Asymptotic estimation theory of change-point problems for time series regression models and its applications

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Abstract

It is important to detect the structural change in the trend of time series model. This paper addresses the problem of estimating change point in the trend of time series regression models with circular ARMA residuals. First we show the asymptotics of the likelihood ratio between contiguous hypotheses. Next we construct the maximum likelihood estimator (MLE) and Bayes estimator (BE) for unknown parameters including change point. Then it is shown that the proposed BE is asymptotically efficient, and that MLE is not so generally. Numerical studies and the applications are also given.

AMS subject classifications: 62M10, 62M15, 62N99

Keywords: Change point, time series regression, asymptotic efficiency, Bayes estimator, maximum likelihood estimator.

1 Introduction

The change point problem for serially correlated data has been extensively studied in the literature. References on various time series models with change-point can be found in the book of Csörgő and Horvath (1997) and the review paper of Kokoszka and Leipus (2000).

Focusing on a change point in the mean of linear process, Bai (1994) derived the limiting distribution of a consistent change-point estimator by least squares method. Later Kokoszka and Leipus (1998) studied the consistency of CUSUM type estimators of mean shift for dependent observations. Their results include long-memory processes. For a spectral parameter change in Gaussian stationary process, Picard (1985) addressed the problem of testing and estimation. Giraitis and Leipus (1990,1992) generalized Picard's results to the case when the process concerned is possibly non-Gaussian.

For a structural change in regression model, a number of authors studied the testing and estimation of change point. It is important to detect the structural change in economic time series because parameter instability is common in this field. For testing structural changes in regression models with long-memory errors, Hidalgo and Robinson (1996) explored a testing procedure with nonstochastic and stochastic regressors. Asymptotic properties of change-point estimator in linear regression models were obtained by Bai(1998), where the error process may include dependent and heteroskedastic observations.

Despite the large body of literature on estimating unknown change-point in time series models, the asymptotic efficiency has been rarely discussed. For the case of independent and identically distributed observations, Ritov (1990) obtained an asymptotically efficient estimator of change point in distribution by a Bayesian approach. Also the asymptotic efficiency of Bayes estimator for change-point was studied by Kutoyants (1994) for diffusion-type process. Dabye and Kutoyants (2001) showed consistency for change-point in a Poisson process when the model was misspecified.

The present paper develops the asymptotic theory of estimating unknown parameters in time series regression models with circular ARMA residuals. The model and the assumptions imposed are explained in Section 2. Also Section 2 discusses the fundamental asymptotics for the likelihood ratio process between contiguous hypotheses. Section 3 provides the asymptotics of the maximum likelihood estimator (MLE) and Bayes estimator (BE) for unknown parameters including change-point. Then it is shown that the BE is asymptotically efficient, and that the MLE is not so generally. Some numerical examples by simulations are given in Section 4. Section 5 is devoted to the investigation of some real time series data. All the proofs are collected in Section 6.

Throughout this paper we use the following notations. A' denotes the transpose of a vector or matrix A and $\chi(\cdot)$ is the indicator function.

2 Asymptotics of likelihood ratio and some lemmas

Consider the following linear regression model

$$\begin{aligned} y_t &= \{\boldsymbol{\alpha}'\chi(t/n \le \tau) + \boldsymbol{\beta}'\chi(t/n > \tau)\}\boldsymbol{z}_t + u_t, \\ &= r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) + u_t, \quad \text{(say)}, \quad t = 1, \dots, n \end{aligned} \tag{2.1}$$

where $\boldsymbol{z}_t = (z_{t1}, \dots, z_{tq})'$ are observable regressors, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_q)'$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_q)'$ are unknown parameter vectors, and $\{u_t\}$ is a Gaussian circular ARMA process with spectral density $f(\lambda)$ and $E(u_t) = 0$. Here τ is an unknown change-point satisfying $0 < \tau < 1$ and $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau) \in \boldsymbol{\Theta} \subset \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}$.

Letting

$$a_{jk}^n(h) = \begin{cases} \sum_{t=1}^{n-h} z_{t+h,j} z_{tk}, & h = 0, 1, \dots \\ \sum_{t=1-h}^{n} z_{t+h,j} z_{tk}, & h = 0, -1, \dots, \end{cases}$$

we will make the following assumptions on the regressors $\{\boldsymbol{z}_t\}$, which are a sort of Grenander's conditions.

Assumption 2.1.

(G.1) $a_{ii}^n(0) = O(n), \quad i = 1, \ldots, q, \quad$ and $\sum_{t=l}^{l+\rho} z_{ti}^2 = O(\rho)$ for any $(1 \le l \le n)$.

(G.2) $\lim_{n\to\infty} z_{n+1,i}^2 / a_{ii}^n(0) = 0, \quad i = 1, \ldots, q.$

(G.3) The limit

$$\lim_{n\to\infty} \frac{a_{ij}^n(h)}{n} = \rho_{ij}(h)$$

exists for every $i, j = 1, \ldots, q$ and $h = 0, \pm 1, \ldots$.
Let $\boldsymbol{R}(h) = \{\rho_{ij}(h); i, j = 1, \ldots, q\}$.

(G.4) $\boldsymbol{R}(0)$ is nonsingular.

From (G.3) there exists a Hermitian matrix function $\boldsymbol{M}(\lambda) = \{M_{ij}(\lambda); i, j = 1, \ldots, q\}$ with positive semidefinite increments such that

$$\boldsymbol{R}(h) = \int_{-\pi}^{\pi} e^{ih\lambda} d\boldsymbol{M}(\lambda). \tag{2.2}$$

Suppose that the stretch of series from model (1) $\boldsymbol{y}_n = (y_1, \cdots, y_n)'$ is available. Denote the covariance matrix of $\boldsymbol{u}_n = (u_1, \cdots, u_n)'$ by Σ_n, and let $\boldsymbol{t}_n = (r_1, \cdots, r_n)'$ with $r_t = r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$. Then the likelihood function based on $\boldsymbol{y}_n$ is given by

$$L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) = \frac{1}{(2\pi)^{n/2} |\Sigma_n|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{y}_n - \boldsymbol{t}_n)' \Sigma_n^{-1} (\boldsymbol{y}_n - \boldsymbol{t}_n)\right]. \tag{2.3}$$

Since we assume that $\{u_t\}$ is a circular ARMA process, it is seen that Σ_n has the following representation

$$\Sigma_n = \boldsymbol{U}_n^* \mathrm{diag}\{2\pi f(\lambda_1), \cdots, 2\pi f(\lambda_n)\} \mathbf{U_n}$$

where $\boldsymbol{U}_n = \{n^{-1/2} \exp(2\pi i t s / n); t, s = 1, \ldots, n\}$ and $\lambda_k = 2\pi k / n$ (see Anderson (1977)). Write

$$F_n(\lambda_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} (y_t - r_t) e^{-it\lambda_k}.$$

Then the likelihood function (2.3) is rewritten as

$$L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) = \frac{1}{(2\pi)^n \{\prod_{k=1}^n f(\lambda_k)\}^{1/2}} \exp\left[-\frac{1}{2} \sum_{k=1}^{n} f(\lambda_k)^{-1} |F_n(\lambda_k)|^2\right]. \tag{2.4}$$

Define the local sequence for the parameters:

$$\boldsymbol{\alpha}_n = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}, \quad \boldsymbol{\beta}_n = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}, \quad \tau_n = \tau + n^{-1}\rho \tag{2.5}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^q$ and $\rho \in \mathbb{R}$. Under the local sequence (2.5) the likelihood ratio process is represented as

$$
\begin{aligned}
Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho) &= \frac{L_n(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \tau_n)}{L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)} \qquad (2.6)\\
&= \exp\left[-\frac{1}{2\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\}\right.\\
&\qquad \left. -\frac{1}{2n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right]
\end{aligned}
$$

where $d_n(\lambda_k) = (2\pi n)^{-1/2}\sum_{t=1}^{n} u_t e^{it\lambda_k}$ and $A(\lambda_k) = A_1 + A_2 + A_3$ with

$$
A_1 = (2\pi f(\lambda_k))^{-1/2} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s e^{-is\lambda_k},
$$

$$
A_2 = -(2\pi n f(\lambda_k))^{-1/2} \sum_{s=1}^{[\tau n+\rho]} \boldsymbol{a}' \boldsymbol{z}_s e^{-is\lambda_k}
$$

and

$$
A_3 = -(2\pi n f(\lambda_k))^{-1/2} \sum_{s=[\tau n+\rho]+1}^{n} \boldsymbol{b}' \boldsymbol{z}_s e^{-is\lambda_k}.
$$

Here note that $d_n(\lambda_k), k = 1, 2, \ldots$ are i.i.d. complex normal random variables with mean 0 and variance $f(\lambda_k)$ (c.f. Anderson (1977)). Henceforth we write the spectral representation of u_t by

$$
u_t = \int_{-\pi}^{\pi} e^{it\lambda} dZ_u(\lambda). \qquad (2.7)
$$

The asymptotic distribution of $Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho)$ is given as follows.

