Monodromy and asymptotic properties of certain multiple integrals

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1. Introduction

The main purpose of this paper is to give an elementary proof (without use of any desingularization result) of a monodromy theorem for integrals of certain multi-valued analytic functions with their singularities on an algebraic manifold, without restrictions on this manifold. In the main theorem (Theorem 1) we shall consider an integral

$$g(y) = \int_{E_n} f(x, y)\, dx, \tag{0}$$

where E_n is the unit Euclidean simplex in $\mathbf{R}^n$, y a parameter in $\mathbf{C}^l$, and where f is regular analytic in a neighbourhood of $E_n \times \{y^0\}$, y^0 being some point in $\mathbf{C}^l$; hence g is defined and regular analytic in a neighbourhood of y^0. Further it will be supposed that f belongs to a certain class of analytic functions on (part of) $\mathbf{C}^{n+l}$, i.e. that f can be continued analytically to a function of the class. Besides being regular analytic (and in general multi-valued) outside some genuine algebraic manifold in $\mathbf{C}^{n+l}$, the functions of such a class (denoted $P_m(\mathbf{C}^{n+l})$, m being an integer $\geqq 1$) are characterized by certain monodromy formulas. To wit, they will be required to satisfy the relation $(T_\gamma^k - \mathrm{id})^m f = 0$ for all loops γ belonging to a certain class and for all germs of f at a point of γ; here T_γ denotes analytic continuation along γ, and k is a positive integer that may depend on γ and on the particular germ of f. Theorem 1 states that if $f \in P_m(\mathbf{C}^{n+l})$, then the function g of (0) belongs to $P_{m+n}(\mathbf{C}^l)$. In particular, if y is a single complex variable, it will follow that g can be continued analytically along any path in $\mathbf{C}$ outside some finite set M of points and that $(T_\gamma^k - id)^{m+n} g = 0$ for any circle γ having only one of the points of M, or all, in its interior; as before k is a positive integer that may depend on the germ of g (and on γ, though this is not necessary here). If f is algebraic, it will, in particular, follow that $g \in P_{1+n}$.

It can be shown that the index $m+n$ of Theorem 1 is the best possible.

The proof of Theorem 1 is by induction over n, whereby one is reduced to prove the theorem in the case $n=1$. In this case one performs the continuation of g by continuation of f and deformation of the path of integration, so as to avoid the singularities of f. In establishing the monodromy formulas, we shall, by the choice of our class of loops, essentially have to treat only the case where y is a single variable and γ a circle around only one singularity of g (chosen to be at ∞). In this case there is a surveyable description of the deformation and of the integrands on the various parts of the path of integration, corresponding to an expression $(T_\gamma^k-\mathrm{id})^{m+1}g$. It will be proved that for some positive k all the mentioned integrands vanish, which will imply Theorem 1. An essential point is, of course, that our class of loops is optimal in the sense that it contains enough loops to ensure that $(T_\gamma^k-\mathrm{id})^{m+1}g=0$ (read as above) for any loop γ in the class.

In Theorem 2 we use Theorem 1 to prove a similar theorem, but now for the integral of a differential k-form over a k-cycle on an algebraic manifold, where both the form and the manifold depend on a parameter, in the latter case algebraically. The coefficients of the differential form are supposed to be of class P_m (the parameter being included among the variables). Theorem 2 states that then the integral is of class P_{m+k} as a function of the parameter. The reduction to Theorem 1 is made by an algebraic triangulation of the cycle.

Further we shall use Theorem 2, in combination with a result of Nilsson [3], to show Theorem 3, which gives an expansion, for the parameter λ large and positive, of a function $e(\lambda)=\int_{f(x)\leqq\lambda}g(x)\,dx$, where f and g are real analytic functions on (part of) $\mathbf{R}^n$ and f real-valued. Further f will be required to be algebraic and g to belong to $P_m(\mathbf{C}^n)$, for some m, and in addition to satisfy a certain growth condition (on $\mathbf{C}^n$). The expansion of $e(\lambda)$ will be of the form $\sum_{j,k}c_{jk}\lambda^{j/s}(\log\lambda)^k$, where s is some positive integer and the constants c_{jk} vanish, when j is large and positive, and when k does not satisfy $0\leqq k\leqq m+n-1$. Since the expansion has a dominating term, it follows that $e(\lambda)$ behaves asymptotically as some $c_{jk}\lambda^{j/s}(\log\lambda)^k$ when $\lambda\to+\infty$.

Theorem 1 is based on an old unpublished result of the author, implying a proof of Theorem 1, except that it was only shown that $g\in P_{2^n}$, instead of $g\in P_{m+n}$. Here I want to acknowledge that I owe the corresponding improvement of the proof mainly to Le Dung Tráng, Jan-Erik Björk, and Dimitris Scarpalézos, in particular the use of a (in general) large number of "intermediate" cycles $\beta^a(\beta')^b$ (see the proof) and the algebraic lemma. In [5] Scarpalézos has also given a variant of the proof. Further, in a private communication of 1971 Pierre Deligne sketched a quite different proof, using resolution of singularities, and giving the correct improvement of the old result. Partly our proof of Theorem 1 generalizes classical methods of Fuchs, Picard, and Lefschetz. Among papers giving general monodromy results let us refer to Grothendieck [1], Landman [2], and Tráng [6] (the latter not using resolution of singularities). We also remark that the present Theorem 3 is an improvement of Theorem 1 of the author's paper [4].

Finally I wish to thank Tráng and Björk for interesting discussions on the subject.

2. The induction class and the main result

We are going to work in complex n-space $\mathbf{C}^n$, with points $z=(z_1, \ldots, z_n)$ etc. Let us start by a few definitions.

Definition. By $K(\mathbf{C}^n)$ we denote the class of all complex-valued (and in general multi-valued) functions f which are defined and regular analytic in a region of the form $\mathbf{C}^n \setminus V$, where $V=V(f)$ is an algebraic manifold in $\mathbf{C}^n$ but not $\mathbf{C}^n$ itself. (It thus suffices to consider manifolds V of the form $p(z)=0$, where p is a complex polynomial $\not\equiv 0$ in n variables.)

Next we define the set of "permissible" paths of continuation in our study of the ramification.

Definition. If V is a subset of $\mathbf{C}^n$, we shall denote by $B(V)$ the class of all closed paths γ in $\mathbf{C}^n \setminus V$ that satisfy the following condition. There is a branch α of an algebraic function $\mathbf{C} \to \mathbf{C}^n$, such that α is regular analytic (and singlevalued) in a pointed neighbourhood of ∞ and that, for all sufficiently large positive real numbers r the closed path $\gamma_r(\alpha)$: $[0, 1] \ni t \mapsto \alpha(re^{2\pi i t})$ is contained in $\mathbf{C}^n \setminus V$ and homotopic to γ in this set (under homotopy for loops). (Thus the γ_r: s are the images under α of the circles $|z|=r$, and $B(V)$ is, but for homotopy, made up of such images corresponding to different α: s.) Note that α must take on only values in $\mathbf{C}^n \setminus V$ in some pointed neighbourhood of ∞.

When γ is a loop in $\mathbf{C}^n$, let T_γ denote analytic continuation along γ. Hence T_γ is a mapping whose domain and range both consist of analytic germs at the point $\gamma(0)$ $(=\gamma(1))$ (we assume throughout that the parameter interval is $[0, 1]$). Clearly T_γ is linear over $\mathbf{C}$.

We are now ready to define our induction class.

