E, p) \sup_h \|f\|_{p,E'} \|h\|_{p,E} = M(\widehat{G}, E, p) \|f\|_{p,E'}. \end{aligned}$$

Thus we conclude $M(G, E', p) \le M(\widehat{G}, E, p)$. Applying this a second time yields

$$M(\widehat{G}, E'', p) \le M(G, E', p) \le M(\widehat{G}, E, p),$$

which upon embedding into the bidual proves the equality

$$M(\widehat{G}, E, p) = M(G, E', p) = M(\widehat{G}, E'', p).$$

The sought result follows when we apply this equality to the group $\widehat{G}$ instead of G.

Remark. The assumption of the Radon–Nikodym property as done in [M2] is clearly superfluous.

A key observation emerges from this and manifests that finite dimensional subspaces carry all information.

Corollary. $M(G, E, p) = M(G, E'', p)$.

Proof. This follows from the preceding proof or simply by two applications of the proposition.

The beginning of a theory for transference of the Hausdorff–Young inequality was attempted in the earlier paper [A2]. As we will see below, the results therein are sufficient to equate **R**-Fourier and **Z**-Fourier type, thus linking [CS] and [M2].

A closer examination of the proof of [A2, Theorem 2] reveals the first line in the next result.

Proposition 1.5. ([A2, Theorems 2, 3, 5, 6])
(1) $M(\mathbf{T},E,p)\leq\liminf_{n\to\infty} M(\mathbf{Z}_n,E,p)$.
(2) $M(\mathbf{Z}_k,E,p)\leq B_p^{-1}M(\mathbf{T},E,p)$ *for all* $k\geq 2$.
(3) $M(\mathbf{R}^n,E,p)\leq M(H,E,p)\leq B_p^{-n}M(\mathbf{R}^n,E,p)$, $H=\mathbf{Z}^n$ *or* $\mathbf{T}^n$.

The third line is simply the straightforward generalization of the corresponding one-dimensional results proved in the earlier paper. The second statement will be refined in Proposition 1.11 below.

In the vein of [M2] and the introduction, the most important conclusion is the following result.

Theorem 1.6.
(1) *The notions of H-Fourier type p for $H=\mathbf{R}$, $\mathbf{T}$ and $\mathbf{Z}$ coincide.*
(2) *Every Banach space satisfies*

$$B_p \sup_n M(\mathbf{Z}_n,E,p)\leq M(H,E,p)\leq \liminf_{n\to\infty} M(\mathbf{Z}_n,E,p),$$

where $H=\mathbf{T}$ or $\mathbf{Z}$.
Hence a necessary and sufficient condition for $\mathbf{Z}$-Fourier type p is that the quantity

$$\Delta_p(E)=\liminf_{k\to\infty} M(\mathbf{Z}_k,E,p)$$

be finite.

Proof. The three type notions coincide by the third inequality in the preceding proposition. For the case $H=\mathbf{T}$ the inequality (2) is just the first and second statements of the proposition. Using duality the case $H=\mathbf{Z}$ appears. This latter case quantifies the equivalence between (periodic) $\mathbf{Z}$-Fourier type and the condition in the statement.

Remark. For definiteness let us record the quantitative behavior that can be extracted from the theorem:

$$\begin{aligned}
M(\mathbf{Z}_k,E,p)&\leq B_p^{-1}\Delta_p(E), \quad k\geq 2,\\
B_p\Delta_p(E)\leq M(\mathbf{Z},E,p)&\leq \Delta_p(E),\\
B_p\Delta_p(E)\leq M(\mathbf{T},E,p)&\leq \Delta_p(E),\\
B_p^2\Delta_p(E)\leq M(\mathbf{R},E,p)&\leq \Delta_p(E).
\end{aligned}$$

Let us next check that the dimension of a torus in no way affects its behaviour under the Hausdorff–Young inequality. This result will play an instrumental part in many of the results whose statements deal with transference procedures.

Theorem 1.7. *$M(\mathbf{Z},E,p)=M(\mathbf{Z}^k,E,p)$ for every $k\in\mathbf{Z}_+$.*

Proof. Due to $M(\mathbf{Z}^{k-1},\mathbf{C},p)=1$, it is obvious that the left-hand side is less than or equal to the right-hand side by an application of Proposition 1.2 with $G_1=\mathbf{Z}$ and $G_2=\mathbf{Z}^{k-1}$. In the other direction we consider a simple function $f:\mathbf{Z}^k\to E$. The key ingredient to make this function into a similarly behaving one on $\mathbf{Z}$ is

$$P_m:\mathbf{T}\longrightarrow\mathbf{T}^k,\quad \theta\longmapsto(\theta,m\theta,\dots,m^{k-1}\theta),\quad m\in\mathbf{Z}_+.$$

This function winds the "thread" $\mathbf{T}$ around the "plate" $\mathbf{T}^k$. The given function f gives rise to its Fourier transform $\hat f:\mathbf{T}^k\to E$. We may from it introduce a new function $\tilde f_m:\mathbf{Z}\to E$ through the relation $\mathcal{F}_{\mathbf{Z}}\tilde f_m=\hat f\circ P_m$. Since the sum $\hat f(\underline{\theta})=\sum_{\underline{n}\in\mathbf{Z}^k} f(\underline{n})e^{-i\underline{n}\cdot\underline{\theta}}$ has finitely many non-zero terms, the form of the embedding P_m shows that $\tilde f_m$ is a simple function. Furthermore the frequencies present in $\hat f\circ P_m$ are expressed as polynomials in the parameter m with the coefficients determined by $\underline{n}\in\mathbf{Z}^k$. Hence there is an integer $m_0\ge1$ such that $\tilde f_m$ assumes exactly the same values as f once $m\ge m_0$. In particular $\|f\|_{\mathbf{Z}^k,p}=\|\tilde f_m\|_{\mathbf{Z},p}$. Since the orbits $P_m(\mathbf{T})$ become dense in $\mathbf{T}^k$, in the sense that each rectangle $\mathrm{I}=\mathrm{I}_1\times\mathrm{I}_2\times\dots\times\mathrm{I}_k$ satisfies

$$\int_{\mathbf{T}}\chi_{\mathrm{I}}\circ P_m(\theta)\,\frac{d\theta}{2\pi}=\int_{\mathbf{T}}\prod_1^k\chi_{\mathrm{I}_j}(m^j\theta)\,\frac{d\theta}{2\pi}\to\prod_1^k\int_{\mathbf{T}}\chi_{\mathrm{I}_j}(\theta)\,\frac{d\theta}{2\pi}=|\mathrm{I}|,\quad\text{as } m\to\infty,$$

we recognize that continuous $g:\mathbf{T}^k\to\mathbf{C}$ satisfy

$$\int_{\mathbf{T}}g\circ P_m\,\frac{d\theta}{2\pi}\to\int_{\mathbf{T}^k}g\,\frac{d\underline{\theta}}{(2\pi)^k},\quad\text{as } m\to\infty.$$

In the present situation that fact yields

$$\|\hat f\|_{\mathbf{T}^k,p'}=\lim_{m\to\infty}\|\hat f\circ P_m\|_{\mathbf{T},p'}\le\liminf_{m\to\infty}M(\mathbf{Z},E,p)\|\tilde f_m\|_{\mathbf{Z},p}=M(\mathbf{Z},E,p)\|f\|_{\mathbf{Z}^k,p}.$$

The usual density arguments extend the inequality to all functions. From this follows $M(\mathbf{Z}^k,E,p)\le M(\mathbf{Z},E,p)$. Hence equality must hold.

By duality we get the best formulation.

Corollary. *$M(\mathbf{T},E,p)=M(\mathbf{T}^k,E,p)$, $k\ge1$.*

Corollary. *$M(\mathbf{T},E,p)=M(\mathbf{T}^\infty,E,p)$ and $M(\mathbf{Z},E,p)=M(\mathbf{Z}^\infty,E,p)$. Here the infinity is interpreted as the direct sum on $\mathbf{Z}$ and the direct product on $\mathbf{T}$.*

Proof. The dual of the direct product is the direct sum of the duals, so it is enough to study $\mathbf{Z}^\infty$. Since the direct sum with infinitely many terms is the limit of direct sums with a finite number of terms, and since the bounds are computed by means of simple functions, which for all purposes can be viewed on some finite product, the statement follows.

