Equivalence of generalized moduli of continuity

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1. Introduction

The generalized modulus of continuity $\omega_\sigma(f, t)$ for a function f on R^n with respect to a measure σ on R^n is defined as

$$\omega_\sigma(f, t) = \sup\{\|\sigma_{(u)} * f\|;\ 0 < u \leqq t\},$$

where $\|\cdot\|$ is some norm, e.g. supremum norm, $*$ denotes convolution, and $\sigma_{(u)}$ is the so-called "dilation" of σ, which is defined by $\sigma_{(u)}(x) = u^{-n}\sigma(x/u)$, or more precisely, $\int \varphi(x)\,d\sigma_{(u)}(x) = \int \varphi(ux)\,d\sigma(x)$ for all continuous functions φ with compact support. The interesting case is when $\int d\sigma = 0$; in this case $\lim_{t\to 0} \omega_\sigma(f, t) = 0$ for every bounded and uniformly continuous function f. The problem is to compare the order of magnitude of $\omega_\sigma(f, t)$ and $\omega_\tau(f, t)$ as $t \to 0$ for given pairs σ, τ. In this paper we study this problem in detail. Specifically we study for which pairs σ, τ the inequality

$$\omega_\tau(f, t) \leqq C\omega_\sigma(f, Bt) \tag{1.1}$$

holds with constants C and B independent of f. Inequalities of this kind have applications to approximation theory. For instance the well-known Jackson and Bernstein theorems concerning trigonometric approximation can easily be deduced from such inequalities.

Our results are formulated in terms of the "order" of the zero at the origin of the Fourier transforms $\hat\sigma(\xi)$ and $\hat\tau(\xi)$ of the measures involved. With each σ there will be associated an ideal $\hat J_0(\sigma)$ in the ring of germs at the origin of Fourier transforms of measures. More exactly $\hat J_0(\sigma)$ is the ideal generated by all the elements $\{\hat\sigma(u\xi);\ u > 0\}$. Our main result states that, provided σ satisfies certain a priori conditions, then (1.1) holds if and only if

$$\hat J_0(\tau) \subset \hat J_0(\sigma). \tag{1.2}$$

In simple cases this condition means that $\hat\sigma$ divides $\hat\tau$ locally at the origin.

The problem of comparing generalized moduli of continuity has been studied by H. S. Shapiro [14]. A typical result of Shapiro states that the estimate

$$\omega_\tau(f, t) \leqq C \int_0^{Bt} \omega_\sigma(f, u) u^{-1} du, \quad t > 0 \tag{1.3}$$

holds if σ and τ satisfy (1.2) and σ satisfies the Tauberian condition (2.11). Thus one of the objects of the present study was to replace (1.3) by the stronger (1.1). A specific motivation for this came from approximation theory. In fact the results on degree of approximation that can be deduced from Shapiro's theorems are somewhat weaker than what is actually known to hold in specific cases. See Section 7 below for details.

As a further application we study the r^{th} order moduli of continuity $\omega_r(f, t)$, $r=1, 2, \ldots$, and more generally the L^p-moduli $\omega_{r,p}(f, t)$ for functions of several variables. From our main theorem one can easily deduce results concerning the equivalence of various possible definitions of the moduli of continuity $\omega_{r,p}(f, t)$. On this point the cases $1<p<\infty$ and $p=1, \infty$ lead to different results (Theorems 6.2 and 6.3).

The method of proof of our main result seems to be of independent interest. An important step is the application of a theorem of Varopoulos on (global) division in certain measure algebras [21].

The plan of the paper is as follows. In Section 2 we give definitions and formulate our results. In Section 3 we prove an estimate for trigonometrical sums in several variables, which is needed in connection with the application of Varopoulos' theorem to our problem. In Section 4 we prove the special case of Varopoulos' theorem which is of interest here. The proof of the main estimate (1.1) is completed in Section 5. In the remaining sections we treat various applications of our results. Higher order moduli of continuity are studied in Section 6, applications to degree of approximation in Section 7, and a couple of other applications are briefly mentioned in Section 8.

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2. Comparison theorems for generalized moduli of continuity

Denote by $M(R^n)$ the set of all complex-valued bounded Borel measures in R^n and by $\dot{C}(R^n)$ the set of complex-valued, bounded, uniformly continuous functions on R^n. The supremum norm in $\dot{C}(R^n)$ is denoted $\|\cdot\|$, and the usual norm in $M(R^n)$ is denoted $\|\cdot\|_M$. The Fourier transform of $\sigma \in M(R^n)$ is denoted $\hat{\sigma}$ and is defined by $\hat{\sigma}(\xi) = \int \exp(-i\langle x, \xi \rangle) d\sigma(x)$, $\xi \in R^n$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. $M(R^n)$ is made into a Banach algebra under convolution, and $L^1(R^n)$ is identified with a subalgebra of $M(R^n)$.

Here are some simple properties of the generalized modulus of continuity. (See also [15], chapter 9.) If $\sigma, \tau \in M(R^n)$, then

$$\omega_{\sigma+\tau}(f, t) \leqq \omega_\sigma(f, t) + \omega_\tau(f, t). \tag{2.1}$$

Writing $\mu = \sigma * \tau$ we have

$$\omega_\mu(f, t) \leqq \|\tau\|_M \, \omega_\sigma(f, t). \tag{2.2}$$

Obviously, for any $a > 0$

$$\omega_{\sigma_{(a)}}(f, t) = \omega_\sigma(f, at). \tag{2.3}$$

If Σ is a finite family of elements of $M(R^n)$ we define $\omega_\Sigma(f, t)$ to be $\sup \{\omega_\sigma(f, t);\, \sigma \in \Sigma\}$.

We denote by $J(\Sigma)$[1)] the ideal in $M(R^n)$ which is generated by all the dilates $\sigma_{(u)}$, $u > 0$, $\sigma \in \Sigma$. If Σ consists of one element $\Sigma = \{\sigma\}$, we write $J(\sigma)$ instead of $J(\Sigma)$. Denoting the elements of Σ by $\sigma^1, \dots, \sigma^J$, each element of $J(\Sigma)$ can be written by means of a finite sum

$$\tau = \sum\nolimits_{k,j} \sigma^j_{(a_{kj})} * \nu^{kj} \tag{2.4}$$

for some measures $\nu^{kj} \in M(R^n)$. We will now introduce an ideal $K(\Sigma)$, which is larger than $J(\Sigma)$, by replacing the sum in (2.4) by an integral. To make this precise we introduce the group $G = R^n \times R_+$ as the (semi-direct) product of the additive group R^n and the multiplicative group R_+ of the positive real numbers. The group G acts on $M(R^n)$ in the following way. Denoting the action of $(h, u) \in R^n \times R_+ = G$ on $\sigma \in M(R^n)$ by $\sigma_{(h,u)}$ we have

$$\sigma_{(h,u)}(x) = \frac{1}{u^n}\, \sigma\left(\frac{x+h}{u}\right), \quad \text{or} \quad \hat{\sigma}_{(h,u)}(\xi) = e^{i\langle h, \xi\rangle} \hat{\sigma}(u\xi).$$

Let $M(G)$ be the algebra of all bounded measures μ on G whose support is contained in $R^n \times [0, B]$ for some B. Now we define $K(\Sigma)$[1)] as the set of all $\tau \in M(R^n)$ which can be written

$$\tau = \sum\nolimits_{j=1}^{J} \int \sigma^j_{(h,u)} \, d\mu^j(h, u) \tag{2.5}$$

for $\mu^j \in M(G)$. The integral is to be interpreted in the weak sense, so that for instance if $J = 1$, (2.5) means the same as

$$\int f \, d\tau = \int_G \int_{R^n} f \, d\sigma_{(h,u)} \, d\mu(h, u) \quad \text{for all} \quad f \in \dot{C}(R^n).$$

[1)] This notation differs from the one used in [6].

Note that (2.5) reduces to (2.4) if each μ^j is a product of an element of $M(R^n)$ and a Dirac measure on R_+, or a sum of such products. Clearly $K(\Sigma)$ is an $M(G)$-module, and in particular an ideal in $M(R^n)$.

It follows from (2.5) that if $\tau\in K(\Sigma)$, then

$$\omega_\tau(f, t) \leqq C\omega_\Sigma(f, Bt). \tag{2.6}$$

(This shows incidentally that ω_Σ depends only on the $M(G)$-module $K(\Sigma)$ and not on the finite generating family Σ.) However, the converse statement is also true, at least if $\hat{\sigma}(0)=0$ for all $\sigma\in\Sigma$ (which is the only interesting case).

Theorem 2.1. *Let the finite subset* $\Sigma\subset M(R^n)$ *be given, let* $\tau\in M(R^n)$, *and assume*

$$\hat{\sigma}(0) = 0 \quad \textit{for every} \quad \sigma\in\Sigma. \tag{2.7}$$

Then (2.6) *holds for some constants* C *and* B *if and only if* τ *belongs to the* $M(G)$-*module* $K(\Sigma)$. *Similarly, if* Σ' *and* Σ *are finite subsets of* $M(R^n)$, Σ *satisfying* (2.7), *then the inequality* $\omega_{\Sigma'}(f, t)\leqq C\omega_\Sigma(f, CT)$ *holds if and only if* $K(\Sigma')\subset K(\Sigma)$.

Proof. For technical reasons we will work with the space $C_0(R^n)$ of functions tending to zero at infinity rather than the space $\dot{C}(R^n)$. The space $M(R^n)$ can be identified with the space of all continuous linear forms on $C_0(R^n)$.

Let σ^j, $j=1, \dots, J$, be the elements of Σ. For each $f\in C_0(R^n)$ and each j consider the function

$$\varphi_j(h, u) = \int f(-x)\, d\sigma^j_{(h,u)}(x) = \sigma^j_{(u)} * f(h), \quad (h, u)\in G. \tag{2.8}$$

Clearly φ_j is continuous. In fact φ_j is uniformly continuous for u in bounded sets, since $\lim_{|h|\to\infty} \varphi_j(h, u)=0$ and $\lim_{u\to 0} \varphi_j(h, u)=0$. The last assertion follows from the fact that $\hat{\sigma}^j(0)=0$ for all j. Set $I=\{u; 0<u\leqq B\}$ and denote by $C_0(R^n\times I)$ the Banach space of all complex-valued uniformly continuous functions on $R^n\times I$ which tend to zero as $u\to 0$ and as $|h|\to\infty$, equipped with the supremum norm. Denote by L_Σ the set of all functions $\varphi=(\varphi_1, \dots, \varphi_J)\in C_0(R^n\times I)^J$ with φ_j of the form (2.8) and $f\in C_0(R^n)$. Assume that (2.6) holds and consider the linear mapping Ψ_0 from L_Σ to the complex numbers defined by

$$\Psi_0(\varphi) = \tau * f(0) = \int f(-x)\, d\tau(x). \tag{2.9}$$

From (2.6) with $t=1$ it follows that Ψ_0 is continuous. By the Hahn—Banach theorem we can extend Ψ_0 to a continuous linear form Ψ defined on all of $C_0(R^n\times I)^J$. But Ψ can be represented by a J-tuple of bounded measures $(\mu^1, \dots, \mu^J)$ on $R^n\times I\subset G$. Combining (2.8) and (2.9) we easily conclude that (2.5) holds, i.e. $\tau\in K(\Sigma)$.

