Positive harmonic functions vanishing on the boundary of certain domains in $\mathbf{R}^n$

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1. Introduction

Let E be a closed, proper subset of the hyperplane $y=0$ in $\mathbf{R}^{n+1}$. A point in $\mathbf{R}^{n+1}$ is, as is customary, denoted by (x, y), where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. We assume that each point of E is regular for Dirichlet's problem in $\Omega = \mathbf{R}^{n+1} \setminus E$. C will in the following be a constant, the value of which may vary from line to line.

Consider the cone $\mathscr{P}_E$ of positive harmonic functions in Ω with vanishing boundary values at each point of E. It is easily seen that $\mathscr{P}_E$ contains a non-zero element (Theorem 1).

According to general Martin theory (see e.g. Helms [8]) each positive harmonic function u in an open set Ω may be represented as an integral

$$u(x) = \int_{\Delta_1} K(x, \xi)\, d\mu(\xi),$$

where Δ_1 denotes the set of minimal points in the Martin boundary of Ω. For each $\xi \in \Delta_1$, the function $x \to K(x, \xi)$ is harmonic and minimal positive in the sense of Martin. We recall that a positive harmonic function $u\colon \Omega \to \mathbf{R}$ is minimal positive, if for each positive harmonic function $v\colon \Omega \to \mathbf{R}$

$$v < u \Rightarrow v = \lambda u \quad \text{for some } \lambda, \quad 0 \leqq \lambda < 1.$$

Now we return to the special setting of this paper, i.e. $\Omega = \mathbf{R}^{n+1} \setminus E$, $E \subset \{y=0\}$. In this situation two cases may occur (Theorem 2):

Case 1. All functions in $\mathscr{P}_E$ are proportional.

Case 2. $\mathscr{P}_E$ is generated by two linearly independent, minimal positive harmonic functions.

Stated in terms of Martin theory: the Martin boundary of Ω has either one or two "infinite" points.

The main aim of this paper is to give conditions on the set E, which determine whether Case 1 or Case 2 occurs. We thereby prove a conjecture made by Kjellberg [13].

2. The existence of functions in $\mathcal{P}_E$. Some lemmas

First we formulate a lemma, which will be quite useful in the sequel.

Lemma 1. *Let* $B=\{(x,y)\in\mathbf{R}^{n+1};\ |x|^2+y^2<1\}$, *the open unit ball in* $\mathbf{R}^{n+1}$. *Suppose that* u *is subharmonic in* B *and that the following estimate holds:*

$$u(x,y)\leqq\frac{1}{|y|^n},\quad (x,y)\in B.$$

Then

$$u(x,y)\leqq C_\varepsilon\quad\text{for}\quad |x|\leqq 1-\varepsilon,\quad (x,y)\in B,$$

where C_ε *only depends on* ε.

Proof. This lemma is a special case of the "log log-theorem" of Beurling and Levinson (Levinson [14]) extended to subharmonic functions in higher dimensions by Domar [4, Th. 2].

Theorem 1. $\mathcal{P}_E$ *contains a non-zero function.*

Proof. Let $D_m=\{(x,y)\in\mathbf{R}^{n+1};\ |x|^2+|y|^2<m^2\}$ and let v_m solve the Dirichlet problem

$$v_m(x,y)=\begin{cases}1 & |x|^2+|y|^2=m^2\\ 0 & (x,y)\in D_m\cap E\end{cases}$$
$$\Delta v_m=0\quad\text{in}\quad D_m\setminus E.$$

We normalize by putting $u_m(x,y)=v_m(x,y)/v_m(0,1)$ and claim that there is a constant C_M, depending only on M such that

$$u_m(x,y)\leqq C_M,\quad (x,y)\in D_M,\quad m\geqq 2M. \tag{2.1}$$

By Harnack's inequality it easily follows that

$$u_m(x,y)\leqq\frac{C_M'}{|y|^n},\quad (x,y)\in D_M. \tag{2.2}$$

Thus an estimate of type (2.1) holds for $(x,y)\in D_M\cap\{|y|\geqq 1\}$. For points $|y|\leqq 1$ we apply Lemma 1 to conclude that (2.1) holds.

The function w_M solving the Dirichlet problem

$$w_M=\begin{cases}C_M & \text{on}\quad |x|^2+y^2=M^2\\ 0 & \text{on}\quad E\cap D_M\end{cases}$$
$$\Delta w_M=0\quad\text{in}\quad D_M\setminus E$$

is a harmonic majorant for all u_m, $m \geqq 2M$ on D_M and therefore $\{u_m\}_{m=1}^{\infty}$ is equicontinuous on each D_M, and we may extract a subsequence converging uniformly on each compact subset to a function u harmonic in Ω. Because of the majorization

$$u_m(x, y) \leqq w_M(x, y), \quad (x, y) \in D_M, \quad m \geqq 2M$$

it follows that u takes the boundary value 0 on E. Since $u(0, 1) = 1$, u is non-zero and Theorem 1 is proved.

Lemma 2. *(Herglotz' theorem.) A positive harmonic function u in the upper halfspace $y > 0$ has the representation*

$$u(x, y) = \varkappa y + C_n \int \frac{y\, d\mu(t)}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}}, \tag{2.3}$$

where $\varkappa \geqq 0$ and μ is a positive measure.

When $u \in \mathscr{P}_E$, $u(x, 0)$ is a continuous function on $\mathbf{R}^n$ and (2.3) reduces to

$$u(x, y) = \varkappa y + C_n \int \frac{y u(t, 0)}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}}\, dt. \tag{2.4}$$

A similar representation also holds in the lower halfspace.