Definition. If m is a positive integer, then $P_m(\mathbf{C}^n)$ will denote the class of all functions f in $K(\mathbf{C}^n)$ such that, with some choice of the corresponding manifold $V(f)$, we have for any path $\gamma \in B(V(f))$ and any determination f_0 of f at $\gamma(0)$ that $(T_\gamma^k - I)^m f_0 = 0$ for some positive integer $k=k(\gamma, f_0)$ (where I denotes the identity operator).

It is clear that $P_m(\mathbf{C}^n) \subseteq P_{m'}(\mathbf{C}^n)$ if $m \leqq m'$ and that $f+g$ belongs to $P_m(\mathbf{C}^n)$ if f and g do. A simple induction argument shows that if $f \in P_m(\mathbf{C}^n)$ and $g \in P_{m'}(\mathbf{C}^n)$, then $fg \in P_{m+m'-1}(\mathbf{C}^n)$.

And if $f \in P_m(\mathbf{C}^n)$, and φ is an algebraic mapping $\mathbf{C}^{n'} \to \mathbf{C}^n$ (in general many-valued) such that the range of φ is not contained in $V(f)$, then the composition $f \circ \varphi$ is in $P_m(\mathbf{C}^{n'})$. Let us also observe that every algebraic function $\mathbf{C}^n \to \mathbf{C}$ is in $P_1(\mathbf{C}^n)$, while $f(z) = \log^m z_1$ is an example of a function in $P_{m+1}(\mathbf{C}^n)$ but not in $P_m(\mathbf{C}^n)$.

Now let us formulate the principal result of this paper.

Let E_n be the n-dimensional Euclidean simplex $x_1 \geqq 0, x_2 \geqq 0, \ldots, x_n \geqq 0$, $x_1 + x_2 + \ldots + x_n \leqq 1$. Let $f(x, y)$ be a (single-valued) complex-valued function of $x \in \mathbf{C}^n$ and $y \in \mathbf{C}^l$ which is regular analytic in a neighbourhood of $\{y^0\} \times E_n$, where y^0 is a given point in $\mathbf{C}^l$. Put

$$g(y) = \int_{E_n} f(x, y)\, dx_1 \ldots dx_n, \tag{1}$$

which defines g as a regular analytic function in a neighbourhood of y^0. We then have

Theorem 1. *Assume that the function f in* (1) *is in $P_m(\mathbf{C}^{n+l})$ (i.e. can be continued analytically to a function in $P_m(\mathbf{C}^{n+l})$). Then the function g defined by* (1) *is in $P_{m+n}(\mathbf{C}^l)$.*

3. Proof of the main theorem

Let us first prove Theorem 1 in the case $n=1$. Then we have to consider a function $g(y) = \int_0^1 f(x, y)\, dx$, where the function f is defined and regular analytic in a neighbourhood of $[0, 1] \times \{y^0\}$ (with $y^0 \in \mathbf{C}^l$) and, moreover, belongs to $P_m(\mathbf{C}^{1+l})$. Thus g is from the beginning defined and regular analytic in a neighbourhood of y^0, and our task is to prove that g can be continued analytically to a function in $P_{m+1}(\mathbf{C}^l)$.

Let $V(f)$: $p(x, y) = 0$ be a manifold of ramification of f, corresponding to f by the definition of the class $P_m(\mathbf{C}^{1+l})$. If $p(x, y)$ does not contain x, $p(x, y) \equiv p(y)$ say, then clearly g is in $K(\mathbf{C}^l)$, and we can take $V(g)$ as the manifold $p(y) = 0$ in $\mathbf{C}^l$. If the path γ in $\mathbf{C}^l$ belongs to $B(V(g))$, then the path $\tilde{\gamma}$: $t \mapsto (0, \gamma(t))$ in $\mathbf{C}^{1+l}$ is in $B(V(f))$, and since obviously we have (extending the integrand from a neighbourhood of $x=0$ to the interval $[0, 1]$ by analytic continuation)

$$(T_\gamma^k - I)^m g(y) = \int_0^1 (T_{\tilde{\gamma}}^k - I)^m f(x, y)\, dx,$$

it follows that g is in $P_m(\mathbf{C}^l)$ and thus also in $P_{m+1}(\mathbf{C}^l)$.

So we can concentrate on the principal case that $p(x, y)$ actually contains x. Then there is a manifold W: $q(y) = 0$ in $\mathbf{C}^l$, where q is a polynomial not identically zero, such that we have: when $y \notin W$ the number of (different) solutions of the equation (in x) $p(x, y)=0$ is constant, $=N$, say, and with convenient enumeration these solutions $\xi_1(y), \dots, \xi_N(y)$ are regular analytic (many-valued) functions of y in $\mathbf{C}^l \setminus W$ and do not coincide with 0 or 1 unless they do so identically, and further the $\xi_j(y)$ s: do not coincide between themselves, when y varies in $\mathbf{C}^l \setminus W$. Now g can be continued analytically along any path in $\mathbf{C}^l \setminus W$. For as y runs through such a path γ, starting at y^0, we can deform the path of integration continuously with respect to y in such a way that it — excepting the endpoints 0 and 1 — never passes through any of the points $\xi_j(y)$. Further f can be continued analytically along the path $t \mapsto (0, \gamma(t))$ (and along $t \mapsto (1, \gamma(t))$, similarly). For at the point $(0, \gamma(0))=(0, y^0)$ we have at the start a regular function element of f, and hence, by Cauchy's formula,

$$f(x, y) = (2\pi i)^{-1} \int_{|z|=\delta} (z-x)^{-1} f(z, y)\, dz, \tag{2}$$

when the positive number δ is sufficiently small and (x, y) sufficiently close to $(0, \gamma(0))$. It is clear that the integral (2) defines an analytic continuation of f along the path $t \mapsto (0, \gamma(t))$, since the point 0 then never coincides with any of the points $\xi_j(y)$, except those coinciding identically with 0, so that the circle of integration in (2) will go free of all the $\xi_j(y)$: s if only δ is sufficiently small. Using this continuation and the way the $\xi_j(y)$: s are defined we see that in the process of deformation of the path of integration (and the corresponding continuation of f) we have all the time a regular branch of f along the whole of this path. Thus the desired continuation of g follows, i.e. we have $g \in K(\mathbf{C}^l)$, with $V(g)=W$.

Now let us turn to the proof of the relevant formulas of monodromy. Then we have to consider a path γ in $B(W)$; that is, apart from homotopy, γ will be of the form γ_r: $t \mapsto \alpha(re^{2\pi i t})$ $(0 \leqq t \leqq 1)$, where all the l coordinates of $\alpha(z)$ are regular analytic branches of algebraic functions of z in a region $|z| > r_0$ in the complex plane, and where γ_r does not pass through any point of $V(g)=W$ when $r > r_0$. We must prove that to every function element g_0 of g (in a neighbourhood of any point $\gamma_r(0)$ (where $r > r_0$)) there is a positive integer k such that $(T_{\gamma_r}^k - I)^{m+1} g_0 = 0$. For then, of course, the corresponding formula follows also for any closed path homotopic to γ_r in $\mathbf{C}^l \setminus V(g)$.