Our main goal is to give conditions for (2.6) in terms of the behaviour at $\xi=0$ of the Fourier transforms of the measures involved. The strongest possible conditions on τ of this sort is of course that $\hat{\tau}$ vanish in a neighbourhood of the origin.

Hence it is natural to begin by asking for which Σ it is true that (2.6) holds for all τ such that $\hat{\tau}(\xi)=0$ in some neighbourhood of the origin. Let Σ be such a set of measures. It follows from Theorem 2.1 that there must exist $\mu^j \in M(G)$ such that

$$1 = \sum_{j=1}^{J} \int \hat{\sigma}^j(u\xi) e^{i\langle \xi, h\rangle} d\mu^j(h, u) \tag{2.10}$$

for ξ outside some compact set. By virtue of Lebesgue's theorem on dominated convergence (2.10) cannot hold if all $\hat{\sigma}^j(\xi)$ tend to zero as $|\xi|\to\infty$. That is to say, (2.10) cannot hold if all the σ^j are absolutely continuous, i.e. belong to $L^1(R^n)$. We can in fact say more: (2.10) cannot hold if there exists $\xi\in R^n\setminus\{0\}$ such that $\lim_{u\to 0} \hat{\sigma}^j(u\xi)=0$ for all j. These observations motivate the following assumption:

(T) *the set of functions* $\{\hat{\sigma}_0(u\xi), u>0, \sigma\in\Sigma\}$ *has no common zero on* $R^n\setminus\{0\}$; *here* σ_0 *denotes the discrete part of* σ.

A bounded measure is called *discrete* if it has the form $\sum_{k=1}^{\infty} c_k \delta_{a_k}$, where $\{a_k\}$ is an arbitrary sequence of points of R^n and $\sum |c_k|<\infty$. The condition

$$\{\hat{\sigma}(u\xi);\ u > 0,\ \sigma\in\Sigma\} \text{ has no common zero on } R^n\setminus\{0\} \tag{2.11}$$

was introduced by H. S. Shapiro in [14] and was called the Tauberian condition. We shall also need a structural assumption:

(S) *if* $n\geqq 2$, *each* $\sigma\in\Sigma$ *can be written* $\sigma=\sigma_0+\sigma_1$, *where* σ_0 *is a discrete measure and* $\sigma_1\in L^1(R^n)$.

We can now formulate our main result.

Theorem 2.2. *Let* Σ *be a finite subset of* $M(R^n)$ *satisfying* (S) *and* (T), *and let* τ *be an element of* $M(R^n)$ *such that* $\hat{\tau}(\xi)=0$ *in some neighbourhood of the origin. Then* $\tau\in K(\Sigma)$, *and hence* $\omega_\tau(f, t)\leqq C\omega_\Sigma(f, Ct)$ *for some* C.

The proof of this theorem will be given in Section 5. It depends heavily on the contents of Sections 3 and 4.

Note that the condition (S) is void if $n=1$.

The Fourier transform of a discrete measure on R cannot vanish on a half-line without vanishing identically. If $n=1$ the condition (T) therefore means simply that the discrete parts of the elements of Σ do not all vanish. This suggests the following reformulation of the condition (T). For simplicity consider the case when Σ has only one element. The measure σ satisfies (T) if and only if for each $\xi\in R^n\setminus\{0\}$ there exists at least one hyperplane $K\subset R^n$, orthogonal to ξ, such that $\sigma(K)\neq 0$. In other words, for each $(n-1)$-dimensional subspace $H\subset R^n$ the measure σ_H, which is defined on the factorspace R^n/H by integrating σ over all hyperplanes parallel to H, must have a non-vanishing discrete part.

Remark. Theorem 2.2 is applicable in the more general case when the supremum norm is replaced by L^p-norm $(1\leqq p<\infty)$ in the definition of the generalized modulus of continuity; denote the object so obtained by $\omega_{\Sigma,p}(f,t)$. In fact it is easily seen that if $\tau\in K(\Sigma)$, then

$$\omega_{\tau,p}(f,t)\leqq C\omega_{\Sigma,p}(f,Bt),\quad f\in L^p(R^n),$$

with $B=\sup\{u;\ (h,u)\in\operatorname{supp}\mu^j\}$ and $C=\sum_j\|\mu^j\|_{M(G)}$.

Using Theorem 2.2 it is easy to strengthen the comparison theorems of H. S. Shapiro as indicated in the introduction. Let $\hat{M}(R^n)$ and $\hat{J}(\Sigma)$ be the set of Fourier transforms of elements of $M(R^n)$ and $J(\Sigma)$ respectively, and let $\hat{M}_0(R^n)$ be the ring of germs at the origin of elements of $\hat{M}(R^n)$. Denote by $\hat{J}_0(\Sigma)$ the image of $\hat{J}(\Sigma)$ under the natural homomorphism $\hat{M}(R^n)\to\hat{M}_0(R^n)$.

Corollary 2.3. *Assume that Σ satisfies* (S) *and* (T) *and that*

$$\hat{\tau}\in\hat{J}_0(\Sigma).$$

Then $\tau\in K(\Sigma)$, hence

$$\omega_\tau(f,t)\leqq C\omega_\Sigma(f,Ct),\quad t>0,\quad f\in\dot{C}(R^n),$$

for some C.

Proof. Take $\psi\in C^\infty(R^n)$ such that $\psi(\xi)=1$ for $|\xi|\leqq\varepsilon$ and $\psi(\xi)=0$ for $|\xi|\geqq 2\varepsilon$. The assumption $\hat{\tau}\in\hat{J}_0(\Sigma)$ implies that $\psi\hat{\tau}=\hat{\tau}_0\in\hat{J}(\Sigma)\subset\hat{K}(\Sigma)$ if ε is small enough. But $\hat{\tau}-\hat{\tau}_0=\hat{\tau}_1$ vanishes in a neighbourhood of the origin, and therefore Theorem 2.2 implies $\tau_1\in K(\Sigma)$. Hence $\tau=\tau_0+\tau_1\in K(\Sigma)$.

Shapiro gives in [14] also an estimate of ω_τ in terms of ω_Σ without assuming that $\{\hat{\sigma};\ \sigma\in\Sigma\}$ divides $\hat{\tau}$ at the origin but assuming only that $\hat{\tau}$ is small at the origin. The smallness property is expressed in terms of homogeneity of a certain degree close to the origin. We shall give a strengthened estimate in this situation, but we prefer to formulate the condition on τ in terms of a certain Besov space. Following Peetre [13] (see also [1], ch. 6) we take a function $\varphi\in L^1(R^n)$ such that $\hat{\varphi}\in C^\infty$, the support of $\hat{\varphi}$ is contained in $1/2<|\xi|<2$, and

$$\sum_{j=0}^{\infty}\hat{\varphi}(\xi 2^j)=1\quad\text{for}\quad 0<|\xi|<1.$$

We shall assume that for some $\gamma>0$ and some C

$$\|\hat{\tau}(\xi)\hat{\varphi}(2^k\xi)\|_{L^1(R^n)}\leqq C2^{-k\gamma},\quad k=1,2,\ldots. \tag{2.12}$$

For $\tau\in M(R^n)$ (2.12) means precisely that τ belongs to the Besov space $\dot{B}^\gamma_{1\infty}$; see [1], ch. 6.3. The set of $\tau\in M(R^n)$ satisfying (2.12) clearly forms an ideal. Of course (2.12) is really a property only of the germ of $\hat{\tau}$ at the origin. Note that if $\hat{\tau}(\xi)$ is positive-homogeneous of degree $\gamma>0$ in some neighbourhood of the origin, then (2.12) holds.

Corollary 2.4. *Assume that Σ satisfies* (S) *and* (T) *and that τ satisfies* (2.12) *for some $\gamma>0$. Then*

$$\omega_\tau(f,t) \leqq Ct^\gamma \int_t^\infty \omega_\Sigma(t,u)u^{-\gamma-1}\,du, \quad t>0,\ f\in \dot{C}(R^n). \tag{2.13}$$

Since $\omega_\Sigma(f,t)$ is increasing, it is obvious that (2.6) implies (2.13) with possibly a larger C. We note also that the right hand side of (2.13) can be estimated by[1)]

$$Ct^\gamma \int_t^1 \omega_\Sigma(f,u)u^{-\gamma-1}\,du + C_1 t^\gamma \|f\|, \quad 0<t<1.$$

Proof of Corollary 2.4. Writing $\tau=\tau_0+\tau_1$ as in the proof of Corollary 2.3 we have $\tau_1\in K(\Sigma)$ by the Theorem 2.2, so we need only estimate ω_{τ_0}. We may assume that $\hat{\tau}_0(\xi)=0$ for $|\xi|>1$. Then, for $|\xi|\neq 0$,

$$\begin{aligned}\hat{\tau}_0(\xi) &= \left(\textstyle\sum_{j=0}^\infty \hat{\varphi}(2^j\xi)\right)^2 \hat{\tau}_0(\xi) = \textstyle\sum_{j=0}^\infty \hat{\varphi}(2^j\xi) \sum_{k=j-1}^{j+1} \hat{\varphi}(2^k\xi)\hat{\tau}_0(\xi)\\ &= \textstyle\sum_{j=0}^\infty 2^{j\gamma}\hat{\varphi}(2^j\xi)\hat{\tau}_0(\xi) \sum_{k=j-1}^{j+1} 2^{-j\gamma}\hat{\varphi}(2^k\xi).\end{aligned}$$

Replace ξ by $t\xi$, take inverse Fourier transform of both members and form convolutions with $f\in\dot{C}(R^n)$. Using (2.2), (2.3) and (2.12) we easily get

$$\omega_{\tau_0}(f,t) \leqq C \textstyle\sum_{j=0}^\infty 2^{-j\gamma}\omega_\varphi(f,2^j t).$$

Since ω_φ is increasing the last sum can be estimated by a constant times

$$\int_1^\infty \omega_\varphi(f,ut)u^{-\gamma-1}\,du.$$

Since $\hat{\varphi}=0$ in a neighbourhood of the origin we have $\omega_\varphi(f,t)\leqq C\omega_\Sigma(f,Ct)$. Using this fact and making a change of variable in the integral we obtain the result.

In some situations one wants the conclusion of Corollary 2.3 to be strengthened to $\omega_\tau(f,t)\leqq C\omega_\Sigma(f,t)$. The next theorem gives general conditions for this to be possible.