Theorem 2.1. *Suppose that Assumption 2.1 holds. Then for all $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau) \in \boldsymbol{\Theta}$, the log-likelihood ratio has the asymptotic representation*

$$
\begin{aligned}
&\log Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho)\\
&= (\boldsymbol{\beta} - \boldsymbol{\alpha})' W_1 + \sqrt{\tau}\boldsymbol{a}' W_2 + \sqrt{1-\tau}\boldsymbol{b}' W_3\\
&\quad -\frac{1}{8\pi^2}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta} - \boldsymbol{\alpha})'\boldsymbol{z}_{s+j}\boldsymbol{z}_s'(\boldsymbol{\beta} - \boldsymbol{\alpha})\\
&\quad -\frac{1}{4\pi}(\sqrt{\tau}\boldsymbol{a} + \sqrt{(1-\tau)}\boldsymbol{b})'\int_{-\pi}^{\pi} f(\lambda)^{-1} d\boldsymbol{M}(\lambda)(\sqrt{\tau}\boldsymbol{a} + \sqrt{(1-\tau)}\boldsymbol{b}) + o_p(1)\\
&= \log Z(\boldsymbol{a}, \boldsymbol{b}, \rho) + o_p(1), \qquad (say),
\end{aligned}
$$

where

$$
W_1 = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} f(\lambda)^{-1} dZ_u(\lambda),
$$

$$
W_2 = \frac{1}{2\pi\sqrt{n\tau}}\int_{-\pi}^{\pi}\sum_{s=1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda}(1 + e^{in\lambda}) f(\lambda)^{-1} dZ_u(\lambda)
$$

and

$$W_3 = \frac{1}{2\pi\sqrt{n(1-\tau)}} \int_{-\pi}^{\pi} \sum_{s=[\tau n+\rho]+1}^{n} \boldsymbol{z}_s e^{is\lambda}(1+e^{-in\lambda}) f(\lambda)^{-1} dZ_u(\lambda).$$

Here W_1, W_2 and W_3 are asymptotically normal with mean $\mathbf{0}$ and covariance matrix V_1, V_2 and V_3, respectively, where

$$V_1 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left| \sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} \right|^2 f(\lambda)^{-1} d\lambda,$$

$$V_2 = V_3 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2 f(\lambda)^{-1} d\boldsymbol{M}(\lambda).$$

Next we present some fundamental lemmas which are useful in the estimation of change point.

Lemma 2.1. *Suppose that Assumption 2.1 holds. Then for any compact set $\mathcal{C} \subset \boldsymbol{\Theta}$, we have*

$$\sup_{\alpha,\beta,\tau \in \mathcal{C}} E_{\alpha,\beta,\tau} Z_n^{1/2}(\boldsymbol{a}, \boldsymbol{b}, \rho) \leq \exp\{-g(\boldsymbol{a}, \boldsymbol{b}, \rho)\}$$

where

$$g(\boldsymbol{a}, \boldsymbol{b}, \rho) = (\boldsymbol{a}', \boldsymbol{b}') \boldsymbol{K} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} + c|\rho|$$

with some positive definite matrix $\boldsymbol{K}$ and $c > 0$.

Lemma 2.2. *Suppose that Assumption 2.1 holds. Then for any compact set $\mathcal{C} \subset \boldsymbol{\Theta}$, there exist $\kappa(\mathcal{C}) = \kappa, B(\mathcal{C}) = B$ such that*

$$\sup_{(\alpha,\beta,\tau)\in\mathcal{C}|a_i|<H,|b_i|<H,\rho_j<H} [\|\boldsymbol{a}_1 - \boldsymbol{a}_2\|^2 + \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|^2 + |\rho_1 - \rho_2|^2]^{-1}$$
$$\times E_{\alpha,\beta,\tau} \left[Z_n^{1/4}(\boldsymbol{a}_2, \boldsymbol{b}_2, \rho_2) - Z_n^{1/4}(\boldsymbol{a}_1, \boldsymbol{b}_1, \rho_1) \right]^4 \leq B(1 + H^{\kappa}).$$

3 Estimation theory

We are interested in the behavior of maximum likelihood estimator (MLE) and Bayes estimator (BE). To introduce these estimators, we need a loss function $w(y), y \in \mathbb{R}^d$ which is

1. nonnegative, continuous at point 0 and $w(0) = 0$, but is not identically 0;
2. symmetric: $w(y) = w(-y)$;
3. the sets $\{y : w(y) < c\}$ are convex for all $c > 0$.

We denote by $\boldsymbol{W}_p$ the class of loss functions satisfying 1-3 with polynomial majorants. The example of such function is $w(y) = |y|^p, p > 0$.

The MLE $\hat{\boldsymbol{\theta}}'_{ML} = (\hat{\boldsymbol{\alpha}}'_{ML}, \hat{\boldsymbol{\beta}}'_{ML}, \hat{\tau}_{ML})$ of $\boldsymbol{\theta}' = (\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau)$ is defined by

$$L(\hat{\boldsymbol{\alpha}}_{ML}, \hat{\boldsymbol{\beta}}_{ML}, \hat{\tau}_{ML}) = \max_{(\alpha,\beta,\tau)\in\boldsymbol{\Theta}} L(\alpha, \beta, \tau) \tag{3.1}$$

The Bayes estimator $\tilde{\boldsymbol{\theta}}'_B = (\tilde{\boldsymbol{\alpha}}'_B, \tilde{\boldsymbol{\beta}}'_B, \tilde{\rho}_B)$ with respect to the quadratic loss function $l(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$ and a prior density $\pi(\cdot)$ is of the form

$$\tilde{\boldsymbol{\theta}}_B = \int_{\boldsymbol{\Theta}} \boldsymbol{\theta} p(\boldsymbol{\theta}|Y_n) d\boldsymbol{\theta} \tag{3.2}$$

where

$$p(\boldsymbol{\theta}|Y_n) = \frac{\pi(\boldsymbol{\theta}) L_n(\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} \pi(\boldsymbol{v}) L_n(\boldsymbol{v}) d\boldsymbol{v}}.$$

We suppose that the prior density is a bounded, positive and continuous function possessing a polynomial majorant on $\boldsymbol{\Theta}$. For $Z(\boldsymbol{u}), \boldsymbol{u} = (\boldsymbol{a}', \boldsymbol{b}', \rho)'$, in Theorem 1, define two random vectors $\hat{\boldsymbol{u}}' = (\hat{\boldsymbol{a}}', \hat{\boldsymbol{b}}', \hat{\rho})$ and $\tilde{\boldsymbol{u}}' = (\tilde{\boldsymbol{a}}', \tilde{\boldsymbol{b}}', \tilde{\rho})$ by relations

$$Z(\hat{\boldsymbol{u}}) = \sup_{\boldsymbol{u}\in\mathbb{R}^{2q+1}} Z(\boldsymbol{u}), \tag{3.3}$$

$$\tilde{\boldsymbol{u}} = \frac{\int_{\mathbb{R}^{2q+1}} \boldsymbol{u} Z(\boldsymbol{u}) d\boldsymbol{u}}{\int_{\mathbb{R}^{2q+1}} Z(\boldsymbol{v}) d\boldsymbol{v}}. \tag{3.4}$$

Theorem 3.1. *Let the parameter set $\boldsymbol{\Theta}$ be an open subset of $\mathbb{R}^{2q+1}$. Then the MLE is uniformly on $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) \in \boldsymbol{\Theta}$, consistent*

$$P - \lim_{n\to\infty} \hat{\boldsymbol{\theta}}_{ML} = \boldsymbol{\theta}$$

and converges in distribution

$$\mathcal{L}_\theta(\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta})\} \underset{d}{\longrightarrow} \mathcal{L}(\hat{\boldsymbol{u}}).$$

For any continuous loss function $w \in \boldsymbol{W}_p$, we have

$$\lim_{n\to\infty} E_{\boldsymbol{\theta}} w((\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta})) = Ew(\hat{\boldsymbol{u}}).$$

A similar theorem for Bayes estimators can be stated as follows.