Lemma 3. *Each function $u \in \mathscr{P}_E$ satisfies the growth estimate*

$$u(x, y) = O(|(x, y)|) \quad \text{as} \quad |(x, y)| \to \infty. \tag{2.5}$$

Proof. We may without loss of generality assume that u is symmetric with respect to the hyperplane $y = 0$. By (2.4) it follows that

$$u(0, R) \leqq R u(0, 1) \tag{2.6}$$

and Harnack's inequality gives the estimate

$$u(x, y) \leqq C \frac{R^{n+1}}{|y|^n} \quad \text{for} \quad |(x, y)| \leqq 2R.$$

We conclude that (2.5) is true for points in the cone $|y| \geqq |x|/4$. For points close to the hyperplane $y = 0$ we argue as follows:

Take a point $(x_0, 0)$ such that $|x_0| = R$ and consider the ball $\{(x, y) \in \mathbf{R}^{n+1};\ |(x, y) - (x_0, 0)| \leqq R\}$. Normalize the coordinates (x, y) by putting

$$\begin{cases} x = x_0 + R\xi \\ y = R\eta. \end{cases}$$

The function $v(\xi, \eta) = U(x_0 + R\xi, R\eta)$ is subharmonic in $|(\xi, \eta)| < 1$ and satisfies there the estimate

$$v(\xi, \eta) \leqq C \frac{R}{|\eta|^n}.$$

By Lemma 1, $v(\xi, \eta) \leqq C'R$ for $|(\xi, \eta)| < 1$, $|\xi| \leqq 3/4$ and it follows that

$$u(x, y) \leqq C'R \quad \text{for} \quad |(x, y)-(x_0, 0)| \leqq \frac{R}{2},$$

where the constant C' depends only on the dimension. Since (2.5) is known to be true in the cone $|y| \geqq |x|/4$, we conclude that (2.5) holds and the proof of Lemma 3 is complete.

Lemma 4. *If $u \in \mathcal{P}_E$ has the representation*

$$u(x,y) = C_n \int \frac{|y|\, u(t, 0)}{(|x-t|^2+y^2)^{\frac{n+1}{2}}}\, dt \tag{2.7}$$

then

$$u(x, y) = o(|(x, y)|) \quad \textit{as} \quad |(x, y)| \to \infty.$$

Proof. The proof is identical to the proof of Lemma 3 except that the initial estimate (2.6) is replaced by $u(0, R) = o(R)$.

3. Some characterizations of the cone $\mathcal{P}_E$

We first state some definitions and results from Friedland & Hayman [5], which will be needed later.

Definition. A function u, $u\colon \mathbf{R}^d \to \mathbf{R}$, has the domain D as a *tract*, if $u > 0$ in D and $u \to 0$ as x approaches any finite boundary point of D from the inside of D.

When u is a subharmonic function in $\mathbf{R}^d$, let $M(r) = \max_{|x|=r} u(x)$ be the maximum modulus.

We recall the definitions of the *order* λ and the *lower order* μ of a subharmonic function

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log^+ M(r)}{\log r}, \quad \mu = \underline{\lim_{r\to\infty}} \frac{\log^+ M(r)}{\log r}.$$

Thus Lemma 3 shows that $\lambda \leqq 1$ for all functions in $\mathcal{P}_E$.

When combining Theorem 1 and Theorem 2 of Friedland & Hayman [5] we obtain

Lemma 5. *Let $\ell(k, d)$ be the infimum of the lower orders of subharmonic functions with k tracts. Then $\ell(k, d) \geqq 2(1-1/k)$.*

Remark. As is seen from the proof in [5], p. 143, for a positive, subharmonic function u with k tracts, it is even true that its maximum modulus $M(r)$ satisfies

$$M(r) \geqq C(u)\, r^{2\left(1-\frac{1}{k}\right)}.$$

Our first characterization of $\mathcal{P}_E$ is given in

Theorem 2. *The cone $\mathcal{P}_E$ is either one- or twodimensional.**

Proof. By Theorem 1, $\dim \mathcal{P}_E \geqq 1$. We suppose $\dim \mathcal{P}_E \geqq 3$, which we will show leads to a contradiction. Then there exist three linearly independent, minimal positive harmonic functions v_1, v_2 and v_3. It follows that the sets

$$\begin{aligned}\Omega_1 &= \{(x, y)\in\Omega;\ v_1(x, y) > v_2(x, y)+v_3(x, y)\}\\ \Omega_2 &= \{(x, y)\in\Omega;\ v_2(x, y) > v_1(x, y)+v_3(x, y)\}\\ \Omega_3 &= \{(x, y)\in\Omega;\ v_3(x, y) > v_1(x, y)+v_2(x, y)\}\end{aligned}$$

are disjoint and non-empty. If say $\Omega_1=\emptyset$, $v_1\leqq v_2+v_3$ in Ω, and then by Kjellberg [12, Th. 1], v_1 is a linear combination of v_2 and v_3, which contradicts the linear independence.

We define

$$w = \max(0, v_1-v_2-v_3,\ v_2-v_1-v_3,\ v_3-v_1-v_2).$$

w is subharmonic in $\mathbf{R}^{n+1}$ and has at least 3 tracts. Lemma 5 now gives that w has lower order $\mu\geqq 2(1-1/3)=4/3$. But this contradicts that $\mu\leqq\lambda\leqq 1$ for all functions in $\mathcal{P}_E$ (Lemma 3), and the proof is finished.

The following theorem, which will be used in the sequel, maybe also illuminates the two cases.

Theorem 3. *Case 1 is characterized by either of the following equivalent conditions:*

(i) *$\mathcal{P}_E$ is one-dimensional;*

(ii) *all functions in $\mathcal{P}_E$ are symmetric with respect to the hyperplane $y=0$;*

(iii) *all functions in $\mathcal{P}_E$ satisfy the growth estimate*

$$u(x, y) = o(|(x, y)|) \quad as \quad |(x, y)| \to \infty.$$

In an analogous manner, we may also give three equivalent characterizations of Case 2.