Corollary. *$M(F,E,p)=M(\mathbf{Z},E,p)$ for each free non-trivial Abelian group F with the discrete topology.*

Proof. When $f: F\to E$ is simple the set $\{x\in F \mid f(x)\neq 0\}$ may be viewed as a subset of some $\mathbf{Z}^k$. A density argument concludes.

Corollary. *$M(\mathbf{R}^n,E,p)\leq M(H,E,p)\leq B_p^{-n}M(\mathbf{R}^n,E,p)$, $H=\mathbf{Z}$ or $\mathbf{T}$.*

Proof. This follows from Proposition 1.5(3) and Theorem 1.7.

Remark. The right-hand inequality is optimal when $E=\mathbf{C}$.

In Section 3 we will have need of a result to switch between $\mathbf{T}^n$ and $\mathbf{Z}^m$. The critical point is that this must be done in a uniform manner that does not change with the dimensions.

Theorem 1.8. *$B_pM(\mathbf{T}^n,E,p)\leq M(\mathbf{Z}^m,E,p)\leq B_p^{-1}M(\mathbf{T}^n,E,p)$, $m,n\geq 1$.*

Proof. This is by now a straightforward computation. In order of appearance we use Theorems 1.7, 1.6, 1.6 and 1.7 again:

$$\begin{aligned} M(\mathbf{T}^n,E,p) &= M(\mathbf{T},E,p)\leq \liminf_{n\to\infty} M(\mathbf{Z}_n,E,p) \\ &\leq B_p^{-1}M(\mathbf{Z},E,p)=B_p^{-1}M(\mathbf{Z}^m,E,p). \end{aligned}$$

The other half is similar or for that matter follows by duality.

A striking use of Theorem 1.7 is to compare the spaces E and E'. Recall that as a corollary of duality $M(\mathbf{R},E,p)=M(\mathbf{R},E',p)$ holds, since $\mathbf{R}$ is essentially selfdual (the standard normalization can be recovered). The important feature is that E *and* E' appear simultaneously. When we consider $\mathbf{T}^n$ the situation changes dramatically. With Proposition 1.5 we can only derive a relation that depends on the dimension. However, the last theorem offers help.

Theorem 1.9. *$B_pM(\mathbf{T}^n,E,p)\leq M(\mathbf{T}^m,E',p)\leq B_p^{-1}M(\mathbf{T}^n,E,p)$, $m,n\geq 1$.*

Proof. Applications of the Theorems 1.7 and 1.8 with duality in between provide the calculation

$$M(\mathbf{T}^n,E,p)=M(\mathbf{T},E,p)=M(\mathbf{Z},E',p)\leq B_p^{-1}M(\mathbf{T}^m,E',p).$$

This is the first half of the statement. For the second half a similar calculation with $M(\mathbf{T}^n,E',p)$ as starting point is performed analogously.

As a last use of Theorem 1.7 let us return to finite groups. The way to interpret Theorem 1.6 is that finite cyclic groups are similar to the circle group itself; they are also from the present viewpoint discrete circles. But how do products of cyclic groups behave? There is a result akin to the one-dimensional theorem.

Theorem 1.10. *Every Banach space satisfies for each $n \geq 1$*

$$B_p^n \sup M(\mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}, E, p) \leq M(H, E, p) \leq \liminf M(\mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}, E, p).$$

Here $H = \mathbf{T}$ or $\mathbf{Z}$ and the supremum is taken over integers $k_1, \ldots, k_n \geq 2$. For limit inferior the condition that all $k_j \to \infty$ is added.

Proof. By the first corollary to Theorem 1.7 we need to prove the inequalities with H replaced by $\mathbf{T}^n$. The right-hand inequality is derived exactly is in [A2, Theorem 2] with the exception that now one deals with step functions on $\mathbf{T}^n$ and approximation with discrete uniform measures on a torus.

The left-hand inequality is but little more laborious. Writing $W_t^{(n)}(\underline{\theta})$ for the n-dimensional heat kernel on $\mathbf{T}^n$, some simple use of the Poisson summation formula tells us that

$$W_t^{(n)}(\underline{\theta}) = \prod_{j=1}^{n} W_t^{(1)}(\theta_j) \quad \text{and} \quad \widehat{W}_t^{(n)}(\underline{m}) = \prod_{j=1}^{n} \widehat{W}_t^{(1)}(m_j).$$

This factorization makes it straightforward to derive the analog of [A2, Lemma 2]:

$$\left(\prod_1^n k_j\right) \sum_{\underline{l} \in \underline{k}\mathbf{Z}^n} |\widehat{\Psi}_t^{(n)}(\underline{l} + \underline{s})|^{p'} \to B_p^{np'}, \quad t \to 0$$

for each $\underline{s} \in \mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}$ viewed as a subset of $\mathbf{Z}^n$. In the formula above we used $\underline{k} = (k_1, \ldots, k_n)$, $\underline{k}\mathbf{Z}^n = \{(k_1 a_1, \ldots, k_n a_n) \,|\, a_j \in \mathbf{Z}\}$, and $\Psi_t^{(n)} = W_t^{(n)} / \|W_t^{(n)}\|_p$. The computation is now conducted in a manner similar to [A2, Theorem 5].

To end this part on structure theorems we will refine Proposition 1.5 and illustrate that a local Hausdorff–Young inequality on $\mathbf{T}$ deserves study in its own right.

Definition. The local Hausdorff–Young bound on $\mathbf{T}$ is given by

$$C_p(t) = \sup\{\|\hat{\phi}\|_{p'} / \|\phi\|_p \mid \phi \colon \mathbf{T} \to \mathbf{C} \text{ has support in an arc of length } \leq t\}.$$

From [A2] we note that $C_p(2\pi/d) = B_p$ for $p' = 2d$ an even integer. In addition $B_p \leq C_p(t) < 1$ for all $p' \leq 2d$, $t \leq 2\pi/d$. Building on this Sjölin [S] established $\lim_{t \to 0} C_p(t) = B_p$ for all p. For general groups a related notion appears at the end of this section.

Proposition 1.11. $M(\mathbf{Z}_m, E, p) \leq C_p(2\pi/m)^{-1} M(\mathbf{T}, E, p)$.

Remark. Since $B_p \leq C_p(t)$ this result is possibly better than the earlier one. However, the two statements coincide as soon as $p' = 2d \leq 2m$.

Proof. Consider an arbitrary $f: \mathbf{Z}_m \to E$. Viewing $\mathbf{T}$ as $[-\pi, \pi]$ we choose any continuous $\phi: \mathbf{T} \to \mathbf{C}$ with support in $[-\pi/m, \pi/m]$. The basis of the proof is the identity

$$\begin{aligned}\|\mathcal{F}_{\mathbf{Z}_m} f\|_{p'}^{p'} \|\hat{\phi}\|_{p'}^{p'} &= \left(\sum_{k=0}^{m-1} \|\hat{f}(k)\|_E^{p'}\right)\left(\sum_{j=-\infty}^{\infty} \sum_{l=0}^{m-1} |\hat{\phi}(jm+l)|^{p'}\right) \\ &= \sum_{l=0}^{m-1} \left(\sum_{k=0}^{m-1} \|\hat{f}(k+l)\|_E^{p'} \sum_{j=-\infty}^{\infty} |\hat{\phi}(jm+k)|^{p'}\right).\end{aligned}$$

Define first $f_l: \mathbf{Z}_m \to E$, $f_l(j) = e^{-2\pi i l j/m} f(j)$, from which follows $\hat{f}_l(k) = \hat{f}(k+l)$ and $\|f_l\|_p = \|f\|_p$, $\|\hat{f}_l\|_{p'} = \|\hat{f}\|_{p'}$ for all l. Consider the mapping

$$\mathrm{P}: L(\mathbf{Z}_m) \longrightarrow C(\mathbf{T}), \quad \mathrm{P}h(\theta) = \sum_{k=0}^{m-1} h(k)\phi\left(\theta - \frac{2\pi k}{m}\right).$$

