Ark. Mat., **37** (1999), 87–120

Criteria of solvability for multidimensional Riccati equations

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Abstract. We study the solvability problem for the multidimensional Riccati equation $-\Delta u=|\nabla u|^q+\omega$, where $q>1$ and ω is an arbitrary nonnegative function (or measure). We also discuss connections with the classical problem of the existence of positive solutions for the Schrödinger equation $-\Delta u-\omega u=0$ with nonnegative potential ω. We establish explicit criteria for the existence of global solutions on $\mathbf{R}^n$ in terms involving geometric (capacity) estimates or pointwise behavior of Riesz potentials, together with sharp pointwise estimates of solutions and their gradients. We also consider the corresponding nonlinear Dirichlet problem on a bounded domain, as well as more general equations of the type $-Lu=f(x,u,\nabla u)+\omega$ where $f(x,u,\nabla u)\asymp a(x)|\nabla u|^{q_1}+b(x)|u|^{q_2}$, and L is a uniformly elliptic operator.

1. Introduction

We study the solvability problem for the generalized Riccati equation

$$-\Delta u=|\nabla u|^q+\omega \tag{1.1}$$

on a domain $\Omega\subset\mathbf{R}^n$, $n\geq 3$, where $q>1$ and ω is a nonnegative function, or a measure $\omega\in M_+(\Omega)$. (Here and in the sequel $M_+(\Omega)$ denotes the class of locally finite positive Borel measures on Ω.) Our results hold, with obvious modifications, also for $n=1$ and $n=2$.

All solutions are understood in the usual weak sense, i.e., u is a solution to (1.1) if $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \nabla u\cdot\nabla\phi\,dx=\int_\Omega |\nabla u|^q\phi\,dx+\int_\Omega \phi\,d\omega \tag{1.2}$$

for all test functions $\phi\in C_0^\infty(\Omega)$. (In the special case when ω is a nonnegative locally integrable function we set $d\omega=\omega(x)\,dx$ in (1.2).)

(1) Partially supported by the NSF and University of Missouri Research Board grants.

Our main goal is to establish necessary and sufficient conditions (with a gap only in the best constants) for the existence of global solutions of (1.1) on $\mathbf{R}^n$, together with sharp pointwise estimates of solutions and their gradients, without any a priori assumptions on $\omega \geq 0$. We show that all weak solutions of (1.1) belong to a function space intrinsically associated with the equation. The characterizations of solvability are given explicitly in terms of the pointwise behavior of the corresponding Riesz potentials as well as in geometric (capacitary) terms.

Analogous criteria and estimates of solutions are obtained for more general semilinear equations of the type $-Lu = f(x, u, \nabla u)$ where $f(x, u, \nabla u) \asymp a(x)|\nabla u|^{q_1} + b(x)|u|^{q_2} + \omega$, and L is a second order uniformly elliptic differential operator. We also characterize the solvability of the corresponding Dirichlet problem on a bounded domain Ω in the case $q > 2$.

We observe that numerous results in the literature on the solvability of semilinear equations contain mostly sufficient conditions which usually are far from being necessary. Some interesting duality theorems for differential inequalities associated with (1.1) can be found in [B]. See also [AP], [KV] where sharp criteria for the existence of positive solutions are found in the case of equations without the gradient term.

It is worthwhile to note that (1.1) with $q = 2$ is intimately related to the problem of the existence of positive solutions for the Schrödinger equation

$$-\Delta v = \omega v, \quad v \geq 0, \tag{1.3}$$

which is easily seen via the substitution $v = e^u$. (The case $n = 1$ is discussed in [Ha] where the references to earlier work on one-dimensional Riccati equations are given.) The exponential substitution requires a change in the corresponding boundary values, e.g., the Dirichlet problem for the Riccati equation with $u = 0$ on $\partial\Omega$ corresponds to the inhomogeneous Dirichlet condition $v \equiv 1$ on $\partial\Omega$.

It is well known that just the existence of a positive solution (without specifying boundary values) is equivalent to the positivity of the Schrödinger operator $-\Delta - \omega$ on $L^2(\Omega)$ for relatively nice ω. (See [Ag], [CZ], [Si] where this equivalence is discussed for potentials ω in L^r_{loc}, $r > n$, or Kato classes.)

The problem of the positivity of the Schrödinger operator $-\Delta - \omega$ on $L^2(\mathbf{R}^n)$ for arbitrary $\omega \in M_+(\mathbf{R}^n)$ was solved in [M1] (see also [M2]) in capacitary terms. Other equivalent characterizations can be found in [S], [H], [MV], and the literature cited there. It follows directly from our results on Riccati equations that the class of all $\omega \in M_+(\mathbf{R}^n)$ such that (1.3) has a global positive solution is essentially the same (up to best constants). In contrast to previous work, the approach of the present paper yields not only the existence, but also new pointwise estimates for solutions of (1.3) and their gradients.

We define the Riesz potential $I_\alpha = (-\Delta)^{-\alpha/2}$ of order α, $0<\alpha<n$, on $\mathbf{R}^n$ by

$$I_\alpha f(x) = c(n,\alpha) \int_{\mathbf{R}^n} f(t)|x-t|^{\alpha-n}\,dt, \tag{1.4}$$

where $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\int_{|x|\geq 1} |x|^{\alpha-n}|f(x)|\,dx<\infty$. Here we have the constant

$$c(n,\alpha) = \pi^{-n/2}2^{-\alpha}\Gamma\big(\tfrac{1}{2}(n-\alpha)\big)\Gamma\big(\tfrac{1}{2}\alpha\big)^{-1}.$$

More generally, for any $\omega\in M_+(\mathbf{R}^n)$ and $f\in L^1_{\mathrm{loc}}(\omega)$ such that

$$\int_{|x|\geq 1} |x|^{\alpha-n}|f(x)|\,d\omega < \infty,$$

we set

$$I_\alpha(f\,d\omega)(x) = c(n,\alpha)\int_{\mathbf{R}^n} f(t)|x-t|^{\alpha-n}\,d\omega(t)$$

and $I_\alpha\omega = I_\alpha(\mathbf{1}\,d\omega)$. The Riesz capacity $\mathrm{Cap}_{\alpha,p}(E)$ of a measurable set $E\subset\mathbf{R}^n$ is defined by

$$\mathrm{Cap}_{\alpha,p}(E) = \inf\{\|f\|^p_{L^p(\mathbf{R}^n)} : I_\alpha f \geq \chi_E,\ f\in L^p_+(\mathbf{R}^n)\}. \tag{1.5}$$

This capacity is associated with the Sobolev space $L^{\alpha,p}(\mathbf{R}^n)$. For integer α and compact sets $E\subset\mathbf{R}^n$ it is well known that $\mathrm{Cap}_{\alpha,p}(E)\asymp\mathrm{cap}_{\alpha,p}(E)$, where $\mathrm{cap}_{\alpha,p}$ is defined by

$$\mathrm{cap}_{\alpha,p}(E) = \inf\bigg\{\int_{\mathbf{R}^n} |\nabla^\alpha h|^p\,dx : \chi_E\leq h\leq 1,\ h\in C_0^\infty(\mathbf{R}^n)\bigg\}. \tag{1.5′}$$

(See [AH], [M2].)

In what follows the capacity $\mathrm{cap}_{1,p}$ with $\alpha=1$ and $1/p+1/q=1$ will play an important role. As we will show, it is intrinsically associated with the equation (1.1). Note that in a parallel theory of the equation

$$-\Delta u = |u|^q + \omega, \tag{1.6}$$

a similar role is played by the capacity $\mathrm{cap}_{2,p}$ (see [AP], [KV]).

The following theorem established in [MV] will play a crucial role in the sequel. (A different proof together with applications to equations of type (1.6) is given in [KV].)

Theorem 1.1. [MV]. *Let $\omega \in M_+(\mathbf{R}^n)$. Let $1<p<\infty$, $1/p+1/q=1$, and $0<\alpha<n$. Then the following statements are equivalent.*

(i) *The inequality*

$$\omega(E) \le C_1 \operatorname{Cap}_{\alpha,p}(E) \tag{1.7}$$

holds for all compact sets $E\subset\mathbf{R}^n$, with a constant C_1 which depends only on p and n.

(ii) *The inequality*

$$\int_E (I_\alpha\omega)^q \, dx \le C_2 \operatorname{Cap}_{\alpha,p}(E) \tag{1.8}$$

holds for all compact sets $E\subset\mathbf{R}^n$, with a constant C_2 which depends only on q and n.

(iii) *The potential $I_\alpha\omega<\infty$ a.e. and*

$$I_\alpha(I_\alpha\omega)^q(x) \le C_3 I_\alpha\omega(x) \quad \text{a.e.} \tag{1.9}$$

Furthermore, the least constants C_1, $C_2^{1/q}$, and $C_3^{1/(q-1)}$ are equivalent, and the constants of equivalence depend only on q, α, and n.

The class of measures characterized by (1.9) turned out to be extremely useful in applications to nonlinear equations. In particular, one of the main results of this paper is the following criterion for the existence of (global) solutions to (1.1) on $\mathbf{R}^n$.

Theorem 1.2. *Let $1<q<\infty$, and let $\omega\in M_+(\mathbf{R}^n)$. Then there exist positive constants C_1, C_2, and C_3 which depend only on q and n such that the following statements hold.*

(i) *If (1.1) has a solution $u\in W^{1,q}_{\text{loc}}(\mathbf{R}^n)$, then $I_1\omega<\infty$ a.e. and*

$$I_1(I_1\omega)^q(x) \le C_1 I_1\omega(x) \quad \text{a.e.} \tag{1.10}$$

(ii) *Conversely, if (1.10) holds with C_2 in place of C_1, then (1.1) has a solution $u\in W^{1,q}_{\text{loc}}(\mathbf{R}^n)$ such that*

$$|\nabla u(x)| \le C_3 I_1\omega(x) \quad \text{a.e.} \tag{1.11}$$

Remark 1.1. Some partial results related to Theorem 1.2 in the case of compactly supported ω were announced without proof in [MV]. They were obtained by the first author, of this paper, using the equivalence of the capacitary inequalities (1.7) and (1.8), which was established by the second and third author in [MV]. Here we give a complete proof based on the inequality (1.9) with $\alpha=1$ which yields a criterion for the existence of any (weak) solution, and also leads to sharp pointwise estimates of solutions and their gradients for arbitrary nonnegative ω.

In Section 2 we will show (see Corollary 2.2) that the space X of $u\in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$ with finite seminorm

$$\|u\|_X = \sup\left\{ \left(\frac{\int_E |\nabla u|^q \, dx}{\mathrm{cap}_{1,p}(E)} \right)^{1/q} : \mathrm{cap}_{1,p}(E) > 0, \ E \text{ compact} \right\}$$

defines a natural function space associated with the nonlinear equation (1.1).

All (weak) solutions $u\in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$ of (1.1) satisfy the estimate

$$\|u\|_X \le p^{p-1}. \tag{1.12}$$

Moreover, we will see that if (1.1) has a solution $u\in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$, then necessarily

$$\omega(E) \le (p-1)^{p-1} \, \mathrm{cap}_{1,p}(E) \tag{1.13}$$

for all compact sets E.

