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FOURIER TRANSFORMS AND THEIR LIPSCHITZ CLASSES

G. SAMPSON AND H. TUY

We define a class of functions A_α for each $\alpha > 0$. We show that the Fourier transform of every function of A_α exists and is Lipschitz of order α. We construct examples to show that the converse is not true in general. However, we show that for a certain class of function k (e.g., $k \in L_2$) if its Fourier transform $\hat{k}$ is Lipschitz of order α then $k \in A_\beta$ for all $\beta < \alpha$.

Boas ([1] and [2]) studied this problem in the case where the function k is nonnegative and gave a complete solution in this case. In connection with this question several authors (e.g. Hirschman [5]; Liang Shin Hahn [4], Drobot, Naparstek and Sampson [3]) have proved mapping properties of convolution operators with kernel k, by studying the behavior of $\hat{k}$. To be more precise, they have proved mapping theorems when $\hat{k} \in \mathrm{Lip}(\alpha)$ with additional conditions on k. In our applications we prove a similar result (see §3, Theorem 4).

Notations and Definitions.

L_{loc} shall denote the set of all Lebesgue measurable functions integrable over all finite intervals. In this paper, the functions $f, g, k, \cdots \in L_{\text{loc}}$.

For $0 < \alpha \leqq 1$, a function f is Lipschitz of order α ($f \in \mathrm{Lip}(\alpha)$) if there is a positive constant A such that

$$\sup_{x \in \mathbf{R}} |f(x+h) - f(x)| \leqq A\,|h|^\alpha.$$

For $\alpha > 1$, we say that a function f is Lipschitz of order α if
(i) $f^{(m)} \in L_\infty$ for all $m < [\alpha]$ and
(ii) $f^{([\alpha])} \in \mathrm{Lip}(\alpha - [\alpha])$.
When we use the symbol

$$\int_a^b g(t, x)\,dt \quad \text{for} \quad -\infty \leqq a < b \leqq \infty.$$

We are assuming that $g(t, x) \in L_{\text{loc}}$ as a function of t for each x and moreover the integral exists in the following sense:

$$(0) \qquad \int_a^b g(t, x)\,dt = \lim_{\substack{\alpha \to a \\ \beta \to b}} \int_\alpha^\beta g(t, x)\,dt.$$