The assumption on the support guarantees $\|\mathrm{P}f_l\|_{p,\mathbf{T}} = m^{1/p} \|f_l\|_p \|\phi\|_p$. Moreover

$$\widehat{\mathrm{P}f_l}(n) = \sum_{k=0}^{m-1} f_l(k) \int_{\mathbf{T}} \phi\left(\theta - \frac{2\pi k}{m}\right) e^{-in\theta} \frac{d\theta}{2\pi} = m\hat{\phi}(n) \mathcal{F}_{\mathbf{Z}_m} f_l(n).$$

There follows an expression for the sequence norm:

$$\|\widehat{\mathrm{P}f_l}\|_{p'} = m\left(\sum_{k=0}^{m-1} \|\mathcal{F}_{\mathbf{Z}_m} f(k+l)\|_E^{p'} \sum_{j=-\infty}^{\infty} |\hat{\phi}(jm+k)|^{p'}\right)^{1/p'}$$

and finally from this

$$\begin{aligned}\|\mathcal{F}_{\mathbf{Z}_m} f\|_{p'}^{p'} \|\hat{\phi}\|_{p'}^{p'} &= m^{-p'} \sum_{l=0}^{m-1} \|\widehat{\mathrm{P}f_l}\|_{p'}^{p'} \leq m^{-p'} M(\mathbf{T}, E, p)^{p'} \sum_{l=0}^{m-1} \|\mathrm{P}f_l\|_p^{p'} \\ &= m^{-p'} M(\mathbf{T}, E, p)^{p'} \sum_{l=0}^{m-1} m^{p'/p} \|f_l\|_p^{p'} \|\phi\|_p^{p'} \\ &= M(\mathbf{T}, E, p)^{p'} \|f\|_p^{p'} \|\phi\|_p^{p'}.\end{aligned}$$

The earlier identities yield

$$\frac{\|\mathcal{F}_{\mathbf{Z}_m} f\|_{p'}}{\|f\|_p} \frac{\|\hat{\phi}\|_{p'}}{\|\phi\|_p} \leq M(\mathbf{T}, E, p).$$

Taking supremum over all possible ϕ shows that

$$\frac{\|\mathcal{F}_{\mathbf{Z}_m} f\|_{p'}}{\|f\|_p} C_p\left(\frac{2\pi}{m}\right) \leq M(\mathbf{T}, E, p).$$

Since this holds for every $f\colon \mathbf{Z}_m \to E$ the proof is finished.

The next goal is results on specific spaces satisfying the Hausdorff–Young inequality. To that end we need a general form of Minkowski's inequality. Generality in the sense of σ-finiteness is not enough: it must be applicable to general regular measures. In striving for complete generality the obstacle of measurability appears. The difficulty is that Borel σ-algebras are not preserved under the formation of products. Explicitly, $\mathcal{B}(X)\otimes\mathcal{B}(Y)$ is contained in but is not necessarily equal to $\mathcal{B}(X\times Y)$. In the usual case of σ-finiteness, however, they do coincide. As I have not been able to find a proof in the literature of the inequality in this generality, I will supply such a proof here.

Proposition 1.12. (Minkowski's inequality) *Let A and B be regular measure spaces and $r\geq 1$. Consider a bounded, lower semicontinuous function $f\colon A\times B\to [0,\infty[$, such that $b\mapsto\int_A f(a,b)\,da$ is bounded. Then*

$$\left[\int_B\left(\int_A f(a,b)\,da\right)^r db\right]^{1/r} \leq \int_A\left[\int_B f(a,b)^r\,db\right]^{1/r} da.$$

Proof. Write $J(b)=\int_A f(a,b)\,da$. By [C, Propositions 7.4.4 and 7.6.4] J is lower semicontinuous. The semicontinuity allows change of order in $(*)$ below (see [C, Exc. 7.6.2]), where Hölder's inequality is also used,

$$\begin{aligned}\|J\|_r^r &= \int_B\int_A J(b)^{r-1} f(a,b)\,da\,db \overset{(*)}{=} \int_A\left(\int_B J(b)^{r-1} f(a,b)\,db\right) da \\ &\leq \int_A\left(\int_B J(b)^r\,db\right)^{1-1/r}\left(\int_B f(a,b)^r\,db\right)^{1/r} da \\ &= \|J\|_r^{r-1}\int_A\left(\int_B f(a,b)^r\,db\right)^{1/r} da.\end{aligned}$$

This inequality is the claimed one as soon as $\|J\|_r<\infty$. To arrive at that fact we multiply J with the characteristic function χ_U, where $U\subseteq B$ is open with compact closure. Now $\chi_U J$ is still lower semicontinuous and $\|\chi_U J\|_r$ is finite. Thus the inequality above proves

$$\|\chi_U J\|_r \leq \int_A \left(\int_B f(a,b)^r \, db \right)^{1/r} da.$$

Finally we use regularity [C, Proposition 7.4.4] of the measure to remove the factor χ_U and hence arrive at the sought inequality.

Remark. There is an alternative assumption that more clearly exhibits the role of measurability. When one considers simple $\mathcal{B}(A)\otimes\mathcal{B}(B)$-measurable and integrable functions, the equality $(*)$ is still valid [C, Exc. 7.6.3] and $\|J\|_r$ may be recovered immediately. By approximation the Minkowski inequality can be continued to integrable and $\mathcal{B}(A)\otimes\mathcal{B}(B)$-measurable functions. The reason is that such functions have supports within rectangles with σ-finite sides.

Proposition 1.13. *Let (X,ν) be a non-zero regular measure space, G an LCA group, and E a Banach space. In case $1 \leq p \leq 2$ and $p \leq r \leq p'$ we have*

$$M(G, L^r(X,E), p) = M(G,E,p).$$

Proof. Let $F = L^r(X,E)$ and consider an arbitrary function $f \in C_c(G\times X, E)$. Notice that $\mathcal{F}_G[x \mapsto f(g,x)](\sigma) = [x \mapsto \mathcal{F}_G f_x(\sigma)]$. There are compact sets $K \subseteq X$ and $L \subseteq G$ such that $x \in K^c$ implies $\mathcal{F}_G f_x(\sigma) = 0$ respectively $t \in L^c$ implies $f_x(t) = 0$. The continuity and boundedness of f as well as of $\mathcal{F}_G f$ guarantees that we may apply Minkowski's inequality where indicated with an asterisk

$$\begin{aligned}
\|\mathcal{F}_G f\|^r_{L^{p'}(\widehat{G},F)} &= \left[\int_{\widehat{G}} \left[\int_X |\mathcal{F}_G f_x(\sigma)|^r \, dx \right]^{p'/r} d\sigma \right]^{r/p'} \\
&\overset{*}{\leq} \int_X \left[\int_{\widehat{G}} |\mathcal{F}_G f_x(\sigma)|^{p'} \, d\sigma \right]^{r/p'} dx \\
&\leq M(G,E,p)^r \int_X \left[\int_G |f_x(t)|^p \, dt \right]^{r/p} dx \\
&\overset{*}{\leq} M(G,E,p)^r \|f\|^r_{L^p(G,F)}.
\end{aligned}$$

The crucial point is that each of the inner integrals in front of an asterisk is over a compact set. Since the considered functions are dense in $L^p(G,F)$, the bound $M(G,F,p) \leq M(G,E,p)$ obtains. However, $L^p(G,F)$ contains a subspace isometric to E, whence the equality follows.

Corollary. *Each Hilbert space H satisfies*

$$M(G,H,p)=M(G,\mathbf{C},p),\quad 1\le p\le 2.$$

Proof. We have $H\simeq L^2(X)$ for a discrete space X.

For the next result recall the spaces $\mathcal{F}L^p(H,E)$ consisting of all strongly measurable functions $u\colon H\to E$ endowed with the norm $\|u\|_{\mathcal{F}L^p(H,E)}=\|\mathcal{F}_H u\|_{L^p(\widehat{H},E)}$.