Remark 1.2. It follows from (1.13) that $q=n/(n-1)$ is a critical exponent for the solvability of (1.1) on $\mathbf{R}^n$. If $1<q\le n/(n-1)$, then $\mathrm{cap}_{1,p}(E)=0$ for all $E\subset\mathbf{R}^n$ (see [AH], [M2]), and hence (1.1) has no global solutions on $\mathbf{R}^n$ provided $\omega\ne 0$.

In the case $n/(n-1)<q<\infty$, the following simple sufficient condition for the solvability of (1.1) can be derived in terms of weak L^r-spaces using the known estimate

$$\mathrm{cap}_{1,p}(E) \ge c(p,n)|E|^{1-p/n}, \quad E \text{ compact}.$$

It is immediate from Theorem 1.2, Theorem 1.1, and the preceding inequality that there exists a constant $C=C(p,n)$ such that (1.1) has a solution if ω is absolutely continuous with respect to Lebesgue measure and $\|\omega\|_{L^{n/p,\infty}}\le C$, where $1/p+1/q=1$.

The solution u whose existence is claimed in Theorem 1.2(ii) satisfies some additional sharp inequalities of Hadamard type (see e.g. [HK, Theorem 4.2]). These estimates are collected in the following theorem. (Note that by Remark 1.2 it suffices to consider the case $n/(n-1)<q<\infty$.)

Theorem 1.3. *Let $n/(n-1)<q<\infty$, $1/p+1/q=1$, and let $\omega\in M_+(\mathbf{R}^n)$.*

(i) *Suppose that* (1.10) *holds with a small enough constant $C=C(q,n)$ as in Theorem* 1.2(ii). *Then there exists a solution u of* (1.1) *such that the following statements hold.*

(a) *If* $1<q<2$, *then*

$$I_2\omega(x)\le u(x)\le CI_2\omega(x)<\infty \quad a.e., \tag{1.14}$$

where C *depends only on* q *and* n, *here* $I_2=(-\Delta)^{-1}$ *is the Newtonian potential of* ω.

(b) *If* $q=2$, *then*

$$u(x)\ge c(n,2)\int_{|t|<1}\frac{d\omega(t)}{|x-t|^{n-2}}+C\log(|x|+1), \tag{1.15}$$

where $C<0$ *depends only on* n.

(c) *If* $q>2$, *then*

$$u(x)\ge c(n,2)\int_{|t|<1}\frac{d\omega(t)}{|x-t|^{n-2}}+C(|x|+1)^{2-p}, \tag{1.16}$$

where $C<0$ *depends only on* n *and* q.

(ii) *For the average values of* $|u|$ *defined by*

$$M_R[u]=\frac{1}{|B_R(0)|}\int_{|t|<R}|u(t)|\,dt, \tag{1.17}$$

it follows that $M_R[u]=O(\phi_q(R))$, *as* $R\to+\infty$, *where*

$$\phi_q(R)=\begin{cases} R^{2-p} & \text{if } q\ne 2,\\ \log R & \text{if } q=2.\end{cases} \tag{1.18}$$

Remark 1.3. The estimates in Theorem 1.2 and Theorem 1.3 are sharp which is easily seen from the equation

$$-\Delta u=|\nabla u|^q+\frac{c}{|x|^p} \tag{1.19}$$

on $\mathbf{R}^n$, where $n/(n-1)<q<\infty$, $1/p+1/q=1$, and $c>0$.

Note that if c is small enough, i.e. $0<c<C(q,n)$, then (1.19) has a solution (not necessarily unique modulo constants)

$$u(x)=C\begin{cases} |x|^{2-p} & \text{if } q\ne 2,\\ \log|x| & \text{if } q=2,\end{cases} \tag{1.20}$$

where the constant $C=C(q,n,c)$ is positive if $p>2$ and negative if $p\le 2$. Obviously, $|\nabla u(x)|=C|x|^{1-p}$ and

$$\int_E |\nabla u|^q\,dx \le C\,\mathrm{Cap}_{1,p}(E)$$

for all compact sets E. (The latter estimate follows easily from Theorem 1.1(iii).)

Remark 1.4. In Section 2 we will obtain a criterion for the existence of *positive* solutions to (1.1), which is similar to Theorem 1.2, but requires additionally that $I_2\omega<\infty$ a.e., in this case there is a solution u such that $I_2\omega\le u\le CI_2\omega$ a.e.

Theorems 1.2 and 1.3 in the case $q=2$ yield new pointwise estimates for positive solutions of the Schrödinger equation (1.3) on $\mathbf{R}^n$ through the connection with the Riccati equation mentioned above (see Corollary 2.9).

In Section 3 we treat the corresponding superlinear Dirichlet problem. For an arbitrary inhomogeneous term $\omega\in M_+(\Omega)$, we prove an analogue of Theorem 1.2 in the case $q>2$. The case $1<q\le 2$ turned out to be more difficult, since our results hold true only with some additional a priori assumptions at the boundary, we do not consider it here.

In Section 4 we demonstrate how our approach works for more general equations of the type

$$-\Delta u = f(x,u,\nabla u)+\omega \tag{1.21}$$

where $f(x,u,\nabla u)\asymp a(x)|\nabla u|^{q_1}+b(x)|u|^{q_2}$. We observe that the solvability problem for nonlinearities of this type does not reduce to a mere combination of the corresponding characterizations for equations of the type (1.1) and (1.6), its solution requires a better understanding of the function spaces and classes of measures involved and is based on additional analytic work.

We establish both necessary and sufficient conditions for the solvability of (1.21) which coincide, in the same sense as above, for constant coefficients $a,b>0$ (Theorem 4.1). Note that these generalizations do not completely cover our previous results on Riccati equations. In the presence of the nonlinear term with $b>0$ the existence of any weak solution necessarily implies the existence of a positive solution. However, if $b=0$, then the restrictions on the inhomogeneous term at infinity are much weaker and hence give rise to nonpositive solutions discussed above.

Most of our results depend only on the estimates of the Green function and its gradient (see [A2], [GW], [HS], [W], [Z]), and hence can easily be carried over to equations with more general uniformly elliptic second order differential operators in place of the Laplacian, as well as certain higher order differential operators.

2. Existence of global solutions for Riccati equations

We first show that all (weak) solutions $u \in W^{1,q}_{\text{loc}}(\mathbf{R}^n)$ of the equation

$$-\Delta u = |\nabla u|^q + \omega \tag{2.1}$$

satisfy the inequality

$$\|u\|_X = \sup\left\{ \left(\frac{\int_E |\nabla u|^q \, dx}{\operatorname{cap}_{1,p}(E)} \right)^{1/q} : \operatorname{cap}_{1,p}(E) > 0, \ E \text{ compact} \right\} \le p^{p-1}, \tag{2.2}$$

where the capacity $\operatorname{cap}_{1,p}$ is defined by (1.5′). Moreover, if (2.1) has a solution $u \in W^{1,q}_{\text{loc}}(\mathbf{R}^n)$, then

$$\sup\left\{ \frac{\omega(E)}{\operatorname{cap}_{1,p}(E)} : \operatorname{cap}_{1,p}(E) > 0, \ E \text{ compact} \right\} \le (p-1)^{p-1}. \tag{2.3}$$

Lemma 2.1. *Let $1<q<\infty$ and $1/p+1/q=1$. Let $\omega \in M_+(\mathbf{R}^n)$. If (1.1) has a solution $u \in W^{1,q}_{\text{loc}}(\mathbf{R}^n)$ then*

$$\int_{\mathbf{R}^n} h^p |\nabla u|^q \, dx \le p^p \int_{\mathbf{R}^n} |\nabla h|^p \, dx \tag{2.4}$$

and

$$\int_{\mathbf{R}^n} h^p \, d\omega \le (p-1)^{p-1} \int_{\mathbf{R}^n} |\nabla h|^p \, dx \tag{2.5}$$

for all $h \in C_0^\infty(\mathbf{R}^n)$, $h \ge 0$.

Proof. Let u be a solution to (2.1) in a weak sense, i.e. (1.2) holds for all $\phi \in C_0^1(\mathbf{R}^n)$. Letting $\phi = h^p$ in (1.2) with $h \in C_0^\infty(\mathbf{R}^n)$, $h \ge 0$, we get

$$\int_{\mathbf{R}^n} \nabla u \cdot \nabla (h^p) \, dx = \int_{\mathbf{R}^n} |\nabla u|^q h^p \, dx + \int_{\mathbf{R}^n} h^p \, d\omega. \tag{2.6}$$

Since $\nabla(h^p) = p h^{p-1} \nabla h$, we have

$$p \int_{\mathbf{R}^n} (\nabla u \cdot \nabla h) h^{p-1} \, dx = \int_{\mathbf{R}^n} |\nabla u|^q h^p \, dx + \int_{\mathbf{R}^n} h^p \, d\omega. \tag{2.7}$$

By Hölder's inequality, together with the inequality $pab - a^q \le (p-1)^{p-1} b^p$ for $a, b > 0$, it follows that

$$\begin{aligned} \int_{\mathbf{R}^n} h^p \, d\omega &= p \int_{\mathbf{R}^n} (\nabla u \cdot \nabla h) h^{p-1} \, dx - \int_{\mathbf{R}^n} |\nabla u|^q h^p \, dx \\ &\le p \left(\int_{\mathbf{R}^n} |\nabla u|^q h^p \, dx \right)^{1/q} \|\nabla h\|_{L^p(\mathbf{R}^n)} - \int_{\mathbf{R}^n} |\nabla u|^q h^p \, dx \\ &\le (p-1)^{p-1} \|\nabla h\|^p_{L^p(\mathbf{R}^n)}, \end{aligned} \tag{2.8}$$

which proves (2.5).

On the other hand, from (2.7) we get

$$\int_{\mathbf{R}^n} |\nabla u|^q h^p\, dx \le p \int_{\mathbf{R}^n} (\nabla u \cdot \nabla h) h^{p-1}\, dx \le p \|\nabla h\|_{L^p(\mathbf{R}^n)} \left(\int_{\mathbf{R}^n} |\nabla u|^q h^p\, dx \right)^{1/q}.$$

Since the right-hand side of the preceding inequality is finite, we obtain

$$\int_{\mathbf{R}^n} |\nabla u|^q h^p\, dx \le p^p \|\nabla h\|^p_{L^p(\mathbf{R}^n)},$$

which proves (2.4). The proof of Lemma 2.1 is complete. □

Minimizing both sides of (2.4) and (2.5) over all $h \in C_0^\infty(\mathbf{R}^n)$ such that $h \ge \chi_E$, where E is a compact subset of $\mathbf{R}^n$, we obtain the following corollary.

Corollary 2.2. *Under the assumptions of Lemma* 2.1 *any solution u of* (2.1) *belonging to $W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$ satisfies* (2.2). *If there is a solution u to* (2.1), *then* (2.3) *holds.*

Let $\omega \in M_+(\mathbf{R}^n)$. The inequality

$$\int_{\mathbf{R}^n} |h|^p\, d\omega \le C \|\nabla h\|^p_{L^p(\mathbf{R}^n)} \tag{2.9}$$

for $h \in C_0^\infty(\mathbf{R}^n)$ is called the *trace inequality.* We will need the following characterization of (2.9) due to V. Maz'ya (see [M2], [AH]).