(I) *$\mathcal{P}_E$ is two-dimensional;*

(II) *there exist non-symmetric functions in $\mathcal{P}_E$;*

(III) *there exists a function $u\in\mathcal{P}_E$ such that $u(x, y)\geqq |y|$.*

For the proof of Theorem 3 we need the following:

Lemma 6. *If $u\in\mathcal{P}_E$ has the representation* (2.7) *(i.e. the constant $\varkappa$ in the representation* (2.4) *is* 0 *both for the upper and lower halfspace), then for all $x\in\mathbf{R}^n$ the function $y\to u(x, y)$ is increasing for $y\geqq 0$.*

* *Added in proof.* An extension of Theorem 2 valid when E is a closed subset of a C^2 hypersurface appears in Ancona [1].

Proof. Let $a>0$ and define

$$v(x, y) = u(x, 2a-y)-u(x, y).$$

v is subharmonic for $y\geqq a$ and $v(x, a)=0$. Form the halfball

$$B\big((0, a); R\big) = \{(x, y)\in \mathbf{R}^{n+1};\ |x|^2+(y-a)^2 < R^2,\ y > a\}.$$

The harmonic measure of the spherical surface of the halfball evaluated at (x, y), $y\geqq a$, is $O(1/R)$ as $R\to\infty$. Since by Lemma 4 $v(x, y)=o(R)$ as $R\to\infty$, it follows that $v(x, y)\leqq 0$, $y\geqq a$. Putting $y=a+h$, $h>0$, we conclude that $u(x, a-h)\leqq u(x, a+h)$ and the proof of Lemma 6 is complete.

Proof of Theorem 3.

(i)$\Rightarrow$(ii): If a non-symmetric function u exists, then $u(x, y)$ and $u(x, -y)$ are linearly independent and hence $\mathcal{P}_E$ cannot be one-dimensional.

(ii)$\Rightarrow$(iii): Since $u\in\mathcal{P}_E$ is symmetric, it has the representation

$$u(x, y) = \varkappa\,|y|+C_n\int\frac{|y|\,u(t, 0)}{\left(|x-t|^2+y^2\right)^{\frac{n+1}{2}}}\,dt.$$

But here $\varkappa=0$. (If $\varkappa>0$ then $u-\varkappa y/2\in\mathcal{P}_E$, which contradicts that all functions in $\mathcal{P}_E$ are symmetric.) By Lemma 4 it now follows that $u(x, y)=o\big(|(x, y)|\big)$ as $|(x, y)|\to\infty$.

(iii)$\Rightarrow$(i): Suppose that $\mathcal{P}_E$ is two-dimensional. Let u_1 and u_2 be two minimal positive harmonic functions, which generate $\mathcal{P}_E$. Then the sets $\Omega_1=\{(x, y);\ u_1(x, y)>u_2(x, y)\}$ and $\Omega_2=\{(x, y);\ u_2(x, y)>u_1(x, y)\}$ are both non-empty, and consequently the function

$$v(x, y) = \max\big(0, u_1(x, y)-u_2(x, y),\ u_2(x, y)-u_1(x, y)\big)$$

is subharmonic and has two tracts.

By the remark following Lemma 5, we conclude that the maximum modulus of v, $M_v(r)$, satisfies

$$M_v(r)\geqq Cr^{2\left(1-\frac{1}{2}\right)} = Cr.$$

We now see from the definition of the function v that

$$\max_{\nu=1,2}\ \max_{|(x, y)|=r} u_\nu(x, y)\geqq Cr,\quad C>0.$$

This contradicts the assumption (iii) and the proof of the equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) is finished.

The proof of the second part, (I)$\Leftrightarrow$(II)$\Leftrightarrow$(III) is essentially contained in the proof of the equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) above.

4. The distinction theorem

We now introduce a function $\beta_E(x)$, $x\in\mathbf{R}^n$, which measures how "thin" the set E is at ∞. Let $0<\alpha<1$ and let K_x be the open cube in $\mathbf{R}^{n+1}$ with center at $(x, 0)$ and side $\alpha|x|$, all sides parallel to the coordinate planes. Let $\Omega_x=K_x\setminus E$. $\beta_E(x)$ is defined as the harmonic measure of ∂K_x in Ω_x, evaluated at the point x, i.e. let w^x solve the Dirichlet problem

$$w^x(\xi)=\begin{cases}1 & \text{on} \quad \partial K_x\\ 0 & \text{on} \quad E\cap\Omega_x\end{cases}$$

$$\Delta w^x=0 \quad \text{on} \quad K_x\setminus E.$$

Then $\beta_E(x)=w^x(x)$.

The following theorem gives a necessary and sufficient condition on E in terms of the function β_E, which determines whether the dimension of the cone $\mathscr{P}_E$ is 1 or 2.

Theorem 4. *Let E and $\mathscr{P}_E$ be as defined in the introduction. Then* $\dim\mathscr{P}_E=1$ *or* 2 *and*

$$\dim\mathscr{P}_E=1 \quad \textit{if and only if} \quad \int_{|x|\geqq 1}\frac{\beta_E(x)}{|x|^n}\,dx=\infty;$$

$$\dim\mathscr{P}_E=2 \quad \textit{if and only if} \quad \int_{|x|\geqq 1}\frac{\beta_E(x)}{|x|^n}\,dx<\infty.$$

The proof will depend on the following simple lemma:

Lemma 7. *Let K be the unit cube in* $\mathbf{R}^{n+1}$.

$$K=\{(x, y)\in\mathbf{R}^{n+1};\ |x_i|\leqq 1,\ i=1,2,\dots,n,\ |y|\leqq 1\}$$

and let $F\subseteqq K\cap\{y=0\}$ be a closed set, all points of which are regular for Dirichlet's problem.