We shall, then, have to study the behaviour of the points $\eta_j(z)=\xi_j(\alpha(z))$ as z varies along circles in the complex plane. Obviously the $\eta_j(z)$: s are regular analytic algebraic functions of z in some (pointed) neighbourhood of ∞. First let us consider the somewhat simplified case where for all large positive r we have that when y is close to $\gamma_r(0)$ $(=\alpha(r))$, then $g_0(y)=\int_0^1 f_0(x, y)\, dx$ (integration along the interval $[0, 1]$), with some branch f_0 of f, and further, that all the points $\eta_j(r)$ are non-real, except if $\eta_j(z)\equiv 0$ or $\equiv 1$. Now each of the functions $\eta_j(z)$ can be expanded in a Puiseux series

$$\eta_j(z) = \sum_{i=-\infty}^{\infty} a_{j,i} z^{i/s}, \tag{3}$$

where s is a positive integer, which can, of course, be taken the same for all j. Studying the variation of the $\eta_j(z)$: s as z runs through a path, we shall assume that for every z the value of $z^{1/s}$ in (3) is chosen the same for all j and i. We also know that the expansions (3) contain only a finite number of non-zero terms with $i>0$.

Now let us divide the $\eta_j(z)$: s into different "rings", in each "ring" taking such $\eta_j(z)$: s for which in the leading term of the expansion (3) the exponent as well as the absolute value of the coefficient (respectively) are the same. It is easily seen that for every sufficiently large real number r there are, corresponding to our "rings", finitely many annuli with centre 0 (where the radii 0 and ∞ are permitted), disjoint but for the boundaries, and covering the whole plane, such that when z runs through the circle $|z|=r$, then all the $\eta_j(z)$: $s\not\equiv 0$ belonging to any of the "rings" will vary in the corresponding annulus. Clearly the radii of the annuli can be chosen as functions of r of the form ar^b, where a and b are real constants. With the help of these annuli we make a deformation of the original path of integration (i.e. the interval [0, 1]) as in Figure 1.

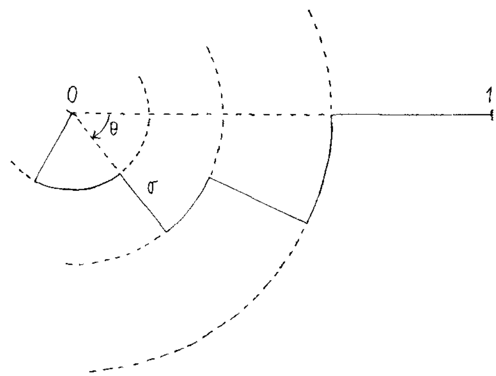

Fig. 1

The angle θ corresponding to the straight piece σ in Picture 1 is taken as $d\cdot\arg(z)$, where d is the exponent of the leading term of the expansions (3) of the $\eta_j(z)$: s belonging to the corresponding "ring". Assume first that for all the $\eta_j(z)$: s different from 0 and 1 the coefficient of the leading term in (3) is non-real. Then it is clear that the deformation described fully suffices for the continuation of g_0 (our function element of g) along γ_r: $t\mapsto\alpha(re^{2\pi it})$ when r is sufficiently large. When z $(=re^{2\pi it})$ has run through the circle $|z|=r$ s times, all the straight pieces will lie on their original places on the real interval [0, 1]. We are now going to show that

$$(T_{\gamma_r}^{sp}-I)^{m+1}g_0=0 \tag{4}$$

for some positive integer p. To calculate the left member of (4), we first consider $T_{\gamma_r}^{sq}g_0$, when q is a positive integer. By the process of continuation, corresponding to the deformation described above, $T_{\gamma_r}^{sq}g_0(y)$ is, for y close to the point $\gamma_r(0)$, a sum of integrals over the straight pieces (now subintervals of $[0, 1]$) and over circles $|x|=\text{const.}$; in both cases the integrated functions are branches of the function $x\mapsto f(x, y)$, and these branches we must now describe more exactly.

Let us do so for a straight piece σ corresponging to the leading exponent μ/s in the expansions (3) and for the circle $\varkappa_r$ (with centre 0) passing through the right endpoint of σ. Here $\varkappa_r$ means the circle run through once in the positive sense, and we use the natural parametrization. For shortness, let us drop the index r a while, writing simply γ, $\varkappa$, etc.

Let $\tilde{\varkappa}$ and $\tilde{\gamma}$ be the following loops in $\mathbf{C}^{1+l}$: $\tilde{\varkappa}(\theta)=(\varkappa(\theta), \gamma(0))$, and $\tilde{\gamma}(t)=(\varkappa(0), \gamma(t))$. Then, clearly, in $T_\gamma^{sq}g_0$ the integrand on σ must be $T_{\tilde{\gamma}}^{sq}T_{\tilde{\varkappa}}^{\mu q}f_0$, f_0 again being the branch of f that we originally have on σ. For as z runs through the circle $|z|=r$ s times, σ rotates $|\mu|q$ times in the negative sense (in the deformation we clearly have to consider only the "rings" corresponding to leading exponents in (3) that are <0). Here we have also used that $T_{\tilde{\varkappa}}$ and $T_{\tilde{\gamma}}$ commute on f_0, i.e. that $T_{\tilde{\varkappa}}T_{\tilde{\gamma}}f_0=T_{\tilde{\gamma}}T_{\tilde{\varkappa}}f_0$. This follows at once from the fact that the points $\eta_j(z)$ keep away from $\varkappa$ so that when y runs through γ we can correspondingly continue f_0 on the whole of $\varkappa$. This commutativity will be essential to our description of integrands, also in the sequel. The same argument also gives the slightly stronger result that $\tilde{\varkappa}$ and $\tilde{\gamma}$ represent commuting elements in the fundamental group of $\mathbf{C}^{1+l}\setminus V(f)$ at $(\varkappa(0), \gamma(0))$.