Theorem 3.2. *The Bayes estimator $\tilde{\boldsymbol{\theta}}_B$, uniformly on $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, is consistent*

$$P_\theta - \lim_{n\to\infty} \tilde{\boldsymbol{\theta}}_B = \boldsymbol{\theta}$$

and converges in distribution

$$\mathcal{L}_\theta(\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\tilde{\boldsymbol{\theta}}_B - \boldsymbol{\theta})\} \underset{d}{\longrightarrow} \mathcal{L}(\tilde{\boldsymbol{u}}).$$

For any continuous loss function $w \in \boldsymbol{W}_p$, we have

$$\lim_{n\to\infty} E_{\boldsymbol{\theta}} w((\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\tilde{\boldsymbol{\theta}}_B - \boldsymbol{\theta})) = Ew(\tilde{\boldsymbol{u}}).$$

Remark. From Theorem 3 and Theorem 1.9.1 of Ibragimov and Has'minski(1981), we can see that the BE is asymptotically efficient such that

$$E\|\hat{\boldsymbol{u}}\|^2 \geq E\|\tilde{\boldsymbol{u}}\|^2.$$

4 Numerical examples

In this section we report some Monte Carlo results for the MLE and BE of an unknown change point. We consider the following time series regression model:

$$y_t = \begin{cases} \boldsymbol{\alpha}'\boldsymbol{z}_t + u_t, \; t = 1, \dots, [\tau n] \\ \boldsymbol{\beta}'\boldsymbol{z}_t + u_t, \; t = [\tau n] + 1 \dots, n, \end{cases} \tag{4.1}$$

where $\{u_t\}$ is a Gaussian AR(1) process generated by

$$u_t = \xi u_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim i.i.d. N(0, \sigma^2).$$

To verify the theoretical results and for comparative purposes, we deal with the following regressors;
Model (I) : $\boldsymbol{z}_t = 1$ (scalar-valued),
Model (II): $\boldsymbol{z}_t = \cos(\nu t)$ (scalar-valued),
Model (III): $\boldsymbol{z}_t = (1, \cos(\nu t))'$.

For simplicity, we assume that the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \xi$ and σ are known and focus on the estimation of unknown change point τ. The error term ε_t's are same across different combinations of parameters and models. The coefficients $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are taken to be (0,2), (1,3) and $((0,1)', (2,3)')$ for the corresponding Models (I), (II) and (III), respectively, and $\nu = \pi/6$.

The MLE and BE with uniform prior of τ are given by

$$\hat{k} = \inf\{k : \max_{1 \le i < n} \{L_n(i/n)\} = L_n(k/n)\}, \qquad \hat{\tau}_{ML} = \hat{k}/n$$

and

$$\tilde{\tau}_B = \frac{\sum_{i=1}^{n-1} \tau_i L_n(\tau_i)}{\sum_{i=1}^{n-1} L_n(\tau_i)}, \qquad \tau_i = i/n, \qquad i = 1, \dots, n-1$$

respectively. Then we compute the mean and the square root of the mean square error (RMSE) for $\hat{\tau}_{ML}$ and $\tilde{\tau}_B$ based on 100 replications.

Table 4.1 summarizes the simulation results for $\xi = 0.7, 0.9$ and $n = 100, 300$. The change point τ is fixed to be 0.5. A closer inspection of Table 1 reveals some interesting characteristics. First, we notice that, in each case, the RMSE of BE is smaller than that of MLE, however mean estimates are almost same for all cases. A change in a cosine trend function seems to increase the bias of a change point estimators, while for $n = 300$, the mean estimates lie in the vicinity of 0.5. The effect of large value of ξ (near unit root) for MLE is particularly significant for Model (I) in view of RMSE.

The histogram of these results are plotted in Figures 4.1 and 4.2 for $\xi = 0.7$ and $\xi = 0.9$, respectively, when $n = 100$. A study of these figures facilitates understanding of the simulation results in Table 4.1. It is obvious that the shape of distributions for MLE and BE is different when $\xi = 0.9$. The former has a fatter tail in general, while the latter has high frequencies around 0.5. For Models (II) and (III), the distributions of MLE and BE are skewed to the right, which causes an increase in bias of an estimator. These facts are verified by comparing the sample coefficient of skewness and the sample kurtosis which are listed in Figures 4.2 and 4.3 together.

It is questioned how large the RMES becomes for different values of ξ and the cases when the change point locates the edge of samples. A perspective view of the result given in Table 1 for the RMSE of Model (I) is shown in Figures 4.3 over a grid of points $\tau = 0.1, \dots, 0.9$ and $\xi = 0.1, \dots, 0.9$. According to this figure, as it is expected, we observe that the RMSE increases as ξ increases. However it seems that the RMSE's are stable and unaffected by τ even though τ is close to 0.1 and 0.9 when ξ is from 0.1 to 0.7. The discrepancy of RMSE between MLE and BE is significantly large for $\xi = 0.9$ and $\tau = 0.5$. As it can be seen from this figure that the BE works better than MLE in terms of RMSE in all cases.

Next, we investigate the effect of the selection of frequency ν in Model (II). The autoregressive parameter ξ is fixed at 0.7. Table 4.2 presents the results . We observe that the precision of the change point estimates deteriorates when ν is close to 0 when $n = 100$. While the consistency is convincing for large n, the RSEM of MLE and BE becomes large as the frequency ν tends to 0.

We summarize the simulation results as follows. First, the performance of BE is better than MLE in terms of RMSE, which is consistent with the theoretical result given in the previous section. Even though we assumed that the parameters except for change point are known, it is expected that similar characteristics will be observed for the cases of unknown parameters. To see these, we will report some real data analysis in next section.

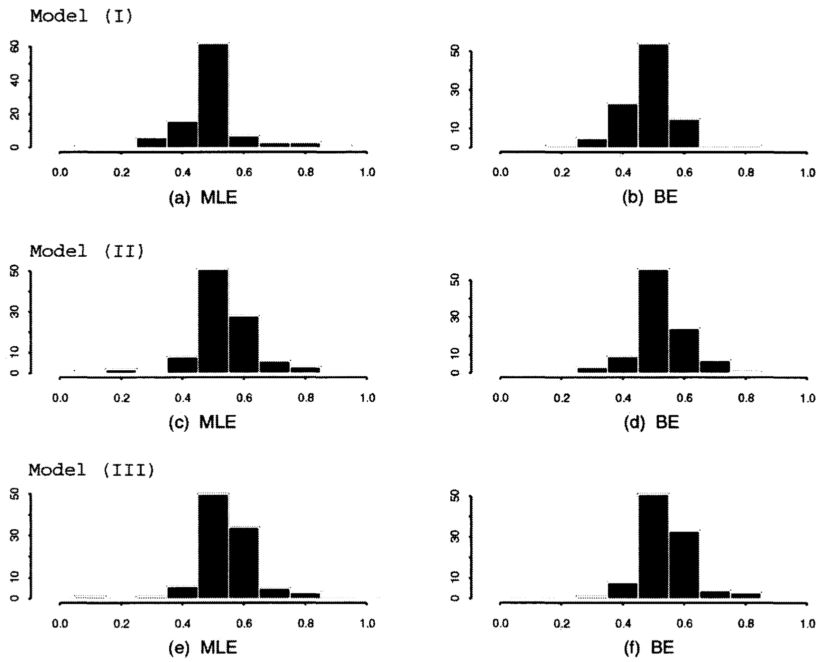

Figure 4.1. Histograms for the results of Table 1 for $\xi = 0.7$ and $n = 100$. The sample coefficient of skewness $\hat{\mu}_1$ and the sample kurtosis $\hat{\mu}_2$ are: (a) $\hat{\mu}_1 = 0.70, \hat{\mu}_2 = 7.12$; (b) $\hat{\mu}_1 = -0.01, \hat{\mu}_2 = 4.68$; (c) $\hat{\mu}_1 = 0.12, \hat{\mu}_2 = 4.74$; (d) $\hat{\mu}_1 = 0.42, \hat{\mu}_2 = 3.56$; (e) $\hat{\mu}_1 = 0.18, \hat{\mu}_2 = 5.55$; (f) $\hat{\mu}_1 = 0.77, \hat{\mu}_2 = 5.10$.

Table 4.1

Average estimates and RMSE of τ when $\tau = 0.5$

	Mean				RMSE			
	$n = 100$		$n = 300$		$n = 100$		$n = 300$	
	MLE	BE	MLE	BE	MLE	BE	MLE	BE
Model (I)								
$\xi = 0.7$	0.4955	0.4893	0.5032	0.4983	0.1121	0.0858	0.0497	0.0422
$\xi = 0.9$	0.4726	0.4924	0.4998	0.5144	0.1981	0.1121	0.1840	0.1220
Model (II)								
$\xi = 0.7$	0.5197	0.5207	0.5000	0.4978	0.1187	0.0854	0.0394	0.0336
$\xi = 0.9$	0.5081	0.5091	0.4984	0.4975	0.1348	0.1058	0.0425	0.0350
Model (III)								
$\xi = 0.7$	0.5311	0.5313	0.4932	0.4940	0.1100	0.0916	0.0337	0.0282
$\xi = 0.9$	0.5314	0.5361	0.4900	0.4885	0.1597	0.1315	0.0538	0.0438