Let ω_y be the harmonic measure of $\{|y|=1\}\cap K$ with respect to $\mathscr{D}=\mathring{K}\setminus F$ and let ω be the harmonic measure of ∂K with respect to $\mathscr{D}$. Then

$$\omega_y(0)\leqq\omega(0)\leqq(n+1)\,\omega_y(0). \tag{4.1}$$

Proof. The left inequality in (4.1) follows just by harmonic majorization. To prove the right inequality, we define ω_i as the harmonic measure of $\{|x_i|=1\}\cap\partial K$ in $\mathscr{D}$. Then

$$\omega(x, y)=\textstyle\sum_{i=1}^n\omega_i(x,y)+\omega_y(x, y),$$

and the desired inequality is a consequence of the inequality

$$\omega_i(0)\leqq\omega_y(0), \quad i=1,2,\dots,n. \tag{4.2}$$

We now prove (4.2) for $i=1$. First we write

$$\omega_1(x, y) = \psi_1(x, y) - w_1(x, y)$$
$$\omega_y(x, y) = \psi_y(x, y) - w_y(x, y),$$

where ψ_1 (ψ_y) is the harmonic measure of $\{|x_1|=1\} \cap K$ ($\{|y|=1\} \cap K$) with respect to $\mathring{K}$. w_1 and w_y solve the following Dirichlet problems for $\mathring{K} \setminus F$:

$$w_1(x, y) = \begin{cases} \psi_1(x, y) & \text{on} \quad F \\ 0 & \text{on} \quad \partial K \end{cases}$$

$$\Delta w_1 = 0 \quad \text{in} \quad \mathring{K} \setminus F$$

$$w_y(x, y) = \begin{cases} \psi_y(x, y) & \text{on} \quad F \\ 0 & \text{on} \quad \partial K \end{cases}$$

$$\Delta w_y = 0 \quad \text{in} \quad \mathring{K} \setminus F.$$

Since by symmetry $\psi_1(0) = \psi_y(0)$, the inequality (4.2) follows from harmonic majorization and the inequality

$$\psi_1(x, 0) \geqq \psi_y(x, 0), \tag{4.3}$$

which in turn is a consequence of

$$\begin{aligned} &\psi_1(x_1, x_2, \ldots, x_n, 0) \geqq \psi_1(0, x_2, \ldots, x_n, 0) \\ &= \psi_y(0, x_2, \ldots, x_n, 0) \geqq \psi_y(x_1, x_2, \ldots, x_n, 0). \end{aligned} \tag{4.4}$$

The two inequalities in (4.4) may easily be proved by reflections of the functions ψ_1 and ψ_y in the hyperplane $x_1 = a$ in analogy with the proof of Lemma 5, and the proof of Lemma 7 is completed.

Proof of Theorem 4. We first prove

$$\int_{|x| \geqq 1} \frac{\beta_E(x)}{|x|^n} \, dx = \infty \Rightarrow \dim \mathcal{P}_E = 1.$$

Suppose on the contrary that $\dim \mathcal{P}_E = 2$. Then according to Theorem 3 there is a function $u \in \mathcal{P}_E$ such that

$$u(x, y) \geqq |y|. \tag{4.5}$$

We now need an estimate of $u(x, 0)$ from below. Recall the notion of the moving cubes K_x, introduced in the definition of the function $\beta_E(x)$.

From (4.5) it follows by harmonic majorization that

$$u(x, 0) \geqq C|x| \omega \left((x, 0), K_x \cap \left\{ |y| = \frac{1}{2} \alpha |x| \right\}, \Omega_x \right),$$

where we have used the standard notation $\omega(\xi, F, \mathscr{D})$ for the harmonic measure of $F \subseteqq \partial\mathscr{D}$ with respect to $\mathscr{D}$ evaluated at ξ. But by Lemma 7, $\omega\big((x, 0), K_x \cap \{|y| = \alpha|x|/2\}, \Omega_x\big) \geqq \beta_E(x)/(n+1)$ and hence

$$u(x, 0) \geqq C|x|\beta_E(x).$$

By Herglotz's theorem (Lemma 2), we have

$$u(0, 1) \geqq C\int_{\mathbf{R}^n} \frac{u(x, 0)}{(|x|^2+1)^{\frac{n+1}{2}}}\,dx \geqq C\int_{\mathbf{R}^n} \frac{|x|\beta_E(x)}{(|x|^2+1)^{\frac{n+1}{2}}}\,dx \geqq C\int_{|x|\geqq 1} \frac{\beta_E(x)}{|x|^n}\,dx = \infty.$$

This contradiction shows that $\dim \mathscr{P}_E = 2$, and this completes the proof of the first implication.

We turn to the proof of the implication

$$\int_{|x|\geqq 1} \frac{\beta_E(x)}{|x|^n}\,dx < \infty \Rightarrow \dim \mathscr{P}_E = 2.$$

Again we argue by contradiction and assume $\dim \mathscr{P}_E = 1$, which by Theorem 3 and Lemma 2 implies that u is represented by a Poisson integral of its boundary values $(\varkappa = 0$ in (2.4)$)$. In particular

$$u(0, R) = C\int_{\mathbf{R}^n} \frac{Ru(x, 0)}{(|x|^2+R^2)^{\frac{n+1}{2}}}\,dx \tag{4.6}$$

and $u(0, R)/R \to 0$ as $R \to \infty$. Choose a sequence $R_k \to \infty$ such that

$$\frac{u(0, r)}{r} \leqq \frac{u(0, R_k)}{R_k} \quad \text{for} \quad r \geqq R_k. \tag{4.7}$$

By Harnack's inequality and Lemma 6 it follows that

$$u(x, 0) \leqq Cu(0, |x|)\beta_E(x). \tag{4.8}$$

The estimate (4.8) inserted in (4.6) gives:

$$u(0, R_k) \leqq C\int_{\mathbf{R}^n} \frac{R_k u(0, |x|)}{(|x|^2+R_k^2)^{\frac{n+1}{2}}}\,\beta_E(x)\,dx.$$

We split the integral into two parts, thereby obtaining

$$u(0, R_k) \leqq C\int_{|x|\leqq R_k} \frac{R_k u(0, |x|)}{(|x|^2+R_k^2)^{\frac{n+1}{2}}}\,\beta_E(x)\,dx + C\int_{|x|\geqq R_k}. \tag{4.9}$$

In the first integral we use the estimates

$$\begin{cases} u(0,|x|) \leqq Cu(0,R_k),\ |x| \leqq R_k & (4.10) \\ \dfrac{1}{(|x|^2+R_k^2)^{\frac{n+1}{2}}} \leqq \dfrac{1}{R_k^{n+1}} & (4.11) \end{cases}$$

and in the second

$$\begin{cases} \dfrac{u(0,|x|)}{|x|} \leqq \dfrac{u(0,R_k)}{R_k}, \quad |x| \geqq R_k & (4.12) \\ \dfrac{1}{(|x|^2+R_k^2)^{\frac{n+1}{2}}} \leqq \dfrac{1}{|x|^{n+1}}. & (4.13) \end{cases}$$

(4.10) is again a consequence of Harnack's inequality and Lemma 6. (4.12) is just a reformulation of (4.7).