Lemma 2.3. *Let $1<p<\infty$ and let $\omega \in M_+(\mathbf{R}^n)$. Then* (2.9) *holds if and only if*

$$\omega(E) \le C \operatorname{cap}_{1,p}(E) \tag{2.10}$$

for all compact sets E. Moreover, the least constants in (2.9) *and* (2.10) *are equivalent, with the constants of equivalence depending only on p and n.*

The following theorem is the main result of this section.

Theorem 2.4. *Let $1<q<\infty$ and let $\omega \in M_+(\mathbf{R}^n)$.*

(i) *If* (2.1) *has a solution, then $I_1\omega<\infty$ a.e. and*

$$I_1(I_1\omega)^q(x) \le C_1 I_1\omega(x), \tag{2.11}$$

where C_1 is a constant which depends only on q and n.

(ii) *Conversely, there exists a constant* C_2 *which depends only on* q *and* n *such that if the inequality*

$$I_1(I_1\omega)^q(x) \le C_2 I_1\omega(x), \tag{2.12}$$

holds, then (2.1) *has a solution* $u \in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$.

(iii) *The solution* u *claimed in* (ii) *satisfies the following inequalities*:

$$|\nabla u(x)| \le C I_1\omega(x), \tag{2.13}$$

where C *depends only on* q *and* n;

if $1<q<2$, *then*

$$I_2\omega(x) \le u(x) \le C I_2\omega(x) \quad a.e., \tag{2.14}$$

where C *depends only on* q *and* n, *here* $I_2=(-\Delta)^{-1}$ *is the Newtonian potential of* ω;

if $q=2$, *then*

$$u(x) \ge c(n,2)\int_{|t|<1} \frac{d\omega(t)}{|x-t|^{n-2}} + C\log(|x|+1), \tag{2.15}$$

where $C<0$ *depends only on* n;

if $q>2$, *then*

$$u(x) \ge c(n,2)\int_{|t|<1} \frac{d\omega(t)}{|x-t|^{n-2}} + C(|x|+1)^{2-p}, \tag{2.16}$$

where $C<0$ *depends only on* n *and* q.

(iv) *Let*

$$M_R[u] = \frac{1}{|B_R(0)|}\int_{|t|<R} |u(t)|\,dt. \tag{2.17}$$

Then $M_R[u]=O(\phi_q(R))$, *as* $R\to+\infty$, *where*

$$\phi_q(R) = \begin{cases} R^{2-p} & \text{if } q\ne 2, \\ \log R & \text{if } q=2. \end{cases} \tag{2.18}$$

Remark 2.1. As was mentioned in the introduction, $q=n/(n-1)$ is a critical exponent for the solvability of (2.1). If $1<q\le n/(n-1)$, then $p\ge n$ and $\mathrm{Cap}_{1,p}(E)=0$

for all sets E. Hence by Corollary 2.3, $\omega=0$ and $\nabla u=0$ a.e., i.e. (2.1) has only a trivial constant solution for $1<q\le n/(n-1)$.

Remark 2.2. The existence of a solution in the space defined by (2.2) can be proved by using the contraction mapping principle and Lemma 2.3 together with Theorem 1.1. However, we prefer to construct a solution using iterations so that we have a good control of the solution and its gradient to prove sharp pointwise estimates.

Proof of Theorem 2.4. Statement (i) follows from Corollary 2.3 and Theorem 1.1. To prove (ii), we will need several lemmas.

Suppose that (2.12) holds with a small enough constant $C_2=C_2(q,n)$ which will be determined later. We may assume (see Remark 2.2) that $n/(n-1)<q<\infty$ and hence $1<p<n$. Then by Theorem 1.1

$$\omega(E)\le C\,\mathrm{Cap}_{1,p}(E) \tag{2.19}$$

for all compact sets E. In particular, for any ball $B=B(x,r)$ of radius r centered at $x\in\mathbf{R}^n$ we have $\mathrm{Cap}_{1,p}(B)=\mathrm{const}\,r^{n-p}$ (see [AH], [M1]), and hence

$$\omega(B(x,r))\le Cr^{n-p}, \tag{2.20}$$

where C is a constant which depends only on q and n. In the same manner, all solutions u of (2.1) satisfy the estimate

$$\int_{B(x,r)}|\nabla u|^q\,dx\le Cr^{n-p} \tag{2.21}$$

where C is a constant which depends only on q and n.

For any measure ω which satisfies (2.20) the Poisson equation $-\Delta u_0=\omega$ has a solution $u_0=K_0\omega$ (see [HK]), where

$$K_0\omega(x)=c(n,2)\left(\int_{|t|\le 1}|x-t|^{2-n}\,d\omega(t)+\int_{|t|>1}[|x-t|^{2-n}-|t|^{2-n}]\,d\omega(t)\right). \tag{2.22}$$

Remark 2.3. We observe that if $I_2\omega<\infty$ a.e. (or, equivalently, $\int_{|t|>1}|t|^{2-n}\,d\omega<\infty$), then $u_0(x)=I_2\omega+\mathrm{const}$. By (2.20) this is true if $1<q<2$, and hence one can use $I_2\omega$ in place of $K_0\omega$ in this case.

Unfortunately, for $q\ge 2$ there are measures ω such that (2.20) and even (2.19) hold but $I_2\omega\equiv+\infty$. However, (2.20) readily implies that, for all $1<q<\infty$, $\int_{|t|>1}|t|^{1-n}\,d\omega<\infty$ and hence $I_1\omega<\infty$ a.e. This fact will be used repeatedly in the sequel.

We will need the following estimates of Hadamard type for solutions of the Poisson equation defined by (2.22). (See [HK].)

Lemma 2.5. *Suppose that $\omega\in M_+(\mathbf{R}^n)$ satisfies* (2.20) *with* $1<p<n$. *Then* (2.22) *defines a solution $u_0=K_0\omega$ to the Poisson equation $-\Delta u_0=\omega$ which is a superharmonic function on $\mathbf{R}^n$ such that the following statements hold.*

(i) *If $p>2$, then $u_0(x)=I_2\omega(x)+\mathrm{const}$.*

(ii) *If $p\le 2$, then*

$$u_0(x)\ge c(n,2)\int_{|t|\le 1}|x-t|^{2-n}\,d\omega(t)-C(q,n)\phi_q(|x|+1), \tag{2.23}$$

where $C(q,n)>0$ and ϕ_q is defined by (2.18).

(iii) *For the average values of $|u_0|$, defined by* (2.17), *$M_R[u_0]=O(\phi_q(R))$, as $R\to+\infty$.*

(iv) *The inequality*

$$|\nabla u_0(x)|\le c(n)I_1\omega(x) \tag{2.24}$$

holds.

Proof. Statements (i) and (ii) can be found in [HK], while (iv) follows by direct differentiation. Note that (2.20) implies that $I_1\omega<\infty$ a.e. (see Remark 2.3).

To prove (iii) note that clearly

$$|u_0(x)|\le CK_1\omega(x),$$

where K_1 is the integral operator with positive kernel defined by

$$\begin{aligned} K_1\omega(x)=&\int_{|t|\le 1}|x-t|^{2-n}\,d\omega(t)+\int_{|x|/2<|t|\le 2|x|}|x-t|^{2-n}\,d\omega(t)\\ &+\int_{1<|t|\le|x|+1}|t|^{2-n}\,d\omega(t). \end{aligned} \tag{2.25}$$

It is easily seen from (2.20) that $M_R[K_1\omega]=O(\phi_q(R))$, and hence

$$M_R[u_0]\le M_R[K_1\omega]=O(\phi_q(R)),$$

as $R\to+\infty$. $\square$

Now we construct a solution to (2.1) under the assumption (2.12). We set $u_0=K_0\omega$ and

$$u_{k+1}=K_0(|\nabla u_k|^q)+K_0\omega,\quad k=1,2,\dots, \tag{2.26}$$

which implies $-\Delta u_{k+1}=|\nabla u_k|^q+\omega$.

Lemma 2.6. *Suppose* $u_0=K_0\omega$ *and* u_k *are defined by* (2.26). *There exists a constant* C_1 *which depends only on* q *and* n *such that if*

$$I_1[(I_1\omega)^q](x)\le C_1 I_1\omega(x)<\infty, \tag{2.27}$$

then the following inequalities hold:

$$|\nabla u_k(x)|\le aI_1\omega(x), \tag{2.28}$$

and

$$|\nabla u_{k+1}(x)-\nabla u_k(x)|\le b\delta^k I_1\omega(x) \tag{2.29}$$

where the constants $a>0$, $b>0$, *and* $0<\delta<1$ *depend only on* q *and* n.

Proof. We first prove (2.28). If follows from Lemma 2.5 that (2.28) holds for $k=0$, i.e. $|\nabla u_0(x)|\le a_0I_1\omega(x)$ where $a_0=c(n)$ is defined by (2.24). Then we show by induction that

$$|\nabla u_k(x)|\le a_kI_1\omega(x). \tag{2.30}$$

By Lemma 2.5 and (2.26) we have

$$|\nabla u_{k+1}(x)|=\big|\nabla K_0|\nabla u_k(x)|^q+\nabla K_0\omega(x)\big|\le c(n)(I_1|\nabla u_k(x)|^q+I_1\omega(x)),$$

where $c(n)$ is the constant in (2.24). By (2.30) and (2.27),

$$I_1|\nabla u_k(x)|^q\le I_1[a_k(I_1\omega)]^q=a_k^qI_1(I_1\omega)^q\le a_k^qC_1I_1\omega.$$

Combining these estimates, we get

$$|\nabla u_{k+1}(x)|\le a_{k+1}I_1\omega(x), \tag{2.31}$$

where a_k are defined by

$$a_{k+1}=c(n)(a_k^qC_1+1),\quad k=0,1,2,\dots,$$

starting from the initial value $a_{-1}=0$. It is easily seen that $\lim_{k\to\infty}a_k=a\le c(n)p$, where a is the smaller root of the equation $x=c(n)(x^qC_1+1)$, provided that $C_1\le q^{-1}p^{1-q}c(n)^{-q}$. This proves (2.28) with $a=c(n)p$.

We next prove by induction that (2.29) holds. We assume as above that $C_1 \le q^{-1}p^{1-q}c(n)^{-q}$ so that (2.28) holds with $a=c(n)p$. Note that $u_1-u_0=K_0|\nabla u_0|^q$, and hence by (2.24) and (2.28)

$$|\nabla u_1-\nabla u_0| \le c(n) I_1 |\nabla u_0(x)|^q \le c(n) a^q I_1 (I_1\omega)^q.$$

Then

$$|\nabla u_1-\nabla u_0| \le b_0 I_1\omega, \tag{2.32}$$

where $b_0=c(n)a^q C_1$ and C_1 is a constant from (2.27).