5 Real data applications

This section is devoted to the application of change point estimation to three data sets (Nile data, U. S. quarterly unemployment rate and international airline ticket sales data) where a visible change point can be observed. Based on these data, we fit (4.1). The estimation procedure is as follows. First, we estimate the unknown parameters by a maximum likelihood method. For fixed $k, q \leq k \leq n-q$, the MLE of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\alpha}}_k = \left(\sum_{t=q}^{k} \boldsymbol{z}_t' \boldsymbol{z}_t\right)^{-1} \sum_{t=q}^{k} \boldsymbol{z}_t' y_t \quad \text{and} \quad \hat{\boldsymbol{\beta}}_k = \left(\sum_{t=k+1}^{n-q} \boldsymbol{z}_t' \boldsymbol{z}_t\right)^{-1} \sum_{t=k+1}^{n-q} \boldsymbol{z}_t' y_t.$$

Then we can estimate the spectral density of the residual process $\{\hat{u}_t = y_t - \{\hat{\boldsymbol{\alpha}}_k' \chi(t \leq k) + \hat{\boldsymbol{\beta}}_k' \chi(t > k)\} \boldsymbol{z}_t\}$ using the following nonparametric estimator

$$\hat{f}_{k,n}(\lambda) = \sum_{l=-M}^{M} w\left(\frac{l}{M}\right) \hat{\Gamma}_{k,n}(l) e^{-il\lambda},$$

where $M = n^{2/5}$, $w(\cdot)$ is a weight function and $\hat{\Gamma}_{k,n}(l) = n^{-1} \sum_{t=1}^{n-l} \hat{u}_t \hat{u}_{t+l}$. Hence the likelihood function is calculated using this spectral estimates. The MLE's of unknown parameters are

$$\hat{k} = \inf\left\{k : \max_{q \leq i \leq n-q} L(\hat{\boldsymbol{\alpha}}_i, \hat{\boldsymbol{\beta}}_i, i/n) = L(\hat{\boldsymbol{\alpha}}_k, \hat{\boldsymbol{\beta}}_k, k/n)\right\}$$

$$\hat{\tau}_{ML} = \hat{k}/n, \quad \hat{\boldsymbol{\alpha}}_{ML} = \hat{\boldsymbol{\alpha}}_{\hat{k}} \quad \text{and} \quad \hat{\boldsymbol{\beta}}_{ML} = \hat{\boldsymbol{\beta}}_{\hat{k}}. \tag{5.1}$$

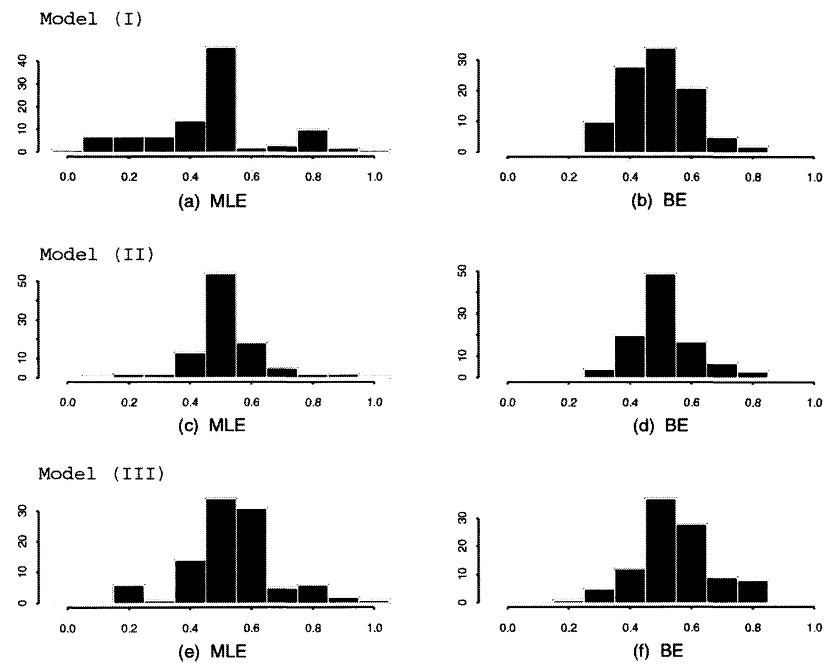

Figure 4.2. Histograms for the results of Table 1 for $\xi = 0.9$ and $n = 100$. The sample coefficient of skewness $\hat{\mu}_1$ and the sample kurtosis $\hat{\mu}_2$ are: (a) $\hat{\mu}_1 = 0.11, \hat{\mu}_2 = 3.25$; (b) $\hat{\mu}_1 = 0.34, \hat{\mu}_2 = 2.71$; (c) $\hat{\mu}_1 = 0.93, \hat{\mu}_2 = 5.50$; (d) $\hat{\mu}_1 = 0.63, \hat{\mu}_2 = 4.00$; (e) $\hat{\mu}_1 = 0.54, \hat{\mu}_2 = 3.51$; (f) $\hat{\mu}_1 = 0.34, \hat{\mu}_2 = 2.95$.

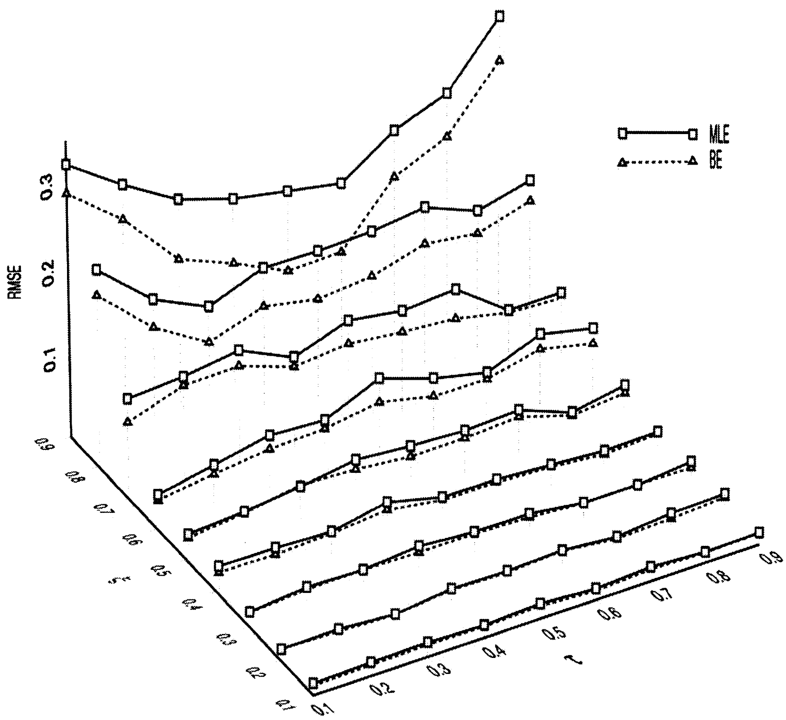

Figure 4.3. RMSE of Model (I) when $n = 100$.

Table 4.2.

Average and RMSE of MLE and BE for τ when $\tau = 0.5$ for Model (II).

	Mean				RMSE			
	$n = 100$		$n = 300$		$n = 100$		$n = 300$	
ν	MLE	BE	MLE	BE	MLE	BE	MLE	BE
$\pi/2$	0.5028	0.5017	0.5005	0.5004	0.0250	0.0201	0.0074	0.0065
$\pi/4$	0.4848	0.4849	0.4944	0.4947	0.0584	0.0496	0.0266	0.0211
$\pi/8$	0.4840	0.4969	0.4857	0.4895	0.1361	0.1217	0.0551	0.0418
$\pi/16$	0.5847	0.5710	0.5183	0.5161	0.2283	0.1629	0.0833	0.0697
$\pi/32$	0.5434	0.5381	0.4613	0.4675	0.2141	0.1715	0.1285	0.1021

Next we compute the Bayes estimator. For simplicity of calculation, we postulate the result that the asymptotic distribution of $\hat{\boldsymbol{\alpha}}_{ML}$ and $\hat{\boldsymbol{\beta}}_B$ are same as $\tilde{\boldsymbol{\alpha}}_B$ and $\tilde{\boldsymbol{\beta}}_B$ (c.f. Kutoyants (1994)). Therefore the Bayes change point estimator $\tilde{\tau}_B$ becomes

$$\tilde{\tau}_B = \frac{\sum_{i=q}^{n-q} \tau_i L_n(\hat{\boldsymbol{\alpha}}_{ML}, \hat{\boldsymbol{\beta}}_{ML}, \tau_i)}{\sum_{i=q}^{n-q} L_n(\hat{\boldsymbol{\alpha}}_{ML}, \hat{\boldsymbol{\beta}}_{ML}, \tau_i)}, \qquad \tau_i = i/n, \qquad i = q, \dots, n-q.$$

Nile data

These data have been investigated by an i.i.d. framework, for details see e.g., Cobb (1978) and Hinkley and Schechtmann (1987). The data consist of readings of the annual flows of the Nile River at Aswan from 1871 to 1970. There was a shift in the flow levels in 1899, which was attributed partly to the weather changes and partly to the start of construction work for a new dam at Aswan. We apply a mean shift model for this data with $z_t = 1$. The MLE gives $\hat{\alpha}_{ML} = 1097.75, \hat{\beta}_{ML} = 849.97$ and $\hat{\tau} = 0.28$ ($\hat{k} = 28$). On the other hand, the BE is $\tilde{\tau}_B = 0.2790(\tilde{k} = [\tilde{\tau}_B n] = 27)$. The original series together with ML trend estimator are plotted in Figure 5.1. Figure 5.2 shows the posterior distribution of τ, which shows strong evidence that the shift occurred in 1898. These results agree with those of the other authors.