After the introduction of the estimates (4.10)—(4.13) and division by R_k, (4.9) takes the form

$$\frac{u(0,R_k)}{R_k} \leqq C\frac{u(0,R_k)}{R_k}\left(\int_{|x|\leqq R_k}\frac{\beta_E(x)}{R_k^n}\,dx+\int_{|x|\geqq R_k}\frac{\beta_E(x)}{|x|^n}\,dx\right). \tag{4.14}$$

But the convergence of

$$\int_{|x|\geqq 1}\frac{\beta_E(x)}{|x|^n}\,dx$$

immediately implies that both integrals in the parenthesis of the right hand side of (4.14) tend to 0 as $R_k\to\infty$. This is a contradiction and also the proof of the second part of Theorem 4 is complete.

5. Some corollaries and applications of Theorem 4

Corollary 1. *Suppose that E omits a one-sided circular cone $\mathscr{K}$ in $\mathbf{R}^n$ for $|x|\geqq R_0$. Then* $\dim \mathscr{P}_E=1$.

Proof. We need only check that

$$\int_{|x|\geqq 1}\frac{\beta_E(x)}{|x|^n}\,dx=\infty.$$

But this is obvious, since if α is chosen small enough, $\beta_E(x)=1$ on the part of a slightly smaller cone $\mathscr{K}'$, where $|x|\geqq 2R_0$.

Let $S_{x_0}(r)$ denote the open ball in $\mathbf{R}^n$ with midpoint x_0 and radius r.

Corollary 2. *Suppose that*

$$m_n(E \cap S_{x_0}(A|x_0|^\alpha)) \geqq \delta > 0$$

for all $x_0 \in \mathbf{R}^n$, $|x_0| \geqq R_0$ *and some* $\alpha < 1/(3n+1)$, $A > 0$, $\delta > 0$.* *Then* $\dim \mathcal{P}_E = 2$. *Here* m_n *denotes n-dimensional Lebesgue measure.*

In particular we deduce

Corollary 2'.

$$m_n(E \cap S_{x_0}(r)) \geqq \eta m_n(S_{x_0}(r))$$

for all $|x_0| \geqq R_0$ *and some* $\eta > 0$, $r > 0$ *implies that* $\dim \mathcal{P}_E = 2$.

For the proof of Corollary 2 we shall need the following lemma about estimation of harmonic measure, which might also be of some independent interest.

Lemma 8. *Let* $E' \subseteqq \{(x, 0) \in \mathbf{R}^{n+1};\ |x| \leqq R\}$, *all points of which are regular for Dirichlet's problem and suppose that*

$$m_n(E' \cap S_{x_0}(h)) \geqq \eta m_n(S_{x_0}(h))$$

for some $\eta > 0$ *and all* x_0, $|x_0| \leqq R - h$. *Let* $\omega(x, y)$ *solve the Dirichlet problem*

$$\omega(x, y) = \begin{cases} 1 & |x|^2 + y^2 = R^2 \\ 0 & (x, 0) \in E' \end{cases}$$

$$\Delta \omega = 0 \quad in \quad \{(x, y) \in \mathbf{R}^{n+1};\ |x|^2 + y^2 < R^2\} \setminus E'.$$

Then

$$\omega(0, 0) \leqq \frac{Ch}{\eta^3 R},$$

where C is an (absolute) constant.

Proof. Without loss of generality we may assume $h = 1$. Let us introduce the following notation:

$$F_m = \left\{x \in \mathbf{R}^n;\ |x_\nu - m_\nu| \leqq \tfrac{1}{2},\ \nu = 1, 2, \ldots, n\right\}, \quad m \in \mathbf{Z}^n$$

$$b_m = \sup_{x \in F_m} \omega(x, 1)$$

$$M_m = \sup_{x \in F_m} \omega(x, 0).$$

Consider the Dirichlet problem for the halfball $\{(x, y) \in \mathbf{R}^{n+1};\ |x|^2 + y^2 \leqq R^2, y \geqq 0\}$. Let $P(x, y;\ x', y')$ be the Poisson kernel for this problem. (x, y) denotes an interior point and (x', y') belongs to the boundary. Put

$$A_{ml}(x) = \int_{F_l \setminus E'} P(x, 1;\ x', 0)\, dx', \quad x \in F_m,$$

* There is no reason to believe that the constant $1/(3n+1)$ is best possible.

where $m, l \in \mathbf{Z}^n$. (A_{ml}) satisfies the following conditions:

$$\begin{cases} \sum_{|l| \leqq R} A_{ml}(x) \leqq 1-\sigma, \quad \sigma = \dfrac{\eta}{C} & (5.1) \\ A_{ml}(x) \leqq \dfrac{C}{|m-l|^{n+1}}. & (5.2) \end{cases}$$

Consider the following Dirichlet problem for the halfball:

$$u(x, y) = \begin{cases} 1 & |x|^2+y^2 = R^2 \\ 0 & (x, 0) \in E' \\ M_m & (x, 0) \in F_m \setminus E', \, m \in \mathbf{Z}^n, \, |m| \leqq R \end{cases}$$

$$\Delta u = 0 \quad \text{in} \quad \{(x, y) \in \mathbf{R}^{n+1}; \; |x|^2+y^2 < R^2, \; y > 0\}.$$

Let $a_{ml} = A_{ml}(x^{(m)})$, where $x^{(m)} \in F_m$ is a point such that $b_m = \omega(x^{(m)}, 1)$.

