A Theorem of Large Deviations for the Equilibrium Prices in Random Exchange Economies

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Abstract

We formulate and prove a theorem concerning the large deviations of equilibrium prices in large random exchange economies.

1 Introduction

We consider an *economic system* (shortly, *economy*) $\mathcal{E}$, where certain *commodities* $j = 1, ..., l$ are traded. Let $R^l_+ =_{\text{def}} \{p = (p^1, ..., p^l) \in R^l;\ p^j \geq 0 \text{ for all } j = 1, ..., l\}$. The elements p of R^l_+ are interpreted as *price vectors* (shortly, *prices*). (We will follow a convention, according to which superscripts always refer to the commodities whereas subscripts refer to the economic agents.)

The *total excess demand function* $Z(p) = (Z^1(p), ..., Z^l(p)) \in R^l$ comprises the total excess demands on the l commodities in the economy at the prices $p \in R^l_+$. Its zeros p^* are called the *equilibrium prices*:

$$Z(p^*) = 0.$$

(In fact, according to Walras' law, we may regard money as an $l+1$'st commodity [the numeraire] having price $p^{l+1} = 1$ and total excess demand $Z^{l+1}(p) = -p \cdot Z(p)$.)

In the classical equilibrium theory the economic variables and quantities are supposed to be deterministic, see [2]. It is, however, realistic to allow uncertainty in an economic model.

We assume throughout this paper that the total excess demand $Z(p)$ is a random variable (for each fixed price p). In particular, it then follows that the equilibrium prices p^* form a *random set.*

The seminal works concerning equilibria of random economies are due to Hildenbrand [5], Bhattacharya and Majumdar [1] and Föllmer [4].

The equilibrium prices in large random economic systems obey (under appropriate regularity conditions) classical statistical limit laws.

The *law of large numbers* [1] states that, as the number n of economic agents increases, the random equilibrium prices (r.e.p.'s) p^*_n become asymptotically equal to deterministic "expected" equilibrium prices:

$$\lim_{n\to\infty} p^*_n = p^*_e.$$

(The subscript n refers to the number of economic agents.)

The *central limit theorem (CLT) for the r.e.p.'s* [1] characterizes the "small deviations" of the r.e.p.'s from their expected values as asymptotically normal:

$$n^{\frac{1}{2}}(p_n^* - p_e^*) \to \mathcal{N} \text{ in distribution,}$$

where $\mathcal{N}$ denotes a multinormal random vector having mean zero.

We argue in this article for the relevance of the theory of large deviations to random equilibrium theory.

To this end, suppose that, an *aposteriori observation of the equilibrium price* is made, and let $\hat{p}$ denote the value of this observation.

If the modeler is concerned with the estimation of the apriori probability of an aposteriori observation $\hat{p}$ of the equilibrium price in a large economy, the use of the CLT requires the apriori model to be "good" in the sense that the observation $\hat{p}$ ought to fall within a narrow range (having the asymptotically negligible order $n^{-\frac{1}{2}} = o(n)$) from its expected value p_e^*.

However, due to the fact that economics is concerned with the (economic) behaviour of human beings, any (predictive) economic model is always to some extent defective. It follows, in particular, that in a large economy an observed equilibrium price $\hat{p}$ may well represent a "large deviation" from its apriori predicted value p_e^* (viz. fall outside the region of validity of the CLT).

The main result of this paper is a *theorem of large deviations (LD's) for the random equilibrium prices.* It yields an exponential estimate for the (apriori small) probabilities of observations of r.e.p.'s "far away" from their expected values. Namely, we prove that, under appropriate regularity conditions, for an arbitrary fixed price p, there exists a constant $i(p) \geq 0$ such that

$$P(p_n^* \approx p) \approx e^{-ni(p)}. \tag{1.1}$$

In accordance with standard LD terminology (see [3]), we refer to the price depending constant $i(p)$ as the *entropy*. In what follows we shall formulate and prove (1.1) as an exact mathematical theorem.

LD theorems for random equilibrium prices were earlier presented in [7],[8]. The version here is of "local type" in that we are concerned with probabilities of observations of r.e.p.'s in small neighborhoods of a given fixed price. Because of this it turns out that the hypotheses of [7],[8] can be somewhat relaxed. Also it becomes possible to give a self-contained proof which does not lean on the general abstract LD theory. Therefore the proof ought to be accessible also to a reader who is not an LD specialist. The basic idea in the proof is to use a centering argument of a type which is commonly used in LD theory.