U. S. quarterly unemployment rates

This data set, ($n = 184$), is analyzed in Tsay (2002) by use of threshold AR model for first differenced series. Here we explain a seasonal trend by employing regression models with trigonometric functions and change point. The regression function is chosen to be $\boldsymbol{z}_t = (1, \cos(\nu t))'$. A Fisher's test for added deterministic periodic component rejects the Gaussian white noise at level .01. We have taken $\nu = 4\pi/184$ which gives the peak in the periodogram. The MLE detected the possible change point $\hat{\tau}_{ML} = 0.49(\hat{k} = 90)$ and corresponding regression coefficients $\hat{\boldsymbol{\alpha}}_{ML} = (4.65, -0.85)'$ and $\hat{\boldsymbol{\beta}}_{ML} = (6.81, -0.94)'$. The BE is $\tilde{\tau}_B = 0.49$ which corresponds to $\tilde{k} = [\tilde{\tau}_B n] = 90$. The estimated trend

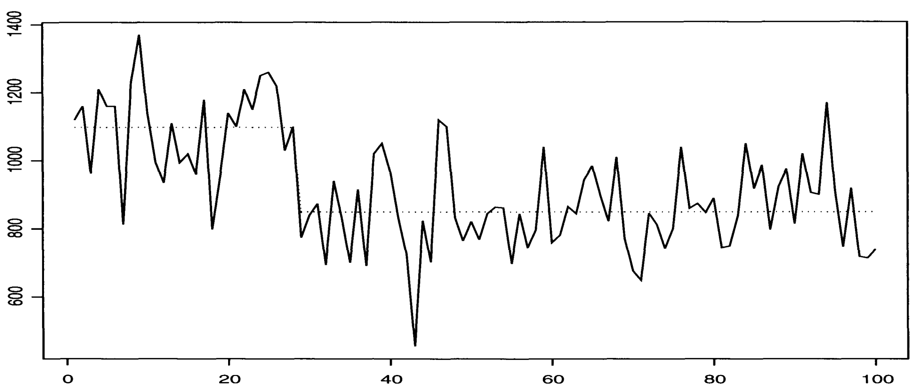

Figure 5.1. Nile data with estimated mean and change point $\hat{k} = 28$ (MLE).

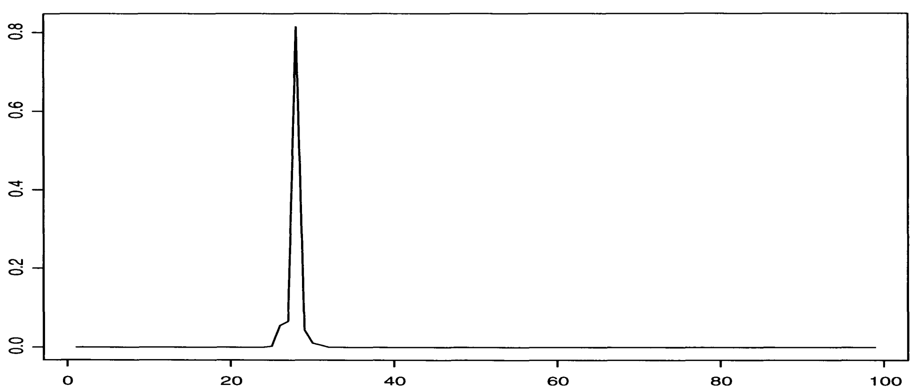

Figure 5.2. Posterior distribution of τ.

function together with original data is shown in Figure 5.3. The posterior distribution for τ is plotted in Figure 5.4. This analysis reveals that the mean level of an unemployment rate increased to about 2% in 3rd quarter of 1970, while the amplitude of long term cyclical trend stayed the same level throughout the period.

International airline ticket sales data

This data have been investigated by fitting a seasonal ARIMA model (Box *et. al.* (1994)). An alternative modeling is deterministic cyclical trend function modeling with a change point for once-differentiated data. The regression function given by $\boldsymbol{z}_t' = (\cos(\nu_1 t), \cos(\nu_2 t), \cos(\nu_3 t))$ is selected by examining the periodogram. There are three frequencies which have comparably large spectrum, namely $\nu_1 = 26\pi/143, \nu_2 = 50\pi/143$ and $\nu_3 = 74\pi/143$. The ML estimators give the $\hat{\alpha}_{ML} = (-7.54, 14.14, 1.43)', \hat{\boldsymbol{\beta}}_{ML} = (-35.76, 37.01, -19.66)'$ and $\hat{\tau}_{ML} = 0.6319(\hat{k} = 91)$. While Bayes estimator is $\tilde{\tau}_B = 0.6216(\tilde{k} = 89)$. As shown in the posterior probability of τ, the change might have occurred from $t = 80$ to 100, which implies the possibility of multiple changes.

6 Proofs

Proof of Theorem 1. From (2.7), we have

$$\log Z_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) \tag{6.1}$$
$$= -\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2} \left\{ d_n(\lambda_k) A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)} \right\} - \frac{1}{2n} \sum_{k=1}^{n} |A(\lambda_k)|^2$$

First we evaluate the first term in (6.2). From (2.7) we have

$$-\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2} \left\{ d_n(\lambda_k) A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)} \right\}$$
$$= -\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2}$$
$$\times \left\{ d_n(\lambda_k) A_1 + d_n(\lambda_k) A_2 + d_n(\lambda_k) A_3 + \overline{d_n(\lambda_k) A_1} + \overline{d_n(\lambda_k) A_2} + \overline{d_n(\lambda_k) A_3} \right\}$$
$$= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 \qquad \text{(say)}.$$

Write the spectral density $f(\lambda)$ in the form

$$f(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} R_f(\lambda) e^{-ij\lambda}$$

where $R_f(j)$'s satisfy $\sum_{j=-\infty}^{\infty} |j|^m |R_f(j)| < \infty$ for any given $m \in \mathbb{N}$. Then, from Theorem 3.8.3 of Brillinger (1975) we may write

$$f(\lambda)^{-1} = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) e^{-ij\lambda}$$

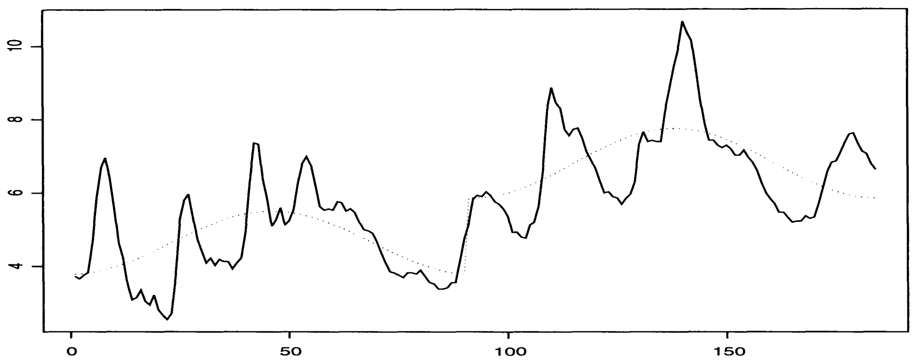

Figure 5.3. U. S. quarterly unemployment rates (1948-1993) with estimated trend and change point $\hat{k} = 90$(MLE).

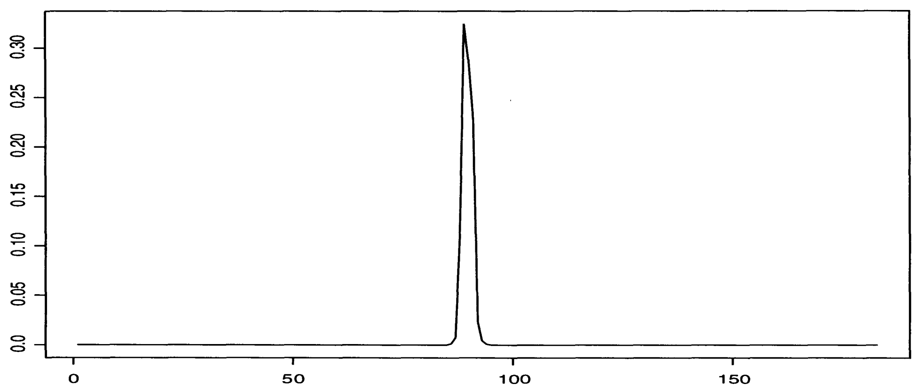

Figure 5.4. Posterior distribution of τ.

