FOURIER ANALYSIS AND DIFFERENTIATION OVER REAL SEPARABLE HILBERT SPACE

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1. Introduction. Let l_2 denote as usual the space of square summable real sequences, the prototype of real separable Hilbert space. It is well known that l_2 possesses no non-trivial, translation invariant Borel measures. However, l_2 does have infinitely many subspaces X, locally compact in the l_2 norm relative topology, which we may call translation spaces and for which such measures φ exist [2]. Here the spaces X are not groups under l_2 vector addition, so the notion of translation invariance must be appropriately modified. For any such X we may of course use the corresponding φ to define over $z \in l_2$ a Fourier transform F of $f \in L_1(X, \mathscr{B}, \varphi)$ by

$$F(z)=\int_X f(x)e^{i(z,x)}d\varphi(x) .$$

However, in order to get the expected inverse formula, it seems necessary to be able to make X into a group—roughly speaking to define a vector in X corresponding to $x+y$ when this l_2 vector sum $\notin X$. This is a severe restriction on our translation spaces X, and the only natural ones still available seem to be essentially modifications of Jessen's infinite torus [9]. With orthogonal coordinates this is the space X_0 defined below, a modified Hilbert cube.

Since X_0 is a locally compact abelian topological group, Fourier analysis upon it becomes standard procedure. We are able to extend some standard one-variable theorems (see [1]), relating Fourier transforms and the operation of differentiation, to the situation here, which seems new. In a summary at the end we discuss the significance of these results as related to the work in functional analysis of Fréchet, Gâteaux, Lévy, Hille, Zorn, Cameron and Martin, and Friedrichs.

2. Fourier integrals on X_0. Let

$$X_0=\{x \in l_2 \mid -h_n<x_n\leq h_n \text{ for integer } n\geq 1\}$$

where the fixed sequence of extended real h_n, $0<h_n\leq+\infty$, has

$$\sum_{n=N+1}^{\infty} h_n^2<+\infty$$

for some fixed integer $N\geq 0$. For simplicity we assume $h_n=+\infty$ for

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