2 Formulation of the LD theorem

We describe now the basic set-up and formulate the large deviation theorem in exact terms.

We will be concerned with a sequence $\mathcal{E}_n$, $n = 1, 2, ...$, of economies. We assume that in the economy $\mathcal{E}_n$ there are N_n *economic agents* labeled as $i = 1, ..., N_n$. We assume that N_n is of the order $O(n)$; namely,

$$N_n \leq An \text{ for some constant } A < \infty. \tag{2.1}$$

Let $(\Omega, P, \mathcal{F})$ be a probability space. We consider a double sequence of R^l-valued maps $\zeta_{in} : \Omega \times R^l_+ \to R^l$, $n = 1, 2, ...$, $i = 1, ..., N_n$, such that, for each fixed n, i and p, the function

$$\zeta_{in}(p) =_{\mathrm{def}} \zeta_{in}(\,\cdot\,, p) : \Omega \to R^l$$

is a random variable (viz. $\mathcal{F}$-measurable). $\zeta_{in}(p)$ is interpreted as the *(random) individual excess demand* by the i'th agent in $\mathcal{E}_n$ at the price p.

Example 2.1. *In a* Cobb-Douglas exchange economy *the individual excess demand by an agent $i \in \mathcal{E}_n$ on commodity j is given by the formula*

$$\zeta^j_{in}(p) = (p^j)^{-1} a^j_{in} \sum_{k=1}^{l} p^k e^k_{in} - e^j_{in},$$

where the parameters $a^j_{in} \geq 0$ satisfy

$$\sum_{j=1}^{l} a^j_{in} = 1 \textit{ for all } i \textit{ and } n,$$

and e^j_{in} denotes the agent's initial endowment *on the commodity j, see e.g. [10]. In a* random Cobb-Douglas exchange economy *the parameters a^j_{in} and e^j_{in} are supposed to be random variables.*

The *random total excess demand* in the economy $\mathcal{E}_n$ is obtained as the sum of the random individual excess demands:

$$Z_n(p) \;=\; \sum_{i=1}^{N_n} \zeta_{in}(p).$$

(In order to indicate its dependence on the size parameter n, we equip henceforth the total excess demand with the subscript n.) For a fixed economy $\mathcal{E}_n$ and for a fixed *realization* $\omega \in \Omega$, a price $p^*_n(\omega)$ at which the total excess demand function vanishes, i.e., such that

$$Z_n(\omega; p^*_n(\omega)) = 0,$$

is called an *equilibrium price for the realization ω in the economy $\mathcal{E}_n$*. We denote by $\pi^*_n(\omega)$ the set of equilibrium prices p^*_n for the realization ω in the economy $\mathcal{E}_n$.

Let

$$C_n(\alpha;p) = \log Ee^{\alpha\cdot Z_n(p)}, \ \alpha \in R^l,$$

denote the *cumulant generating function (c.g.f.)* of the random total excess demand $Z_n(p)$, $p \in R^l_+$, and let

$$c(\alpha;p) = \limsup_{n\to\infty} n^{-1}C_n(\alpha;p).$$

We denote

$$i(p) = - \inf_{\alpha\in R^l} c(\alpha;p)$$

and call it the *entropy* (associated with the price p). Note that, due to the fact that $c(0;p) \equiv 0$ it follows that $i(p) \geq 0$ always.

Recall that a c.g.f. is always a convex function. Consequently, $C_n(\alpha;p)$ as well as the limit $c(\alpha;p)$ are convex functions (of the variable α). Thus in particular, if

$$\frac{\partial c}{\partial \alpha}(\alpha(p);p) = 0 \text{ for some } \alpha(p) \in R^l, \text{ cf. the hypothesis (H1),} \tag{2.2}$$

then it follows that

$$i(p) = -c(\alpha(p);p). \tag{2.3}$$

The zeros p_e^* of the entropy function $i(p)$ will be called *expected equilibrium prices*:

$$i(p_e^*) = 0.$$

Under appropriate regularity conditions these are the same as the zeros of the *mean excess demand function* $\mu(p)$, defined by

$$\mu(p) = \lim_{n\to\infty} n^{-1}Z_n(p).$$

Proposition 2.1. *Suppose that*

(2.4) *there is a unique* $\alpha(p)$ *such that* $\frac{\partial c}{\partial \alpha}(\alpha(p);p) = 0$, *and*

(2.5) $c(\alpha;p)$, $\alpha \in R^l$ *is differentiable at* $\alpha = 0$.