It follows that for $(T_\gamma^{sp}-I)^{m+1}g_0$, where p is a positive integer, the integrand on σ is $((T_{\tilde{\gamma}}^s T_{\tilde{\varkappa}}^\mu)^p-I)^{m+1}f_0$. Thus, if the path $\tilde{\gamma}^s\tilde{\varkappa}^\mu$ is in $B(V(f))$, we can conclude that there is a positive integer k such that the integrand on σ vanishes when $p=k$ and thus also when k divides p, since the polynomial $(t^p-1)^{m+1}$ is then divisible by $(t^k-1)^{m+1}$. So let us see that a certain class of loops of the form $\tilde{\gamma}^a\tilde{\varkappa}^b$ (a, b being integers) is contained in $B(V(f))$ (in the more intricate study below of the contributions from the circles it will not be enough with just $\tilde{\gamma}^s\tilde{\varkappa}^\mu$). Let μ'/s be the leading exponent in (3) for the "ring" next outside the "ring", corresponding to the straight piece σ. Then we have $\mu\leqq\mu'$, and clearly we have only to consider the case $\mu<\mu'$ (since straight pieces σ corresponding to the same leading exponent in (3) move with the same angular speed in the deformation). We now state that $\beta=\tilde{\gamma}^s\tilde{\varkappa}^\mu$ and $\beta'=\tilde{\gamma}^s\tilde{\varkappa}^{\mu'}$ are both in $B(V(f))$ when r is large enough, and, more generally, that this is also true for $\beta^d(\beta')^{d'}$ whenever d, d' are integers $\geqq 0$. To see this for β, form $\tilde{\alpha}(z)=(2az^\mu, \alpha(z^s))$, where a is the modulus of the coefficients of the leading terms in (3) corresponding to the "ring" of σ, and where, of course, α is still the algebraic mapping defining the loops γ_r: $\gamma_r(t)=\alpha(re^{2\pi it})$. It is clear that in some pointed neighbourhood of infinity $\tilde{\alpha}$ is regular analytic and single-valued and does not take on values in $V(f)$. For by the expansions (3) (and the definition of "rings") it follows that for every j we have either $|\eta_j(z^s)|>2a|z^\mu|$ or $|\eta_j(z^s)|<2a|z^\mu|$ when $|z|$ is large enough. Further, when the positive number r_1 is large enough, the loop $t\mapsto\tilde{\alpha}(r_1e^{2\pi it})$ is homotopic to the loop $\beta=\tilde{\gamma}_r^s\tilde{\varkappa}_r^\mu$ in $\mathbf{C}^{1+l}\setminus V(f)$. For if we choose $r_1=r^{1/s}$, then we have $\tilde{\alpha}(r_1e^{2\pi it})=(2ar^{\mu/s}e^{2\pi i\mu t}, \alpha(re^{2\pi ist}))$. But clearly one representative of the homotopy class of β is the loop $t\mapsto(\varkappa(0)e^{2\pi i\mu t}, \alpha(re^{2\pi ist}))$, which by an obvious deformation in the first coordinate only (using the properties of the "rings") is seen to be homotopic to the loop $t\mapsto\tilde{\alpha}(r_1e^{2\pi it})$. This proves that $\beta\in B(V(f))$, as asserted. For β' $(=\tilde{\gamma}^s\tilde{\varkappa}^{\mu'})$ the proof is similar, but with the above $\tilde{\alpha}$ replaced by $\tilde{\alpha}'(z)=(bz^{\mu'}/2, \alpha(z^s))$, where b is the absolute value of the leading in coefficients (3), corresponding to the "ring" next outside that of σ. Finally, for $\beta^d(\beta')^{d'}$ (where d and d' are integers >0) we can take $\tilde{\alpha}_{d,d'}(z)=(z^{d\mu+d'\mu'}, \alpha(z^{ds+d's}))$. For when $|z^{ds+d's}|=r_1$ we have $|z^{d\mu+d'\mu'}|=r_1^c$, with $c=(d\mu+d'\mu')/(ds+d's)$, and hence $\mu/s<c<\mu'/s$. So we can use the same argument as above, since these inequalities mean that $z^{d\mu+d'\mu'}$ will lie strictly between the points $\eta_j(z^{ds+d's})$ of two "rings".

Thus we have proved that for all non-negative integers d, d' we have $\beta^d(\beta')^{d'}\in B(V(f))$ (trivially β^d belongs to $B(V(f))$, since β does, and similarly for $(\beta')^{d'}$; these cases were considered above only for $d=1$ and $d'=1$). Further we have shown that the contribution to $(T_\gamma^{sp}-I)^{m+1}g_0$ from the straight piece σ vanishes, if only p is divisible by some positive integer k. Now let us turn to the contribution from the circle $\varkappa$. Let λ be the positive integer $\mu'-\mu$, with μ, μ' as above. When z runs through the circle $|z|=r$ s times, then in our deformation the number of times that $\varkappa$ is run through increases by λ. Using the above description of the integrands on the straight piece σ, and then continuing along $\varkappa^\lambda$, we see that, in $T^{sq}g_0$, on $\varkappa^\lambda$ we have to integrate (the continuation along $\varkappa^\lambda$ of) the sum

$$\sum_{j=0}^{q-1}T_{\tilde{\varkappa}}^{\lambda j}T_{\tilde{\gamma}}^{sq}T_{\tilde{\varkappa}}^{\mu q}f_0=T_{\tilde{\gamma}}^{sq}T_{\tilde{\varkappa}}^{\mu q}\sum_{j=0}^{q-1}T_{\tilde{\varkappa}}^{\lambda j}f_0=\varphi_q(T_{\tilde{\gamma}}, T_{\tilde{\varkappa}})f_0,$$

where φ_q is the rational function

$$\varphi_q(\xi,\eta)=\xi^{sq}\eta^{\mu q}\sum_{j=0}^{q-1}\eta^{\lambda j}=\xi^{sq}\eta^{\mu q}(\eta^{\lambda q}-1)/(\eta^\lambda-1)=((\xi^s\eta^{\mu'})^q-(\xi^s\eta^\mu)^q)/(\eta^\lambda-1).$$

From the binomial theorem it follows that in $(T_\gamma^{sp}-I)^{m+1}g_0$ our integrand on $\varkappa^\lambda$ is $\Psi(T_{\tilde{\gamma}}, T_{\tilde{\varkappa}})f_0$, where ψ is the rational function

$$\Psi(\xi,\eta)=\sum_{j=1}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}\varphi_{pj}(\xi,\eta)$$

$$=(\eta^\lambda-1)^{-1}\left(\sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}(\xi^s\eta^{\mu'})^{pj}-\sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}(\xi^s\eta^\mu)^{pj}\right)$$

$$=(\eta^\lambda-1)^{-1}\big(((\xi^s\eta^{\mu'})^p-1)-((\xi^s\eta^\mu)^p-1)^{m+1}\big).$$

So we have

$$(5)\qquad \Psi(\xi,\eta)=\frac{(\xi^s\eta^{\mu'})^p-(\xi^s\eta^{\mu})^p}{\eta^\lambda-1}\sum_{j=0}^m\big((\xi^s\eta^{\mu'})^p-1\big)^j\big((\xi^s\eta^{\mu})^p-1\big)^{m-j}.$$

Since $\lambda=\mu'-\mu$, the first factor to the right in (5) is equal to η^μ times a polynomial. By the lemma at the end of this section the polynomials $(X-1)^j(Y-1)^{m-j}$ (where $0\leqq j\leqq m$) all belong to the ideal in the polynomial ring $Q[X, Y]$ generated by the polynomials $(X^aY^b-1)^m$, where a, b are integers $\geqq 0$. Hence, replacing X by $(\xi^s\eta^{\mu'})^p$ and Y by $(\xi^s\eta^\mu)^p$, and letting $Q_\eta(\xi,\eta)$ be the ring of all rational functions of the form $h(\xi\ \eta)/\eta^j$, where $h\in Q[\xi,\eta]$ and $j\in Z$ (the integers), we get from (5) that ψ belongs to the ideal in $Q_\eta(\xi,\eta)$ generated by the rational functions $((\xi^s\eta^{\mu'})^{ap}(\xi^s\eta^\mu)^{bp}-1)^m$, where a, b are integers $\geqq 0$. Replacing ξ by $T_{\tilde\gamma}$ and η by $T_{\tilde\varkappa}$ and using that, as proved earlier, every $\beta^a(\beta')^b$ is in $B\big(V(f)\big)$ (where $\beta=\tilde\gamma^s\tilde\varkappa^\mu$ and $\beta'=\tilde\gamma^s\tilde\varkappa^{\mu'}$), and that $T_{\tilde\varkappa}$ is invertible, we conclude that $\psi(T_{\tilde\gamma}, T_{\tilde\varkappa})f_0=0$ if p is divisible by some positive integer k'. For we only have to involve finitely many pairs (a, b), and for each of them we know that $\big((\beta^a(\beta')^b)^p-I\big)^{m+1}f_0=0$, if only p is divisible by some positive integer $k_{a,b}$. So we have shown that in $(T_\gamma^{sp}-I)^{m+1}g_0$ the contribution from the circle $\varkappa$ vanishes, if only $k'|p$.