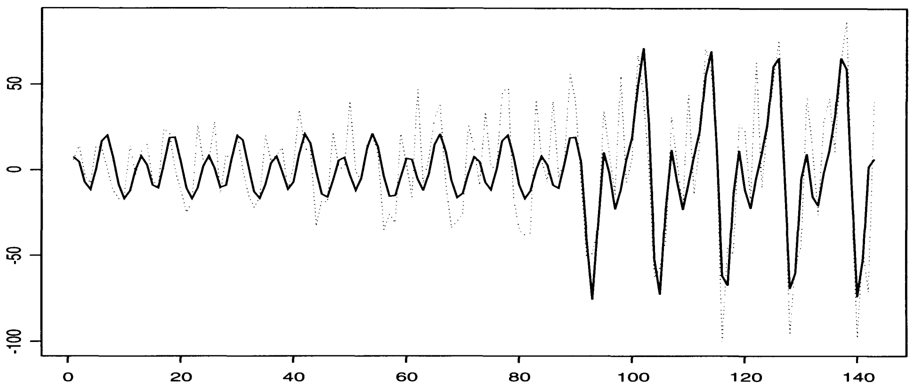

Figure 5.5. The international airline ticket sales, once -differentiated data (dotted line) with estimated trend and change point $\hat{k} = 91$(black line).

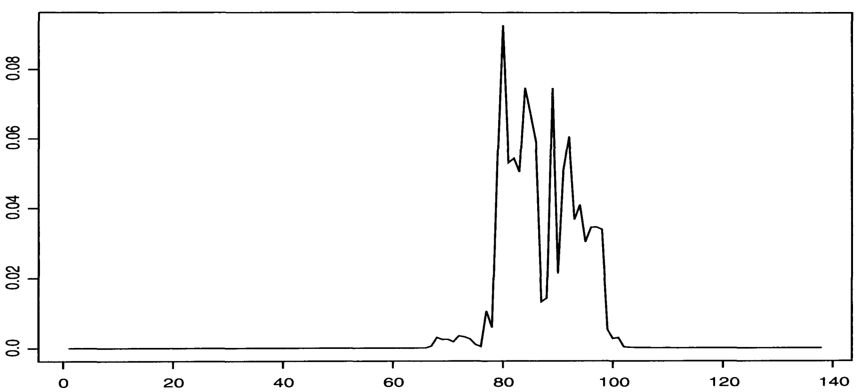

Figure 5.6. Posterior probabilities of τ.

where $\Gamma(j)$'s satisfy for any given $m \in \mathbb{N}$

$$\sum_{j=-\infty}^{\infty} |j|^m |\Gamma(j)| < \infty.$$

Then E_1 can be written as

$$\begin{aligned}
E_1 &= -\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2} d_n(\lambda_k) A_1 \\
&= -\frac{1}{4n\pi} \sum_{k=1}^{n} f(\lambda_k)^{-1} \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s)\lambda_k} \\
&= -\frac{1}{4n\pi} \sum_{k=1}^{n} \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) e^{-ij\lambda_k} \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s)\lambda_k} \\
&= -\frac{1}{4n\pi} \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s-j)\lambda_k}
\end{aligned}$$

It is well known that

$$\sum_{k=1}^{n} e^{i(t-s-j)\lambda_k} = \begin{cases} n \text{ if } \ t-s-j = 0 \pmod{n} \\ 0 \text{ otherwise.} \end{cases} \tag{6.2}$$

Since $-[\tau n + \rho] \leq t - s \leq [(1-\tau)n]$ and $\Gamma(j)$ satisfies $\sum_j |j|^m |\Gamma(j)| < \infty$ for any given m, we have

$$\sum_{|j| \geq n} |\Gamma(j)| \leq \frac{1}{n^m} \sum_{|j| \geq n} (j)^m |\Gamma(j)| = o(n^{-m}).$$

Hence we have only to evaluate E_1 for $l = 0$ of $t - s - j = ln$. Thus E_1 is

$$\begin{aligned}
E_1 &= -\frac{1}{4\pi} \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \boldsymbol{z}_s u_t \frac{1}{n} \sum_{k=1}^{n} e^{i(t-s-j)\lambda_k} \\
&\simeq -\frac{1}{8\pi^2} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \boldsymbol{z}_s \{u_{s+j}\} \equiv \tilde{E}_1 \quad \text{(say)}.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{E}_1 &= -\frac{1}{8\pi^2} \sum_{j=-\infty}^{\infty} \Gamma(j) (\boldsymbol{\beta}-\boldsymbol{\alpha})' \sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s \int_{-\pi}^{\pi} e^{ij\lambda} e^{is\lambda} dZ_u(\lambda) \qquad (6.3) \\
&= -\frac{1}{4\pi} (\boldsymbol{\beta}-\boldsymbol{\alpha})' \int_{-\pi}^{\pi} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} f(\lambda)^{-1} dZ_u(\lambda) \\
&= \frac{1}{2} (\boldsymbol{\beta}-\boldsymbol{\alpha})' W_1 \quad \text{(say)},
\end{aligned}$$

where $Z_u(\lambda)$ is the spectral measure of u_t defined by (2.7). Let $\sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} = \boldsymbol{A}(\lambda; h, \rho)$. we observe

$$E(W_1 W_1^*) \longrightarrow \frac{1}{4\pi^2}\int_{-\pi}^{\pi} \boldsymbol{A}(\lambda; h, \rho)\boldsymbol{A}^*(\lambda; h, \rho) f(\lambda)^{-1} d\lambda \quad \text{as} \quad n \to \infty.$$

Recalling that $\{u_t\}$ is Gaussian, we have

$$W_1 \xrightarrow[D]{} N\left(\mathbf{0}, \frac{1}{4\pi^2}\int_{-\pi}^{\pi} \boldsymbol{A}(\lambda; h, \rho)\boldsymbol{A}^*(\lambda; h, \rho) f(\lambda)^{-1} d\lambda\right) \tag{6.4}$$

Similarly we obtain

$$E_4 \sim \frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{\alpha})' W_1. \tag{6.5}$$

Next we calculate the second term E_2 that is

$$\begin{aligned}
E_2 &= -\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2} d_n(\lambda_k) A_2 \\
&= \frac{1}{4n\pi} \sum_{k=1}^{n} f(\lambda_k)^{-1} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{s=1}^{[\tau n+\rho]} u_t \boldsymbol{a}' \boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4n\pi}\frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \Gamma(j) e^{-ij\lambda_k} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{s=1}^{[\tau n+\rho]} \boldsymbol{a}' u_t \boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4\pi}\frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{s=1}^{[\tau n+\rho]} \boldsymbol{a}' u_t \boldsymbol{z}_s \frac{1}{n} \sum_{k=1}^{n} e^{i\lambda_k(t-s-j)}.
\end{aligned}$$

Here note that $n-1 \geq t-s \geq -[\tau n]$. Because of (6.2) we have only to evaluate E_2 for $l = 0, 1$ of $t - s - j = ln$. Then

$$E_2 \simeq \frac{1}{4\pi}\frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) \frac{\boldsymbol{a}'}{\sqrt{n}} \sum_{s=1}^{[\tau n+\rho]} (u_{s+j} + u_{s+j+n}) \boldsymbol{z}_s = \tilde{E}_2 \qquad \text{(say)}.$$

Similarly as in $\tilde{E}_1$,

$$\tilde{E}_2 = \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{a}'}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\int_{-\pi}^{\pi}e^{is\lambda}e^{ij\lambda}(1+e^{in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \tag{6.6}$$

$$= \frac{1}{4\pi}\frac{\boldsymbol{a}'}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\int_{-\pi}^{\pi}e^{is\lambda}\left(\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{is\lambda}\right)(1+e^{in\lambda})dZ_u(\lambda)\boldsymbol{z}_s$$

$$= \frac{\boldsymbol{a}'}{4\pi\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s\int_{-\pi}^{\pi}e^{is\lambda}(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda)$$

$$= \frac{\boldsymbol{a}'}{4\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda)$$

$$= \frac{\sqrt{\tau}\boldsymbol{a}'}{2}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi\sqrt{n\tau}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda)$$

$$= \frac{\sqrt{\tau}\boldsymbol{a}'}{2}W_2 \qquad \text{(say)},$$

where

$$W_2 \xrightarrow[Dt]{} N\left(\boldsymbol{0}, \frac{1}{2\pi}\int_{-\pi}^{\pi}2f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\right), \tag{6.7}$$

which follows from the Riemann-Lebesgue theorem and Grenander's conditions (G.1) - (G.4). Similarly we obtain.

$$E_5 \simeq \frac{\sqrt{\tau}\boldsymbol{a}'}{2}W_2. \tag{6.8}$$

Next

$$E_3 = -\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}f(\lambda_k)^{-1/2}d_n(\lambda_k)A_3$$

$$= \frac{1}{4n\pi}\sum_{k=1}^{n}f(\lambda_k)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}u_t\boldsymbol{b}'\boldsymbol{z}_s e^{i(t-s)\lambda_k}$$

$$= \frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'u_t\boldsymbol{z}_s e^{i(t-s)\lambda_k}$$

$$= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'u_t\boldsymbol{z}_s\frac{1}{n}\sum_{k=1}^{n}e^{i\lambda_k(t-s-j)}.$$