Then

(2.6) $\mu(p) = \frac{\partial c}{\partial \alpha}(0;p)$, *and*

(2.7) $i(p) = 0$ *if and only if* $\mu(p) = 0$.

Proof of Proposition 2.1 That (2.5) implies (2.6) is a standard fact in LD theory (see e.g. [3]).

In order to prove (2.7) assume first that $i(p) = 0$, i.e.,

$$c(\alpha(p);p) = \min_{\alpha\in R^l} c(\alpha;p) = 0.$$

Since $c(0;p) \equiv 0$, it follows from the uniqueness of $\alpha(p)$ that $\alpha(p) = 0$. Therefore

$$\mu(p) = \frac{\partial c}{\partial \alpha}(0;p) = 0.$$

Suppose conversely that

$$\mu(p) = \frac{\partial c}{\partial \alpha}(0;p) = 0.$$

Again, due to uniqueness, $\alpha(p) = 0$ so that

$$i(p) = -c(\alpha(p);p) = -c(0;p) = 0,$$

indeed. □

Example 2.2. *Suppose that $N_n \equiv n$ and $\zeta_{in}(p) = \zeta_i(p)$ for $i = 1, ..., n$, where $\zeta_i(p)$, $i = 1, 2, ...$, is a sequence of i.i.d. random variables (for each fixed price p). In this case*

$$C_n(\alpha;p) \;\equiv\; nc(\alpha;p), \tag{2.8}$$

and therefore

$$c(\alpha;p) \;=\; \log Ee^{\alpha\cdot\zeta_1(p)}$$

is equal to the c.g.f. of the individual excess demand $\zeta_1(p)$. Moreover, due to the classical LLN for i.i.d. random variables, the mean excess demand is equal to the expectation of the individual excess demand:

$$\mu(p) = E\zeta_1(p).$$

Let us now fix a price $p \in R^l_+$. We formulate the following set of hypotheses. (The abbreviation "w.p.1" means the same as "with probability 1", and the phrase "eventually" means "for all sufficiently big n".)

(H1) $\exists \alpha = \alpha(p) \in R^l : \frac{\partial c}{\partial \alpha}(\alpha(p);p) = 0$;

(H2) $c(\alpha(p);p) = \lim_{n\to\infty} n^{-1}C_n(\alpha(p);p)$;

(H3) $\exists A_1(p) < \infty,\ \varepsilon_1(p) > 0 : |\zeta'_{in}(q)| \leq A_1(p)$ w.p.1, for all i and n, for $|q-p| < \varepsilon_1(p)$;

(H4) $\exists A_2(p) < \infty,\ \varepsilon_2(p) > 0 : |\zeta''_{in}(q)| \leq A_2(p)$ w.p.1, for all i and n, for $|q-p| < \varepsilon_2(p)$;

(H5) $\exists A_{-1}(p) < \infty : |(n^{-1}Z'_n(p))^{-1}| \leq A_{-1}(p)$ w.p.1., for all n.

Remarks.

(i) Condition (H4) implies condition (H3).

(ii) Suppose that $\zeta_{in}(p) = \zeta_i(p)$, where $\zeta_i(p)$, $i = 1, 2, ...$, are i.i.d. as before. Now, due to (2.8), the hypothesis (H2) is trivially true. Also it turns out that in this case hypothesis (H5) can be replaced by the simpler hypothesis

(H5') $\mu'(p)$ is non-singular,

see [9].

Theorem 2.1. *(i) Suppose that the hypotheses (H1-3) hold true. Then there exists a constant $M_0(p) < \infty$ such that*

$$P(\pi_n^* \cap U(p, \varepsilon) \neq \emptyset) < e^{-n(i(p) - M_0(p)\varepsilon)}$$

eventually, for all $0 < \varepsilon < \varepsilon_1(p)$.