Applying the above results to all the (finitely many) straight pieces and circles of the contour of integration, we find that all the contributions to $(T_\gamma^{sp}-I)^{m+1}g_0$ vanish, if only p is divisible by some positive integer k_0, and thus we have $(T_\gamma^{sp}-I)^{m+1}g_0=0$ then. This proves the theorem, under the simplifying conditions imposed above.

Now let us see that these assumptions can be removed. For one thing we have supposed that in dividing the points $\eta_j(z)$ into "rings" according to the expansions (3) none of the leading coefficients was real. Now instead let us make the much weaker assumption that for none of the $\eta_j(z)$: s all the coefficients in (3) are real, unless $\eta_k(z)$ is identically equal to 0 or 1, i.e. we demand that all the $\eta_j(z)$: s are non-real when z is large and positive, with the mentioned (possible) exceptions. Then one must in general refine the deformation. So, in every "group" of points $\eta_j(z)$ having the same leading real coefficient in (3), make a partition into "rings" of second order, in the same way as for the "rings" above, but now according to the second non-zero term in the expansions (3). If we make the same deformation as above (thus with respect to the "rings" of first order), this will in general not suffice to avoid the points $\eta_j(z)$. But these points can cross the contour of integration only at the straight pieces, since they stay in the annuli described above. Let σ be one of the straight pieces in our deformation. If, in the corresponding "ring" of first order, there is no point $\eta_j(z)$ the leading coefficient of which (in (3)) is real (or even positive), then we do not have to deform σ further, since the points $\eta_j(z)$ of our "ring" cannot cross the contour (and the deformations with respect to all the other "rings" will all be made within the corresponding annuli). On the other hand, if there are points $\eta_j(z)$ of our "ring" with a positive leading coefficient in (3), then this coefficient must be common (and hence the whole leading term), i.e. these $\eta_j(z)$: s all belong to the same "group" (of first order). Further, this common leading term $\xi(z)$ (say) will be a point lying on σ throughout the process of deformation. From our assumptions on the $\eta_j(z)$: s it follows that $\eta_j(z) \neq \xi(z)$ for all j and large complex z. Now, on the above deformation of first order we superpose a deformation of the straight piece σ. This deformation is made in the same way as that of first order, but now with respect to the "rings" of second order in the "group" that goes with $\xi(z)$, and, of course, with $\xi(z)$ as the centre of the annuli. To be sure, σ now lies on both sides of the centre $\xi(z)$, (unless $\xi(z) \equiv 1$), but we use the obvious generalization to a double-sided deformation. Also, we perform simultaneously the corresponding deformations of all the straight pieces in the deformation of first order. A typical picture of the deformation with respect to "rings" of both first and second order is given in *Figure 2*.

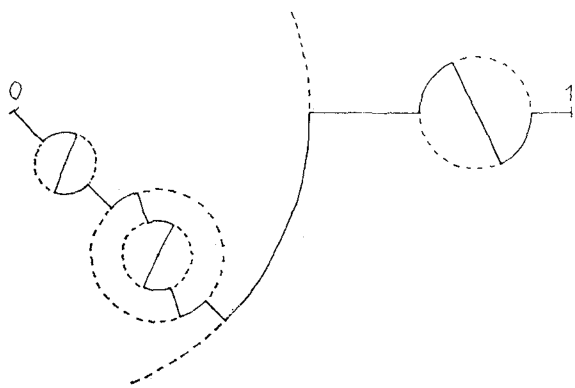

Fig. 2

If all the second coefficients in (3) are non-real when the first one is real, then the described deformation is sufficient. If not, we go on in the same way, getting "rings" and "groups" of order 3, 4, ... Since there is only a finite number of points $\eta_j(z)$, and none of them has got all its coefficients real (except when $\eta_j(z) \equiv 0$ or $\equiv 1$), the process of repeated subdivision will stop after a finite number of steps. It is clear that the simultaneous deformation with respect to all the "rings" of different orders is such that no point $\eta_j(z)$ ever crosses the contour of integration, when $|z|$ is large enough. So this deformation suffices to give the continuation of our integral along the loops γ_r. It is also easy to see that the above argumentation for the vanishing of the integrand holds also in the present more general case. For instance, we can apply the linear mapping $x \mapsto x/\xi(z)-1$ in x separately to be able to use the same description (apart from double-sidedness) of the deformation around a centre $\xi(z)$ as in the simplified case. At this, the crucial loops in the latter case correspond, by the inverse mapping to loops δ_r of the form $t \mapsto \beta(re^{2\pi it}) \in \mathbf{C}^{1+l}$, with β algebraic, regular analytic, and single-valued in a pointed neighbourhood of ∞. Further, it is in the nature of the deformation that every such δ_r does not intersect the critical manifold $V(f)$, when r is large enough. But this is precisely what it takes to make these loops δ_r belong to $B(V(f))$. So we see that the whole proof works also in the more general case that we have now considered. (To handle this case, it is also possible, using a similar argument, to divide the path of integration into a finite number of parts, for each of which one has the simplified situation treated earlier. even with only one "ring" to have to deform for.)

Thus it remains only (in the case $n=1$) to remove the assumption that when z is large and positive none of the points $\eta_j(z)$ lies on the real line, except possibly at 0 or 1 (and in that case identically). Now $\eta_j(z)-\eta_k(z)$, $\eta_j(z)-0$, and $\eta_j(z)-1$ are for all j and k, algebraic functions of z. Using their Puiseux expansions, we can easily see that when z is positive and sufficiently large, the (deformed, since we have started the continuation of g at an arbitrary point in $\mathbf{C}^l$) contour of integration can be composed of a finite number of straight line segments, each having endpoints that are algebraic functions of z (e.g. of the form $\eta_j(z)+cz^a$, where c is a complex and a a real constant), and where the points $\eta_j(z)$ keep away from the straight lines prolonging the segments (unless $\eta_j(z)\equiv 0$ or $\equiv 1$ and the segment is one of the extreme ones). For each of these segments we then have the simplified situation already treated (using entire linear mappings in x, depending algebraically on z, one easily reduces oneself to the case that the endpoints of the segment are 0 and 1). Performing the deformation of all these segments at the same time, in the manner described above, we find also in the general case that if only the integer k is divisible by some positive integer k_0, then for large r $(T^k_{\gamma_r}-I)^{m+1}g_0$ is the integral of the zero function along some contour, and thus vanishes. This ends the proof in the case $n=1$.

To prove the theorem in the case of a general n, we use induction over n. Thus, assume that Theorem 1 is true when $n=p$, and consider the case $n=p+1$. Then we have

$$g(y) = \int_{E_{p+1}} f(x, y)\, dx = \int_{E_p} f_1(x_1, \dots, x_p, y)\, dx_1 \dots dx_p$$

with

$$f_1(x_1, \dots, x_p, y) = \int_0^{1-x_1-\dots-x_p} f(x, y)\, dx_{p+1}$$

$$= (1-x_1-\dots-x_p)\cdot\int_0^1 f(x_1, \dots, x_p, t(1-x_1-\dots-x_p), y)\, dt.$$

Clearly f_1 is regular analytic in a neighbourhood of $E_p\times\{y^0\}$, and the last integrand is so (as a function of $(x_1, \dots, x_p, t, y_1, \dots, y_l)$) in a neighbourhood of $E_p\times[0, 1]\times\{y^0\}$. Further, since $f\in P_m$, it follows from the remarks of Section 2 that the last integrand is a function in $P_m(\mathbf{C}^{p+1+l})$, for of course $(x_1, \dots, x_p, t(1-x_1-\dots-x_p), y)$ is not identically contained in any genuine algebraic sub-manifold of $\mathbf{C}^{p+1+l}$. From the result in the case $n=1$ we conclude that $f\in P_{m+1}(\mathbf{C}^{p+1})$. By the induction hypothesis it then follows that $g\in P_{m+p+1}(\mathbf{C}^l)$. This completes the induction and by that the proof of Theorem 1, as soon as the following lemma (referred to above) has been proved.

