Subalgebras of Orlicz spaces and related algebras of analytic functions

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1. Introduction

An Orlicz function φ is a real-valued function defined as $[0, \infty)$ satisfying the condition (a) φ is non-decreasing (b) $\varphi(0)=0$ and φ is continuous at 0 and (c) φ is not identically zero. In addition φ satisfies the Δ_2-condition at ∞ provided for some C and x

(1.0.1) $$\varphi(2x) \leqq C\varphi(x) \quad x \geqq X$$

or equivalently, for some C

(1.0.2) $$\varphi(2x) \leqq C\big(\varphi(x)+1\big) \quad 0 \leqq x < \infty.$$

If φ satisfies the Δ_2-condition at ∞ then if (S, Σ, ν) in a finite measure space we may define the Orlicz space $L_\varphi = L_\varphi(S, \Sigma, \nu)$ to be the set of all complex-valued Σ-measurable functions f an S such that

$$\int_S \varphi(|f|)\, d\nu < \infty.$$

As usual in L_φ we identify two functions which differ only on a set of ν-measure zero. L_φ is then an F-space (complete metrizable topological vector space) if we take for a base of neighborhoods of 0 the sets $B(\varepsilon; r)$ $(\varepsilon>0, r>0)$ where $f \in B(\varepsilon, r)$ if and only if

$$\int_S \varphi(r|f|)\, d\nu \leqq \varepsilon.$$

In this topology $f_n \to 0$ if and only if $f_n \to 0$ in ν-measure and

$$\int_S \varphi(|f_n|)\, d\nu \to 0.$$

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If φ satisfies the condition $\varphi(x)>0$ if and only if $x>0$ then we need not insist that $f_n \to 0$ in ν-measure here and the sets $B(\varepsilon; 1)$ form a base for the topology. In fact it is always possible to replace φ by an equivalent function ψ (so that $L_\psi = L_\varphi$) with this property.

In this paper we wish to consider the special case when L_φ becomes an algebra (under pointwise multiplication); in this case we shall say that L_φ is an *Orlicz algebra.* If S is not a finite union of ν-atoms then it is not difficult to see that a necessary and sufficient condition for this to occur is that for some C, X

$$\varphi(x^2) \leqq C\varphi(x) \quad x \geqq X \tag{1.0.3}$$

or equivalently, for some C

$$\varphi(x^2) \leqq C(\varphi(x)+1) \quad 0 \leqq x < \infty. \tag{1.0.4}$$

Two typical examples are given by $\varphi(x)=x(1+x)^{-1}$ (corresponding to the algebra L_0 of all ν-measurable functions) and $\varphi(x)=\log_+ x$. It is easy to see that under condition (1.0.3) L_φ is an F-algebra, (i.e. multiplication is jointly continuous) and possesses an identity.

Let us observe at this point that (1.0.3) implies the existence of some $p>0$ and $A<\infty$ such that

$$\varphi(x^t) \leqq A(t^p+1)(\varphi(x)+1) \quad t \geqq 0, \quad x \geqq 0 \tag{1.0.5}$$

and hence that for some $A, B<\infty$

$$\varphi(x) \leqq A + B(\log_+ x)^p \quad x \geqq 0. \tag{1.0.6}$$

From (1.0.6) we can see that L_φ is in general non-locally convex. There has been very little study of Orlicz algebras. The special case of L_0 has been studied by Bunger [2], Peck [7] and Williamson [15].

Our aim in this paper is to study closed subalgebras (containing the identity) of an Orlicz algebra L_φ. If we take Σ_0 to be a sub-σ-algebra of Σ then $L_\varphi(S, \Sigma_0, \nu)$ is an example of a subalgebra of L_φ; we shall call such subalgebras *elementary.*

We can now state the basic problems of this paper; for this suppose (S, Σ, ν) has no atoms.

Problem 1. *For which Orlicz functions φ is it true that every closed subalgebra of $L_\varphi(S, \Sigma, \nu)$ is elementary?*

Problem 2. *For which Orlicz functions φ is it true that every closed self-adjoint subalgebra of $L_\varphi(S, \Sigma, \nu)$ is elementary?*

Here a subalgebra A is self-adjoint if $f \in A$ implies $\bar{f} \in A$. Problem 2 is in fact equivalent for Problem 1 for the *real* Orlicz space L_φ.

The answers to these problems do not depend on the measure space S, and one may take $S=(0, 1)$ with Lebesgue measure on the Borel sets. In fact we may reduce the problem to considering whether the sub-algebra generated by a single element f of L_φ is always elementary. This in turn depends only on the distribution of f, and enables us to restate Problem 1 and 2.

To do this we denote the polynomials on $\mathbf{C}$ by $\mathscr{P}$. If μ is a finite Borel measure on $\mathbf{C}$ then $\mathscr{P}\subset L_\varphi(\mu)$ provided

$$\int_{\mathbf{C}} \varphi(|z|)\, d\mu(z) < \infty. \tag{1.0.7}$$

We then denote by $A_\varphi(\mu)$ the closure of $\mathscr{P}$ in $L_\varphi(\mu)$. It is not difficult to see that $A_\varphi(\mu)$ is elementary if and only if $A_\varphi(\mu)=L_\varphi(\mu)$. Now we restate Problems 1 and 2

Problem 1′. *For which Orlicz functions φ does there exist a finite Borel measure on $\mathbf{C}$ satisfying* (1.0.7) *such that $A_\varphi(\mu)\neq L_\varphi(\mu)$?*

Problem 2′. *As 1′ except we require μ supported on $\mathbf{R}\subset\mathbf{C}$.*

Let us mention two examples. If we take $\varphi(x)=\log_+ x$ and take for μ normalized Haar measure on the unit circle $\Gamma\subset\mathbf{C}$ then $A_\varphi(\mu)$ can be identified with the *Hardy algebra* N^+ (cf. [11]) of all functions analytic unit disc Δ of bounded characteristic and satisfying

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\log_+|f(re^{i\theta})|\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log_+|f(e^{i\pi})|\,d\theta$$

(where $f(e^{i\theta})$ are the boundary values of f on Γ). This space has been extensively studied by Roberts and Stoll [9]) and Yanagihara [16], [17]. Thus if $\varphi(x)=\log_+ x$, $L_\varphi(S)$ possesses non-elementary subalgebras (clearly $N^+\neq L_\varphi(\mu)$, since it has continuous linear functionals).

On the other hand if we take $\varphi(x)=x/(1+x)$ the same construction only leads to $A_\varphi(\mu)=L_0(\mu)$ (as was shown to the author by Joel Shapiro). In fact a reasonably simple argument using Runge's theorem shows that $L_0(S)$ has no non-elementary closed sub-algebras. Williamson [15] shows that $L_0(0, 1)$ has a dense subalgebra which is a field.

Let us now say that a closed subset E of $\mathbf{C}$ is φ-elementary if whenever μ is a finite Borel measure supported on E, satisfying (1.0.7), we have $A_\varphi(\mu)=L_\varphi(\mu)$. We can now ask the broader question

Problem 3. *For a given set E characterize those φ such that E is φ-elementary.*

In this paper we investigate four special cases including $E=\mathbf{C}$ and $E=\mathbf{R}$ which correspond to Problems 1′ and 2′.

Our main results are as follows.

(1) $E=\Gamma$. Then E is φ-elementary if and only if

$$\liminf_{x\to\infty} \frac{\varphi(x)}{\log_+ x} = 0 \tag{1.0.8}$$

(2) $E=\bar{\Delta}$. We do not have the complete answer. We show that $\bar{\Delta}$ is φ-elementary if $\varphi(x)=\log_+ \log_+ x$, but not φ-elementary if $\varphi(x)=(\log_+ \log_+ x)^p$ where $p>2$. As $\bar{\Delta}$ is compact it is not difficult to show that if E is φ-elementary and $\psi(x) \leqq C(\varphi(x)+1)$ for all x then E is ψ-elementary. Hence E is not φ-elementary for $\varphi(x)=(\log_+ x)^p$ for any $p, 0<p<\infty$.

(3) $E=\mathbf{R}$. Again we do not have a complete characterization. We show that $\mathbf{R}$ is φ-elementary if φ is concave function of $\log_+ \log_+ x$ and

$$\sum_{n=1}^{\infty} \frac{\varphi(e^{[n]})}{\varphi(e^{[n+1]})} < \infty \tag{1.0.9}$$

where $e^{[1]}=e$ and $e^{[n]}=\exp(e^{[n-1]})$, $n \geqq 2$. On the other hand if

$$\int_0^{\infty} \frac{d\varphi(x)}{\varphi(e^x)} < \infty \tag{1.0.10}$$

then $\mathbf{R}$ is not φ-elementary. In particular if $\varphi(x)=\log_+ \ldots \log_+ x$ with any number of iterates then $\mathbf{R}$ is not φ-elementary. Thus for $\mathbf{R}$ to be φ-elementary φ must grow very slowly indeed; contrast the case $E=\bar{\Delta}$.

(4) $E=\mathbf{C}$. Again (1.0.10) is sufficient for C to be not φ-elementary; we also show that if for some $C, X<\infty$

$$\varphi(e^x) \leqq C\varphi(x) \quad x \geqq X \tag{1.0.11}$$

Then $\mathbf{C}$ is φ-elementary (and so, of course, every closed subalgebra of $L_\varphi(S)$ is elementary).

These results are given in Sections 3, 4 and 5 with applications to Orlicz algebras in Section 6. In Section 2 we develop some general results on $A_\varphi(\mu)$ and introduce the notion of an analytic algebra. We hope to continue the study of $A_\varphi(\mu)$ in a later paper.

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2. Subalgebras of Orlicz algebras

Suppose $L_\varphi(S, \Sigma, \nu)$ is an Orlicz algebra and that A is a closed subalgebra of L_φ containing 1. Then as we have seen in the introduction we call A elementary if for some sub-σ-algebra Σ_0 of Σ we have $A=L_\varphi(S,\Sigma_0,\nu)$. In addition we shall call A *analytic* if $\dim A>1$ and A has the property that if $p\in A$ and $p^2=p$ then either $p=0$ or $p=1$. Of course A cannot be both elementary and analytic.

If $f\in L_\varphi$ denote by $\operatorname{Alg}(f)$ the closed subalgebra generated by 1 and f. We shall say f is elementary or analytic according as $\operatorname{Alg}(f)$ is elementary or analytic. These properties only depend on the distribution of f i.e. the Borel measure μ an $\mathbf{C}$ given by

$$\mu(B) = \nu(f^{-1}(B)) \quad B\in\mathscr{B}$$

where $\mathscr{B}$ denotes the Borel sets of $\mathbf{C}$.

Thus we shall instead consider a Borel measure μ on $\mathbf{C}$ satisfying (1.0.7) and define $A_\varphi(\mu)$ to be elementary if $A_\varphi(\mu)=L_\varphi(\mu)$ and analytic if $\dim A_\varphi>1$ and if $p\in A_\varphi$ and $p^2=p$ then $p=0$ or 1. A_φ is elementary or analytic precisely as z is elementary or analytic in $L_\varphi(\mu)$.