Similarly,

$$u_{k+1}-u_k = K_0(|\nabla u_k|^q - |\nabla u_{k-1}|^q)$$

and by (2.24)

$$|\nabla u_{k+1}-\nabla u_k| \le c(n) I_1\big(\big||\nabla u_k|^q - |\nabla u_{k-1}|^q\big|\big).$$

Using the inequality $|r^q-s^q| \le q|r-s| \max(r,s)^{q-1}$ with $r=|\nabla u_k|$ and $s=|\nabla u_{k-1}|$ together with (2.28) we have

$$\big||\nabla u_k|^q - |\nabla u_{k-1}|^q\big| \le qa|\nabla u_k-\nabla u_{k-1}|(I_1\omega)^{q-1}. \tag{2.33}$$

From this we obtain

$$|\nabla u_{k+1}-\nabla u_k| \le c(n) qa I_1[|\nabla u_k-\nabla u_{k-1}|(I_1\omega)^{q-1}]. \tag{2.34}$$

Suppose

$$|\nabla u_k-\nabla u_{k-1}| \le b_k I_1\omega.$$

Then by (2.34)

$$|\nabla u_{k+1}-\nabla u_k| \le c(n) qab_k I_1(I_1\omega)^q.$$

Using (2.27), we see by induction that

$$|\nabla u_{k+1}-\nabla u_k| \le b_{k+1} I_1\omega,$$

where $b_{k+1} \le c(n)qaC_1b_k$ and C_1 is a constant in (2.27). Thus

$$b_{k+1} \le (c(n)qaC_1)^{k+1} b_0,$$

where $b_0=c(n)a^qC_1$. Choosing C_1 in (2.27) so that $\delta=c(n)qaC_1<1$, we complete the proof of (2.29). □

Lemma 2.7. *Let K_1 be defined by* (2.25). *Then under the assumptions of Lemma* 2.6,

$$|u_{k+1}(x)-u_k(x)| \le c\delta^k K_1\omega_1(x), \tag{2.35}$$

where $d\omega_1=(I_1\omega)^q\,dx$, and the constants $c>0$ and $0<C<1$ depend only on q and n.

Proof. We have

$$|u_{k+1}(x)-u_k(x)| = \big|K_0(|\nabla u_k|^q-|\nabla u_{k-1}|^q)\big| \le CK_1\big||\nabla u_k|^q-|\nabla u_{k-1}|^q\big|,$$

where C depends only on n. Combining this estimate with (2.33), we have

$$|u_{k+1}(x)-u_k(x)| \le CqaK_1[|\nabla u_k-\nabla u_{k-1}|(I_1\omega)^{q-1}].$$

Applying Lemma 2.6, we obtain

$$|u_{k+1}(x)-u_k(x)| \le c\delta^k K_1[(I_1\omega)^q],$$

where c depends only on n and q, and $0<\delta<1$. This proves (2.35). □

Now we are in a position to complete the proof of Theorem 2.4. Suppose that (2.27) holds with a constant $C_1=C_1(q,n)$ small enough so that the estimates (2.28), (2.29), and (2.31) are valid. Let

$$u(x)=u_0(x)+\sum_{k=0}^{\infty}(u_{k+1}(x)-u_k(x)), \tag{2.36}$$

where $u_0=K_0\omega$ and u_k are defined by (2.26). By Lemma 2.7,

$$|u_{k+1}(x)-u_k(x)| \le c\delta^k K_1\omega_1(x),$$

where $d\omega_1=(I_1\omega)^q\,dx$ and $0<\delta<1$. Hence $u(x)=\lim_{k\to\infty} u_k(x)$ and

$$|u(x)| \le cK_1\omega_1(x) \quad \text{a.e.} \tag{2.37}$$

Note that by Theorem 1.1 it follows that ω_1 satisfies $\omega_1(E)\le C\,\mathrm{Cap}_{1,p}(E)$ for all compact sets E. In particular, $\omega_1(B(0,R))\le CR^{n-p}$. This implies that $K_1\omega_1\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $M_R[u]=O(\phi_q(R))$ (see Lemma 2.5). Moreover, by Lemma 2.6,

$$|\nabla u_{k+1}(x)-\nabla u_k(x)| \le b\delta^k I_1\omega(x)$$

and hence $|\nabla u(x)|\le cI_1\omega(x)$, where the constants depend only on q and n. Thus $u\in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$ and by Theorem 1.1

$$\|u\|_X=\sup\biggl\{\biggl(\frac{\int_E|\nabla u|^q\,dx}{\mathrm{Cap}_{1,p}(E)}\biggr)^{1/q}:\mathrm{Cap}_{1,p}(E)>0\biggr\}<C(q,n).$$

Let $\phi\in C_0^\infty(\mathbf{R}^n)$ be an arbitrary test function. Since

$$\nabla u(x)=\lim_{k\to\infty}\nabla u_k(x)=\nabla u_0(x)+\sum_{k=0}^{\infty}(\nabla u_{k+1}(x)-\nabla u_k(x))\quad\text{a.e.},$$

we have

$$\int_{\mathbf{R}^n}\nabla\phi\cdot\nabla u_k\,dx\to\int_{\mathbf{R}^n}\nabla\phi\cdot\nabla u\,dx,\qquad\int_{\mathbf{R}^n}\phi|\nabla u_k|^q\,dx\to\int_{\mathbf{R}^n}\phi|\nabla u|^q\,dx,$$

as $k\to\infty$, by the dominated convergence theorem. By (2.26),

$$\int_{\mathbf{R}^n}\nabla\phi\cdot\nabla u_{k+1}\,dx=\int_{\mathbf{R}^n}\phi|\nabla u_k|^q\,dx+\int_{\mathbf{R}^n}\phi\,d\omega.$$

Letting $k\to\infty$ in the preceding inequality, we obtain

$$\int_{\mathbf{R}^n}\nabla\phi\cdot\nabla u\,dx=\int_{\mathbf{R}^n}\phi|\nabla u|^q\,dx+\int_{\mathbf{R}^n}\phi\,d\omega.$$

Thus $u\in W^{1,q}_{\mathrm{loc}}(\mathbf{R}^n)$ is a (weak) solution to (2.1). The estimates (2.15)–(2.18) follow from (2.37) and Lemma 2.5. The proof of Theorem 2.4 is complete, except for the estimates (2.14) in the case $1<q<2$ which are discussed in Corollary 2.8 below in the context of the existence of positive solutions. $\square$

Corollary 2.8. *Under the assumptions of Theorem* 2.4, *if* (2.1) *has a solution* $u\ge0$ *a.e., then* $I_2\omega<\infty$ *a.e. and* (2.11) *holds. Conversely,* (2.12) *together with* $I_2\omega<\infty$ *a.e. implies that* (2.1) *has a solution* u *such that*

$$I_2\omega\le u\le CI_2\omega\quad\text{a.e.},\tag{2.38}$$

where C *depends only on* q *and* n.

Proof. If (2.1) has a nonnegative solution $u<\infty$ a.e., then u is superharmonic on $\mathbf{R}^n$ and $u=I_2|\nabla u|^q+I_2\omega+\text{const}$ (see [HK]). Hence $I_2\omega<\infty$ a.e., and (2.11) holds by Theorem 2.4.

Conversely, if $I_2\omega<\infty$ a.e. and (2.12) holds, then by Remark 2.3 one can use the Newtonian potential I_2 in place of K_0 in the definition of u_k (see the proof of Theorem 2.4). Hence setting $u_0=I_2\omega$ and

$$u_{k+1}=I_2|\nabla u_k|^q+I_2\omega, \quad k=0,1,\dots,$$

we see that Lemma 2.7 holds with I_2 in place of the operator K_1 defined by (2.25). This gives

$$u\le u_0+\sum_{k=0}^{\infty}|u_{k+1}-u_k|\le CI_2\omega,$$

where C depends only on q and n. The lower estimate in (2.38) is obvious since $u=\lim_{k\to\infty}u_k$ a.e., and $u_k\ge I_2\omega$ for all k. □

The next corollary gives new pointwise estimates for positive solutions of the Schrödinger equation

$$-\Delta v=\omega v, \quad v\ge 0, \tag{2.39}$$

which, as was mentioned in the introduction, is equivalent to (2.1) with $q=2$ and $u=\log v$.

Corollary 2.9. *Suppose* $\omega\in M_+(\mathbf{R}^n)$.

(i) *If* (2.38) *has a nonnegative (weak) solution* v, *then* $I_1\omega<\infty$ *a.e., and there exists a constant* $C_1=C_1(n)$ *such that*

$$I_1[(I_1\omega)^2](x)\le C_1I_1\omega(x) \quad a.e. \tag{2.40}$$

Furthermore,

$$\int_E|\nabla\log v|^2\,dx\le 4\,\mathrm{cap}_{1,2}(E) \quad and \quad \omega(E)\le\mathrm{cap}_{1,2}(E) \tag{2.41}$$

for all compact sets E.

(ii) *Conversely, there exists a constant* $C_2=C_2(n)$ *such that if* (2.40) *holds with* C_2 *in place of* C_1, *then there exists a positive solution* v *to* (2.39) *which satisfies the following estimates:*

$$|\nabla\log v(x)|\le CI_1\omega(x), \tag{2.42}$$

and

$$v(x)\ge C(|x|+1)^c \tag{2.43}$$

for some $C>0$ *and* $c<0$.

If in addition $I_2\omega<\infty$ *a.e., then there is a solution* v *such that*

$$I_2\omega(x)\le\log v(x)\le CI_2\omega(x) \tag{2.44}$$

for some $C>0$, *where all constants depend only on* n.

3. Solvability of the Dirichlet problem

In this section we consider the nonlinear Dirichlet problem

$$\begin{cases} -\Delta u = |\nabla u|^q + \omega & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for $q>1$ and $\omega \in M_+(\Omega)$ on a domain $\Omega \subset \mathbf{R}^n$. In what follows we assume for simplicity that Ω is bounded, and the boundary of Ω is smooth enough (satisfies the exterior sphere condition).

Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. We say that $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ is a solution to (3.1) in a weak sense if $\int_\Omega |\nabla u|^q \delta(x)\, dx < \infty$ and

$$-\int_\Omega u \Delta h\, dx = \int_\Omega |\nabla u|^q h\, dx + \int_\Omega h\, d\omega \tag{3.1$'$}$$

for all $h \in C^2(\overline{\Omega})$ such that $h=0$ on $\partial\Omega$.

Since $|h(x)| \le C\delta(x)$ for $x \in \Omega$, this definition is applicable to all $\omega \in M^+(\Omega)$ such that $\int_\Omega \delta\, d\omega < \infty$. We remark that this implies $G\omega(x) = \int_\Omega G(x,y)\, d\omega(y) < \infty$ a.e. on Ω, where $G(x,y)$ is the Green function of the Laplacian on Ω.

An equivalent definition of the solvability of (3.1), for the same class of ω, is that there exists $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ such that

$$u(x) = \int_\Omega G(x,y) |\nabla u(y)|^q \, dy + \int_\Omega G(x,y)\, d\omega(y) \tag{3.1$''$}$$

a.e. on Ω. Note that $u \in L^1(\Omega)$ follows from (3.1$''$) and the assumptions

$$\int_\Omega |\nabla u|^q \delta\, dx < \infty, \qquad \int_\Omega \delta\, d\omega < \infty, \tag{3.2}$$

which are clearly necessary for the right-hand side of (3.1$''$) to be finite a.e.