We need the following local variant of the Tauberian condition:

(T_{loc}) *the family of functions $\{\hat{\sigma}(t\xi)\}_{0<t<1,\sigma\in\Sigma}$ has no common zero on $R^n\setminus\{0\}$.*

This condition means that the restriction of $\{\hat{\sigma}(\xi);\ \sigma\in\Sigma\}$ to an arbitrarily small neighbourhood of the origin satisfies (T). The condition is of course satisfied by $\Sigma=\{\sigma\}$, if $\hat{\sigma}(\xi)$ is different from zero in some pointed neighbourhood of the origin.

[1)] There is an error in the formulation of Theorem 2 and Theorem 4 in [5]; in the expression $t^\gamma \int_t^1 u^{-\gamma}\omega_\sigma(f,u)u^{-1}du$ the integral should be taken from t to infinity. The same remark applies to formula (4.8) page 43 in [6].

Corollary 2.5. *Assume that Σ satisfies* (S), (T), (T_{loc}) *and that the ideal $\hat{J}_0(\Sigma)$ has a finite set of generators. Then, for any $B>0$ there exists a constant C_B such that*

$$\omega_\Sigma(f, Bt) \leqq C_B \omega_\Sigma(f, t), \quad t>0, \quad f\in \dot{C}(R^n). \tag{2.14}$$

Proof. To simplify the notation we prove the corollary only in the case when Σ consists of one element σ. Take $\psi\in C^\infty(R^n)$ such that $\psi(\xi)=1$ for $|\xi|<1/2$, $\psi(\xi)=0$ for $|\xi|>1$, let δ be a (small) positive number to be chosen later, and define the measures σ_i, $i=0, 1, 2$, by

$$\hat{\sigma}(\xi) = \hat{\sigma}(\xi)\psi(\xi/\delta)+\hat{\sigma}(\xi)(1-\psi(\delta\xi))$$
$$+\hat{\sigma}(\xi)\big(\psi(\delta\xi)-\psi(\xi/\delta)\big) = \hat{\sigma}_0(\xi)+\hat{\sigma}_1(\xi)+\hat{\sigma}_2(\xi).$$

It will be enough to prove

$$\omega_{\sigma_i}(f, Bt) \leqq C_B \omega_\sigma(f, t), \quad t>0, \quad f\in \dot{C}(R^n), \tag{2.15}$$

for $i=0, 1, 2$. Let $\hat{\sigma}(b_j\xi)$, $j=1, \dots, N$, $b_j>0$, be a set of generators for $\hat{J}_0(\sigma)$. It is obviously possible to choose all $b_j\leqq 1$. Then, for an arbitrary $B>0$, there exists $\delta>0$ and $h_j\in \hat{M}(R^n)$ such that

$$\hat{\sigma}(B\xi) = \textstyle\sum_{j=1}^N h_j(\xi)\hat{\sigma}(b_j\xi), \quad \text{for} \quad |\xi|<\delta. \tag{2.16}$$

Multiplying this identity by $\psi(\xi/\delta)$ we see that (2.15) holds for $i=0$.

Let τ be the measure defined by $\hat{\tau}(\xi)=1-\psi(\xi)$. Then $\omega_\tau(f, t)\leqq C\omega_\sigma(f, Ct)$ by Theorem 2.2, hence

$$\omega_{\tau_{(1/C)}}(f, t) \leqq C\omega_\sigma(f, t).$$

Now $\hat{\tau}_{(1/C)}(\xi)=1$ for $|\xi|>C$, hence if $\delta<1/2C$ we have $\hat{\sigma}_1=\hat{\sigma}_1\hat{\tau}_{(1/C)}$, which gives (2.15) for $i=1$.

The measure σ_2, finally, satisfies $\hat{\sigma}_2(\xi)=0$ outside the set $\delta/2<|\xi|<1/\delta$, hence $\hat{\sigma}_2(B\xi)=0$ outside the set $K=\{\xi;\ B\delta/2<|\xi|<B/\delta\}$. The condition ($T_{loc}$) implies that the family of functions $\{\hat{\sigma}(t\xi)\}_{0<t<1}$ has no common zero on the set K. Using Wiener—Levy's theorem and a partition of unity it is then easy to prove that $\hat{\sigma}_2(B\xi)$ has a representation of the form (2.16) in all of R^n with $b_j\leqq 1$. This shows that (2.15) holds for $i=2$. The proof is complete.

Combining Corollaries 2.3 and 2.5 we get for instance the following result. Assume that Σ satisfies (S), (T) and (T_{loc}) and that $\hat{\tau}\in\hat{J}_0(\Sigma)$. Then $\omega_\tau(f, t)\leqq C\omega_\Sigma(f, t)$ for some C.

The assumptions of Corollary 2.5 are usually very easy to verify for specific choices of Σ. For instance, if $\sigma\in M(R)$ and $\hat{\sigma}$ is real analytic in some neighbourhood of the origin — this is of course the case if σ has compact support — then $\hat{\sigma}(\xi)$ has a zero at the origin of some definite order, say k, and then $\hat{J}_0(\sigma)$ is generated by the single function (more precisely the germ) ξ^k. More generally, assume that

$\sigma \in M(R^n)$ and that $\hat{\sigma}$ is real analytic in a neighbourhood of the origin. Since the ring O_0 of real analytic germs in n variables is Noetherian, the ideal generated in O_0 by all the germs $\hat{\sigma}(t\xi)$, $t>0$, must be finitely generated. Using this fact it is immediately seen that $\hat{J}_0(\sigma)$ is finitely generated.

The assumption that $\hat{J}_0(\sigma)$ is finitely generated cannot be omitted from Corollary 2.5. To see this take $\sigma \in M(R)$ such that $\hat{\sigma} \in C^\infty$, $\hat{\sigma}(\xi) = \exp(-1/|\xi|)$ for $0<|\xi|<1/2$ and $\hat{\sigma}(\xi)=1$ for $|\xi|>1$. Then $\sigma = \delta_0 + \sigma_1$, where $\hat{\sigma}_1$ is infinitely differentiable and has compact support, hence $\sigma_1 \in L^1(R^n)$. Thus all hypotheses of Corollary 2.5 are satisfied except the assumption that $\hat{J}_0(\sigma)$ be finitely generated. On the other hand, it follows from the proof of Theorem 2.1 (see (2.17) below) that the estimate

$$\omega_\sigma(f, 2t) = \omega_{\sigma(2)}(f, t) \leqq C\omega_\sigma(f, t)$$

cannot hold, since $\hat{\sigma}_{(2)}(\xi)/\hat{\sigma}(\xi) = \hat{\sigma}(2\xi)/\hat{\sigma}(\xi)$ is unbounded near the origin.

We finally consider the question of finding useful necessary conditions for the estimate (1.1). First of all it follows immediately from Theorem 2.1 that (1.1) implies

$$|\hat{\tau}(\xi)| \leqq C_1 \sup_{\substack{0<u\leqq B \\ 1\leqq j\leqq J}} |\hat{\sigma}^j(u\xi)|, \quad \xi \in R^n. \tag{2.17}$$

In fact it is easily seen from the proof of Theorem 2.1 that (2.17) must hold with the same B as in (1.1). The fact that (1.1) implies (2.17) was proved in [7] (Theorem 2) with different methods. The result in [7], however, is somewhat stronger, the constant C being allowed to depend on f in the hypothesis.

Next we shall prove that if the Fourier transforms $\hat{\sigma}$, $\sigma \in \Sigma$, are assumed to have some regularity property at the origin, then $\tau \in K(\Sigma)$ implies $\hat{\tau} \in \hat{J}_0(\Sigma)$. This will give us a partial converse of Corollary 2.3. Assume first that each $\hat{\sigma}^j$ is locally homogeneous of some positive degree, say q_j. Then for small ξ we have $\hat{\sigma}^j_{(h,u)}(\xi) = u^{q_j}\hat{\sigma}^j(\xi)e^{i\langle \xi, h\rangle}$. Thus (2.5) gives for small ξ

$$\hat{\tau}(\xi) = \sum_{j=1}^J \hat{\sigma}^j(\xi) \int_G u^{q_j} e^{i\langle \xi, h\rangle} d\mu^j(h, u).$$

This shows that $\hat{K}(\Sigma)$ is locally generated (as an $\hat{M}(R^n)$-ideal) by $\hat{\sigma}^1, \dots, \hat{\sigma}^J$. Or, stated in another way, $\hat{K}(\Sigma)$ and $\hat{J}(\Sigma)$ have the same image in $\hat{M}_0(R^n)$ under the mapping $\hat{M}(R^n) \to \hat{M}_0(R^n)$. The same conclusion holds of course under the more general assumption that $\hat{J}_0(\Sigma)$ is generated by a finite number of homogeneous functions. Combining this fact with Corollary 2.3 we get the following result.

Corollary 2.6. *Assume that Σ satisfies* (S) *and* (T) *and that $\hat{J}_0(\Sigma)$ is generated by a finite number of elements that are positive-homogeneous functions of some positive degree. Then*

$$\omega_{\Sigma'}(f, t) \leqq C\omega_\Sigma(f, Ct)$$

if and only if $\hat{J}_0(\Sigma')\subset\hat{J}_0(\Sigma)$. *Assume in addition that* Σ *satisfies* ($\mathrm{T}_{\mathrm{loc}}$). *Then*

$$\omega_{\Sigma'}(f, t) \leqq C\omega_\Sigma(f, t)$$

if and only if $\hat{J}_0(\Sigma')\subset\hat{J}_0(\Sigma)$.

Let us call the two moduli of continuity ω_σ and ω_τ *equivalent*, if $\omega_\sigma(f, t)\leqq C\omega_\tau(f, Ct)$ and vice versa.

We wish to point out that the two statements

$$\omega_{\sigma_i}(f, t) \leqq Ct^\gamma \quad \text{for some} \quad C,$$

$i=1, 2,$ may be equivalent even if the ω_{σ_i}, $i=1, 2,$ are inequivalent. For instance, this situation occurs if $\sigma_i\in M(R)$ and $\hat{\sigma}_i(\xi)$ vanishes at $\xi=0$ precisely of the order β_i, where $\gamma<\beta_1<\beta_2$.

For verification that the hypotheses of Corollary 2.6 are fulfilled the following fact is often useful.