By harmonic majorization

$$b_m \leqq \sum_{|l| \leqq R} a_{ml} M_l + C_m, \tag{5.3}$$

where C_m is the maximum of the harmonic measure of $|x|^2+y^2=R^2$, $y>0$, evaluated at $(x, 1)$, $x \in F_m$.

Clearly the following inequality holds:

$$C_m \leqq \min\left(\frac{C}{R-|m|}, 1\right) = g_m. \tag{5.4}$$

We also need the estimate

$$M_m \leqq b_m + g_m. \tag{5.5}$$

It follows by a reflection of the harmonic function $\omega(x, y)$ in the hyperplane $y=1/2$ analogous to that in the proof of Lemma 6. Thus, define

$$v(x, y) = \omega(x, y) - \omega(x, 1-y).$$

v is superharmonic in

$$S_m = \left\{(x, y); \; |x-m|^2 + \left(y-\tfrac{1}{2}\right)^2 \leqq \left(R-|m|-\tfrac{1}{2}\right)^2, y \geqq \tfrac{1}{2}\right\}$$

$$v\left(x, \tfrac{1}{2}\right) = 0$$

and $v(x, y) \geqq -1$ on the spherical surface of S_m. Since the harmonic measure of the spherical surface is $O\left((R-|m|)^{-1}\right)$ at the point $(m, 1)$, we conclude that (5.5) holds.

When the estimates (5.4) and (5.5) are introduced in (5.3), it takes the form

$$b_m \leqq \sum_l a_{ml}(b_l + g_l) + g_m. \tag{5.6}$$

From this inequality we wish to conclude that

$$b_m \leqq \frac{C_\sigma}{R-|m|}.$$

In order to do so, we show that the matrix $A=(a_{ml})$ is a strict contraction ($\|A\|<1$) with respect to the weighted l^∞-norm

$$\begin{cases} \|b\| = \max\limits_{|m|\leqq R} |b_m|\varkappa_m \\ \varkappa_m = \max(R-|m|, K), \end{cases}$$

when $K=K(\sigma)$ is large enough.

Evidently, for $\|b\|\leqq 1$

$$(Ab)_m \leqq (1-\sigma)\frac{1}{K},$$

so we only have to verify that

$$(Ab)_m \leqq (1-\delta)\frac{1}{R-|m|}, \quad |m| \leqq R-K, \delta > 0.$$

Splitting the sums in three parts, we have

$$(Ab)_m \leqq \sum\nolimits_{|l|\leqq R} a_{ml}\frac{1}{\varkappa_l} \leqq \sum\nolimits_{|l-m|\leqq\frac{\sigma}{2}(R-|m|)} a_{ml}\frac{1}{R-|l|}$$

$$+\sum\nolimits_{\frac{\sigma}{2}(R-|m|)\leqq|l-m|\leqq R-|m|-1} \frac{C}{|l-m|^{n+1}}\frac{1}{R-|l|}+\sum\nolimits_{|l-m|\geqq R-|m|-1}\frac{C}{|l-m|^{n+1}}\frac{1}{K}$$

$$= S_1+S_2+S_3.$$

Using (5.1), the first sum is estimated as follows:

$$S_1 \leqq \frac{1-\sigma}{1-\frac{\sigma}{2}}\cdot\frac{1}{R-|m|} \leqq \left(1-\frac{\sigma}{2}\right)\frac{1}{R-|m|}.$$

The second sum is estimated by an integral

$$S_2 \leqq C\int_{\frac{\sigma}{2}(R-|m|)\leqq|t|\leqq R-|m|-1} \frac{1}{|t|^{n+1}}\frac{1}{R-|t+m|}\,dt$$

$$\leqq C\int_{\frac{\sigma}{2}(R-|m|)\leqq r\leqq R-|m|-1} \frac{1}{r^{n+1}}\,r^{n-1}\frac{1}{R-r-|m|}\,dr$$

$$\leqq C\left[\frac{1}{\sigma(R-|m|)^2}+\frac{\log(R-|m|)}{(R-|m|)^2}\right].$$

Similarly the third sum is estimated by an integral

$$S_3 \leqq C \int_{r \geqq R-|m|-1} \frac{1}{K} \frac{1}{r^{n+1}} r^{n-1} \, dr \leqq \frac{C}{K} \frac{1}{R-|m|}.$$

If $K \geqq C'/\sigma^2$ for some constant C', it follows that

$$\|A\| \leqq 1 - \frac{\sigma}{4}.$$

We now return to the inequality (5.6), which implies the vector inequality

$$b \leqq A(b + C_1 K r) + C_2 K r \tag{5.7}$$

with

$$r_m = \min\left(\frac{1}{K}, \frac{1}{R-|m|}\right).$$

Clearly $\|r\| \leqq 1$. Since A is a contraction it follows that

$$b \leqq Ab + CKr,$$

from which we conclude

$$b \leqq CK(I-A)^{-1} r.$$

Consequently

$$\|b\| \leqq CK \|(I-A)^{-1}\| \leqq \frac{C}{\sigma^3}.$$

By the definition of the norm

$$b_0 \leqq \frac{C}{R\sigma^3}.$$

Remembering the inequality (5.5) and the fact that $\sigma \sim \eta$, we finally get the required estimate

$$\omega(0, 0) \leqq \frac{C}{R\eta^3}$$

and the proof of Lemma 8 is finished.

Proof of Corollary 2. We choose $\eta = C/|x|^{n\alpha}$ and $h = |x|^\alpha$ in Lemma 8 and thereby obtain the estimate

$$\beta_E(x) \leqq C \frac{|x|^{(3n+1)\alpha}}{|x|}.$$

Since $\alpha < 1/(3n+1)$, the integral $\int_{|x| \geqq 1} \beta_E(x)/|x|^n \, dx$ converges.

Theorem 4 now implies that $\dim \mathcal{P}_E = 2$, which is the desired conclusion.