Proposition 1.14. *Let G and H be LCA groups and E a Banach space. Under the assumptions $p\le r\le p'$, $1\le p\le 2$ an operator bound holds:*

$$M(G,\mathcal{F}L^r(H,E),p)=M(G,E,p).$$

Proof. Since $\mathcal{F}L^r(H,E)\simeq L^r(\widehat{H},E)$ isometrically the preceding proposition proves the statement.

These two propositions are valid in the range $p\le r\le p'$. If one tries to get beyond those tame exponents the behaviour drastically changes.

Examples. Here we distinguish the cases $(*)$ $1\le r\le p$ and $(**)$ $p'\le r$. In order to illustrate the mechanisms determining the order of growth of $M(\mathbf{T}^n,l^r,p)$, we need a local Hausdorff–Young inequality; in the setting of $\mathbf{T}$ it has been discussed by the author in the second part of [A2]. To be general we define $\bar{n}=\bar{n}(G)$ as the supremum of all n admitting a function $f\in C_c(G)$ such that n translates of it have disjoint supports. Next, $LM(G,\mathbf{C},p)$ is taken as the supremum of $\|\hat{f}\|_{p'}/\|f\|_p$ as $n\to\bar{n}$ of all possible admissible functions f. The basic examples are these:

(L1) $\bar{n}(G)=\operatorname{card} G$, $LM(G,\mathbf{C},p)=1$ for discrete groups G.

(L2) $\bar{n}(\mathbf{T}^n)=\infty$, $LM(\mathbf{T}^n,\mathbf{C},p)\ge B_p^n$.

(L3) $\bar{n}(\mathbf{R}^n)=\infty$, $LM(\mathbf{R}^n,\mathbf{C},p)=B_p^n$.

They originate from the usual Hausdorff–Young inequality and from [A2] (see also [S] for more on (L2)). Let, furthermore, the letter $d\le\infty$ denote the dimension of the Lebesgue space in question, whereas its companion $\bar{d}$ equals $\min(d,\bar{n}(G))$. Some basic calculations on the bound $M(G,E,p)$ are collected for reference:

(M1) $M(G,L^r(X,\nu),p)\le M(G,\mathbf{C},p)\|\nu\|^{1/r-1/p}\|\operatorname{Id}\|_{L^r(\nu)\to L^p(\nu)}$ in case $(*)$.

(M2) $LM(G,\mathbf{C},p)\bar{d}^{1/r-1/p}\le M(G,L^r(X,\nu),p)$ when $(*)$ holds.

(M3) $M(\mathbf{Z},l^r(X),p)=d^{1/r-1/p}$ under condition $(*)$.

(M4) $B_p d^{1/r-1/p}\le M(\mathbf{T},l^r(X),p)\le d^{1/r-1/p}$ in case $(*)$.

(M5) $M(\mathbf{T},l^r(X),p)=d^{1/p'-1/r}$, provided $(**)$ holds.

(M6) $B_p d^{1/p'-1/r}\le M(\mathbf{Z},l^r(X),p)\le d^{1/p'-1/r}$, given $(**)$.

(M7) $M(\mathbf{Z}_n,l^r(X),p)=\max\{1,k^{1/r-1/p},k^{1/p'-1/r}\}$ for $k=\min(n,d)$ and $r\ge 1$.

(M8) $M(\mathbf{R}^n,l^r(X),p)=B_p^n\max\{1,d^{1/r-1/p},d^{1/p'-1/r}\}$ for $r\ge 1$ arbitrary.

The reasoning behind the various claims will mostly be sketched. Recall Hardy's characterization of equality in the Hausdorff–Young inequality on $\mathbf{Z}$: only the point masses on $\mathbf{Z}$ yield equality. By analysing when Minkowski's inquality is an equality and applying this knowledge to the proof as conducted in Proposition 1.13, one quickly establishes a basic characterization.

Proposition. *The Hausdorff–Young inequality on $L^p(\mathbf{Z}, l^r_m)$ is an equality in the cases:*

(1) *for $p<r\le p'$ if and only if $f\colon \mathbf{Z}\to l^r_m$ is a point mass;*

(2) *for $r=p$ if and only if $f=(f_1, \dots, f_m)$, $f_k\colon \mathbf{Z}\to\mathbf{C}$, is such that each f_k is a point mass. The support of the coordinate functions f_k possibly depends on k.*

Remark. The counterparts on $\mathbf{T}$ are of course the exponential functions that dualize the special solutions above. After [L] we know that on $\mathbf{R}^n$ the corresponding functions are the Gaussian maximizers.

These special functions and their analogs yield the lower bound (M2) in the list of examples. The upper bound (M1) is contained in the inequality below. Using Minkowski's inequality we get

$$\begin{aligned}
\|\hat f\|_{p',r} &= \left[\int_{\widehat G}\left(\int_X |\hat f|^r\,d\nu\right)^{p'/r} d\mu\right]^{1/p'} \le \left[\int_X\left(\int_{\widehat G} |\hat f|^{p'}\,d\mu\right)^{r/p'} d\nu\right]^{1/r}\\
&\le M(G,\mathbf{C},p)\left[\int_X\left(\int_G |f|^p\,d\eta\right)^{r/p} d\nu\right]^{1/r}\\
&\le M(G,\mathbf{C},p)\|\nu\|^{1/r-1/p}\left[\int_G\int_X |f|^p\,d\nu\,d\eta\right]^{1/p}\\
&\le M(G,\mathbf{C},p)\|\nu\|^{1/r-1/p}\|\operatorname{Id}\|_{L^r(\nu)\to L^p(\nu)}\|f\|_{p,r}.
\end{aligned}$$

Together they prove (M3) and (M4). After dualization also (M5), (M6) and (M8). The remaining piece (M7) just needs a slight correction in case $n\le d$.

2. The factorization theorem

This section is devoted to a proof previously left unfinished. The main result mentioned earlier was Theorem 1.1. Before embarking on the proof itself it should be remarked that the labour would be very much lessened were we only to consider open and compact subgroups.

The basic representation in Theorem 2.1 is taken from [Re], [R], [Bo]. The key facts from the integration theory of locally compact spaces and regular measures are conveniently found in Cohn [C] and Reiter [Re]. Throughout this section we let

E denote a fixed Banach space and p a fixed exponent obeying $1<p\leq 2$. In order to enhance readability we at times use the multiplicative notation for residue classes, i.e., tH denotes the same as $t+H$.

Consider the subspace $K_H \subseteq C(G)$ consisting of all functions invariant with respect to H-translation. Each $f\in K_H$ may be identified with a unique function in $C(G/H)$. This identification will be denoted $\tilde{}$ (cf. the comments after the theorem). The point of deviation from [R], [Bo] is that the presentation by these authors involves a choice of representative for xH in G, in place of the present tilde. The price we pay is a slightly heavier notation.

Theorem 2.1. (Weil's formula [Re, 3.3.3i]) *Let G be a locally compact Abelian group and H a closed subgroup. There are then Haar measures μ_G, μ_H and $\mu_{G/H}$ such that for every $f\in C_c(G)$*

$$\int_G f(x)\,d\mu_G(x) = \int_{G/H}\left(\int_H f(x+h)\,d\mu_H(h)\right)^{\sim}(xH)\,d\mu_{G/H}(xH).$$

Proof. When $f\in C_c(G)$ the function $g\colon G\to\mathbf{C}$, $x\mapsto\int_H f(x+h)\,d\mu_H(h)$ is well defined and continuous. Obviously it is translation invariant for $x\in H$. Consequently the object $\int_{G/H}\tilde{g}(xH)\,d\mu_{G/H}(xH)$ is meaningful. However,

$$f\in C_c(G)\longmapsto \int_{G/H}\left(\int_H f(x+h)\,d\mu_H(x)\right)^{\sim}(xH)\,d\mu_{G/H}(xH)$$

constitutes a positive translation invariant functional on $C_c(G)$. Hence the claim follows.

In the following text two different operators $\tilde{}$ will be used. One will act on functions with arguments in G and the other on functions with arguments in $\widehat{G}$. As there is no risk of misinterpretation we will not distinguish the two with different symbols. In addition the very same notation will be used in the vector space valued case as well as for lower semicontinuous functions which are translation invariant over a subgroup. This latter use is implied in the following generalizations of Theorem 2.1.