Let $1/p + 1/q = 1$. For any compact set $E \subset \Omega$, we set

$$\operatorname{cap}_{1,p,\Omega}(E) = \inf\left\{ \int_\Omega |\nabla h|^p\, dx : 1 \ge h \ge \chi_E,\ h \in C_0^\infty(\Omega) \right\}. \tag{3.3}$$

Let $\mathcal{Q} = \{Q\}$ be a Whitney decomposition of Ω into a family of cubes Q with disjoint interiors such that $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$, $\operatorname{dist}(Q, \partial\Omega) \asymp \operatorname{diam} Q$ (see [St]). Then it is easily seen (cf. [M2]) that $\operatorname{cap}_{1,p,\Omega}(E) \asymp \operatorname{cap}_{1,p}(E)$ for any compact set $E \subset Q$ with the constants of equivalence independent of Q, here $\operatorname{cap}_{1,p}$ is the capacity on $\mathbf{R}^n$ defined by (1.5$'$). In particular, $\operatorname{cap}_{1,p,\Omega}(Q) \asymp |Q|^{1-p/n}$.

Theorem 3.1. *Let $1<q<\infty$ and $1/p+1/q=1$. Let Ω be a bounded smooth domain in $\mathbf{R}^n$. Let $\omega\in M_+(\Omega)$. Then there exist positive constants C_1 and C_2 such that the following statements hold.*

(i) *If* (3.1) *has a solution $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$, then the inequality*

$$\omega(E)\le C\,\mathrm{cap}_{1,p,\Omega}(E) \tag{3.4}$$

holds for all compact sets $E\subset\Omega$ with a constant $C<C_1(q,n)$.

(ii) *Let $2<q<\infty$. If* (3.4) *holds with $C<C_2(q,n,\Omega)$, then* (3.1) *has a solution u.*

(iii) *The solution u whose existence is claimed in* (ii) *satisfies the inequality*

$$|\nabla u(x)|\le CI_1\omega(x) \quad a.e.\ on\ \Omega, \tag{3.5}$$

with a constant which depends only on q, n, and Ω.

Remark 3.1. One of the assumptions on Ω we need below is that the following Hardy inequality holds:

$$\int_\Omega \frac{|h(x)|^p}{\delta(x)^p}\,dx\le C\|h\|^p_{W^{1,p}(\Omega)} \tag{3.6}$$

for all $h\in C_0^\infty(\Omega)$, where $1<p<\infty$. It is known that (3.6) is valid for a wide class of Ω (see [A1], [N]).

Proof of Theorem 3.1. We will need several lemmas. The following lemma is proved in the same manner as Lemma 2.1 for $\Omega=\mathbf{R}^n$.

Lemma 3.2. *Let $1<p<\infty$ and let Ω be a bounded domain in $\mathbf{R}^n$. Let $\omega\in M_+(\Omega)$. Suppose the equation*

$$-\Delta u=|\nabla u|^q+\omega \quad on\ \Omega$$

has a solution $u\in W^{1,q}_{\mathrm{loc}}(\Omega)$ in a weak sense, i.e.,

$$-\int_\Omega \nabla u\cdot\nabla h\,dx=\int_\Omega |\nabla u|^q\,h\,dx+\int_\Omega h\,d\omega$$

for all $h\in C_0^\infty(\Omega)$. Then the inequality

$$\omega(E)\le C\,\mathrm{cap}_{1,p,\Omega}(E)$$

holds for all compact sets $E\subset\Omega$, with a constant C which depends only on q, n, and Ω.

Let $\{\phi_Q\}$ be a partition of unity associated with the Whitney decomposition of Ω defined above: $\phi_Q\in C_0^\infty(Q^*)$, $\phi_Q\ge 0$, $\sum_Q\phi_Q=1$, and $|\nabla\phi_Q|\le C/\mathrm{diam}\,Q$. Note that $\mathrm{dist}\,(Q^*,\partial\Omega)\asymp\mathrm{diam}\,Q$ and $1\le\sum_Q\chi^*_Q\le C(n)$, here $Q^*=(1+\varepsilon)Q$, $0<\varepsilon<\frac14$, (see [St]).

Lemma 3.3. *Let $1<p<\infty$ and let Ω be a bounded domain in $\mathbf{R}^n$ such that (3.6) holds. Let $\omega\in M_+(\Omega)$. Then the following statements are equivalent.*

(i) *The inequality*

$$\int_\Omega |h|^p\, d\omega \le C \int_\Omega |\nabla h|^p\, dx \tag{3.7}$$

holds for all $h\in C_0^\infty(\Omega)$, with a constant C which depends only on p, n, and Ω.

(ii) *The inequality*

$$\omega(E)\le C\,\mathrm{cap}_{1,p,\Omega}(E) \tag{3.8}$$

holds for all compact sets $E\subset\Omega$, with a constant C which depends only on p, n, and Ω.

(iii) *The inequality*

$$\omega(E)\le C\,\mathrm{cap}_{1,p}(E), \quad E\subset Q, \tag{3.8$'$}$$

holds with a constant C which depends only on p, n, and Ω, here $\mathrm{cap}_{1,p}$ *is the capacity on $\mathbf{R}^n$ defined by* $(1.5')$.

Proof of Lemma 3.3. The equivalence of (i) and (ii) is known [M2]. Clearly, (ii) $\Rightarrow$ (iii), since as was mentioned above $\mathrm{cap}_{1,p,\Omega}(E)\asymp\mathrm{cap}_{1,p}(E)$ for $E\subset Q$.

To prove (iii) $\Rightarrow$ (i) we observe that if $(3.8')$ holds for all $E\subset Q$, then by the preceding remarks

$$\int_Q |h|^p\, d\omega \le C\int_{Q^*} |\nabla h|^p\, dx$$

for all $h\in C_0^\infty(\Omega)$ with a constant C which depends only on q, n, and Ω. Hence applying this inequality with $h\phi_Q$ in place of h we have

$$\int_\Omega |h|^p\, d\omega \le C\sum_Q \int_Q |h\phi_Q|^p\, d\omega \le C\sum_Q \int_{Q^*} |\nabla(h\phi_Q)|^p\, dx.$$

Since

$$\int_{Q^*} |\nabla(h\phi_Q)|^p\, dx \le C\int_{Q^*} |\nabla h|^p\, dx + C\int_{Q^*} \frac{|h|^p}{\delta^p}\, dx,$$

it follows by the properties of $\mathcal{Q}$ and (3.6) that

$$\int_\Omega |h|^p\, d\omega \le C\int_\Omega |\nabla h|^p\, dx$$

for all $h\in C_0^\infty(\Omega)$. $\square$

In the following two lemmas we make use of the assumption $2<q<\infty$ in a crucial way.

Lemma 3.4. *Let $\Omega\subset\mathbf{R}^n$, $n\geq 2$, denote a bounded domain with boundary $\partial\Omega$ which is C^1 embedded in $\mathbf{R}^n$. Assume further that ω is a measure on Ω such that* (3.8) *holds. Then, if $1<p<2$, we have $\int_\Omega \delta\, d\omega<\infty$, where $\delta(x)$ is the distance from $x\in\Omega$ to $\partial\Omega$, and there is a constant k depending only on p, n, and Ω such that for $a\in\overline{\Omega}$*

$$\int_{B(a,r)} \delta\, d\omega \leq kCr^{n+1-p}, \quad r>0, \tag{3.9}$$

where C is the constant in (3.8).

Proof of Lemma 3.4. We may assume that $a\in\partial\Omega$ and it is also enough to prove (3.9) for $r<\frac{1}{2}R$, where R is some fixed positive constant depending on Ω. We may then assume (see [St, pp. 180–190]), that $\bar{\Omega}\cap B(a,R)$ is a special Lipschitz domain, i.e. such that if (x,y) is a point in $\bar{\Omega}\cap B(a,R)$ then

$$\{(x,y): x\in\mathbf{R}^{n-1},\ \varphi(x)\leq y\in\mathbf{R},\ |x|\leq R\}$$

with the point a corresponding to $(0,0)$ and φ a Lipschitz function such that $\varphi(0)=0$ and $|\varphi(x')-\varphi(x'')|\leq L|x'-x''|$, where the constant L is the same for all balls $B(a,R)$. Furthermore, the sets $\partial\Omega\cap B(a,R)$ are parametrized as

$$\{(x,y): \varphi(x)=y,\ |x|\leq R\}.$$

Let $\eta\in W_0^{1,p}(B(a,2r))$, $0\leq\eta\leq 1$, $\eta=1$ on $B(a,r)$, and let η be extended as 0 outside $B(a,2r)$ in such a way that $|\nabla\eta|\leq(1/r)\chi_{B(a,2r)}$.

By Lemma 3.2 it follows that (3.8) is equivalent to (3.7), and hence (see [M2]) to the inequality

$$\int_\Omega |h|^p\, d\omega \leq A\int_\Omega |\nabla h|^p\, dx, \quad h\in W_0^{1,p}(\Omega),$$

for $1<p<\infty$. Furthermore, A is equivalent to the constant C in (3.8).

The preceding inequality gives with the test function

$$h(x,y)=(y-\varphi(x))^{1/p}\eta(x,y),$$

where $\delta(x)\leq y-\varphi(x)$, that

$$\int_{B(a,r)} \delta\, d\omega \leq \int_\Omega h^p\, d\omega \leq A\int_{\Omega\cap B(a,2r)} |\nabla h|^p\, dx.$$

Since

$$|\nabla h|^p \le 2^{p-1}(y-\varphi(x))|\nabla\eta|^p + 2^{p-1}(1/p)^p(y-\varphi(x))^{-(p-1)}|\nabla(y-\varphi(x))|^p\eta,$$

and the Lipschitz condition implies that $|\nabla(y-\varphi(x))|\le\sqrt{1+L^2}$ and, when $\varphi(0)=0$, also $-L|x|\le\varphi(x)\le L|x|$, we obtain the estimate

$$\begin{aligned}\int_{\Omega\cap B(a,2r)}(y-\varphi(x))|\nabla\eta|^p\,dx\,dy &\le \frac{1}{r^p}\int_{|x|\le 2r}dx\int_{\varphi(x)}^{2r}(y-\varphi(x))\,dy\\ &= \frac{1}{2r^p}\int_{|x|\le 2r}(2r-\varphi(x))^2\,dx \le k_1 r^{n+1-p}.\end{aligned}$$

Since $p<2$, we have

$$\begin{aligned}\int_{\Omega\cap B(a,2r)}\frac{|\nabla(y-\varphi(x))|^p}{(y-\varphi(x))^{p-1}}\eta\,dx\,dy &\le (1+L^2)^{p/2}\int_{|x|\le 2r}dx\int_{\varphi(x)}^{2r}\frac{dy}{(y-\varphi(x))^{p-1}}\\ &= \frac{(1+L^2)^{p/2}}{2-p}\int_{|x|\le 2r}(2r-\varphi(x))^{2-p}\,dx \le k_2 r^{n+1-p},\end{aligned}$$

which proves the lemma. □

Lemma 3.5. *Suppose* $1<p<2$ *and* $\omega\in M_+(\Omega)$. *Suppose the inequality* (3.8) *holds, i.e.*

$$\omega(E)\le C\,\mathrm{cap}_{1,p,\Omega}(E)$$

for all compact sets $E\subset\Omega$. *Then a similar inequality holds for the measure* $d\omega_1 = |\nabla G\omega|^q\,dx$, *i.e.*

$$\int_E |\nabla G\omega|^q\,dx \le C\,\mathrm{cap}_{1,p,\Omega}(E) \tag{3.10}$$

for all compact sets $E\subset\Omega$. *Furthermore,* $\int_\Omega \delta\,d\omega_1<\infty$.