Proposition 2.7. *If* $\hat{\sigma}(\xi)$ *is real analytic in some neighbourhood of the origin, then* $\hat{J}_0(\sigma)$ *is generated by a finite number of homogeneous polynomials.*

Proof. This statement can be proved with wellknown methods, so we will only briefly scetch the proof. Let O_0 be the ring of power series in n variables that are convergent in some neighbourhood of the origin, and let $\mathcal{M}$ be the unique maximal ideal in O_0. The essential point in the proof is Krull's theorem (Corollary 1 of Theorem 12, ch. IV in [23]), which implies the following: if L is an ideal in O_0 and $v\in L+\mathcal{M}^k$ for every k, then $v\in L$. Let $f\in O_0$ and let G_f be the ideal in O_0 generated by all the dilates of f. We will prove that G_f is generated by a finite number of homogeneous polynomials; the assertion easily follows from this. Let $f=\sum_{m=1}^\infty q_m$, where q_m is homogeneous of degree m. Since O_0 is Noetherian, G_f is generated by a finite number of the dilates

$$f(t\xi) = \sum_{m=1}^\infty t^m q_m(t\xi). \tag{2.18}$$

Forming suitable linear combinations we find that $q_m\in G_f+\mathcal{M}^k$ for every m and k, hence $q_m\in G_f$ by Krull's theorem. On the other hand, it is obvious from (2.18) that G_f is generated modulo $\mathcal{M}^k$ by $q_1, \ldots, q_k$. Since the polynomial ring P is Noetherian, the sequence of ideals $L_k=(q_1, \ldots, q_k)$ in P is finite, i.e. $L_N=L_{N+1}=\ldots$ for some N. Krull's theorem therefore implies that G_f is generated by $q_1, \ldots, q_N$. This completes the proof.

3. An estimate for the average of trigonometrical sums

We shall consider trigonometrical sums of the form

$$p(\xi) = \sum_{j=1}^{\infty} c_j \exp(i\langle a_j, \xi\rangle), \quad \xi \in R^n, \tag{3.1}$$

where a_j are arbitrary points of R^n, c_j are complex numbers, and $\sum_{j=1}^{\infty} |c_j| < \infty$. Set for $\omega \in S^{n-1} = \{\xi \in R^n;\ |\xi| = 1\}$ and $T > 0$

$$F_1(\omega, T) = \frac{1}{2T} \int_{-T}^{T} |p(\omega t)|\, dt.$$

The main purpose of this section is to prove the following lemma.

Lemma 3.1. *Assume that for no $\omega \in S^{n-1}$ the function $R \ni t \to p(t\omega)$ is identically zero. Then there exists number $\alpha > 0$ such that*

$$F_1(\omega, T) \geqq \alpha, \quad \omega \in S^{n-1}, \quad T > 1.$$

In the case $n=1$ the situation is quite simple. In this case there exists a number L such that the mean of $|p(\xi)|$ over any interval of length at least L is greater than some positive number α. This follows easily from the fact that $p(\xi)$ is almost periodic in the sense of Bohr. On the other hand, let

$$p(\xi) = p(\xi_1, \xi_2) = e^{i\xi_1} + e^{-i\xi_1} = 2\cos \xi_1.$$

Then $p(\xi) = 0$ on the line $\xi_1 = \pi/2$, and it is easily seen that there are arbitrarily large intervals I lying on some ray through the origin such that the mean of $|p(\xi)|$ over I is arbitrarily small. Note also that the period of $t \to p(\omega t)$ becomes unbounded as ω tends to $(0, 1)$.

Proof of Lemma 3.1. We will prove that the function F_2 defined by

$$F_2(\omega, T) = \frac{1}{2T} \int_{-T}^{T} |p(\omega t)|^2\, dt \tag{3.2}$$

is bounded away from zero for large T. This obviously implies the assertion of the lemma, since $p(\xi)$ is bounded. We first find a convenient expression for $F_2(\omega, T)$. The function $q(\xi) = |p(\xi)|^2$ can be written

$$q(\xi) = d_0 + \sum_{k=1}^{\infty} d_k \exp(i\langle b_k, \xi\rangle).$$

where $b_k \neq 0$ for all k, $\sum_1^{\infty} |d_k| < \infty$ and $d_0 = \sum_j |c_j|^2 > 0$, since $p(\xi) \neq 0$. Integrating the expression for $q(\xi)$ we get

$$F_2(\omega, T) = d_0 + \sum_{k=1}^{\infty} d_k K(T\langle b_k, \omega\rangle),$$

where $K(t)=(\sin t)/t$. Hence, for each fixed ω

$$\lim_{T\to\infty} F_2(\omega, T) = d_0+\sum_{k\in I(\omega)} d_k \tag{3.3}$$

where

$$I(\omega) = \{k;\ \langle b_k, \omega\rangle = 0\}.$$

The proof of the lemma will be carried out by induction over n. If $n=1$ the sum in the right member of (3.3) disappears, so that $\lim_{T\to\infty} F_2(\omega, T)=d_0>0$, which implies the assertion for this case. Assume next that $n\geqq 2$ and that the assertion of the lemma is proved for trigonometric sums of the form (3.1) defined in R^{n-1}. Choose N so large that $\sum_{k>N} |d_k|<d_0/4$, and set

$$E = \{\xi\in R^n;\ \langle b_k, \xi\rangle = 0 \text{ for some } k,\ 1 \leqq k \leqq N\}.$$

Denote by $d(\omega, E)$ the Euclidean distance from ω to E. If we set $\beta=\min\{|b_k|;\ 1\leqq k\leqq N\}$, we have

$$|\langle b_k, \omega\rangle| = |b_k|\,|\omega|\,|\cos(b_k, \omega)| \geqq \beta d(\omega, E),$$

$$\omega\in S^{n-1}, \quad k = 1, \ldots, N.$$

Choose A so that

$$|K(t)| \sum_{k=1}^{N} |d_k| < d_0/4 \quad \text{for} \quad |t| > A.$$

Assume that $\omega\in S^n\setminus E$ and that $T>A/(\beta d(\omega, E))$. Then

$$\Big|\sum_{k=1}^{N} d_k K(T\langle b_k, \omega\rangle)\Big| < d_0/4.$$

Hence,

$$F_2(\omega, T) \geqq d_0-\frac{d_0}{4}-\frac{d_0}{4} = \frac{d_0}{2}, \quad \text{if} \quad T> A/(\beta d(\omega, E)). \tag{3.4}$$

Next we consider $\omega\in E$. E is a finite union of hyperplanes in R^n. If we consider the restriction of $p(\xi)$ to each of those hyperplanes and use the induction assumption, we conclude that there exists $\alpha_0>0$ such that

$$F_2(\omega, T) \geqq \alpha_0, \quad \omega\in S^{n-1}\cap E, \quad T> 1. \tag{3.5}$$

Finally we will consider ω in a small neighbourhood of E. Since $q(\xi)$ is uniformly continuous we can choose $\delta>0$ such that

$$|q(\omega_0 t)-q(\omega t)| < \alpha_0/2$$

whenever $\omega, \omega_0\in S^{n-1}$ and $|(\omega-\omega_0)t|<\delta$. Hence

$$|F_2(\omega_0, T)-F_2(\omega, T)| < \alpha_0/2, \quad \text{if} \quad |\omega-\omega_0|\,T < \delta. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$F_2(\omega, T) \geqq \alpha_0/2 \quad \text{if} \quad 1 < T < \delta/(2d(\omega, E)). \tag{3.7}$$

For arbitrary $B \geqq 1$ we have $F_2(\omega, BT) \geqq (1/B) F_2(\omega, T)$. Hence we get from (3.7)

$$F_2(\omega, T) \geqq \alpha_0/(2B) \quad \text{if} \quad 1 < T < \delta B/(2d(\omega, E)). \tag{3.8}$$

If we choose B so large that $\delta B/2 > A/\beta$, we may combine (3.4) and (3.8) to obtain

$$F_2(\omega, T) \geqq \min\left(d_0/2, \alpha_0/(2B)\right) \quad \text{if} \quad T > 1.$$

Thus the proof is complete.

Corollary 3.2. *Under the assumptions of the lemma* $\lim_{T\to\infty} F_1(T, \omega)$ *is bounded away from zero for* $\omega \in S^{n-1}$, *i.e. the expression in the right member of* (3.3) *is bounded away from zero.*

Note that the last statement is an immediate consequence of (3.3) if $p(\xi)$ is a trigonometric polynomial, i.e. the sum (3.1) is finite.

Actually we shall need below an estimate for the mean of $|p(\xi)|$ on a more general set of line segments. If $p(\xi)$ is a trigonometrical sum of the form (3.1) and $A \geqq 1$ we set

$$G_A(\xi) = \int_1^A |p(\xi t)|\, dt, \quad \xi \in R^n.$$

Lemma 3.3. *Assume that for no* $\omega \in S^{n-1}$ *the function* $t \to p(t\omega)$ *is identically zero. Then, as* $A \to \infty$ $G_A(\xi)$ *tends to infinity uniformly for* $|\xi| \geqq 1$, *hence in particular there exists an* A *such that*

$$G_A(\xi) \geqq 1 \quad \textit{for} \quad |\xi| \geqq 1.$$

Proof. Applying Lemma 3.1 to the function $r(\xi) = p(\xi)p(-\xi)$ we obtain, if $\sup |p(\xi)| \leqq C$,

$$0 < \alpha \leqq \frac{1}{2T} \int_{-T}^{T} |p(t\omega)\, p(-t\omega)| dt$$

$$\leqq \frac{C}{2T} \int_0^T |p(t\omega)|\, dt + \frac{C}{2T} \int_{-T}^{0} |p(-t\omega)|\, dt = \frac{C}{T} \int_0^T |p(t\omega)|\, dt.$$

Hence, if $|\xi| \geqq 1$ and $T > 1$,

$$G_A(\xi) = A \cdot \frac{1}{A|\xi|} \int_0^{A|\xi|} \left|p(t\xi/|\xi|)\right| dt - \frac{1}{|\xi|} \int_0^{|\xi|} \left|p(t\xi/|\xi|)\right| dt \geqq A(\alpha/C) - C.$$

Taking A sufficiently large we obtain the desired result.

4. Wiener algebras of measures

For the proof of Theorem 2.2 we need to know that a certain closed subalgebra B of $M(R^n)$ has the following property: every $\sigma \in B$, whose Fourier transform is bounded away from zero is invertible as an element of B. The well-known example of Wiener and Pitt [22] shows that the full algebra $M(R^n)$ does not have this property (see [10], § 32). However, here we will consider a somewhat stronger property, which is more natural from ring-theoretic point of view. Denote by $(\sigma_1, \dots, \sigma_r)$ the ideal in B generated by $\sigma_1, \dots, \sigma_r$. We will always assume that B is a closed subalgebra of $M(R^n)$ with unit. We shall say that B is a *Wiener algebra*, if the following condition is satisfied

(4.1) *if* $\sigma_j \in B$, $j=1, \dots, r$, *and* $\sum_{j=1}^r |\hat{\sigma}_j(\xi)| \geqq \varepsilon > 0$ *for all* $\xi \in R^n$, *then* $(\sigma_1, \dots, \sigma_r) = B$.

If V is an affine subspace of R^n we denote by L_V^1 the set of elements of $M(R^n)$ which are supported by V and are absolutely continuous with respect to the surface measure on V. Let $A_0 = A_0(R^n)$ be the set of all finite sums of elements of any of the sets L_V^1, $V \subset R^n (0 \leqq \dim (V) \leqq n)$, and let $A = A(R^n)$ be the smallest closed algebra containing A_0. Note that A_0 is a (non-closed) subalgebra of A.