Proposition. *The same representation is valid for compactly supported, continuous Banach space valued functions.*

Theorem 2.2. (Weil's formula [Re, 3.3.3iii]) *The representation in Theorem 2.1 holds also in case $f\colon G\to[0,\infty]$ is lower semicontinuous.*

The present integral representation is a necessary technical tool to factorize the group in a form suitable for the Hausdorff–Young inequality. The idea of the proof is to relativize the Fourier transform on G to the subgroup H and its cosets. Then Weil's formula can be used separately on H and G/H in order to establish the Hausdorff–Young inequality on G starting from its validity on H and G/H. The main reason that the following preparations turn out to be rather tedious is that no simplifying assumptions are made. Most notably we avoid any form of σ-finiteness. As a consequence we must first establish a few preliminary lemmas in order to be sure we may use a version of Minkowski's integral inequality.

Lemma 2.3. *The function* $(\sigma,t)\mapsto S_\sigma(t)=\int_H f(t+h)\overline{\sigma(t+h)}\,dh$, $\widehat{G}\times G\to E$ *is continuous for each* $f\in C_c(G,E)$.

Proof. We fix once and for all $U_0\subset G$ to be a neighbourhood of zero, such that $\overline{U}_0$ is compact. Since the group addition is continuous from $G\times G$ to G, the set $K_0=\operatorname{supp} f+\overline{U}_0$ is compact.

Consider next a pair $(\sigma_0,t_0)\in\widehat{G}\times G$. We want to prove continuity at that point. Pick $\varepsilon>0$ arbitrary and put

$$\delta=\varepsilon\Big/\left(1+|H\cap(-t_0+K_0)|+2\int_H\|f(t_0+h)\|_E\,dh\right),$$

where $|\cdot|$ denotes measure in H. That $\delta>0$ follows from $f\in C_c(G,E)$.

Precisely as in the complex valued case there is a symmetric neighbourhood $U\subseteq U_0$ of zero such that

$$x-y\in U \quad\Longrightarrow\quad \|f(x)-f(y)\|<\delta.$$

In addition we may assume that U also fulfills

$$x,y\in K_0,\ x-y\in U \quad\Longrightarrow\quad |\sigma_0(x)-\sigma_0(y)|<\delta,$$

since σ_0 is continuous. Finally we may choose an open set U^* in $\widehat{G}$ containing σ_0 such that $\sigma\in U^*$ implies $|\sigma(x)-\sigma_0(x)|<\delta$ for all $x\in K_0$.

With these choices any (σ,t) in $U^*\times(t_0+U)$ satisfies

$$\begin{aligned}|S_\sigma(t)-S_{\sigma_0}(t_0)|&\le\int_H\|f(t+h)-f(t_0+h)\|_E\,dh\\&\quad+\int_H\|f(t_0+h)\|_E(|\overline{\sigma(t+h)}-\overline{\sigma_0(t+h)}|+|\overline{\sigma_0(t+h)}-\overline{\sigma_0(t_0+h)}|)\,dh\\&\le\int_H\|f(t+h)-f(t_0+h)\|_E\,dh+2\delta\int_H\|f(t_0+h)\|_E\,dh.\end{aligned}$$

But $H\cap\operatorname{supp}[f(t+h)-f(t_0+h)]\subseteq H\cap(-t_0-U+\operatorname{supp} f)\subseteq H\cap(-t_0+K_0)$, whence

$$|S_\sigma(t)-S_{\sigma_0}(t_0)|\le\delta|H\cap(-t_0+K_0)|+2\delta\int_H\|f(t_0+h)\|_E\,dh<\varepsilon.$$

Therefore the function is continuous at every point.

Corollary. (1) *$(\sigma, tH)\mapsto\widetilde{S}_\sigma(tH)$ is continuous from $\widehat{G}\times G/H$ to E.*
(2) *$(\sigma H^\perp, tH)\mapsto\|\widetilde{S}_\sigma(tH)\|_E^{\sim}(\sigma H^\perp)$ is continuous from $\widehat{G}/H^\perp\times G/H$ to $\mathbf{R}$.*

Proof. (1) is immediately clear in view of the identification $^\sim$. When $\eta\in H^\perp$ an easy calculation gives $\widetilde{S}_{\sigma\eta}(tH)=\overline{\eta(tH)}\widetilde{S}_\sigma(tH)$. The continuity from $\widehat{G}\times G/H$ to $[0,\infty[$ for $(\sigma, tH)\mapsto\|\widetilde{S}_\sigma(tH)\|_E$ is consequently carried over to the function in (2).

Lemma 2.4. *The function $\sigma H^\perp\mapsto(\int_{G/H}\|\widetilde{S}_\sigma(tH)\|_E^p\,dtH)^\sim$ is continuous and bounded.*

Proof. We fix one $\sigma_0\in\widehat{G}$ and consider an arbitrary $\varepsilon>0$. We may then choose an open neighbourhood U of σ_0 such that

$$\sigma\in U \quad\Longrightarrow\quad |\sigma(x)-\sigma_0(x)|<\varepsilon \quad \text{for all } x\in\operatorname{supp} f.$$

Considering $\sigma\in U$ we get that

$$\|\widetilde{S}_\sigma(tH)-\widetilde{S}_{\sigma_0}(tH)\|_E\le\int_H\|f(t+h)\|_E|\sigma(t+h)-\sigma_0(t+h)|\,dh\le\varepsilon\int_H\|f(t+h)\|_E\,dh,$$

whence

$$\int_{G/H}\|\widetilde{S}_\sigma(tH)-\widetilde{S}_{\sigma_0}(tH)\|_E^{\sim p}\,dtH\le\varepsilon^p\int_{G/H}\biggl(\int_H\|f(t+h)\|_E\,dh\biggr)^{\sim p}dtH=\varepsilon^pM,$$

where $M<\infty$, since $(\int_H\|f(t+h)\|_E\,dh)^{\sim p}\in C_c(G/H)$. We deduce

$$\begin{aligned}\biggl|\biggl(\int_{G/H}\|\widetilde{S}_\sigma(tH)\|_E^p\,dtH\biggr)^{1/p}-\biggl(\int_{G/H}\|\widetilde{S}_{\sigma_0}(tH)\|_E^p\,dtH\biggr)^{1/p}\biggr|&\\ \le\biggl(\int_{G/H}\bigl|\|\widetilde{S}_\sigma(tH)\|-\|\widetilde{S}_{\sigma_0}(tH)\|\bigr|^p\,dtH\biggr)^{1/p}&\\ \le\biggl(\int_{G/H}\|\widetilde{S}_\sigma(tH)-\widetilde{S}_{\sigma_0}(tH)\|^p\,dtH\biggr)^{1/p}&\le\varepsilon M^{1/p}.\end{aligned}$$

This says that $\sigma\mapsto\int_{G/H}\|\widetilde{S}_\sigma(tH)\|^p\,dtH$ is continuous. Recalling from the corollary that $\|\widetilde{S}_\sigma(tH)\|$ depends only on $\sigma H^\perp$ (and of course tH), the continuity in the statement follows. As for boundedness the compact support of the function f used in S_σ produces a compact subset $K=(\operatorname{supp} f)+H$ of G/H such that the integration in the function we study here really takes place over the fixed compact K at all times. This guarantees boundedness.

We are now in a position to prove that the Hausdorff–Young inequality can be studied on each factor alone.

Theorem 2.5. *For each closed subgroup H of an LCA group G*

$$M(G,E,p) \leq M(G/H,E,p)M(H,E,p).$$

Proof. Consider a function $f \in C_c(G,E)$. Weil's formula provides us with

$$\mathcal{F}_G f(\sigma) = \int_{G/H} \left(\int_H f(x+h)\overline{\sigma(x+h)}\, dh \right)^{\sim} (xH)\, dxH,$$

$$\|f\|_p = \left[\int_{G/H} \left(\int_H \|f(x+h)\|_E^p\, dh \right)^{\sim} (xH)\, dxH \right]^{1/p}.$$

For a successful proof we must prepare the choice of measures.