Lemma. (Tráng—Björk—Scarpalézos) *Let m be a positive integer, and let $(j_1, k_1), \dots, (j_N, k_N)$ be pairs of integers >0, with $N\geqq 2m-2$ that are pairwise non-proportional (over the rationals). Further let a, b be integers $\geqq 0$ such that $a+b\geqq m$. Then the polynomial $(X-1)^a(Y-1)^b$ belongs to the ideal I in the polynomial ring $Q[X, Y]$ (Q being the field of rationals), generated by the polynomials $(X-1)^m, (Y-1)^m, (X^{j_1}Y^{k_1}-1)^m, \dots, (X^{j_N}Y^{k_N}-1)^m$.*

Proof. The proof is by induction, using the induction hypothesis $H(i)$: $(X-1)^a(Y-1)^b\in I$ whenever a, b are integers $\geqq 0$ such that $a+b\geqq i$. Now $H(2m)$ is trivially true, since when $a+b\geqq 2m$ either a or b is $\geqq m$. In the general induction step we have to prove that if $H(i)$ is true for an $i>m$, then $H(i-1)$ is true. For then the induction will give that $H(m)$ is true, which is the statement of the lemma.

To get a uniform description of the polynomials that generate I, write $(j_0, k_0)=(1, 0)$ and $(j_{N+1}, k_{N+1})=(0, 1)$. Changing variables in the polynomials: $s=X-1$, $t=Y-1$ (which defines an automorphism of $Q[X, Y]$), we have to prove that when $a+b\geqq i-1$ we have $s^a t^b\in\tilde I$, where $\tilde I$ is the ideal in $Q[s, t]$ generated by the polynomials $((s+1)^{j_r}(t+1)^{k_r}-1)^m$ $(r=0, \dots, N+1)$. Using the induction hypothesis, that this is true when $a+b\geqq i$, we get from the binomial expansions that $((s+1)^{j_r}(t+1)^{k_r}-1)^{i-1}-(j_r s+k_r t)^{i-1}$ belongs to $\tilde I$, and hence, since $i-1\geqq m$, that $(j_r s+k_r t)^{i-1}\in\tilde I$ (for $r=0, \dots, N+1$). That is, we have

$$\sum_{\nu=0}^{i-1}\binom{i-1}{\nu}(j_r/k_r)\, s^\nu t^{i-1-\nu}\in\tilde I \quad (r=1, \dots, N+1),$$

and also $s^{i-1}\in\tilde I$. Since the numbers j_r/k_r are by assumption all different, and since $N+2\geqq 2m\geqq i$, it follows by linear elimination that $s^\nu t^{i-1-\nu}\in\tilde I$ whenever $0\leqq\nu\leqq i-1$, which implies $H(i-1)$.

4. Integrals on variable algebraic manifolds

In this section we shall use Theorem 1 to obtain a result for the integral of a differential form with coefficients in the class P_m over a cycle on a variable algebraic manifold.

Thus, let $p_1(x, y), \dots, p_r(x, y)$ be real polynomials in $x\in\mathbf{R}^n$, where $r<n$, and in $y\in\mathbf{R}^l$, and let R_y be that part of the algebraic manifold (in $\mathbf{R}^n$) defined by $p_1(x, y)=p_2(x, y)=\dots=p_r(x, y)=0$, where the differentials $d_x p_1(x, y), \dots, d_x p_r(x, y)$ are linearly independent. Assume that for some $y^0\in\mathbf{R}^l$ the set R_{y^0} is non-empty and that we are given a (singular) k-cycle γ_0 on R_{y^0}. Further suppose that we have an exterior differential k-form $\omega_y(x)$ in an open neighbourhood Ω of γ_0, depending on the parameter y. Then we can write (uniquely)

$$\omega_y(x) = \textstyle\sum_I f_I(x, y)\, dx_I, \tag{6}$$

where the sum is taken over all strictly increasing k-tuples $I=(i_1, \dots, i_k)$ with $1\leqq i_j\leqq n$ for all j, and where $dx_I=dx_{i_1}\wedge\dots\wedge dx_{i_k}$. Let us assume that all the coefficients $f_I(x, y)$ are real analytic in the pair (x, y) in a set $\Omega\times\Omega_1$, where Ω_1 is an open neighbourhood of y^0. Further we require that the restriction of $\omega_y(x)$ to $R_y\cap\Omega$ is closed for all y near y^0. Clearly we can, for y close to y^0, define $\gamma(y)$ as a k-cycle on R_y in such a way that $\gamma(y^0)=\gamma_0$ and that $\gamma(y)$ varies continuously with y in the uniform sense. Then the function

$$g(y) = \int_{\gamma(y)} \omega_y(x) \tag{7}$$

is defined in a neighbourhood of y^0 (in $\mathbf{R}^l$), and it is easily seen that it is real analytic in such a neighbourhood. The germ of g at y^0 is clearly independent of the particular way $y\mapsto\gamma(y)$ of deforming γ_0, since $\omega_y(x)$ is closed on R_y. We now have the following theorem.

Theorem 2. *Assume that all the coefficients f_I in* (6) *are in $P_m(\mathbf{C}^{n+1+l})$, i. e. every local element of them can be continued analytically to a function in $P_m(\mathbf{C}^{n+1+l})$ (where the critical manifolds $V(f_I)$ may intersect $\{\gamma_0\}\times\{y^0\}$). Then the function g defined by* (7) *is in $P_{m+k}(\mathbf{C}^l)$.*

Remark. Theorem 2 easily gives a corresponding result for complex algebraic manifolds. Then, of course, one lets y-space be $\mathbf{C}^l$ and assumes $\omega_y(x)$ also to be holomorphic with respect to y in a neighbourhood (in $\mathbf{C}^{l+n}$) of $\{y^0\}\times\{\gamma_0\}$. For, in order to apply Theorem 2, we consider the R_y: s as manifolds in $\mathbf{R}^{2n}$ and first restrict y to a neighbourhood of y^0 in the set $y^0+\mathbf{R}^l$ (clearly the function g, as defined by (7) also in the complex case, will be holomorphic in a neighbourhood of y^0).