The result which we need can be formulated as follows.

Theorem 4.1. (Varopoulos [21]). *A is a Wiener algebra.*

Varopoulos proves in fact a result about general locally compact groups, which contains Theorem 4.1 as a special case. An extension of Theorem 4.1 in a different direction was obtained by Björk [2], who proved that if A_{an} is the set of all finite sums of absolutely continuous measures with respect to surface measures on arbitrary real analytic manifolds, then the closed algebra $\bar{A}_{an}$ is a Wiener algebra.

Since Varopoulos' paper is not very easy to read, we have included a proof of Theorem 4.1 here.

Let us recall some basic notions from the theory of Banach algebras. We denote the maximal ideal space of a Banach algebra B by $\mathcal{M}_B$. The elements of $\mathcal{M}_B$ may be considered as multiplicative linear forms on B. Considering $\mathcal{M}_B$ as a subset of B^*, the dual space of B, we may provide $\mathcal{M}_B$ with the topology induced by the weak-star topology on B^*. If B is a subalgebra of $M(R^n)$ there is a natural mapping $\pi\colon R^n \to \mathcal{M}_B$ induced by the Fourier transform: for $\xi \in R^n$, $\pi(\xi)$ is the linear form $\mu \to \hat{\mu}(\xi)$.

We shall use the well known fact that B is a Wiener algebra if and only if $\pi(R^n)$ is dense in $\mathcal{M}_B$. For the proof one observes that (4.1) can be formulated

$$\sigma_j \in B, \quad \sum_{j=1}^r |\hat{\sigma}_j(\xi)| \geqq \varepsilon > 0 \Rightarrow \sum_{j=1}^r |\lambda(\sigma_j)| > 0 \quad \text{for every} \quad \lambda \in \mathcal{M}_B.$$

We refer the reader to [10] for details.

Let B be any closed subalgebra of $M(R^n)$ containing $L^1(R^n)$. Then the following holds:

(4.2) *if* $\gamma\in\mathcal{M}_B$ *does not annihilate* $L^1(R^n)$, *then there exists* $\theta\in R^n$ *such that* $\gamma(\mu)=\hat{\mu}(\theta)$ *for all* $\mu\in B$.

To prove the statement consider first the restriction γ_0 of γ to $L^1(R^n)$. By a well-known theorem $\gamma_0(f)=\hat{f}(\theta), f\in L^1(R^n)$, for some $\theta\in R^n$. Next let $\mu\in B$ and choose $f\in L^1(R^n)$ such that $\hat{f}(\theta)\neq 0$. Then, since $g=\mu*f\in L^1(R^n)$ we have both $\gamma(g)=\hat{g}(\theta)=\hat{\mu}(\theta)\hat{f}(\theta)$ and $\gamma(g)=\gamma(\mu)\hat{f}(\theta)$, which proves the statement.

Finally we shall need the following lemma.

Lemma 4.2. *Let* $a_j\in R^n$, $j=1, \dots, s$ *be linearly independent over the set* Q *of rational numbers. Then the set of all* $\eta\in R^n$, *such that the numbers*

$$\langle\eta, a_j\rangle, \quad j = 1, \dots, s,$$

are linearly independent over Q, *forms a dense subset of* R^n.

Proof. For each non-zero $q=(q_1, \dots, q_s)\in Q^s$, let F_q be the set of all $\eta\in R^n$ which are orthogonal to $\sum_{j=1}^{s} q_j a_j$. Since the latter vector is different from zero by the assumption, F_q must be a hyperplane. Since Q^s is denumerable, the set

$$F=\cup\{F_q;\ q\in Q^s\setminus 0\}$$

has no interior point by the Baire category theorem. But the set of η mentioned in the lemma is equal to the complement of F, hence the proof is complete.

Proof of Theorem 4.1. It is enough to prove that $\pi(R^n)$ is dense in $\mathcal{M}_A$. Let γ be an arbitrary element of $\mathcal{M}_A$, let $\mu_1, \dots, \mu_r\in A$, and let $\varepsilon>0$. Our task is to find $\xi\in R^n$ such that

$$|\gamma(\mu_j)-\hat{\mu}_j(\xi)| < \varepsilon, \quad j = 1, \dots, r. \tag{4.3}$$

We may assume that all $\mu_j\in A_0$. It is even enough to prove (4.3) for arbitrary $\varepsilon>0$ and arbitrary μ_j of the form

$$\mu_j = f_j*\delta_{a_j},$$

where $f_j\in L^1_{K_j}$, K_j some linear subspace of R^n, and $a_j\in R^n$.

Let P_γ be the set of all subspaces $K\subset R^n$ such that γ does not annihilate all of L^1_K. We claim that P_γ contains a largest element. In fact, let $H_i\in P_\gamma$, $i=1, 2$, and let $v_i\in L^1_{H_i}$, $\gamma(v_i)\neq 0$. Then $v=v_1*v_2\in L^1_{H_1+H_2}$, and $\gamma(v)\neq 0$. Hence P_γ is closed under the operation of forming linear hull of subspaces. To construct the largest element of P_γ we need now only take any element of P_γ with largest dimension.

Denote by H the largest element of P_γ. Assume that $K_j\subset H$ for $j=1, \dots, s$ and $K_j\not\subset H$ for $s<j\leqq r$. Then by (4.2) there exists $\theta\in R^n$ such that

$$\gamma(\mu_j) = \hat{f}_j(\theta)\gamma(\delta_{a_j}) \quad \text{for} \quad 1 \leqq j \leqq s,$$

Furthermore $\gamma(\mu_j)=0$ for $j>s$ by the choice of H. Set $\xi=\theta+\eta$ where η is a vector orthogonal to H, which will be chosen later. Then $\hat{f}_j(\theta+\eta)=\hat{f}_j(\theta)$ for $j\leqq s$, hence

$$\hat{\mu}_j(\xi)=\hat{f}_j(\theta)\exp(-i\langle\theta+\eta, a_j\rangle), \quad j\leqq s.$$

Hence it is enough to find η such that

$$|\gamma(\delta_{a_j})-\exp(-i\langle\theta+\eta, a_j\rangle)|<\varepsilon, \quad j\leqq s, \tag{4.4}$$

and

$$|\hat{f}_j(\theta+\eta)|<\varepsilon, \quad j>s. \tag{4.5}$$

Multiplying (4.4) by $\exp(i\langle\theta, a_j\rangle)$ we get

$$|\gamma_1(\delta_{a_j})-\exp(-i\langle\eta, a_j\rangle)|<\varepsilon, \tag{4.6}$$

where $\gamma_1(\delta_a)$ is defined by

$$\gamma_1(\delta_a)=e^{i\langle\theta,a\rangle}\gamma(\delta_a), \quad a\in R^n.$$

But $\gamma_1(\delta_a)=1$ for all $a\in H$ by (4.2) and the choice of H. Hence the expression on the left hand side of (4.6) depends only on the residue class $\bar{a}_j$ of a_j in R^n/H.

Since both $\bar{a}\to\gamma(\delta_a)$ and $\bar{a}\to\exp(-i\langle\eta, a\rangle)$ are group homomorphisms from R^n/H into the circle, it is easy to see that it is enough to prove (4.6) for arbitrary $\varepsilon>0$ and for arbitrary a_j such that $\bar{a}_j$, $j=1, \dots, s$ are linearly independent over Q. Next we note that, by an obvious modification of Lemma 4.2, the set $\bar{E}$ of all $\eta\in H^\perp$ (=orthogonal complement of H) such that $\langle\eta, a_j\rangle$, $j=1, \dots, s$, are linearly independent over Q forms a dense subset of $H^\perp$. (Note that the canonical isomorphism between R^n and its dual identifies $H^\perp$ with the dual of R^n/H.) Hence we may choose $\eta_0\in H^\perp\cap\bar{E}$ such that

$$\eta_0\notin K_j^\perp, \quad s<j\leqq r.$$

Indeed, since $K_j\not\subset H$ for $j>s$, we have $K_j^\perp\not\supset H^\perp$, i.e. $H^\perp\cap K_j^\perp$ is a proper subspace of $H^\perp$ for each such j. Set $\langle\eta_0, a_j\rangle=\alpha_j$, $j<s$, and let c_j be arbitrary complex numbers such that $|c_j|=1$. Since the α_j are linearly independent over Q, it is well known that we can choose $t\in R$ such that

$$|c_j-e^{-it\alpha_j}|<\varepsilon, \quad j\leqq s$$

(cf. [11], p. 60). There exists in fact arbitrarily large t with this property. Taking $c_j=\gamma_1(\delta_{a_j})$ and $\eta=t\eta_0$ we obtain (4.6). Moreover, since $\eta_0\notin K_j^\perp$, $j>s$, we have

$$\lim_{t\to\infty}\hat{f}_j(\theta+t\eta_0)=0, \quad j>s.$$

Hence (4.5) is satisfied if t is sufficiently large. This completes the proof of Theorem 4.1.

5. Proof of Theorem 2.2.

In this section we complete the proof of Theorem 2.2 using the results of Sections 3 and 4.

Assume that Σ satisfies (S) and (T). Since $K(\Sigma)$ is an ideal and $K(\Sigma)$ is invariant under dilation, it is enough to prove that $K(\Sigma)$ contains a measure π such that $\hat{\pi}(\xi)=1$ outside some compact set. It is even enough to prove that $\hat{K}(\Sigma)$ contains an element which is invertible outside some compact set.

Assume first that all the elements of Σ are discrete measures and that $n\geqq 2$. By (T) $K(\Sigma)$ must contain a discrete measure σ such that $\hat{\sigma}(\xi)\geqq 0$ and

$$\{\hat{\sigma}(u\xi);\ u>0\} \text{ has no common zero on } R^n\setminus\{0\}; \tag{5.1}$$

we may for instance take $\hat{\sigma}(\xi)=\sum_j \hat{\sigma}_j(\xi)\overline{\hat{\sigma}_j(\xi)}$. Define ϱ by

$$\hat{\varrho}(\xi)=\int_1^B \hat{\sigma}(u\xi)\,du. \tag{5.2}$$

Clearly $\varrho\in K(\Sigma)$. By Lemma 3.3 $\hat{\varrho}(\xi)$ is bounded away from zero outside some compact set if B is large enough. Moreover, we claim that ϱ belongs to the algebra A considered in Section 4. To see this it is enough to consider the case when σ is a Dirac measure δ_a for some $a\in R^n$. But in this case $\varrho\in L^1_V$ where V is the line $\{ta;\ t\in R\}$, which proves the assertion. By Theorem 4.1 A is a Wiener algebra. Thus $\hat{\varrho}(\xi)$ is invertible outside a compact set. This completes the proof of Theorem 2.2 in the case when all the elements of Σ are discrete.

