VOLUME IN TERMS OF CONCURRENT CROSS-SECTIONS

HERBERT BUSEMANN

1. Of the two expressions

$$|M| = \frac{1}{2}\int_0^{2\pi} r^2(\omega)\,d\omega = \frac{1}{2}\int_0^{2\pi}\left(\int_{-r(\omega-\pi/2)}^{r(\omega+\pi/2)} |\rho|\,d\rho\right)d\omega$$

for the area $|M|$ of a plane domain M, given in polar coordinates ρ, ω by the inequalities $0 \leq \rho \leq r(\omega)$, $0 \leq \omega \leq 2\pi$, the first has the well-known extension

$$(1) \qquad |M| = \frac{1}{n}\int_{\Omega_n} r^n(u)\,d\omega_u^n$$

to n dimensions. Here Ω_n is the surface of the unit sphere in the n-dimensional Euclidean space, $d\omega_u^n$ is its area element at the point u, and M is given by $0 \leq \rho \leq r(u)$, $u \in \Omega_n$.

In the second expression, $|\rho|$ may be interpreted as (1-dimensional) volume of the simplex with one vertex at the origin z and the other at a variable point $p = (\rho, \omega \pm \pi/2)$ in the cross-section of M with the line normal to ω. The purpose of the present note is *the proof and the application of the following extension of this second expression to* $n-1$ *sets* $M_1, \cdots, M_{n-1}$ *in* E_n:

$$(2) \qquad |M_1| \cdots |M_{n-1}|$$

$$= \frac{(n-1)!}{2}\int_{\Omega_n}\left(\int_{M_1(u)} \cdots \int_{M_{n-1}(u)} T(p_1, \cdots, p_{n-1}, z)\,dV_{p_1}^{n-1} \cdots dV_{p_{n-1}}^{n-1}\right) d\omega_u^n.$$

Here $M_j(u)$ is the cross-section of M_j with the hyperplane $H(u)$ through z normal to the unit vector u, the point p_j varies in $M_j(u)$, the differential $dV_{p_j}^{n-1}$ is the ($(n-1)$-dimensional) volume element of $M_j(u)$ at p_j, and $T(p_1, \cdots, p_{n-1}, z)$ is the volume of the simplex with vertices $p_1, \cdots, p_{n-1}, z$.

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