MULTIPLICATIVE CLOSURE AND THE WALSH FUNCTIONS

PAUL CIVIN

1. **Introduction**. In his memoir [1] *On the Walsh functions,* N. J. Fine utilized the algebraic character of the Walsh functions by interpreting them as the group characters of the group of all infinite sequences of 0's and 1's, the group operation being addition mod 2 of corresponding elements. In this note, we investigate some properties of real orthogonal systems which are multiplicatively closed. We show that any infinite system of the stated type is isomorphic with the group of the Walsh functions. Furthermore, under hypotheses stated in Theorem 3, there is a measurable transformation of the interval $0 \leq x < 1$ into itself, which carries the Walsh functions into the given system of functions.

As is well known, the Walsh functions are linear combinations of the Haar functions. B. R. Gelbaum [2] gave a characterization of the latter functions in which the norm of certain projection operators and the linear closure in L of the set of functions played an essential role. Although the characteristic features of the Walsh functions are, aside from orthogonality, totally distinct from those for the Haar functions, there is some similarity of proof technique in establishing the characterization.

For the sake of completeness, we define the Rademacher functions $\{\phi_n(x)\}$ as follows:

$$(1.1)\qquad \phi_0(x) = \begin{cases} 1 \ (0 \leq x < 1/2) \\ -1 \ (1/2 \leq x < 1) \end{cases} ; \quad \phi_0(x+1) = \phi_0(x).$$

$$(1.2)\qquad \phi_n(x) = \phi_0(2^n n) \qquad (n = 1, 2, \cdots).$$

The Walsh functions, as ordered by Paley [3], are then given by

$$(1.3)\qquad \psi_0(x) = 1,$$

and

$$(1.4)\qquad \text{if } n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k} \qquad (n_1 < n_2 < \cdots < n_k), \qquad \text{then}$$

Received September 21, 1951.
Pacific J. Math. 2(1952), 291-295