We define the *spectrum* of A_φ, $\operatorname{Spec} A_\varphi$ to be the set of $\lambda\in\mathbf{C}$ such that for some (unique) continuous multiplicative linear functional $\theta\in A_\varphi^*$ we have

$$\theta(z) = \lambda$$

so that if $f\in\mathscr{P}$

$$\theta(f) = f(\lambda).$$

The following proposition is easy and we omit the proof.

Proposition 2.1. *If $A_\varphi(\mu)$ is elementary then* $\operatorname{Spec} A_\varphi$ *coincides with the set of atoms of μ and is at most countable.*

Proposition 2.2. *Let $D=\{z\colon |z-a|<r\}$ be an open disc in* $\mathbf{C}$. *Suppose D intersects* $\operatorname{Spec} A_\varphi(\mu)$ *in a set of planar measure* 0. *Then $1_D\in A_\psi(\mu)$ (where $1_D(z)=1$ if $z\in D$ and $1_D(z)=0$ if $z\notin D$).*

Proof. For $0<t<r$, let

$$C_t = \{\zeta\in\Gamma\colon\ a+t\zeta\in\operatorname{Spec} A_\psi(\mu)\}.$$

Then, by an application of Fubini's theorem, C_t has (Haar) m-measure 0 in Γ for almost every t, $0<t<r$.

Now we recall (1.0.6)

$$\varphi(x) \leqq A+B(\log_+ x)^p \quad x\geqq 0$$

for some A, B, p. Hence

$$\int_0^r \int_{\mathbf{C}} \varphi\left(\frac{1}{|t-|z-a||}\right) d\mu(z)\, dt$$
$$\leqq \int_{\mathbf{C}} \int_0^r A+B(\log_+ |t-|z-a||^{-1})^p\, dt\, d\mu(z)$$
$$< \infty$$

since the inner integral is bounded independent of $z\in\mathbf{C}$. Hence for almost every t, $0<t<r$ we have both that C_t is of measure 0 and

$$\int_{\mathbf{C}} \varphi\left(\frac{1}{|t-|z-a||}\right) d\mu(z) < \infty. \tag{2.2.1}$$

For such t we show $1_{D_t}\in A_\varphi$ where $D_t=\{z\colon |z-a|<t\}$. For each $n\in\mathbf{N}$ let ω be a primitive nth root of 1. Since $m(C_t)=0$

$$m(C_t\cup\omega C_t\cup\ldots\cup\omega^{n-1}C_t)=0$$

and so for some $\zeta=\zeta_n\in\Gamma$, we have $\omega^k\zeta\notin C_t$ for $1\leqq k\leqq n$.

For $1\leqq k\leqq n$,

$$\int_{\mathbf{C}} \varphi\left(\frac{1}{|z-a-t\omega^k\zeta|}\right) d\mu(z) \leqq \int_{\mathbf{C}} \varphi\left(\frac{1}{|t-|z-a||}\right) d\mu(z)$$
$$< \infty$$

so that $(z-a-t\omega^k\zeta)^{-1}\in L_\varphi$. However $a+t\omega^k\zeta\notin \operatorname{Spec} A_\varphi$ so that there exists a sequence $f_n\in\mathscr{P}$ with $f_n\to 1$ in L_φ but $f_n(a+t\omega^k\zeta)=0$. Thus $(z-a-t\omega^k\zeta)^{-1}f_n\in\mathscr{P}$ and $(z-a-t\omega^k\zeta)^{-1}f_n\to(z-a-t\omega^k\zeta)^{-1}\in A_\varphi$. Now if

$$h_n(z)=\prod_{k=1}^n \frac{\zeta t}{t\omega^k\zeta+a-z}=\frac{\zeta^n t^n}{\zeta^n t^n-(z-a)^n}$$

then $h_n\in A_\varphi$.

If $z\in D_t$ then

$$|1-h_n(z)|=\frac{|z-a|^n}{|\zeta^n t^n-(z-a)^n|}\leqq\frac{|z-a|^n}{t^n-|z-a|^n}$$

so that $h_n(z)\to 1$ and

$$|1-h_n(z)|\leqq\left(1-\left(\frac{|z-a|}{t}\right)^n\right)^{-1}\leqq\left(1-\frac{|z-a|}{t}\right)^{-1}.$$

If $|z-a|>t$ then

$$|h_n(z)|\leqq\frac{t^n}{|z-a|^n-t^n}$$

so that $h_n(z) \to 0$ and

$$|h_n(z)| \leqq \left(\frac{|z-a|}{t} - 1\right)^{-1}.$$

From (2.2.1) we have $\mu\{z: |z-a|=t\}=0$ and so $h_n \to 1_{D_t}$ μ-a.e. and

$$\varphi(|h_n(z) - 1_{D_t}(z)|) \leqq \varphi\left(\frac{t}{|t-|z-a||}\right) \quad \mu\text{-a.e.}$$

By the Dominated Convergence Theorem, $h_n \to 1_{D_t}$ and so $1_{D_t} \in A_\varphi$.

Now we can find $t_n \to r$ with $1_{D_{t_n}} \in A_\varphi$ and so $1_D \in A_\varphi$.

Before our next theorem we remark that $\operatorname{Spec} A_\varphi$ is a Borel set, indeed an F_σ-set. To see this let V_n be a base of closed neighborhoods of 0 in A_φ and let $E_n = \{\lambda \in \mathbf{C};\ |f(\lambda)| \leqq 1 \text{ for } f \in \mathscr{P} \cap V_n\}$. Then each E_n is closed in $\mathbf{C}$ and $\cup E_n = \operatorname{Spec} A_\varphi$.

Theorem 2.3. *The following conditions are equivalent:*

(i) $A_\varphi(\mu)$ *is non-elementary*

(ii) $\operatorname{Spec} A_\varphi(\mu)$ *has positive planar measure*

(iii) $\operatorname{Spec} A_\varphi(\mu)$ *is uncountable*

(iv) *There is a Borel set B with $\mu(B) > 0$ such that $A_\varphi(\mu|B)$ is analytic.*

Proof. We shall denote by $\mathscr{B}_0$ the set of all Borel subsets of $\mathbf{C}$ with $1_B \in A_\varphi(\mu)$. Then $\mathscr{B}_0$ is a sub-σ-algebra of $\mathscr{B}$ and contains all μ-null sets, and clearly $L_\varphi(\mathscr{B}_0;\ \mu) \subset A_\varphi(\mu)$.

(i)$\Rightarrow$(ii): If $\operatorname{Spec} A_\varphi(\mu)$ has measure zero, then by Proposition 2.2, $\mathscr{B}_0$ contains all open discs and so $\mathscr{B}_0 = \mathscr{B}$. This implies $A_\varphi = L_\varphi$.

(ii)$\Rightarrow$(iii): Immediate.

(iii)$\Rightarrow$(iv): We can find $\lambda \in \operatorname{Spec} A_\varphi(\mu)$ which is not a μ-atom. We define

$$\theta(f) = f(\lambda) \quad f \in \mathscr{P}$$

and we also denote by θ the unique continuous extension of θ to A_φ. Then θ is continuous on $L_\varphi(\mathscr{B}_0;\ \mu)$ and is a multiplicative linear functional. Hence there is an atom B of $\mathscr{B}_0$ such that if $C \in \mathscr{B}_0$

$$\begin{aligned} \theta(1_C) &= 1 \quad \text{if} \quad C \supset B \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We shall show that $A_\varphi(\mu|B)$ is analytic. First suppose $\dim A_\varphi(\mu|B) = 1$. Then z is constant μ-a.e. on B so that there exists $\lambda_1 \in B$ such that $\mu\{\lambda_1\} = \mu(B)$. Now if $f \in \mathscr{P}$, $f 1_B = f(\lambda_1) 1_B$ and so $\theta(f) = \theta(f 1_B) = f(\lambda_1)$. Hence $\lambda_1 = \lambda$ and we have contradicted our assumption.

Next suppose $1_A \in A_\varphi(\mu|B)$ is an idempotent and suppose $f_n \in \mathscr{P}$ and $f_n \to 1_A$ in $A_\varphi(\mu|B)$. Then $f_n 1_B$ converges in $A_\varphi(\mu)$ to $1_{A\cap B}$ and so $A \cap B \in \mathscr{B}_0$. Hence either $\mu(A\cap B) = \mu(B)$ or $\mu(A \cap B) = 0$, so that $1_A = 0$ or $1_A = 1$ in $A_\varphi(\mu|B)$.

(iv)$\Rightarrow$(i): Suppose $A_\varphi(\mu|B)$ is analytic; then B is not a μ-atom. Choose $C\in\mathscr{B}$ with $0<\mu(C)<\mu(B)$. Then $1_C\in L_\psi(\mu)$, but $1_C\notin A_\varphi(\mu)$ since if $f_n\in\mathscr{P}$ and $f_n\to 1_C$ then $f_n\to 1_C$ in $A_\varphi(\mu|B)$.

Let us call a subset E of $\operatorname{Spec} A_\varphi$ *equicontinuous* if the evaluations $f\to f(\lambda)$ are equicontinuous for $\lambda\in E$; of course $\operatorname{Spec} A_\varphi$ is an increasing union of equicontinuous sets, and equicontinuous sets are necessarily bounded.

If $f\in A_\varphi$ then there is a sequence $g_n\in\mathscr{P}$ such that $g_n\to f$ in A_φ and pointwise μ-a.e. Hence if for $\lambda\in\operatorname{Spec} A_\varphi$ we denote by θ_λ the corresponding multiplicative linear functional on A_φ we have

$$\theta_\lambda(f)=f(\lambda)\quad \mu\text{-a.e.}\quad \lambda\in\operatorname{Spec} A_\varphi.$$

Hence by choosing a representative suitably from the equivalence class of f we may suppose

$$\theta_\lambda(f)=f(\lambda)\quad \lambda\in\operatorname{Spec} A_\varphi.$$

We shall make this assumption in the future.

It now follows that each $f\in A_\varphi$ is a uniform limit of polynomials on equicontinuous subsets of $\operatorname{Spec} A_\varphi$, and is hence continuous on such sets.

Theorem 2.4. *Suppose $A_\varphi(\mu)$ is analytic. The μ is supported on $\overline{\operatorname{Spec} A_\varphi}$.*

Proof. Suppose $\lambda\notin\overline{\operatorname{Spec} A_\varphi}$, and let D_r be the open disc of radius r and centre λ. For small enough r, $D_r\cap\operatorname{Spec} A_\varphi=\emptyset$ and so by 2.2, $1_{D_r}\in A_\varphi$. Hence either $1_{D_r}=0$ or $1_{D_r}=1$ in $A_\varphi(\mu)$.

If for all $r>0$ $1_{D_r}=1$, then $\mu\{\lambda\}=\mu(\mathbf{C})$ and so $\dim A_\varphi=1$ contradicting the analyticity of A_φ. Thus for some $r>0$, $1_{D_r}=0$ i.e. $\mu(D_r)=0$ and $\lambda\notin\operatorname{supp}\lambda$.