(ii) Suppose that the hypotheses (H1-2,4-5) hold true. Then there exists a constant $M_1(p) < \infty$ such that

$$P(\pi_n^* \cap U(p, \varepsilon) \neq \emptyset) > e^{-n(i(p) + M_1(p)\varepsilon)}$$

eventually, for all $\varepsilon > 0$.

Let us call a price $p \in R_+^l$ *non-expected*, if the entropy $i(p) > 0$. Under the conditions (2.4-5) this is equivalent to p not being a zero of the mean excess demand $\mu(p)$:

$$\mu(p) \neq 0.$$

By using *Borel-Cantelli lemma* we obtain the following corollary of part (i) of the LD theorem:

Corollary 2.1. *Suppose that the hypotheses (H1-3) hold true. Let $p \in R_+^l$ be a non-expected price. Then*

$$\pi_n^* \cap U(p, \varepsilon) = \emptyset \textit{ eventually, w.p.1, for all } 0 < \varepsilon < \varepsilon_1(p).$$

3 Proof of the LD theorem

For the proof of the upper bound (i) we need two lemmas. The first is of standard type in LD theory.

We define the following sequence of probability measures:

$$P_{n;p}(d\omega) = e^{\alpha(p) \cdot Z_n(\omega;p) - C_n(\alpha(p);p)} P(d\omega), \quad n = 1, 2, ...$$

Lemma 3.1. *Suppose that hypotheses (H1-2) hold true. Then for each $\delta > 0$, there exists a constant $\eta = \eta(\delta; p) > 0$ such that*

$$P_{n;p}(|Z_n(p)| \geq n\delta) < e^{-n\eta(\delta;p)} \textit{ eventually.}$$

Proof of Lemma 3.1. Let $t > 0$ be arbitrary. By Chebyshev's inequality we have for the j'th component of the total excess demand:

$$\begin{aligned} P_{n;p}(Z_n^j(p) \geq n\delta) &\leq e^{-tn\delta} E_{n;p} e^{tZ_n^j(p)} \\ &= e^{C_n(\alpha(p)+te_j;p)-C_n(\alpha(p);p)-n\delta t} \end{aligned}$$

where e_j denotes the j'th unit vector in R^l. Due to (H1) and (H2),

$$\begin{aligned} \limsup_{n\to\infty} n^{-1} \log P_{n;p}(Z_n^j(p) \geq n\delta) &\leq c(\alpha(p)+te_j;p) - c(\alpha(p);p) - \delta t \\ &= \delta(t)t - \delta t \end{aligned}$$

where $\delta(t) \to 0$ as $t \to 0$. By choosing t small enough we thus see that

$$\limsup_{n\to\infty} n^{-1} \log P_{n;p}(Z_n^j(p) \geq n\delta) < 0.$$

By symmetry, we have also

$$\limsup_{n\to\infty} n^{-1} \log P_{n;p}(Z_n^j(p) \leq -n\delta) < 0,$$

which completes the proof of Lemma 1. □

Lemma 3.2. *Suppose that the hypotheses (H1-2) hold true. Then, for all $\delta > 0$, we have:*

$$e^{-n(i(p)+2|\alpha(p)|\delta)} < P(|Z_n(p)| < n\delta) < e^{-n(i(p)-2|\alpha(p)|\delta)} \text{ eventually.}$$

Proof of Lemma 3.2. Recalling (2.3) we see that it suffices to prove that

$$\limsup_{n\to\infty} |n^{-1} \log P(|Z_n(p)| < n\delta) - c(\alpha(p);p)| \leq |\alpha(p)|\delta. \tag{3.1}$$

Due to Lemma 1,

$$\frac{1}{2} < 1 - e^{-n\eta(\delta;p)} < P_{n;p}(|Z_n(p)| < n\delta) \leq 1 \text{ eventually,}$$

and hence, in view of the definition of the probability measure $P_{n;p}(\cdot)$:

$$\frac{1}{2} < e^{-C_n(\alpha(p);p)} E(e^{\alpha(p)\cdot Z_n(p)}; |Z_n(p)| < n\delta) \leq 1 \text{ eventually.}$$

Now clearly,

$$|\log E(e^{\alpha(p)\cdot Z_n(p)}; |Z_n(p)| < n\delta) - \log P(|Z_n(p)| < n\delta)| \leq |\alpha(p)|n\delta,$$

whence

$$-\log 2 - |\alpha(p)|n\delta < \log P(|Z_n(p)| < n\delta) - C_n(\alpha(p);p) \leq |\alpha(p)|n\delta \text{ eventually,}$$

from which the claim (3.1) follows by letting $n \to \infty$. □

Now we are able to prove the upper bound inequality (i).