Proof of Theorem 2. Let us choose (oriented) Euclidean k-simplices $S_1, \ldots, S_q$ lying in an open subset of $\mathbf{R}^n$, where short distance projection along the normal planes $\{x^0+t_1 \operatorname{grad}_x p_1(x^0, y^0)+\ldots+t_r \operatorname{grad}_x p_r(x^0, y^0);\ t_j\in\mathbf{R},\ \text{all } j\}$ (with x^0 on R_{y^0}) is uniquely defined and real analytic when y is sufficiently close to y^0, and where the projections of the simplices S_i constitute a k-cycle on R_y, homotopic to $\gamma(y)$ in $R_y \cap \Omega$, Ω being the above-mentioned neighbourhood of γ_0. In the definition of g replacing $\gamma(y)$ by the cycle made up of the projections of the S_i: s on R_y, and mapping each S_i affinely onto the unit simplex E_k we can write

$$g(y)=\int_{E_k} F(\xi, y)\, d\xi,$$

where $F(\xi, y)$ is a sum of terms of the form $\varphi(\xi, y) f_I(\psi(\xi, y), y)$, where f_I is one of the coefficients of $\omega_y(x)$ (in (6)), while φ and ψ are algebraic and real analytic in a neighbourhood of $E_k\times\{y^0\}$. Clearly we can, if necessary, in a convenient way add a new variable to y (e.g. in the equations of the manifold) to achieve that $(\psi(\xi, y), y)$ does not belong identically to the critical manifold $V(f_I)$ in any of these terms, and we assume that this has been done. Then F is in $P_m(\mathbf{C}^{k+l})$, by the observations in Section 2. Further F also extends to a regular holomorphic function in a neighbourhood (in $\mathbf{C}^{k+l}$) of $E_k\times\{y^0\}$. Thus, by Theorem 1, g can be extended to a function in $P_{m+k}(\mathbf{C}^l)$. Since we have possibly added a new variable to y (y_1, say), it remains to show that the function $g_1(y_1, \ldots, y_{l-1})=g(y_1, \ldots, y_{l-1}, y_1^0)$ is in $P_{m+k}(\mathbf{C}^{l-1})$ (we may suppose that the value $y_1=y_1^0$ corresponds to the original function g, now called g_1). Put $y'=(y_1, \ldots, y_{l-1})$, and let the critical manifold $V(g)$ be given by the equation $p(y', y_1)=0$. We can write $p(y', y_1)=(y_1-y_1^0)^j p_1(y', y_1)$, where j is an integer $\geqq 0$ and p_1 a polynomial such that $p_1(y', y_1^0)$ does not vanish identically as a polynomial in y'. Now let us prove that $g_1\in P_{m+k}(\mathbf{C}^{l-1})$, with $V(g_1)$ equal to the manifold W: $p_1(y', y_1^0)=0$.

When y' is close to $(y')^0$, we have by Cauchy's formula

$$g_1(y')=(2\pi i)^{-1}\int_{|\eta-y_1^0|=\delta} (\eta-y_1^0)^{-1} g(y', \eta)\, d\eta, \tag{8}$$

if the positive number δ is small enough. It is easy to see that by (8) we can continue g_1 analytically along any path in $C^{l-1}\setminus W$; we must only take δ so small that $p_1(y', \eta)\neq 0$ for all y' on the path and all η with $|\eta-y_1^0|=\delta$. To see that g_1 satisfies a monodromy formula of the required kind, we have to consider an algebraic mapping $C\ni z\mapsto\alpha(z)\in\mathbf{C}^{l-1}$, which is regular analytic and single-valued in a pointed neighbourhood of ∞, and where further $\alpha(z)\notin W$ when $|z|$ is sufficiently large. Continuing g_1 along the loops β_r: $t\mapsto\alpha(re^{2\pi i t})$, by the use of (8), and wanting to prove that $(T^i_{\beta_r}-I)^{m+k} g_1\to 0$, it will be enough to show that we can choose the corresponding radius δ of (8) of the form r^{-N}, with N a positive integer, such that the loop $\hat{\beta}_r$: $t\mapsto(\beta_r(t), y_1^0+r^{-N}e^{-2\pi i N t})$ lies in $\mathbf{C}^l\setminus V(g)$ for all sufficiently large r. For then it is in $B(V(g))$, so that $(T^i_{\beta_r}-I)^{m+k}g_0=0$ for some positive integer i, where g_0 denotes the function element of g appearing in (8) for the arbitrary element of g_1 for which we want to show the monodromy formula. Since it is in the nature of the method of continuation that we have all the time a single-valued element of g on the circle of integration, continuation along $\hat{\beta}_r$ gives the same result as along the loop $t\mapsto(\beta_r(t), y_1^0+r^{-N})$, that is, the loop run through by the point $\eta=1$ in (8) in the continuation. It follows that we have $(T^i_{\beta_r}-I)^{m+k}g_{10}=0$ for any element g_{10} of g_1 and for some positive integer $i\ (=i(g_{10}))$. So it only remains to see that the integer N can be chosen as stated. We know that $h(z)=p_1(\alpha(z), y_1^0)$ is different from zero when $|z|$ is large enough. Expanding $h(z)$ and $\alpha(z)$ in Puiseux series we see that if we choose N large enough we also have $p_1(\alpha(z), y_1^0+\varrho z^{-N})\neq 0$, and thus also $p(\alpha(z), y_1^0+\varrho z^{-N})\neq 0$ when $|z|$ is large enough and $0<\varrho\leqq 1$. This implies exactly the properties needed, and so Theorem 2 is proved.

5. The asymptotic behaviour of certain integrals

Let us start this section by defining a subclass $AP_m(\mathbf{C}^n)$ of $P_m(\mathbf{C}^n)$, by imposing a growth condition (the same as in Nilsson [3]).

Definition. When m is a positive integer, $AP_m(\mathbf{C}^n)$ will denote the class of all functions f in $P_m(\mathbf{C}^n)$ which also satisfy the following condition. There are real constants $a=a(f)$ and $b=b(f)$, a choice $V(f)\colon p(z)=0$ of the critical manifold of f, and further to every pair (k_1, k_2) of positive integers and every function element f_0 of f at some point z^0 in $\mathbf{C}^n\setminus V(f)$ another real constant $C=C(k_1, k_2, f_0, z^0)$ such that

$$|f(z)|\leqq C(|z|+1)^a|p(z)|^{-b} \tag{9}$$

for all $z\in\mathbf{C}^n\setminus V(f)$ and every determination of f at z which can be obtained from f_0 by continuation from z^0 to z along a path consisting of at most k_1 pieces, each being a regular algebraic path $t\mapsto z(t)$ such that the defining polynomials of the coordinate functions $t\mapsto z_j(t)$ can all be taken of total degree $\leqq k_2$.

As an application of Theorem 2 of this paper and Theorem 1 of Nilsson [3] we now give the following theorem.

Theorem 3. *Let f be a real-valued function defined in an open subset Ω of R^n, and assume that f is regular analytic and algebraic in Ω. Suppose that there is a real number λ_0 such that the set $M_\lambda=\{x\in\Omega;\ \lambda_0\leqq f(x)\leqq\lambda\}$ is compact for all real numbers λ. Let the complex-valued function g be defined and real analytic in Ω, and, moreover, suppose that g is in $AP_m(\mathbf{C}^n)$. Put*

$$e(\lambda)=\int_{M_\lambda} g(x)\,dx \quad (\lambda\in\mathbf{R}).$$

Then we have $e(\lambda)=\sum_{j=0}^{m+n-1}(\log\lambda)^j e_j(\lambda)$ $(\lambda\in\mathbf{R})$, *where each of the functions* $e_j(\lambda)$ *has, for* λ *large and positive, a Puiseux expansion* $\sum_{k=-\infty}^{\infty} a_{jk}\lambda^{k/s}$, *where* s *is a positive integer, and where* $a_{jk}\neq 0$ *for at most finitely many positive* k.

Further (as a consequence of this expansion of $e(\lambda)$*) there is a rational number* c *and an integer* p, *with* $0\leqq p\leqq m+n-1$, *and a complex constant* C *such that*

$$e(\lambda)=C(1+o(1))\lambda^c(\log\lambda)^p \quad (\lambda\to+\infty), \tag{10}$$

and this formula can be differentiated any number of times, in each step taking the derivative of $1+o(1)$ as $o(1)\lambda^{-1}$.