Theorem 2.5. *Suppose $A_\varphi(\mu)$ is analytic and $E\subset\operatorname{Spec} A_\varphi$ is closed equicontinuous set. Then $A_\varphi(\mu)\cong A_\varphi(\mu|\mathbf{C}\setminus E)$ and so $A_\varphi(\mu|\mathbf{C}\setminus E)$ is also analytic.*

Proof. We suppose $\varphi(x)>0$ for $x>0$. Then there exists $\varepsilon>0$ such that if

$$\int_{\mathbf{C}}\varphi(|f|)\,d\mu\leqq\varepsilon$$

$$\sup_{z\in E}|f(z)|\leqq 1. \tag{2.5.1}$$

We shall show that on $\mathscr{P}$, $A_\varphi(\mu)$ and $A_\varphi(\mu|\mathbf{C}\setminus E)$ induce the same topology. Suppose, on the contrary, that the $A_\varphi(\mu|\mathbf{C}\setminus E)$ topology is weaker. Then there is a sequence $f_n\in\mathscr{P}$ such that

$$\int_{\mathbf{C}\setminus E}\varphi(|f_n|)\,d\mu\to 0 \tag{2.5.2}$$

but

$$\int_{\mathbf{C}}\varphi(|f_n|)\,d\mu=\delta \tag{2.5.3}$$

where $0<\delta\leqq\varepsilon$. It may further be supposed that if ϱ is any F-norm on $A_\varphi(\mu|\mathbf{C}\setminus E)$ inducing the topology that $\varrho(f_n)\leqq 2^{-n}$.

It will be enough to show $f_n(z)\to 0$ for any $z\in E$. Indeed if so then by (2.5.1) and the Bounded Convergence Theorem

$$\int_E \varphi(|f_n|)\,d\mu\to 0$$

and this leads with (2.5.2) and (2.5.3) to a contradiction.

Suppose then that for some $\lambda\in E$, $f_n(\lambda)\not\to 0$. Then we may suppose by selecting a subsequence that $f_n(\lambda)\to\alpha\neq 0$, where $|\alpha|\leqq 1$.

Since $\{f_n\}$ is uniformly bounded by 1 an E, f_n has a weak limit point g in $L_2(E,\mu)$ and there is a sequence h_n of convex combinations $h_n\in\mathrm{Co}\{f_n, f_{n+1},\ldots\}$ such that

$$h_n(z)\to g(z)\quad \mu\text{-a.e.}\quad z\in E.$$

Now $\varrho(h_n)\leqq 2\cdot 2^{-n}$ so that

$$\int_{\mathbf{C}\setminus E}\varphi(|h_n|)\,d\mu\to 0$$

and

$$\int_E \varphi(|g-h_n|)\,d\mu\to 0.$$

Thus h_n converges in $A_\varphi(\mu)$ to a function G where

$$G(z)=g(z)\quad \mu\text{-a.e.}\quad z\in E$$

$$G(z)=0\quad z\notin E$$

and of course $G(\lambda)=\alpha$.

Now let $A=\{f\in A_\varphi\colon f|_{\mathbf{C}\setminus E}=0\ \ \mu\text{-a.e.}\}$. Then A is a closed subspace of $L_\varphi(\mu)$ contained in $L_\infty(\mu)$. Hence A is also closed in $L_2(\mu)$ and by a theorem of Grothendieck [4], $\dim A<\infty$. We shall show that $\dim A=1$ and $A=A_\varphi$ thus reaching a contradiction. Suppose $H\in A$; then $H^n\in A$ for all n and so H satisfies some polynomial equation. Let p be the polynomial of minimal degree such that $p(H)=0$. Then if p has two non-trivial co-prime factors p_1 and p_2 we can find polynomials v_1 and v_2 such that

$$v_1(z)p_1(z)+v_2(z)p_2(z)\equiv 1$$

and so

$$1=v_1(H)p_1(H)+v_2(H)p_2(H).$$

Also $v_1(H)p_1(H)$ and $v_2(H)p_2(H)$ are idempotents so that we may suppose $v_1(H)p_1(H)=1$ and $v_2(H)p_2(H)=0$. Then $p_2(H)=v_1(H)p_1(H)p_2(H)=0$ and this contradicts the minimality of p. We conclude $p(z)=c(z-w)^m$ for some c, $w\in\mathbf{C}$ and $m\in\mathbf{N}$. Thus

$$(H-w)^m=0$$

and so $H=w$ is a constant. Since A is non-trivial ($G \in A$), we have $A=\mathbf{C}1$ and $G=\alpha 1$; thus $\mu(\mathbf{C}\setminus E)=0$, and $A=A_\varphi$ and we have a contradiction.

For our final theorem of this section we define the convolution $\mu * \nu$ of two finite Borel measures on $\mathbf{C}$ by

$$\int_{\mathbf{C}} f(z)\, d\mu * \nu(z) = \int_{\mathbf{C}} \int_{\mathbf{C}} f(uv)\, d\mu(u)\, d\nu(v)$$

for f continuous and of compact support. If $\mu \geqq 0$, $\nu \geqq 0$ this equality extends to positive Borel functions f with both sides possibly infinite.

Theorem 2.6. *Suppose $A_\varphi(\mu)$ is analytic and ν is a finite positive Borel measure such that*

$$\int_{\mathbf{C}} \varphi(|z|)\, d\nu(z) < \infty.$$

Then if supp $\nu\setminus\{0\}$ is connected, $A_\varphi(\mu * \nu)$ is analytic.

Proof. We shall suppose $\varphi(x)>0$ whenever $x>0$ for convenience. First observe

$$\begin{aligned}\int \varphi(|z|)\, d\mu * \nu(z) &= \int_{\mathbf{C}} \int_{\mathbf{C}} \varphi(|uv|)\, d\mu(u)\, d\nu(v) \\ &\leqq C \int_{\mathbf{C}} \int_{\mathbf{C}} (\varphi(|u|)+\varphi(|v|)+1)\, d\mu(u)\, d\nu(v) \\ &< \infty\end{aligned}$$

since $\varphi(uv) \leqq C(\varphi(u)+\varphi(v)+1)$ for some constant C. Thus $A_\varphi(\mu * \nu)$ is well-defined.

Since Spec A_φ has positive planar measure there exists ε, $0<\varepsilon<1$ such that if $|z-1|<\varepsilon$, z Spec $A_\varphi \cap$ Spec $A_\varphi \neq \emptyset$.

Now let us suppose B is a Borel set and $1_B \in A_\varphi(\mu * \nu)$; we shall show that either $1_B=1$ or $1_B=0$. There is a sequence $f_n \in \mathscr{P}$ with

$$\int_{\mathbf{C}} \Big(\int_{\mathbf{C}} \varphi(|1_B(uv)-f_n(uv)|)\, d\mu(u) \Big)\, d\nu(v) \to 0.$$

By passing to a subsequence we may suppose that for some Borel set F with $\nu(\mathbf{C}\setminus F)=0$, we have

$$\int_{\mathbf{C}} \varphi(|1_B(uv)-f_n(uv)|)\, d\mu(u) \to 0 \quad v \in F.$$

For $v \in F$, $f_n(uv)$ converges to an idempotent $e(v)=0$ or 1 in $A_\varphi(\mu)$. For $z \in$ Spec A_φ,

$$\lim_{n\to\infty} f_n(zv) = g(v) \quad v \in F$$

where $g(v) \in \{0, 1\}$. Let $F_0=\{v \in F\colon\ g(v)=0\}$ and $F_1=\{v \in F\colon\ g(v)=1\}$. Then $\bar F_0 \cup \bar F_1 \supset$ supp ν; if both F_0 and F_1 are non-empty there exists a non-zero $\lambda \in \bar F_0 \cap \bar F_1$. Pick $\lambda_0 \in F_0$ and $\lambda_1 \in F_1$ with

$$\left| \frac{\lambda_i}{\lambda} - 1 \right| \leqq \frac{\varepsilon}{3} \quad i = 0, 1.$$

Then $\left|\frac{\lambda_1}{\lambda_0}-1\right|<\varepsilon$ and so there exists $z\in\operatorname{Spec} A_\varphi\cap\lambda_1\lambda_0^{-1}\operatorname{Spec} A_\varphi$. Thus

$$\lim_{n\to\infty} f_n(\lambda_0 z) = g(\lambda_0) = 0.$$

But also

$$\lim_{n\to\infty} f_n(\lambda_1(\lambda_0\lambda_1^{-1}z)) = g(\lambda_1) = 1.$$

This contradiction shows either $F_0=\emptyset$ or $F_1=\emptyset$. If $F_1=\emptyset$ then $e(v)=0$ in A_φ (since it is an idempotent and A_φ is analytic), for all $v\in F$ and hence $1_B=0$ in $A_\varphi(\mu*v)$; if $F_0=\emptyset$ equally $e(v)=1$ for all $v\in F$ and hence $1_B=1$ in $A_\varphi(\mu*v)$.

Corollary 2.6. (i) *If* $\mathbf{C}$ *is not* φ*-elementary there is a rotation invariant measure* μ *an* $\mathbf{C}$ *such that* $A_\varphi(\mu)$ *is analytic*

(ii) *If* $\bar{\Delta}$ *is not* φ*-elementary then* $A_\varphi(\sigma)$ *is analytic for planar measure* σ *on* $\bar{\Delta}$.

Proof. (i) Follows easily by convolving with Haar measure m on Γ.

(ii) Let μ be a measure on $\bar{\Delta}$ such that $A_\varphi(\mu)$ is analytic. Let l be linear measure on [0, 1]. Then $l*m*\mu$ is rotation invariant and $A_\varphi(l*m*\mu)$ is analytic. In polar co-ordinates

$$d(l*m*\mu)(z) = w(r)\,dr\,d\theta \quad r>0$$

where w is monotone decreasing. If we let $R=\inf\{s\colon w(s)=0\}$ then $A_\varphi(\bar{\mu})$ is analytic where

$$d\bar{\mu} = w\left(\frac{r}{R}\right) dr\,d\theta.$$

(since $A_\varphi(\bar{\mu})\cong A_\varphi(\mu)$). Now $\operatorname{supp}\bar{\mu}=\bar{\Delta}$ and so $\overline{\operatorname{Spec} A_\varphi(\bar{\mu})}\supset\bar{\Delta}$.

Now for any $r<1$ there exists $z_0\in\operatorname{Spec} A_\varphi(\bar{\mu})$ with $|z_0|>r$. Clearly by rotation invariance the set $(z_0 w\colon w\in\Gamma)$ is equicontinuous and by the maximum modulus principle for $f\in\mathscr{P}$

$$\max_{|z|\leqq r}|f(z)| \leqq \max_{|w|=1}|f(wz_0)|$$

so that $r\bar{\Delta}\subset\operatorname{Spec} A_\varphi(\bar{\mu})$ and is equicontinuous. In particular, $A_\varphi(\bar{\mu}|\mathbf{C}\setminus\frac{1}{2}\bar{\Delta})$ is analytic. As $w(r)\leqq w(\frac{1}{2}R)$ for $\frac{1}{2}R\leqq r\leqq R$,

$$\mu|\mathbf{C}\setminus\frac{1}{2}\bar{\Delta}\leqq 2w\left(\frac{R}{2}\right)\sigma$$

and hence $\operatorname{Spec} A_\varphi(\sigma)\supset\Delta$, and the sets $r\Delta$ $(0<r<1)$ are equicontinuous on $A_\varphi(\sigma)$. If $f\in A_\varphi(\sigma)$ then f is analytic on Δ and $A_\varphi(\sigma)$ contains no non-trivial idempotents.