Next we consider the case when Σ is arbitrary satisfying (S) and (T) and $n\geqq 2$. Then $K(\Sigma)$ must contain a measure of the form $\sigma=\sigma_0+\sigma_1$, where $\sigma_1\in L^1(R^n)$, $\hat{\sigma}_0\geqq 0$ and σ_0 satisfies (5.1). For instance the measure σ defined by

$$\hat{\sigma}=\sum_j \hat{\sigma}^j\overline{\hat{\sigma}^j}=\sum_j(\hat{\sigma}_0^j+\hat{\sigma}_1^j)(\bar{\hat{\sigma}}_0^j+\bar{\hat{\sigma}}_1^j) \tag{5.3}$$

is easily seen to have these properties. Again defining ϱ by (5.2) and choosing B large we get $\hat{\varrho}=\hat{\varrho}_0+\hat{\varrho}_1$ where $\hat{\varrho}_0$ is invertible outside a compact set and $\varrho_1\in L^1(R^n)$. Hence $\hat{\varrho}$ is also invertible outside a compact set.

Finally we consider the case $n=1$; in this case we have no structural assumption (S). Again we write $\sigma^j=\sigma_0^j+\sigma_1^j$, where σ_0^j is the discrete part of σ^j, and define σ by (5.3). The only problem is that σ_1^j and σ_1 may not belong to $L^1(R)$. But (T) is easily seen to imply that $\sigma=c\delta_0+\lambda$, where $c\neq 0$ and λ is free from mass at the origin. This in turn implies that ϱ, defined by (5.2), will have the form $\varrho=b\delta_0+\varrho_1$, where $b\neq 0$ and $\varrho_1\in L^1(R)$. This completes the proof.

6. Applications to moduli of continuity of higher order

The r^{th} order modulus of continuity $\omega_r(f, t)$ may be defined as follows. For $z \in R^n \setminus \{0\}$, let Δ_z be the measure $\delta_z - \delta_0$, and for r positive integer set $\Delta_z^r = \Delta_z * \dots * \Delta_z$ (r factors). Then set for $f \in \dot{C}(R^n)$

$$\omega_r(f, t) = \sup \{\|\Delta_z^r * f\|;\ z \in R^n,\ |z| \leqq t\}, \quad t > 0.$$

The following properties of $\omega_r(f, t)$ are well-known. For any B there exists a constant C_B such that

$$\omega_r(f, Bt) \leqq C_B \omega_r(f, t), \quad t > 0. \tag{6.1}$$

Moreover, if $r < s$

$$\omega_r(f, t) \leqq Ct^r \Big(\int_t^1 u^{-r-1} \omega_s(f, u)\, du + \|f\| \Big), \quad 0 < t < 1. \tag{6.2}$$

The constants are independent of f, but depend on r and n. For $r > s$ there is the trivial estimate $\omega_r(f, t) \leqq 2^{r-s} \omega_s(f, t)$. The inequalities (6.1) and (6.2) for $n > 1$ are immediate consequences of the same inequalities for $n = 1$. For the latter case proofs are given in [19], Section 3.3.2. It may be noted that for $n = 1$ (6.1) is a special case of Corollary 2.5, and (6.2) is a special case of Corollary 2.4.

We will now consider some alternative definitions of $\omega_r(f, t)$. Let Δ_j be the difference measure with respect to the j^{th} variable, i.e. $\Delta_j = \delta_{e_j} - \delta_0$, where $e_1, \dots, e_n$ is the natural basis in R^n, and set $\Delta^\alpha = \Delta_1^{\alpha_1} * \dots * \Delta_n^{\alpha_n}$ for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. For any positive integer r set

$$\Sigma_r = \{\Delta^\alpha;\ |\alpha| = r\}.$$

Then Σ_r is a finite subset of $M(R^n)$ and we may consider the generalized modulus of continuity $\omega_{\Sigma_r}(f, t)$.

Theorem 6.1. *The moduli of continuity ω_r and ω_{Σ_r} are equivalent in the sense that*

$$C^{-1} \omega_r(f, t) \leqq \omega_{\Sigma_r}(f, t) \leqq C \omega_r(f, t), \quad t > 0, \tag{6.3}$$

for some constant C, depending only on r and n.

Before we prove this theorem we shall consider still another way of defining $\omega_r(f, t)$. This time we want to use a finite number of directional derivatives of order r. Let E be a finite subset of the unit sphere $S^{n-1} \subset R^n$ and set $\Gamma_r(E) = \{\Delta_z^r;\ z \in E\}$. We shall prove the following statement.

Theorem 6.2. *The moduli of continuity ω_r and $\omega_{\Sigma_r(E)}$ are equivalent in the sense that*

$$C^{-1} \omega_r(f, t) \leqq \omega_{\Gamma_r(E)}(f, t) \leqq C \omega_r(f, t), \quad t > 0, \tag{6.4}$$

if and only if the set of homogeneous polynomials in $\xi_1, \dots, \xi_n$

(6.5) $\{\langle z, \xi \rangle^r;\ z \in E\}$ *spans the vector space of all homogeneous polynomials of degree r.*

Proof of Theorem 6.1. Let us first compare $\omega_{\Gamma_r(E)}$ and ω_{Σ_r}. It is easily seen that Σ_r satisfies the condition (T) of Section 2, and that $\Gamma_r(E)$ satisfies the same condition if and only if

(6.6) $$E \text{ spans } R^n \text{ as a vector space.}$$

It is also clear that $\hat{J}_0(\Sigma_r)$ and $\hat{J}_0(\Gamma_r(E))$ are generated by a finite number of homogeneous elements. Hence Corollary 2.6 implies that

(6.7) $$C^{-1}\omega_{\Sigma_r}(f, t) \leqq \omega_{\Gamma_r(E)}(f, t) \leqq C\omega_{\Sigma_r}(f, t)$$

if and only if

(6.8) $$\hat{J}_0(\Sigma_r) = \hat{J}_0(\Gamma_r(E)).$$

The set of monomials $\{\xi^\alpha;\ |\alpha|=r\}$ forms a set of generators for $\hat{J}_0(\Sigma_r)$, and similarly $\{\langle z, \xi\rangle^r;\ z\in E\}$ forms a set of generators for $\hat{J}_0(\Gamma_r(E))$ (cf. (6.9) and (6.10) below). Thus it is obvious that (6.5) implies (6.8). Conversely, if (6.8) holds, then for any $z\in E$ there exists $h_\alpha \in \hat{M}_0(R^n)$ such that $\langle z, \xi\rangle^r = \sum_{|\alpha|=r} \xi^\alpha h_\alpha(\xi)$. Since h_α are continuous this implies $\langle z, \xi\rangle^r = \sum \xi^\alpha h_\alpha(0)$, i.e. (6.5) holds.

To prove the second inequality in (6.3) we now choose any finite set E such that (6.7) holds and then observe that the second inequality of (6.4) is trivial. To study the first inequality we take an arbitrary $z\in S^{n-1}$ and set $\tau=\Delta_z^r$; we need to prove that

$$\omega_\tau(f, t) \leqq C\omega_{\Sigma_r}(f, t)$$

for some C independent of z. Partition τ in the usual fashion $\hat{\tau}=\hat{\tau}_0+\hat{\tau}_1$ where $\hat{\tau}_0=0$ for $|\xi|>1$ and $\hat{\tau}_0=1$ for $|\xi|<1/2$. By Corollary 2.6 (or Corollary 2.3 combined with Corollary 2.5 if you prefer) we have $\omega_{\tau_1}(f, t) \leqq C\omega_{\Sigma_1}(f, t)$. To verify that C may be chosen independent of z we choose μ such that $\hat{\mu}=0$ near the origin and $\hat{\mu}=1$ for $|\xi|>1$, estimate ω_μ in terms of ω_{Σ_r} and observe that $\hat{\tau}_1=\hat{\tau}_1\hat{\mu}$. It remains to estimate ω_{τ_0}, where

$$\hat{\tau}_0(\xi) = \psi(\xi)\hat{\Delta}_z^r(\xi) = \psi(\xi)(e^{-i\langle z, \xi\rangle}-1)^r$$

and ψ is infinitely differentiable and vanishes for $|\xi|>1$. It will be enough to show that

(6.9) $$\hat{\tau}_0(\xi) = \sum_{|\alpha|=r} \hat{\varrho}_\alpha(\xi)\hat{\Delta}^\alpha(\xi),$$

for some measures ϱ_α whose norms are bounded with respect to $z\in S^{n-1}$. That this is possible is rather obvious for continuity and compactness reasons, but we prefer to give an explicit estimate as follows. First observe that we can easily construct measures π_α such that $\|\pi_\alpha\|_M \leqq C$ and

(6.10) $$\langle z, \xi\rangle^r = \sum_{|\alpha|=r} \hat{\pi}_\alpha(\xi)\xi^\alpha.$$

In fact we can take $\hat{\pi}_\alpha(\xi)$ as constant functions. Finally, to pass from (6.10) to (6.9) we observe that the function $h(v)=(\exp(-iv)-1)/v$ is infinitely differentiable on

R and $h(v)\neq 0$ for $|v|\leqq 1$, hence $\psi(\xi)\xi^{\alpha}/\hat{\Delta}^{\alpha}(\xi)$ is infinitely differentiable, and similarly

$$\hat{\tau}_0(\xi)/\langle z, \xi\rangle^r = \psi(\xi)h(\langle z, \xi\rangle)^r$$

is infinitely differentiable and has bound $\hat{M}(R^n)$-norm. This completes the proof of Theorem 6.1.

Proof of Theorem 6.2. We have already seen that (6.5) is a necessary and sufficient condition for (6.8). Hence the assertion follows from Theorem 6.1 together with Corollary 2.6.

As a natural generalization of the higher order moduli of continuity ω_r one may consider ω_σ, where σ is an arbitrary finite sum

$$\sigma = \sum_{k=1}^{s} \alpha_k \delta_{b_k}, \tag{6.11}$$

$\alpha_k \in R$, $b_k \in R^n$, $\sum \alpha_k = 0$. Using the reformulation of the condition (T) given after Theorem 2.2 it is easy to check whether a given σ satisfies (T). To describe $\hat{J}_0(\sigma)$ write $\hat{\sigma} = \sum_{j=1}^{\infty} q_j$, where q_j is homogeneous of degree j. By the proof of Proposition 2.7 $\hat{J}_0(\sigma)$ is generated by $q_1, \dots, q_N$ for some N. Let us give one example of the applications of Corollary 2.6 to this situation. We claim that if σ is an arbitrary measure of the form (6.11) satisfying (T), then

$$\omega_r(f, t) \leqq C\omega_\sigma(f, t)$$

for r sufficiently large. To prove this we need only check that $\hat{J}_0(\Sigma_r) \subset \hat{J}_0(\sigma)$. It is enough to prove the corresponding inclusion between ideals in the ring P of polynomials. Let I_σ be the ideal in P generated by $q_1, q_2, \dots$, and let L be the ideal of all polynomials vanishing at the origin. We have to prove that $L^r \subset I_\sigma$ if r is large enough. But this follows from Hilbert's Nullstellensatz, because the assumption (T) means that the ideal I_σ has no common zero other than the origin.