Since $[(1-\tau)n] \geq t-s \geq 1-n$, we have only to evaluate E_3 for $l = 0, -1$ of $t-s-j = ln$. Hence

$$E_3 \simeq \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}(u_{s+j}+u_{s+j-n})\boldsymbol{z}_s = \tilde{E}_3$$

Similarly as in $\tilde{E}_2$ we have

$$
\begin{aligned}
\tilde{E}_3 &= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\int_{-\pi}^{\pi} e^{is\lambda}e^{ij\lambda}(1+e^{-in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{1}{4\pi}\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\int_{-\pi}^{\pi} e^{is\lambda}\left(\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{is\lambda}\right)(1+e^{-in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{\boldsymbol{b}'}{4\pi\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s\int_{-\pi}^{\pi} e^{is\lambda}(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\boldsymbol{b}'}{4\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi\sqrt{n(1-\tau)}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}W_3,
\end{aligned}
\tag{6.9}
$$

where

$$
W_3 \xrightarrow[D]{} N\left(\boldsymbol{0}, \frac{1}{2\pi}\int_{-\pi}^{\pi} 2f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\right) \tag{6.10}
$$

Similarly we obtain.

$$
E_6 = \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}W_3 \tag{6.11}
$$

Hence from (6.4), (6.5), (6.7), (6.8), (6.10) and (6.11), we have

$$
\begin{aligned}
&-\frac{1}{2\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\} \\
&\simeq (\boldsymbol{\beta}-\boldsymbol{\alpha})'W_1 + \sqrt{\tau}\boldsymbol{a}'W_2 + \sqrt{1-\tau}\boldsymbol{b}'W_3.
\end{aligned}
\tag{6.12}
$$

Next we evaluate the second term in (6.2), which is

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}|A(\lambda_k)|^2 \\
&= -\frac{1}{2n}\sum_{k=1}^{n}(A_1+A_2+A_3)\overline{(A_1+A_2+A_3)} \\
&= -\frac{1}{2n}\sum_{k=1}^{n}(|A_1|^2+|A_2|^2+|A_3|^2+A_1\overline{A_2}+A_1\overline{A_3}+A_2\overline{A_3}++A_2\overline{A_1}+A_3\overline{A_1}+A_3\overline{A_2}).
\end{aligned}
$$

We have

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}|A_1|^2 \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{f(\lambda_k)}\left(\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha})e^{i(t-s)\lambda_k} \\
&= -\frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha})\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k} \\
&= -\frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_{s+j}\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha}).
\end{aligned}
\tag{6.13}
$$

Next we have

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}|A_2|^2 \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda}\left(-\frac{1}{\sqrt{n}}\sum_{t=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
&= -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{t=1}^{[\tau n+\rho]}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\left\{\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}\right\}.
\end{aligned}
$$

Note that $[\tau n] \geq t-s \geq -[\tau n]$. Similarly we have

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}|A_2|^2 \\
&\simeq -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{a} = -\frac{\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\frac{1}{n\tau}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{a} \\
&= -\frac{\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{a} = -\frac{\tau}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{a} \\
&= -\frac{\tau}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{a}
\end{aligned}
\tag{6.14}
$$

Also we obtain

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}|A_3|^2 \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{f(\lambda_k)}\left(-\frac{1}{\sqrt{n}}\sum_{t=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
&= -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{t=[\tau n+\rho]+1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\left\{\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}\right\} \\
&\simeq -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{b} \\
&= -\frac{1-\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\frac{1}{n(1-\tau)}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{b} \\
&= -\frac{1-\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
&= -\frac{1-\tau}{4\pi}\boldsymbol{b}'\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
&= -\frac{1-\tau}{4\pi}\boldsymbol{b}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}.
\end{aligned}
\tag{6.15}
$$

The fourth term becomes

$$
\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n}A_1\overline{A_2} \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\left(\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t e^{it\lambda}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
&= \frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{\sqrt{n}}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=h+1}^{[\tau n+\rho]}\sum_{s=1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}
\end{aligned}
$$

From $1-\rho \le t-s \le [\tau n]+\rho-1$, $t-s-j=0$, it is seen that

$$
\begin{aligned}
-\frac{1}{2n}\sum_{k=1}^{n}A_1\overline{A_2} &\simeq \frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{\sqrt{n}}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_{t-j}'\boldsymbol{a} \\
&= O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
\tag{6.16}
$$

Similarly we observe

$$\frac{1}{2n}\sum_{k=1}^{n} A_1\overline{A_3} \simeq O(n^{-1/2}), \qquad \frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_1} \simeq O(n^{-1/2}) \tag{6.17}$$
$$\text{and} \qquad \frac{1}{2n}\sum_{k=1}^{n} A_3\overline{A_1} \simeq O(n^{-1/2}).$$

Now we evaluate

$$\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_3} \\
&= -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\left(-\frac{1}{\sqrt{n}}\sum_{t=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_t e^{it\lambda}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
&= -\frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{n}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=1}^{[\tau n+\rho]}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{a}'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{b}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}.
\end{aligned}$$

Since $-n+1 \le t-s \le -1$, we have only to evaluate for $t-s-j = 0, -n$.

$$\begin{aligned}
&-\frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_3} \\
&\simeq -\frac{1}{4\pi}\frac{1}{2\pi}\sqrt{\tau(1-\tau)}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{1}{\sqrt{\tau n}}\sum_{t=1}^{[\tau n+\rho]}\frac{1}{\sqrt{(1-\tau)n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{a}'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{b}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k} \\
&\simeq -\frac{\sqrt{\tau(1-\tau)}}{4\pi\cdot 2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
&= -\frac{\sqrt{\tau(1-\tau)}}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}.
\end{aligned} \tag{6.18}$$

Similarly we have

$$-\frac{1}{2n}\sum_{k=1}^{n} A_3\overline{A_2} \simeq -\frac{\sqrt{\tau(1-\tau)}}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}. \tag{6.19}$$

From the equations from (6.14) to (6.19) together with (6.4), (6.7), (6.10) and (6.13) complete the proof of Theorem 1.

Proof of Lemma 1. From Hannan (1970) and Anderson (1977) the joint density of $d_n(\lambda_1), \cdots, d_n(\lambda_n)$ is given by

$$p(d_n(\lambda_1), \cdots, d_n(\lambda_n)) = C_n\prod_{k=1}^{k}\exp(-\overline{d_n(\lambda_k)}f(\lambda_k)^{-1}d_n(\lambda_k)) \tag{6.20}$$

where $C_n = \pi^{-n} \prod_{k=1}^n f(\lambda_k)^{-1}$. Using this,

$$
\begin{aligned}
&EZ_n^{1/2}(\boldsymbol{a}, \boldsymbol{b}, \rho) \\
&= E \exp\left[-\frac{1}{4\sqrt{n}} \sum_{k=1}^n f(\lambda_k)^{-1/2} \left\{ d_n(\lambda_k) A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)} \right\} \right] \exp\left[-\frac{1}{4n} \sum_{k=1}^n |A(\lambda_k)|^2 \right] \\
&= \int \cdots \int C_n \exp\left(-\sum_{k=1}^n \overline{d_n(\lambda_k)} f(\lambda_k)^{-1} d_n(\lambda_k) \right) \\
&\quad \times \exp\left(-\frac{1}{4\sqrt{n}} \sum_{k=1}^n f(\lambda_k)^{-1/2} \left\{ d_n(\lambda_k) A(\lambda_k) + \overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)} \right\} \right) \\
&\quad \times \exp\left(-\frac{1}{4n} \sum_{k=1}^n |A(\lambda_k)|^2 \right) \boldsymbol{d}(d_n(\lambda_1) \cdots d_n(\lambda_n)) \\
&= \int \cdots \int C_n \exp\left[-\sum_{k=1}^n \left(f(\lambda_k)^{-1/2} d_n(\lambda_k) + \frac{\overline{A(\lambda_k)}}{4\sqrt{n}} \right) \left(\overline{f(\lambda_k)^{-1/2} d_n(\lambda_k)} + \frac{A(\lambda_k)}{4\sqrt{n}} \right) \right] \\
&\quad \times \exp\left[\frac{1}{16n} \sum_{k=1}^n |A(\lambda_k)|^2 - \frac{1}{4n} \sum_{k=1}^n |A(\lambda_k)|^2 \right] \boldsymbol{d}(d_n(\lambda_1) \cdots d_n(\lambda_n) \\
&= \exp\left(-\frac{3}{16n} \sum_{k=1}^n |A(\lambda_k)|^2 \right).
\end{aligned}
$$