Proof of Lemma 3.5. Assume that (3.8), or equivalently (3.8′), holds. Let Q be a Whitney family of cubes associated with Ω. By Lemma 3.3 it is enough to prove

$$\int_E |\nabla G\omega|^q\,dx \le C\,\mathrm{cap}_{1,p}(E) \tag{3.11}$$

for all $E\subset Q$ with a constant which depends only on q, n, and Ω.

We will use the following known estimates for the gradient of the Green kernel (see [W], [GW]):

$$|\nabla_x G(x,y)| \le C\frac{\delta(y)}{|x-y|^n}, \quad |\nabla_x G(x,y)| \le C\frac{1}{|x-y|^{n-1}} \tag{3.12}$$

for all $x, y \in \Omega$, $x \neq y$.

To prove (3.11), for a fixed cube $Q \in \mathcal{Q}$, we define the measures ν_1 and ν_2 by

$$d\nu_1 = \chi_{Q^*}\, d\omega, \quad d\nu_2 = (1-\chi_{Q^*})\, d\omega. \tag{3.13}$$

We show that (3.11) holds with ν_1 and ν_2 in place of ω.

Since $\operatorname{supp} \nu_1 \subset Q^*$, it follows that

$$\nu_1(F) \le C \operatorname{cap}_{1,p,\Omega}(F \cap Q^*) \le C \operatorname{cap}_{1,p}(F)$$

for all compact sets $F \subset \mathbf{R}^n$. Hence by Theorem 1.1, for any $E \subset Q$ we have

$$\int_E (I_1 \nu_1)^q \, dx \le C \operatorname{cap}_{1,p}(E) \asymp C \operatorname{cap}_{1,p,\Omega}(E),$$

where I_1 is the Riesz potential of order $\alpha = 1$. Then, by the second estimate in (3.12),

$$\int_E |\nabla G \nu_1|^q \, dx \le C \int_E (I_1 \nu_1)^q \, dx \le C \operatorname{cap}_{1,p,\Omega}(E).$$

We now prove a similar inequality for ν_2. By the first estimate in (3.12) it follows

$$|\nabla G \nu_2(x)| \le C \int_{(Q^*)^c} \frac{\delta(y)}{|x-y|^n} \, d\nu(y).$$

Let x_Q be the center of Q. Then for all $x \in Q$ and $y \in (Q^*)^c$ we have $|x-y| \asymp |x_Q - y|$. Hence for any $E \subset Q$

$$\int_E |\nabla G \nu_2|^q \, dx \le C|E| \int_{(Q^*)^c} \frac{\delta(y)}{|x_Q - y|^n} \, d\omega(y).$$

Thus the desired estimate reduces to

$$|E| \left(\int_{(Q^*)^c} \frac{\delta(y)}{|x-y|^n} \, d\omega(y) \right)^q \le C \operatorname{cap}_{1,p}(E) \tag{3.14}$$

for $E \subset Q$. From the known isoperimetric inequalities (see [AH], [M2])

$$\left(\frac{|E|}{|Q|} \right)^{1-p/n} \le C \frac{\operatorname{cap}_{1,p}(E)}{\operatorname{cap}_{1,p}(Q)} \tag{3.15}$$

we see that it suffices to prove (3.14) for $E=Q$, i.e.

$$|Q|^{(p-1)/n}\int_{(Q^*)^c}\frac{\delta(y)}{|x_Q-y|^n}\,d\omega(y)\le C, \tag{3.16}$$

with a constant C which depends only on p, n, and Ω.

Let $d\mu(y)=\delta(y)\,d\omega(y)$ and let $B(x,r)$ be a ball of radius r centered at $x\in\Omega$. By Lemma 3.4 $\mu(B(x,\varrho))\le C\varrho^{n-p+1}$ for $\varrho>\delta(x)$ and $1<p<2$. Then letting $r=\delta(x_Q)$, we get

$$\begin{aligned}|Q|^{(p-1)/n}\int_{(Q^*)^c}\frac{\delta(y)}{|x_Q-y|^n}\,d\omega(y) &\le Cr^{p-1}\int_r^\infty\frac{\mu(B(x_Q,\varrho))}{\varrho^{n+1}}\,d\varrho\\ &\le Cr^{p-1}\int_r^\infty\frac{d\varrho}{\varrho^p}\le C<\infty,\end{aligned}$$

which proves (3.11). The estimate $\int_\Omega\delta_1\,d\omega<\infty$ follows from this and Lemma 3.4. □

Now we are in a position to complete the proof of Theorem 3.1. The necessity of (3.8) follows from Lemma 3.2. The sufficiency is proved by the contraction mapping principle in the Banach space X of functions u such that $u\in L^{1,q}_{\mathrm{loc}}(\Omega)\cap L^1(\Omega)$ with norm

$$|||u|||_X=\|u\|_{L^1(\Omega)}+\sup\left\{\left(\frac{\int_E|\nabla u|^q\,dx}{\operatorname{cap}_{1,p,\Omega}(E)}\right)^{1/q}:\operatorname{cap}_{1,p,\Omega}(E)>0\right\}.$$

Note that it follows from Lemma 3.5 that $|||u|||_X<\infty$ implies $\int_\Omega\delta|\nabla u|^q\,dx<\infty$, where $q>2$.

Letting $u=\beta v$ where β is a positive constant, we see that the equation $u=G|\nabla u|^q+G\omega$ has a solution if and only if the equation $v=Av+f$ is solvable, where $Av=\beta^{q-1}G(|\nabla v|^q)$ and $f=\beta^{-1}G\omega$. We apply the contraction mapping principle to the equation $v=Av+f$ on the unit ball of X, where $\beta>0$ is chosen small enough so that A is a contraction.

By Lemma 3.4 and Lemma 3.5 we have that $|||f|||_X\le1$ if (3.4) holds with the constant $C<C_2(q,n,\Omega)$. Furthermore, as in the proof of Lemma 2.6, it follows from Lemma 3.5 that $|||Af-Ag|||_X\le c|||f-g|||_X$ with $c<1$ for any f and g in the unit ball of X. Thus there exists u such that $|||u|||_X\le1$ and

$$u=G|\nabla u|^q+G\omega\quad\text{a.e.}$$

The pointwise estimate $|\nabla u|\le CI_1\omega$ can be proved in the same manner as in the case $\Omega=\mathbf{R}^n$ in Section 2, we omit the details. The proof of Theorem 3.1 is complete. □

4. Some generalizations

In this section we demonstrate how our approach works for more general superlinear inhomogeneous equations of the type

$$-\Delta u = f(x, u, \nabla u), \tag{4.1}$$

where $f(x, u, \nabla u) \asymp a(x)|\nabla u|^{q_1} + b(x)|u|^{q_2} + \omega(x)$, and $q_1, q_2 > 1$.

As was mentioned in the introduction, we are interested in sharp solvability results with close sufficiency and necessity conditions, for an arbitrary nonnegative inhomogeneous term $\omega \neq 0$.

For simplicity, we consider in detail the solvability problem on $\mathbf{R}^n$ for the equation

$$-\Delta u = a|\nabla u|^{q_1} + b|u|^{q_2} + \omega, \tag{4.2}$$

with bounded coefficients $a, b \geq 0$ and arbitrary $\omega \in M_+(\mathbf{R}^n)$. (The necessity part of our results is proved for constant a and b.)

The solvability of (4.2) is understood in a weak sense, i.e. there exists $u \in W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n) \cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} \nabla u \cdot \nabla \phi \, dx = \int_{\mathbf{R}^n} a|\nabla u|^{q_1} \phi \, dx + \int_{\mathbf{R}^n} b|u|^{q_2} \phi \, dx + \int_{\mathbf{R}^n} \phi \, d\omega \tag{4.3}$$

for all $\phi \in C_0^\infty(\mathbf{R}^n)$. We will actually show that under certain assumptions (for $b \neq 0$) there exists a *nonnegative* solution $u \in W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n) \cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$ satisfying (4.3), or equivalently

$$u = I_2(a|\nabla u|^{q_1}) + I_2(bu^{q_2}) + I_2\omega + c \quad \text{a.e.}, \tag{4.4}$$

where $u \geq 0$, $c \geq 0$ and $I_2 = (-\Delta)^{-1} = I_1^2$ is the Newtonian potential.

Theorem 4.1. *Let* $1 < q_i < \infty$ *and* $1/p_i + 1/q_i = 1$, $i = 1, 2$. *Let* $\omega \in M_+(\mathbf{R}^n)$. *Then there exist positive constants* C_j, $j = 1, \dots, 6$, *which depend only on* q_i *and* n *such that the following statements hold.*

(i) *If equation* (4.2) *with constant coefficients* $a, b > 0$ *has a solution* u *belonging to* $W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n) \cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$, *then*

$$I_1(I_1\omega)^{q_1}(x) \leq \frac{C_1}{a} I_1\omega(x) < \infty \quad \textit{a.e.}, \tag{4.5}$$

and

$$I_2(I_2\omega)^{q_2}(x) \leq \frac{C_2}{b} I_2\omega(x) < \infty \quad \textit{a.e.} \tag{4.6}$$

(ii) *Conversely, let* $\tilde{a}=\|a\|_{L^\infty}$ *and* $\tilde{b}=\|b\|_{L^\infty}$ $(0<a,b<\infty)$. *Then if the inequalities* (4.5) *and* (4.6) *hold with the constants* C_3 *and* C_4 *in place of* C_1 *and* C_2 *and* $\tilde{a}$ *and* $\tilde{b}$ *in place of* a *and* b, *respectively, then* (4.2) *has a solution* $u\in W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n)\cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$ *such that the following inequalities hold*:

$$|\nabla u(x)|\le C_5 I_1\omega(x), \quad I_2\omega(x)\le u(x)\le C_6 I_2\omega(x) \quad a.e. \tag{4.7}$$

Remark 4.1. As we will see below, any solution $u\in W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n)\cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$ of (4.2) (with constant coefficients a and b) satisfies the estimates

$$\int_E (a|\nabla u|^{q_1}+b|u|^{q_2})\,dx+\omega(E)\le a^{1-p_1}C(q_1)\operatorname{cap}_{1,p_1}(E),$$
$$\int_E (a|\nabla u|^{q_1}+b|u|^{q_2})\,dx+\omega(E)\le b^{1-p_2}C(q_2,n)\operatorname{cap}_{2,p_2}(E),$$

for all compact sets E, here $\operatorname{cap}_{\alpha,p}$ is the capacity of order $\alpha=1,2$ defined by $(1.5')$. In particular, a nontrivial global solution to (4.2) may exist only if $n/(n-1)<q_1<\infty$ and $n/(n-2)<q_2<\infty$.