To this end, note first that, due to the hypotheses (2.1), (H3) and the *mean value theorem*, we can conclude that the event

$$\pi_n^* \cap U(p,\varepsilon) \neq \emptyset$$

implies the event

$$|Z_n(p)| \leq AA_1(p)n\varepsilon \text{ w.p.1, for all } n \geq 1,\ 0 < \varepsilon < \varepsilon_1(p).$$

Thus, in view of Lemma 2,

$$P(\pi_n^* \cap U(p,\varepsilon) \neq \emptyset) \leq P(|Z_n(p)| < AA_1(p)n\varepsilon) < e^{-n(i(p)-M_0(p)\varepsilon)} \text{ eventually,}$$

where the constant $M_0(p) \; = \; 2AA_1(p)|\alpha(p)|$.

For the lower bound we need the following lemma which is a straightforward corollary of Theorem XIV in [6].

Lemma 3.3. *Suppose that $f : R_+^l \to R^l$ has bounded second derivative in an ε-neighborhood of the price p:*

$$|f''(q)| \leq M < \infty \text{ for } |q-p| < \varepsilon.$$

Moreover, suppose that the derivative $f'(p) \in R^{l\times l}$ is non-singular, and

$$|f'(p)^{-1}| < \min\{\frac{\varepsilon}{2|f(p)|}, \frac{1}{4M\varepsilon}\}.$$

Then

$$f(q) = 0 \;\; \text{for some } |q-p| < \varepsilon.$$

*Proof of Lemma 3.3*Let

$$g(h) \;=\; f'(p)^{-1}(f(p+h)-f(p)),\ |h| < \varepsilon.$$

Then $g(0) = 0$, $g'(0) = I$ (= the identity), and

$$|g''(h)| \leq M|f'(p)^{-1}|.$$

It follows that

$$|g'(h_1) - g'(h_2)| < 2\varepsilon M|f'(p)^{-1}| < \frac{1}{2}.$$

Let

$$z \doteq -f'(p)^{-1}f(p).$$

Then $|z| \;=\; |f'(p)^{-1}||f(p)| \;<\; \frac{\varepsilon}{2}$ and hence by setting $s \;=\; \frac{1}{2}$ in [L: Lemma XIV.1.3] we can conclude that there exists a unique $|h| < \varepsilon$ satisfying $g(h) = z$, viz. $f(p+h) = 0$.

Now we are able to prove the lower bound inequality (ii). To this end, let

$$f(p) = n^{-1} Z_n(p)$$

in Lemma 3. Due to (H4) and (H5), we have

$$M = A_2(p)$$

and

$$|f'(p)^{-1}| \leq A_{-1}(p).$$

Note that, by monotonicity, it suffices to prove the assertion for small $\varepsilon > 0$ only. Thus we may assume that

$$\varepsilon < \min \left\{ \varepsilon_2(p), \frac{1}{4A_2(p)A_{-1(p)}} \right\},$$

where $\varepsilon_2(p)$ is as in (H4). Now, in view of Lemma 3 it follows that, if

$$|n^{-1} Z_n(p)| < \frac{\varepsilon}{2A_{-1}(p)},$$

then

$$n^{-1} Z_n(q) = 0 \ \text{ for some } |q - p| < \varepsilon,$$

viz.

$$\pi_n^* \cap U(p, \varepsilon) \neq \emptyset.$$

Finally, by Lemma 2

$$\begin{aligned} P(\pi_n^* \cap U(p,\varepsilon) \neq \emptyset) &\geq P(|n^{-1} Z_n(p)| < \frac{\varepsilon}{2A_{-1}(p)}) \\ &> e^{-n(i(p) + M_1(p)\varepsilon)} \text{ eventually,} \end{aligned}$$

where the constant

$$M_1(p) \ = \ |\alpha(p)| \frac{1}{A_{-1}(p)}.$$

This completes the proof of the theorem.

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