Claim. ([Re, 5.5.4]) *The Haar measures involved can be chosen in such a way that the Plancherel formula holds with standard normalization for each of the pairs* $(\mu_G, \mu_{\widehat{G}}), (\mu_H, \mu_{\widehat{G}/H^\perp})$ *and* $(\mu_{G/H}, \mu_{H^\perp})$. *Hence Theorems* 2.1 *and* 2.2 *hold for each setting* $(\mu_G, \mu_H, \mu_{G/H})$ *and* $(\mu_{\widehat{G}}, \mu_{H^\perp}, \mu_{\widehat{G}/H^\perp})$ *simultaneously.*

Accepting this fact we simply denote every integration by means of the respective variable. Employing the technical notation $S_\sigma(x) = \int_H f(x+h)\overline{\sigma(x+h)}\, dh$ one finds $\mathcal{F}_G f(\sigma) = \int_{G/H} \widetilde{S}_\sigma(xH)\, dxH$.

For $\eta \in H^\perp$ we then see that

$$S_{\sigma\eta}(x) = \overline{\eta(x)} \int_H f(x+h)\overline{\sigma(x+h)}\, dh = \overline{\eta(x)} S_\sigma(x) \tag{1}$$

from which follows $\int \widetilde{S}_{\sigma\eta}(xH)\, dxH = \int \overline{\widetilde{\eta}(xH)}\, \widetilde{S}_\sigma(xH)\, dxH$. This should be interpreted as a restriction identity

$$H^\perp \longrightarrow E, \quad \eta \longmapsto \mathcal{F}_G f(\sigma\eta) = \mathcal{F}_{G/H} \widetilde{S}_\sigma(\eta). \tag{$*$}$$

At this point we finally know all we need in order to derive the proper inequality. The mapping $\sigma \mapsto \|\mathcal{F}_G f(\sigma)\|_E^{p'}$ is a continuous function on $\widehat{G}$. Consequently Theorem 2.2 proves

$$\|\mathcal{F}_G f\|_{p'}^{p'} = \int_{\widehat{G}} \|\mathcal{F}_G f\|_E^{p'}\, d\sigma = \int_{\widehat{G}/H^\perp} \left(\int_{H^\perp} \|\mathcal{F}_G f(\sigma\eta)\|^{p'}\, d\eta \right)^{\sim} d\sigma H^\perp.$$

Using the formalism in $(*)$ we may apply the Hausdorff–Young inequality on G/H to find

$$\|\mathcal{F}_G f\|_{p'}^{p'} = \int_{\widehat{G}/H^\perp} \left(\int_{H^\perp} \|\mathcal{F}_{G/H} \widetilde{S}_\sigma(\eta)\|_E^{p'}\, d\eta \right)^{\sim} (\sigma H^\perp)\, d\sigma H^\perp$$

$$\leq M(G/H,E,p)^{p'} \int_{\widehat{G}/H^\perp} \left(\int_{G/H} \|\widetilde{S}_\sigma(xH)\|_E^p\, dxH \right)^{\sim p'/p} d\sigma H^\perp.$$

In the light of Lemma 2.4 Minkowski's inequality applied to $\|\widetilde{S}_\sigma(xH)\|^{\sim}(\sigma H)^p$ and $r=p'/p$ yields

$$\begin{aligned}\|\mathcal{F}_G f\|_{p'}^{p'} &\le M(G/H,E,p)^{p'}\left[\int_{G/H}\left(\int_{\widehat{G}/H^\perp}\|\widetilde{S}_\sigma(xH)\|_E^{\sim}(\sigma H^\perp)^{p'}\,d\sigma H^\perp\right)^{p/p'}dxH\right]^{p'/p}\\ &= M(G/H,E,p)^{p'}\\ &\quad\times\left[\int_{G/H}\left(\int_{\widehat{G}/H^\perp}\|\mathcal{F}_H(f(x+\cdot))(\sigma H^\perp)\|_E^{p'}\,d\sigma H^\perp\right)^{p/p'}dxH\right]^{p'/p}.\end{aligned}$$

The equality is due to the fact that $\eta\in H^\perp$ calls for $\|\widetilde{S}_{\sigma\eta}(xH)\|\overset{(1)}{=}\|\widetilde{S}_\sigma(xH)\|=\|\int_H f(x+h)\overline{\sigma(h)}\,dh\|=\|\mathcal{F}_H f(x+\cdot)(\sigma|_H)\|$. The inner integral allows another application of the Hausdorff–Young inequality, but on H this time,

$$\begin{aligned}\|\mathcal{F}_G f\|_{p'} &\le M(G/H,E,p)M(H,E,p)\left[\int_{G/H}\left(\int_H\|f(x+h)\|_E^p\,dh\right)^{\sim}(xH)\,dxH\right]^{1/p}\\ &= M(G/H,E,p)M(H,E,p)\|f\|_p.\end{aligned}$$

Since the compactly supported functions are dense in $L^p(G,E)$, the claimed inequality follows.

3. Factorization into a subgroup and its quotient

Section 1 also postponed two results on factorization from below, expressing a perfect dependence under additional circumstances. The second can be viewed as dual to the first.

Lemma 3.1. *Let H be an open subgroup of an LCA group G. Then $H^\perp$ is compact. It is furthermore possible to choose the Haar measures in a way such that $\mu_{H^\perp}(H^\perp)=1$, $\mu_G|_H=\mu_H$ and $\mu_{\widehat{H}}=\mu_{\widehat{G}/H^\perp}\phi^{-1}$, where $\phi\colon\widehat{H}\to\widehat{G}/H^\perp$ is the natural isomorphism.*

Proof. Based on $\widehat{H^\perp}\simeq\widehat{\widehat{G}}/(H^\perp)^\perp\simeq G/H$, compactness of $H^\perp$ is equivalent to openness of H. We may now normalize the Haar measures according to the claim in the last section and in addition take $\mu_{H^\perp}(H^\perp)=1$ and consequently $1=\|1\|_{2,H^\perp}^2=\|\hat{1}\|_{2,G/H}^2=\mu_{G/H}(\{0\})$.

Next let $U\subseteq H$ be open in H and hence also in G. In particular χ_U is lower semicontinuous and Theorem 2.2 yields

$$\mu_G(U)=\int_{G/H}\left(\int_H\chi_U(x+h)\,d\mu_H(h)\right)^{\sim}(xH)\,dxH=\int_H\chi_U(h)\,d\mu_H(h)=\mu_H(U).$$

Regularity of the respective measures applied to a Borel set $A \subseteq H$ proves

$$\mu_G(A) = \inf\{\mu_G(U) \mid A \subseteq U \subseteq H,\ U \text{ open}\} = \inf \mu_H(U) = \mu_H(A).$$

Hence the trace of μ_G over H coincides with μ_H.

To determine the image measure in the statement one considers $H^\perp$-translation invariant, continuous, and nonnegative functions g. Using $\mu_{H^\perp}(H^\perp)=1$ and Theorem 2.2 we find that

$$\int_{\widehat{G}} g \, d\mu_{\widehat{G}} = \int_{\widehat{G}/H^\perp} \tilde{g}(\sigma H^\perp) \, d\mu_{\widehat{G}/H^\perp}(\sigma H^\perp).$$

In particular we can apply this to $|\mathcal{F}_G f|^2$ when $f \in C_c(H) \subset C_c(G)$, to get

$$\begin{aligned} \int_{\widehat{H}} |\mathcal{F}_H f|^2 \, d\mu_{\widehat{H}} &= \int_H |f|^2 \, d\mu_H = \int_G |f|^2 \, d\mu_G = \int_{\widehat{G}} |\mathcal{F}_G f|^2 \, d\mu_{\widehat{G}} \\ &= \int_{\widehat{G}/H^\perp} |\mathcal{F}_G f|^{\sim 2} \, d\mu_{\widehat{G}/H^\perp} = \int_{\widehat{G}/H^\perp} |\mathcal{F}_H f|^2 \circ \phi^{-1} \, d\mu_{\widehat{G}/H^\perp}. \end{aligned}$$

The last equality is due to the trace formula which yields $\mathcal{F}_G f(\sigma) = \mathcal{F}_H f(\sigma|_H)$. Remembering that $\mathcal{F}_H C_c(H)$ is dense in $L^2(\widehat{H})$, one immediately deduces

$$\int_{\widehat{H}} \chi_A \, d\mu_{\widehat{H}} = \int_{\widehat{G}/H^\perp} \chi_A \circ \phi^{-1} \, d\mu_{\widehat{G}/H^\perp}.$$

This proves that the natural isomorphism $\phi: \widehat{H} \to \widehat{G}/H^\perp$ has the required property.