We have now essentially proved the conjecture of Kjellberg [13], which in our notation may be stated as follows:

Corollary 3. *If $E \subseteqq \mathbf{R}^n$ has the property that there are numbers R and $\varepsilon > 0$, such that each ball in $\mathbf{R}^n$ of radius R contains a subset of E of n-dimensional Lebesgue measure ε, then a function $u \in \mathscr{P}_E$ has the representation:*

$$u(x, y) = cy + \varphi(x, y), \quad y > 0,$$

where φ is bounded in $\mathbf{R}^{n+1}$.

What remains to prove is the boundedness of φ. We first note that u is bounded on the hyperplane $y=0$. This is a consequence of the estimate $u(x, y) = O(|(x, y)|)$ (Lemma 3) and the estimate of harmonic measure in Lemma 8. That φ is bounded now follows immediately, since φ is the Poisson integral of the bounded function $u(x, 0)$ (cf. (2.4)) and Kjellberg's conjecture is completely proved.

We shall now confine ourselves to the case, when $n=1$ (the complex plane) and to regularly distributed intervals, where sharper results may be obtained.

Theorem 5. *Let p be a real number, $p \geqq 1$, and put*

$$E = \bigcup_{m=-\infty}^{\infty} [\operatorname{sign}(m) \cdot |m|^p - d_m, \operatorname{sign}(m) \cdot |m|^p + d_m],$$

where $\{d_m\}_{m=-\infty}^{\infty}$, $0 < d_m < 1/2$, is a sequence of real numbers such that

$$\log d_k \sim \log d_m \quad \text{if} \quad k \sim m, \tag{5.8}$$

$k, m \to \infty$ or $k, m \to -\infty$.

Then

(i) $\dim \mathscr{P}_E = 1$ *if and only if* $\sum_{m \neq 0} \frac{\log d_m}{m^2} = \infty$;

(ii) $\dim \mathscr{P}_E = 2$ *if and only if* $\sum_{m \neq 0} \frac{\log d_m}{m^2} < \infty$.

Proof. We first prove

$$\sum_{m \neq 0} \frac{-\log d_m}{m^2} = \infty \Rightarrow \dim \mathscr{P}_E = 1.$$

Again we intend to apply Theorem 4. Without loss of generality we may assume

$$\sum_{m=1}^{\infty} \frac{-\log d_m}{m^2} = \infty.$$

To estimate the harmonic measure $\beta_E(t)$ for $t > 0$, we use the auxiliary function $\log |\sin \pi z^{1/p}|$, defined for $\operatorname{Re} z > 0$, where the branch of $z^{1/p}$ is chosen, which is positive for real positive z. On the circle $|z-k^p|=d_k$ we have

$$|\sin \pi z^{\frac{1}{p}}| \leqq |z-k^p| \cdot \max_{|z-k^p|\leqq d_k} \pi \left|\cos \pi z^{\frac{1}{p}} \cdot \frac{1}{p} z^{\frac{1}{p}-1}\right| \leqq d_k \cdot 2\pi \frac{1}{p} k^{1-p}.$$

Consider the square

$$R_t = \{(x, y);\ |x-t| \leqq \tfrac{1}{2} t,\ |y| \leqq \tfrac{1}{2} t\}.$$

It follows that

$$u_1(z) = \log|\sin \pi z^{\frac{1}{p}}| + \min_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \log\frac{1}{d_k} - C \leqq 0$$

on

$$\bigcup_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \{z\in\mathbf{C};\ |z-k^p| = d_k\}$$

and

$$u_1(z) \leqq Ct^{\frac{1}{p}} + \min_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \log\frac{1}{d_k} \quad \text{on} \quad R_t.$$

Therefore, for $t\in I_m=\{t\in\mathbf{R};\ (m+1/4)^p\leqq t\leqq(m+3/4)^p\}$,

$$\beta_E(t) \geqq C\frac{u_1(t)}{t^{\frac{1}{p}} + \min_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \log\frac{1}{d_k}} \geqq C\frac{\min_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \log\frac{1}{d_k} - C_1}{t^{\frac{1}{p}} + \min_{\frac{1}{2}t\leqq k^p\leqq\frac{3}{2}t} \log\frac{1}{d_k}}.$$

Because of the assumption (5.8)

$$\beta_E(t) \geqq C\frac{\log\frac{1}{d_m} - C_1}{m+\log\frac{1}{d_m}} \quad \text{for} \quad t\in I_m.$$

Using this estimate we get

$$\int_{|t|\geqq 1} \frac{\beta_E(t)}{|t|}\,dt \geqq \int_1^\infty \frac{\beta_E(t)}{t}\,dt$$

$$\geqq C\sum_{m=1}^\infty \int_{I_m} \frac{1}{t} \frac{\log\frac{1}{d_m} - C_1}{m+\log\frac{1}{d_m}}\,dt \geqq C\sum_{m=1}^\infty \frac{m^{p-1}}{m^p} \cdot \frac{\log\frac{1}{d_m} - C_1}{m+\log\frac{1}{d_m}}.$$

To show that the sum in the right hand side is divergent, we divide the situation up into two cases:

Case 1. $d_m \leqq e^{-m/2}$ only for finitely many m. Then for some N_0,

$$\int_{|t|\geqq 1} \frac{\beta_E(t)}{|t|}\,dt \geqq C_1 \sum_{m=N_0}^{\infty} \frac{\log \frac{1}{d_m} - C_2}{m^2} = \infty.$$

Case 2. $d_m \leqq e^{-m/2}$ for infinitely many m. We choose a subsequence m_j such that

$$m_{j+1} \geqq 4m_j.$$

But (5.8) implies that

$$\log d_k \leqq -Cm \quad \text{for} \quad \tfrac{1}{2}m \leqq k \leqq 2m.$$

Hence

$$\int_{|t|\geqq 1} \frac{\beta_E(t)}{|t|}\,dt \geqq C \sum_j \sum_{\frac{1}{2}m_j \leqq k \leqq 2m_j} \frac{1}{m_j} - C_1 \sum_{m=1}^{\infty} \frac{1}{m^2} = \infty$$

and the proof of the first part of Theorem 5 is complete.

