ON CERTAIN SERIES EXPANSIONS INVOLVING WHITTAKER FUNCTIONS AND JACOBI POLYNOMIALS

PETER HENRICI

1. INTRODUCTION

1. 1. Outline of the paper. By substituting polar coordinates in the partial differential equation

$$(1)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{4\mu+1}{x}\frac{\partial u}{\partial x}+\frac{4\nu+1}{y}\frac{\partial u}{\partial y}+k[4\lambda-k(x^2+y^2)]u=0\ ,$$

and separating variables, one is led in a natural way to certain combinations of Whittaker functions and Jacobi polynomials (called for brevity J.-W. functions in this paper). With a view towards deriving some functional relations involving hypergeometric functions, we develop in the first part of the paper a technique for the construction of expansions of arbitrary regular analytic solutions of (1) in terms of these J.-W. functions. The method of our investigation consists in setting up a one-to-one correspondence between the class of even analytic functions of one complex variable regular in a circle around the origin and a certain class E of regular solutions of (1). This correspondence associates with a solution $u(x, y) \in E$ the function $u(x, -ix)$ obtained by considering $u(x, y)$ on the (imaginary) characteristic $x-iy=0$ of (1).[1] Since the maps of the even powers of a single variable in this correspondence are shown to be the J.-W. functions mentioned above, the expansion problem in question is reduced to the problem of finding the Taylor expansion of a given analytic function of one variable.

Applying this technique to some special solutions of (1), we are led to three expansions involving various kinds of hypergeometric functions. The first of them contains a number of well-known theorems on special functions as special cases, namely, among others, Bateman's addition theorem in the theory of Bessel functions, Ramanujan's formula for the product of two confluent hypergeometric series, and Erdélyi's addition theorem (with respect to the parameters) for the product of two M-functions. The second application gives rise to

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[1] This procedure is related to Bergman's operator method in the theory of elliptic partial differential equations with regular coefficients; see the remark at the end of § 4.