3. $A_\varphi(\mu)$ for measures with bounded support

Suppose D is a compact subset of $\mathbf{C}$; we shall seek conditions on φ such that D is φ-elementary. If D is not φ-elementary then D supports a measure μ such that $A_\varphi(\mu)$ is analytic by the results of Section 2.

If D is nowhere dense and fails to separate the plane, Mergelyan's theorem shows that $C(D)\subset A_\varphi(\mu)$ for any measure μ and so $A_\varphi(\mu)$ is elementary (see Stout [12] p. 287).

Theorem 3.1. *Suppose D is a simple closed curve (i.e. D is homeomorphic to Γ). Then a necessary and sufficient condition that D be φ-elementary is that*

$$\liminf_{x\to\infty}\frac{\varphi(x)}{\log_+ x}=0. \tag{3.1.1}$$

Proof. If (3.1.1) fails to hold then for any measure μ on D the $L_\varphi(\mu)$-topology on $\mathscr{P}$ is stronger than that of $L_\psi(\mu)$ where $\psi(x)=\log_+ x$ (of course, since D is compact (1.0.7) is automatic for any Orlicz function). Hence $\operatorname{Spec} A_\varphi(\mu)\supset\operatorname{Spec} A_\psi(\mu)$ and it suffices to show that there exists μ so that $A_\psi(\mu)$ is non-elementary. Let Ω be the bounded component of $\mathbf{C}\setminus D$ and pick $w\in\Omega$; let μ be a harmonic measure for w, so that μ is supported on $\partial\Omega=D$. Then $w\in\operatorname{Spec} A_\psi(\mu)$ since

$$\log_+|f(w)|\leqq\int_D\log_+|f(z)|\,d\mu(z)\quad f\in\mathscr{P}.$$

However w is not an atom of μ and so $A_\psi(\mu)$ is non-elementary.

Conversely, if (3.1.1) holds suppose D supports a measure μ so that $A_\varphi(\mu)$ is analytic. We define an Orlicz function θ by

$$\begin{aligned}\theta(x)&=\varphi(e^x)\quad 1\leqq x<\infty\\ &=0\qquad\quad 0\leqq x<1.\end{aligned}$$

Then $\liminf_{x\to\infty}\dfrac{\theta(x)}{x}=0$ and so the *real* Orlicz space $L_{\theta,\mathbf{R}}(v)$ has trivial dual if v is a measure without atoms ([10], [13]).

We show first that μ has no atoms. Let $A(D)$ be the uniform algebra consisting of all uniform limits of polynomials in D. If $a\in D=\partial\Omega$ then a is a peak point for $A(D)$ ([12] p. 296) i.e. there exists $g\in A(D)$ with $g(a)=1$ and $|g(z)|<1$ for $z\in D$ with $z\neq a$. Then $g\in A_\varphi(\mu)$ and $g^n\to h$ in $A_\varphi(\mu)$ where $h(a)=1$ and $h(z)=0$ $z\neq a$, $z\in D$. Since h is an idempotent $h=0$ ($h=1$ implies $\dim A_\varphi(\mu)=1$) and so $\mu\{a\}=0$. Thus $L_{\theta,\mathbf{R}}(\mu)$ has trivial dual.

Now pick $w\in\operatorname{Spec} A_\varphi(\mu)$. Since $A(D)\subset A_\varphi(\mu)$, it is clear that $w\in\operatorname{Spec} A(D)=\bar\Omega$. By Walsh's theorem ([12] p. 285), $A(D)$ is a Dirichlet algebra i.e. $\operatorname{Re} A(D)$ is dense n $C_R(D)$. For $f\in\operatorname{Re} A(D)$ define

$$\beta(f)=\operatorname{Re} g(w)\quad\text{where}\quad\operatorname{Re} g=f\quad\text{on } D.$$

Then β is well-defined and

$$|\beta(f)| \leqq \|f\|_D$$

since $\operatorname{Re} g$ is harmonic; β is also a positive linear functional. We shall show β is continuous in the L_θ-topology and since $\operatorname{Re} A(D)$ is dense in L_θ this is a contradiction.

Suppose $f_n \in \operatorname{Re} A(D)$ and $f_n \to 0$ in $L_\theta(\mu)$. Then $e^{f_n} \to 1$ in μ-measure. Let $B_n = \{z \in D: f_n(z) \leqq 1\}$; then

$$1_{B_n} e^{f_n} \leqq e$$

and so by the Bounded Convergence Theorem

$$\int_{B_n} \varphi(e^{f_n})\, d\mu \to 0.$$

On $D - B_n$

$$\varphi(e^{f_n}) = \theta(f_n)$$

and hence $e^{f_n} \to 1$ in $L_{\varphi,\mathbf{R}}(\mu)$. Now suppose $g_n \in A(D)$ and $\operatorname{Re} g_n = f_n$ on D. Then on D

$$|e^{g_n}| = e^{f_n}$$

and so $|e^{g_n}| \to 1$ in $L_\varphi(\mu)$. Thus e^{g_n} is bounded in $A_\varphi(\mu)$ and so for some $M < \infty$

$$|e^{g_n(w)}| \leqq M \quad n \in \mathbf{N}$$

or

$$e^{\beta(f_n)} \leqq M \quad n \in \mathbf{N}$$

i.e.

$$\beta(f_n) \leqq \log M \quad n \in \mathbf{N}.$$

Thus β is bounded above on any null sequence and is continuous and we have reached our contradiction.

Lemma 3.2. *Suppose $\beta > 2$. Then there is a nondecreasing function G defined on $[0, 1]$ such that*

(1) $G(0) = 0$.

(2) *There is a decreasing sequence $\{a_n: n = 0, 1, 2, \dots\}$ with $a_0 = 1$ and such that G is constant on each interval $[a_n, a_{n-1})$.*

(3) $G(x) \leqq \frac{1}{2} \quad 0 \leqq x \leqq 1$

(4) *If $H(x) = \int_0^x G(t)\, dt$, then*

$$\lim_{x \to 0} \frac{G(x)}{H(x)} \left(\log \frac{1}{H(x)}\right)^{-\beta} = 0 \tag{3.2.1}$$

$$\int_0^1 \left(\frac{G(x)}{H(x)}\right)^2 \left(\log \frac{1}{H(x)}\right)^{-\beta} dx < \infty. \tag{3.2.2}$$

Proof. Define

$$F(x) = \exp(-x^{-\alpha}) \quad x > 0$$

where $\alpha(\beta-2)>1$. Then

$$F'(x) = \alpha x^{-(1+\alpha)} F(x)$$

$$F''(x) = \alpha x^{-(2+\alpha)} F(x)\big(\alpha x^{-\alpha}-(\alpha+1)\big).$$

Choose $\delta>0$ so that $F'(\delta)<\frac{1}{2}$ and $F''(x)>0$ for $0<x<\delta$. Then define $a_n\to 0$ so that

$$F'(a_n) = 2^{-n}F'(\delta) \quad n = 1, 2, 3, \dots$$

and $a_0=0$. Now define

$$G(x) = 2^{-n}F'(\delta) \quad \text{for} \quad a_n \leqq x < a_{n-1}$$

with $G(0)=0$. Then

$$\tfrac{1}{2}F'(x) \leqq G(x) \leqq F'(x) \quad 0 \leqq x \leqq \delta$$

$$\tfrac{1}{2}F(x) \leqq H(x) \leqq F(x) \qquad 0 \leqq x \leqq \delta.$$

Thus

$$\int_0^\delta \left(\frac{G(x)}{H(x)}\right)^2 \left(\log\frac{1}{H(x)}\right)^{-\beta} dx \leqq 4\int_0^\delta \alpha^2 x^{\alpha\beta-2(1+\alpha)}\,dx < \infty$$

while $G(x)\leqq\frac{1}{2}$, $H(x)\leqq\frac{1}{2}$ for all x so that

$$\int_0^1 \left(\frac{G(x)}{H(x)}\right)^2 \left(\log\frac{1}{H(x)}\right)^{-\beta} dx < \infty.$$

Also

$$\begin{aligned} \frac{G(x)}{H(x)}\left(\log\frac{1}{G(x)}\right)^{-\beta} &\leqq 4\alpha x^{-(1+\alpha)+\alpha\beta} \\ &\leqq 4\alpha x^{\alpha} \\ &\to 0 \quad \text{as} \quad x\to 0. \end{aligned}$$

Theorem 3.3. *Suppose $\beta>2$ and $\varphi(x)=(\log_+\log_+ x)^\beta$. Then there is a closed nowhere dense subset D of $\bar{\Delta}$ of planar measure zero and a finite positive measure supported on D so that $A_\varphi(\mu)$ is analytic.*

Proof. We shall simply show the existence of a measure μ so that $A_\varphi(\mu)$ is non-elementary. To do this define G as in Lemma 3.2 and ϱ be the Borel measure on [0, 1] so that

$$\varrho[0, x] = \int_0^x \frac{1}{H(r)}\left(\log\frac{1}{H(r)}\right)^{-\beta} dG(r).$$

Then ϱ is a finite measure since if $a>0$

$$\varrho[a,1]=\int_a^1 \frac{1}{H(r)}\left(\log\frac{1}{H(r)}\right)^{-\beta} dG(r)$$

$$=\left[\frac{G(r)}{H(r)}\left(\log\frac{1}{H(r)}\right)^{-\beta}\right]_a^1$$

$$+\int_a^1\left(\frac{G(r)}{H(r)}\right)^2\left[\left(\log\frac{1}{G(r)}\right)^{-\beta}-\beta\left(\log\frac{1}{G(r)}\right)^{-\beta-1}\right]dr$$

and letting $a\to 0$ we see from (3.2.1) and (3.2.2) that ϱ is finite. Also ϱ is supported on the set $\{0, a_n\colon n=0,1,2,\ldots\}$.

Now define the measure μ on $\bar{\Delta}$ by

$$\int_{\bar{\Delta}} F(z)\,d\mu(z)=\frac{1}{2\pi}\int_0^1\int_0^{2\pi} F\big((1-r)e^{i\theta}\big)\,d\theta\,d\varrho(r).$$

Then μ is supported on a countable union of circles and hence $\operatorname{supp}\mu=D$ satisfies the hypotheses of the theorem.