Recalling the definition of likelihood process in (2.7), we have

$$
\exp\left(-\frac{3}{16n} \sum_{k=1}^n |A(\lambda_k)|^2 \right) = \exp\left(-\frac{3}{16n} \sum_{k=1}^n |A_1 + A_2 + A_3|^2 \right) \tag{6.21}
$$

From the proof of Theorem 1 and Assumption (G.1), the first term in (6.21) is bounded by

$$
\begin{aligned}
&-\frac{1}{16n} \sum_{k=1}^n (A_1 \overline{A_1}) \qquad (6.22) \\
&\simeq -\frac{3}{16} \frac{1}{8\pi^2} \sum_{t=[\tau n]+1}^{[\tau n+\rho]} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_t \Gamma(t-s) \boldsymbol{z}_s (\boldsymbol{\beta} - \boldsymbol{\alpha}) \\
&\le -\frac{3}{16} \frac{1}{8\pi^2} \sum_{t=[\tau n]+1}^{[\tau n+\rho]} \{(\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_t\}^2 \times \min_{\lambda} f(\lambda)^{-1} \\
&= -[O(\rho)]
\end{aligned}
$$

for $\rho > 0$. We have already shown in (6.17) and (6.18) that

$$
\frac{1}{16n} \sum_{k=1}^n \left\{ A_1 \overline{(A_2 + A_3)} \right\} = O(n^{-1/2}) \tag{6.23}
$$

$$
\text{and} \quad \frac{1}{16n} \sum_{k=1}^n \{ \overline{A_1} (A_2 + A_3) \} = O(n^{-1/2}).
$$

Furthermore, from the proof of Theorem 1 we can find a positive definite matrix $\boldsymbol{K}$ so that

$$\frac{3}{16n}\sum_{k=1}^{n}(A_2+A_3)\overline{(A_2+A_3)} \simeq (\boldsymbol{a}', \boldsymbol{b}')\boldsymbol{K}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \tag{6.24}$$

Hence (6.23)-(6.24) implies the required result.

Proof of Lemma 2. Let $\boldsymbol{\theta}_1' = (\boldsymbol{\alpha}_1', \boldsymbol{\beta}_1', \tau_1)'$ and $\boldsymbol{\theta}_2' = (\boldsymbol{\alpha}_2', \boldsymbol{\beta}_2'\tau_2)'$ are some given values in $\boldsymbol{\Theta}$, and are the forms of $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}_1, \boldsymbol{\beta}_1 = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}_2, \tau_1 = \tau + n^{-1}\rho_1, \boldsymbol{\alpha}_2 = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}_2, \boldsymbol{\beta}_2 = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}_1$ and $\tau_2 = \tau + n^{-1}\rho_2$. Denoting $A(\lambda_k)$ under $\boldsymbol{\theta}_i$ as $A(\boldsymbol{a}_i, \boldsymbol{b}_i, \rho_i; \lambda_k)$ we set

$$\begin{aligned}
\Delta_{1n} &= A(\boldsymbol{a}_1, \boldsymbol{b}_1, \rho_1; \lambda_k) - A(\boldsymbol{a}_2, \boldsymbol{b}_2, \rho_2; \lambda_k)\\
\Delta_{2n} &= |A(\boldsymbol{a}_1, \boldsymbol{b}_1, \rho_1; \lambda_k)|^2 - |A(\boldsymbol{a}_2, \boldsymbol{b}_2, \rho_2; \lambda_k)|^2
\end{aligned}$$

and

$$Y_n = \exp\left[-\frac{1}{8\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\{d_n(\lambda_k)\Delta_{1n} + \overline{d_n(\lambda_k)}\ \overline{\Delta_{1n}}\} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right].$$

The process Y_n is written as

$$Y_n = \left(\frac{L_n(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \tau_2)}{L_n(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \tau_1)}\right)^{1/4}. \tag{6.25}$$

Then we observe

$$\begin{aligned}
&E_{\alpha,\beta,\tau}\left|Z_n^{1/4}(\boldsymbol{a}_1, \boldsymbol{b}_1, \rho_1) - Z_n^{1/4}(\boldsymbol{a}_2, \boldsymbol{b}_2, \rho_2)\right|^{1/4}\\
&= E_{\alpha_1,\beta_1,\tau_1}(1 - Y_n)^4\\
&= E\left(1 - 4Y_n + 6Y_n^2 - 4Y_n^3 + Y_n^4\right)
\end{aligned}$$

We have

$$\begin{aligned}
-4EY_n &= -4E\exp\left(-\frac{1}{8\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\{d_n(\lambda_k)\Delta_{1n} + \overline{d_n(\lambda_k)}\ \overline{\Delta_{1n}}\} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right)\\
&= -4\int\cdots\int C_1\exp\left[-\sum_{k=1}^{n}\left\{\frac{d_n(\lambda_k)}{f_k^{1/2}} + \frac{\overline{\Delta_{1n}}}{8\sqrt{n}}\right\}\left\{\frac{\overline{d_n(\lambda_k)}}{f_k^{1/2}} + \frac{\Delta_{1n}}{8\sqrt{n}}\right\}\right]\\
&\quad\times\exp\left[\frac{1}{64n}\sum_{k=1}^{n}\Delta_{1n}\overline{\Delta_{1n}} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right]\boldsymbol{d}(d_n(\lambda_1)\cdots d_n(\lambda_n))\\
&= -4\exp\left[\frac{1}{64n}\sum_{k=1}^{n}\Delta_{1n}\overline{\Delta_{1n}} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right] = -4\exp(\eta + \gamma) \quad \text{say}
\end{aligned}$$

Similarly, we obtain

$$6EY_n^2 = 6\exp(4\eta + 2\gamma), \qquad -4EY_n^3 = -4\exp(9\eta + 3\gamma)$$

and

$$EY_n^4 = \exp(16\eta + 4\gamma).$$

Hence

$$E[1-Y_n]^4 = 1 - 4e^{\eta+\gamma} + 6e^{4\eta+2\gamma} - 4e^{9\eta+3\gamma} + e^{16\eta+4\eta}. \tag{6.26}$$

Using the following expansion for small y

$$e^y \simeq 1 + y$$

we have

$$\begin{aligned} E[1-Y_n]^4 &= 1 - 4(1+\eta+\gamma) + 6(1+4\eta+2\gamma) - 4(1+9\eta+3\gamma) + (1+16\eta+4\gamma) \\ &\quad + O(\eta^2) + O(\gamma^2) + O(\eta\gamma) \\ &= 0 + O(\eta^2) + O(\gamma^2) + O(\eta\gamma) \end{aligned}$$

which implies that the Taylor expansion of (6.26) starts with the linear combinations of second order terms of η^2, γ^2 and $\eta\gamma$. Here we need to evaluate the asymptotics of η and γ in (6.26). Assume that without loss of generality $\rho_1 \geq \rho_2$, then

$$\begin{aligned} \Delta_{1n} &= \frac{1}{\sqrt{2\pi}} f(\lambda_k)^{-1/2} \sum_{s=[\tau n+\rho_2]+1}^{[\tau n+\rho_1]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s e^{-is\lambda_k} \\ &\quad - \frac{1}{\sqrt{2\pi n}} f(\lambda_k)^{-1/2} \left(\sum_{s=1}^{[\tau n+\rho_1]} (\boldsymbol{a}_1 - \boldsymbol{a}_2)' \boldsymbol{z}_s e^{-is\lambda_k} + \sum_{s=[\tau n+\rho_1]+1}^{n} (\boldsymbol{b}_1 - \boldsymbol{a}_2)' \boldsymbol{z}_s e^{-is\lambda_k} \right). \end{aligned}$$

Using the similar argument in proof of Lemma 1, we observe

$$\eta = O\left[(\rho_1 - \rho_2)\right] + O\left[((\boldsymbol{a}_1 - \boldsymbol{a}_2)', (\boldsymbol{b}_1 - \boldsymbol{b}_2)')\boldsymbol{K} \begin{pmatrix} \boldsymbol{a}_1 - \boldsymbol{a}_2 \\ \boldsymbol{b}_1 - \boldsymbol{b}_2 \end{pmatrix}\right],$$

which is written as

$$\eta = O[(\rho_2 - \rho_1)] + O(\|\boldsymbol{a}_1 - \boldsymbol{a}_2\|) + O(\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|).$$

Analogously we have

$$\gamma = O[(\rho_2 - \rho_1)] + O(\|\boldsymbol{a}_1 - \boldsymbol{a}_2\|) + O(\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|),$$

which completes the proof.

Proof of Theorem 2. The proof follows from Theorem 1, Lemmas 1 and 2 of this paper and Theorem 1.10.1 of Ibragimov and Has'minski (1981).

Proof of Theorem 3. The properties of the likelihood ratio $Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho)$ established in Theorem 1, Lemmas 1 and 2 allow us to refer to Theorem 1.10.2 of Ibragimov and Has'minski (1981).

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