Note that for this part we only need a one-sided condition on E, e.g.

$$E \cap [0, \infty) = \bigcup_{k=1}^{\infty} [k^p - d_k,\, k^p + d_k],$$

where $\sum_{k=1}^{\infty} (-\log d_k)/k^2 = \infty$ and (5.8) holds.

Now turn to the proof of the implication

$$\sum_{m\neq 0} \frac{-\log d_m}{m^2} < \infty \Rightarrow \dim \mathscr{P}_E = 2.$$

We first prove $\int_1^{\infty} (\beta_E(t)/t)\,dt < \infty$.

For $m \geqq 1$ define

$$\varepsilon_m = m - m\left(1 - \frac{d_m}{m^p}\right)^{\frac{1}{p}} \sim \frac{1}{p}\frac{d_m}{m^{p-1}}. \tag{5.9}$$

The function which we will use to estimate the harmonic measure is $F(z^{1/p})$, harmonic for $\operatorname{Re} z > 0$, $z \notin \bigcup_{k=1}^{\infty} [k^p - d_k,\, k^p + d_k]$, where

$$F(w) = \log|\pi w| + \sum_{m=1}^{\infty} \left\{ \frac{1}{2\varepsilon_m} \int_{m-\varepsilon_m}^{m+\varepsilon_m} \log\left|1 - \frac{w}{t}\right| dt + \log\left|1 + \frac{w}{m}\right| \right\}.$$

Again the branch of $z^{1/p}$ is chosen, which is positive for real positive z.

A simple computation shows that for real w

$$\frac{1}{2\varepsilon_m} \int_{m-\varepsilon_m}^{m+\varepsilon_m} \log|w - t|\,dt \geqq \log \varepsilon_m.$$

Moreover

$$\left|\frac{1}{2\varepsilon_m}\int_{m-\varepsilon_m}^{m+\varepsilon_m}\log|w-t|\,dt-\log|w-m|\right| \tag{5.10}$$
$$\leqq \frac{\varepsilon_m^2}{6}\sup_{m-\varepsilon_m\leqq t\leqq m+\varepsilon_m}\frac{1}{|w-t|^2}\leqq\frac{\varepsilon_m^2}{|w-m|^2}$$

for $|w-m|\geqq 1$.

In particular, for $w=0$, (5.10) becomes

$$\left|\frac{1}{2\varepsilon_m}\int_{m-\varepsilon_m}^{m+\varepsilon_m}\log t-\log m\right|\leqq\frac{\varepsilon_m^2}{m^2}. \tag{5.11}$$

We shall need a lower bound for $F(w)$ when $w\in[m-\varepsilon_m, m+\varepsilon_m]$:

$$F(w) = \log\left|\frac{\sin\pi w}{(w+m+1)(w-m)(w-m-1)}\right|+\sum_{k=m-1}^{m+1}\frac{1}{2\varepsilon_k}\int_{k-\varepsilon_k}^{k+\varepsilon_k}\log|t-w|\,dt$$
$$+\sum_{\substack{k=1\\ k\neq m-1,m,m+1}}^{\infty}\left\{\frac{1}{2\varepsilon_k}\int_{k-\varepsilon_k}^{k+\varepsilon_k}\log|t-w|\,dt-\log|z-k|\right\}$$
$$+\sum_{k=1}^{\infty}\left\{\log k-\frac{1}{2\varepsilon_k}\int_{k-\varepsilon_k}^{k+\varepsilon_k}\log t\,dt\right\}$$
$$\geqq \min_{w\in[m-\varepsilon_m, m+\varepsilon_m]}\log|\pi\cos\pi w|-2\log 2+\log\varepsilon_{m-1}+\log\varepsilon_m+\log\varepsilon_{m+1}$$
$$-\sum_{\substack{k=1\\ k\neq m-1,m,m+1}}^{\infty}\frac{\varepsilon_k^2}{|w-k|^2}-\sum_{k=1}^{\infty}\frac{\varepsilon_k^2}{k^2}$$
$$\geqq -3\max_{m-1\leqq k\leqq m+1}\log\frac{1}{\varepsilon_k}-10.$$

Consider again the square

$$R_t=\left\{(x,y);\ |x-t|\leqq\tfrac{1}{2}t,\ |y|\leqq\tfrac{1}{2}t\right\}.$$

It follows that

$$u_2(z)=F(z^{1/p})+3\max_{\frac{1}{4}m\leqq k\leqq\frac{7}{4}m}\log\frac{1}{\varepsilon_k}+10\geqq 0 \quad \text{on} \quad R_t$$

and

$$u_2(z)\geqq Ct^{\frac{1}{p}} \quad \text{on} \quad |y|=\tfrac{1}{2}t,\ |x-t|\leqq\tfrac{1}{2}t.$$

Furthermore $F(z)$ is bounded above for real z. Using Lemma 7 and (5.8) we conclude that for $m^p \leqq t \leqq (m+1)^p$

$$\beta_E(t) \leqq C \frac{\max\limits_{\frac{1}{4}m \leqq k \leqq \frac{7}{4}m} \log \frac{pk^{p-1}}{d_k} + C_1}{t^{\frac{1}{p}}} \leqq C \frac{\log p + (p-1)\log m + \log \frac{1}{d_m} + C_1}{m}.$$

Hence

$$\int_1^\infty \frac{\beta_E(t)}{t} \leqq C \sum_{m=1}^\infty \frac{(m+1)^p - m^p}{m^p} \frac{\log p + (p-1)\log m + \log \frac{1}{d_m} + C_1}{m}$$

$$\leqq C' \sum_{m=1}^\infty \frac{-\log d_m}{m^2} + C' \sum_{m=1}^\infty \frac{\log p + (p-1)\log m + C_1}{m^2} < \infty.$$

The proof of $\int_{-\infty}^{-1} (\beta_E(t)/|t|)\,dt < \infty$ is completely analogous and thus the proof of Theorem 5 is finished.

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