If $n=1$ we can easily analyze a more general situation than the preceding one. Let σ be an arbitrary element of $M(R)$ with compact support and non-vanishing discrete part. Then $\hat{\sigma}$ is real analytic, so that $\hat{J}_0(\sigma)$ must be generated by ξ^r, where r is the order of the zero of $\hat{\sigma}(\xi)$ at $\xi=0$, hence ω_σ is equivalent to ω_r.

We will now consider L^p-moduli of continuity of higher order. Define for $f \in L^p(R^n)$ $(1 \leqq p < \infty)$

$$\omega_{r,p}(f, t) = \sup \{\|\Delta_z^r * f\|_{L^p};\ |z| \leqq 1\}.$$

We wish to compare $\omega_{r,p}$ with the generalized moduli of continuity $\omega_{\Sigma_r, p}$ and $\omega_{\Gamma_r(E), p}$. We note first that Theorem 6.1 is valid for $1 \leqq p < \infty$ as well. This is an immediate consequence of the remark after Theorem 2.2 and the arguments in the proof of Theorem 6.1. However, concerning $\omega_{\Gamma_r(E), p}$ the situation changes radically as we turn to the case $p < \infty$. In fact we have the following theorem

Theorem 6.3. *Assume* $1<p<\infty$. *Then* $\omega_{r,p}$ *and* $\omega_{\Gamma_r(E),p}$ *are equivalent in the sense that*

(6.12) $$C^{-1}\omega_{r,p}(f,t) \leqq \omega_{\Gamma_r(E)}(f,t) \leqq C\omega_{r,p}(f,t)$$

if and only if E *spans* R^n *as a vector space.*

Proof. The necessity of the condition is obvious. The second inequality is trivial (for any E). To consider the first inequality assume E spans R^n. Working as in the proof of Theorem 6.1 we define τ, τ_0, and τ_1 and estimate $\omega_{\tau_1,p}$ using Corollary 2.6 and the remark after Theorem 2.2 (this remark applies of course also to Corollary 2.5). It remains to estimate $\omega_{\tau_0,p}$. In doing this we may assume that $E=\{\Delta_1, \dots, \Delta_n\}$. Let $m_p(R^n)$ denote the set of multipliers on $\hat{L}^p(R^n)$. It is enough to prove that there exist functions $h_i \in m_p(R^n)$ such that

$$\hat{\tau}_0(\xi) = \sum_{i=1}^n h_i(\xi)\hat{\Delta}_i^r(\xi).$$

Reasoning as in the proof of Theorem 6.1 we see that it is sufficient to find $h_i \in m_p(R^n)$ such that

(6.13) $$\langle z, \xi\rangle^r = \sum_{i=1}^n h_i(\xi)\xi_i^r.$$

Let us choose

$$h_i(\xi) = \langle z, \xi\rangle^r \xi_i^r / \sum_{i=1}^n \xi_i^{2r}.$$

Then (6.13) holds. To see that $h_i \in m_p(R^n)$ for $1<p<\infty$ we observe that h_i are infinitely differentiable outside the origin and positive-homogeneous of degree zero (see e.g. Theorem 6.1.6 in [1]). Noting that the $m_p(R^n)$-norm of h can be estimated in terms of

$$\sup\{|D^\alpha h(\xi)|;\ |\xi| = 1,\ |\alpha| \leqq n+1\}$$

we conclude that in fact C can be taken independent of z for $|z|=1$. This completes the proof.

7. Applications to degree of approximation

In this section we study the approximation process $f \to \mu_{(t)} * f$ where μ is a kernel in $M(R^n)$ with integral equal to 1. We want to compare the order of magnitude of $\|f-\mu_{(t)}*f\|$ as $t\to 0$ with that of the moduli of continuity $\omega_r(f,t)$.

We first formulate a so-called direct theorem.

Theorem 7.1. *Let* $\mu\in L^1(R^n)$, $\int d\mu=1$, *and let* r *be a positive integer. Then the estimate*

(7.1) $$\|\mu_{(t)}*f-f\| \leqq C\omega_r(f,t), \quad t>0, \quad f\in\dot{C}(R^n),$$

holds if and only if

$$1-\hat{\mu}\in\hat{J}_0(\Sigma_r). \tag{7.2}$$

Furthermore, if $1-\hat{\mu}$ *satisfies* (2.12) *for some real* γ, $0<\gamma<r$, *then*

$$\|\mu_{(t)}*f-f\| \leqq Ct^{\gamma}\int_t^{\infty} u^{-\gamma-1}\omega_r(f,u)\,du, \quad t>0, \quad f\in\dot{C}(R^n). \tag{7.3}$$

Proof. The direct part of the first statement follows from Corollary 2.3, the second statement from Corollary 2.4. The converse part of the first statement follows from Corollary 2.6. Note that $\hat{J}_0(\Sigma_r)$ is generated by the set of homogeneous polynomials $\{\xi^{\alpha};\ |\alpha|=r\}$.

The condition (7.2) is usually very easy to verify in specific cases. If $\hat{\mu}$ is $(r-1)$-times continuously differentiable near the origin, then it is obviously necessary for the validity of (7.2) that all derivatives of $1-\hat{\mu}$ of order $\leqq r-1$ vanish at the origin. If $\hat{\mu}$ is s-times continuously differentiable, where $s>r+(n/2)$, then this condition is also sufficient.

In formulating the inverse theorem it is convenient to introduce the "modulus of approximation" $E_k(f,t)$ for $k\in L^1(R^n)$ and $\int k(x)\,dx=1$

$$\begin{aligned} E_k(f,t) &= \omega_{\delta_0-k}(f,t) \\ &= \sup\{\|k_{(u)}*f-f\|;\ 0<u<t\}, \quad t>0. \end{aligned}$$

Theorem 7.2. *Let* $k\in L^1(R^n)$, $\int k\,dx=1$, *and let* r *be a positive integer. Then there exists a constant* C, *such that*

$$\omega_r(f,t) \leqq Ct^{r}\int_t^{\infty} u^{-r-1}E_k(f,u)\,du, \quad t>0, \quad f\in\dot{C}(R^n). \tag{7.4}$$

Moreover, if

$$\hat{J}_0(\Sigma_r)\subset\hat{J}_0(\delta_0-k), \tag{7.5}$$

i.e. if the ideal in $\hat{M}_0(R^n)$ *which is generated by the set of functions* $\{1-\hat{k}(t\xi)\}_{t>0}$ *contains all homogeneous polynomials of degree* r, *then*

$$\omega_r(f,t) \leqq CE_k(f,t), \quad t>0, \quad f\in\dot{C}(R^n). \tag{7.6}$$

Proof. Let $\Sigma=\{\delta_0-k\}$ and let τ be any of the measures $\Delta^{\alpha}\in\Sigma_r$, $|\alpha|=r$. The measure δ_0-k obviously satisfies (S) and (T), and Δ^{α} satisfies (2.12) with $\gamma=r$. Thus we obtain (7.4) from Corollary 2.4. Similarly, the second statement of the theorem follows from Corollary 2.3 together with (6.1).

Remark. If $\hat{J}_0(\delta_0-k)$ is generated by a finite number of positive-homogeneous functions, then Corollary 2.6 shows that (7.5) is in fact also necessary for the validity of (7.6). We can also state, without any extra condition on k, that

$$\sup_{0<u<1} |1-\hat{k}(u\xi)|/|\xi|^r$$

must be bounded in some neighbourhood of the origin if (7.6) holds (cf. (2.17)).

We wish to point out that it is much easier to prove Theorems 7.1 and 7.2 directly than to prove the more general Theorem 2.2. This is so because the hardest part of Theorem 2.2 is the inversion outside a compact set of a measure ϱ of the form (5.2) under the conditions given in Theorem 2.2. And this step is quite easy if $\Sigma=\Sigma_r$ or $\Sigma=\{\delta_0-k\}$.

In most applications the ideal $\hat{J}_0(\delta_0-\mu)$ is easy to describe. In fact, assume that

$$\hat{\mu}(\xi) = 1+q(\xi)h(\xi) \tag{7.7}$$

near the origin, where $q(\xi)$ is a positive-homogeneous function in $\hat{M}_0(R^n)$ and $h\in\hat{M}_0(R^n)$, $h(0)\neq 0$. Then $\hat{J}_0(\delta_0-\mu)$ is generated by the single element q. The following lemma is often useful.

Lemma 7.3. *Assume that the measure $\mu\in M(R^n)$ is positive, radial (i.e. depending only on $|x|$), that $\int d\mu=1$, $\mu\neq\delta_0$, and that $\int |x|^{(n/2)+3}d\mu<\infty$. Then the ideal $\hat{J}_0(\delta_0-\mu)$ is generated by the single element $|\xi|^2$.*

Proof. The last assumption implies that $\hat{\mu}\in C^{[n/2]+3}$ (here C^k denotes the set of k-times continuously differentiable functions). For symmetry reasons all first derivatives of $\hat{\mu}$ vanish at the origin. The same is true of all mixed second order derivatives. Positivity together with $\mu\neq\delta_0$ implies $\partial^2\hat{\mu}(0)/\partial\xi_j^2=-\int x_j^2 d\mu<0$, and by symmetry this quantity is independent of j. This shows that $\hat{\mu}$ near the origin has the form

$$\hat{\mu}(\xi) = 1-a\,|\xi|^2(1+r(\xi)),$$

where $a>0$ and $r(0)=0$. Since $\hat{\mu}\in C^{[n/2]+3}$ we have $r\in C^{[n/2]+1}$. This is known to imply that $r(\xi)$ belongs locally to $\hat{L}^1(R^n)$. Thus we have shown that $\hat{\mu}$ satisfies (7.7) with $q(\xi)=|\xi|^2$, which implies the statement of the lemma.

Several kernels considered in the literature satisfy the conditions of the lemma. Let us mention for instance the kernels (appropriate normalizing constants are denoted by c_n) $c_n\exp(-|x|^2)$, $c_n\exp(-|x|)$, the mean value kernels $c_n\chi_{B^n}$ and $c_n\chi_{S^{n-1}}$ (here χ_{B^n} and $\chi_{S^{n-1}}$ denote the characteristic functions of the unit ball and the unit sphere, respectively).