It follows from the known relations between Riesz capacities (see [AH, Theorem 5.5.1]) that, for $p_1=2p_2$, the inequality

$$\operatorname{cap}_{1,p_1}(E)\le C\operatorname{cap}_{2,p_2}(E)$$

holds for compact sets $E\subset\mathbf{R}^n$, with a constant C which depends only on p_1, p_2, and n. In this case the second term on the right hand side of (4.2) is "dominated" by the first one. In all other cases the contributions of the nonlinearities involving $|\nabla u|^{q_1}$ and $|u|^{q_2}$ are generally not comparable.

Remark 4.2. Similar inequalities with weighted capacities hold for variable a and b. Unfortunately, they generally are not sufficient for the solvability of (4.2). (See [KV] and [VW] where this problem is considered for equations without the gradient term.)

Proof of Theorem 4.1. To prove statement (i), notice that as in the proof of Lemma 2.1 it follows from (4.3), with h^{p_1} in place of ϕ, that for any $h\in C_0^\infty(\mathbf{R}^n)$, $h\ge 0$, we have

$$\int_{\mathbf{R}^n}(a|\nabla u|^{q_1}+b|u|^{q_2})h^{p_1}\,dx+\int_{\mathbf{R}^n}h^{p_1}\,d\omega\le p_1\biggl(\int_{\mathbf{R}^n}|\nabla u|^{q_1}h^{p_1}\,dx\biggr)^{1/q_1}\|\nabla h\|_{L^{p_1}(\mathbf{R}^n)}.$$

From this (see the proof of Lemma 2.1 and Corollary 2.2) we have

$$\int_E|\nabla u|^{q_1}\,dx\le a^{-p_1}C(q_1)\operatorname{cap}_{1,p_1}(E),$$

and

$$\int_E b|u|^{q_2}\,dx+\omega(E)\le a^{1-p_1}C(q_1)\operatorname{cap}_{1,p_1}(E)$$

for all compact sets E. This proves the first estimate in Remark 4.1, by Theorem 1.1 the preceding inequality also yields (4.5).

The proof of (4.6) is a little more technical. Notice that by Theorem 1.1 it suffices to obtain an equivalent capacitary estimate

$$\omega(E)\le b^{1-p_2}C(q_2,n)\operatorname{cap}_{2,p_2}(E) \tag{4.6$'$}$$

for all compact sets E. Suppose $u\in W^{1,q_1}_{\mathrm{loc}}(\mathbf{R}^n)\cap L^{q_2}_{\mathrm{loc}}(\mathbf{R}^n)$ is a solution of (4.2) so that (4.3) holds. We first prove the inequality

$$\int_{B_r}|u|^{q_2}\,dx\le b^{-p_2}C(q_2,n)r^{n-2p_2},$$

together with (4.6$'$) for any ball $B_r=B_r(x_0)$ of radius r centered at x_0. Without loss of generality we set $x_0=0$.

Let $\psi\in C_0^\infty(\mathbf{R}^n)$ be a cut-off function such that $\psi\ge0$, $\psi(x)=1$ if $|x|<1$ and $\psi(x)=0$ if $|x|>2$. We will also need the inequalities $|\nabla\psi(x)|^s\le C(s)\psi(x)^{s-1}$ and $|\Delta\psi(x)|^s\le C(n,s)\psi(x)^{s-1}$ for any $s>1$. For $r>0$ set $\psi_r(x)=\psi(x/r)$, then we have $|r\nabla\psi_r(x)|^s\le C(s)\psi_r(x)^{s-1}$ and $|r^2\Delta\psi_r(x)|^s\le C(n,s)\psi_r(x)^{s-1}$ for $r<|x|<2r$. Now using (4.3) with ψ_r in place of ϕ we get

$$\int_{\mathbf{R}^n}\nabla u\cdot\nabla\psi_r\,dx=-\int_{\mathbf{R}^n}u\,\Delta\psi_r\,dx=\int_{\mathbf{R}^n}(a|\nabla u|^{q_1}+b|u|^{q_2})\psi_r\,dx+\int_{\mathbf{R}^n}\psi_r\,d\omega.$$

In particular,

$$-\int_{\mathbf{R}^n}u\Delta\psi_r\,dx\ge\int_{\mathbf{R}^n}b|u|^{q_2}\psi_r\,dx.$$

By Hölder's inequality, the preceding estimate implies

$$\int_{\mathbf{R}^n}b|u|^{q_2}\psi_r\,dx\le\left(\int_{\mathbf{R}^n}|u|^{q_2}\psi_r\,dx\right)^{1/q_2}\left(\int_{\mathbf{R}^n}|\Delta\psi_r|^{p_2}\psi_r^{1-p_2}\,dx\right)^{1/p_2}.$$

Since

$$\int_{\mathbf{R}^n}|\Delta\psi_r|^{p_2}\psi_r^{1-p_2}\,dx\le\int_{r<|x|<2r}r^{-2p_2}\,dx=C(n)r^{n-2p_2},$$

we have

$$\int_{\mathbf{R}^n}b^{p_2}|u|^{q_2}\psi_r\,dx\le C(q_2,n)r^{n-2p_2}.$$

From this and the preceding estimates it follows

$$\int_{B_r}(a|\nabla u|^{q_1}+b|u|^{q_2})\,dx+\omega(B_r)\leq b^{1-p_2}C(q_2,n)r^{n-2p_2},$$

which in particular gives (4.6′) for $E=B_r(x_0)$.

Now we prove (4.6′) for arbitrary compact E. Let $h=I_2 g$, where $g\in C_0^\infty(\mathbf{R}^n)$, $g\geq 0$. Then $h\geq 0$ and $\Delta h=-g\in C_0^\infty(\mathbf{R}^n)$. Note that $h(x)\asymp(|x|+1)^{2-n}$ and hence by the estimate $\int_{B_r}|u|^{q_2}\,dx\leq Cr^{n-2p_2}$ proved above, we have that $\int_{\mathbf{R}^n}h^{p_2}|u|^{q_2}\,dx<\infty$.

We next prove the inequality

$$\left|\int_{\mathbf{R}^n}u\Delta(h^{p_2})\,dx\right|\leq C(q_2,n)\left(\int_{\mathbf{R}^n}h^{p_2}|u|^{q_2}\,dx\right)^{1/q_2}\|\Delta h\|_{L^{p_2}(\mathbf{R}^n)}$$

following the argument in [AP]. Clearly,

$$\int_{\mathbf{R}^n}u\Delta(h^{p_2})\,dx=p_2\int_{\mathbf{R}^n}[(p_2-1)h^{p_2-2}|\nabla h|^2+h^{p_2-1}\Delta h]u\,dx.$$

Since $h=-I_2(\Delta h)$, then by Hedberg's inequality [He] we have

$$|\nabla h|^2\leq c(n)hM(\Delta h),$$

where M is the Hardy–Littlewood maximal operator. Applying this estimate together with Hölder's inequality and the maximal inequality in $L^{p_2}(\mathbf{R}^n)$, we have

$$\begin{aligned}\left|\int_{\mathbf{R}^n}u\Delta(h^{p_2})\,dx\right|&\leq C(q_2,n)\left(\int_{\mathbf{R}^n}M(\Delta h)h^{p_2-1}|u|\,dx+\int_{\mathbf{R}^n}|\Delta h|\,h^{p_2-1}|u|\,dx\right)\\&\leq C(q_2,n)\left(\int_{\mathbf{R}^n}h^{p_2}|u|^{q_2}\,dx\right)^{1/q_2}\|\Delta h\|_{L^{p_2}(\mathbf{R}^n)}.\end{aligned}$$

The rest of the proof can be completed by making use of (4.3) with $\psi_r(I_2 g)^{p_2}$, where $I_2 g\geq\chi_E$, in place of ϕ, and repeating the estimates used above, or by the following approximation argument as in [AP]. Choose a sequence of functions $\phi_m\in C_0^\infty(\mathbf{R}^n)$ such that $\phi_m\to h^{p_2}$ in $C^2(\mathbf{R}^n)$, and use (4.3) with ϕ_m in place of ϕ, so that

$$-\int_{\mathbf{R}^n}u\Delta\phi_m\,dx=\int_{\mathbf{R}^n}\nabla u\cdot\nabla\phi_m\,dx=\int_{\mathbf{R}^n}(a|\nabla u|^{q_1}+b|u|^{q_2})\phi_m\,dx+\int_{\mathbf{R}^n}\phi_m\,d\omega.$$

Letting $m\to\infty$, we have

$$-\int_{\mathbf{R}^n}u\Delta(h^{p_2})\,dx=\int_{\mathbf{R}^n}(a|\nabla u|^{q_1}+b|u|^{q_2})h^{p_2}\,dx+\int_{\mathbf{R}^n}h^{p_2}\,d\omega.$$

From this and the preceding estimates it follows

$$C(q_2,n)\left(\int_{\mathbf{R}^n} h^{p_2}|u|^{q_2}\,dx\right)^{1/q_2}\|\Delta h\|_{L^{p_2}(\mathbf{R}^n)} \geq \int_{\mathbf{R}^n}(a|\nabla u|^{q_1}+b|u|^{q_2})h^{p_2}\,dx + \int_{\mathbf{R}^n} h^{p_2}\,d\omega.$$

Hence

$$\int_{\mathbf{R}^n} h^{p_2}|u|^{q_2}\,dx \leq b^{-p_2}C(q_2,n)\|\Delta h\|^{p_2}_{L^{p_2}(\mathbf{R}^n)},$$

and

$$\int_{\mathbf{R}^n} ah^{p_2}|\nabla u|^{q_1}\,dx + \int_{\mathbf{R}^n} h^{p_2}\,d\omega \leq b^{1-p_2}C(q_2,n)\|\Delta h\|_{L^{p_2}(\mathbf{R}^n)}.$$

Minimizing over $h=I_2 g\geq\chi_E$, we obtain $\omega(E)\leq b^{1-p_2}C(q_2,n)\,\mathrm{Cap}_{2,p_2}(E)$, where Cap_{2,p_2} is the Riesz capacity defined by (1.5), which is equivalent to cap_{2,p_2}. This by Theorem 1.1 yields (4.6), as well as the second estimate in Remark 4.1.

To prove statement (ii) of Theorem 4.1, we will need the following lemma.

Lemma 4.2. *Let $1<q<\infty$ and $1/p+1/q=1$. Let $\omega\in M_+(\mathbf{R}^n)$. Then the inequality*

$$I_2(I_2\omega)^q(x)\leq C_1 I_2\omega(x) \quad a.e. \tag{4.8}$$

holds if and only if

$$I_1(I_2\omega)^q(x)\leq C_2 I_1\omega(x) \quad a.e., \tag{4.9}$$

provided $I_2\omega(x)<\infty$ a.e. Moreover, $C_1\leq C_2\leq C(q,n)C_1$.

Remark 4.3. It is not difficult to see that, more generally, for all $0<\alpha\leq\beta<n$ and $\omega\in M_+(\mathbf{R}^n)$ the inequality

$$I_\alpha(I_\beta\omega)^q(x)\leq CI_\alpha\omega(x)<\infty \quad \text{a.e.}$$

is equivalent to

$$I_\beta(I_\beta\omega)^q(x)\leq CI_\beta\omega(x)<\infty \quad \text{a.e.}$$

In the case $\alpha>\beta$ the second inequality obviously implies the first one, but the converse is not true.

