# EXTENSION OF FUNCTIONS ON FULLY NORMAL SPACES

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1. **Introduction.** Starting from the recent discovery of A. H. Stone that metric spaces are "paracompact" [12] (paracompactness means that every open covering has a refinement only a finite number of whose members meet a suitable neighborhood of each point [5]), J. Dugundji has been able to extend to metric spaces certain techniques in the theory of retracts which were hitherto applicable at most to separable metric spaces [6]. The cornerstone of his method is a theorem (see 2.4, below) according to which a continuous function on a closed set $A$ of a metric space $X$ with values in a convex (= "locally convex") topological linear space $L$ may be extended to the whole space $X$, indeed without enlarging the convex hull of the image. Essentially, the possibility of doing this for a locally separable metric space $X$ is implicitly given by a procedure for the real valued case in [10].

One of the problems to which we address ourselves in this paper is that of determining whether the assumption that $X$ is metric can be reduced to $X$ is merely paracompact. The answer (see 6, below) is *no*. However, we have fairly general results which imply that if $L$ is metric and complete (and $X$ is paracompact) then the extension is possible (4.1, below). Our proof utilizes a process of extending a pseudo-metric on $A$ to all of $X$, which is ultimately based on a theorem of Hausdorff. We generalize Hausdorff's theorem (3.2 and 3.4) and incidentally show how Dugundji's result enables one to construct a short proof of Hausdorff's theorem.

None of these extension theorems can properly be regarded as a true generalization of Tietze's extension theorem, which deals with mappings on normal spaces with values on the line or in the Hilbert cube, since there exist normal, not fully normal spaces. In order to provide a generalization of Tietze's theorem, we have shown by way of application that the Hilbert cube may be replaced by any compact convex subset of a normed linear space (4.3).

S. Kakutani [10] has introduced the notion of "simultaneous extension regarded as a linear positive operation," in his case of real valued functions on a locally separable metric space $X$: this means that it is possible so to extend

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