Let k be any kernel in $L^1(R^n)$ satisfying the conditions of Lemma 7.3. Then first of all $\hat{J}_0(\delta_0-k)=\hat{J}_0(\Sigma_2)$ if the dimension $n=1$, and $\hat{J}_0(\delta_0-k)\subset\hat{J}_0(\Sigma_2)$ for any n. On the other hand, the converse inclusion is not valid if $n>1$, since for instance the function $\xi_1\xi_2$ is contained in $\hat{J}_0(\Sigma_2)$ but not in the ideal generated by $|\xi|^2$. Thus for instance

$$\omega_2(f, t) \leqq CE_k(f, t) \tag{7.8}$$

holds for $n=1$, but due to the remark following Theorem 7.2 this estimate does not hold for $n>1$. Finally we note that the inclusion

$$\hat{J}_0(\Sigma_3) \subset \hat{J}_0(\delta_0 - k)$$

holds for arbitrary n, since $p(\xi)/|\xi|^2$ belongs to $\hat{M}_0(R^n)$ for any homogeneous polynomial p in n variables of degree 3.

The assumptions of Lemma 7.3 also imply saturation of order 2 and a characterization of the saturation class (see [4]).

The Cauchy—Poisson kernel in n dimensions is defined by

$$k(x) = c_n/(1+x^2)^{1+n/2}, \tag{7.9}$$

where c_n is chosen so that the integral of k is 1. This kernel does not satisfy the last assumption of Lemma 7.3. Its Fourier transform is $\hat{k}(\xi)=\exp(-|\xi|)$, so that the ideal $\hat{J}_0(\delta_0-k)$ is generated by the function $|\xi|$. Hence none of the ideals $\hat{J}_0(\delta_0-k)$ and $\hat{J}_0(\Sigma_1)$ contains the other, and therefore none of $E_k(f, t)$ and $\omega(f, t)$ dominates the other. However, since $\hat{J}_0(\Sigma_2)\subset\hat{J}_0(\delta_0-k)$ we have

$$\omega_2(f, t) \leqq CE_k(f, t)$$

and hence (or by Theorem 7.2) (7.4) holds with $r=1$. Also, by Theorem 7.1 we have (7.3) for $\gamma=1$ and any r. On the other hand, considering L^p-norms, $1<p<\infty$, instead of supremum norms we get a different situation. In fact we have for $1<p<\infty$ with obvious notation and $k(x)$ defined by (7.9)

$$C^{-1}\omega_p(f, t) \leqq E_{k,p}(f, t) \leqq C\omega_p(f, t), \quad t>0, \quad f\in L^p(R^n).$$

The reason for this is that the ideal in the ring of germs at the origin of elements of $m_p(R^n)$ generated by $|\xi|$ coincides with the ideal generated by ξ_j, $j=1, 2, \ldots, n$ (cf. the proof of Theorem 6.3).

Numerous estimates closely related to those of Theorems 7.1 and 7.2 have been given in the literature for specific kernels and sometimes for various classes of kernels. In one dimension results of this kind can be found for instance in the book by Butzer and Nessel [9], where many references to earlier literature are given. As an example of a text treating the several variable case we mention Nikol'skiĭ's book [12].

Sharp results on trigonometric approximation of periodic functions are also easy to deduce from our general theorems. Denote by $C^*(R^n)$ the set of continuous functions on R^n, 2π-periodic in each variable. Let T_m be the set of trigonometric polynomials in n variables of total degree $\leqq m$, and set

$$E^*(f, m) = \inf\{\|f-p\|;\ p\in T_m\}.$$

Then we have the following well-known result (see e.g. [19]).

Theorem 7.4. (Jackson's theorem.) *There exists a constant C depending only on r and the dimension n, such that*

$$(7.10)\qquad E^*(f, m) \leqq C\omega_r(f, 1/m), \quad m = 1, 2, \dots, \quad f\in C^*(R^n).$$

A strong inverse theorem for trigonometric approximation is Stečkin's theorem [17]. A several variable version of Stečkin's theorem is given in Timan's book. This result can be formulated as follows.

Theorem 7.5. *There exists a constant C depending only on r and the dimension, such that*

$$\omega_r(f, 1/m) \leqq C\frac{1}{m^r}\sum_{j=0}^{m} j^{r-1}E^*(f, j), \quad f\in C^*(R^n), \quad m = 1, 2, \dots.$$

Proof of Theorems 7.4 and 7.5. Let K_a be the cube in R^n consisting of all points ξ such that $|\xi_1|+\dots+|\xi_n|\leqq a$. Take a function $\psi\in L^1(R^n)$ such that $\hat\psi\in C^\infty$, $\hat\psi=1$ in $K_{1/2}$ and $\hat\psi=0$ outside K_1, and set $\lambda=\delta_0-\psi$. We claim that $\omega_\lambda(f, 1/m)$ has the same order of magnitude as $E^*(f, m)$; more exactly

$$(7.11)\quad C^{-1}\omega_\lambda(f, 1/2m) \leqq E^*(f, m) \leqq \omega_\lambda(f, 1/m), \quad m = 1, 2, \dots, \quad f\in C^*(R^n),$$

for some C. To prove this note first that the function $p_m=f*\psi_{(1/m)}$ is 2π-periodic in each variable and that the Fourier transform in the sense of the theory of distributions of p_m satisfies $\hat p_m=\hat f\hat\psi_{(1/m)}$, hence $\hat p_m=0$ in the complement of K_m, since $\hat\psi_{(1/m)}(\xi)=\hat\psi(\xi/m)$ vanishes in that set. This shows that $p_m\in T_m$. Since $f*\lambda_{(1/m)}=f-p_m$ the second part of (7.11) now follows. Let $p\in T_m$ and set $g_t=\lambda_{(t)}*p$. Then $\hat g_t$ must vanish in the interior of $K_{1/2t}$ since $\hat\lambda_{(t)}$ vanishes in that set. But g_t is also an element of T_m, hence $\hat g_t$ vanishes outside K_m. This shows that g_t must be identically zero if $1/2t>m$. Hence, if $t<1/2m$ and $p\in T_m$

$$\|\lambda_{(t)}*f\| = \|\lambda_{(t)}*(f-p)\| \leqq \|\lambda\|_M\|f-p\|,$$

which proves the first inequality in (7.11).

To complete the proof of Theorem 7.4 we now invoke Corollary 2.3 with $\tau=\lambda$ and $\Sigma=\Sigma_r$. The basic assumption $\hat J_0(\lambda)\subset\hat J_0(\Sigma_r)$ is trivially fulfilled since $\hat J_0(\lambda)=\{0\}$.

To deduce Theorem 7.5 we use Corollary 2.4 with $\Sigma=\{\lambda\}$ and τ equal to any one of the elements of Σ_r. From the estimate of ω_r in terms of ω_λ so obtained it is easy to deduce the desired inequality.

Intimately connected with (7.11) is a well-known estimate for Peetre's K-functional. Let us briefly describe the problem. For further information we refer the reader to [20] or [1] (Theorem 6.7.3). Letting $\|\cdot\|$ denote the supremum norm as above we define the norm

$$\|f\|_m = \|f\| + \sup_{|\alpha|=m} \|D^\alpha f\|$$

(we write $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$), and let A and B be the corresponding Banach spaces of continuous and C^m-functions, respectively. The K-functional is defined for $f \in A$ by

$$K(f, t) = K(f, t;\ A, B) = \inf\{\|f_0\| + t\|f_1\|_m;\ f_0 + f_1 = f\}, \quad t > 0.$$

The estimate that we have in mind is

$$C^{-1}K(f, t) \leqq \omega_m(f, t^{1/m}) + t\|f\| \leqq CK(f, t), \quad 0 < t < 1.$$

It is quite easy to prove the second inequality. The first inequality, however, is usually proved by means of a rather complicated formula which defines a particular decomposition $f = f_0 + f_1$ (cf. [20] p. 78). We will give a simple proof of this estimate here.

As in the proof of Theorem 7.5 take $\psi \in L^1(R^n)$ such that $\hat{\psi} \in C^\infty$, $\hat{\psi} = 1$ for $|\xi| < 1$, $\hat{\psi} = 0$ for $|\xi| > 2$, and set $\lambda = \delta_0 - \psi$. Set $s = t^{1/m}$ and choose

$$f_0 = \lambda_{(s)} * f;\ \ f_1 = f - f_0 = \psi_{(s)} * f.$$

Since $\hat{\lambda}$ vanishes near the origin we have

$$\|f_0\| \leqq \omega_\lambda(f, s) \leqq C\omega_m(f, s).$$

To estimate $\|f_1\|_m$ we take any multi-index α, $|\alpha| = m$, and observe that

$$D^\alpha f_1 = (D^\alpha \psi_{(s)}) * f = s^{-m} \varphi_{(s)} * f, \tag{7.12}$$

where $\varphi = D^\alpha \psi$. Since $\hat{\varphi}(\xi) = (i\xi)^\alpha \hat{\psi}(\xi)$ we have $\hat{\varphi} \in \hat{J}_0(\Sigma_m)$, and hence $\omega_\varphi \leqq C\omega_m$, Letting α vary we get from (7.12)

$$\begin{aligned} t\|f_1\|_m &\leqq ts^{-m}\, \omega_\varphi(f, s) + t\|f_1\| \\ &\leqq C\omega_m(f, s) + Ct\,\|f\|, \end{aligned}$$

which completes the proof.

8. Some further applications

We will conclude by mentioning two more applications of the results and methods of this paper.

In [16] an application to a problem on the modulus of continuity of holomorphic functions is described. Consider functions f continuous on the closed unit disc and holomorphic in the open disc. It was proved by Tamrazov [18] that the modulus of continuity of f can be estimated in terms of the modulus of continuity of its restriction f_0 to the boundary of the unit disc as follows

$$\omega(f, t) \leqq C\omega(f_0, t).$$

By contrast, it has long been known that for *harmonic* functions in the unit disc there is no better estimate than

$$\omega(f, t) \leqq Ct \int_t^\infty u^{-2}\omega(f_0, u)\, du.$$

Shapiro gives a new proof of Tamrazov's result using some of the ideas of this paper. This proof shows that — after a transformation to a half-plane — the phenomena in question can be understood in terms of the behaviour at the origin of the Fourier transform of the kernel $k(x)=(\pi(1+x^2))^{-1}$ associated with the Poisson kernel $P(x, y)=(1/y)k(x/y)$, $y>0$, $x\in R$.

Finally we wish to mention the extension of Theorem 2.2 to vector-valued measures, which is given in [6]. The purpose of this extension was to obtain a new proof and a new understanding of a theorem on directional moduli of continuity of vector-valued functions given earlier by the author [3].

By making appropriate definitions it was possible to give the extension to the vector-valued case a formulation very similar to Corollary 2.3. For instance, the condition that $\hat{\tau}$ belongs to the ideal $\hat{J}_0(\Sigma)$ in the ring $\hat{M}_0(R^n)$ is replaced by the condition that the (germ at the origin of the) vector-valued function $\hat{\tau}$ belongs to a certain *submodule* of the module $\hat{M}_0(R^n)^m$ over the ring $\hat{M}_0(R^n)$. The submodule in question is constructed from the set Σ of vector-valued measures in an analogous fashion as the ideal $\hat{J}_0(\Sigma)$ is constructed above.

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