Proof of Lemma 4.2. Applying I_1 to both sides of (4.9) and taking into account that $I_2=I_1^2$, we see that (4.9) $\Rightarrow$ (4.8) with $C_1\leq C_2$.

In the opposite direction, suppose that (4.8) holds. Then by Theorem 2.1 in [MV] the "testing inequality" of E. Sawyer type

$$\int_B (I_2\omega_B)^q\,dx \le C(q,n)C_1\omega(B) \tag{4.10}$$

holds for all balls B, here $d\omega_B=\chi_B\,d\omega$. It is well known (see e.g. [MV]) that (4.10) implies the estimate

$$\omega(B) \le C(q,n)C_1^{p-1}r^{n-2p} \tag{4.11}$$

for all balls $B=B_r(x)$ of radius r, which we will need below. The proof of the implication (4.10) $\Rightarrow$ (4.9) is based on the same idea as in Theorem 2, [VW]. Let $d\nu=(I_2\omega)^q\,dx$. Then

$$I_1[(I_2\omega)^q](x)=C(q,n)\int_0^\infty \frac{\nu(B_t(x))}{t^{n-1}}\,\frac{dt}{t}.$$

For a fixed $x\in\mathbf{R}^n$ and $t>0$ we estimate $\nu(B_t(x))$. We set $d\omega_1=\chi_{B_{2t}(x)}\,d\omega$ and $d\omega_2=(1-\chi_{B_{2t}(x)})\,d\omega$, so that $\omega=\omega_1+\omega_2$, and hence

$$\nu(B_t(x)) \le 2^{q-1}\left(\int_{B_t(x)} (I_2\omega_1)^q\,dy+\int_{B_t(x)} (I_2\omega_2)^q\,dy\right).$$

By (4.10), $\int_{B_t(x)}(I_2\omega_1)^q\,dy\le C(q,n)C_1\omega(B_{2t}(x))$. To estimate the second term note that for all $y\in B_t(x)$ obviously $I_2\omega_2(y)\asymp I_2\omega_2(x)$ with the constants of equivalence depending only on q and n. Hence

$$\int_{B_t(x)} (I_2\omega_2)^q\,dy \le C(q,n)t^n(I_2\omega_2(x))^q.$$

Since

$$I_2\omega_2(x)=C(q,n)\int_{2t}^\infty \frac{\omega(B_r(x))}{r^{n-2}}\,\frac{dr}{r},$$

by combining the preceding inequalities we obtain

$$\nu(B_t(x)) \le C(q,n)\left(\omega(B_t(x))+t^n\int_{2t}^\infty \frac{\omega(B_r(x))}{r^{n-2}}\,\frac{dr}{r}\right).$$

From this it follows

$$\begin{aligned}I_1[(I_2\omega)^q](x) &\le C(q,n)C_1\int_0^\infty \frac{\omega(B_{2t}(x))}{t^{n-1}}\,\frac{dt}{t}+C(q,n)\int_0^\infty\left(\int_t^\infty \frac{\omega(B_r(x))}{r^{n-2}}\,\frac{dr}{r}\right)^q dt\\ &=I+II.\end{aligned}$$

Clearly,

$$I = C(q,n)C_1 \int_0^\infty \frac{\omega(B_{2t}(x))}{t^{n-1}} \frac{dt}{t} \le C(q,n)C_1 I_1\omega(x).$$

On the other hand, by the Hardy inequality

$$II = C(q,n) \int_0^\infty \left(\int_t^\infty \frac{\omega(B_r(x))}{r^{n-2}} \frac{dr}{r} \right)^q dt \le C(q,n) \int_0^\infty \frac{\omega(B_r(x))^q}{r^{(n-2)q}} \, dr.$$

By (4.11) we have

$$\omega(B_r(x))^q \le C(q,n)C_1 \omega(B_r(x)) r^{(n-2p)(q-1)}.$$

Hence

$$II \le C(q,n)C_1 \int_0^\infty \frac{\omega(B_r(x))}{r^{n-1}} \frac{dr}{r} = C(q,n)C_1 I_1\omega(x).$$

Thus

$$I_1[(I_2\omega)^q](x) \le C_2 I_1\omega(x),$$

where $C_2 = C(q,n)C_1$. □

We now complete the proof of statement (ii) of Theorem 4.1. Suppose that (4.5) and (4.6) hold with the constants C_1 and C_2 small enough depending only on q_i and n. By applying I_1 to both sides of (4.5) we get

$$I_2(I_1\omega)^{q_1} \le \frac{C_1}{a} I_2\omega. \tag{4.12}$$

Also, by Lemma 4.2 it follows that (4.6) is equivalent to

$$I_1(I_2\omega)^{q_2} \le \frac{C_2}{b} I_1\omega. \tag{4.13}$$

We set $u_0 = I_2\omega$ and

$$u_{k+1} = I_2(a|\nabla u_k|^{q_1}) + I_2(b u_k^{q_2}) + I_2\omega \tag{4.14}$$

for $k = 1, 2, \ldots$.

We need simultaneous pointwise estimates of $u_k(x)$ and $|\nabla u_k(x)|$ based on (4.5), (4.6), (4.12), and (4.13), similar to those established in Lemma 2.6 in the case of the Riccati equation.

Since $|\nabla u_0| \le c(n) I_1\omega$, by (4.12) and (4.6) we have

$$u_1 \le a I_2(I_1\omega)^{q_1} + b I_2(I_2\omega)^{q_2} + I_2\omega \le C I_2\omega,$$

where C depends only on q and n. As in the proof of Lemma 2.6, using repeatedly (4.12) and (4.6) we get by induction

$$u_k(x) \le A I_2 \omega(x), \tag{4.15}$$

where A depends on q and n, provided C_1 and C_2 are small enough.

Clearly, it follows from (4.14) that

$$|\nabla u_{k+1}| \le c(n)(aI_1|\nabla u_k|^{q_1} + bI_1 u_k^{q_2} + I_1\omega). \tag{4.16}$$

Arguing as above and applying (4.5) and (4.13) we get

$$|\nabla u_1| \le c(n)[aI_1(I_1\omega)^{q_1} + bI_1(I_2\omega)^{q_2} + I_1\omega] \le CI_1\omega.$$

Then again by induction we obtain

$$|\nabla u_k(x)| \le AI_1\omega(x), \tag{4.17}$$

where A depends on q and n, provided C_1 and C_2 are chosen small enough. Using (4.15) and (4.17), we proceed as in the case of the Riccati equation to obtain simultaneously the inequalities

$$|\nabla u_{k+1}(x) - \nabla u_k(x)| \le B\delta^k I_1\omega(x) \tag{4.18}$$

and

$$|u_{k+1}(x) - u_k(x)| \le B\delta^k\, I_2\omega(x), \tag{4.19}$$

where the constants A, B, and $0<\delta<1$ depend only on q and n. By the same argument as in the proof of Theorem 2.4, the preceding estimates yield that

$$u(x) = u_0(x) + \sum_{k=0}^{\infty}(u_{k+1}(x) - u_k(x))$$

is a solution of (4.2), and the estimates (4.7) hold. $\square$

It follows from the known estimates of Green functions of uniformly elliptic differential operators L mentioned above (see [A2], [HS], [GW], [W]) that our main results remain true for equations of the type (4.1) with L in place of the Laplacian.

References

[AH] ADAMS, D. R. and HEDBERG, L. I., *Function Spaces and Potential Theory,* Springer-Verlag, Berlin–Heidelberg, 1996.

[AP] ADAMS, D. R. and PIERRE, M., Capacitary strong type estimates in semilinear problems, *Ann. Inst. Fourier* (*Grenoble*) **41**:1 (1991), 117–135.

[Ag] AGMON, S., On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds, in *Methods of Functional Analysis and Theory of Elliptic Equations* (Greco, D., ed.), pp. 19–52, Liguori, Naples, 1983.

[A1] ANCONA, A., On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$, *J. London Math. Soc.* **34** (1986), 274–290.

[A2] ANCONA, A., First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains, *J. Anal. Math.* **72** (1997), 45–92.

[B] BARAS, P., Semilinear problem with convex nonlinearity, in *Recent Advances in Nonlinear Elliptic and Parabolic Problems* (Bénilan, P., Chipot, M., Evans, L. C. and Pierre, M., eds.), Pitman Research Notes in Math. Sciences **208**, pp. 202–215, Longman, Harlow, 1989.

[BP] BARAS, P. and PIERRE, M., Singularités éliminables pour des équations semi-linéaires, *Ann. Inst. Fourier* (*Grenoble*) **34**:1 (1984), 185–206.

[CZ] CHUNG, K. L. and ZHAO, Z., *From Brownian Motion to Schrödinger's Equation,* Springer-Verlag, Berlin, 1995.

[GW] GRÜTER, M. and WIDMAN, K.-O., The Green function for uniformly elliptic equations, *Manuscripta Math.* **37** (1982), 303–342.

[H] HANSSON, K., Imbedding theorems of Sobolev type in potential theory, *Math. Scand.* **45** (1979), 77–102.

[Ha] HARTMAN, P., *Ordinary Differential Equations,* Republ. 2nd ed., Birkhäuser, Boston, Mass., 1982.

[HK] HAYMAN, W. K. and KENNEDY, P. B., *Subharmonic Functions, Vol.* I, Academic Press, London–New York–San Francisco, 1976.

[He] HEDBERG, L. I., On certain convolution inequalities, *Proc. Amer. Math. Soc.* **36** (1972), 505–510.

[HS] HUEBER, H. and SIEVEKING, M., Uniform bounds for quotients of Green functions on $C^{1,1}$-domains, *Ann. Inst. Fourier* (*Grenoble*) **32**:1 (1982), 105–117.

[KV] KALTON, N. J. and VERBITSKY, I. E., Nonlinear equations and weighted norm inequalities, to appear in *Trans. Amer. Math. Soc.*

[L] LIONS, P. L., On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441–467.

[M1] MAZ'YA, V. G., On the theory of the n-dimensional Schrödinger operator, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1145–1172 (Russian).

[M2] MAZ'YA, V. G., *Sobolev Spaces*, Springer-Verlag, Berlin–Heidelberg–New York, 1985.

[MV] MAZ'YA, V. G. and VERBITSKY, I. E., Capacitary estimates for fractional integrals, with applications to partial differential equations and Sobolev multipliers, *Ark. Mat.* **33** (1995), 81–115.

[N] NYSTRÖM, K., Integrability of Green potentials in fractal domains, *Ark. Mat.* **34** (1996), 335–381.

[S] SCHECHTER, M., Hamiltonians for singular potentials, *Indiana Univ. Math. J.* **22** (1972), 483–503.

[Si] SIMON, B., Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.

[St] STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.

[VW] VERBITSKY, I. E. and WHEEDEN, R. L., Weighted inequalities for fractional integrals and applications to semilinear equations, *J. Funct. Anal.* **129** (1995), 221–241.

[W] WIDMAN, K.-O., Inequalities for the Green function and boundary continuity of the gradients of solutions of elliptic differential equations, *Math. Scand.* **21** (1967), 13–67.

[Z] ZHAO, Z., Green function for Schrödinger operator and conditioned Feynman–Kac gauge, *J. Math. Anal. Appl.* **116** (1986), 309–334.

Received June 5, 1997

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