Also define ν on $\bar{\Delta}$ by

$$\int_{\bar{\Delta}} F(z)\,d\nu(z)=\frac{1}{2\pi}\int_0^1\int_0^{2\pi} F\big((1-r)e^{i\theta}\big)\,d\theta\,dG(r).$$

We shall first consider $A_{\log}(\nu)$ and show that this is non-elementary. Indeed if $w\in\Delta$ and $f\in\mathscr{P}$

$$\log_+|f(w)|\leqq\frac{1}{2\pi}\frac{r+|w|}{r-|w|}\int_0^{2\pi}\log_+|f(re^{i\theta})|\,d\theta\qquad |w|<r<1$$

and so for $0<t<1-|w|$

$$(1-|w|-t)\log_+|f(w)|\leqq\frac{1}{\pi}\int_0^{2\pi}\log_+\left|f\big((1-t)e^{i\theta}\big)\right|d\theta.$$

Integrating with respect to dG over $[0,1-|w|]$ we have

$$H(1-|w|)\log_+|f(w)|\leqq 2\int_{\bar{\Delta}}\log_+|f(z)|\,d\nu(z).$$

Thus if

$$\int_{\bar{\Delta}}\log_+|f(z)|\,d\nu(z)\leqq\tfrac{1}{2}\tag{3.3.1}$$

then

$$\log_+|f(w)|\leqq\frac{1}{H(1-|w|)}.$$

This shows $\operatorname{Spec} A_{\log}(\nu)\supset\Delta$. Next we show that the identity map $\mathscr{P}\to\mathscr{P}$ from $A_\varphi(\mu)$ to $A_{\log}(\nu)$ is continuous. Indeed, if it is not then there is a sequence

$f_n \to 0$ in $A_\varphi(\mu)$, bounded away from 0 in $A_{\log}(\nu)$ but each satisfying (3.3.1), since (3.3.1) defines a neighborhood of 0 in $A_{\log}(\nu)$.

Clearly $f_n \to 0$ in ν-measure. We shall show that

$$\int_{\bar{\Delta}} \log_+ |f_n(z)|\, d\nu(z) \to 0,$$

and this will give a contradiction.

First we observe that there is an $X < \infty$ such that if $x \geqq X$ and $e^e \leqq t \leqq x$

$$\frac{\log_+ t}{(\log_+ \log_+ t)^\beta} \leqq \frac{\log_+ x}{(\log_+ \log_+ x)^\beta}.$$

Choose $R > 0$ so that

$$\exp\left(\frac{1}{H(R)}\right) \geqq X.$$

Then

$$|f_n(re^{i\theta})| \leqq \exp\left(\frac{1}{H(R)}\right) \quad 0 \leqq r \leqq 1-R$$

and hence by the Dominated Convergence Theorem

$$\int_{|z| \leqq 1-R} \log_+ |f_n(z)|\, d\nu(z) \to 0.$$

Similarly if $B_n = \{z\colon\ |z| > 1-R,\ |f_n(z)| > e^e\}$ then

$$\int_{(1-R<|z|\leqq 1)\setminus B_n} \log_+ |f_n(z)|\, d\nu(z) \to 0.$$

Finally

$$\int_{B_n} \log_+ |f_n(z)|\, d\nu(z) \leqq \int_{B_n} \frac{1}{H(1-|z|)} \left(\log \frac{1}{H(1-|z|)}\right)^{-\beta} (\log_+ \log_+ |f_n(z)|)^\beta\, d\nu(z)$$

$$= \int_{B_n} (\log_+ \log_+ |f_n(z)|)^\beta\, d\mu(z) \to 0.$$

Thus we have a contradiction and so $\Delta \subset \operatorname{Spec} A_\varphi(\mu)$, and $A_\varphi(\mu)$ is non-elementary.

Theorem 3.4. *Let $\varphi(x) = \log_+ \log_+ x$. Then $\bar{\Delta}$ is φ-elementary.*

Proof. It suffices to show that $A_\varphi(\sigma)$ is not analytic where σ is planar measure on Δ. Indeed since $\frac{1}{2}\, \bar{\Delta}$ is then equicontinuous we may consider $A_\varphi\left(\sigma \,|\, \mathbf{C} \setminus \frac{1}{2} \bar{\Delta}\right)$ i.e. planar measure in the annulus $\frac{1}{2} \leqq |z| \leqq 1$, D say.

Therefore suppose $A_\varphi\left(\sigma \,|\mathbf{C} \setminus \frac{1}{2} \bar{\Delta}\right)$ is analytic. We start from an example of Polya and Szego ([8] pp. 115—116); cf. Hayman [5] p. 81). There is an entire function E such that for some constant M_0 we have

$$|E(z)-e^{e^z}| \leqq M_0 \quad \operatorname{Re} z \geqq 0 \quad |\operatorname{Im} z| \leqq \pi$$
$$|E(z)| \leqq M_0 \qquad \text{otherwise.}$$

Let $M_1 \geqq M_0$ be chosen so that

$$|E(z)| \leqq M_1 \quad |z| \leqq M.$$

For any $n \in \mathbf{N}$ and $0 \leqq \theta < 2\pi$ we define

$$f_{n,\theta}(z) = \frac{1}{n} E(e^{i\theta} E(nz)).$$

First observe that for any choice of θ_n, the sequence f_{n,θ_n} converges to 0 in σ-measure on the annulus D. Indeed if $B_n = \left\{z\colon |f_{n,\theta}(z)| > \frac{1}{n} M_1\right\}$ then for $z \in B_n$ we have $|E(nz)| > M$ and so nz belongs to the strip $\operatorname{Re} w \geqq 0$, $|\operatorname{Im} w| \leqq \pi$. Hence $|\operatorname{Im} z| \leqq \frac{\pi}{n}$ and clearly $\sigma(B_n) = O\left(\frac{1}{n}\right)$ independent of θ.

Next we shall show

$$\sup_{|z| \leqq 1/2} |f_n(z)| \to \infty$$

as $n \to \infty$ uniformly in θ. Indeed for any $y \geqq 0$

$$\left|\exp\left(\exp\left(n\left(\frac{1}{4} + tiy\right)\right)\right)\right| = \exp\left(e^{\frac{1}{4}n} \cos ny\right)$$

$$\operatorname{Arg}\left(\exp\left(\exp\left(n\left(\frac{1}{4} + iy\right)\right)\right)\right) = e^{\frac{1}{4}n} \sin ny \bmod 2\pi.$$

Hence there is a constant C independent of y and n so that

$$\left|\operatorname{Arg} E\left(n\left(\frac{1}{4} + iy\right)\right) - e^{\frac{1}{4}n} \sin ny\right| \leqq C \exp\left(-e^{\frac{1}{4}n} \cos ny\right)$$

for $0 \leqq y \leqq \frac{\pi}{n}$. Hence for large enough n given θ, there exists $y_n(\theta)$ with $0 \leqq y_n \leqq \frac{\pi}{4n} < \frac{1}{4}$ and

$$E\left(e^{i\theta} n\left(\frac{1}{4} + iy_n\right)\right) \in \mathbf{R}$$

and

$$E\left(e^{i\theta} n\left(\frac{1}{4} + iy_n\right)\right) \geqq e^{e^{\frac{1}{16}n}}.$$

Hence

$$\left|f_{n,\theta}\left(\frac{1}{4}+iy_n(\theta)\right)\right| \to \infty$$

uniformly in θ and so

$$\sup_{|z|\leqq 1/2} |f_{n,\theta}(z)| \to \infty.$$

As $\frac{1}{2}\bar{\Delta}$ is equicontinuous we must conclude that no sequence $f_{n_k,\theta_k}\to 0$ with $n_k\to\infty$. This implies that for some $\varepsilon>0$, and $N\in\mathbf{N}$,

$$\int_D \log_+ \log_+ |f_{n,\theta}(z)|\, d\sigma(z) \geqq \varepsilon$$

for $n\geqq N$ and $0\leqq\theta<2\pi$.

Thus

$$\int_0^{2\pi}\int_D \log_+ \log_+ |f_{n,\theta}(z)|\, d\sigma(z)\, d\theta \geqq 2\pi\varepsilon.$$

Suppose $n\geqq N$ and $n\geqq M_1 e^{-e}$. Then if $\log_+ \log_+ |f_{n,\theta}(z)|>0$ we have $|E(e^{i\theta}E(nz))|>M_1$. Let

$$G_n = \{(\theta, z)\colon\ |E(e^{i\theta}E(nz))| > M_1\}$$

so that

$$I_n = \int_{G_n} \log_+ \log_+ |E(e^{i\theta}E(nz))|\, d\sigma(z)\, d\theta \geqq 2\pi\varepsilon.$$

For $z\in D$ let $G_n(z)=\{\theta\colon (\theta, z)\in G_n\}$. Then

$$\int_{G_n(z)} d\theta \leqq \frac{C}{|E(nz)|}$$

where C is independent of n and z. Also

$$|E(e^{i\theta}(E(nz)))| \leqq M_0+e^{e|E(nz)|}$$

Hence

$$\begin{aligned} I_n &\leqq C\int_{|E(nz)|>M} \log_+ \log_+ (M_0+e^{e|E(nz)|})|E(nz)|^{-1}\, d\sigma(z) \\ &\leqq C'\int_{|E(nz)|>M_0} d\sigma(z) \end{aligned}$$

where C' is independent of n. Thus $I_n=O\left(\frac{1}{n}\right)$ and we have a contradiction.

Remarks. The author has been unable to decide whether $\bar{\Delta}$ is φ-elementary $\varphi(x)=(\log_+ \log_+ x)^\beta$ with $1<\beta\leqq 2$. Since we are dealing with a bounded set we may deduce that if φ grows faster than $(\log_+ \log_+ x)^\beta$ for $\beta>2$, then $\bar{\Delta}$ is not φ-elementary (e.g. $\varphi(x)=(\log_+ x)^p$ where $0<p<\infty$); equally if φ grows slower than $\log_+ \log_+ x$ then $\bar{\Delta}$ is φ-elementary.

4. Measures supported on unbounded sets

Suppose φ is an Orlicz function satisfying (1.0.3). Suppose also that μ is a finite positive measure supported on $\mathbf{R}_+$ whose support is unbounded and such that

$$\int \varphi(x)\, d\mu(x) < \infty. \tag{4.0.1}$$

Then we define $\Lambda_\varphi(\mu)$ to be the space of entire functions f such that

$$\int \varphi(M(f;\, r))\, d\mu(r) < \infty \tag{4.0.2}$$

where

$$M(f;\, r) = \max_{|z|=r} |f(z)|.$$

If we denote by $\mathscr{E}$ the space of entire functions (equipped with the topology of uniform convergence on compacta), then we may regard M as a map $M\colon \mathscr{E} \to L_0(\mu)$ defined by

$$M(f)(r) = M(f;r)$$

and M satisfies the conditions

$$M(f) \geqq 0 \quad f \in \mathscr{E} \tag{4.0.3}$$

$$M(f+g) \leqq M(f)+M(g) \quad f, g \in \mathscr{E} \tag{4.0.4}$$

$$M(\alpha f) = |\alpha|\, M(f) \quad f \in \mathscr{E}, \quad \alpha \in \mathbf{C}. \tag{4.0.5}$$

From (4.0.3)—(4.0.5) we can see that we may induce a metrizable vector topology on $\Lambda_\varphi(\mu)$ by taking as a base of neighborhoods of 0 sets of the form $M^{-1}(V)$ where V is a neighborhood of 0 in $L_\varphi(\mu)$.