Lemma 3.2. *Let H be an open subgroup of an LCA group G and let $g \in C_c(H,E)$ be extended to be zero on $G \setminus H$. Denote the extension by f. Then*

$$\frac{\|\mathcal{F}_H g\|_{p',\widehat{H}}}{\|g\|_{p,H}} = \frac{\|\mathcal{F}_G f\|_{p',\widehat{G}}}{\|f\|_{p,G}}.$$

Proof. As remarked earlier, each of the quotients are independent of the chosen Haar measures, as long as Plancherel's formula is valid. Hence we may use the normalization from Lemma 3.1. The restriction $\mu_G|_H = \mu_H$ implies

$$\|f\|_{p,G} = \|g\|_{p,H}, \quad \mathcal{F}_G f(\sigma) = \mathcal{F}_H g(\sigma|_H), \quad \sigma \in \widehat{G}.$$

An application of the second identity together with Theorem 2.2 and Lemma 3.1 shows that

$$\begin{aligned} \|\mathcal{F}_G f\|_{p',\widehat{G}}^{p'} &= \int_{\widehat{G}/H^\perp} \|\mathcal{F}_H g(\sigma|_H)\|^{p'\sim}(\sigma H^\perp) \, d\mu_{\widehat{G}/H^\perp}(\sigma H^\perp) \\ &= \int_{\widehat{G}/H^\perp} \|\mathcal{F}_H g\|^{p'} \circ \phi^{-1} \, d\mu_{\widehat{G}/H^\perp} = \int_{\widehat{H}} \|\mathcal{F}_H g\|^{p'} \, d\mu_{\widehat{H}}. \end{aligned}$$

The two expressions relating norms combine to yield

$$\frac{\|\mathcal{F}_H g\|_{p',\widehat{H}}}{\|g\|_{p,H}} = \frac{\|\mathcal{F}_G f\|_{p',\widehat{G}}}{\|f\|_{p,G}},$$

which concludes the lemma.

Proposition 3.3. *When H is an open subgroup of an LCA group G the following bound obtains*:

$$M(H,E,p) \le M(G,E,p).$$

Proof. Let $g \in C_c(H,E)$. We extend it to $f \in C_c(G,E)$ through the prescription $f|_H = g$ and $f|_{H^c} = 0$. According to Lemma 3.2 and the Hausdorff–Young inequality

$$\frac{\|\mathcal{F}_H g\|_{p'}}{\|g\|_p} = \frac{\|\mathcal{F}_G f\|_{p'}}{\|f\|_p} \le M(G,E,p),$$

and the proposition follows.

Remark. The main obstacle for general closed subgroups is that the extension need not be continuous any longer.

Proposition 3.4. *If the subgroup H is compact in G, the quotient group obeys*

$$M(G/H,E,p) \le M(G,E,p).$$

Proof. Based on $\widehat{H} \simeq \widehat{G}/H^\perp$, one sees that H is compact if and only if $H^\perp$ is open in $\widehat{G}$. Hence duality can be applied and Proposition 3.3 for $H^\perp$ shows

$$M(G/H,E,p) = M(H^\perp,E',p) \le M(\widehat{G},E',p) = M(G,E,p).$$

4. Fourier type and group reductions

One way to describe the geometry of a particular Banach space is to prove that it satisfies the Hausdorff–Young inequality for some group. This makes for a far more specific instrument than the usual type theory. As was mentioned earlier, information of this kind has implications in interpolation theory. In the first section a few results point in the direction that it should be possible to test only a few groups in order to conclude the necessary properties. This section is devoted to the proofs that smaller classes of groups indeed determine the geometry of the space.

Definition.

(1) A Banach space E is said to be of *G-Fourier type* p in case $M(G,E,p)$ is finite.

(2) The space possesses *strict* G-Fourier type p when $M(G,E,p)\leq 1$.

(3) It is said to be of *universal* (strict) Fourier type p in case it is of (strict) G-Fourier type for each group G. If there is a common operator bound, i.e., $\sup_G M(G,E,p)<\infty$, we say that E has *uniform* Fourier type p.

Remark. It must be pointed out that this terminology differs from that of Milman [M2]. There Fourier type is the present strict type whereas weak Fourier type in that paper corresponds to our Fourier type. I have chosen a new phrasing, in the first place to keep the analogy with (Rademacher) type theory and secondly as an indication towards the fact that the strict property leads to a very easy theory once factorization is available. Presently we will see that universal and uniform Fourier type coincide as notions.

Proposition 4.1. *If $G\simeq\mathbf{R}^n\times G_1$ is the decomposition of G with n maximal, then*

$$M(\mathbf{T},E,p)M(G_1,E,p)\geq M(G,E,p)\geq B_p^n \max\{M(G_1,E,p),\ M(\mathbf{T},E,p)\}.$$

Hence a Banach space is of each G-Fourier type if and only if it enjoys the property for every group with a compact-open subgroup.

Proof. The result follows from the factorization theorem for direct sums in conjunction with the proposition that compares $M(\mathbf{R}^n,E,p)$ to $M(\mathbf{T},E,p)$.

Proposition 4.2. *In order to decide $M(G,E,p)<\infty$ it is sufficient to examine at most three groups: two compact and one discrete or vice versa. These three are determined by the group G alone.*

Proof. It is sufficient to check $\mathbf{T}$ and G_1. Denoting the open and compact subgroup of G_1 with H, factorization proves finiteness of $M(H,E,p)$ and $M(G_1/H,E,p)$ to be necessary and sufficient. The discrete group is G_1/H. As an alternative to $\mathbf{T}$ one may test $\mathbf{Z}$.

Theorem 4.3. *Let G be a discrete LCA group. Then*

$$M(G,E,p)=\sup M(G_0,E,p),$$

where supremum is taken over all finitely generated subgroups G_0 of G.

Proof. Start with an arbitrary simple function $f\colon G\to E$. The subgroup G_0 generated by the support of f is finitely generated. Lemma 3.2 says that

$$\frac{\|\mathcal{F}_{G_0}f\|_{p',\widehat{G}_0}}{\|f\|_{p,G_0}}=\frac{\|\mathcal{F}_G f\|_{p',\widehat{G}}}{\|f\|_{p,G}}.$$

Upon application of the appropriate suprema one finds

$$M(G,E,p)=\sup_{G_0}\sup_{f\text{ simple}}\frac{\|\mathcal{F}_{G_0}f\|_{p',\widehat{G}_0}}{\|f\|_{p,G_0}}=\sup_{G_0}M(G_0,E,p).$$

As a corollary we get a result on the geometry of the space. Let us denote by $\mathcal{D}$ the collection of discrete Abelian groups and by $\mathcal{K}$ the collection of compact Abelian groups.

Corollary 4.4. $\sup_{G\in\mathcal{D}}M(G,E,p)=\sup_{n,H\text{ finite}}M(\mathbf{Z}^n\times H,E,p)$.

Proof. It is known [G] that a finitely generated Abelian group is isomorphic to $\mathbf{Z}^n\times H$ for some natural number n and some finite Abelian group H. All these are included in $\mathcal{D}$ when discretely topologized. Hence the theorem proves the equality.

Corollary 4.5. $\sup_{G\in\mathcal{K}}M(G,E,p)=\sup_{n,H\text{ finite}}M(\mathbf{T}^n\times H,E,p)$.

Proof. This follows from the first corollary and duality.

Observation. Again we see how the Hausdorff–Young inequality is determined by finite dimensional conditions. The theorem and its corollaries express that it is always enough to check at most a countable number of well behaved groups to decide on even the most intricate group.

These corollaries will allow us a reasonable characterization of universal strict Fourier type. Let us introduce two properties of Banach spaces.