If $c\neq 0$, *we have actually* $0\leqq p\leqq m+n-2$. *In particular, if* g *is algebraic and if* $e(\lambda)$ *tends to either* 0 *or* ∞ *at least with some negative or positive (respectively) power of* λ, *then* (10) *is valid with* $0\leqq p\leqq n-1$.

Proof. Put $\Omega_0=\{x\in\Omega;\ f(x)>\lambda_0\}$ and $G=\{x\in\Omega_0;\ df(x)=0\}$, and let us see (using standard methods in algebraic geometry) that $f(G)$ is a finite set. For this it is, in view of Sard's theorem, sufficient to show that G has only a finite number of connected components. To do so, let us consider non-trivial algebraic equations (in y) $p(x,y)=\sum_0^M p_j(x)y^j$ and $q(x,y)=\sum_0^N q_k(x)y^k$ satisfied by $f(x)$ and $h(x)=|\mathrm{grad}\, f(x)|^2$, respectively, when $x\in\Omega_0$. Of course we can take the polynomials p_j and q_k real. Also put $r(x,y)=p(x,y+\lambda_0)=\sum_0^M r_l(x)y^l$. By a linear change of coordinates we can achieve that for every $(x_2,\dots,x_n)\in\mathbf{R}^{n-1}$ neither $p_M(x)$ $(=r_M(x))$ nor $q_N(x)$ vanishes identically as a polynomial in x_1. Then the equation (in y) $p(x,y)=0$ has, for any $x\in\mathbf{R}^n$, exactly M complex solutions (possibly infinite and possibly coinciding) defined by considering y as an algebraic function of x_1 separately, and $f(x)$ is one of these solutions for every $x\in\Omega_0$. The corresponding statement is valid for the equation $q(x,y)=0$ and $h(x)$, and, of course, for $r(x,y)=0$ and $f(x)-\lambda_0$. Let F be the (finite) class of the algebraic manifolds (in the natural compactification $C\mathbf{R}^n$ to a closed ball in $\mathbf{R}^n$, say) $\{x;\ \partial^\mu p_j(x)/\partial x_1^\mu=0\}$, $\{x;\ \partial^\nu q_k(x)/\partial x_1^\nu=0\}$, $\{x;\ \partial^\sigma r_l(x)/\partial x_1^\sigma=0\}$, and the infinity plane (where μ, j, ν, k, σ, l vary in the non-negative integers). Then there is a (finite) triangulation of $C\mathbf{R}^n$ such that each of the manifolds of F is a union of faces (of some dimension) of the simplices of the triangulation (cf. van der Waerden [7], app. of Chap. 4). From this we can see that $\mathbf{R}^n$ is the disjoint union of a finite number of connected sets L such that the number of roots of the equation $p(x,y)=0$ equal to infinity is constant on every L, as well as the number of them equal to λ_0, and that the number of roots of the equation $q(x,y)=0$ equal to zero is also constant on every L, and that further all the roots of both equations vary continuously on L. Now Ω_0 must be the union of such sets L. For we have that on any L all the finite roots of $p(x,y)=0$ are locally bounded, and those different from λ_0 are locally bounded away from λ_0. So, assuming that L intersects Ω_0 as well as its complement, we choose a point $\alpha\in L$ in the closure of $L \cap \Omega_0$ but not in Ω_0, we get a contradiction, since α must lie on the boundary of Ω_0, so that either $f(x) \to +\infty$ or $f(x) \to \lambda_0$ when $x \to \alpha$, $x \in \Omega_0$.

Again by the properties of the sets L, we have that $h(x) = |\text{grad } f(x)|^2$ on every $L \subset \Omega_0$ either vanishes identically or has no zero at all. So $G = \{x \in \Omega_0;\ df(x) = 0\}$ is the union of some of the sets L (which are finitely many and connected).

Let us suppose that the number λ_0 of the present theorem is larger than $\max f(G)$, which is no restriction, since changing λ_0 means only the addition of a constant to $e(\lambda)$, giving a new expansion of $e(\lambda)$ of the same kind (and the rest of the theorem will follow from the expansion). It is then clear that $e(\lambda)$ is a regular analytic function of the real variable when $\lambda > \lambda_0$, and that then

$$de(\lambda)/d\lambda = \int_{S_\lambda} g\, dx/df, \tag{11}$$

where S_λ is the (real analytic) manifold $f(x) = \lambda$. Now we can apply Theorem 2 to the integral (11). For S_λ is an orientable compact C^∞ manifold in $\mathbf{R}^n$ of dimension $n-1$, and hence, as is well known, an $(n-1)$-cycle. Also, it varies continuously with λ. Further the differential form $g dx/df$, which we can regard as a differential form on an open subset of $\mathbf{R}^n$ is clearly closed on every S_λ and has coefficients which are regular analytic in a neighbourhood of S_λ and belong to $AP_m(\mathbf{C}^n)$ and hence also to $AP_m(\mathbf{C}^{n+1})$, including λ as a variable. For it is evident that the product of an algebraic function on $\mathbf{C}^n$ and a function in $AP_m(\mathbf{C}^n)$ is in $AP_m(\mathbf{C}^n)$. So by Theorem 2 we get that $de/d\lambda$ is in $P_{m+n-1}(\mathbf{C})$. Further Theorem 1 of Nilsson [3] gives that $de/d\lambda$ satisfies the growth condition for $AP_{m+n-1}(\mathbf{C})$. For the requirement of that theorem of linear finiteness of the integrand is not used in proving that the integral obeys the growth condition. Thus the function $de/d\lambda$ has the following properties:

i) *There is a positive constant k such that $de/d\lambda$ can be continued analytically along any path in the complex region $|\lambda| > k$.*

ii) *There is a positive integer s such that in this region we have $(T^s - I)^{m+n-1}(de/d\lambda) = 0$, where T is continuation one round in the positive sense along circles $|\lambda| = \text{constant}$.*

iii) *To every branch E of $de/d\lambda$ in the region $|\lambda| > k$, $-\pi < \arg \lambda < \pi$ there are real constants a and K such that $|E(\lambda)| \leq K|\lambda|^a$ for all λ in the region.*

By a simple induction argument it follows from the properties (i) and (ii) that $de(\lambda)/d\lambda$ can be written in the form $\sum_{j=0}^{m+n-2} (\log \lambda)^j h_j(\lambda)$, where the h_j: s are (in general many-valued) analytic functions in the region $|\lambda| > k$, all satisfying $(T^s - I) h_j = 0$. Further each $h_j(\lambda)$ is a (finite) linear combination of terms of the form $(\log \lambda)^i E(\lambda)$, where i is an integer and E a branch of $de/d\lambda$. So it follows from (iii) that the h_j: s grow at most polynomially, and thus their Puiseux expansions have only a finite number of terms with a positive exponent. Integrating $de/d\lambda$

term-wise (using these expansions of the h_j: s), we get the desired expansion of $e(\lambda)$, since the exponents of the logarithms increase by at most one at the integration. The asymptotic properties of $\varepsilon(\lambda)$ follow at once from the expansion, except the refinement in the case $c\neq 0$ (in (10)). But in this case the leading term in the expansion of $e(\lambda)$ is $C\lambda^c (\log \lambda)^p$, where $c\neq 0$, and where we have used the notations of (10). So it must have come from integrating the term $Cc\lambda^{c-1} (\log \lambda)^p$ in the expansion of $de(\lambda)/d\lambda$, and thus we have $0\leqq p\leqq m+n-2$. This completes the proof of Theorem 3.

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Received September 15, 1979

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