Proposition 4.1. (i) *The inclusion map* $\Lambda_\varphi(\mu) \to \mathscr{E}$ *is continuous*
(ii) $\mathscr{P}$ *is dense in* $\Lambda_\varphi(\mu)$ *and hence* $\Lambda_\varphi(\mu)$ *is separable.*
(iii) $\Lambda_\varphi(\mu)$ *is complete and hence is an F-space.*

Proof. (i) Suppose $f_n \to 0$ is $\Lambda_\varphi(\mu)$, and that $R>0$. We claim $M(f_n;\, R) \to 0$. Indeed, if $M(f_n;\, R) \geqq \varepsilon$ then $M(f_n;\, r) \geqq \varepsilon$ for $r \geqq R$ and so $\mu\{r\colon M(f_n;\, r) \geqq \varepsilon\} \geqq \mu[R, \infty)$ as $M(f_n;\, r) \to 0$ in μ-measure we see that $M(f_n;\, R) \to 0$. Hence $f_n \to 0$ in $\mathscr{E}$.

(ii) If $f \in \Lambda_\varphi(\mu)$ has Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

then we define

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

and

$$\sigma_N(z) = \frac{1}{N}(S_1(z) + \dots + S_N(z)).$$

Then $\sigma_n \in \mathscr{P}$ and

$$M(\sigma_n; r) \leqq 2M(f; r) \quad n = 1, 2, \dots$$
$$M(f-\sigma_n; r) \to 0 \qquad \text{pointwise.}$$

Hence by the Dominated Convergence Theorem $M(f-\sigma_n;\ r) \to 0$ in $L_\varphi(\mu)$ i.e. $\sigma_n \to f$ in $\Lambda_\varphi(\mu)$.

(iii) If f_n is Cauchy in $\Lambda_\varphi(\mu)$, then f_n converges to some f in $\mathscr{E}$. Now if $n \in \mathbf{N}$

$$M(f-f_n;\ r) = \lim_{m\to\infty} M(f_m - f_n;\ r)$$

and hence, bearing in mind that φ need not be continuous,

$$\varphi\big(M(f-f_n;\ r)\big) \leqq \liminf_{m\to 0} \varphi\big(2M(f_m-f_n;\ r)\big).$$

By Fatou's lemma

$$\int \varphi\big(M(f-f_n;\ r)\big)\, d\mu(r) \leqq \liminf_{m\to\infty} \int \varphi\big(2M(f_m-f_n;\ r)\big)\, d\mu(r)$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

Thus $M(f-f_n;\ r) \to 0$ in $L_\varphi(\mu)$ and we see $f \in \Lambda_\varphi(\mu)$.

We can now give our first result which is a criterion for $\mathbf{C}$ to be φ-elementary.

Theorem 4.2. *Suppose that for some* $C < \infty$

$$\varphi(e^x) \leqq C\big(\varphi(x)+1\big) \quad 0 \leqq x < \infty. \tag{4.2.1}$$

Then $\mathbf{C}$ *is* φ*-elementary.*

Proof. We shall suppose on the contrary that $\mathbf{C}$ supports a measure μ so that (1.0.7) holds and $\Lambda_\varphi(\mu)$ is analytic. From Corollary 2.6 we may suppose μ is rotation invariant so that

$$d\mu = d\nu(r)\, \frac{d\theta}{2\pi}$$

for some measure ν supported on $\mathbf{R}_+$. Since $\varphi(x) = O\,(\log_+ \log_+ x)$ it is clear that μ has unbounded support; otherwise $\bar{\Delta}$ would not be φ-elementary. Hence ν has unbounded support.

We use the same function E as in Theorem 3.4. We claim that for any $n \in N$, $E_n \in \Lambda_\varphi(\mu)$, where $E_n(z) = E(nz)$

$$M(E_n; r) \leqq M_0 + e^{e^{nr}}$$

and hence

$$\varphi\big(M(E_n; r)\big) \leqq A\big(\varphi(e^{e^{nr}})+1\big)$$

for constant A. However

$$\varphi(e^{e^{nr}}) \leqq C\big(\varphi(e^{nr})+1\big)$$
$$\leqq C^2\varphi(nr) + C + 1$$

and hence

$$\int \varphi(M(E_n;\, r))\, dv(r) < \infty$$

i.e. $E_n \in A_\varphi(v)$. Thus there is a sequence $\{f_m\}$ of polynomials with $f_m \to E_n$ in $A_\varphi(v)$ i.e. $f_m(z) \to E_n(z)$ pointwise and

$$\int \varphi(M(f_m - E_n;\, r))\, dv(r) \to 0 \quad \text{as} \quad n \to \infty.$$

Now

$$\int_0^\infty \int_0^{2\pi} \varphi(|f_m(re^{i\theta}) - E_n(re^{i\theta})|) \frac{d\theta}{2\pi}\, dv(r) \to 0$$

i.e. $f_m \to E_n$ in $A_\varphi(\mu)$.

Next we claim $\frac{1}{n} E_n \to 0$ in $A_\varphi(\mu)$. Indeed $\frac{1}{n} E_n(z) \to 0$ unless $z \in \mathbf{R}_+$, so that $\frac{1}{n} E_n(z) \to 0$ in μ-measure.

Now for constant B independent of r, we have $|E(re^{i\theta})| \leqq M_0$ except on a set of θ of measure at most $B(r+1)^{-1}$. Thus

$$\int_0^{2\pi} \varphi\left(\frac{1}{n} |E_n(re^{i\theta})|\right) d\theta \leqq \frac{B}{1+nr} \varphi\left(\frac{1}{n} (e^{e^{nr}} + M_0)\right) + \varphi\left(\frac{M_0}{n}\right)$$

$$\leqq \frac{B'}{1+nr} (\varphi(e^{e^{nr}}) + 1) + \varphi\left(\frac{M_0}{n}\right)$$

where B' is again independent of r and n. Thus

$$\int_0^{2\pi} \varphi\left(\frac{1}{n} |E_n(re^{i\theta})|\right) d\theta \leqq \frac{B'C^2 \varphi(nr)}{1+nr} + \frac{B'}{1+nr} (C+2) + \varphi\left(\frac{M_0}{n}\right).$$

The right-hand side is uniformly bounded in r and tends to 0 pointwise. We conclude

$$\int_0^\infty \int_0^{2\pi} \varphi\left(\frac{1}{n} |E_n(re^{i\theta})|\right) \frac{d\theta}{2\pi}\, dv(r) \to 0$$

i.e. $\frac{1}{n} E_n \to 0$ in $A_\varphi(\mu)$.

However $A_\varphi(\mu)$ is analytic and $\operatorname{Spec} A_\varphi(\mu)$ is rotation invariant. Hence there exists $\alpha \in \operatorname{Spec} A_\varphi(\mu)$ with $\alpha > 0$. Thus

$$\frac{1}{n} E(\alpha n) \to 0$$

and hence

$$\frac{1}{n} e^{e^{\alpha n}} \to 0.$$

This contradiction proves the theorem.

The only examples where we know where $\mathbf{C}$ is not φ-elementary have the property that $\mathbf{R}$ is also not φ-elementary. We now proceed to study this case.

Proposition 4.3. *Suppose μ is a finite positive measure supported on $\mathbf{R}$ and that*

$$\inf_n \int \varphi\left(\frac{|x|^n}{n!}\right) d\mu(x) = 0. \tag{4.3.1}$$

Then $A_\varphi(\mu)$ is elementary.

Proof. If for any n we have

$$\int \varphi\left(\frac{|x|^n}{n!}\right) d\mu(x) = 0.$$

Then μ has bounded support and by the Stone—Weierstrass Theorem $\mathscr{P}$ is dense in $C(\operatorname{supp}\mu)$ and hence $A_\varphi(\mu)$ is elementary. Otherwise we may suppose that for some sequence $n_k \to \infty$

$$\int \varphi\left(\frac{|x|^{n_k}}{n_k!}\right) d\mu(x) \to 0.$$

Now for $0 \leqq \alpha \leqq 1$ consider

$$S_k(x) = e^{i\alpha x} - \left(1 + i\alpha x + \frac{(i\alpha x)^2}{2!} + \dots + \frac{(i\alpha x)^{n_k-1}}{(n_k-1)!}\right).$$

By applying Taylor's theorem to the real and imaginary parts of S_k separately we see

$$|S_k(x)| \leqq \frac{2|x|^{n_k}}{(n_k)!} \quad x \in \mathbf{R}$$

and hence $S_k \to 0$ in $L_\varphi(\mu)$. Thus $e^{i\alpha x} \in A_\varphi$ for $0 \leqq \alpha \leqq 1$ and hence for all α.

Now suppose f is bounded and continuous on $\mathbf{R}$, and that $n \in \mathbf{N}$. Then there is a linear combination g_n of functions of the form $e^{imx/n}$ (with $m \in N$) such that

$$|f(x) - g_n(x)| \leqq \frac{1}{n} \quad |x| \leqq n\pi.$$

If $\sup |f(x)| = \|f\|_\infty$ then $|g_n(x)| \leqq \|f\|_\infty + \frac{1}{n}$ for all x. Thus

$$\int \varphi(|f - g_n|)\, d\mu(x) \leqq \varphi\left(\frac{1}{n}\right) \mu[-n\pi, n\pi] + \varphi\left(2\|f\|_\infty + \frac{1}{n}\right) \mu\{x\colon |x| > n\pi\}$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Hence $f \in A_\varphi(\mu)$ and $A_\varphi(\mu) = L_\varphi(\mu)$.

Our next result shows how to construct analytic algebras in R and is a partial converse to the preceding proposition.

Theorem 4.4. *Suppose μ is a measure supported on $\mathbf{R}_+$ such that $d\mu(x)=w(x)\,\dfrac{dx}{x}$ where*

(4.4.1) $w(x)=0 \quad 0\leqq x<1.$

(4.4.2) $w(1)>0$ *and w is monotone decreasing for* $1\leqq x<\infty$.

(4.4.3) *For some constant* $c>0$

$$w(x^2)\geqq cw(x) \quad 1\leqq x<\infty$$

(4.4.4)
$$\int_1^\infty \varphi(x)w(x)\frac{dx}{x}<\infty$$

(4.4.5)
$$\int_1^\infty \varphi\left(\frac{x^n}{n!}\right)w(x)\frac{dx}{x}\geqq \varepsilon>0 \quad n=1,2,3\,\ldots.$$

Then $A_\varphi(\mu)$ is analytic.

Proof. We shall show that $A_\varphi(\mu)=\Lambda_\varphi(\mu)$ and this will show that $A_\varphi(\mu)$ consists of functions which are entire and hence $A_\varphi(\mu)$ is analytic.

Step 1. Suppose $f\in\Lambda_\varphi(\mu)$ and

$$g(z)=f(z^2).$$

Then

$$M(g;r)=M(f;r^2)$$

and

$$\int_0^\infty \varphi(M(g;r))\frac{w(r)}{r}\,dr=\int_0^\infty \varphi(M(f;r^2))\frac{w(r)}{r}\,dr$$

$$=\int_0^\infty \varphi(M(f;r))w(\sqrt{r})\frac{dr}{2r}$$

$$\leqq \frac{1}{2c}\int_0^\infty \varphi(M(f;r)w(r)\frac{dr}{r}<\infty.$$

Hence $g\in\Lambda_\varphi(\mu)$.