Definition.

(1) A Banach space E is said to have the *property* (P) (with exponent p) if each prime number k gives $M(\mathbf{Z}_k,E,p)=1$.

(2) It has *property* (F) when $\sup_{H\text{ finite}}M(H,E,p)<\infty$.

Fact. *The Banach space E has property* (P) *if and only if its dual E' has. Likewise with* (F).

Proof. By duality $M(H,E,p)=M(H,E',p)$ for each finite Abelian group.

Proposition 4.6. *The following are equivalent for a Banach space E.*

(1) *It has property* (P).

(2) *It satisfies* $\sup_{G\in\mathcal{D}}M(G,E,p)=1$.

(3) $\sup_{G\in\mathcal{K}}M(G,E,p)=1$ *holds.*

Proof. That the condition (2) implies (1) is obvious. In the other direction the condition on prime cyclic groups proves by factorization that $M(H,E,p)=1$ for each finite group H. In addition, Theorems 1.6(2) and 1.7 prove $M(\mathbf{Z}^n,E,p)=1$ for each n. Factorization then guarantees $M(\mathbf{Z}^n \times H, E, p)=1$ and Corollary 4.4 shows that (2) holds.

Using duality as always we see that (3) is equivalent to (2) with the change that E' should replace E in (2). By the preceding argument E satisfies (3) if and only if E' has property (P). By the fact above this is the case precisely when (1) holds for E itself.

Theorem 4.7. *A Banach space is of universal strict Fourier type p if and only if it enjoys property* (P) *with exponent p.*

Proof. We need only assume (P), the other direction being trivial. From Proposition 4.6 we conclude $M(G, E, p)=1$ for each compact or discrete group. In particular $M(\mathbf{R}^n, E, p) \le M(\mathbf{T}, E, p)=1$.

Consider a group G_1 with an open and compact subgroup H. Then G_1/H is discrete and factorization produces

$$M(G_1, E, p) \le M(H, E, p) M(G_1/H, E, p) = 1.$$

On grounds of Proposition 4.1 we are done.

One should wonder whether property (P) can be stated as a single condition. In fact it can be expressed with a single group. One such group will emerge presently.

The next step is of course to rid ourselves of 'strict' in the theorem and find the corresponding result. To begin with we will dispose of the universal notion altogether. In order to improve the bounds in the key lemma below let us note an improvement of Proposition 1.5(1). It is not strictly needed for the lemma but enables a final enhancement of Corollaries 4.4 and 4.5.

Lemma 4.8. *For each finite group H*

$$M(\mathbf{T}^n \times H, E, p) \le \liminf M(\mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n} \times H, E, p),$$

when all $k_j \to \infty$.

Proof. Consider a step function $f\colon \mathbf{T}^n \times H \to E$, i.e., $f = \sum_{l=1}^{L} \alpha_l \chi_{A_l}$ for $\alpha_l \in E$, $A_l = I_l \times \{h_l\}$, $h_l \in H$ and I_l is a rectangle in $\mathbf{T}^n$. For fixed h_l the corresponding I_l are disjoint.

We next construct $f_{\underline{k}}\colon \mathbf{Z}_{\underline{k}} \times H \to E$, where $\underline{k}=(k_1, \ldots, k_n)$, $\mathbf{Z}_{\underline{k}} = \mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}$, through the prescription

$$f_{\underline{k}}(\underline{j}, h) = f(e^{2\pi i (j_1/k_1)}, \ldots, e^{2\pi i (j_n/k_n)}, h).$$

It is somewhat tedious but perfectly straightforward (see [A2]) to derive, as all k_j tend to infinity, that

$$\|f_{\underline{k}}\|_{L^p(\mathbf{Z}_{\underline{k}}\times H,E)} \longrightarrow \|f\|_{L^p(\mathbf{T}^n\times H,E)},$$
$$\mathcal{F}_{\mathbf{Z}_{\underline{k}}\times H} f_{\underline{k}}(\underline{j},\eta) \longrightarrow \mathcal{F}_{\mathbf{T}^n\times H} f(\underline{j},\eta), \quad \eta\in\widehat{H}.$$

Proceeding analogously to [A2, Theorem 2] one finds

$$\|\mathcal{F}_{\mathbf{Z}_{\underline{k}}\times H} f\|_{L^{p'}(\mathbf{Z}^n\times H,E)} \le \liminf M(\mathbf{Z}_{\underline{k}}\times H,E,p)\|f\|_{L^p(\mathbf{T}^n\times H,E)},$$

which establishes the result.

Theorem 4.9. $\sup_{G\in\mathcal{D}} M(G,E,p) = \sup_{H \text{ finite}} M(H,E,p) = \sup_{G\in\mathcal{K}} M(G,E,p)$.

Proof. In view of Lemma 4.8 the rightmost equality follows from Corollary 4.5 (every finite group is trivially compact). Dualizing, the lemma also yields the left side thanks to Corollary 4.4.

Lemma 4.10. *The two notions of universal and uniform Fourier type coincide.*

Proof. Clearly universal Fourier type is a weaker notion than uniform Fourier type. Hence suppose the Banach space E enjoys universal Fourier type p. We want to find a uniform bound on $M(G,E,p)$.

Consider the group $\mathbf{Q}/\mathbf{Z}$ and write $\mathcal{Q}$ for the direct sum of a countable number of copies of $\mathbf{Q}/\mathbf{Z}$. We supply $\mathcal{Q}$ with the discrete topology. Let $M=M(\mathcal{Q},E,p)$. Since $\mathbf{Q}/\mathbf{Z}$ contains an isomorph of every finite cyclic group, the large group $\mathcal{Q}$ contains a copy of each finite group. Trivially each finite subgroup H is open in $\mathcal{Q}$. Hence

$$M(H,E,p)\le M$$

by Proposition 1.3. In particular by Theorems 1.6 and 1.7

$$M(\mathbf{Z}^n,E,p)\le M,\ \ M(\mathbf{T}^n,E,p)\le M, \quad n\ge 1.$$

Theorem 4.9 gives for each compact or discrete group

$$M(G,E,p)\le M.$$

Continuing as in the last part of Theorem 4.7, we find for each LCA group

$$M(G,E,p)\le M(\mathbf{T}^n,E,p)M(G_1/H,E,p)M(H,E,p)\le M^3.$$

Thus we have arrived at uniform Fourier type.

Theorem 4.11. *A Banach space E is of uniform Fourier type p if and only if it has property* (F).

Proof. This proof is entirely contained in the proof of the preceding lemma. The only difference is that the bound M is extracted from the (F)-property and the group $\mathcal{Q}$ does not enter the picture at all.

Remarks.

(1) In retrospect the proof of Lemma 4.10 shows that the properties (P) and (F) may be replaced with the corresponding bound for the single group $\mathcal{Q}$. But with $\mathcal{Q}$ being such a horrid group, nothing is really won through a restatement. As an alternative $\mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_4 \oplus ...$ will do once discretely topologized, but it is still a very unwieldy entity.

(2) The essential difference between (P) and (F), i.e., strict type and uniform type respectively, is the dimensionality. With strict type there is no problem in forming any kind of direct sum of groups, whereas without the constant 1 it is hard to get higher dimensional groups with a bounded operator norm. The factorization simply does not allow control of growth of the norm as the number of factors grows.

(3) The lesson Theorem 4.11 presents, is that a space not of every G-Fourier type has to have the bound $M(H, E, p)$ arbitrarily large for finite groups.

To understand better the geometric influence of the group on the Banach space there are two questions that point in directions not completely answerable by this paper.

Question 4.12. Is it always true that $M(G^n, E, p) \leq M(G, E, p)$ for infinite groups and all integers $n \geq 1$?

Question 4.13. Is there an infinite dimensional Banach space with the property $M(\mathbf{Z}_2^\infty, E, p) \neq M(\mathbf{Z}_3^\infty, E, p)$? Does Walsh–Fourier type p force Fourier type, i.e., does finiteness of $M(\mathbf{Z}_2^\infty, E, p)$ imply the same for $M(\mathbf{T}, E, p)$? These three groups seem to be the most obvious source where counterexamples may be found.

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