

TRANSLATION INVARIANT MEASURE OVER SEPARABLE HILBERT SPACE AND OTHER TRANSLATION SPACES

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1. **Introduction.** We consider the problem of defining a nontrivial, translation-invariant Borel measure over real separable Hilbert space. As noted by Loewner [4], this is not possible; but instead of relinquishing as he does the real number system for a non-Archimedean ordered field for the values of a "measure," we shall consider several topological subspaces of Hilbert space arising frequently in analysis. These are locally compact; and using either the Kolmogoroff stochastic processes construction [2], or else following the Haar measure construction [1] or [5], we can get a nontrivial, essentially translation-invariant Borel measure. However, since the special subspaces considered are not groups under translation, and do not even contain a group germ, the usual Haar measure construction must be modified in a special fashion, and the precise translation invariance obtained is somewhat restrictive. Actually we carry through this modified Haar measure construction for the more general situation of a locally compact translation space, which is defined as an appropriate subspace of an Abelian topological group. The results are collected in a summary at the end.

2. **Formulation of the problem.** Let

$$\ell_2 = \left\{ x = \{x_n\} \mid \sum_{n=1}^{\infty} (x_n)^2 < +\infty, x_n \text{ real} \right\},$$

the square summable real sequences and thus the real separable Hilbert space prototype. Since ℓ_2 is a subset of R_∞ , the countably infinite Cartesian product of the real line $(-\infty, \infty)$, we have available on ℓ_2 as well as the ℓ_2 norm metric topology also the product topology defined relatively from R_∞ . Under these two topologies we shall consider the ℓ_2 -subsets

$$X = \{ x \in \ell_2 \mid |x_n| \leq h(n) \text{ for all } n \},$$

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