Step 2. Suppose $f\in\Lambda_\varphi(\mu)$ and $f(z)=\sum_{n=0}^\infty a_n z^n$. Then $a_n x^n\to 0$ pointwise and $|a_n|x^n\leqq M(f;x)$. Hence by the Dominated Convergence Theorem

$$\int_0^\infty \varphi(|a_n|x^n)w(x)\frac{dx}{x}\to 0.$$

It follows that $|a_n|\leqq(n!)^{-1}$ eventually and so

(4.4.6)
$$\sup_{z\in\mathbf{C}}|f(z)|e^{-|z|}<\infty.$$

Step 3. From (4.4.6) and Step 1 we deduce

$$\sup_{z\in\mathbf{C}} |f(z^{2^n})|e^{-|z|} < \infty \quad n\in\mathbf{N}$$

so that

$$\|f\|_\alpha = \sup_{z\in\mathbf{C}} e^{-|z|^\alpha} |f(z)| < \infty \quad \alpha > 0.$$

It follows that the norms $f\to\|f\|_\alpha$ are continuous on $\Lambda_\varphi(\mu)$ for $\alpha>0$.

Our aim will be to show that on $\mathscr{P}$ the $\Lambda_\varphi(\mu)$-topology and the $A_\varphi(\mu)$-topology agree, and hence that $\Lambda_\varphi(\mu)=A_\varphi(\mu)$. It is trivial that the $\Lambda_\varphi(\mu)$ topology is stronger than the A_φ-topology.

If it is strictly stronger then we may find a sequence $f_n\in\mathscr{P}$ such that $f_n\to 0$ in A_φ, f_n is bounded away from 0 in Λ_φ and $\|f_n\|_\alpha\leqq 1$ where $\alpha=1/15$.

Step 4. Since $\|f_n\|_\alpha\leqq 1$, the set $\{f_n\}$ is relatively compact in $\mathscr{E}$ and has a cluster point g. We show that $g=0$. Indeed for some subsequence $f_{n_k}\to g$ pointwise. Since $f_{n_k}\to 0$ is μ-measure we have $g(x)=0$ for $1\leqq x<\infty$. Since g is entire $g=0$. We deduce that $f_n\to 0$ in $\mathscr{E}$ and hence that

$$\|f_n\|_{2\alpha}\to 0.$$

Step 5. We may pass to a subsequence (still labelled f_n) such that

$$\|f_n\|_{2\alpha}\leqq 2^{-n}$$

and $\sum \varepsilon_n f_n$ converges in $L_\varphi(\mu)$ for every choice of $\varepsilon_n=\pm 1$.

Step 6. Let $\varepsilon_n=\pm 1$ be given. Then $h=\sum_{r=1}^\infty \varepsilon_n f_n$ exists in $\mathscr{E}$ and $\|h\|_{2\alpha}\leqq 1$. The series also converges in μ-measure to a function in $L_\varphi(\mu)$, which we may take to be h (by selection of representative in the equivalence class).

Step 7. We show $h\in\Lambda_\varphi(\mu)$. Let E be the subset of $(1,\infty)$ such that

$$E=\{x\colon \log|h(x)| > \cos(5\pi\alpha)\log M(h;x)\}.$$

Then by a theorem of Barry [1]

$$\liminf_{r\to\infty}\frac{1}{\log r}\int_{E\cap[1,r)}\frac{dt}{t}\geqq 1-\frac{2}{5}=\frac{3}{5}.$$

Hence for some $1<R<\infty$ and all $r\geqq R$

$$\int_{E\cap[1,r)}\frac{dt}{t}\geqq\frac{11}{20}\log r.$$

Choose $\beta_0=R$ and then $\beta_n=\beta_{n-1}^2$, $n=1,2,3,\ldots$.

Then

$$\int_{E\cap[1,\beta_n)} \frac{dt}{t} \geqq \frac{11}{20} \log \beta_n$$

$$\int_{E\cap[1,\beta_{n-1})} \frac{dt}{t} \leqq \frac{1}{2} \log \beta_n$$

and so

$$\int_{E\cap[\beta_{n-1},\beta_n)} \frac{dt}{t} \geqq \frac{1}{20} \log \beta_n.$$

Now

$$\int_{\beta_{n-1}}^{\beta_n} \varphi\big(M(h;\, t)\big) w(t) \frac{dt}{t} \leqq \varphi\big(M(h;\, \beta_n)\big) \int_{\beta_{n-1}}^{\beta_n} w(t) \frac{dt}{t}$$

$$\leqq \frac{1}{2} \varphi\big(M(h;\, \beta_n)\big) w(\beta_{n-1}) \log \beta_n$$

$$\leqq \frac{1}{2c^2} \varphi(M(h;\, \beta_n) w(\beta_{n+1}) \log \beta_n.$$

For $x \in E\cap[\beta_n, \beta_{n+1})$

$$\log |h(x)| > \cos 5\pi\alpha \log M(h;\, \beta_n)$$

$$= \frac{1}{2} \log M(h;\, \beta_n)$$

so that $\varphi\big(M(h;\, \beta_n)\big) \leqq C\big(\varphi(|h(x)|)+1\big)$ by (1.0.4).

Hence

$$\varphi\big(M(h;\, \beta_n)\big) \int_{E\cap[\beta_n,\beta_{n+1})} w(t) \frac{dt}{t} \leqq C \int_{\beta_n}^{\beta_{n+1}} \big(\varphi(|h(t)|)+1\big) \frac{w(t)}{t}\, dt$$

so

$$\frac{1}{20} \varphi\big(M(h;\, \beta_n)\big) w(\beta_{n+1}) \log \beta_{n+1} \leqq C \int_{\beta n}^{\beta_{n+1}} \big(\varphi(|h(t)|)+1\big) \frac{w(t)}{t}\, dt.$$

Combining we have

$$\int_{\beta_{n-1}}^{\beta_n} \varphi\big(M(h;\, t)\big) \frac{w(t)}{t}\, dt \leqq \frac{10C}{c^2} \int_{\beta_n}^{\beta_{n+1}} \big(\varphi(|h(t)|)+1\big) \frac{w(t)}{t}\, dt$$

so by summing we deduce $h \in \Lambda_\varphi$.

Step 8. Thus $\Sigma\, \varepsilon_n f_n$ converges *pointwise* in $\Lambda_\varphi(\mu)$ for every $\varepsilon_n = \pm 1$. Since Λ_φ is a separable F-space we may apply the Orlicz—Pettis Theorem ([3], [6]) to deduce that Σf_n converges in $\Lambda_\varphi(\mu)$ and hence $f_n \to 0$ which produces the desired contradiction.

Theorem 4.5. *The following conditions on an Orlicz function φ satisfying* (1.0.4) *are equivalent:*

(i) *R is not φ-elementary.*

(ii) *There is a finite positive measure μ on $[1, \infty)$ such that*

$$\int \varphi(x)\,d\mu(x) < \infty$$

and

$$\inf_n \int \varphi\left(\frac{x^n}{n!}\right) d\mu(x) > 0$$

(iii) *If $a = \sup [x\colon \varphi(x) = 0]$ then there is a finite positive measure ν supported on $[a, \infty)$ such that*

$$\liminf_{t\to\infty} \int_t^\infty \frac{\varphi(x^t)}{\varphi(x)}\,d\nu(x) > 0.$$

Proof. (i)$\Rightarrow$(ii) Proposition 4.3,

(ii)$\Rightarrow$(i) We shall use (1.0.5). Let ϱ be the measure on $\mathbf{R}_+$ given by

$$\frac{d\varrho}{dx} = x^{-p-2} \quad x \geqq 1$$

$$= 1 \quad 0 \leqq x < 1$$

and consider $L_\varphi(\mathbf{R}_+ \times \mathbf{R}_+)$ in the product $\mu \times \varrho$ measure. Define $f \in L_\varphi(\mathbf{R}_+ \times \mathbf{R}_+)$ by

$$f(x, y) = x^y.$$

Then $f \in L_\varphi$ since

$$\varphi(|f|) \leqq A(y^p + 1)(\varphi(x) + 1).$$

Clearly $|f| \geqq 1$ a.e. and f has a distribution whose density u is given by

$$u(x) = \int_0^\infty F(x^{1/t})\,x^{1/t-1} \frac{d\varrho}{dt}\,dt$$

where $F(x) = \mu[x, \infty)$.

If $u(x) = w(x)/x$ then

$$w(x) = \int_0^\infty F(x^{1/t})\,x^{1/t} \frac{d\varrho}{dt}\,dt$$

$$= \int_1^\infty F(\xi) \frac{\xi \log \xi}{\log x} \frac{d\varrho}{dt}\left[\frac{\log x}{\log \xi}\right] d\xi$$

after the substitution $\xi = x^{1/t}$. Hence w is monotone decreasing and also

$$w(x^2) = \int_1^\infty F(\xi) \frac{\xi \log \xi}{2 \log x} \frac{d\varrho}{dt}\left[\frac{\log x^2}{\log \xi}\right] d\xi$$

$$\geqq 2^{-p-3} w(x).$$

It remains to establish that w satisfies (4.4.5) and then Theorem 4.4 can be applied. To do this observe

$$\int_1^\infty \varphi\left(\frac{x^n}{n!}\right) w(x)\frac{dx}{x} = \int_0^\infty\int_0^\infty \varphi\left(\frac{f(x,y)^n}{n!}\right) d\mu(x)\,d\varrho(y)$$

$$\geqq \int_0^\infty\int_0^\infty \varphi\left(\frac{x^{ny}}{n!}\right) d\mu(x)\frac{dy}{y^{p+2}}$$

$$\geqq \int_0^\infty \varphi\left(\frac{x^n}{n!}\right) d\mu(x)\int_1^\infty \frac{dy}{y^{p+2}}.$$

(ii)⇒(iii). If μ is given by (ii), let ϱ be the distribution of x^2 in $L_\varphi(\mu)$. Then let

$$dv = \varphi(x)\,d\varrho \quad \text{on} \quad [a,\infty).$$

Thus v is a finite measure supported on $[a,\infty)$. Now if $n\leqq t<n+1$,

$$\int_t^\infty \frac{\varphi(x^t)}{\varphi(x)}\,dv(x) = \int_t^\infty \varphi(x^t)\,d\varrho(x)$$

$$\geqq \int_{n+1}^\infty \varphi(x^n)\,d\varrho(x)$$

$$\geqq \int_{n+1}^\infty \varphi\left(\frac{x^n}{(n+1)^n}\right) d\varrho\,(x).$$

By the Bounded Convergence Theorem

$$\int_1^{n+1} \varphi\left(\frac{x^n}{(n+1)^n}\right) d\varrho(x) \to 0$$

and hence

$$\liminf_{t\to\infty}\int_t^\infty \frac{\varphi(x^t)}{\varphi(x)}\,dv(x) \geqq \liminf_{n\to\infty}\int_1^\infty \varphi\left(\frac{x^n}{(n+1)^n}\right) d\varrho(x)$$

$$\geqq \liminf_{n\to\infty}\int_1^\infty \varphi\left(\frac{x^n}{(2n)!}\right) d\varrho(x)$$

$$= \liminf_{n\to\infty}\int_1^\infty \varphi\left(\frac{x^{2n}}{(2n)!}\right) d\mu(x) > 0.$$

(iii)⇒(ii). Let

$$d\varrho(x) = \frac{1}{\varphi(x)}\,dv(x) \quad x\geqq 1+a$$

so that (since $\varphi(1+a)>0$) ϱ is a finite positive measure supported as $[1+a,\infty)$ and

$$\int \varphi(x)\,d\varrho(x) < \infty.$$

Now if $x \geqq n$

$$\frac{x^{2n}}{n!} \geqq \frac{n^n}{n!} x^n \geqq x^n$$

so that

$$\int_0^\infty \varphi\left(\frac{x^{2n}}{n!}\right) d\varrho(x) \geqq \int_n^\infty \frac{\varphi(x^n)}{\varphi(x)} d\mu(x).$$

Hence

$$\liminf_{n\to\infty} \int_0^\infty \varphi\left(\frac{x^{2n}}{n!}\right) d\varrho(x) > 0.$$

Let μ be the distribution of x^2. Then (ii) follows for μ, since clearly it is impossible that the integral should vanish for any n, as μ has unbounded support.

Although Theorem 4.5 gives a necessary and sufficient criterion for $\mathbf{R}$ to be φ-elementary, it does not appear easy to convert this to a purely analytic condition on φ. We do however give some conditions which are either necessary or sufficient.

Corollary 4.6. *If* $\mathbf{R}$ *is not* φ*-elementary*

(4.6.1) $$\lim_{n\to\infty} \sup_{x\geqq n} \frac{\varphi(x^n)}{\varphi(x)} = \infty.$$

Proof. This is immediate from the Bounded Convergence Theorem.

Corollary 4.7. *Suppose* $\varphi(x)=\psi(\log_+ \log_+ x)$, *where* ψ *is a concave function on* R_+, *and that* R *is not* φ*-elementary. If* $x_n (n\geqq 0)$ *is any sequence such that* $x_n \geqq e^{x_{n-1}}$ *for* $n\in\mathbf{N}$ *then*

(4.7.1) $$\sum_{n=1}^\infty \frac{\varphi(x_{n-1})}{\varphi(x_n)} < \infty.$$

Proof. The hypotheses ensure that $\varphi(x^t)/\varphi(x)$ is a decreasing function of t for $x > e^e$. Indeed

$$\varphi(x^n) = \psi(\log\log x + \log n)$$

and since $\log\psi$ is also concave we have that $\log\varphi(x^n) - \log\varphi(x)$ decreases with x. If ν is chosen to satisfy (iii), let $F(x)=\nu[x,\infty)$. Then for $e^e < T < \infty$ and $\varepsilon > 0$ we have for all $t\geqq T$.

$$\int_t^\infty \frac{\varphi(x^t)}{\varphi(x)} d\nu(x) \geqq \varepsilon.$$

Now

$$\begin{aligned}\varphi(e^{t^2}) &= \psi(2\log t)\\ &\leqq 2\psi(\log t)\\ &= 2\varphi(e^t)\end{aligned}$$

i.e. $\varphi(e^{t^2})/\varphi(e^t) \leqq 2$.

Hence

$$\int_t^\infty \frac{\varphi(x^t)}{\varphi(x)}\,d\nu(x) \leqq \int_t^{e^t} \frac{\varphi(x^t)}{\varphi(x)}\,d\nu(x) + 2\nu\;[e^t, \infty).$$

Again for some T_1 and all $t \geqq T_1$

$$\int_t^{e^t} \frac{\varphi(x^t)}{\varphi(x)}\,d\nu(x) \geqq \frac{\varepsilon}{2}.$$

Then

$$\frac{\varphi(t^t)}{\varphi(t)}\,\nu[t, e^t) \geqq \frac{\varepsilon}{2}$$

i.e.

$$\nu[t, e^t) \geqq \frac{\varepsilon}{2}\,\frac{\varphi(t)}{\varphi(t^t)}.$$

However

$$\begin{aligned}\varphi(t^t) &\leqq \varphi(e^{t^2})\\ &\leqq 2\varphi(e^t)\end{aligned}$$

so that

$$\nu[t, e^t) \geqq \frac{\varepsilon}{4}\,\frac{\varphi(t)}{\varphi(e^t)} \quad \text{for} \quad t \geqq T_1.$$

Since ν is finite we deduce (4.7.1).

We now have a positive result

Corollary 4.8. *Suppose φ is unbounded, continuous and that*

(4.8.1) $$\int_0^\infty \frac{d\varphi(x)}{\varphi(e^x)} < \infty.$$

Then **R** *is not φ-elementary.*

Proof. Note first that the integral can only diverge at ∞; indeed if $a = \sup(s\colon \varphi(s) = 0)$, then

$$\int_0^\infty \frac{d\varphi(x)}{\varphi(e^x)} = \int_a^\infty \frac{d\varphi(x)}{\varphi(e^x)}$$

and $\varphi(e^a) > 0$.

We can define a Borel measure ϱ on $[a, \infty)$ such that

$$\varrho[x, \infty) = \frac{1}{\varphi(e^x)} \qquad a \leqq x < \infty$$

ϱ is then finite with total mass $\varphi(e^a)^{-1}$. Now define ν so that

$$d\nu(x) = \varphi(x)\,d\varrho(x) \quad a \leqq x < \infty.$$

We claim ν is finite. Indeed for $b<\infty$

$$\int_a^b d\nu(x) = \int_a^b \varphi(x)\, d\varrho(x)$$
$$= \int_a^b \varphi(x)\, d\left[\frac{1}{\varphi(e^a)} - \frac{1}{\varphi(e^x)}\right]$$
$$= \left[-\frac{\varphi(x)}{\varphi(e^x)}\right]_a^b + \int_a^b \frac{d\varphi(x)}{\varphi(e^x)}.$$

Thus

$$\int_a^b d\nu(x) \leqq \frac{\varphi(a)}{\varphi(e^a)} + \int_a^\infty \frac{d\varphi(x)}{\varphi(e^x)}$$

so that ν is finite. Also ν satisfies the conditions of 4.5 (iii). We have for $t \geqq a$

$$\int_t^\infty \frac{\varphi(x^t)}{\varphi(x)}\, d\nu(x) = \int_t^\infty \varphi(x^t)\, d\varrho(x)$$
$$\geqq \frac{\varphi(t^t)}{\varphi(e^t)} \geqq 1.$$

Corollary 4.9. *If φ is continuous and there exists $\alpha>1, X<\infty, c>0$ such that*

(4.9.1) $$\varphi(e^x) \geqq c\varphi(x)^\alpha \quad x \geqq X.$$

Then **R** *is not φ-elementary.*

Proof. By 4.8. Contrast Theorem 4.2.

Examples. The function $\varphi(x)=\log_+ \dots \log_+ x$ with m-iterates of $\log_+$ satisfies (4.9.1) for any finite m and hence **R** is not φ-elementary. On the other hand the function $\varphi(x)=m$ where m is the least integer such that $\log_+ \dots \log_+ x \leqq 1$ for m-iterates of $\log_+$, is an example of an unbounded function such that **C** is φ-elementary, by Theorem 4.2.

5. Applications to Orlicz algebras

Theorem 5.1. *Suppose (S, Σ, μ) is a diffuse finite measure space, and $L_\varphi(S, \Sigma, \mu)$ is an Orlicz algebra. In order that any closed sub-algebra of $L_\varphi(S, \Sigma, \mu)$ containing 1 be elementary it is sufficient that for some $C<\infty$.*

(5.1.1) $$\varphi(e^x) \leqq C\big(\varphi(x)+1\big) \quad 0 \leqq x < \infty$$

and necessary that either φ be bounded or

(5.1.2) $$\int_0^\infty \frac{d\varphi(x)}{\varphi(e^x)} = \infty.$$

Proof. If (5.1.2) fails then L_φ contains a single real element which generates a non-elementary algebra.

If (5.1.1) holds then every element of L_φ is elementary. We show that this means that every closed sub-algebra A is also elementary. Indeed let $\Sigma_0=\{B\in\Sigma\colon 1_B\in A\}$; Σ_0 is a sub-σ-algebra of Σ. If $f\in A$ then f is elementary and hence for any open set U in $\mathbf{C}$, $1_{f^{-1}(U)}\in A$ i.e. f is is Σ_0-measurable. Thus $L_\varphi(S,\Sigma_0,\mu)\subset A\subset L_\varphi(S,\Sigma_0,\mu)$.

Theorem 5.2. *Under the hypotheses of Theorem* 5.1 *condition* (5.1.2) *is necessary in order that every closed self-adjoint sub-algebra of* L_φ *containing* 1 *be elementary. A sufficient condition that every such sub-algebra is elementary is that* $\varphi(x)=\psi(\log_+\log_+ x)$ *where* ψ *is concave and*

$$\sum_{n=1}^{\infty}\frac{\varphi(x_{n-1})}{\varphi(x_n)}=\infty \tag{5.2.3}$$

for some sequence $(x_n\colon n\geqq 0)$ *satisfying* $x_n\geqq e^{x_{n-1}}$ *for all* n.

Conditions (5.1.2) *and* (5.2.3) *are also respectively necessary and sufficient that every closed sub-algebra of the real Orlicz algebra* $L_{\varphi,\mathbf{R}}$ *is elementary.*

Proof. As for Theorem 5.1.

Our final result observes that a closed subalgebra A with identity of an Orlicz algebra cannot be a field. In this context, we point out that Williamson [15] showed that $L_0(0,1)$ has a dense subalgebra which is a field and Waelbroeck [14] has given an example of an F-algebra which is a field. See also Turpin [13].

Theorem 5.3. *Let* A *be a closed subalgebra of an Orlicz algebra* $L_\varphi(S,\Sigma,\mu)$ *which contains the identity* 1 *and is a field. Then* $A=\mathbf{C}1$.

Proof. Suppose $f\in A$ and $f\notin\mathbf{C}1$. Let B be the closed subalgebra of A generated by all rational functions in f. Then the proof of Proposition 2.2 can be used to show that $1_D\circ f\in B$ for every open disc D in $\mathbf{C}$. Hence $1_D\circ f=1$ or 0 for each such disc. This again implies $f\in\mathbf{C}1$ which is a contradiction.

6. Concluding remarks

It is possible to develop the study of the spaces $A_\varphi(\mu)$ to a much greater extent than we have attempted here. In particular, we propose to study spaces $A_\varphi(\mu)$ when μ is supported on the real line or is rotation invariant with unbounded support in a subsequent paper. There we shall examine questions relating to the equality $A_\varphi(\mu)=A_\varphi(\mu)$ (if μ is supported in R) and also attempts to characterize for given φ these measures μ of which $A_\varphi(\mu)$ is analytic.

The main aim of this paper has been to establish conditions on φ so that a given set $E(=\Gamma, \bar{\Delta}, \mathbf{C}, \mathbf{R})$ supports a measure μ for which $A_\varphi(\mu)$ is analytic. Our results have only been partially successful. Of particular interest are the cases $\mathbf{C}$ and $\mathbf{R}$ where our necessary conditions and our sufficient conditions are very close but do not match. It would be very interesting to plug that gap.

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